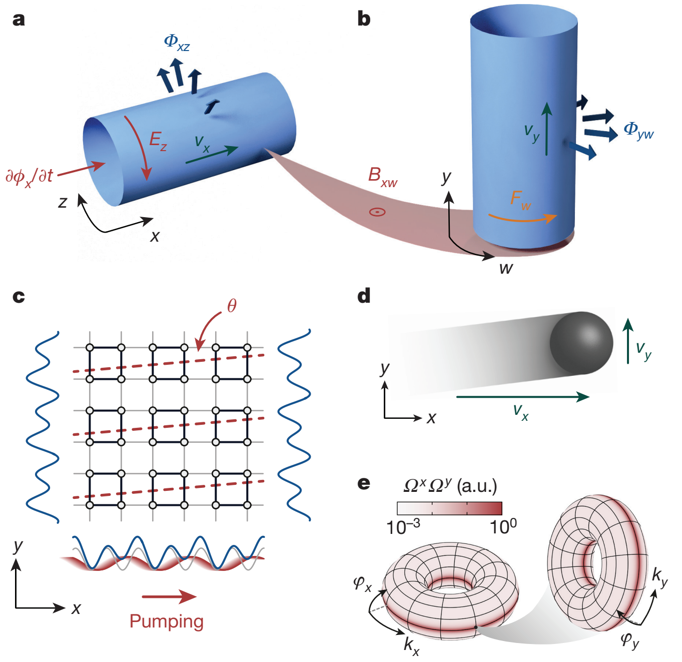

Figure 1 | 4D quantum Hall system and the corresponding 2D topological charge pump. a, A 2D quantum Hall system on a cylinder pierced by a uniform magnetic flux Φ_{xz} (blue arrows). Threading a magnetic flux $\phi_x(t)$ through the cylinder creates an electric field E_z on the surface (red arrows), resulting in a linear Hall response along x with velocity v_x (green arrow). **b**, A 4D quantum Hall system can be composed of two 2D quantum Hall systems in the x–z and y–w planes. A weak magnetic perturbation B_{xw} in the x–w plane couples the two systems and generates a Lorentz force F_w (orange arrow) for particles moving along x. This force induces an additional nonlinear Hall response in the y direction with velocity v_y (green arrow). **c**, A dynamical version of the 4D quantum Hall system can be realized by using a topological charge pump in a 2D superlattice (blue potentials). Such a superlattice is created by superimposing two lattices with periods d_s (grey) and $d_l > d_s$ (red) along both x and y, depicted here for $d_l = 2d_s$, as in the experiment. The black circles show the lattice sites that are formed by the potential minima, and the black (grey) lines indicate strong (weak) tunnel coupling between neighbouring sites. The system is modulated periodically by moving the long lattice adiabatically along x, mimicking the perturbing electric field E_z in the 4D model. The magnetic perturbation B_{xw} maps onto a small tilt angle θ of the long lattice along y with respect to the corresponding short lattice. In this case, the shape of the double wells along y depends on the position along x. The dashed red lines indicate the potential minima of the tilted long lattice. **d**, The pumping shifts the cloud of atoms (grey) in the x direction (with velocity v_x), as per the quantized linear response of a 2D quantum Hall system. For non-zero θ, the two orthogonal axes are coupled, leading to an additional quantized nonlinear response with 4D topology in the perpendicular y direction (with velocity v_y). **e**, The velocity of the nonlinear response is determined by the product of the Berry curvatures $\Omega^x\Omega^y$ (see Methods; a.u., arbitrary units), depicted here for the lowest subband with $d_l = 2d_s$ and lattice depths as in Fig. 3. The left (right) torus shows a cut at $k_y = 0$, $\varphi_y = \pi/2$ ($k_x = \pi/(2d_l)$, $\varphi_x = \pi/2$) through the generalized 4D Brillouin zone spanned by k_x, φ_x, k_y and φ_y.

is equivalent to threading the flux ϕ_x in the 4D model, leading to a quantized motion along x (the linear response; Fig. 1c, d). The magnetic perturbation B_{xw} corresponds to a transverse phase φ_y that depends linearly on x and thereby couples the motion in the x and y directions (see Methods). We realize this by tilting the long y lattice relative to the short one by an angle $\theta \ll 1$ (Fig. 1c) such that $\varphi_y(x) = \varphi_y^{(0)} + 2\pi\theta x/d_{l,y}$ to first order in θ. When φ_x is varied, the motion along x changes φ_y and—analogously to the Lorentz force in 4D—induces a quantized nonlinear response along y, which is equivalent to the nonlinear Hall response of a 4D quantum Hall system[6] (Fig. 1c, d).

