(578) $\vec{u}(t)$ diff. eq. $\vec{u} \times \vec{u}' = \vec{v} \Rightarrow \boxed{\vec{v} = \vec{v}(t)}$ twice diff. $\vec{v}$ and $\vec{v}'$ never 0 or parallel

scalar product rules [illegible]

differentiate with t

$$\vec{u}' \times \vec{u}' + \vec{u} \times \vec{u}'' = \vec{u} \times \vec{u}'' = \vec{v}'$$

[illegible] cross product $= 0$

$\vec{u}$ [illegible] $\vec{v}'$ are perp.

P.S. $\vec{a} \times \vec{b} = \vec{c}$ [illegible] $\vec{a}$ [illegible] $\vec{b}$ [illegible] perp

[illegible] $\vec{u} \times \vec{u}'' = \vec{v}' \Rightarrow \vec{u}$ [illegible] $\vec{v}$ [illegible] perp.

$\Rightarrow \vec{u}$ is parallel to $\vec{v} \times \vec{v}'$

$\vec{u}$ [illegible] $\vec{v}'$ [illegible] $\vec{v}$ [illegible] perp [illegible] $\Rightarrow \vec{v}' \times \vec{v} = \vec{k}$ [illegible] $\vec{v}'$ [illegible] $\vec{v}$ [illegible] perp. $\Rightarrow \vec{k}$ [illegible] $\vec{u}$ [illegible] parallel

$\vec{u} = f \cdot (\vec{v} \times \vec{v}')$ for some scalar function $f(t)$

const [illegible] const

$$\vec{u}' = f'(v \times v') + f(v \times v')' = f'(v \times v') + f(v' \times v' + v \times v'')$$

$f(t) = \lambda$ [illegible]

$$f(\vec{v} \times \vec{v}') \times [f' \vec{v} \times \vec{v}' + f \vec{v}' \times \vec{v}' + f \vec{v} \times \vec{v}''] =$$

$$= f^2 (\vec{v} \times \vec{v}') \times (\vec{v} \times \vec{v}'')$$

cab — bac formula

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$$

[illegible]: $a = v \times v'$, $b = v$, $c = v''$

$(v \times v') \cdot v = 0$ [illegible] $v \times v'$ [illegible] v [illegible] perp [illegible] dot product [illegible] $\cos(90°) = 0$

[illegible]

$$f^2 (\vec{v}'' \cdot (\vec{v} \times \vec{v}') \cdot \vec{v} - \vec{v} \cdot (\vec{v} \times \vec{v}') \cdot \vec{v}'') = f^2$$

$$f^2 ((\vec{v} \times \vec{v}') \cdot \vec{v}'') \vec{v} = \vec{v}$$

[illegible] $v \neq 0$

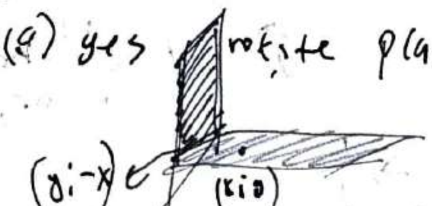
$$f = \frac{1}{(\vec{v} \times \vec{v}') \cdot \vec{v}''}$$

$(a \times b) \cdot c \neq 0$ if linearly independent

$\neq 0$ only solved if $(\vec{v}, \vec{v}'$ and $\vec{v}'')$ consists of linearly independent vectors

$$\vec{u} = f(t)(\vec{v} \times \vec{v}') \Rightarrow \boxed{\vec{u} = \frac{1}{(\vec{v} \times \vec{v}') \cdot \vec{v}''} \cdot (\vec{v} \times \vec{v}')}$$

(279) Bijection ~~of~~ f of (a) plane with itself

(a) yes  rotate plane perp. to itself

(e) 3D space for A, B ims $A B$ and $f(A)$ $f(B)$ on perp.

if yes $(\vec{a} - \vec{b})(f(\vec{a}) - f(\vec{b})) = 0$

~~[illegible]~~ vectors [illegible]

P.S. $\vec{a}$ = Aorigins $\binom{0}{0}$ to $A \cdot \vec{a}$

~~[illegible]~~ $g(\vec{p}) = f(\vec{p}) - f(\vec{0})$

[illegible] f maps the origins to itself, [illegible] $g(\vec{p})$ [illegible]

[illegible] $\vec{b} = (0,0,0)$ ~~$(\vec{a})(f(\vec{a}) - f(\vec{b})) = 0$~~

~~$\vec{a} \cdot f(\vec{b}) - \vec{b} \cdot f(\vec{a}) = 0$~~

$\underbrace{\vec{a} \cdot f(\vec{a})}_{=0} + \vec{a} \cdot f(\vec{b}) + \vec{b} f(\vec{a}) + \underbrace{\vec{b} f(\vec{b})}_{=0} = 0$

$\vec{a} \cdot f(\vec{b}) - \vec{b} f(\vec{a}) = 0$

for any vectors $\vec{a}, \vec{b}, \vec{c}$, m and m, n real numbers

(1) $m(\vec{a} \cdot f(\vec{b}) + \vec{b} \cdot f(\vec{a})) = 0$

(2) $n \cdot (\vec{a} \cdot f(\vec{c}) + \vec{c} \cdot f(\vec{a})) = 0$

(3) $a \cdot f(m\vec{b} + n\vec{c}) + (m\vec{b} + n\vec{c}) \cdot f(\vec{a}) = 0$

$(1) + (2) - (3) = \vec{a} \cdot mf(\vec{b}) + nf(\vec{c}) - f(m\vec{b} + n\vec{c}) = 0$

$f(m\vec{b} + n\vec{c}) = m \cdot f(\vec{b}) + nf(\vec{c})$

$\Leftrightarrow$ f is linear and determined by images of unit vectors $i = (1,0,0)$. $j = (0,1,0)$ and $k = (0,0,1)$ ..

If $f(\vec{i}) = (a_1, a_2, a_3)$ $f(\vec{j}) = (b_1, b_2, b_3)$ and $f(\vec{k}) = (c_1, c_2, c_3)$

for $\vec{x}$ vector

$$f(\vec{x}) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \vec{x}$$

CAB - BAC identity

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{b}\cdot\vec{a})\vec{c}$$

~~a × (b × c)~~

$$\vec{a}\times(\vec{b}\times\vec{c}) = A(BC - CB) - (BC - CB)A$$

A, B, C skew matrix for which $A^T = -A$, so $A = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$

bijection from $\mathbb{R}^3$ to $f(x,y,z)$

(5B) $f(\vec{v}\cdot\vec{w}) = [f(\vec{v}), f(\vec{w})]$

bijection $su(2)$ of 2×2 matrix

$\mathrm{Tr}(A) = 0$, $\bar{A}^T = -A$ matrix

$su2$ generally has

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

a is imaginary, b and c are complex

trace $\to \mathrm{tr}(A) = 0$

$a + (-a) = 0$

$\boxed{c = \bar{b}}$

$c = a + ibK$

$b = a - ibK$

$$\bar{A}^T = \begin{bmatrix} -a & c \\ b & a \end{bmatrix}$$

define

$$f(x,y,z) = \begin{bmatrix} -iz & y - ix \\ y + ix & iz \end{bmatrix}$$

$$\bar{f}^T(x,y,z) = \begin{bmatrix} iz & y+ix \\ y-ix & -iz \end{bmatrix} = -f(x,y,z)$$

(5D) on $so(3)$ * A to $\vec{a} = (a_1, a_2, a_3)$: $SO(3)$

B to $\vec{b} = (b_1, b_2, b_3)$

i = row, j = column

then $(i\ j)$ entry of $A * B = (-1)^{i+j} a_{4-j} b_{4-i}$

Prove $CBA - BCA = (A*C)B - (A*B)C$

$so(3) \Rightarrow A^T = -A$ so

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

B - similarly

$$A\cdot B = \begin{bmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 & a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 & a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 \end{bmatrix}$$

$\mathrm{tr}(AB) = a_3b_3$

$= (a_2b_3 - a_3b_2) + (a_1b_2 - a_2b_1) + (a_3b_1 - a_1b_3)$

$A * B = AB - \frac{1}{2}\mathrm{tr}(AB)$ ... prove ...

$$\underbrace{CBA - BCA}_{\text{CAB - BAC}(1)} + \underbrace{ABC - ACB}_{\text{CAB - BAC}(2)} = \underbrace{-\tfrac{1}{2}\mathrm{tr}(AC)B}_{(1)} + \underbrace{\tfrac{1}{2}\mathrm{tr}(AB)C}_{(2)}$$

$\vec{a}\cdot\vec{b} = -\frac{1}{2}\mathrm{tr}(AB)$

3:20

559 $f(x,y)$ xy plane

$$\frac{\partial f}{\partial x} + a f \cdot \frac{\partial f}{\partial y} = 0$$

$$\cancel{\partial f + f \cdot \frac{\partial x}{\partial y} = 0 \Rightarrow \frac{\partial y}{\partial x} = \frac{\partial f}{f} = \frac{1}{f} \cdot \partial f}$$

$$f(x) \cdot f'\left(\frac{a}{x}\right) = x \Rightarrow f\left(\frac{a}{x}\right) \cdot f'(x) = \frac{a}{x} \qquad g'(x) = f'(x) \cdot f\left(\frac{a}{x}\right) + f(x) \cdot f'\left(\frac{a}{x}\right)$$

$$g(x) = f(x) \cdot \frac{x}{f(x)} = x$$

$f: (0,\infty) \to (0,\infty)$

558 $$\boxed{f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}}$$; define $g(x) = f(x) \cdot f'\left(\frac{a}{x}\right)$

If ~~g is const function~~ $\Rightarrow \boxed{g'(x) = 0}$

$f(x) \cdot f'\left(\frac{t}{x}\right) = \text{const}$ f is differentiable

for $x \to a$

$$\boxed{f\left(\frac{a}{x}\right) f'(x) = g\left(\frac{a}{x}\right)}$$

$$\cancel{f(x) \cdot f'\left(\frac{a}{x}\right) = g(x)} \quad f\left(\frac{a}{x}\right) \cdot f'(x) = \frac{a}{x}$$

$$g'(x) = f'(x) \cdot f\left(\frac{a}{x}\right) + f(x) \cdot f'\left(\frac{a}{x}\right)\left(-\frac{a}{x^2}\right) = f'(x) \cdot f\left(\frac{a}{x}\right) - \frac{a}{x^2} f\left(\frac{a}{x}\right) f(x) =$$

$$= \frac{a}{x} - \frac{a}{x} = 0$$

$$b = g(x) = f(x) \cdot f\left(\frac{a}{x}\right) = f(x) \cdot \frac{a}{x} \cdot \frac{1}{f'(x)} \Rightarrow \frac{f'(x)}{f(x)} = \frac{a}{bx}$$

$$\ln f(x) = \frac{a}{b} \ln x + \ln c$$

$$\boxed{f(x) = c \cdot x^{\frac{a}{b}}}$$

$$c \cdot \frac{a}{b} \cdot \frac{a^{\frac{a}{b}-1}}{x^{\frac{a}{b}-1}} = \frac{x}{c \cdot x^{\frac{a}{b}}}$$

$$c^2 a^{\frac{a}{b}} = b$$

$c = \text{const}$

$$b = a^{\frac{a}{b}} \qquad f_0(x) = \sqrt{b}\left(\frac{x}{\sqrt{a}}\right)^{\frac{a}{b}}$$

(5.7.7) $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a} \neq \vec{0}$ prove $\vec{a}+\vec{b}+\vec{c} = 0$

$(\vec{a}\times\vec{b}) - (\vec{b}\times\vec{c}) = 0$

[illegible]

$\vec{b}\times\vec{c} = -(\vec{c}\times\vec{b})$

1) $\underbrace{(\vec{a}\times\vec{b}) + (\vec{c}\times\vec{b})}= 0$

$\vec{b}\times(a+c) = \underbrace{b\times a + b\times c}$

$1 \Rightarrow \vec{b}\times(\vec{a}+\vec{c}) = 0$ [illegible] b and $a+c$ parallel

[illegible] $\times$ cross [illegible] $=0$

$\vec{b} = \lambda(\vec{a}+\vec{c})$

scalar const [illegible] $\vec{b}$ $\vec{a}$ $\vec{c}$

2) [illegible] $\vec{c}\times(\vec{a}+\vec{b}) = 0$

$\vec{c}\times(\vec{a}+\lambda\vec{a}+\lambda\vec{c}) = 0$

$\vec{c}\times\vec{a} + \lambda(\vec{c}\times\vec{a}) + \underbrace{\lambda\,\vec{c}\times\vec{c}}_{0} = 0$ [illegible]

$(1+\lambda)(\vec{c}\times\vec{a}) = 0$

$\vec{c}$ $\vec{a}$ [illegible] $\vec{c}$ ps $\vec{a}$ [illegible] parallel

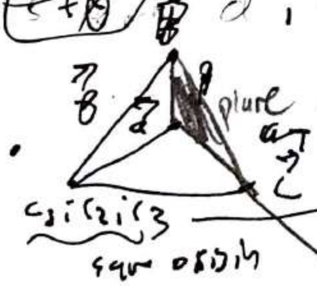
$\boxed{\lambda = -1}$

$\Rightarrow \vec{a}+\vec{b}+\vec{c} =$ $\quad \Rightarrow \vec{b} = -1(\vec{a}+\vec{c}) = -\vec{a}-\vec{c}$

$= \vec{a} - (\vec{a}+\vec{c}) + \vec{c} = 0$

(5.7.8) $\vec{a},\vec{b},\vec{c}$ linear indep. same origin

$\underline{\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}}$ is perp to [illegible].



[illegible] $\vec{b}-\vec{a}$ [illegible] endpoints [illegible]

same origin [illegible]

same origin $(0,0,0)$

$\underbrace{\vec{b}-\vec{a}}$ [illegible] $\underbrace{\vec{c}-\vec{a}}$ [illegible] plane [illegible].

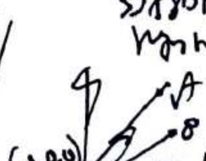
[illegible] perp. [illegible]

$(\vec{b}-\vec{a})\times(\vec{c}-\vec{a}) =$

$= \vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{a}\times\vec{c} - \underbrace{\vec{a}\times\vec{a}}_{0}$

$\vec{a}\times\vec{c} = -(\vec{c}\times\vec{a})$ [illegible]

$= \vec{b}\times\vec{c} + \vec{c}\times\vec{a} + \vec{a}\times\vec{b}$



$\vec{A} = A - 0 = A$

and $\vec{B}-\vec{B} = A - B$

(75) $\vec{b}$ and $\vec{c}$ are perp.

$\vec{a}$ is ~~$\vec{o}$~~ $\vec{b}$ [illegible]

$\vec{x} \cdot \vec{a} = m$

$\vec{x} \times \vec{b} = \vec{c}$

$$\vec{a} \times (\vec{x} \times \vec{b}) = \vec{a} \times \vec{c}$$

CAB – [illegible]

$$(\vec{a} \cdot \vec{b})\vec{x} - (\vec{a} \cdot \vec{x})\vec{b} = \vec{a} \times \vec{c}$$

$$\vec{x} = \frac{m}{\vec{a} \cdot \vec{b}}\vec{b} + \frac{1}{\vec{a} \cdot \vec{b}}\vec{a} \times \vec{c}$$

(74) form triangle ([illegible] triangle [illegible]) ; [illegible] [illegible] $\vec{a} + \vec{b} + \vec{c} = 0$

[illegible] $\vec{u} + \vec{v} + \vec{w} = 0$

$\vec{u}$ $\vec{c}$ = [illegible], [illegible] $\vec{u} = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$ [illegible] $\vec{c}$-[illegible] $= 0$

[illegible] $\vec{u} \cdot \vec{c} = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{c}) - (\vec{c} \cdot \vec{a})(\vec{b} \cdot \vec{c})$ $= 0$

[illegible] $\cdot$ c-[illegible] [illegible] [illegible] [illegible].

[illegible] c-[illegible] [illegible] [illegible].

$(\vec{b} \cdot \vec{c}) \cdot (\vec{a} \cdot \vec{c})$ [illegible] [illegible]

$(\vec{c} \cdot \vec{a})(\vec{b} \cdot \vec{c})$ $\vec{b} \cdot \vec{c}$ [illegible]

[illegible] c-c [illegible]

[illegible] $\vec{a}$ ~~[illegible]~~

~~[illegible]~~

u [illegible] c

[illegible] ~~orthogonal~~. perp.

