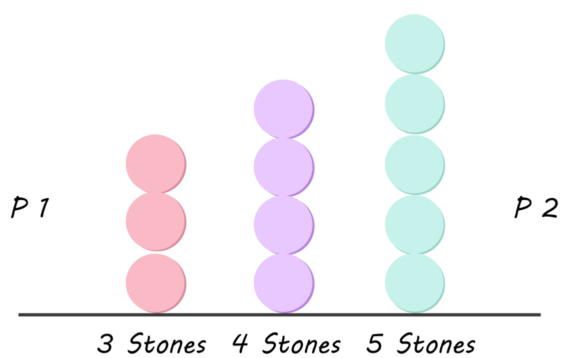
Proof of Nim Game & Sprague Grundy Theorem

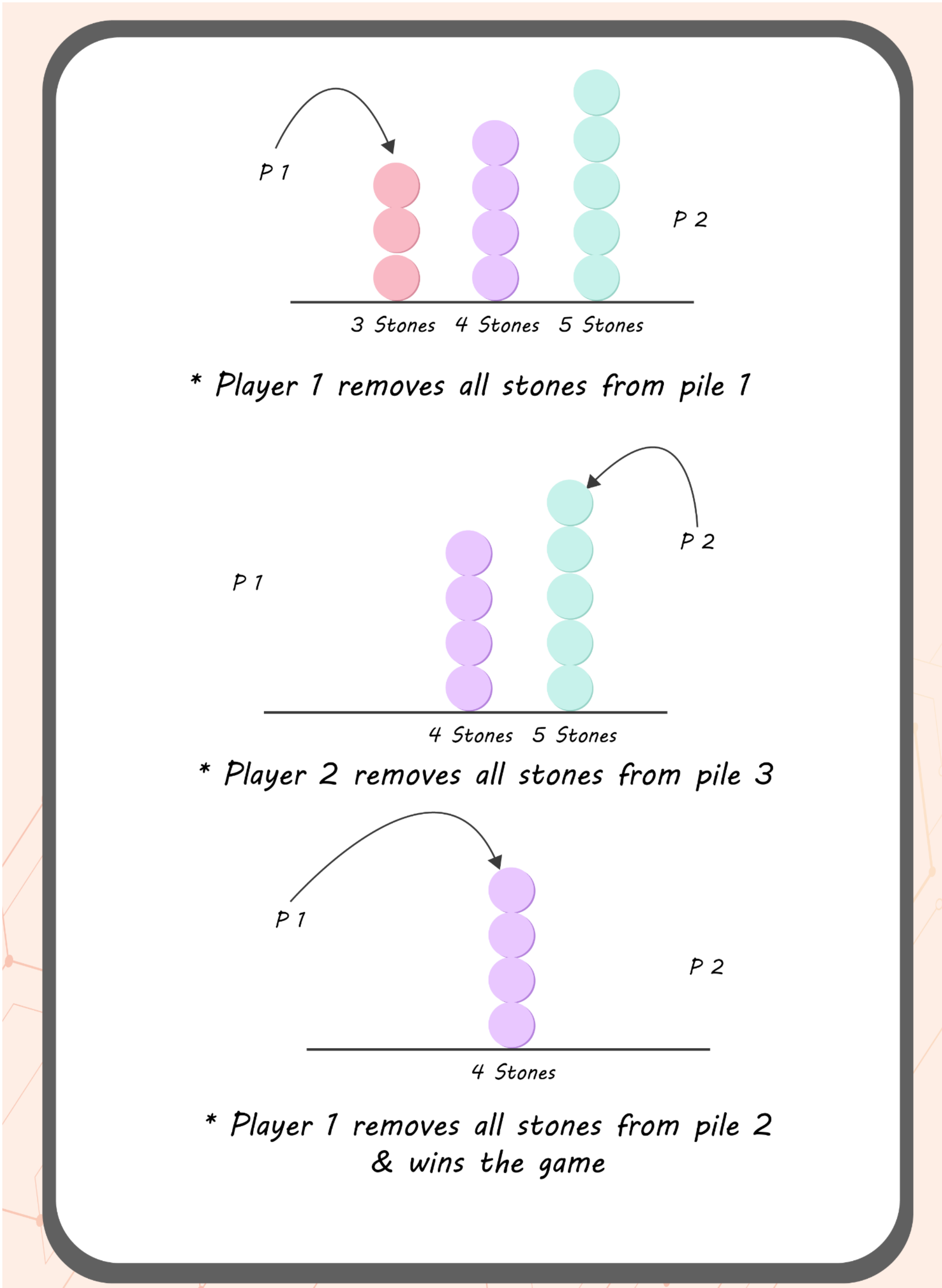
Nim Game

In this game, we are given numerous piles of stones and there are two players. The objective of the game is to remove a certain number of stones from one pile at a time in one move. The player may choose how many stones he/she wants to remove at their own convenience. The player who removes the last stone wins the game.