For a uniformly populated band in an infinite system, the centre-of-mass (COM) displacement during one cycle $\varphi_x = 0 \to 2\pi$ is

$$\nu_1^x a_x \boldsymbol{e}_x + \nu_2 \theta \frac{a_x}{d_{l,y}} a_y \boldsymbol{e}_y$$

with a_x (a_y) the size of the superlattice unit cell and $\boldsymbol{e}_x$ ($\boldsymbol{e}_y$) the unit vector along x (y) (see Methods). The first term describes the quantized linear response along x. It is proportional to the first Chern number of the pump (ν_1^x; denoted ν in ref. 31), which is obtained by integrating the Berry curvature

$$\Omega^x(k_x, \varphi_x) = i(\langle \partial_{\varphi_x} u | \partial_{k_x} u\rangle - \langle \partial_{k_x} u | \partial_{\varphi_x} u\rangle)$$

over the generalized 2D Brillouin zone spanned by the quasi-momentum k_x and φ_x. Here, $|u(k_x, \varphi_x)\rangle$ denotes the eigenstate of a given non-degenerate band at k_x and φ_x. Because ν_1^x can take only integer values, the motion is quantized[25]. The second term is the nonlinear response in the y direction. It is quantified by a 4D integer topological invariant, the second Chern number of the pump (denoted $\mathcal{V}$ in ref. 31):

$$\nu_2 = \frac{1}{4\pi^2} \oint_{\mathrm{BZ}} \Omega^x \Omega^y \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\varphi_x \mathrm{d}\varphi_y$$

where BZ indicates the generalized 4D Brillouin zone (Fig. 1e). Therefore, the nonlinear response is also quantized and has intrinsic 4D symmetries that result from the higher-dimensional non-commutative geometry.

We implement a 2D topological charge pump by using bosonic ^{87}Rb atoms that form a Mott insulator in isolated planes of a 3D optical lattice with superlattices along x and y, with $d_s \equiv d_{s,x} = d_{s,y}$ and $d_l \equiv d_{l,x} = d_{l,y} = 2d_s$ (see Methods), creating double-well potentials along x and y (Fig. 1c). In the tight-binding limit, this implementation realizes a 2D Rice–Mele model[27] in each plane with dimerized on-site energies and tunnel couplings between neighbouring sites in both directions (see Methods). The corresponding unit cell is a four-site plaquette, $a_x = a_y = 2d_s$, and the lowest band splits into four subbands.

In the experiment, we study the nonlinear response of the lowest subband, for which $\nu_2 = +1$ for $d_l = 2d_s$. Our main results are: (i) the observation of a 4D-like bulk response; (ii) the local probing of its 4D geometric properties; and (iii) the revealing of the 4D quantum Hall effect by demonstrating the quantization of the response. As the initial state, a quarter-filled Mott insulator that uniformly occupies the lowest subband is prepared at $\varphi_x = 0$ (see Methods). The pumping is performed along x by adiabatically varying φ_x; we examine the resulting motion of the atoms. We probe the system locally by using a small cloud of atoms that extends over approximately 20 sites in the x direction. In this case, the variation in $\bar{\Omega}^y(\varphi_y)$ over the cloud is negligible and the y displacement per cycle is given by $\bar{\Omega}(\varphi_y^{(0)})\theta a_x a_y / d_l$, with

$$\bar{\Omega} = \frac{1}{2\pi} \oint \Omega^x \Omega^y \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}\varphi_x$$

(see Methods). From this local response, the quantized nonlinear response of an infinite system can be reconstructed by sampling all $\varphi_y^{(0)} \in [0, 2\pi)$, thereby integrating over the entire 4D Brillouin zone. To probe the motion of the cloud, we measure its COM position as a function of φ_x. Because the nonlinear response results from two weak perturbations, the displacement per cycle is typically only a fraction of d_l. It is therefore too small to be resolved experimentally, because the number of experimental cycles is limited by heating. However, for suitable lattice parameters, signatures of the nonlinear drift—the key feature of the 4D Hall effect—can be seen at $\varphi_y^{(0)} = \pi/2$ (Fig. 2), at which $\bar{\Omega}$ is strongly peaked (see Fig. 1e). Unlike the linear response, this motion depends on θ, demonstrating the intrinsically 4D character of the nonlinear response, which results from the two independent perturbations in orthogonal subspaces. This result demonstrates the existence of this dynamical, transverse, bulk phenomenon directly.