Metric Space — Definition and Axioms

Definition

A metric space is a pair (U, D) where

$$D : U \times U \to \mathbb{R}_{\geq 0}$$

is a distance function satisfying, for all $x, y, z \in U$:

1. $D(x, y) \geq 0$ — nonnegativity
2. $D(x, x) = 0$ — identity (zero self-distance)
3. $x \neq y \Rightarrow D(x, y) > 0$ — distinct points are apart
4. $D(x, y) = D(y, x)$ — symmetry
5. $D(x, z) \leq D(x, y) + D(y, z)$ — triangle inequality

Symbols and Components

Symbol	Meaning	Type / Role
U	The set of all objects we can measure distances between. Examples: $\mathbb{R}$, $\mathbb{R}^n$, or a set of strings, points, images, etc.	Set
D	The distance function. Takes two elements of U and returns a nonnegative real number.	Function $D : U \times U \to \mathbb{R}_{\geq 0}$
x, y, z	Arbitrary elements (points) in U. Used universally ("for all").	Variables
$\mathbb{R}$	The set of real numbers.	Standard mathematical set
$\mathbb{R}_{\geq 0}$	The set of nonnegative real numbers $\{r \in \mathbb{R} \mid r \geq 0\}$.	Codomain of D
$U \times U$	Cartesian product — all ordered pairs (x, y) with $x, y \in U$.	Domain of D
"for all $x, y, z \in U$"	Universal quantifier — conditions hold for every triple of points in U.	Logical quantification
$D(x, z) \leq D(x, y) + D(y, z)$	Triangle inequality — direct path no longer than detour via y.	Core axiom

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Function Type Summary

$$D : U \times U \longrightarrow \mathbb{R}_{\geq 0}$$

Input: two elements $x, y \in U$
Output: a real number $r = D(x, y) \geq 0$
Meaning: "the distance between x and y"

Construction Context

Here U is our set of points — for example, take $U = \mathbb{R}$, the real line.
Then:

$$U \times U = \{(x, y) \mid x, y \in \mathbb{R}\}$$

This is the set of all ordered pairs of real numbers such as $(1, 3)$, $(4, 7)$, $(-2, 5)$.
D takes such a pair (x, y) and returns their distance on the number line.

Example distance function:

$$D(x, y) = |x - y|$$

Verification Examples for Each Axiom

(1) Nonnegativity — $D(x, y) \geq 0$

Satisfying:
$U = \mathbb{R}$, $D(x, y) = |x - y|$
Example: $D(2, 5) = |2 - 5| = 3 \geq 0$

Violating:
$U = \mathbb{R}$, $D(x, y) = x - y$
Example: $D(2, 5) = 2 - 5 = -3 < 0$

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(2) Identity — $D(x, x) = 0$

Satisfying:
$U = \mathbb{R}$, $D(x, y) = |x - y|$
Example: $D(4, 4) = |4 - 4| = 0$

Violating:
$U = \mathbb{R}$, $D(x, y) = |x - y| + 1$
Example: $D(4, 4) = |4 - 4| + 1 = 1 \neq 0$

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(3) Distinctness — $x \neq y \Rightarrow D(x, y) > 0$

Satisfying:
$U = \mathbb{R}$, $D(x, y) = |x - y|$
Example: $D(2, 5) = |2 - 5| = 3 > 0$

Violating:
$U = \mathbb{R}$, $D(x, y) = 0$ for all x, y
Example: $D(2, 5) = 0$ despite $2 \neq 5$

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(4) Symmetry — $D(x, y) = D(y, x)$

Satisfying:
$U = \mathbb{R}$, $D(x, y) = |x - y|$
Example: $D(2, 5) = |2 - 5| = 3$, $D(5, 2) = |5 - 2| = 3$

Violating:
$U = \mathbb{R}$, $D(x, y) = y - x$
Example: $D(2, 5) = 3$, $D(5, 2) = -3 \neq 3$

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(5) Triangle Inequality — $D(x, z) \leq D(x, y) + D(y, z)$

Satisfying:
$U = \mathbb{R}$, $D(x, y) = |x - y|$
Example: $x = 2$, $y = 4$, $z = 6$
$D(2, 6) = 4 \leq D(2, 4) + D(4, 6) = 2 + 2 = 4$

Violating:
$U = \mathbb{R}$, $D(x, y) = (x - y)^2$
Example: $x = 2$, $y = 3$, $z = 5$
$D(2, 5) = 9$, $D(2, 3) + D(3, 5) = 1 + 4 = 5$
$9 > 5 \Rightarrow$ triangle inequality violated

Each property (1–5) must hold simultaneously for D to be a valid metric on U.