[illegible]

$\vec{v}$ [illegible] a-[illegible]

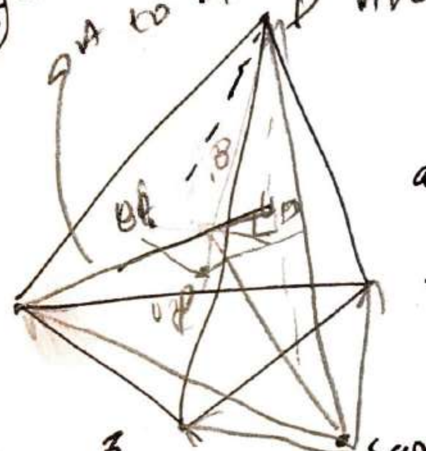
$\vec{w}$ [illegible] b-[illegible] $\Rightarrow$ [illegible] [illegible] [illegible] $\Rightarrow$

[illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] [illegible] a, b, c

[illegible] [illegible] [illegible] $\Rightarrow$ [illegible] [illegible].

(536) height from A to BDC · [illegible] D ABC



$\vec{r_a} = \vec{a} + \lambda\left(\underbrace{\frac{\vec{b}+\vec{c}+\vec{d}}{3} - \vec{p}}_{\text{directional vector}}\right)$

altitude from A to △BDC

$G_a = \frac{b+c+d}{3}$ - centroid

$G_b = \frac{a+c+d}{3}$ ...

$\therefore \lambda = 3$

$\vec{r_a} = \underbrace{\vec{a}+\vec{b}+\vec{c}+\vec{d}}_{\text{same for all 4 vertices}} - 3\vec{p}$

(Q.E.D.)

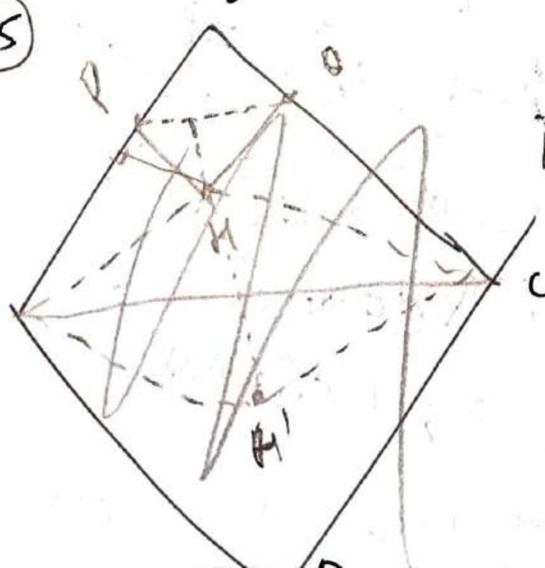
for converse ([illegible])

If 4 altitudes ~~are concurrent~~ intersect at point H

$\vec{r_a}' = \underbrace{\left(\frac{\vec{b}+\vec{c}+\vec{d}}{3}\right)}_{\text{position}} + \lambda'\underbrace{(\vec{a}-\vec{h})}_{\text{directional}}$

$\vec{r} = \vec{a} + \lambda\vec{b}$

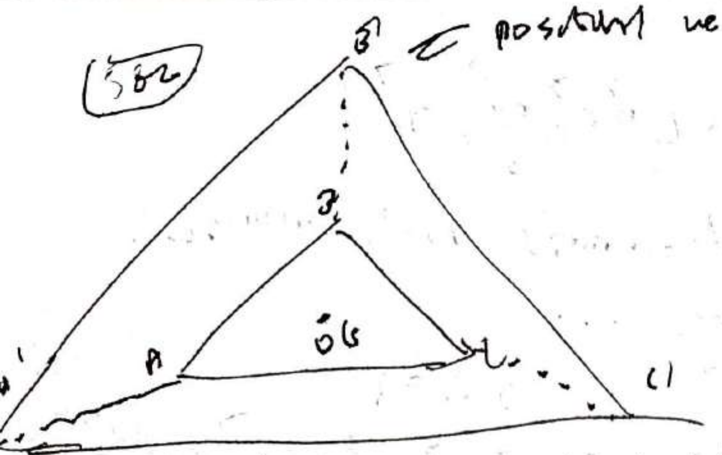
(535)



HC ⊥ BP

H'C ⊥ HQ

582 ← position vectors

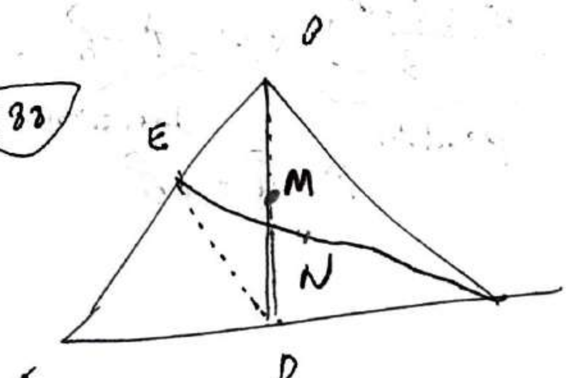


Δ's have same centroid ⇒

$\vec{A}+\vec{B}+\vec{C}=\vec{A'}+\vec{B'}+\vec{C'}$, form triangle with AA', BB'... so $AA'+BB'+CC' \Rightarrow$

$\Leftrightarrow \vec{AA'}+\vec{BB'}+\vec{CC'}=0$

583



$S_{BCDE}=4\cdot S_{\triangle AMN}$

$S_{\triangle AMN}=\frac{1}{2}||\vec{AM}\times\vec{AN}||=$

$=\frac{1}{2}||\frac{\vec{BD}}{2}\times\frac{\vec{EC}}{2}||=$ AB, AC

$=\frac{1}{8}||\vec{BD}\times\vec{EC}||=$ AE, AC [illegible]

$=\frac{1}{8}||(\vec{AB}+\vec{AD})\times(\vec{AE}+\vec{AC})||=$

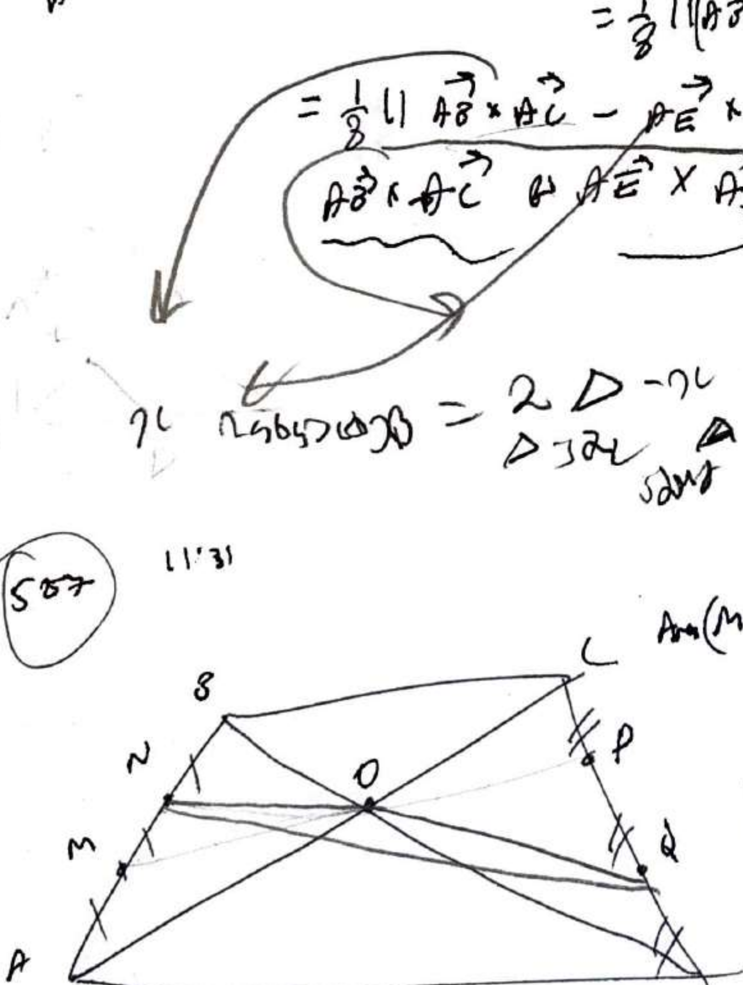
$=\frac{1}{8}||\vec{AB}\times\vec{AC}-\vec{AE}\times\vec{AD}||=\frac{1}{2}||\vec{AM}\times\vec{AN}||$

$\vec{AB}\times\vec{AC}$ & $\vec{AE}\times\vec{AD}$ are perp to the plane of triangle.

[illegible] $=2\,\Delta$ [illegible] vectors [illegible]

Δ [illegible] Area [illegible]

587 11:31



Area(MOP) = Area(NOQ)

$\frac{1}{2}||\vec{ON}\times\vec{OQ}||=\frac{1}{2}||(\frac{1}{3}\vec{OA}+\frac{2}{3}\vec{OB})\times(\frac{2}{3}\vec{OD}+\frac{1}{3}\vec{OC})||$

$\vec{OA}\,||\,\vec{OC}$

$\vec{OB}\,||\,\vec{OD}$

[illegible]

$\vec{OA}\times\vec{OC}=0$

$=\frac{1}{2}(\vec{OA}\times\vec{OD}+\vec{OB}\times\vec{OC})$

[illegible]

MOP [illegible]

$\vec{ON}=\frac{2}{3}\vec{OA}+\frac{1}{3}\vec{OB}$ [illegible]

$X=\frac{\mu\cdot A+\lambda\cdot B}{\mu+\lambda}$

μ & λ [illegible]

(589) [illegible] (1) [illegible]

midpoints of sides are $M\left(\frac{a}{2};\frac{b}{2}\right)$

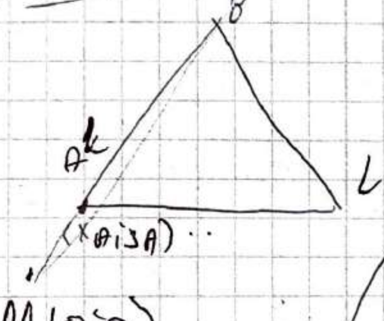
$N\left(\frac{c}{2};\frac{b}{2}\right)$; $P\left(\frac{c}{2};\frac{d}{2}\right)$; $Q\left(\frac{a}{2};\frac{d}{2}\right)$

MP and NQ have same midpoint

$MP=\left(\dfrac{\frac{a}{2}-\frac{c}{2}}{2};\dfrac{\frac{b}{2}-\frac{d}{2}}{2}\right)$ $NQ=\left(\dfrac{\frac{a}{2}-\frac{c}{2}}{2};\dfrac{\frac{b}{2}-\frac{d}{2}}{2}\right)$

[illegible] $\left(\frac{a+c}{4};\frac{b+d}{4}\right)$ [illegible] parallelogram.

(590) M [illegible] origin



centroids of MAB, MAC, MBC

$G_A=\left(\dfrac{(x_A-0)+(x_B-0)}{3};\dfrac{y_B+y_A}{3}\right)$

$G_B=\left(\dfrac{x_A+x_C}{3};\dfrac{y_A+y_C}{3}\right)$

$G_C=\left(\dfrac{x_B+x_C}{3};\dfrac{y_B+y_C}{3}\right)$

[illegible] vectors [illegible]
substant ~~coordinat~~ of [illegible] (reflection, centroid)

$G_{ABC}=\dfrac{(x_A+x_B+x_C;y_A+y_B+y_C)}{3}$ — coordinates of A, B, C

[illegible] , 3-[illegible]

[illegible] triangle G_A G_B G_C [illegible] $\triangle ABC$-[illegible] reflection-[illegible]

$(x_A+x_B+x_C;y_A+y_B+y_C)$-[illegible]

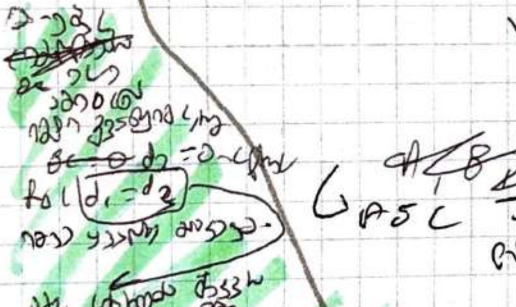
[illegible] $\frac{1}{3}$-[illegible] respect to origin.

[illegible]

[illegible] $d_1=d_2$ [illegible]

FP; PL; PK
[illegible] (FKL [illegible])

(591)



$d_1;d_2;d_3$ are sides of triangles. triangle [illegible]

$d_1<d_2+d_3$

$d_2<d_1+d_3$

$d_3<d_1+d_2$

$d_2+d_3-d_1>0$

$F(d)=d_2+d_3-d_1$ F(d) [illegible] function of coordinates (x; y) of [illegible] to line

~~$f(x)=ax+by+c$~~

$d_1;d_2;d_3$ [illegible] perp to lines ($ax+by+c$ [illegible]) AB; BC; AC.

[illegible] $f(d)$-[illegible] $F(x;y)$, [illegible]

K-[illegible] BC-C [illegible]

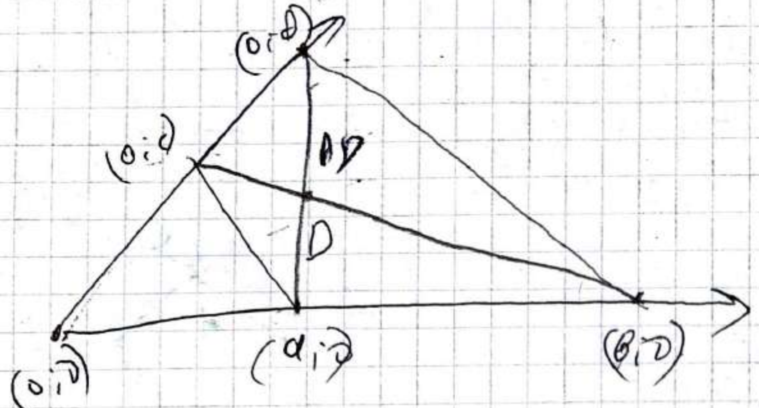
$d_1=d_3$

[illegible] bisector [illegible] K

[illegible] $f(x;y)=0$. [illegible] $d_1=d_2+d_3$, [illegible] $F(d)$ BC-C? ($d_1=d_3$) [illegible] [illegible] $d_2=0$

4:40

coordinate axis be sides of quadrilateral.



line eq: $ax + by = c$

AD diagonal $= \frac{d-0}{0-a} = \frac{-d}{a}$ slope. line: $y = -\frac{d}{a}x + c$

in standard form

$\frac{1}{a}x + \frac{1}{d}y = 1$

BC diag. $-\frac{c}{b}$; $y = -\frac{c}{b}x + c$

$\frac{x}{b} + \frac{y}{c} = 1$

BC [illegible] AD-C [illegible] D [illegible]

$$\left.\begin{aligned}\frac{x}{b} + \frac{y}{c} &= 1\\ \frac{x}{a} + \frac{y}{d} &= 1\end{aligned}\right| \Rightarrow x \; y$$

D [illegible] $\left(\frac{ab(c-d)}{ac-bd}; \frac{cd(a-b)}{ac-bd}\right)$

AD [illegible] BC $-$ diagonal midpoint $\frac{a}{2}; \frac{c}{2}$

$\frac{b}{2}; \frac{d}{2}$

[illegible] diagonal midpoint $= \left(\frac{ab(c-d)}{2(ac-bd)}; \frac{cd(a-b)}{2(ac-bd)}\right)$

[illegible] are collinear $) \Rightarrow$

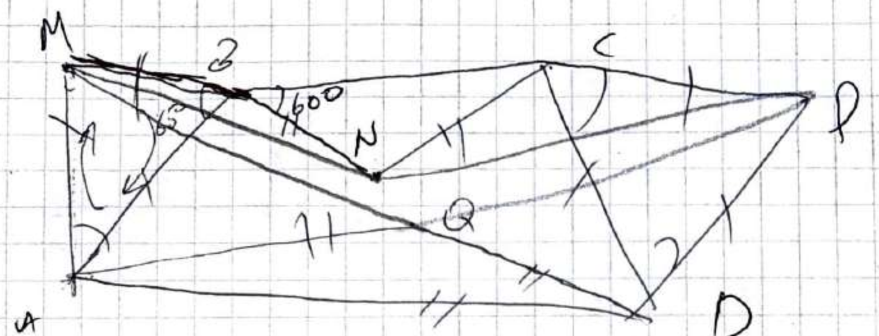
$(x_1; y_1)$; $(x_2; y_2)$ (x_3, y_3) collinear,

$$\frac{1}{2(ac-bd)}\begin{vmatrix} a & c & 1 \\ b & d & 1 \\ ab(c-d) & cd(a-b) & ac-bd \end{vmatrix} = 0$$

if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

[illegible]

