Numerical Analysis

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Chapter 1

Introduction

1.1 Why Numerical Analysis?

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1.3 Condition Number of a Problem

Consider a function in one variable $f : \mathbb{R} \to \mathbb{R}$. Condition number for a function $f(x)$ tells about the error amplification of a function $f(x)$ i.e., for a given error in input x, how much is the error in the output $f(x)$.

Absolute Condition Number κ_{abs} of the function $f(x)$ is defined as:

$$\kappa_{\text{abs}} = \frac{\text{Absolute Change in Output}}{\text{Absolute Change in Input}} = \lim_{\delta x \to 0} \left| \frac{f(x+\delta x) - f(x)}{x + \delta x - x} \right| = |f'(x)| \quad (1.1)$$

Relative Condition Number κ_r of the function $f(x)$ is defined as:

$$\kappa_r = \frac{\text{Relative Change in Output}}{\text{Relative Change in Input}} = \lim_{\delta x \to 0} \frac{\left| \frac{f(x+\delta x) - f(x)}{f(x)} \right|}{\left| \frac{x+\delta x - x}{x} \right|} = \left| \frac{x}{f(x)} f'(x) \right| \quad (1.2)$$

Now what if the function has multiple inputs? Or What if the function has multiple outputs?

Examples:-

1. Input 2 numbers $a, b \in \mathbb{R}$ and then find $f(a, b) = a + b$?. This problem takes 2 inputs- a, b and one output $f(a, b)$.

2. Find the roots of a polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. We are inputting the vector $\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \end{bmatrix}^T$ and the output is x in this case.

3. Given a matrix $A \in \mathbb{R}^{m\times n}$. Input a vector $\mathbf{x} \in \mathbb{R}^{n\times 1}$ and then find $f(\mathbf{x}) = A\mathbf{x} \in \mathbb{R}^{m\times 1}$?

4. Solve the linear system $A\mathbf{x} = \mathbf{b}$ where $A \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^{n\times 1}$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$. Inputs are $A, \mathbf{b}$ and output is a vector $\mathbf{x}$.

To accommodate these cases, a generalized definition of a (relative) condition number κ_r for a function $f : X \to Y$ where $X \subset \mathbb{R}^{m\times 1}$ and $Y \subset \mathbb{R}^{n\times 1}$ is shown below:

$$\kappa_r = \lim_{r\to 0} \sup_{\|x\|_q \le r} \frac{\frac{\|f(x+\delta x)-f(x)\|_p}{\|f(x)\|_p}}{\frac{\|\delta x\|_q}{\|x\|_q}} \tag{1.3}$$

where $p, q \in \mathbb{N}$ and $\|.\|_p$ denotes the vector $p-$ norm.

1.3.1 Vector Norms(Recap)

For a vector x in the vextor space X over a field F, $\|.\| : F \to R$ is defined such that:

1. $\|x\| \ge 0 \quad \forall x \in X$.
2. $\|\alpha x\| = |\alpha|, \quad \forall x \in X, \quad \alpha \in F$
3. $\|x + y\| \le \|x\| + \|y\|, \quad \forall x, y \in X$.
4. $\|x\| = 0 \iff x = 0$

Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$. Different possible vector norms which satisfy the above conditions are:

1. Euclidean norm (2-norm)
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{1.4}$$
2. Supremum norm(max. norm)
$$\|x\|_{\max} = \|x\|_\infty = \max_{1\le i\le n} |x_i| \tag{1.5}$$
3. 1-norm
$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \tag{1.6}$$

4. p-norm

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \tag{1.7}$$

NOTE:- Supremum norm of x is p-norm of x as $p \to \infty$

Proof:- From the definition,

$$\lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \leq \left(n \max_{1\leq i\leq n} |x_i|^p\right)^{\frac{1}{p}} = n^{\frac{1}{p}} \max_{1\leq i\leq n} |x_i|$$

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \geq \left(\max_{1\leq i\leq n} |x_i|^p\right)^{\frac{1}{p}} = \max_{1\leq i\leq n} |x_i|$$

From the above 2 inequalities, we can say that:

$$\max_{1\leq i\leq n} |x_i| \leq \|x\|_p \leq n^{\frac{1}{p}} \max_{1\leq i\leq n} |x_i|$$

