METHODS

Experimental specifications. The experiments were conducted using arrays of evanescently coupled waveguides fabricated in borosilicate glass using femtosecond-laser-writing technology[30,31]. The waveguides are all identical in refractive index and dimension, but the inter-waveguide separation was modulated to realize the off-diagonal 2D model (equation (1)). In all cases, we observe the output image (after 15 cm of propagation) over a range of wavelengths (1,510–1,590 nm) in increments of 5 nm, and then average the output intensities over all wavelengths (Figs 2, 3). We note that the bandgap remains open over this range. We perform the averaging over wavelength to minimize sensitive interference effects due to fabrication imperfections.

Model implementation with waveguide arrays. The diffraction of paraxial light through the structures is governed by the paraxial Schrödinger equation[30,35]:

$$i\partial_z\psi = -\frac{1}{2k_0}\nabla^2\psi - \frac{k_0\Delta n}{n_0}\psi$$

where the wavefunction $\psi(x, y, z)$ corresponds to the electric-field envelope, $\boldsymbol{E}(x, y, z) = \psi(x, y, z)\exp(ik_0z - i\omega t)\hat{\boldsymbol{E}}_0$, $\nabla^2 = \partial_x^2 + \partial_y^2$ is the transverse Laplacian, $\Delta n(x, y, z)$ is the change in refractive index relative to the background index n_0, and $k_0 = 2\pi n_0/\lambda$ is the wavenumber in the background medium. For an array of single-mode, weakly coupled waveguides, the evolution generated by the paraxial Schrödinger equation can be described using tight-binding theory, whereby light hops between the bound modes of adjacent waveguides. The hopping amplitude t associated with a given waveguide separation can be obtained by numerically computing the two lowest eigenvalues E_1 and E_2 of the full equation for a system consisting of two waveguides; the hopping amplitude is then $t = (E_1 - E_2)/2$.

To perform this computation for our waveguides, we used a best-fitting Gaussian model for the variation in the waveguide refractive index: $\Delta n(x, y) = \delta n\exp\left(-x^2/\sigma_x^2 - y^2/\sigma_y^2\right)$, with $\delta n = 2.8\times10^{-3}$, $\sigma_x = 3.50$ μm and $\sigma_y = 5.35$ μm. These parameters were obtained by calibrating over a set of 1D test arrays. Using this profile and a background index of $n_0 = 1.473$, we obtain a model of the form $t(s) = A\exp(-\gamma s)$ for the dependence of the hopping amplitudes on the waveguide separation s. Here $A = A(\lambda)$ and $\gamma = \gamma(\lambda)$ are wavelength-dependent parameters plotted in Extended Data Fig. 1. We obtain these parameters by computing the couplings along the x and y directions separately for different values of s (15–35 μm) at wavelengths of 1,450–1,650 nm and then fitting the average of the x and y couplings to a model of the form given above for the hopping amplitudes. We then used this model to solve for the waveguide spacings that are required to implement the modulated hopping amplitudes defined by the Hamiltonian in equation (1).

To provide a clearer picture of the waveguide configurations used in our photonic system, we include an illustration of a 1D pump in Extended Data Fig. 1b. Varying the waveguide spacings along the propagation direction allows us to control the hopping amplitudes in a way that implements a sweep of the pump parameter ϕ_x. To obtain the full 2D array, we consider additional copies of such a structure stacked vertically along the y direction, with the vertical spacings determined by the hopping amplitudes associated with the y direction.

The decoupled model. Here we examine how the bulk response in an analogous electronic system (that is, one in which states are filled up to a given Fermi level) explains the behaviour of the boundary states. The model in equation (1) decomposes along the x and y directions into a sum of two independent off-diagonal Harper models, $H_x(\phi_x)$ and $H_y(\phi_y)$ (compare with equation (1))[6–8,36]

$$H(\phi_x, \phi_y) = H_x(\phi_x) + H_y(\phi_y) \tag{2}$$

Each $H_i(\phi_i)$ is a one-parameter family of 1D Hamiltonians, that is, a 1D topological pump. Treating the parameter ϕ_i as a Bloch momentum associated with an additional spatial dimension $\tilde{i} \in \{v, w\}$, we perform a dimensional extension of this model and obtain a model that describes the 4D integer quantum Hall system on a lattice with nearest-neighbour hopping in the i direction and next-nearest-neighbour hopping in the $\tilde{i}$ direction[37,38].