(600) convex quadrilateral ABCD



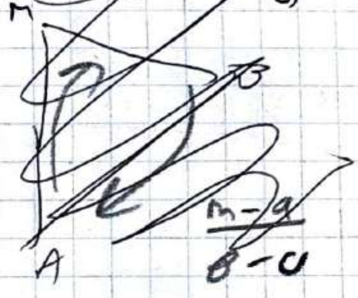
stage MNPQ

[illegible] ? ([illegible]) [illegible]

[illegible]

$$\frac{m-a}{b-a} = \frac{n-c}{b-c} = \frac{p-c}{d-c} = \frac{q-a}{d-a} = \varepsilon$$

[illegible] $b-c$, $b-a$ [illegible]



[illegible]

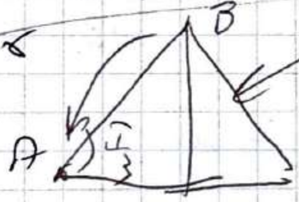
$$e^{\frac{i\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \quad w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

[illegible]

[illegible] convex quadrilateral [illegible]

$\frac{\pi}{3}$ re [illegible]

~~[illegible]~~ ~~$A = B \cdot e^{\frac{i\pi}{3}}$~~



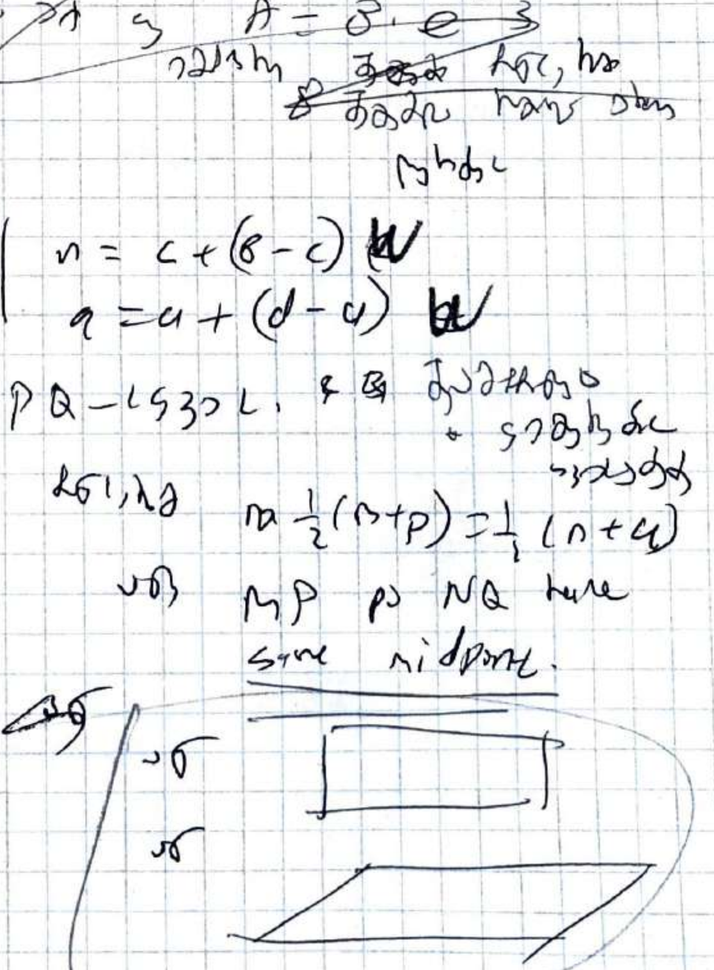
[illegible]

$$m = a + (b-a)w \qquad n = c + (b-c)w$$
$$p = c + (d-c)w \qquad q = a + (d-a)w$$

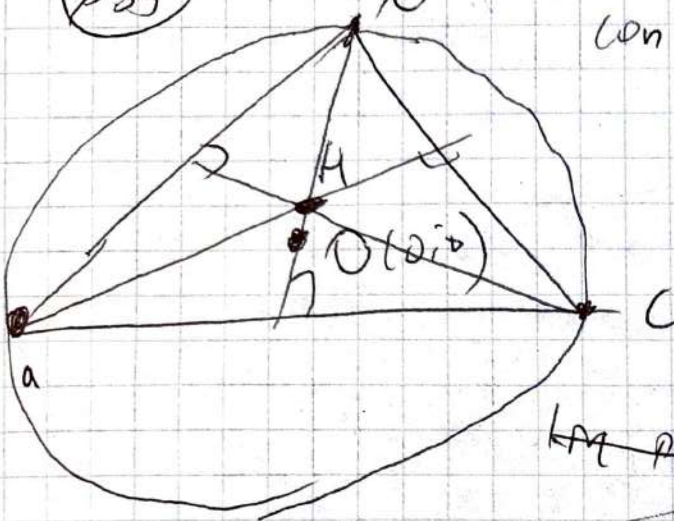
~~[illegible]~~ MNPQ – [illegible]

[illegible] $\frac{1}{2}(m+p) = \frac{1}{2}(n+q)$

[illegible] MP [illegible] NQ have same midpoint.



8g9



orthocenter [illegible]

centroid G

circumcenter O [illegible]

$$\frac{OG}{OH} = \frac{1}{2}$$

coord. of centroid = $\dfrac{a+b+c}{3}$

~~line passing~~

[illegible] O',

line ~~passing through A and H~~

is ~~perp to B and C~~ line

$\frac{1}{3}\cdot(a+b+c)$ [illegible]

& [illegible] orthocenter.

[illegible]

[illegible] $\triangle$ [illegible]

[illegible], orthocenter [illegible]

3.([illegible])

& [illegible] centroid

[illegible]

a to a+b+c [illegible]

line

from B to C

$$(a+b+c) - (a) = b+c$$

$$c-b$$

arg. of $\left(\dfrac{b-c}{b+c}\right) =$

$$\arg(-s) = \pm\frac{\pi}{2}$$

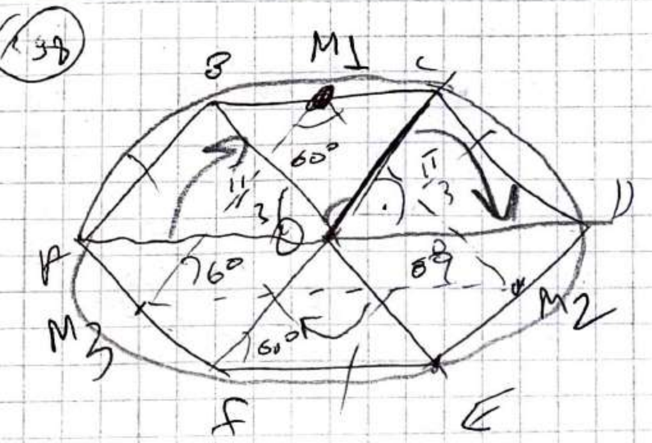
[illegible]

[illegible]

[illegible]

O G H

(598)



[illegible]:

if $AB = CD = EF = r$

M_1, M_3, M_2 [illegible] ტოლგვერდა [illegible]

B-ს კოორდ = ~~[illegible]~~

ვთქვათ A-ს კოორდ = a $\Rightarrow B = a\cdot e^{\frac{i\pi}{3}}$

[illegible] შემობრუნებით [illegible]

C-ს კოორდ D-ს $D = y\cdot e^{\frac{i\pi}{3}}$

midpoints იქნება $= M_1 = \frac{1}{2}(a e^{\frac{i\pi}{3}} + y)$ E-ს კოორდ.

$= M_2 = \frac{1}{2}(y e^{\frac{i\pi}{3}} + E)$

$= M_3 = \frac{1}{2}(E\cdot e^{\frac{i\pi}{3}} + a)$

[illegible] M_2M_3 და M_1M_2 [illegible]

$$\frac{M_3 - M_2}{M_1 - M_2} = e^{\frac{i\pi}{3}}$$

ე.ი. $\frac{i\pi}{3}$-ით შემობრუნება [illegible]

(603) $A_0, A_1 \dots A_n$ vertices of regular n-gon

$A_0A_1 \cdot A_0A_2 \cdots A_0A_{n-1} = n$

A-ების კოორდ $A_k = e^{\frac{2\pi i\cdot k}{n}}$, სადაც $A_0 = e^{\frac{2\pi i\cdot 0}{n}} = e^0 = 1$

A_0A_1 ის distance $\prod(z-\epsilon^i)$

$|(1-\epsilon)(1-\epsilon^2)\cdots(1-\epsilon^{n-1})|$

in general

$$(z-\epsilon)(z-\epsilon^2)\cdots(z-\epsilon^{n-1}) = \frac{1}{z-1}(z-1)(z-\epsilon)\cdots(z-\epsilon^{n-1})$$

$$= \frac{1}{z-1}(z^n - 1) = z^{n-1} + z^{n-2} + \dots + 1$$

$z = 1$ ჩასმით [illegible] $\underbrace{1+1+1\dots}_{n} = n$

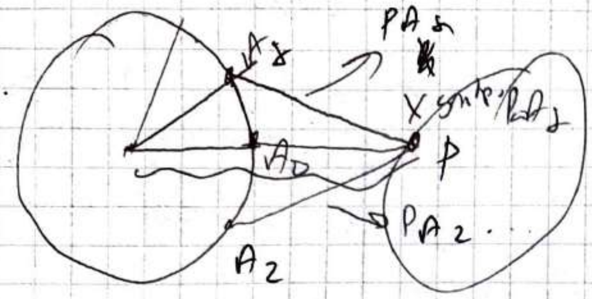
II [illegible] [illegible]

[illegible] A_0 [illegible] between O ~~[illegible]~~ and P

[illegible] $(P\cdot A_0)\cdot(P\cdot A_1)\,(\ldots)\cdots(PA_{n-1}) = x^n - 1$

$A_0A_1 \cdot A_0A_2 \ldots A_0A_{n-1} = \cancel{\ldots}$

$= \cancel{x^{n-1} + x^{n-2} + \ldots + 1}$



[illegible]

A_0A_1 [illegible] $= \dfrac{x^n-1}{x-1}$

$\cancel{A}$ $\dfrac{x^n-1}{x-1} \Rightarrow$ [illegible] $\overline{x-1}$??

[illegible]

[illegible] A_0 [illegible] distance

$x \to 1$ [illegible] P [illegible]

x [illegible] real coordinate

$x \to 1$ [illegible] A_0-[illegible]

$x^n - 1$ [illegible]

[illegible]

~~$P\cdot(A_0, A_1 \ldots A_{n-1}$ [illegible]) [illegible]~~

[illegible] $r = 1$ [illegible] $x^n - 1$.

$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} \ldots)$

[illegible] $A_1, A_2 \ldots A_n$ radius $= 1$ circumcircle. find max value

$\prod_{k=1}^{n} PA_k$ as P ranges over circumcircle

complex number [illegible]

$\prod_{k=1}^{n} |z - \epsilon^k|$ as z ranges over unit disk

$= \underbrace{|z^n - 1|}_{\text{triangle}} \le |z^n| + 1 = 2$

$z =$ radius max

[illegible] $|z| = r$ max

how z is n^{th} root of -1

[illegible]

P [illegible]

P-[illegible] real value [illegible], ~~[illegible]~~ (disk [illegible])

z [illegible]

[illegible] complex number [illegible]

[illegible]

u:46

complex numbers in plane geometry, how to angles ($\frac{\pi}{4}$: $\frac{\pi}{3}$: $\frac{\pi}{2}$ ...) of regular polygons.

line passing through z_1 and z_2 $\quad z = t \cdot z_1 + (1-t) z_2$

θ between 2 lines ($z_1 z_2$) and ($z_3 z_4$) arg of $\frac{z_1 - z_2}{z_3 - z_4}$

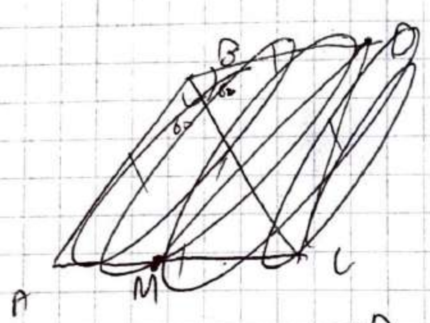
length of segment determined by z_1 and z_2 $\quad |z_1 - z_2|$

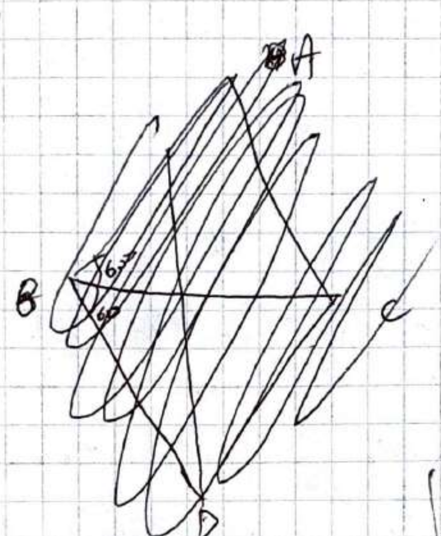
vertices of regular $\underline{n\text{-gon}}$ can be chosen, up to scaling factor $1, \epsilon, \epsilon^2 \dots \epsilon^{n-1}$ where $\epsilon = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

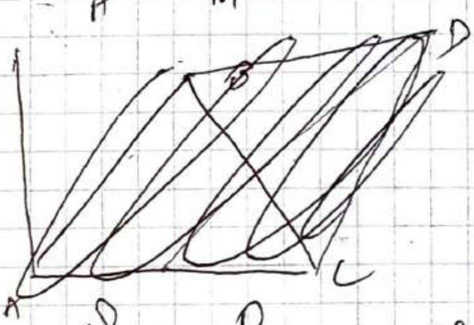
n^{th} root of unity

into 1.

ABC, BCD equilateral $\triangle$-s shown







BM and NC angle

B, M, N, C

$\triangle BBC$ & $\triangle BCD$

sharing one side $\overline{BC}$

BM and CN is $\frac{\pi}{3}$

$\theta = ?$

A B-C $\quad$ angle $= 0$
CA-B $= 1$ $\Big\}$ > ... say $B = e^{\frac{i\pi}{3}}$

A-C angle ... $e^{\frac{i\pi}{3}}$

so D-C $\quad e^{-\frac{i\pi}{3}}$

N's coordinate $= t \cdot e^{\frac{i\pi}{3}}$

ND and AC equations

ND: $\quad z = \alpha \cdot t e^{\frac{i\pi}{3}} + (1-\alpha) e^{-\frac{\pi}{3} i}$ $\Big\}$ intersection ...

parameter

AC: $\quad z = \beta e^{\frac{i\pi}{3}} + (1-\beta) \cdot 1$

$\alpha \cdot t \frac{1+i\sqrt{3}}{2} + (1-\alpha)\frac{1-i\sqrt{3}}{2} = \beta \cdot \frac{1+i\sqrt{3}}{2} + (1-\beta)$

Re and Im equality $\quad \alpha \cdot t + (1-\alpha) = \beta + 2(1-\beta) \quad \alpha t - (1-\alpha) = \beta$

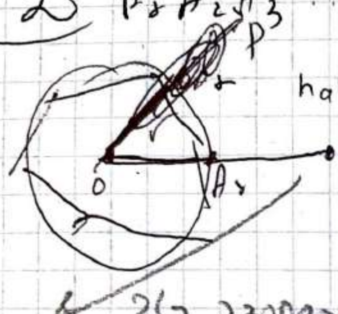
BM angle ... same line ea. so

$\boxed{\alpha = \frac{1}{t}}$ $\Rightarrow$ M's coordinate $e^{\frac{i\pi}{3}} + (1 - \frac{1}{t}) e^{-\frac{i\pi}{3}}$

angle between BM and CN

$$\frac{e^{\frac{i\pi}{3}} + (1 - \frac{1}{t}) e^{-\frac{i\pi}{3}}}{t e^{\frac{i\pi}{3}} - 1} = \cdots \frac{1}{t} e^{-\frac{i\pi}{3}}$$

<u>Mko 2</u> $A_1 A_2 A_3 \dots A_n$ regular polygon



half line OA_1 chose point P such, A_1 is between O and P.

Prove $\prod_{i=1}^{n} PA_i = PO^n - r^n$

real where ... Half line ... A_1 ... P - ... st-gp.

vertices in complex plane such $A_i = r \cdot \epsilon^i$

ϵ is n^{th} root of unity

coordinate of P is real number $r x$ real number

$$\prod_{i=1}^{n} PA_i = \prod_{i=1}^{n} |rx - r\epsilon^i| = r^n \prod_{i=1}^{n} |x - \epsilon^i| =$$

distance from P to A_i

$$= r^n \left| \prod_{i=1}^{n} (x - \epsilon^i) \right|$$

$x^n - 1$ ...