As $p \to \infty$, $\quad n^{\frac{1}{p}} \max_{1\leq i\leq n} |x_i| \to \max_{1\leq i\leq n} |x_i|$. Therefore, by using sandwich theorem, we can say that

$$\|x\|_p = \max_{1\leq i\leq n} |x_i|$$

1.3.2 Examples on finding Condition number

1. Let $f(a, b) = a + b$. Find the condition number of this problem?

The inputs are a, b. Let the inputs have an error δa, δb respectively.

$$\text{Relative error in input} = \frac{\left\| \begin{bmatrix} a + \delta a \\ b + \delta b \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right\|_p}{\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_p}$$

For simplicity, let us consider 2-norm. Any norm can be used in fact. Therefore,

$$\text{Relative error in input} = \frac{\sqrt{\delta a^2 + \delta b^2}}{\sqrt{a^2 + b^2}}$$

The output $f(a+\delta a, b+\delta b) = a+b+\delta a+\delta b$. Therefore,

$$\text{Relative Error in output} = \frac{|(a+b+\delta a+\delta b)-(a+b)|}{|a+b|} = \frac{|\delta a+\delta b|}{|a+b|}$$

The relative condition number is:

$$\kappa_r = \lim_{r\to 0} \sup_{\left\|\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}\right\|_2 \le r} \frac{\frac{|\delta a+\delta b|}{|a+b|}}{\frac{\sqrt{\delta a^2+\delta b^2}}{\sqrt{a^2+b^2}}}$$

$$\implies \kappa_r = \lim_{r\to 0} \sup_{\left\|\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}\right\|_2 \le r} \frac{|\delta a+\delta b|}{\sqrt{\delta a^2+\delta b^2}} \cdot \frac{\sqrt{a^2+b^2}}{|a+b|}$$

To calculate

$$\lim_{r\to 0} \sup_{\left\|\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}\right\|_2 \le r} \frac{|\delta a+\delta b|}{\sqrt{\delta a^2+\delta b^2}}$$

we assume that $\delta a = \alpha\cos\theta$ and $\delta b = \alpha\sin\theta$ where $\alpha > 0$ and $0 \le \theta < 2\pi$.

Therefore, we have:

$$\lim_{r\to 0} \sup_{\left\|\begin{bmatrix}\delta a\\ \delta b\end{bmatrix}\right\|_2 \le r} \frac{|\delta a+\delta b|}{\sqrt{\delta a^2+\delta b^2}} = \lim_{r\to 0}\sup_{\alpha<r} \frac{|\alpha\cos\theta+\alpha\sin\theta|}{\alpha} = \lim_{r\to 0}\sup_{\alpha<r} |\cos\theta+\sin\theta| = \sqrt{2}$$

Thus, the condition number for adding 2 numbers is:

$$\kappa_r = \frac{\sqrt{2(a^2+b^2)}}{|a+b|} \le \sqrt{2} \text{ (if } a,\ b > 0)$$

(as $|a+b| \ge \sqrt{a^2+b^2}$ for $a, b \in \mathbb{R}^+$)

For $a, b > 0$, we can clearly see that the condition number is bounded above by $\sqrt{2}$. In other words, **addition is well-conditioned.**

By performing a similar exercise, we can show that the **subtraction is ill-conditioned** as for $\frac{a}{b} \to 1$, $\kappa_r \to \infty$.

Multiplication and division operations are also ill-conditioned.

2. Condition number on finding roots of the polynomial $x^2 - 2x + 1$.

Chapter 2

Numerical Linear Algebra

2.1 Columnspace, Nullspace and all

Consider a matrix $A \in \mathbb{R}^{m \times n}$ defined as:

$$A = \begin{bmatrix} - & r_1^T & - \\ - & r_2^T & - \\ & \vdots & \\ - & r_m^T & - \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

where $r_i \in \mathbb{R}^{n \times 1}$ for $1 \leq i \leq m$ are the rows and $a_i \in \mathbb{R}^{m \times 1}$ for $1 \leq i \leq n$ are the columns of A.

