of the transverse superlattice phase φ_y, this term also contains the magnetic perturbation, that is, a weak homogeneous magnetic field in the x–w plane

$$\hat{H}_{yw} = -\sum_{\boldsymbol{m}} J_y \hat{a}^\dagger_{\boldsymbol{m}+\boldsymbol{e}_y}\hat{a}_{\boldsymbol{m}} + \text{h.c.} \\ -\sum_{\boldsymbol{m}} \frac{\Delta_y^{(0)}}{4} \mathrm{e}^{i(\widetilde{\Phi}_{yw} m_y + \widetilde{\Phi}_{xw} m_x)} \hat{a}^\dagger_{\boldsymbol{m}+\boldsymbol{e}_w}\hat{a}_{\boldsymbol{m}} + \text{h.c.}$$

with $\widetilde{\Phi}_{xw} = -2\pi\theta d_{s,x}/d_{l,y}$. The strength of the perturbing magnetic field is then

$$B_{xw} = -\frac{\Phi_0}{d_{s,w} d_{l,y}}\theta$$

where $d_{s,w}$ is the lattice spacing along w. For $\delta J_\mu^{(0)} \neq 0$, the third contribution ($\hat{H}_{\delta J}$) leads to the appearance of additional next-nearest-neighbour tunnel coupling elements in the x–z and y–w planes, with amplitudes of $\delta J_x^{(0)}/4$ and $\delta J_y^{(0)}/4$, respectively. The individual 2D models without the magnetic perturbation B_{xw} then correspond to the Harper–Hofstadter–Hatsugai model[36] with a uniform magnetic flux Φ_{xz} and Φ_{yw}, the same flux as for $\delta J_\mu^{(0)} = 0$.

Transport properties of a 2D topological charge pump. When the pump parameter φ_x is changed slowly, a particle that is initially in an eigenstate $|u(k_x, \varphi_x(t=0), k_y, \varphi_y)\rangle$ of the 2D superlattice Hamiltonian $\hat{H}_{2D}$ (equation (1)) will adiabatically follow the corresponding instantaneous eigenstate $|u(k_x, \varphi_x(t), k_y, \varphi_y)\rangle$. In absence of a tilt ($\theta = 0$), the particle acquires an anomalous velocity $\Omega^x \partial_t \varphi_x \boldsymbol{e}_x$ during this evolution, analogously to the linear Hall response in a quantum Hall system. In this case, the Berry curvature Ω^x is defined in a 4D generalized Brillouin zone (k_x, φ_x, k_y, φ_y)

$$\Omega^x(k_x, \varphi_x, k_y, \varphi_y) = i(\langle \partial_{\varphi_x} u | \partial_{k_x} u\rangle - \langle \partial_{k_x} u | \partial_{\varphi_x} u\rangle)$$

For a homogeneously populated band, the COM displacement along x during one cycle, obtained by integrating the average anomalous velocity over one period, can be expressed as an integral of the Berry curvature over the 2D generalized Brillouin zone spanned by k_x and φ_x. It is therefore determined by the first Chern number of the pump

$$\nu_1^x = \frac{1}{2\pi}\oint \Omega^x \,\mathrm{d}k_x \mathrm{d}\varphi_x$$

When a tilt is present ($\theta \neq 0$), this motion along x leads to a change in φ_y. This induces an additional anomalous velocity in the y direction, giving rise to the nonlinear response. Neglecting the contribution from the group velocity (which averages to zero for a homogeneously populated band), we obtain for a given eigenstate

$$v_y(k_x, \varphi_x, k_y, \varphi_y) = \Omega^y \partial_t \varphi_y = \frac{2\pi}{d_{l,y}}\theta\, \Omega^x \Omega^y \partial_t \varphi_x \qquad (2)$$

The distribution of $\Omega^x\Omega^y$ in the 4D generalized Brillouin zone is shown in Fig. 1e for the lattice parameters used for the measurements in Figs 3 and 4. It exhibits pronounced peaks around $\varphi_x \in \{\pi/2, 3\pi/2\}$ and $\varphi_y \in \{\pi/2, 3\pi/2\}$. For $d_l = 2d_s$, $\Omega^x\Omega^y$ is π-periodic in both φ_x and φ_y because the corresponding eigenstates are related by a gauge transformation, owing to the translational symmetry of the superlattice potential[38].

For a small cloud that homogeneously populates a single band, as in the experiment, the variation in $\Omega^x\Omega^y$ over the size of the cloud along x (L_x) due to the position dependence of φ_y is negligible for $L_x \ll d_{l,y}/\theta$. The average velocity for the nonlinear response can then be calculated by averaging equation (2) over both quasi-momenta k_x and k_y. The COM displacement after a complete cycle can be determined by integrating the velocity over one period. We can thus express the change in the COM position per cycle as

$$\delta y_{\text{COM}} = \underbrace{\frac{1}{2\pi}\oint \Omega^x\Omega^y \,\mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\varphi_x}_{\overline{\Omega}(\varphi_y)} \theta \frac{a_x}{d_{l,y}} a_y \qquad (3)$$

If the number of pump cycles is small, then the change in φ_y as a result of the linear pumping response can be neglected and the nonlinear displacement per cycle is very well approximated by $\delta y_{\text{COM}} \approx \overline{\Omega}(\varphi_y^{(0)})\,\theta\, a_x a_y / d_{l,y}$.