ϵ^i ... n^{th} root of unity

$w^n = 1$

$w^n - 1 = 0$

$x^n - 1 = 0$

$x^n - 1 = \prod (x - \epsilon_i)$

$= r^n (x^n - 1) =$

$= (rx)^n - r^n =$

$= (PO)^n - r^n$

(592) A_i , A_j

A_iA_j midpoint [illegible] $M_{ij} = \left(\frac{x_i+x_j}{2}, \frac{y_i+y_j}{2}\right)$

midpoint-ის [illegible] x coordinate x [illegible] [illegible] y coordinate y [illegible]

$A_1 \quad A_2 \quad A_3 \quad \dots \quad A_n$

midpoints (A_1A_2), (A_1A_3) ... (A_1A_n) ~~[illegible]~~ are all $\rightarrow (n-1)$ [illegible]

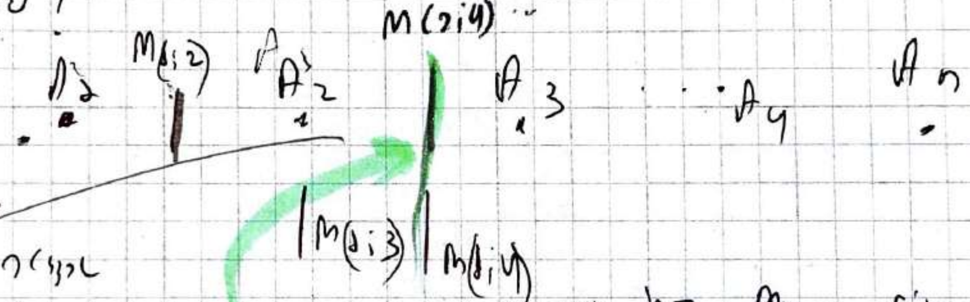
different from (A_2A_n) (A_3A_n) ... $(A_{n-1}A_n)$ $\rightarrow (n-2)$ [illegible].

1) [illegible] A_1 [illegible]-fix-[illegible] (I ~~coord~~ point)

2) A_n [illegible]fix-[illegible] ([illegible] point)

[illegible] at least $(n-1)+(n-2) = 2n-3$ [illegible] midpoints

[illegible]?



$A_2, A_3 \dots$ [illegible]

[illegible], [illegible] [illegible] [illegible], [illegible] A_4 [illegible]-fix-[illegible] [illegible] midpoint-[illegible].

A_1-[illegible] [illegible] [illegible] [illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] [illegible] A_1-[illegible] [illegible] fix-[illegible]

[illegible] [illegible] [illegible] [illegible] (A_2, A_3, ...) [illegible] $A_2, A_3 \dots$ [illegible]-fix[illegible].

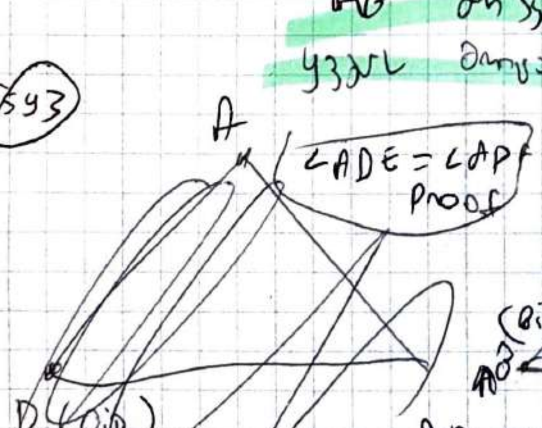
[illegible] [illegible] [illegible]

[illegible] [illegible] [illegible] (A_1, A_2)

[illegible] [illegible] [illegible] [illegible] A_n [illegible] [illegible]-fix-[illegible] [illegible]

[illegible] [illegible] A_{n-1} [illegible] [illegible] ($A_{\dots}, A_n$)

(593)



$\angle ADE = \angle ADF$ Proof

DF slope $= \frac{a\dots(c-b)}{bc(a-b)}$

[illegible] PF slope $= -$ PE slope. [illegible] DE [illegible] DF are symmetric with respect to y-axis

$\angle ADE = \angle ADF$

D(0,0) origin [illegible]

AD [illegible] BC [illegible] axis-[illegible]

~~D [illegible] [illegible] [illegible] [illegible]~~

[illegible] equation line ~~ax+by=c~~ $mx + by = bm$

intersection E-[illegible]

AC equation $ax + cy = ac$

$E\left(\frac{bc(a-m)}{ab-cm}, \frac{am(b-c)}{ab-cm}\right)$ · DE slope $= \frac{am(b-c)}{bc(a-m)}$

(597) $xy = 1$ 4 points $x_1; x_2; x_3; x_4$

if lie on circle $x_1 x_2 x_3 x_4 = 1$ [illegible]

↑ system-ის det-ის [illegible]

$y = \frac{1}{x}$ ⇒ points $\left(x_i, \frac{1}{x_i}\right)$ lie on circle ⇒

there are numbers A, B, C, a for

$$x_i^2 + \frac{1}{x_i^2} + 2x_iA + 2\frac{1}{x_i}B + C = 0$$

for $i = 1, 2, 3, 4$

from $x^2 + y^2 = r^2$

$(x_i)^2 + \left(\frac{1}{x_i}\right)^2$

equation of circle in plane

$$Ax^2 + By^2 + 2Cx + 2Dxy + E = 0$$

$$x_i \cdot 2A + \frac{1}{x_i} \cdot 2B + C \cdot 1 + \left(x_i^2 + \frac{1}{x_i^2}\right) \cdot 1 = 0$$

და $2A; 2B; C$ the unknown

ამოხსნას, რომ სისტემის [illegible] det ≠ 0

$$\begin{vmatrix} x_1^2 + \frac{1}{x_1^2} & x_1 & \frac{1}{x_1} & 1 \\ x_2^2 + \frac{1}{x_2^2} & x_2 & \frac{1}{x_2} & 1 \\ x_3 & \cdots & & \\ x_4 & \cdots & & \end{vmatrix} =$$

მივიღეთ, რომ [illegible] და det-ი $x_1 x_2 x_3 x_4$ [illegible]

$= \left| \; ??? \; \right|$

[596]

[illegible]

$$-\frac{\sqrt{2}-1}{2} \le \sqrt{1-x^2} - (px+q) \le \frac{\sqrt{2}-1}{2}$$

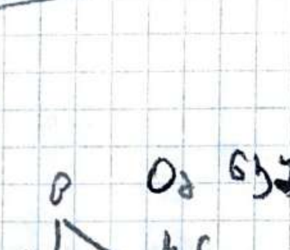
$$\underbrace{\sqrt{1-x^2}}_{\text{circle radius}=1} - \frac{\sqrt{2}-1}{2} \le \underbrace{px+q}_{\text{the eq. [illegible]}} \le \sqrt{1-x^2} + \frac{\sqrt{2}-1}{2} = \sqrt{1-x^2} - \left(-\frac{\sqrt{2}-1}{2}\right)$$

at $x \in (0,1)$

$O_1\left(0, \frac{\sqrt{2}-1}{2}\right)$

$O_2\left(0, -\frac{\sqrt{2}-1}{2}\right)$

$A\left(1, \frac{\sqrt{2}-1}{2}\right)$ $B\left(0, \frac{\sqrt{2}+1}{2}\right)$



AB distance = 1

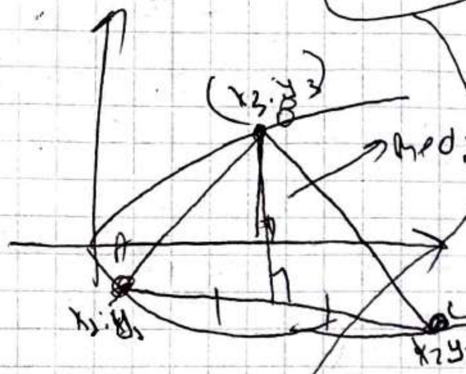
BO_2 distance $= \sqrt{2} = AB$

$\triangle BO_2A$ = isosceles [illegible] O_2 და A-ში have to be equal.

(8-12)

Centers of eq. triangles inscribed in parabola.

$y^2 = 4px$



median $= \dfrac{y - y_3}{x - x_3} = \dfrac{y_1 + y_2 - 2y_3}{x_1 + x_2 - 2x_3}$

$\dfrac{y_1 + y_2 - 2y_3}{x_1 + x_2 - 2x_3} \cdot \dfrac{y_2 - y_1}{x_2 - x_1} = -1$

slope of AC

$(x_1 - x_2)(x_1 + x_2 - 2x_3) + (y - y_1)(y_1 + y_2 - 2y_3) = 0$

$(\div\, y_1 - y_2)$

$(y_1 + y_2)(y_1^2 + y_2^2 - 2y_3^2) + 16p^2(y_1 + y_2 - 2y_3) = 0$

$(\div\, y_2 - y_3)$

$-\ (y_2 + y_3)(y_2^2 + y_3^2 - 2y_1^2) + 16p^2(y_2 + y_3 - 2y_1) = 0$

$y_1^3 - y_3^3 + (y_1 - y_3)(y_2^2 - 2y_1 y_3) + 48p^2(y_1 - y_3) = 0$

$(\div\, y_1 - y_3)$

$y_1^2 + y_2^2 + y_3^2 + 3(y_1 y_2 + y_2 y_3 + y_3 y_1) + 48p^2 = 0$

center of triangle formula:

$\cancel{x = \dfrac{y_1^2 + y_2^2 + y_3^2}{12p}}$

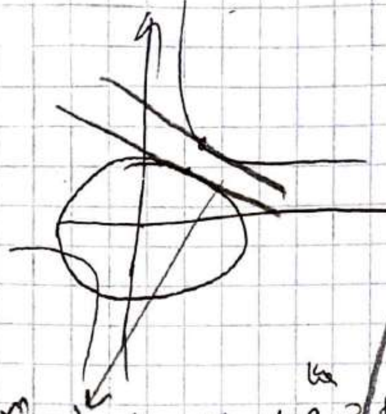
$y = \dfrac{y_1 + y_2 + y_3}{3}$ $\quad R \, 1 \to 3$

~~formula~~ $= x = \dfrac{x_1 + x_2 + x_3}{3} = \dfrac{y_1^2}{4p} + \dfrac{y_2^2}{4p} + \dfrac{y_3^2}{4p} \left(\dfrac{1}{3}\right)$

$x = \dfrac{y_1^2 + y_2^2 + y_3^2}{12p}$

$y^2 = \dfrac{4p}{9}(x - 8p)$

[illegible] 2



$xy = 4$ P.

[illegible] Q $x^2 + 4y^2 = 4$ Q

distance from P to Q $\to 1$

~~hyperbola~~ to hyperbola $xy = 4$ [illegible]

tangent at $(2; 2)$ [illegible] $\boxed{x = y}$

$y = 4 - x$ [illegible] tangent [illegible]

[illegible] slope -1 (x_0, y_0) [illegible] ellipse-[illegible]

[illegible] distance [illegible]

[illegible] tangent [illegible]

$$\frac{d}{dx}\left(x^2 + 4y^2 = 4\right)\frac{d}{dx}$$

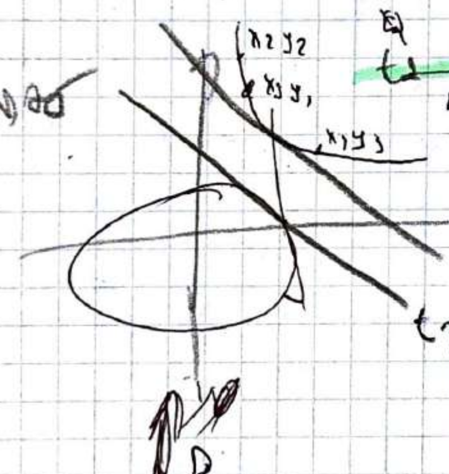
$$x \cdot x_0 + 4y\, y_0 = 4$$

$$\text{slope} = -\frac{x_0}{4y_0} = -1 \quad (1) \Rightarrow \quad x_0^2 + 4y_0^2 = 4$$

$$(1) \Rightarrow x_0 = \frac{4}{\sqrt5} \quad \text{და} \quad y_0 = \frac{1}{\sqrt5}$$

[illegible] tangent to ellipse $y = \sqrt5 - x$

distance between $y = \sqrt5 - x$ და $y = 4 - x$ $\boxed{\text{is } 4 - \frac{\sqrt5}{2} > 1}$



t_1 [illegible]

ellipse-[illegible] $xy = 4$

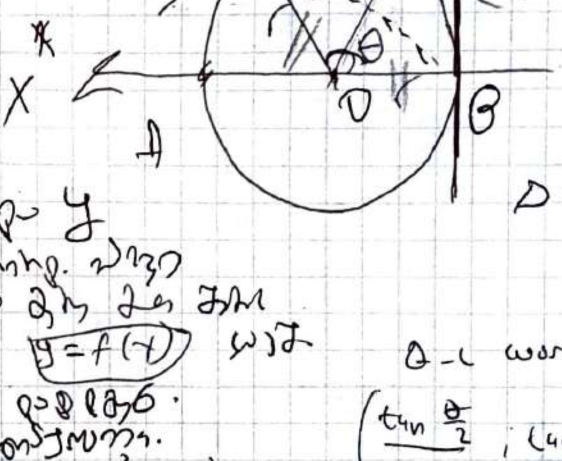
hyperbola-[illegible] (x_2, y_2) (x_1, y_1) (x_3, y_3)

~~tangent~~ [illegible] t_2 [illegible]

[illegible] distance > [illegible]

[illegible]

locus of [illegible] Q, [illegible]



Q-[illegible] $\boxed{r = 1}$ $\to$ [illegible] P θ, [illegible]

$\to$ [illegible] θ [illegible] tangent [illegible]

$$~~AP~~\; MP = OP = \tan\frac{\theta}{2}$$

$$\triangle POM\text{-}[illegible]\; MP = \tan\frac{\theta}{2}$$

$$\triangle PQM\text{-}[illegible] \quad PQ = \cancel{\frac{\tan\frac{\theta}{2}}{\sin\theta}}$$

$$\sin\theta = \frac{PM}{QP} \quad (1) \Rightarrow QP = \frac{\tan\frac{\theta}{2}}{\sin\theta}$$

X და y [illegible]

$\boxed{y = f(x)}$ [illegible]

Q-[illegible] coordinates [illegible]

$$\left(\frac{\tan\frac{\theta}{2}}{\sin\theta};\ \tan\frac{\theta}{2}\right) = \Big(\underbrace{\frac{1}{1+\cos\theta}}_{X};\ \underbrace{\frac{\sin\theta}{1+\cos\theta}}_{y}\Big)$$

[illegible]

$$y^2 = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2 = \frac{1-\cos^2\theta}{(1+\cos\theta)^2} =$$

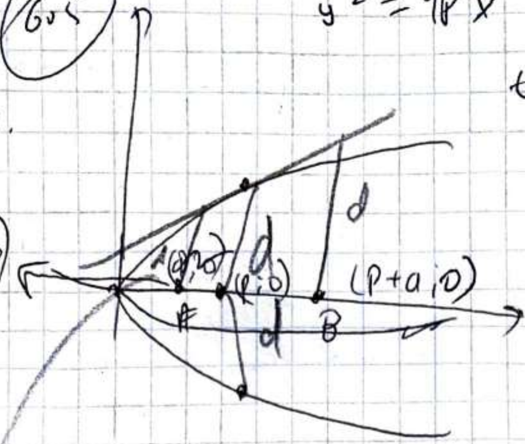
9.39

(605)

$y^2 = 4px$

A & B -
equal distance
from $(p,0)$

$(p-a,0)$



tangent of slope m
to parabola

$y - y_1 = m(x - x_1)$

$y = mx + c$

$(mx+c)^2 = 4px$

tangent
$m^2x^2 + (2mc - 4p)x + c^2 = 0$

$D = 0$ $\quad (2mc-4p)^2 - 4(m^2)(c^2) = 0$

$\boxed{c = \frac{p}{m}}$ $\quad mx + \frac{p}{m}$

distance

A & B - as const

$\left| \frac{m(p \pm a) + \frac{p}{m}}{\sqrt{1+m^2}} \right|$

distance from
(x_0, y_0) to line
$y - mx - \frac{p}{m}$ is $d = \frac{|Ax_0 + By_0 + c|}{\sqrt{A^2+B^2}}$

$A(p+a, 0)$; $B(p-a, 0)$

$d_A = \frac{|-m(p+a) + 0 - \frac{p}{m}|}{\sqrt{m^2+1}}$ $\quad d_B = \frac{|m(p-a) + \frac{p}{m}|}{\sqrt{m^2+1}}$

diff. of squares of distances are

$$\frac{m^2(p+a)^2 + 2p(p+a) + \frac{p^2}{m^2} - \left(m^2(p-a)^2 + 2p(p-a) + \frac{p^2}{m^2}\right)}{1+m^2} = 4pa$$

m - doesn't play

(606) PQ

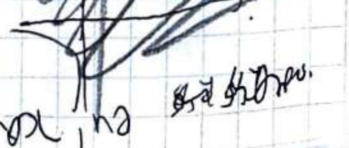
asymptotes are parallel to parabola. $xy = 1$

$xy = 1$

asymptote

If $P(x_1, y_1)$, $Q(x_2, y_2)$ are ... then ... coordinate $xy = 1$ ...