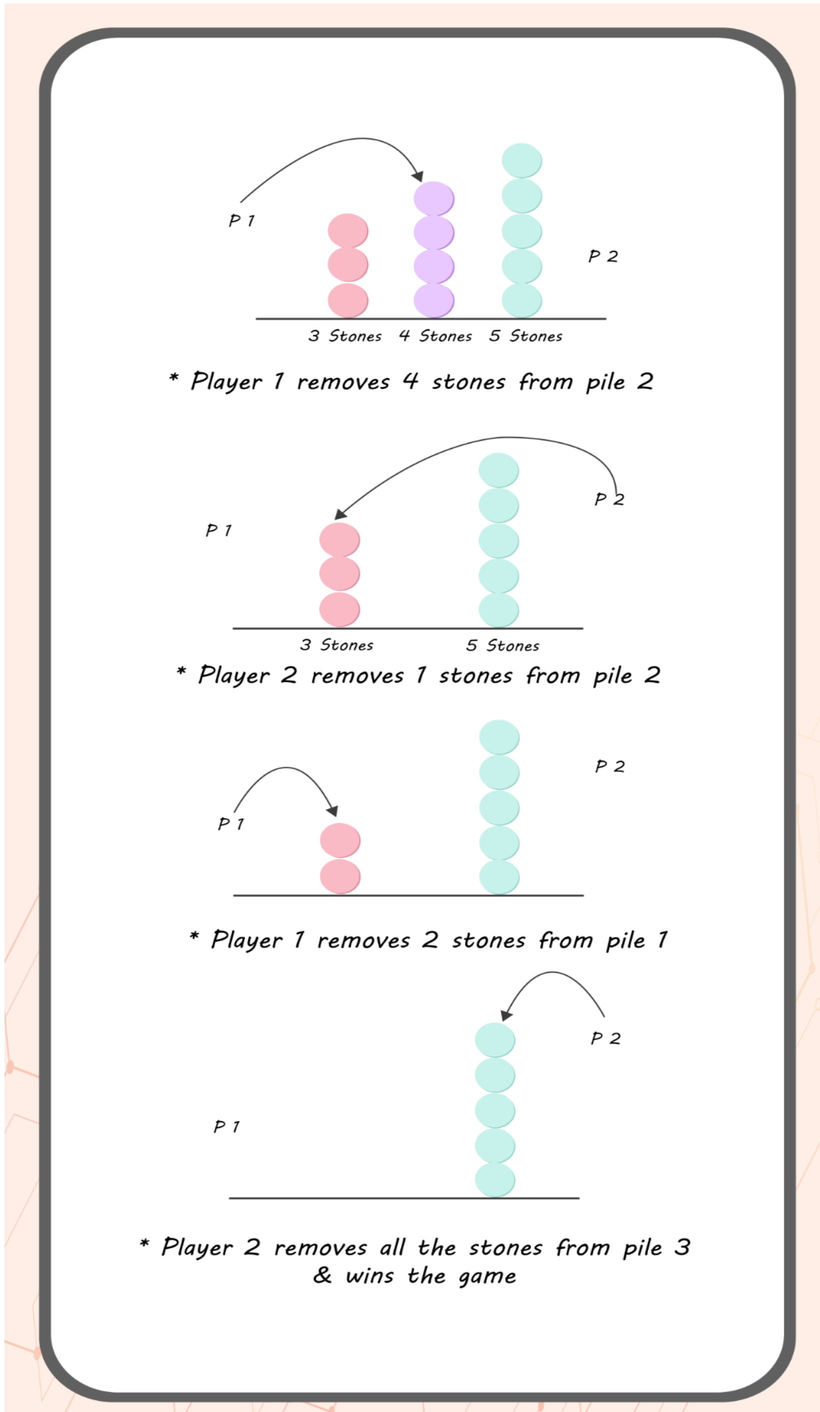
Let's take a look at the image given below for an example:



Now let's discuss one of the many possibilities in which **player 1 wins** the game.



