ℓ_p Norms — Product Metrics and Induced Distances

Definition

Let (U_i, D_i) be metric spaces for $i = 1, \ldots, d$.
Their **product space** is

$$\hat{U} = U_1 \times U_2 \times \cdots \times U_d$$

For some p with $1 \leq p \leq \infty$, define the **product metric** as

$$\hat{D}_p(x, y) = \Big(\sum_{i=1}^{d} D_i(x_i, y_i)^p \Big)^{1/p}$$

If all $U_i = \mathbb{R}$ and $D_i(a, b) = |a - b|$, then $\hat{U} = \mathbb{R}^d$ and

$$D_p(x, y) = \|x - y\|_p$$

is the standard ℓ_p **distance**.

Common ℓ_p Norms

Name	Symbol	Definition	Unit Ball Shape
Manhattan (L_1)	$\|x\|_1$	$\sum_{i=1}^{d} \|x_i\|$	Diamond
Euclidean (L_2)	$\|x\|_2$	$\Big(\sum_{i=1}^{d} x_i^2 \Big)^{1/2}$	Circle / Sphere
Chebyshev (L_∞)	$\|x\|_\infty$	$\max_i \|x_i\|$	Square / Cube

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Induced Distance Functions

Metric	Formula	Interpretation
ℓ_1 **Distance (Manhattan)**	$D_1(x, y) = \sum_i \|x_i - y_i\|$	Grid-based “taxicab” distance
ℓ_2 **Distance (Euclidean)**	$D_2(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$	Straight-line distance
ℓ_∞ **Distance (Chebyshev)**	$D_\infty(x, y) = \max_i \|x_i - y_i\|$	Largest coordinate difference

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Properties

1. Each $\|\cdot\|_p$ defines a **valid norm** on $\mathbb{R}^d$.

2. The induced distance $D_p(x, y) = \|x - y\|_p$ is a **metric** satisfying all metric axioms.

3. Changing p changes the geometry of the space:
   - Small $p \to$ sharper shapes (L_1)
   - Large $p \to$ rounder shapes (L_2)
   - $p \to \infty \to$ box-shaped geometry (L_∞)
4. All ℓ_p norms are **equivalent** in finite-dimensional spaces:

$$c_1\|x\|_p \le \|x\|_q \le c_2\|x\|_p$$

   for constants $c_1, c_2 > 0$ depending on dimension d.

Geometric Intuition

p	Equation of Unit Ball in $\mathbb{R}^2$	Shape
1	$\{\lvert x_1\rvert + \lvert x_2\rvert = 1\}$	Diamond
2	$\{x_1^2 + x_2^2 = 1\}$	Circle
∞	$\{\max(\lvert x_1\rvert, \lvert x_2\rvert) = 1\}$	Square

As p increases, the ℓ_p ball transitions from a diamond ($p = 1$) to a circle ($p = 2$) to a square ($p = \infty$). This affects neighborhood shape, volume growth, and nearest-neighbor relationships.

Construction Context

Clarkson (2006) defines these as **product metrics**, meaning ℓ_p distances arise naturally by combining d metric coordinates (U_i, D_i) into one higher-dimensional space.
Thus, $(\mathbb{R}^d, D_p)$ is a **metric space** for all $1 \le p \le \infty$.