Columnspace of a matrix A is the span(linear combination) of columns of A. Also called as Range of A.

$$\text{Range}(A) = \text{Columnspace}(A) = \{Ax : x \in \mathbb{R}^{n \times 1}\} \tag{2.1}$$

$$Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} a_i x_i$$

Rowspace of a matrix A is the span(linear combination) of rows of A.

$$\text{Rowspace}(A) = \{A^T y : y \in \mathbb{R}^{m \times 1}\} \tag{2.2}$$

$$A^T y = \begin{bmatrix} | & | & & | \\ r_1 & r_2 & \cdots & r_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} r_i y_i$$

Nullspace of a matrix A is defined as follows:

$$\text{Nullspace}(A) = \{z \in \mathbb{R}^{n\times 1} : Az = 0\} \tag{2.3}$$

NOTE:-

1. A linear system $Ax = b$ has a solution ONLY IF $b \in \text{Range}(A)$.
2. Dimension of Range(A) is the number of linearly independent columns of A or the column rank of A. Similarly, the dimension of Rowspace(A) is the row rank or the number of independent rows of A.
3. For a matrix A, row rank = column rank = rank$\leq \min(m, n)$.
4. Nullspace of a matrix is orthogonal to row space of a matrix.i.e, Given any vector $z \in \text{Nullspace}(A)$ and $w \in \text{Rowspace}(A)$, z is orthogonal to w.

Proof:- Let $w \in \text{Rowspace}(A)$, then $\exists y \in \mathbb{R}^{n\times 1}$ such that $w = A^T y$.

Also as $z \in \text{Nullspace}(A)$, we have $Az = 0$.

Therefore,

$$\langle w, z\rangle = w^T z = y^T Az = 0$$

Thus, nullspace of a matrix is orthogonal to row space of a matrix.

5. **Rank-Nullity Theorem:** Dimension of Nullspace(A)+Rank(A) $= n =$ No. of columns of A

2.2 Matrix Norms

Consider a matrix $A \in \mathbb{R}^{m\times n}$. Just like how we have defined a vector norm, we could have defined an "element wise matrix norm" as follows:

$$\|A\|_p^* = \left(\sum_{i=1}^{n} |A_{ij}|^p\right)^{\frac{1}{p}} \tag{2.4}$$

But this definition of norm does not satisfy the **submultiplicative property**. We are interested in this property as this dictates the convergence of iterative schemes.

A matrix norm is said to be submultiplicative if for any matrices $A \in \mathbb{R}^{m\times k}$ and $B \in \mathbb{R}^{k\times n}$, we have

$$|AB\| \leq \|A\|\|B\| \tag{2.5}$$

Consider the case

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

and $p \to \infty$, Therefore we have

$$\|A\|_\infty^* = \max_{1\le i\le m, 1\le j\le n} |A_{ij}| = 2$$

.

$$A^2 = \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Therefore,

$$\|A^2\|_\infty^* = 8$$

We can clearly see that:

$$\|A^2\|_\infty^* = 8 \geq \|A\|_\infty^* \cdot \|A\|_\infty^* = 2 \times 2 = 4$$

which violates the submultiplicative property.

Hence, we define a p-norm of matrix which satisfies submultiplicative property as follows.

$$\|A\|_p = \sup_{x\in\mathbb{R}^n \ \{\mathbf{0}\}} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|y\|_p=1} \|Ay\|_p \tag{2.6}$$

p-norms are submultiplicative.

PROOF:- From the definition of p-norm,

$$\begin{aligned}
\|AB\|_p &= \sup_{x\in\mathbb{R}^n \ \{\mathbf{0}\}} \frac{\|ABx\|_p}{\|x\|_p} \\
&= \sup_{x\in\mathbb{R}^n \ \{\mathbf{0}\}} \frac{\|ABx\|_p}{\|Bx\|_p} \frac{\|Bx\|_p}{\|x\|_p} \\
&\leq \left[\sup_{x\in\mathbb{R}^n \ \{\mathbf{0}\}} \frac{\|ABx\|_p}{\|Bx\|_p} \right] \cdot \left[\sup_{x\in\mathbb{R}^n \ \{\mathbf{0}\}} \frac{\|Bx\|_p}{\|x\|_p} \right] \\
&= \|A\|_p \cdot \|B\|_p
\end{aligned}$$

$$\therefore \|AB\|_p \leq \|A\|_p \cdot \|B\|_p$$

Using this property, we can say that

$$\|A^n\|_p \leq \|A\|_p^n \tag{2.7}$$

$\|A\|_1$ = Maximum of column sum of absolute values.

$\|A\|_\infty$ = Maximum of row sum of absolute values.

2.3 Condition number of Matrix vector products

Consider a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $x \in \mathbb{R}^{n\times 1}$. Assume that there is no error in representing A. We are interested in finding the condition number of the Matrix-Vector product $f(x; A) = Ax$.