Now let's discuss one of the many possibilities in which **player 2 wins** the game.


P 1
P 2
3 Stones
4 Stones
5 Stones
* Player 1 removes 4 stones from pile 2
P 1
P 2
3 Stones
5 Stones
* Player 2 removes 1 stones from pile 2
P 1
P 2
* Player 1 removes 2 stones from pile 1
P 1
P 2
* Player 2 removes all the stones from pile 3
& wins the game

Nim Formula:

Game theory says that we can predict the outcome of the game even before it is played.

We have to calculate a **Nim Sum** which is equal to the **cumulative XOR of the stones in each pile at the initial stage.** If the value of Nim Sum is **zero** and player 1 and player 2 are both playing optimally, then player 1 always loses. If the Nim Sum is **non-zero** and player 1 and player 2 are both playing optimally then player 1 always wins.

In the above case Nim Sum = 3 ^ 4 ^ 5.

Proof. The key to the proof is the presence of a symmetric **strategy for the opponent**. We show that a once in a position with the xor-sum equal to zero, the player won't be able to make it non-zero in the long term — if they transition to a position with a non-zero xor-sum, the opponent will always have a move returning the xor-sum back to zero.

We will prove the theorem by mathematical induction.

For an empty Nim (where all the piles are empty i.e. the multiset is empty) the xor-sum is zero and the theorem is true.

Now suppose we are in a non-empty state. Using the assumption of induction (and the acyclicity of the game) we assume that the theorem is proven for all states reachable from the current one.

Then the proof splits into two parts: if for the current position the xor-sum, $s=0$, we have to prove that this state is losing, i.e. all reachable states have xor-sum $t\neq0$. If $s\neq0$, we have to prove that there is a move leading to a state with $t=0$.

- Let $s=0$ and let's consider any move. This move reduces the size of a pile x to a size y. Using elementary properties $\oplus$, we have

$$t=s\oplus x\oplus y=0\oplus x\oplus y=x\oplus y$$

Since $y<x$, $y \oplus x$ can't be zero, so $t \neq 0$. That means any reachable state is a winning one (by the assumption of induction), so we are in a losing position.

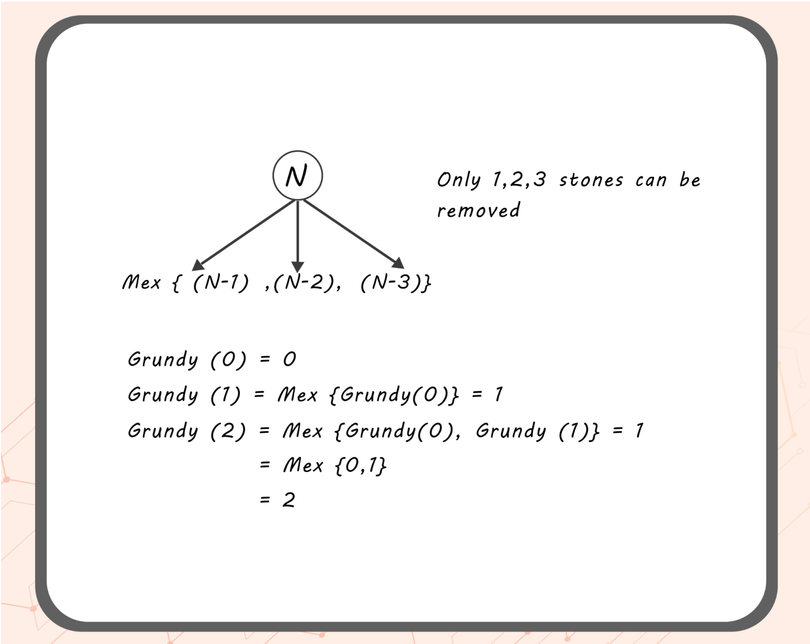
- Let $s \neq 0$. Consider the binary representation of the number s. Let d be the number of its leading (biggest value) non-zero bit. Our move will be on a pile whose size's bit number d is set (it must exist, otherwise the bit wouldn't be set in s). We will reduce its size x to $y=x \oplus s$. All bits at positions greater than d in x and y match and bit d is set in x but not set in y. Therefore, $y<x$, which is all we need for a move to be legal. Now we have:

