

CME241

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Assignment-1 Solution

1 Defining the MDP

Defining the MDP:

Rewards=

$$\begin{aligned} &0, t \neq T \\ &-e^{-aW_T}/a, t = T \end{aligned}$$

States= t, W_t to $(t+1, W_{t+1})$;
with a discount factor of γ

2 Bellman Optimality Condition

We see that the Bellman Optimality recursive equation is:

$$V^*(t, W_t) = \max_{x_t} (E[R(t, W_t) + \gamma V^*(t+1, W_{t+1})])$$

for $t \neq T$,

$$V^*(t, W_t) = \gamma * \max_{x_t} (E[V^*(t+1, W_{t+1})])$$

3 Substitute Solution

We use the wealth process:

$$W_{t+1} = (1+s)x_t + (1+r)(W_t - x_t)$$

where $s \sim N(\mu, \sigma^2)$

Substituting $V^* = -b_t \exp\{-c_t W_t\}$,

$$-b_t \exp\{-c_t W_t\} = \gamma \max_{x_t} [E[-b_{t+1} \exp\{-c_{t+1}((1+s)x_t + (1+r)(W_t - x_t))\}]]$$

Simplifying, we get:

$$RHS = \gamma \max_{x_t} [-b_{t+1} \exp\{-c_{t+1}\}(1+r)(W_t - x_t) \exp\{-c_{t+1}(1+r)x_t\}]$$

;

The computed expectation is:

$$\exp\{((c_{t+1}x_t\sigma)^2 - 2c_{t+1}x_{t+1}(1+\mu))/2\}$$

Putting the expectation in the main formula, differentiate w.r.t x_t to optimize RHS:

$$(1+r) + (c_{t+1}x_t\sigma^2) - (1+\mu) = 0$$

Simplifying,

$$x_t^* = (\mu - r)/(c_{t+1}\sigma^2)$$

Therefore, the net equation is:

$$-b_t \exp\{-c_t W_T\} = -\gamma b_{t+1} \exp\{-c_{t+1}(1+r)W_t - (\mu - r)^2/(2\sigma^2)\}$$

Therefore, by term comparison:

$$\begin{aligned} b_t &= -\gamma b_{t+1} \exp\{-(\mu - r)^2/2\sigma^2\} \\ c_t &= c_{t+1}(1+r) \end{aligned}$$

Now at t=T:

$$\begin{aligned} b_T &= 1/a \\ c_T &= a \end{aligned}$$

Therefore, if we follow the recursive steps:

$$b_t = (-\gamma \exp\{-(\mu - r)^2/2\sigma^2\})^{T-t}/a$$

$$c_t = (1+r)^{T-t}a$$

$$x_t = (\mu - r)/\sigma^2(1+r)^{T-t}a$$