CME241

Saakaar Bhatnagar

Assignment-1 Solution

1 Defining the MDP

We can write the stochastic differential equation for the wealth process as

$$dW_t = (\pi_t * (\mu - r) + r) * W_t * dt + \sigma * \pi * W_t * dB_t$$

Defining the MDP:

Rewards=

$$0, t \neq T$$

$$-e^{-aW_T}/a, t = T$$

States= t, W_t to $(t+1, W_{t+1})$;
with a discount factor of γ

2 Bellman Optimality Condition

We see that the Bellman Optimality recursive equation is:

$$V^*(t, W_t) = \max_{x_t}(E[R(t, W_t) + \gamma V^*(t+1, W_{t+1})])$$

for $t \neq T$,

$$V^*(t, W_t) = \gamma * \max_{x_t}(E[V^*(t+1, W_{t+1})])$$

3 Subsitute Solution

We use the wealth process:

$$W_{t+1} = (1+s)x_t + (1+r)(W_t - x_t)$$

where s $\sim N(\mu, \sigma^2)$
Substituting $V^* = -b_t \exp\{-c_t W_t\}$,

$$-b_t \exp\{-c_t W_t\} = \gamma \max_{x_t}[E[-b_{t+1} \exp\{-c_{t+1}((1+s)x_t + (1+r)(W_t - x_t))\}]]$$

Simplifying, we get:

$$RHS = \gamma \max_{x_t}[-b_{t+1} \exp\{-c_{t+1}\}(1+r)(W_t - x_t) \exp\{-c_{t+1}(1+r)x_t\}]$$

;

The computed expectation is:

$$\exp\big\{((c_{t+1}x_t\sigma)^2 - 2c_{t+1}x_{t+1}(1+\mu))/2\big\}$$

Putting the expectation in the main formula, differentiate w.r.t x_t to optimize RHS:

$$(1+r) + (c_{t+1}x_t\sigma^2) - (1+\mu) = 0$$

Simplifying,

$$x_t^* = (\mu - r)/(c_{t+1}\sigma^2)$$

Therefore, the net equation is:

$$-b_t \exp\{-c_t W_T\} = -\gamma b_{t+1} \exp\big\{-c_{t+1}(1+r)W_t - (\mu - r)^2/(2\sigma^2)\big\}$$

Therefore, by term comparison:

$$b_t = -\gamma b_{t+1} \exp\big\{-(\mu - r)^2/2\sigma^2\big\}$$

$$c_t = c_{t+1}(1+r)$$

Now at t=T:

$$b_T = 1/a$$

$$c_T = a$$

Therefore, if we follow the recursive steps:

$$b_t = (-\gamma \exp\big\{-(\mu - r)^2/2\sigma^2\big\})^{T-t}/a$$

$$c_t = (1+r)^{T-t}a$$

$$x_t = (\mu - r)/\sigma^2(1+r)^{T-t}a$$