(x_1, y_1) $\quad (x_2, y_2)$



$y - y_1 = \frac{y_2 - y_1}{x_1 - x_2}(x - x_2)$ $\quad xy = 1 = x_1y_1 = x_2y_2$

(612) $y^2 = 2px$ · focus F. $n \geq 3$ [illegible] polygon $A_1 A_2 \dots A_n$ with corner F

$\boxed{f = \frac{p}{2}}$

[illegible] x-axis [illegible]

Half-lines intersect the parabola at $B_1, B_2 \dots B_n$.

prove:

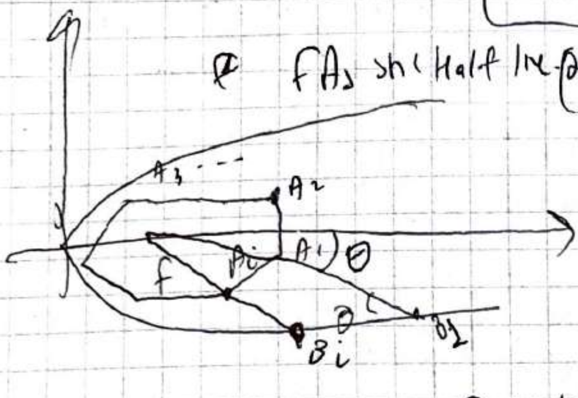
$FB_1 + FB_2 + \cdots FB_n \geq np$

$\boxed{FB_i = t_i}$

FA_i [illegible] Half line [illegible] · α between FB_i with x-axis [illegible]

Half line [illegible]

$\alpha_i = \alpha + (i-1)\frac{2\pi}{n}$



regular polygon [illegible] angle $= \frac{2\pi i}{n}$

coord. of points on ray originating at F making θ with x-axis is

$\left(\frac{p}{2} + t\cos\theta,\ t\sin\theta\right)$

[illegible]

$\left(\frac{p}{2} + t_i\cos\alpha_i,\ t_i\sin\alpha_i\right)$

[illegible] $\cos$, $\sin$ [illegible] $-1 \leq \cos \leq 1$ [illegible]

B_i belongs to parabola ~~[illegible]~~ [illegible]

$$y^2 = 2px$$

$$(t_i\sin\alpha_i)^2 = 2p\left(\frac{p}{2} + t_i\cos\alpha_i\right)$$

$\boxed{FB_1 = t_1}$ $\boxed{t_i = \frac{p}{1-\cos\alpha_i}}$

$FB_1 + FB_2 + \cdots \geq np$ (p [illegible])

$t_1 + t_2 + \cdots \geq n\cancel{p}$

$$\frac{1}{1-\cos\alpha_1} + \frac{1}{1-\cos\alpha_2} \cdots \frac{1}{1-\cos\alpha_n} \geq n$$

[illegible] Cauchy [illegible]

$$(1-\cos\alpha_1) + (1-\cos\alpha_2) + \cdots (1-\cos\alpha_n) = n - \sum_{i=1} \cos\left(\alpha + \frac{2(i-1)\pi}{n}\right)$$

$\sum \cos(\alpha+\beta)$

$\cos(\alpha)$ [illegible]

$$= n - \cos\alpha \cdot \sum_{i=1}^{n} \cos\left(\frac{2(i-1)\pi}{n}\right) \quad = 0$$

$$+ \sin\alpha \sum_{i=1}^{n} \left(\frac{2(i-1)\pi}{n}\right) = n$$

$= 0$

$\sum \cos(\alpha)\cdot\sum\cos(\beta)$

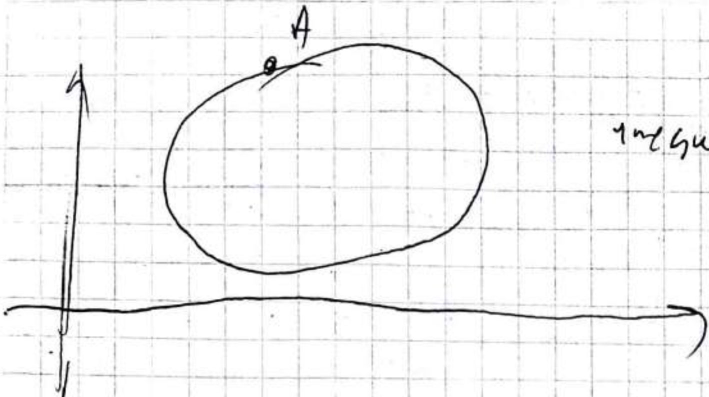
$= \cos(\alpha)\sum 1 \cdot \cos\beta$

Cauchy - Schwarz [illegible]

$$\left(\frac{1}{1-\cos\alpha_1} + \frac{1}{1-\cos\alpha_2} + \cdots\right) \geq \frac{1}{(1-\cos\alpha_1) + (1-\cos\alpha_2) \cdots}$$

$$\cdots (1-\cos\alpha_n) \geq \frac{n^2}{n} = n$$

(6.11)



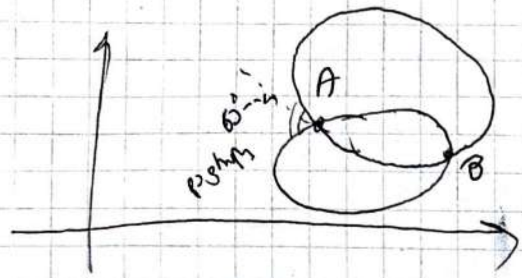
intqual $\{p_i,\alpha\}$,ვა ბრუ ...

n-gon ...
ellipse ...
და წრე ...

~~ ... ~~

$x^2 + ay^2 = b$

$x^2 + y^2 + cx + dy + e = 0$



$(1-a)y_i^2 + cx_i + dy_i + (ae+b) = 0$

ა $y_i^2 = -\frac{c}{1-a}x_i - \frac{d}{1-a}y_i - \frac{e+b}{1-a}$

$c \neq 0$, ... 2 ...

parabola-ზე ... , ა

Parabola-ს ... და vertical-ზე

$y^2 = 4px$. $\underbrace{x^2 + y^2 + c'x + d'y + e' = 0}_{\text{წრე}}$

ellipse

$\frac{1}{16p^2}y_i^4 + y_i^2 + \frac{c'}{4p}y_i^2 + d'y_i + e' = 0$

4 ... x ... unique x ...

ა result ... $n > 4$ ...

(6.09) $T_i\ (4p\alpha_i^2 ; 4p\alpha_i)$; $y^2 = 4px$ ... $(x = \alpha_i)$

Parabolic tangent ... eq: $yy_0 = 2p(x + x_0)$

$2\alpha_i y = x + 4p\alpha_i^2$

If P_{ij} is intersection of tangent(i) და tangent(j)

$2\alpha_j y = x + 4p\alpha_j^2$ ... $(4p\alpha_i\alpha_j\ ,\ 2p(\alpha_i + \alpha_j))$

Area of $T_1T_2T_3 = \pm\frac{1}{2}\begin{vmatrix} 4p\alpha_1^2 & 4p\alpha_1 & 1 \\ 4p\alpha_2^2 & 4p\alpha_2 & 1 \\ 4p\alpha_3^2 & 4p\alpha_3 & 1 \end{vmatrix} = \pm 8p^2 \begin{vmatrix} \alpha_1^2 & \alpha_1 & 1 \\ \alpha_2^2 & \alpha_2 & 1 \\ \alpha_3^2 & \alpha_3 & 1 \end{vmatrix} =$

(609) ამოხ.

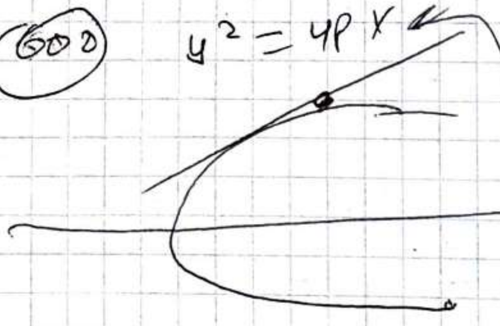
$$8p^2\,|(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)|$$

Area of $P_{12}P_{23}P_{31}$ ტანგენტების [illegible]

$$\pm\frac{1}{2}\begin{vmatrix} 4p\alpha_1\alpha_2 & 2p(\alpha_1+\alpha_2) & 1 \\ 4p\alpha_2\alpha_3 & 2p(\alpha_2+\alpha_3) & 1 \\ 4p\alpha_3\alpha_1 & 2p(\alpha_3+\alpha_1) & 1 \end{vmatrix} =$$

$$= 4p^2\,|(\alpha_1-\alpha_3)(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)|$$

(600) $y^2=4px$



ϕ

ტანგენტის ფორმულა

$\cancel{yy_0=2p(x+x_0)}$

$\cancel{y=}$ $y-y_0=m(x-x_0)$

$k(3)$

$$(mx-mx_0+y_0)^2-4px=0$$

$$D=0;\quad m^2x_0-my_0+p=0$$

(1) $m_1+m_2=\dfrac{y_0}{x_0}$ ტანგენტის [illegible]

$m_1\cdot m_2=\dfrac{p}{x_0}$; if $\phi=90^\circ$

then $m_1m_2=-1$ $\;-\dfrac{1}{m}$ m

$\dfrac{p}{x_0}=-1$

$\cancel{p=x_0}$

$\boxed{x=-p}$

$\phi=A-B$ ([illegible])

$\cancel{\tan\phi}$

$\boxed{\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}}$ if $\phi\neq90^\circ$

$$\tan\phi=\frac{m_1-m_2}{1+m_1m_2}=\frac{m_1-m_2}{1+\frac{p}{x_0}}$$

(1) $m_1+m_2=\dfrac{y_0}{x_0}$

$m_1-m_2=\tan\phi+\dfrac{p_0}{x_0}\tan\phi$.

ვა square and substract ვიღებთ $m_1m_2=\dfrac{y_0^2}{4x_0^2}-\left(1+\dfrac{p}{x_0}\right)^2\tan^2\phi=\dfrac{p}{x_0}$

(2)-ს ჩავსვამთ [illegible]

$$-y^2+(x+p)^2\tan^2\phi+4px=0$$

(תרגיל 2) $3^{3^n}+1$ is product of $2n+1$ primes (not nec. distinct)

אינדוקציה $n=1$ קל

$$3^{3^{n+1}}+1 = \left(3^{3^n}+1\right)\cdot y$$

$3^{3^{n+1}}$

$3^{3^n}-1$ מכפלה של $2n+1$

$3^{3^{n+1}}-1$ $2(n+1)+1 = 2n+3$

$$y = \frac{3^{3^n\cdot 3}+1}{3^{3^n}+1}$$

$$\boxed{y = 3^{2\cdot 3^n}-3^{3^n}+1}$$ אם y לא composite.

$3^{3^n}+1$ מכפלה של $2n+1$

$2n+3$

$$3^{2\cdot 3^n}-3^{3^n}+1 = \left(3^{3^n}+1\right)^2 - 3\cdot 3^{3^n} = \left(3^{3^n}+1\right)^2-\left(3^{\frac{3^n+1}{2}}\right)^2$$

completing square

a^2-b^2

$(a-b)(a+b)$

$$3^{3^n}+1-3^{\frac{3^n+1}{2}} \quad \text{ו} \quad 3^{3^n}+1-3^{\frac{3^n+1}{2}}$$

אז composite אצל y.

(תרגיל 33) $4n+3$ primes.

3 prime

$\cancel{4(n+1)+3}$ $4(n-1)+3$

$4n-1$

$p_1, p_2 \ldots p_n$

$p = 4n-1$

consider

$M = 4p_1p_2\cdots p_n - 1$

$4n-1$ prime divisor

$p_1, p_2 \cdots$ (-1)

(131) $P(n)\ n(n-1)^4+1 = n^5-4n^4+6n^3-4n^2+n+1$

[illegible] [illegible]: any rational zero of polynomial
<u>must</u> be ~~[illegible]~~ $\frac{p}{q}$, p = const [illegible] $c=1$

q = leading coeff. [illegible] n^5 [illegible]. $q=1$

[illegible] [illegible] [illegible] [illegible] [illegible]

1 [illegible] −1 [illegible]

$P(1)$ − [illegible] $\neq 0$ | [illegible] no integer zeros
$P(-1)$ − [illegible] $\neq 0$

$$P(n) = \underbrace{(n^2+an+1)}_{[illegible]}\underbrace{(n^3+bn^2+cn+1)}_{[illegible]}$$

[illegible]

[illegible] [illegible] [illegible]

$$a+b=-4$$
$$c+ab+1=6$$
$$b+ac+1=-4$$
$$a+c=1$$

$$b-c=-5$$
$$(b-c)(a-1)=10$$
$$a-1=-2$$
$a=-1$; $b=-4+1=-3$
$c=2$

$$\Rightarrow n(n-1)^4+1 = \underbrace{(n^2-n+1)(n^3-3n^2+2n+1)}_{\text{integers } > 1}$$

724) $\frac{n^3-3n^2+4}{2n-1} \to$ [illegible]

$8 \cdot \frac{n^3-3n^2+4}{2n-1} = 4n^2-10n-5+\frac{27}{2n-1}$

$2n-1$ to divide 27.

$2n-1 = \pm1; \pm3; \pm9; \pm27$

$n = -13; -4; -1; 1; 2; 5; 14$

[illegible] $\frac{n^3-3n^2+4}{2n-1}$ [illegible] $\underbrace{4n^2-10n-5}$ [illegible]

[illegible] n [illegible]

[illegible] $\frac{27}{2n-1} - 2$ [illegible]

725) $P = 1! \cdot 2! \cdot 3! \cdots 100!$ $\quad (2k)! = (2k-1)! \cdot 2k$

$P = (1!)^2 \cdot 2 \cdot (3!)^2 \cdot 4 \cdot (5!)^2 \cdot 6 \cdots (99!)^2 \cdot 100 =$

$= (1! \cdot 3! \cdot 5! \cdots 99!)^2 \cdot 2 \cdot 4 \cdot 6 \cdots 100 =$

$= (1! \, 3! \cdot 5! \cdots 99!)^2 \cdot 2^{50} \cdot 50! =$

$= (1! \, 3! \cdots 2^{25})^2 \cdot 50!$ $\quad 50!$ [illegible]

[illegible] factorials [illegible]

$(2k)! = (2k-1)! \cdot 2k$ [illegible]

[illegible] $k=1$-[illegible]

$2! = 2(1!)\ 2$

$4! = (3!) \cdot 4$

[illegible] $(2k-1)$

726) $a_1; a_2; a_3$ $\quad \gcd(a_i; a_j) = \gcd(i; j)$

Prove $a_i = i$ for all a_i [illegible]

For any integer m: $\gcd(a_m; a_{2m}) = \gcd(2m; m) = m$

so m divides a_m

a_m [illegible]

$(a_n; a_m)$ [illegible] $m : a_n - i$ [illegible] $\Longleftarrow$

[illegible] m-[illegible] m [illegible] gcd.

m [illegible] $\gcd(a_m; a_n)$

$(\Rightarrow)$ $\gcd(a_m; a_n) = \gcd(m; n)$

so a_n has the same divisors as n.

$ax - by = gcd(a,b)$

$ax - by = 1$ [illegible] are [illegible] co-prime

$P(x)$ $\quad P(7) = 5 \quad P(15) = 9$

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_0$

$P(7) = a_n 7^n + a_{n-1} 7^{n-1} + \dots a_0 = 5$

$P(15) = \dots \quad = 9$

$9 - 5 = 4 = P(15) - P(7) = a_n(15^n - 7^n) + a_{n-1}(15^{n-1} - 7^{n-1}) + \dots a_1(15-7)$

$15^k - 7^k$ for any k

$a_1 \cdot 8$

$15 \equiv 7 \mod 8$

$\boxed{15^k \equiv 7^k \bmod 8}$ [illegible]

$15^k - 7^k \equiv 7^k - 7^k \equiv 0 \pmod 8$

$15^k - 7^k : 8$ [illegible]

[illegible] $4 \not : 8$ [illegible]

into 2

$(36a+b)(a+36b)$ can't be power of 2

contrad.

