Case study 1:-

1. a) Asset A → $\frac{0.10+0.05+0.15}{3} = 0.1000$

Asset B → $\frac{0.20+0.10+0.25}{3} = \frac{0.55}{3} = 0.1833$

Asset C → $\frac{0.15+0.05+0.20}{3} = \frac{0.40}{3} = 0.1333$

b) Centered data:-

	Asset A	Asset B	Asset C
T_1	0.0000	0.0167	0.0167
T_2	-0.0500	-0.0833	-0.0833
T_3	0.0500	0.0667	0.0667

c). Covariance Matrix

	Asset A	Asset B	Asset C
Asset A	0.0025	0.00375	0.00375
Asset B	0.00375	0.00583	0.00583
Asset C	0.00375	0.00583	0.00583

$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$ or $\frac{1}{n-1}\sum(x_i-\bar{x})(y_i-\bar{y})$

$\mathrm{Cov}(X,X) = E[X^2] - (E[X])^2$ or $\frac{1}{n-1}\sum(x_i-\bar{x})^2$

As we have calculated $(X_i - \bar{X})$ values for each entry in b), we can take those values to calc. covariance matrix in c)

$\frac{1}{n-1} \Sigma (X_i - \bar{X})^2$ - for Asset A $= \frac{1}{3-1} \left((-0.05)^2 + (0)^2 + (0.05)^2\right)$

$= \frac{1}{2} \times (0.0025 \times 2) = 0.0025$

for cov. b/w Asset A & B $\rightarrow \frac{1}{n-1} \Sigma (X_i - \bar{X})(Y_i - \bar{Y}) = \frac{1}{2}\Big(0 \times 0.0167$

$+ (-0.05)(-0.0833)$

$+ (0.05)(0.0667)\Big)$

$= \frac{1}{2}(0.0075) = \underline{0.00375}$

similarly we can calculate all, so I have written answer in covariance matrix.

2. SVD

d) $A = U \Sigma V^T$

U → matrix of eigenvectors of AA^T

V^T → ~~transpose~~ matrix of eigenvectors of A^TA.

Σ → Diagonal matrix containing singular values (sqrts of eigenvalues)

$$AA^T = CC^T = \begin{bmatrix} 0.0025 & 0.00375 & 0.00375 \\ 0.00375 & 0.00583 & 0.00583 \\ 0.00375 & 0.00583 & 0.00583 \end{bmatrix} \begin{bmatrix} 0.0025 & 0.00375 & 0.00375 \\ 0.00375 & 0.00583 & 0.00583 \\ 0.00375 & 0.00583 & 0.00583 \end{bmatrix}$$

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$\underline{A.A^T = C.C^T}$

$$\begin{bmatrix} 0.000034375 & 0.0000531 & 0.0000531 \\ 0.0000531 & 0.0000820403 & 0.0000820403 \\ 0.0000531 & 0.0000820403 & 0.0000820403 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda I$$

$$\begin{bmatrix} 0.000034375-\lambda & 0.0000531 & 0.0000531 \\ 0.0000531 & 0.00008204-\lambda & 0.00008204 \\ 0.0000531 & 0.00008204 & 0.00008204-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

eigen values $\rightarrow \det|A-\lambda I| = 0.$

$$(0.000034375-\lambda)\left((0.00008204-\lambda)^2-(0.00008204)^2\right)$$
$$-0.0000531\left[(0.0000531\times(0.00008204-\lambda)) - 0.0000531\times 0.00008204\right]$$
$$+0.0000531\left[0.0000531\times 0.00008204 - 0.0000531\times(0.00008204-\lambda)\right]$$

$$= (0.000034375-\lambda)(0.00008204\times(-2\lambda)+\lambda^2) + 2\times(0.0000531)^2\lambda$$
$$-5.84\times10^{-9}\lambda + 0.000034375\lambda^2$$
$$+0.00016408\lambda^2 - \lambda^3 + 5.63922\times10^{-9}\lambda$$

$$-\lambda^3 + 0.000198455\lambda^2 \overset{-0.00078}{\cancel{5.63922}}\times10^{-9}\lambda = 0.$$
$$x^2 - 0.000198455x \overset{+0.00028}{\cancel{5.63922}}\times10^{-9} = 0$$

singular ~~eigen~~ values $\rightarrow$ $\sigma_1 = 1.4073\times10^{-2}$
$\downarrow$ $\sigma_2 = 7.3915\times10^{-5}$
sqrt of eigen values $\sigma_3 = \sim 0.$

eigen values
$\lambda_1 = 1.99\times10^{-4}$
$\lambda_2 = 5.46\times10^{-9}$
$\lambda_3 = 1.02\times10^{-71}$
$\sim 0.$