From the definition of condition number, we can write the condition number κ_r of the matrix vector product as:

$$\kappa_r = \lim_{r\to 0} \sup_{\|\delta x\|_q \le r} \frac{\frac{\|A(x+\delta x)-Ax\|_p}{\|Ax\|_p}}{\frac{\|x+\delta x-x\|_q}{\|x\|_q}}$$

For simplicity let us choose $p = q$. Therefore,

$$\kappa_r = \lim_{r\to 0} \sup_{\|\delta x\|_p \le r} \frac{\|A\delta x\|_p}{\|\delta x\|_p} \frac{\|x\|_p}{\|Ax\|_p}$$

From the definition of matrix p-norm, we can say that:

$$\lim_{r\to 0} \sup_{\|\delta x\|_p \le r} \frac{\|A\delta x\|_p}{\|\delta x\|_p} = \|A\|_p$$

Therefore the condition number of the matrix vector product is:

$$\kappa_r = \frac{\|A\|_p \|x\|_p}{\|Ax\|_p} \tag{2.8}$$

From Sub-multiplicative property, as $\|Ax\|_p \ge \|A\|_p \|x\|_p$, we can show that $\kappa_r \ge 1$.

2.4 Solving Linear Systems

Chapter 3

Interpolation

Chapter 4

Parts

You can add parts to organize one or more book chapters together. Parts can be inserted at the top of an .Rmd file, before the first-level chapter heading in that same file.

Add a numbered part: `# (PART) Act one {-}` (followed by `# A chapter`)

Add an unnumbered part: `# (PART\*) Act one {-}` (followed by `# A chapter`)

Add an appendix as a special kind of un-numbered part: `# (APPENDIX) Other stuff {-}` (followed by `# A chapter`). Chapters in an appendix are prepended with letters instead of numbers.

Chapter 5

Footnotes and citations

5.1 Footnotes

Footnotes are put inside the square brackets after a caret `^[]`. Like this one [1].

5.2 Citations

Reference items in your bibliography file(s) using `@key`.

For example, we are using the **bookdown** package [Xie, 2023] (check out the last code chunk in index.Rmd to see how this citation key was added) in this sample book, which was built on top of R Markdown and **knitr** [Xie, 2015] (this citation was added manually in an external file book.bib). Note that the `.bib` files need to be listed in the index.Rmd with the YAML `bibliography` key.

The RStudio Visual Markdown Editor can also make it easier to insert citations: https://rstudio.github.io/visual-markdown-editing/#/citations

[1] This is a footnote.

Chapter 6

Blocks

6.1 Equations

Here is an equation.

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k} \tag{6.1}$$

You may refer to using `\@ref(eq:binom)`, like see Equation (6.1).

6.2 Theorems and proofs

Labeled theorems can be referenced in text using `\@ref(thm:tri)`, for example, check out this smart theorem 6.1.

Theorem 6.1. *For a right triangle, if c denotes the* length *of the hypotenuse and a and b denote the lengths of the* ***other*** *two sides, we have*

$$a^2 + b^2 = c^2$$

Read more here https://bookdown.org/yihui/bookdown/markdown-extensions-by-bookdown.html.

6.3 Callout blocks

The R Markdown Cookbook provides more help on how to use custom blocks to design your own callouts: https://bookdown.org/yihui/rmarkdown-cookbook/custom-blocks.html

Chapter 7

Sharing your book

7.1 Publishing

HTML books can be published online, see: https://bookdown.org/yihui/bookdown/publishing.html

7.2 404 pages

By default, users will be directed to a 404 page if they try to access a webpage that cannot be found. If you'd like to customize your 404 page instead of using the default, you may add either a `_404.Rmd` or `_404.md` file to your project root and use code and/or Markdown syntax.

7.3 Metadata for sharing

Bookdown HTML books will provide HTML metadata for social sharing on platforms like Twitter, Facebook, and LinkedIn, using information you provide in the `index.Rmd` YAML. To setup, set the `url` for your book and the path to your `cover-image` file. Your book's `title` and `description` are also used.

This `gitbook` uses the same social sharing data across all chapters in your book-all links shared will look the same.

Specify your book's source repository on GitHub using the `edit` key under the configuration options in the `_output.yml` file, which allows users to suggest an edit by linking to a chapter's source file.

Read more about the features of this output format here:

https://pkgs.rstudio.com/bookdown/reference/gitbook.html

Or use:

```
?bookdown::gitbook
```

Bibliography

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