[illegible] a and b are ~~coprey~~ coprime ~~with~~ $a < b$

(1) $36a + b = 2^m$ $\quad$ (2) $a + 36b = 2^n$

(1) + (2) $\quad 37(a+b) = 2^m(2^{n-m} + 1)$

(1) − (2) $\quad$ ~~35(a+b)~~ $35(a-b) = 2^m(2^{n-m} - 1)$

$a - b : 2^m \quad$ [illegible] $a + b - 4 : 2^m$

$a - b \equiv 0 \bmod 2^m$

$a + b \equiv 0 \bmod 2^m$

$a+b = 2^m k \quad a - b = 2^m l$

$a = \frac{(a+b)+(a-b)}{2} \; ; \; b = \frac{a+b-(a-b)}{2}$

$a = 2^{m-1}(k+l)$

$b = 2^{m-1}(k-l)$ [illegible] $a, b : 2^{m-1}$ [illegible]

[illegible] a, b coprime [illegible]

[illegible] $m = 1$, [illegible]

[illegible] a [illegible] b [illegible]

[illegible] ([illegible]) $1 - 1$ [illegible]

[illegible] $36a + b = 2^1$ [illegible]

722 $f(1)=2$

$f(f(n)) = f(n)+n$.

let x_1 be golden ratio of positive root $x^2-x-1=0$

$$\lfloor x_1 \lfloor x_1 n + \tfrac{1}{2}\rfloor + \tfrac{1}{2}\rfloor = \lfloor x_1 + \tfrac{1}{2}\rfloor + n$$

f(n) st ~~f(n) = ...~~ $f(n) = [x_1 n + \tfrac{1}{2}]$

$x_1 = \frac{1+\sqrt{5}}{2} > 1$, $\Rightarrow f$ increasing

$$\lfloor (x_1 - 1)\lfloor x_1 n + \tfrac{1}{2}\rfloor + \tfrac{1}{2}\rfloor = n$$

$x(x) - x - 1 = 0$

$x(x) = x+1$

$$\left(x_1 \lfloor x_1 n + \tfrac{1}{2}\rfloor\right)$$

721 $\lfloor nx\rfloor \geq \frac{[x]}{1} + \frac{[2x]}{2} + \frac{[3x]}{3} + \cdots \frac{\lfloor nx\rfloor}{n}$

$f(x) = [nx] - \frac{[x]}{1} - \cdots \frac{[nx]}{n}$

~~$f(x) = f(x+1)$~~ $nx+n$

$f(0) = 0$

$\lfloor k(x+1)\rfloor = \lfloor kx + k\rfloor = [kx]+k$

$f(x+1) = [n(x+1)] = [nx]+n$

$\frac{\lfloor k(x+1)\rfloor}{k} = \frac{[kx]+k}{k} = \frac{[kx]}{k} + 1$

$\sum^{n} \frac{\lfloor k(x+1)\rfloor}{k} = \sum \frac{[kx]}{k} + 1 = \sum \frac{[kx]}{k} + n$

$f(x+1) = \lfloor nx\rfloor + n - \sum \frac{[kx]}{k} + n = f(x)$ $= [nx] - \cdots$ $f(x)$

$f(0) = 0$.

(74) $n > 1$ $2^n - 1$ for no integer n

710 — 713

α, β: $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

(723) f, g - $N \to N$ partition.

$g(n) = f(f(kn)) + 1$

$\left\{ \begin{array}{l} f(n) ; n \geq 1 \\ f'(n) ; n \geq 1 \end{array} \right.$ $\begin{array}{l} g(n) ; n \geq 1 \\ g'(n) \geq 1 \end{array}$ → sequences.

$f(n) ; g(n)$ are unique to $f'(n) ; g'(n)$

n_0, has

$f(n_0) \neq f'(n_0)$

$g(n_0) \neq g'(n_0)$ or

$f(n)$ $g(n)$
$f'(n)$ $g'(n)$ } positive numbers

$\cancel{f(n_0) = g'(n_0)}$

$\cancel{f(n_0)}$

(a) $f(n_0) = g'(n_0)$

or

$f'(n_0) = g(n_0)$

$\to$ (a)

$\cancel{f'(n_0) = g(n_0)}$ $f(n_0) = g'(n_0) = f'(f'(kn_0)) + 1 =$

$= f(f(kn_0)) + 1 = g(n_0)$

$f'(kn_0) < f(n_0)$

$f'(kn_0) = f(kn_0)$

$f(n_0) = g(n_0)$

X.

take α - positive root of eq. $kx^2 - kx - 1 = 0$

$\beta = k \cdot \alpha^2$ $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

sequences $f(n) = \lfloor \alpha n \rfloor$ — strictly increasing and

$g(n) = \lfloor \beta n \rfloor$

$k\alpha^2 = k\alpha + 1$

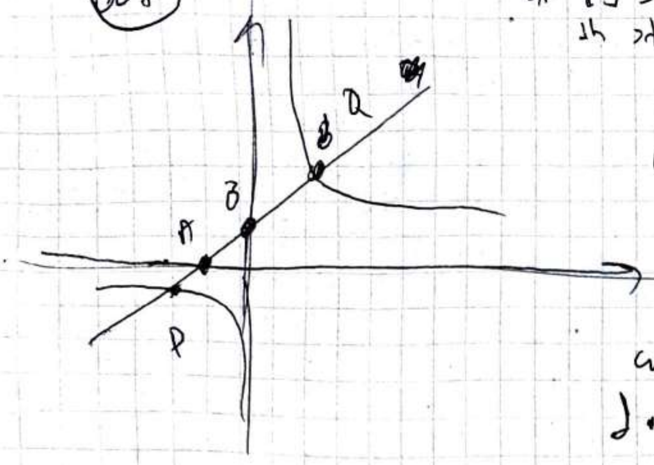
$g(n) = \lfloor k\alpha^2 n \rfloor = \lfloor (k\alpha + 1)n \rfloor = \lfloor k\alpha \cdot n \rfloor + n$

$\cancel{\lfloor \alpha \cdot k \cdot n \rfloor + n = \lfloor \alpha \lfloor \alpha k n \rfloor \rfloor + 1}$

$\lfloor \alpha \cdot kn \rfloor + n = \lfloor \alpha \lfloor \alpha kn \rfloor \rfloor + 1$ proved

$\lfloor (\alpha - 1) \lfloor \alpha \cdot kn \rfloor \rfloor = n - 1$

(607)



$xy = 1$ [illegible]

[illegible]

line-ის formula

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

~~asymptotes~~

[illegible] line [illegible]

[illegible] (+) [illegible]

1. [illegible] x და y-ის [illegible]

2. [illegible] hyperbolas, vertical horizontal [illegible]

A და B-ს [illegible]

$A(a,0)$ და $B(0,b)$

P და Q-ს [illegible]

$y = \frac{1}{x}$ [illegible] in eq. of line, [illegible]

$$x^2 - ax + \frac{a}{b} = 0$$

$$\boxed{x_1 + x_2 = a}$$ [illegible]

$x = \frac{1}{y}$ [illegible]

$$y^2 - by + \frac{b}{a} = 0$$

$$\boxed{y_1 + y_2 = b}$$

$P(x_1, y_1)$

$Q(x_2, y_2)$

$$AP^2 = (x_1 - a)^2 + (y_1 - 0)^2 = (a - x_2 - a)^2 + (b - y^2) =$$

$$= x_2^2 + (b - y^2) = BQ^2$$

$x_1 + x_2 = a$, [illegible]

$x_1 = a - x_2$

[illegible] BQ^2 [illegible] $x_1 + x_2 = a$, $y_1 + y_2 = b$ [illegible]

= [illegible]

(614) [illegible] $\sin$-[illegible]

$$\int \frac{d}{a + b\cos x + c\sin x} \to d(\cos\alpha \cdot \cos x + \sin\alpha \cdot \sin x) = d\cos(x - \alpha)$$

[illegible]

$$r(\cos(x - \alpha)) = r(\cos x \underbrace{\cos\alpha}_{\frac{b}{d}} + \sin x \underbrace{\sin\alpha}_{\frac{c}{d}})$$

$$\boxed{r = d}$$

$r = \sqrt{b^2 + c^2}$

$d = \sqrt{b^2 + c^2}$

$$\int \frac{d}{a + \cos(x - \alpha)} dx$$

$u = x - \alpha$

$$\int \frac{du}{a + d\cos u}, \quad \boxed{t = \tan\frac{u}{2}} = \frac{2}{a+d}\int \frac{1}{1 + \frac{a-d}{a+d}t^2}$$

if $a = d$ [illegible] $= \frac{1}{a}\tan\frac{x - \alpha}{2}$

if $\frac{a-d}{a+d} > 0$

$$\frac{2}{\sqrt{a^2 - d^2}}\arctan\left(\sqrt{\frac{a-d}{a+d}}\tan\frac{x-\alpha}{2}\right)$$

(1.5) (1) $x+y = z+u$
(2) $2xy = zu$ } a m_{max} such that $m \le \frac{x}{y}$

$(x \ge y)$

(1) ⇒ (2) $2xy = z(x+y-z)$

$$z^2 - xz - yz + 2xy = 0$$

$: y^2$

$$\left(\frac{z}{y}\right)^2 - \left(\frac{z}{y}\right)\left(\frac{x}{y}\right) - \left(\frac{z}{y}\right) + 2\left(\frac{x}{y}\right) = 0$$

$$\to z^2 - zx - z + 2x = 0$$

conditions $m \le \frac{x}{y}$ and $x \ge y$

[illegible] $m \le X$ $\qquad \frac{x}{y} \ge 1$ $\qquad X \ge 1$

Hyperbola equation (rational curve [illegible])

[illegible]

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$D = B^2 - 4AC$. if $D > 0$ then Hyperbola

if $D = 0$ parabolla.

if $D < 0$ ellipse ([illegible] $A = C$ [illegible])

~~[illegible]~~

rational then, $z = tX$ [illegible]

$$x(t^2 x - tX - t + 2) = 0$$

$X = 0$, $y = 0$ the origin | $X = \frac{t-2}{t^2-t} \Rightarrow y = \frac{t^2-2t}{t^2-t}$

[illegible] parametric hyperbola-ს.

[illegible]

~~[illegible]~~ [illegible]

$(X; Z)$ [illegible]

asymptotes $z = 2$ [illegible] $z = X - 2$

[illegible] X_{min} on curve $g(X, z) = z^2 - zX - z + 2X = 0$

X_{min} [illegible] min-ის [illegible] $(X \ge 1)$-ის

[illegible] X_{min}

Lagrange-ის

$$1 = \lambda(-z+2)$$
$$0 = \lambda(2z - X - 1)$$

$z = \frac{X+1}{2}$ [illegible] $g(X, z)$: $X = 3 \pm 2\sqrt{2}$

$m \le X$

~~$m \le X$~~

$\boxed{m_{max} = 3 + 2\sqrt{2}}$

[illegible] $X > 1$-ის

[illegible] $X = 3 + 2\sqrt{2}$

(622) $x^2+y^2=1$ [illegible] $r=2a(1+\cos\theta)$

transf. of plane. [illegible] pole / origin [illegible]

[illegible] circle, [illegible] origin [illegible] $(x-1)^2+(y-0)^2=1$ [illegible] $|z-1|=1$

[illegible] $(0,0)$ [illegible] if ϕ maps [illegible] $r=2a(1+\cos\theta)$

equation of cardioid $|\phi^{-1}(z)-1|=1$

[illegible] $|\phi(\phi^{-1}(z))-1|=1$ [illegible] $\boxed{|z-1|=1}$

original [illegible]

$r=a\cdot 2\cdot\cos^2\frac{\theta}{2}$

$\sqrt{r}=\sqrt{2a}\cdot\cos\frac{\theta}{2}$; $r=\sqrt{2a}\cdot\sqrt{r}\cdot\cos\frac{\theta}{2}$

1 [illegible] $r\cdot 1$ [illegible] $r-\sqrt{2a}\cdot\sqrt{r}\cdot\cos\frac{\theta}{2}=0$

$r\left(\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right)$ ← eq. of circle [illegible]

$-\sqrt{2a}\sqrt{r}\cos\frac{\theta}{2}+1-1=\left(\sqrt{r}\cos\frac{\theta}{2}\right)^2-\sqrt{2a}\sqrt{r}\cos\frac{\theta}{2}+$

$(\sqrt{r}\cdot\cos\frac{\theta}{2})^2=r\cos^2\frac{\theta}{2}$

$(\sqrt{r}\cdot\sin\frac{\theta}{2})^2=r\cdot\sin^2\frac{\theta}{2}$

[illegible] square

$+1+(\sqrt{r}\sin\frac{\theta}{2})^2-1$; $a=2$ [illegible] (perfect square [illegible])

$(\sqrt{r}\cdot\cos\frac{\theta}{2}-1)^2+(\sqrt{r}\sin\frac{\theta}{2})^2=1$

$\cancel{(x-1)^2}$ [illegible] $|1-$ [illegible] $z|$ $\cancel{\phi(z)=z^2}$ [illegible]

~~[illegible]~~

$|\sqrt{z}-1|=1$, $\phi(z)=z^2$

$z=r\cos t$

$z=r\sin t$

(624) curve 1) planar

$$a_1t^n+b_1t^p=x-c_1$$
$$a_2t^n+b_2t^p=y-c_2$$
$$a_3t^n+b_3t^p=z-c_3$$

$$\begin{vmatrix} a_1 & b_1 & x-c_1\\ a_2 & b_2 & y-c_2\\ a_3 & b_3 & z-c_3\end{vmatrix}=0$$

016 $r = 1 + \cos\theta$ is vertical [illegible]

017

$AM = CD$ find locus of M [illegible]

$AB = a$

$AD = \cancel{a\cos\theta}\ \frac{a}{\cos\theta}$

$AC = a\cos\theta$

~~$AM = AD - AC$~~

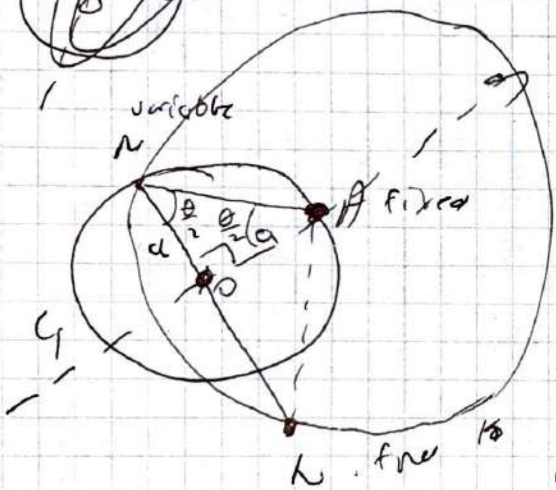
$CD = AD - AC = AM$

$AM = \frac{a}{\cos\theta} - a\cos\theta = \frac{a\sin^2\theta}{\cos\theta}$

018

MO [illegible] polar axis.