Sorted in descending order

Substituting each eigen value, we obtain eigen vectors as $\rightarrow$

$$U = \begin{bmatrix} +0.416 & 0.909 & 0 \\ +0.643 & -0.294 & -0.707 \\ +0.643 & -0.294 & 0.707 \end{bmatrix}$$

column
wise eigen vectors

e) Principal component directions -

$PC1 = (+0.416, +0.643, +0.642)$

$PC2 = [0.909, -0.294, -0.294]$

$PC3 = [0, -0.707, 0.707]$

f) Based on eigen vectors, proportion of total variance explained by each principal component →

$$\frac{\lambda_1}{\Sigma\lambda} = \frac{\cancel{1.4093 \times 10^{-2}}}{\cancel{1.4093 \times 10^{-2}} + 7.3915 \times 10^{-5}} \qquad \frac{1.99 \times 10^{-4}}{5.46 \times 10^{-9} + 1.99 \times 10^{-4}}$$

$$\approx 99.9972\%$$

$= \cancel{99.48\%}$

$$\frac{\lambda_2}{\Sigma\lambda} = \cancel{\frac{7.3915 \times 10^{-5}}{1.4093 \times 10^{-2}}} = \cancel{0.52\%} \qquad \frac{5.46 \times 10^{-9}}{5.46 \times 10^{-9} + 1.99 \times 10^{-4}}$$

$$\approx 0.00274\%$$

$$\frac{\lambda_3}{\Sigma\lambda} = \sim 0\%$$

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Q3. g) Significance of principal components in terms of Market Trends

→ PC1 explains most of the variance, means it represents the dominant trends in asset returns.

- The first principal component weights all assets positively, suggesting that they move together in a correlated fashion (likely due to a common market factor).

→ PC2 has very low variance, means any deviation from PC1 is minor.

h) Behaviours of Assets

→ PC1 suggests that all assets move in the same direction. A +ve shift in PC1 means all assets increase

→ PC2, on the other hand captures smaller differences b/w them, especially how Asset A behaves compared to the others.

- If an asset has a high weight in PC1, it means it moves in sync with overall market whereas if it has strong presence in PC2, it might behave differently from the rest, showing more independent trends.

i) Recommendations for Investment or Risk Management.

1) Diversification

Since PC1 explains almost all variance, it means assets move very similarly. This high correlation makes portfolio vulnerable to market-wide shocks. To reduce risk, investors should consider adding assets that aren't highly correlated with these 3 stocks.

Risk management:- As risk is mostly market driven, (since PC1 drives returns), hedging strategies should focus on reducing exposure to PC1 movements (eg-using derivatives)

Investment strategy

- If an investor wants broad market exposure, they should focus on assets that have high PC1 weights, since these closely follow market's movement.

- If looking for unique, asset specific opportunities, they should analyze PC2, which captures the smaller differences b/w assets.
- To truly diversify, investors should look for assets that contribute ~~to~~ more to PC2 or PC3, as these represent independent trends & reduce dependence on PC1.

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CASE STUDY 2

Given n vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$ and $w \in \mathbb{R}^m$

To find : $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ such that $L = \left\| \sum_{i=1}^{n} x_i v_i - w \right\|$.

is minimized $\Rightarrow$ L represents how far $x_1 v_1 + x_2 v_2 + \ldots$ is from w.

Let A be $m \times n$ matrix then.

$A = [v_1 \ v_2 \ \ldots \ v_n]$ (each v_i is an m-dimensional column).

A is how we can move around inside the subspace spanned by the vectors v_i.

Each component x_i is the "coefficient" that says how much v_i we include.

L is a norm $\Rightarrow$ $L \geq 0$.

(a) To show $\|Ax - w\|$ can be solved directly if A is diagonal.

Let A is a $n \times n$ diagonal matrix $(m = n)$ (it may extend with zero padding).

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & a_n \end{pmatrix}$$

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$$||Ax - w||_2 = \sqrt{\sum_{i=1}^{n} (a_i x_i - w_i)^2} \quad \text{(Euclidean norm)}.$$

$$||Ax - w||_2^2 = \sum_{i=1}^{n} (a_i x_i - w_i)^2.$$

Partial derivative wrt x_i

$$\frac{\partial}{\partial x_i} (a_i x_i - w_i)^2 = 2a_i (a_i x_i - w_i) = 0$$

$$\Rightarrow a_i x_i - w_i = 0 \quad (\text{assuming } a_i \neq 0).$$

$$\Rightarrow \quad x_i = \begin{cases} \frac{w_i}{a_i} & \text{if } a_i \neq 0 \\ 0 & \text{if } a_i = 0. \end{cases}$$

$\therefore$ A being a diagonal matrix, minimization problem is trivial to solve.