The response of a large system with $L_x \gg d_{l,y}/\theta$ can be obtained by averaging equation (3) over $\varphi_y(x) \in [0, 2\pi)$, yielding

$$\delta y_{\text{COM}} = \frac{1}{2\pi}\oint \overline{\Omega}(\varphi_y)\,\theta \frac{a_x}{d_{l,y}} a_y \,\mathrm{d}\varphi_y = \nu_2 \theta \frac{a_x}{d_{l,y}} a_y$$

where the second Chern number ν_2 is calculated by integrating $\Omega^x\Omega^y$ over the entire 4D generalized Brillouin zone

$$\nu_2 = \frac{1}{4\pi^2}\oint_{\text{BZ}} \Omega^x\Omega^y \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\varphi_x \mathrm{d}\varphi_y$$

Note that to probe the intrinsic transport properties of the unperturbed system, both fields that generate the response have to be small perturbations such that the evolution remains adiabatic and the energy gap to the excited subbands remains open, which protects the topological invariants. Nonetheless, going beyond this limit can result in additional exciting phenomena. For example, a configuration with $\theta = \pi/4$ can lead to spatial frustration and the resulting model might enable the observation of quantized electric quadrupole moments similar to those proposed previously[29].

Pump path. Varying the pump parameter φ_x periodically modulates the tight-binding parameters $\delta J_x(\varphi_x)$ and $\Delta_x(\varphi_x)$ that describe the superlattice along x (equation (1)). For $d_l = 2d_s$, the modulation of δJ_x and Δ_x is out of phase and the system therefore evolves along a closed trajectory in the δJ_x–Δ_x parameter space (Extended Data Fig. 2a). This pump path encircles the degeneracy point ($\delta J_x = 0$, $\Delta_x = 0$), at which the two lowest subbands of the Rice–Mele model touch. This singularity can be interpreted as the source of the non-zero Berry curvature Ω^x in the generalized Brillouin zone, which gives rise to the linear pumping response. All pump paths that encircle the degeneracy can be continuously transformed into one another without closing the gap to the first excited subband and are thus topologically equivalent with respect to the linear response; that is, the value of ν_1^x does not change.

Similarly, the tight-binding parameters δJ_y and Δ_y depend on the phase of the transverse superlattice φ_y. For a large cloud, all possible values of φ_y, and thus δJ_y and Δ_y, are sampled simultaneously (Extended Data Fig. 2b). During a pump cycle, the system therefore traces out a closed surface in the 4D parameter space of δJ_x, Δ_x, δJ_y and Δ_y (Extended Data Fig. 2c). In this parameter space, the two lowest subbands touch in the two planes ($\delta J_x = 0$, $\Delta_x = 0$) and ($\delta J_y = 0$, $\Delta_y = 0$), which intersect at a single point at the origin (Extended Data Fig. 2d). Analogously to the linear response, this degeneracy generates the non-zero Berry curvatures Ω^x and Ω^y, which cause the nonlinear motion in the y direction. Owing to the 4D character of the parameter space, the 4D pump path can enclose the degeneracy (Extended Data Fig. 2e). Whenever this is the case, the topology of the cycle does not change and the value of ν_2 remains the same.

To visualize the pump path in the 4D parameter space in Extended Data Fig. 2, we apply the following transformation

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \delta J_x/\delta J_x^{(0)} \\ \Delta_x/\Delta_x^{(0)} \\ \delta J_y/\delta J_y^{(0)} \\ \Delta_y/\Delta_y^{(0)} \end{pmatrix} \qquad (4)$$

where the tight-binding parameters are normalized by their respective maximum values. The degeneracy planes are then given by $r_1 = -r_2$, $r_3 = -r_4$ and $r_1 = r_2$, $r_3 = r_4$, respectively; that is, they become perpendicular planes in (r_1, r_2, r_3) space.

Lattice configuration. All experiments were performed in a mutually orthogonal retro-reflected 3D optical lattice consisting of superlattices along x and y and a simple lattice in the z direction. Each superlattice is created by superimposing two standing waves: a short lattice with wavelength $\lambda_s = 767$ nm and a long lattice with $\lambda_l = 2\lambda_s$. The vertical lattice along z is formed by a standing wave with $\lambda_z = 844$ nm.

Initial state preparation for band-mapping measurements. For all sequences, a quarter-filled Mott insulator consisting of about 5,000 ^{87}Rb atoms was prepared with one atom localized in the ground state of each unit cell, creating a uniform occupation of the lowest subband in the 2D superlattice. To this end, a Bose–Einstein condensate was loaded from a crossed dipole trap into the lattice by first ramping up the blue-detuned short lattices along x and y to $3.0(1)E_{r,s}$ over 50 ms to lower the initial density of the cloud of atoms. These lattices were then switched off again within 50 ms, while the vertical lattice and both long lattices were increased to $30(1)E_{r,z}$ and $30(1)E_{r,l}$, respectively, with $\varphi_x = 0.000(5)\pi$ and $\varphi_y = \varphi_y^{(0)}$. Subsequently, doubly occupied lattice sites were converted to singly occupied ones (see below), creating a Mott insulator with unit filling and a negligible fraction of doublons. Each lattice site was then split into a four-site plaquette by ramping up the short lattices along x and y to their final depth of $7.0(2)E_{r,s}$ and decreasing the long lattices to $20.0(6)E_{r,l}$ over 5 ms.

Removing doubly occupied sites. After preparing the Mott insulator with unit filling in the long lattices, sites containing two atoms were converted to singly occupied ones using microwave-dressed spin-changing collisions and a