019



C_2 C_2 centre $= A$

find locus of L

pole [illegible] O [illegible] axis OA [illegible]

[illegible]

$r = a$ [illegible]

$AM = AL = 2a\sin\frac{\theta}{2}$

$\triangle LAB$ ~~$\angle LAB$~~

$\angle LMA = \frac{\pi}{2} - \frac{\theta}{2}$

$LM = 2AM\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2\cdot 2a\sin\frac{\theta}{2}\sin\frac{\theta}{2} = 4a^2\sin^2\frac{\theta}{2}$

$OL = LM - OM$

$r = a - 4a\sin^2\frac{\theta}{2} = a(2\cos\theta - 1)$

[illegible] polar [illegible] $= r$. OL [illegible]

(541) $f(x+y)+f(x-y)=2f(x)g(y)$

$|f(x)|$ can be 0 to 1 [illegible] M

$\boxed{|f(x)|\le 1}$, then $g(y)\le 1$ Proof

$2|f(x)|\,|g(x)|=|f(x+y)+f(x-y)|\le|f(x+y)|+|f(x-y)|\le 2M$

$2|f(x)\cdot g(x)|=2|f(x)|\cdot|g(x)|$

$|g(y)|\le\dfrac{M}{|f(x)|}\le\dfrac{M}{M}=1$

$|f(x)|$ [illegible]

$|g(y)|\le 1$, [illegible] $\sin x$ [illegible] $\cos x$

[illegible]

$f(x)$ [illegible] $|\ |$ [illegible] $g(y)$ [illegible]

$g(y)$ [illegible] max [illegible]

$0\le|g(y)|\le\dfrac{f(x-y)+f(x+y)}{|f(x)|}$ [illegible]

[illegible]

[illegible] substitution

(538) $f(x+y)+f(y+z)+f(z+x)\ge 3f(x+2y+3z)$

$y=0;\ z=0$ [illegible] $f(x)+f(0)+f(x)\ge 3f(x)$

$\boxed{f(0)\ge f(x)}$ $f(0)\ge f(x)$ [illegible] $f(0);f(x)$ [illegible]

$x=\frac{x}{2};\ y=\frac{x}{2};\ z=-\frac{x}{2}$

$f\left(\frac{x}{2}+\frac{x}{2}\right)+f(0)+f(0)\ge 3f(0)$

~~$f(x)\ge f(0)$~~

$f(0)\le f(x)$

[illegible] $f(0)=f(x)$ $\Rightarrow f(x)$ is constant

(537) $f(x)=f\left(\dfrac{x}{1+x}\right)$; $f(-1)=f\left(-\frac{1}{2}\right)=f\left(-\frac{1}{3}\right)=\dots \lim_{n\to\infty} f\left(-\frac{1}{n}\right)=f(0)$

[illegible] $f(x)=f(0)$

$f\left(-\frac{1}{2}\right)=\dfrac{-\frac{1}{2}}{1-\frac{1}{2}}=-\frac{1}{3}$

$f(x_1)=f(x)$ recurrence [illegible]

$x_1=\dfrac{x}{1+x}$

$x_2=\dfrac{x}{1+x_1}=\dfrac{x}{(1+x)(1+2x)}$

$x_n=\dfrac{x}{1+nx}$ for $n\to\infty$ [illegible]

for $x\in(0;-1;-\frac{1}{2}\dots-\frac{1}{n})$

[illegible] has [illegible] fixed

$f\left(\dfrac{x}{1+x}\right)=f\left(\dfrac{\frac{x}{1+x}}{1-\frac{x}{1+x}}\right)=f(x)$

[illegible] $x\notin(0,-1,-\frac{1}{2}\dots-\frac{1}{n})$

$\Rightarrow$ ~~$f(x)=f\left(\frac{x}{1+nx}\right)$~~

$f(1)=f\left(\dfrac{1}{1+1}\right)=f\left(\dfrac{1}{2}\right)$ $f(2)=f\left(\dfrac{2}{1+2x}\right)$

$f(2)=f\left(\dfrac{2}{1+2}\right)=f\left(\dfrac{2}{3}\right)\dots$

$f(x)=f\left(\dfrac{x}{1+nx}\right)$ as $n\to\infty$ $=f(0)$

[illegible]: when $x\in(0;-1\dots\frac{1}{n})$ $f(x)=f(0)$

when $x\notin(0;-1\dots\frac{1}{n})$ $f(x)=f(2)$ [illegible]

~~$f(2)$~~ $=\dfrac{2}{1+2\left(\frac{1}{2}\right)}$ $f(2)=f\left(\dfrac{2}{1+2}\right)$

$f(2)=f\left(\dfrac{2}{1+n\cdot x}\right)$

[illegible] $x\in(0;-1;-\frac{1}{2}\dots-\frac{1}{n})$ [illegible] $f(2)=f\left(\dfrac{2}{1+n\left(-\frac{1}{n}\right)}\right)=\emptyset$ [illegible]

530

(1) $f(z)+zf(1-z)=1+z$ complex functions

$z \to z-1$ [illegible] $f(1-z)$ (1)-ში და (2) ში [illegible]

(2) $f(1-z)+(1-z)f(z)=2-z$

[illegible] (1) და (2) $(1-z+z^2)f(z)=1-z+z^2$

$f(z)=1$ ~~$e^{\pm\frac{\pi i}{3}}$~~

თუ $\boxed{1-z+z^2=0}$

მაშ $z=e^{\pm\frac{\pi i}{3}}$

$\alpha=e^{\frac{\pi i}{3}}$ [illegible]

~~$\bar{\alpha}=\alpha^2=1-\alpha$~~ $1-\alpha+\alpha^2=0$

[illegible] $\bar{\alpha}$ $\bar{\alpha}=\alpha^2=1-\alpha$

$f(\alpha)+\alpha\cdot f(\bar{\alpha})=1+\alpha$

~~$\alpha\bar{\alpha}^2=1$~~

$\boxed{\alpha\cdot\bar{\alpha}=1}$

$f(\bar{\alpha})=\frac{1+\alpha-f(\alpha)}{\alpha}=\bar{\alpha}+1-\bar{\alpha}f(\alpha)$

მაშ $\frac{1}{\alpha}=\bar{\alpha}$

$f(z)=1$; $f(e^{\frac{\pi i}{3}})=\beta$

$f(\bar{\alpha})=f(e^{-\frac{\pi i}{3}})=\bar{\alpha}+1-\bar{\alpha}\cdot\beta$

535 $f(x^2-y^2)=(x-y)(f(x)+f(y))$

$x=y$ $f(0)=0$. $x=-1$ და $y=0$ $f(1)=-f(-1)$

$x=a$; $y=1$ $x=a$ $y=-1$

$f(a^2-1)=(a-1)(f(a)+f(1))$

$f(a^2-1)=(a+1)(f(a)-f(1))$ [illegible]

$f(a)=f(1)\cdot a$

მაგ. $f(3)=f(1)\cdot 3$

[illegible] $(-1)^2$ და $(1)^2$ [illegible]

$f(1)=-f(-1)$ [illegible] $y=1$ და $y=-1$ [illegible]

$y=kx+b$ [illegible]

$f(0)=0 \Rightarrow b=0$

$y=kx$

(523) $$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \ln\frac{b}{a}$$

e^a და b [illegible]

a და $b = y$

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \int_0^\infty \int_a^b e^{-xy}\,dy\,dx = \int_a^b \int_0^\infty e^{-xy}\,dx\,dy = \int_a^b \frac{1}{y}\,dy = \ln\frac{b}{a}$$

$$\int_a^b \frac{e^{y\cdot x}}{x} = e^{-xy}\,dy\,dx$$

[illegible] integral [illegible] [illegible] Tonelli's theorem [illegible] [illegible], $\int_a^b \int_d^c \to \int_d^c \int_a^b$ [illegible] [illegible] [illegible].

[illegible] 523-[illegible] [illegible] [illegible] y [illegible], y [illegible] [illegible]

y-[illegible] limits 1 და (0 [illegible] ∞) ($\lim f(y) = 0$) limit[illegible] [illegible] $f(1) - f(0) = 0$

[illegible] 1 [illegible] [illegible]

(522) $$\int_0^\infty e^{-sx} x^{-1} \sin x\,dx = \arctan(s^{-1})$$

$$f(x,y) = e^{-sxy} \sin x$$

$$\int_0^\infty \int_1^\infty |f(x,y)|\,dy\,dx = \int_0^\infty \int_1^\infty e^{-sxy} |\sin x|\,dy\,dx = \frac{1}{s}\int_0^\infty e^{-sx} x^{-1} |\sin x|\,dx$$

$$\int_0^\infty \int_1^\infty f(x,y)\,dy\,dx = \int_1^\infty \left(\int_0^\infty e^{-sxy} \sin x\,dx\right) dy =$$

$$= \int_1^\infty \frac{1}{s^2y^2+1}\,dy = \frac{1}{s}\int_0^\infty e^{-sx} x^{-1} |\sin x|\,dx$$

formula $\int_0^\infty e^{-cx} \sin x\,dx = \frac{1}{1+c^2}$ $c = -sy$

$$b = \int_1^\infty \frac{s}{s^2y^2+1} = \arctan\frac{\pi}{2} - \arctan s \ldots$$

[illegible] integral[illegible] [illegible] [illegible] [illegible] $\int_a^b \int_d^c \to \int_d^c \int_a^b$ [illegible] [illegible] 2-[illegible] [illegible] [illegible] [illegible] [illegible]

11:52

① If $\int_c^d \int_a^b |f(x,y)|\,dx\,dy < \infty$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\cdot dx$$

$$I = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx$$

$$\int_{-\infty}^{\infty} e^{-xt^2}\,dt = \int_{-\infty}^{\infty} e^{-(\sqrt{x}t)^2}\,dt = \quad \boxed{u = \sqrt{x}t}$$

$$= \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}\cdot\frac{1}{\sqrt{x}} = \sqrt{\frac{\pi}{x}}$$

consider $f(x,t) = e^{-xt^2} e^{-x}$

$$I = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\,dx = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^{\infty} e^{-xt^2} e^{-x}\,dt\,dx$$

$$\int_{-\infty}^{\infty} e^{-xt^2} = \frac{1}{\sqrt{x}}\cdot\sqrt{\pi}$$

$$\boxed{\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-xt^2}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty \int_{-\infty}^{\infty} e^{-(t^2+1)x}\,dt\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \int_0^\infty e^{-(t^2+1)x}\,dx\cdot dt =$$

change of variable

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{t^2+1}\,dt = \frac{\pi}{\sqrt{\pi}} = \sqrt{\pi}$$

② $f(t) = \sum_{n=1}^{\infty} \frac{1}{x^2+n^4}$ find $\int_0^\infty f(t)\,dt$

$F(x,n)$

$$\int_0^x f(t)\,dt = \int_0^x \sum_{n=1}^{\infty} f(t,n)\,dt = \sum_{n=1}^{\infty} \int_0^x \frac{1}{t^2+n^4}\,dt = \sum_{n=1}^{\infty} \frac{1}{n^2} \arctan\frac{x}{n^2}$$

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^2}\arctan\left(\frac{x}{n^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2}\cdot\frac{\pi}{2} = \frac{\pi^2}{6}\cdot\frac{\pi}{2} = \frac{\pi^3}{12}$$

③ $|x| < 1$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{1}{t}\ln(1-t)\,dt$$

∞ $\frac{1}{1-r}$

$$\frac{x^n}{n}$$

$$\sum_{n=0}^{\infty} \int_0^x t^{n-1}\,dt = \int_0^x \left(\sum_{n=0}^{\infty} t^{n-1}\right) dt = \int_0^x \left(t + t^2 + t^3 \dots\right) dt =$$

$$= \int_0^x \frac{1}{1-t}\,dt$$

$$\sum_{n=0}^{\infty} \int_0^x t^{n-1}\,dt = \int_0^x \frac{1}{1-t}\,dt = -\ln(1-x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \int_0^x \frac{t^{n-1}}{n}\,dt =$$

$$= \int_0^x \boxed{\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}}\,dt = -\int_0^x \frac{1}{t}\ln(1-t)\,dt \qquad \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = -\frac{1}{x}\ln(1-x)$$

(542) $3f(2x+1) = f(x) + 5x$ [illegible] [illegible] linear [illegible]

$f(2x+1) = \cancel{\frac{f(x)}{3}} \; \cancel{\frac{ax+b}{3} + \frac{5x}{3} = \frac{ax}{3} + \frac{b}{3} + \frac{5x}{3} = \frac{a+5}{3}x + \frac{b}{3}}$

[illegible] $f(x) = ax + b$

$f(2x+1) = a(2x+1) + b = 2ax + a + b$

(1) [illegible] $3(f(2x+1)) = 6ax + 3a + 3b$ $\cancel{f(x) = ax+b}$

$\cancel{= ax + b + 5x}$

(2) [illegible] $f(x) + 5x = ax + b + 5x$

(1) → (2) $\Rightarrow a = 1$; $b = -\frac{3}{2}$;

$\Rightarrow \boxed{f(x) = x - \frac{3}{2}}$

1 solution [illegible]

[illegible] input $2x+1$ [illegible] polynomial [illegible] linear [illegible] $f(2x+1)$ $f(x)$ input

$f(x)$

$g(x) = \frac{1}{3} g\left(\frac{x-1}{2}\right)$

for $x = -1$

$g(-1) = \frac{1}{3} g(-1) \Rightarrow g(-1) = 0$

$x = \frac{x-1}{2} \Rightarrow 2x = x - 1$
$x = -1$

[illegible] function [illegible] $g(x)$ [illegible] $= 0$ [illegible]

$g(x) = f(x) - \left(x - \frac{3}{2}\right)$ [illegible] original [illegible]

$\cancel{3g(2x+1) = g(x)}$

$3 \cdot g(2x+1) = g(x) \Rightarrow 3 \cdot g(x) = g\left(\frac{x-1}{2}\right)$

[illegible] x [illegible] (that's from oxford ...)

$\cancel{2x+1}$ $\frac{2x+1-1}{2} = x$ [illegible] $\frac{x-1}{2} = x$

$\cancel{3g\left(\frac{x-1}{2}\right) = g(x)}$

define recurrence $x_0 = x$; $x_{n+1} = \frac{x_n - 1}{2}$

[illegible] define [illegible]

① $f(x) = kx$ [illegible] $f(k) = \frac{k2+k}{3}$

② $g(x) = 3 \ldots$ ([illegible] $g\left(\frac{x-1}{2}\right) = \ldots$

① [illegible] 2 [illegible] $f(x)$ [illegible] k [illegible] recurrence

② $g(x) = $ [illegible]
$g\left(\frac{x-1}{2}\right) = $ [illegible] | $g(x) = \ldots$
$g(-1) = \ldots$

now $\frac{x_n - 1}{2}$ [illegible]

$g(2x+1)$ [illegible] recurrence [illegible] limit [illegible]

$x_{n+1} = \frac{x_n - 1}{2}$

$x_{n+2} = \cancel{\frac{x_{n+1}-1}{2} \ldots \frac{x_n}{4}}$

convergent? [illegible] limit? [illegible]

$\cancel{x_{n+1} - x_n}$

$|x_{n+m} - x_m| < \varepsilon$

$x_1 = \frac{x}{2}$
$x_2 = \frac{x}{4}$
$x_3 = \frac{x}{8}$

$\Rightarrow x_{n+m} = \frac{x}{2^{n+m}}$

$x_m = \frac{x}{2^m}$

$L = \frac{L-1}{2}$

$L = -1$

$\Rightarrow g(x) = \frac{1}{3} g(x_1) = \frac{1}{9} g(x_2) = \cdots \frac{1}{3^n} g(x_n)$

$= \frac{1}{3^n} (\lim g(x_n))$

$= \frac{1}{3^n}(-1)$

$|x_{n+m} - x_m| = \left|\frac{x}{2^{n+m}} - \frac{x}{2^m}\right| = \left|\frac{x}{2^m}\left(\frac{1}{2^n} - 1\right)\right|$

$= \left|\frac{x}{2^{n+m}} - \frac{x}{2^m}\right| \leq \frac{|x|}{2^n}$

$0 \leq \frac{1}{2^n} \leq 1$

convergent (???)

[illegible]

$\Rightarrow g(x) = 0$

$g(x) = f(x) - \left(x - \frac{3}{2}\right) = 0$

$f(x) = x - \frac{3}{2}$ [illegible]

[illegible] $x=0, y=0$ ...

$f((x-y)^2) = f(x)^2 - 2x f(y) + y^2$ find functions [illegible] $x=y$ ...

for $y=0$ $\quad f(x^2) = f(x^2) - 2x f(0)$ (1)

for $x=0$, $f(y^2) = f(0)^2 + y^2$ (2)

(2)-ში $y=0$ $f(0)=0$ ან $f(0)=1$ [illegible]

(1) და (2)-ს $x=y$ [illegible] $f(x^2) - 2x f(0) = f(0)^2 + x^2$

$f(x)^2 = (x+f(0))^2$ [illegible]

$f(y) = \dfrac{f(x)^2 - f((x-y))^2 + y^2}{2x} = \dfrac{(x+f(0))^2 - (x-y+f(0))^2 + y^2}{2x}$

$= y + f(0)$

$\Rightarrow f(y) = y + f(0)$

$f(y) = y + 0$ ან $f(y) = y+1$

$f(t) = x$ ან $f(x) = x+1$

და $f(t)^2 = (t + f(0))^2$

სადაც $\boxed{t = x-y}$

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(543) $f: (0;\infty) \to (0;\infty) \to f(t) \geq x$

(i) $f(f(f(x))) + 2x = f(3x)$

$f(3x) \geq 2x \Leftrightarrow \boxed{f(x) \geq \frac{2x}{3}}$ [illegible]

[illegible]

1) [illegible] $f(t)$-სა და x-ს [illegible] $f(f(f(x)))$ [illegible]

~~$f(f(f(x))) + 2x = f(3x)$~~ $\quad f(\underset{\text{input}}{x}) \geq kx$

~~$f(f(t)) \geq f(kx)$~~

$f(\underbrace{f(x)}_{\text{input}}) \geq k f(x) \geq k^2 x$

$f(ff(x)) \geq k^3 x$

$k^3 x + 2x \leq f(3x) \Leftrightarrow f(3x) \geq k^3 x + 2x$

$f(x) \geq k \frac{3+2}{3}$ ✓

ე.ი. თუ $f(x) \geq kx$ -ისთვის $f(x) \geq \frac{k^3+2}{3} x$

AM-GM [illegible]

$\boxed{k_1 = \frac{2}{3}}$ $\quad k_{n+1} = \frac{k_n^3+2}{3} = \frac{k_n^3+1+1}{3} \geq k_n$ $\quad k_{n+1} \geq k_n$ ე.ი. sequence increasing

$f(x) \geq x$ [illegible] $f(t) \leq x$ [illegible] ან $f(t) - x \leq 0$

ე.ი. $k_{n+1} \geq k_n$, [illegible] $\boxed{k_n < 1}$ [illegible]

$k_1 < 1$ და $\frac{k_2+2}{3} < 1$ [illegible], ე.ი. bounded [illegible]

~~$f(kx)$~~ $f(3^n x) =$

$L = \frac{L^3+2}{3}$;

$L \in [0;1]$ $(L=1)$

$\Rightarrow \lim k_n = 1$ $\Leftarrow$ [illegible] limit [illegible]

(ii)

ე.ი. $\boxed{f(x) \geq kx \geq 1x}$

$f(3x) \geq 2x + f(x)$

iteratively $f(3^n x) - f(x) \geq (3^n - 1)x$ და $f(x) - x \leq f(3^n x) - 3^n(x)$ for $n \to \infty$

$\boxed{f(x) = x}$ $\quad f(x) - x \leq 0$ $\quad f(x) \leq x$

(571) n = odd (θ Real) $\frac{\theta}{\pi}$ is irrational

$a_k = \tan\left(\theta + \frac{k\pi}{n}\right)$; $k = 1, 2, 3 \dots n$.