(b) $A = U \Sigma V^T$.

$$A\vec{x} - w = U\Sigma V^T \vec{x} - w = U\Sigma V^T \vec{x} - UU^T w$$

$$= U(\Sigma V^T \vec{x} - U^T w).$$

if A is diagonal $\Rightarrow$ A is a symmetric matrix
$\Rightarrow$ spectral decomposition applies to it.

$$A = Q \Lambda Q^T$$

We know, for A being a diagonal matrix its eigenvectors are standard basis vectors

$\Rightarrow Q = I$ and $Q^T = I$

$U = Q$, $V^T = Q^T$

$\Rightarrow U = I$ and $V = I$

$\Rightarrow A = I \Sigma I^T$ meaning the rotation parts U and V are simply the identity.

$\cancel{A\vec{x} - w = I(\Sigma I}$ $\Sigma = \Lambda$.

then

$$A\vec{x} - w = I(\Lambda I \vec{x} - I \cdot \vec{\omega}) = (\Lambda \vec{x} - \vec{\omega}).$$

The eigen values of A (diagonal matrix): $\det(A - \lambda I = 0$

$$\Rightarrow \begin{bmatrix} a_1 - \lambda & 0 & 0 & \cdots & 0 \\ 0 & a_2 - \lambda & 0 & \cdots & \vdots \\ \vdots & 0 & & \ddots & \\ 0 & \cdots & & & a_n - \lambda \end{bmatrix} = 0 \Rightarrow \begin{matrix} a_1 = \lambda_1 \\ a_2 = \lambda_2 \\ a_n = \lambda_n. \end{matrix}$$

$$\Rightarrow (a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda) \cdots (a_n - \lambda) = 0$$

$$\therefore A\vec{x} - w = \Lambda \vec{x} - \vec{\omega}$$

$\therefore$ $\min_{\vec{x}} \| y \Lambda \vec{x} - w \|$ is identical in structure to the diagonal case of part (a) and reduces to the same special case.

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(c) Given (x_i, y_i) data points $i \in [1, n]$ in $\mathbb{R}^2$.

Goal: find a, b to best fit line $y = ax + b$.

$$\sum_{i=1}^{n} (ax_i + b - y_i)^2 \to \text{minimize}$$

To ~~be~~ obtain $ax_i + b - y_i$ where x_i and y_i are constants and a, b are unknown

$\Rightarrow$ ~~(a b)~~ a, b must be components of $\vec{x}$ in previous form.

$\bar{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ then let $A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$

and $w = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ax_1 + b \\ \vdots \\ ax_n + b \end{bmatrix}$

$$A\vec{x} - \vec{w} = \begin{bmatrix} ax_1 + b - y_1 \\ ax_2 + b - y_2 \\ \vdots \\ ax_n + b - y_n \end{bmatrix}$$

Then $L = \|A\vec{x} - \vec{w}\|_2 = \sqrt{\sum_{i=1}^{n} (ax_i + b - y_i)^2}$

Then for minimizing L we need to minimize

$\|A\vec{x} - w\|_2^2 \Rightarrow$ minimize $\sum_{i=1}^{n} (ax_i + b - y_i)^2$.

$\therefore$ finding the best fit line is equivalent to

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minimizing $||A\vec{x} - w||^2$ (previous question).

(d). finding best fit for quadratic equation:

$y = ax^2 + bx + c$ given data points $(x_1, y_1), \dots, (x_n, y_n)$.

then unknowns are a, b, c.

$$\Rightarrow \vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \vec{x} \text{ is of } 3\times1 \text{ form.}$$

~~to get 3x1~~

$$~~\Rightarrow \bar{A}~~ \quad \Rightarrow A = \begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \quad A\vec{x} = \begin{bmatrix} ax_1^2 + bx_1 + c \\ \vdots \\ ax_n^2 + bx_n + c \end{bmatrix}$$

and
$$\vec{w} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

$L = ||A\vec{x} - \vec{w}||_2$, then minimizing L

$\Rightarrow$ minimizing $L^2 \Rightarrow$ minimizing $\sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2$

$\Rightarrow$ finding a, b, c s.t ~~y =~~ $\bar{y} = ax^2 + bx + c$ is best fit to data points.