$$t=s \oplus x \oplus y=s \oplus x \oplus (s \oplus x)=0$$

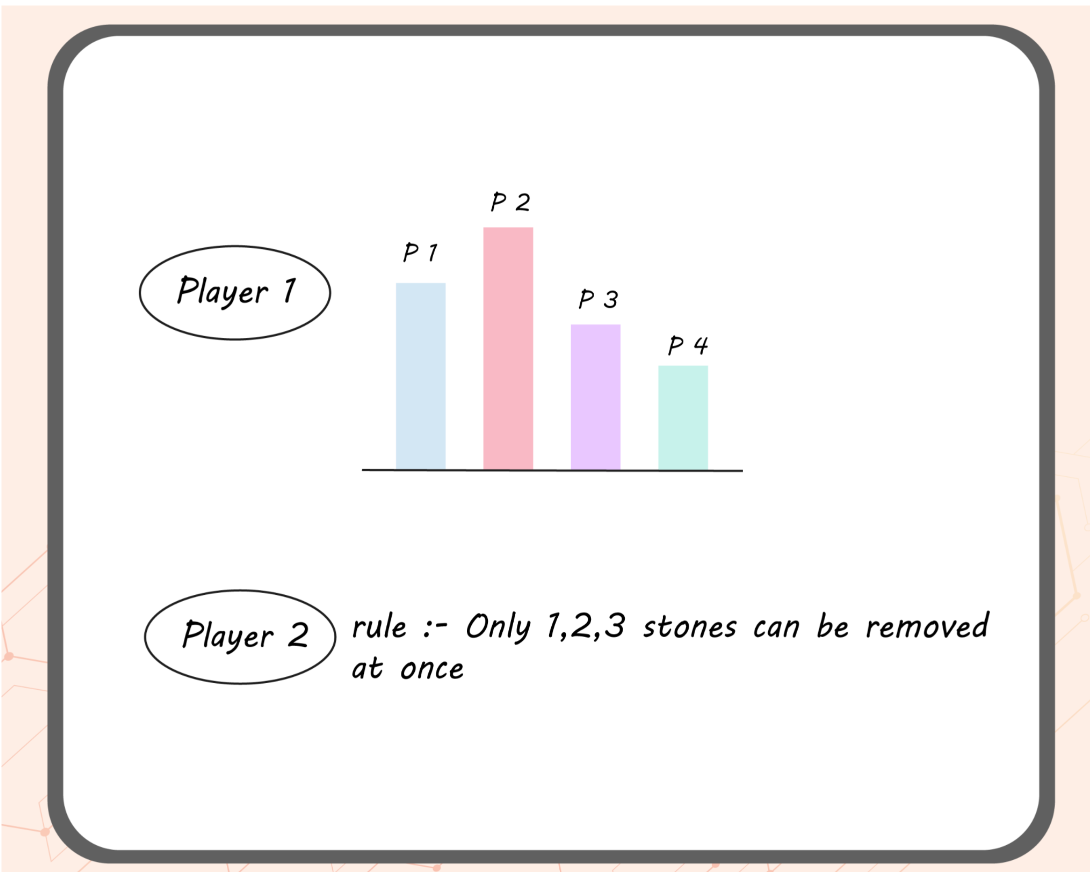
This means we found a reachable losing state (by the assumption of induction) and the current state is winning.

Sprague Grundy Theorem

We have already studied the game of nim and how the value of Grundy(N) is calculated but what if the rules of the games are modified such that in one turn a player can only remove 1, 2, or 3 stones.



In such a game it would be very difficult to calculate the outcome of this game. So Sprague- Grundy Theorem states that instead of taking the cumulative XOR of the piles of stones initially present, we take the cumulative XOR of the grundy value of the piles.



If **Cumulative XOR (Grundy (p1), Grundy (p2), Grundy (p3), Grundy (p4))** is Non-Zero, then Player 1 always wins. Else, Player 2 always wins.

Proof. We will use proof by induction.

For vertices without a move, the value x is the mex (minimum excludent) of an empty set, which is zero. That is correct since an empty Nim is losing.

Now consider any other vertex v. By induction, we assume the values x_i corresponding to its reachable vertices are already calculated.

Let *p = mex of {x1,…,xk}*. Then we know that for any integer $i \in [0,p)$ there exists a reachable vertex with Grundy value i. This means v is **equivalent to a state of the game of Nim with increases with one pile of size** p. In such a game we have transitioned to piles of every size smaller than p and possibly transitions to piles with sizes greater than p. Therefore, p is indeed the desired Grundy value for the currently considered state.

References

https://cp-algorithms.com/game_theory/sprague-grundy-nim.html