ℓ_p Norms — Product Metrics and Induced Distances

Definition

The ℓ_p norm is a function that assigns a non-negative real number to each vector in $\mathbb{R}^d$:

$$|\cdot|_p : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$$

This describes a mapping that assigns every vector in d-dimensional space a real number greater than or equal to zero.

For a vector $x = (x_1, x_2, \ldots, x_d)$, the ℓ_p norm measures its size or length according to the parameter p.

Formally:

$$|x|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}, \qquad 1 \leq p < \infty$$

This formula takes each component's absolute value, raises it to the power p, sums these values, and then takes the pth root of that sum. The result is a single non-negative real number representing the vector's length.

and for $p = \infty$:

$$|x|_\infty = \max_i |x_i|$$

The norm evaluates to the maximum absolute component of the vector, which captures the greatest deviation across all dimensions.

Explanation

- $|\cdot|_p$ is the *symbol* for the ℓ_p norm — it is a function name.
- The dot $\cdot$ means "insert any vector here."
- $\mathbb{R}^d$ is the d-dimensional space of vectors $(x_1, x_2, \ldots, x_d)$.
- $\mathbb{R}_{\geq 0}$ denotes the set of all non-negative real numbers — possible lengths.
- The summation symbol $\sum_i$ means "add up all components $i = 1, \ldots, d$."
- $|x_i|$ is the absolute value of the i-th coordinate (negatives don't matter).
- The exponent p controls how strongly large components influence the result.
- The power $1/p$ takes the p-th root, giving a single real number: the vector's length.

Example

Let $x = (3, 4)$ and $p = 2$:

$$|x|_2 = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Calculating using the $p = 2$ norm sums squares of each component and then takes the square root, resulting in the straight-line distance from the origin to the point $(3, 4)$.

Common ℓ_p Norms

Name	Symbol	Definition	Unit Ball Shape
Manhattan (L_1)	$\|x\|_1$	$\sum_{i=1}^{d} \|x_i\|$	Diamond
Euclidean (L_2)	$\|x\|_2$	$\left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$	Circle / Sphere
Chebyshev (L_∞)	$\|x\|_\infty$	$\max_i \|x_i\|$	Square / Cube

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Induced Distance Functions

Metric	Formula	Interpretation
ℓ_1 **Distance (Manhattan)**	$D_1(x, y) = \sum_i \|x_i - y_i\|$	Grid-based "taxicab" distance
ℓ_2 **Distance (Euclidean)**	$D_2(x, y) = \sqrt{\sum_i (x_i - y_i)^2}$	Straight-line distance
ℓ_∞ **Distance (Chebyshev)**	$D_\infty(x, y) = \max_i \|x_i - y_i\|$	Largest coordinate difference

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Distance Examples

Manhattan Distance Example

Consider points $A = (1, 2)$ and $B = (4, 6)$:

$$D_1(A, B) = |1 - 4| + |2 - 6| = 3 + 4 = 7$$

The Manhattan distance sums the absolute differences of each coordinate, representing distance if movement is only allowed along grid lines.

Euclidean Distance Example

For the same points:

$$D_2(A, B) = \sqrt{(1 - 4)^2 + (2 - 6)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

The Euclidean distance measures the shortest straight-line distance between the points.

Chebyshev Distance Example

Again, for points $A = (1, 2)$ and $B = (4, 6)$:

$$D_\infty(A, B) = \max(|1 - 4|, |2 - 6|) = \max(3, 4) = 4$$

The Chebyshev distance considers the maximum absolute difference across any coordinate, reflecting the fewest moves needed in any direction assuming diagonal movement is allowed.

Properties

1. Each $|\cdot|_p$ defines a **valid norm** on $\mathbb{R}^d$.

2. The induced distance $D_p(x, y) = |x - y|_p$ is a **metric** satisfying all metric axioms.

3. Changing p changes the geometry of the space:
   - Small $p \to$ sharper, diamond-like shapes (L_1)
   - Larger $p \to$ rounder shapes (L_2)
   - $p \to \infty \to$ box-shaped geometry (L_∞)

4. All ℓ_p norms are **equivalent** in finite-dimensional spaces:

$$c_1|x|_p \leq |x|_q \leq c_2|x|_p$$

   for constants $c_1, c_2 > 0$ depending on dimension d.
   This relation shows the norms behave similarly in finite dimensions and no norm can be arbitrarily large compared to another.

Geometric Intuition

p	Equation of Unit Ball in $\mathbb{R}^2$	Shape
1	$\lvert x_1\rvert + \lvert x_2\rvert = 1$	Diamond
2	$x_1^2 + x_2^2 = 1$	Circle
∞	$\max(\lvert x_1\rvert, \lvert x_2\rvert) = 1$	Square

With increasing p, the shape of the unit ball transitions from a diamond to a circle to a square, reflecting different "distance" concepts.

Construction Context

According to Clarkson (2006), ℓ_p distances naturally arise as **product metrics**.
Each coordinate space (U_i, D_i) contributes its own metric, and combining them yields a single metric space:

$$\hat{U} = U_1 \times U_2 \times \cdots \times U_d, \quad \hat{D}_p(x, y) = \left(\sum_{i=1}^{d} D_i(x_i, y_i)^p \right)^{1/p}.$$

This defines the combined distance in product spaces, generalizing the ℓ_p norms to coordinate-wise metrics.

If $U_i = \mathbb{R}$ and $D_i(a,b) = |a-b|$, then
$\hat{U} = \mathbb{R}^d$ and $\hat{D}_p(x,y) = |x-y|_p$.
Hence $(\mathbb{R}^d, D_p)$ is a **metric space** for all $1 \le p \le \infty$.

Example: Euclidean Distance Visualization

For $p = 2$, the distance between two points $p = (p_1, p_2)$ and $q = (q_1, q_2)$ in $\mathbb{R}^2$ is

$$d(p,q) = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}.$$

This formula computes the straight-line distance as the length of the hypotenuse of a right triangle formed by the horizontal and vertical differences.