For $b = 1/3$, the spectrum of the 1D pump (2D quantum Hall) system consists of three bands (Extended Data Fig. 2a). Each band n has an associated non-zero first Chern number (denoted ν_1 in ref. 34) of

$$\nu_n = \frac{1}{2\pi i}\int_0^{2\pi} C_n(\phi_i, k_i)\mathrm{d}\phi_i\mathrm{d}k_i$$

which is an integral over the Berry curvature (also known as the Chern density) of the filled nth band

$$C_n(\phi_i, k_i) = \mathrm{tr}\left\{P_n\left[\frac{\partial P_n}{\partial\phi_i}, \frac{\partial P_n}{\partial k_i}\right]\right\}$$

where we have defined the spectral projector P_n onto all states in the nth band. Energy gaps in the 2D Hall effect are also characterized by first Chern numbers. The first Chern number of a spectral gap is the sum of first Chern numbers of the bands below that gap in energy. The first Chern number of the bandgap manifests through the quantization of the Hall conductance in response to an applied in-plane electric field; for example, in our case $I_x = (e^2/h)E_v\sum_n\nu_n$, where I_x denotes the current density along the x direction, E_v is an electric field along the v direction, and the sum is over all filled bands. This quantized bulk response has corresponding edge states; that is, gapless boundary states appear in a finite sample (as many as the sum of the Chern numbers of bands below a given gap) and carry the transverse quantized conductance[39].

As discussed in the main text, the eigenstates of the full Hamiltonian (equations (1) and (2)) are tensor products of the eigenstates of the two independent Harper models $|\psi_{mn}\rangle = |\psi_m\rangle\otimes|\psi_n\rangle$, where m enumerates the states in the x–v plane and n those in the y–w plane. Their associated energies are $E_{mn} = E_m + E_n$, so that each pair of bands from the decoupled models yields a band of the 2D pump (4D quantum Hall) model. Therefore, in a finite system, because each constituent 1D pump has bulk and boundary modes, the tensor product eigenstates can be categorized as bulk–bulk, bulk–boundary and boundary–boundary. A colour-coded illustration of the resulting band structure is shown in Extended Data Fig. 2b.

The resulting Minkowski sum spectrum is not always gapped: depending on the amplitudes $\tilde{t}_i$ and λ_i, the joint spectrum may not be gapped. Consequently, if the gaps are closed, then we can no longer discuss the topology of the combined spectrum because any small perturbation will mix the states from the different bands. When the spectral gaps are open, the bulk–boundary and boundary–boundary modes lie for some ϕ_i at energies within the gaps and for others at energies in the bulk bands. Therefore, the boundary–boundary (2D corner) modes that overlap with the bulk are generally expected to become finite-lifetime resonances upon the introduction of higher-neighbour hoppings that destroy the tensor product structure of the eigenstates. Nonetheless, the in-gap bulk–boundary and boundary–boundary modes are protected for arbitrary perturbations that do not close the gap and are the surface states associated with a non-zero second Chern number.

Second Chern number. Let us consider an energy E_j in the jth gap of the 2D pump (4D quantum Hall) system (Extended Data Fig. 2b). The second Chern number (denoted ν_2 in ref. 34) associated with this gap is

$$\mathcal{V}_j = -\frac{1}{8\pi^2}\int\varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left\{P_j\frac{\partial P_j}{\partial k_\mu}\frac{\partial P_j}{\partial k_\nu}P_j\frac{\partial P_j}{\partial k_\rho}\frac{\partial P_j}{\partial k_\sigma}\right\}\mathrm{d}^4\boldsymbol{k}$$

where $P_j(\boldsymbol{k})$ is the projector onto the subspace spanned by the eigenstates at Bloch momentum $\boldsymbol{k} = (\phi_x, \phi_y, k_x, k_y)$ with energies below the gap. The subscripts of $\boldsymbol{k}$ mark the vector component. Using the decomposition of H discussed above, $\mathcal{V}_j$ can be written in terms of the first Chern numbers ν_n of the Harper models as[7]

$$\mathcal{V}_j = \sum_{\text{band pairs } m,n \text{ with } E_{mn}<E_j}\nu_n^{xv}\nu_m^{yw}$$

where ν_n^{xv} and ν_m^{yw} are the first Chern numbers associated with the nth band in the x–v plane and mth band in the y–w plane, respectively. Combining this result with the first Chern numbers shown in Extended Data Fig. 2a, the second Chern numbers associated with the lower and upper gaps of the 2D pump (4D quantum Hall) Hamiltonian are $\mathcal{V} = +1$ and -1, respectively. Although the Hamiltonian that governs our photonic system does not decompose in the way discussed above, owing to the presence of higher-neighbour couplings, the upper and lower gaps remain open (see Fig. 1) and, as a result, the associated second Chern numbers remain unchanged.

Bulk responses and their corresponding edge phenomena. Measuring the second Chern number via the bulk response directly requires both an external electric and magnetic field to be applied. However, the presence of the second Chern number implies the presence of surface states, irrespective of the application of external fields. In this section, we explain the relationship between the presence of the surface states in the model and the second Chern number, from the point of view of topological pumping.

The second Chern number of the jth bandgap has an associated quantized nonlinear bulk response

$$I_\alpha = \frac{\mathcal{V}_j}{2}\frac{e^2}{h\Phi_0}\varepsilon_{\alpha\beta\gamma\delta}B_{\beta\gamma}E_\delta$$

where I_α denotes the current density along the α direction, Φ_0 is the flux quantum, E_δ is an electric-field perturbation along the δ direction, $B_{\beta\gamma}$ is a magnetic-field perturbation in the β–γ plane, and $\varepsilon_{\alpha\beta\gamma\delta}$ is a Levi–Civita symbol that highlights