the 4D non-commutative nature of the response. The second Chern number $\mathcal{V}_j$ is defined as the sum over all bands up to the jth of a 4D volume integral over a generalized 4D Berry curvature of the given band.

In our spinless case, we can write the 4D Berry curvature in terms of the 2D Berry curvatures that exist in the two orthogonal planes associated with the independent models[6,8,27–29]. Let us consider these orthogonal planes to be x–v and y–w. In addition, for the choice of boundary conditions in our experiment, let us focus on the responses in the direction $\alpha = x$ and study their bulk–edge correspondence. The responses in the $\alpha = y$ direction will be similar. Having fixed the response direction, there are various choices for the orientation of the perturbing fields in four dimensions. These can be split into density-type responses and Lorentz-type responses.

Density-type response. Consider the case where the extrinsic perturbing field $B_{\beta\gamma}$ is set in a plane for which there is a non-trivial Berry curvature from the underlying model. For responses in the $\alpha = x$ direction, this occurs when $\beta\gamma = yw$. Correspondingly, the orientation of the electric-field perturbation is $\delta = v$. Owing to the non-trivial intrinsic Berry curvature in the x–v plane, E_v also generates a 2D quantum Hall-like response, and the bulk response is

$$I_v = I_y = I_w = 0$$
$$I_x = \frac{e^2}{h}\nu^{xv}\tilde{n}E_v + \frac{e^2}{h\Phi_0}\mathcal{V}_j E_v B_{yw}$$

where ν^{xv} contains the sum over first Chern numbers of filled bands. It is now apparent why we denote this response as 'density-like'. The bulk response has a 2D quantum Hall-like response, multiplied by a particle-density factor $\tilde{n}$ that results from the integration over the 4D volume. The second Chern number response here can be understood[40] to be a Streda formula correction to $\tilde{n}$.

To support such a response in finite-sized systems, the corresponding edge phenomena must manifest a band of modes that traverse the gap. The density of this edge band is modulated by the magnetic-field perturbation. In addition, from the response to E_v we conclude that the in-gap band is dispersive with respect to k_v. Repeating this argument for the density-type response in the y direction, we expect an additional in-gap band that is dispersive with respect to k_w.

In 2D topological pumping, we generate the electric field E_v using Faraday's law of induction, that is, by modulating ϕ_x. Correspondingly, the density-type quantized 4D quantum Hall response implies that within a full 0–2π cycle of ϕ_x a band of states (corresponding to $\tilde{n}$) must cross the gap and appear ν^{xv} times on either side of the x-direction open boundary conditions. The density of this band is modulated by the external magnetic-field perturbation and thus accommodates the density-type second Chern number response. Following the same arguments, similar bands must appear upon scans of ϕ_y to support the response in the $\alpha = y$ direction. In the photonic experiment, we excite these edge bands directly (as well as, inevitably, in-bulk bands) and show that they truly carry modes from one side of the sample to the other in both the x and y directions.

From the above discussion, it is apparent that the observation of edge-to-edge pumping implies that a full band spectrum supports density-type second Chern bulk responses, and it suffices to see these responses as a function of scans of ϕ_x and ϕ_y. In terms of edge physics, adding a perturbing B_{yw} field is not illustrative: the intrinsic field has already set up the conditions (via the density response) for a current that arises from both the first and second Chern numbers.

Lorentz-type responses. Consider the case where the extrinsic perturbing field $B_{\beta\gamma}$ is set in a plane for which there is no Berry curvature from the underlying model. For responses in the $\alpha = x$ direction, this occurs when $\beta\gamma \in \{vy, vw\}$. Correspondingly, the orientation of the electric-field perturbation is $\delta \in \{w, y\}$. We are interested in 2D topological pumping, that is, in generating the electric field using Faraday's law of induction; consequently, we do not apply the electric-field perturbation in the y direction. Because we cannot apply a B_{vw} perturbation between the two dynamical axes of the pump, we are left with the response

$$I_x = \mathcal{V}_j \frac{e^2}{h\Phi_0}\varepsilon_{xvyw}B_{vy}E_w$$

Because E_w is generated by pumping ϕ_y, this response means that $\mathcal{V}_j$ charge-carrier modes must appear within the gap every $1/B_{vy}$ cycles on each side of the x axis.

In the 2D model, the Lorentz-type magnetic-field perturbation enters (in the correct gauge) as a spatial modulation of the model, by changing the modulated hopping:

$$t_x(\phi_x) \rightarrow \tilde{t}_x + \lambda_x\cos(2\pi b_x x + 2\pi B_{vy} y + \phi_x)$$
$$t_y(\phi_y) \rightarrow \tilde{t}_y + \lambda_y\cos(2\pi b_y y + \phi_y)$$

In the y–w plane, a first Chern number bulk response occurs as a function of scans of ϕ_y, leading to a gradual change in the coordinate y. Therefore, owing to the magnetic-field perturbation B_{vy}, as ϕ_y is scanned a slow modulation of the potential in the x direction also occurs.

This is a slow modulation that would mean that $1/B_{vy}$ cycles of ϕ_y generate in the same time a full scan of ϕ_x from 0 to 2π (see also the bulk-pumping experiment in cold atoms[34]). In a finite-sized system, this Lorentz-type bulk response implies that boundary modes must appear and cross the gap in response to a joint modulation of both pump parameters ϕ_x and ϕ_y: this is precisely the corner mode shown in black in Fig. 1c and Extended Data Fig. 2b, and for these gaps $|\mathcal{V}_j| = 1$.

Data availability. The data that support the findings of this study are available from the corresponding authors on reasonable request.

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