Prove: $\frac{a_1 + a_2 \dots a_n}{a_1 a_2 \dots a_n \dots}$

$\left(\frac{1+ix}{1-ix}\right)^n = w^{2n}$

$\frac{1+ix}{1-ix} = e^{2i \arctan x}$ ახსენით (x ნამდ.) $\mathbb{R}$-ზე ~~ნამდვილ~~ ... ყველა

$\left(e^{2i\arctan x}\right)^n = w^{2n}$; $e^{2in\cdot\arctan x} = (\cos\theta + i\sin\theta)^{2n} = \left(e^{i\theta}\right)^{2n} = e^{2in\theta}$

$2n \cdot \arctan x = 2n\theta \ (+ 2\pi k)$

$\arctan x = \theta + \frac{k\pi}{n}$

$\boxed{x = \tan\left(\theta + \frac{k\pi}{n}\right)}$ $x = a_k$ და

tan (ფესვების ჯამის) და ნამრავლის გამოყენება.

და $a_k = \tan\left(\theta + \frac{k\pi}{n}\right)$ $\frac{a_1 + a_2 + a_3 \dots a_n}{a_1 \cdot a_2 \cdot a_3 \dots}$

Proved; ახლა ვაჩ.

x-ის polynomial-ის ფესვების ჯამი და ნამრავლი

$\left(\frac{1+ix}{1-ix}\right)^n = w^{2n}$

$(1+ix)^n - w^{2n}(1-ix)^n = 0$

$(1+ix)^n = \sum_{j=0}^{n} \binom{n}{j} (ix)^j = \sum_{j=0}^{n} \binom{n}{j} i^j x^j$

$(w^{2n}) \cdot (1-ix)^n = \sum_{j=0}^{n} \binom{n}{j} (-ix)^j = \left(\sum_{j=0}^{n} \binom{n}{j} (-1)^j i^j x^j\right) \cdot w$

და $\sum_{j=0}^{n} \binom{n}{j} i^j x^j - \sum_{j=0}^{n} \binom{n}{j} w^{2n} (-1)^j i^j x^j =$

$= \sum_{j=0}^{n} \binom{n}{j} i^j \left(x^j - w^{2n} \cdot (-1)^j \cdot x^j\right)$ x გავიტანოთ

$= \sum_{j=0}^{n} \binom{n}{j} i^j \left(1 - w^{2n} (-1)^j\right) x^j$ მივიღებთ polynomial x^n ხარისხის

If j is even $(-1)^j = 1$ და $1 - w^{2n}$

და odd -ის -1 და $1 - w^{2n}(-1) = 1 + w^{2n}$

და $\binom{n}{0}$ $i^0 (1 - w^{2n}) x^0 = 1 - w^{2n}$ $j=0$ even

$\binom{n}{1}$ $i^1 (1 + w^{2n}) x^1 = n \cdot i (1 + w^{2n}) x$ $j=1$

$\binom{n}{2}$ $i^2 (1 - w^{2n}) x^2$. $j=2$ ...

n = odd-ის ... ბოლო $j = n$

$\binom{n}{n} i^n (1 - w^{2n}$ ან $1 + w^{2n}) x^n$

. ვიეტა-ს $\sum$ფესვების $= \frac{-n \cdot i (1-w^{2n})}{i^n (1+w^{2n})}$

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$\cos a \cdot \cos 2a \cdot \cos 3a \cdots \cos(999)a$ $\quad a = \frac{2\pi}{1999}$ $\quad \to$ odd $\quad 1999 = 2\cdot 999+1$

[illegible] $S = \cos a \cdot \cos 2a \cdots \cos(na)$ $\quad \to$ odd

generally [illegible] odd [illegible] $\quad a = \frac{2\pi}{2n+1}$

[illegible] $z = e^{ia} \Rightarrow S = 2^{-n}\prod_{k=1}^{n}(z^k + z^{-k})$

~~$\cos(n)(a) = \frac{1}{2}$~~

$\cos n = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

$z^k + z^{-k} = z^{2n+1-k} + z^{-(2n+1-k)}$

$S^2 = 2^{-2n}\prod_{k=1}^{2n}(z^k + z^{-k}) = 2^{-2n}\cdot\prod_{k=1}^{2n} z^{-k}\cdot\prod_{k=1}^{2n}(1+z^{2k})$

$= z^{-(1+2+\cdots 2n)}$

$= z^{-n(2n+1)}$

~~$\prod_{k=1}^{2n}(1+z^{2k}) = \prod_{n=1}^{2n}(1+z^k)$~~

~~$z^{n+1} - 1 = (z-1)\prod_{k=1}^{2n}(z - \varepsilon^k)$~~

~~$\prod(1+z) = 2$~~ $\quad 2\left(1+\frac{2\pi}{?}\right)$ $\quad t = z^k$

873 $f(n) = x^n \sin(nA) + y^n \sin(nB) + z^n \sin(nC)$

$A+B+C = k\pi$. Prove $f(1) = f(2) = 0 \to$

[illegible] $p = xe^{iA}$, $q = y\cdot e^{iB}$; $r = z\cdot e^{iC}$

$f(n) = p^n + q^n + r^n = x^n(e^{iA})^n + \cdots$

$= x^n(\cos(nA) + i\sin(nA))$ [illegible]

[illegible] $f(n) = \operatorname{Im}(f(n))$

$f(n) = 0$ [illegible] $f(n)$ [illegible] ~~real~~ real for all n.

$f(1) = f(2) = 0$ [illegible] $f(1)$ [illegible] $f(2)$ are real. [illegible] 2 [illegible] induction [illegible]. ~~$f(1) = 0$ $f(2)$~~

$f(1)$ [illegible] $f(2)$ [illegible] also real.

$f(k)$ is real prove $f(n+1)$

Real $\to a = p+q+r = f(1)$ Real. Real $b = pq+qr+rp = \frac{1}{2}\underbrace{(f(1)^2 - f(2))}_{\text{Real}}$ A [illegible] formula.

Real $\to c = p\cdot q\cdot r = x\cdot y\cdot z\, e^{i(A+B+C)}$ Real.

[illegible] p, q, r [illegible] 3 [illegible] Real [illegible] [illegible] p, q, r [illegible]

$P(t) = t^3 - at^2 + bt - c$ Real coeff.

$f(n+1) = p^{n+1} + q^{n+1} + r^{n+1}$

$= a\cdot(p^n + q^n + r^n) - b(p^{n-1} + q^{n-1} + r^{n-1}) + c(p^{n-2} + q^{n-2} + r^{n-2})$

$= a\cdot f(n) - bf(n-1) + cf(n-2)$ [illegible]

(5.7.4) $\phi(t)$ $\quad P = \int_0^\infty e^{-(t+i\phi(t))}\,dt$ and $Q = \int_0^\infty e^{-2(t+i\phi(t))}\,dt$

ensure $4P^2 - 2Q$ is real/positive

Prove $|4P^2 - 2Q| \le 3$

$e^{-t}\cdot \underbrace{e^{i\phi(t)}}_{\cos i\phi(t) + \sin i\phi(t)}$

$$4P^2 - 2Q = 4\left(\int_0^\infty e^{-t}\cdot\cos\phi(t)\right)^2 - 4\int_0^\infty e^{-t}\sin\phi(t)\,dt^2 - 2\cdot\int_0^\infty e^{-2t}\cos 2\phi(t)\,dt.$$

$$4\left(\int_0^\infty e^{-t}\cdot\cos\phi(t)\,dt\right)^2 = \left(\int_0^\infty e^{-\frac{1}{2}t}\, e^{-\frac{1}{2}t}\cdot\cos\phi(t)\,dt\right)^2$$

$$\le \int_0^\infty e^{-t}\,dt\cdot\int_0^\infty e^{-t}\cos^2\phi(t)\,dt$$

$$= \int_0^\infty e^{-t}\cdot\cos^2\phi(t)\,dt$$

$$4P^2 - 2Q \le 4\int_0^\infty e^{-t}\cdot\cos^2\phi(t) - 2\int_0^\infty e^{-2t}\cos 2\phi(t)\,dt$$

$$= 4\int_0^\infty (e^{-t} - e^{-2t})\cdot\cos^2\phi(t)\,dt + 1$$

$$\le 4\int_0^\infty (e^{-t} - e^{-2t})\,dt + 1 = 3$$

equality holds, if $\cos^2\phi(t) = 1$ const for all t.

$f: [-1,1]$ cont. function

$f(2x^2 - 1) = 2x\cdot f(x)$ for all $x \in [-1,1]$

$[-1,1]$ range (trig cos/sin) show f is identically $= 0$

$2x^2 - 1$ should remind us of $2\cos^2 t - 1 = \cos 2t$, $x = \cos t$ for $t \in [0,\pi]$

$f(\cos 2t) = 2\cos t\, f(\cos t)$

$t \in [0,\pi]$ for $t = 0$ $\quad f(1) = f(-1) = 0$

$g(t) = \frac{f(\cos t)}{\sin t}$ double angle formula

$$g(2t) = \frac{f(2\cos^2 t - 1)}{\sin 2t} = \frac{2\cos t\cdot f(\cos t)}{2\sin t\cos t} = \frac{f(\cos t)}{\sin t} = g(t)$$

$g(t + 2\pi) = \cdots = g(t)$

$g(t + \frac{2\pi}{6}) = g(t)$

$g(1 + \frac{2\pi}{t})$ on $t = 2^k$

$$g\left(1 + \frac{n\pi}{2^k}\right) = g(2^{k+1} + 2n\pi) = g(2^{k+1}) = g(1)$$

g is cont. everywhere except when $\sin = 0$ (at π's multiples)

$1 + \frac{n\pi}{2^k}$ is dense, so g const or const). $f(\cos t) = c\cdot\sin t$

$\cos(t) = x$

$f(x) = c\sqrt{1 - x^2}$ for $x \in [-1,1]$

$\underbrace{f(2x^2 - 1)}_{\text{even}} = \underbrace{2x\cdot f(x)}_{\text{odd}}$ $\quad$ f is even function.

$c = 0$

$x+y+z=xyz$

$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz$$

if $xyz=0$, გამოდის.

if not.

$:4xyz$ ასე მივიღებთ

$\tan \le \infty$

$$\frac{1-y^2}{2y}\cdot\frac{1-z^2}{2z}+\frac{1-z^2}{2z}\cdot\frac{1-x^2}{2x}+\frac{1-x^2}{2x}\cdot\frac{1-y^2}{2y}=1$$

$\tan A$ -ს გამო.

$x=\tan A$

$y=\tan B$

$z=\tan C$

$$\boxed{\frac{1-\tan^2(x)}{2\tan x}=\frac{1}{\tan 2x}=\cot(2x)}$$

$$\cot 2B\cdot\cot 2C+\cot 2C\cdot\cot 2A+\cot 2A\cot 2B=1$$

~~$\tan 2A+\tan 2B+\tan 2C=$~~ $\frac{1}{\tan 2B\tan 2C}+\frac{1}{\tan 2C}\ldots$

$$\boxed{\tan(2A+2B+2C)=0}$$

$$\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$$

(575) $\sqrt{abc}+\sqrt{(1-a)(1-b)(1-c)}\le 1$ for $a,b,c\in[0,1]$

ვიყენებ. ან ავიღო $\sin^2$ და ვიღებ $1-a=1-\sin^2\theta=$ $=\cos^2\theta$

ან $\sin(\ldots)$. ან

$\sqrt{ab}+\sqrt{(1-a)(1-b)}$ ვაჩვენოთ რომ ≤ 1

$a,b\in[0,\frac{\pi}{2}]$

$\sin a\cdot\sin B+\cos a\cdot\cos B\le 1$; $\cos(a-B)\le 1$

true. x^3 და $x^{\frac{1}{2}}$ ... ...

(576) $x^3-3x=\sqrt{x+2}$ if $x>2$ ... ...

$x=2\cos a$

$\cos 3A=4\cos^3A-3\cos A$

$x^3-3x>4x-3x=x>\sqrt{x+2}$

$2\cos 3a=\sqrt{2(1+\cos a)}=$

$=2\cos\frac{a}{2}$

$x<-\ldots$

$-2\le x\le 2$

$2\cdot\cos(a)-2$ ...

$x=2\cos a$ $a\in[0;\pi]$

$2\cos 3a-2\cos\frac{a}{2}=0$

$2\sin(A)\sin B=\cos(A-B)-\cos(A+B)$

$$2\sin\frac{7a}{4}\sin\frac{5a}{4}=0$$

$a=0$, $a=\frac{4\pi}{7}$ და $\frac{4\pi}{5}$

solutions

$x=2;\ 2\cos\frac{4\pi}{7};\ -\frac{1}{2}(1+\sqrt{5})$

$x=2\cdot 2$

$x\ge 2$

662) $\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \dots = 2^{\frac{n}{2}} \cdot \cos\frac{n\pi}{4}$

$(1+i)^n = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

[illegible] [illegible] [illegible]

real the parts

[illegible] $(1+i)^n$

663) $\binom{n}{1}\cos x + \binom{n}{2}\cos 2x + \dots \binom{n}{n}\cos nx = ?$

$= S_1$

$S_2 = \binom{n}{1}\sin x + \binom{n}{2}\sin 2x + \dots \binom{n}{n}\sin nx$

[illegible], using euler's formula ~~[illegible]~~

~~[illegible]~~

$1 + S_1 + iS_2 = \binom{n}{0} + \binom{n}{1}e^{ix} + \binom{n}{2}e^{2ix} + \dots \binom{n}{n}e^{inx}$

euler's form. $S_1 = \underbrace{\sum_{k=1}^{n}\binom{n}{k}\cos kx}_{\text{re part}}$ $\quad S_2 = \underbrace{\sum_{k=1}^{n}\binom{n}{k}\sin kx}_{\text{im part}}$

$e^{ikx} = \cos kx + i\sin kx$ [illegible] $\cos(2x) + \cos(4x)$, $\sin(2x) + \sin(4x)$ [illegible]

e^{ikx} [illegible] $\binom{n}{1}$, $\binom{n}{2}$ … [illegible] $\binom{n}{k} e^{ikx}$

[illegible] $= \binom{n}{k}(\cos kx + i\sin kx)$

$\sum_{k=0}^{n}\binom{n}{k}e^{ikx} = 1 + S_1 + iS_2$

[illegible] $(e^{ix\cdot k}) = (e^{ix})^k$ $\quad$ ~~$\cos 2x$~~ [illegible] e^{ikx} [illegible] e^{ikx} [illegible] $(e^{ix})^k$ [illegible]

$\sum_{k=0}^{n}\binom{n}{k}(e^{ix})^k = (1+e^{ix})^n$

$= (1+\cos x + i\sin x)$

$\cos(2x)$ [illegible], $\sin(2x)$ [illegible], x [illegible] $\frac{x}{2}$

$= 2\cdot\cos^2\left(\frac{x}{2}\right) + i\cdot 2\cdot\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

$= 2\cdot\cos\left(\frac{x}{2}\right)\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)$

$= 2\cdot\cos\left(\frac{x}{2}\right)\cdot\boxed{e^{i\frac{x}{2}}}$

$\sum_{k=0}^{n}\binom{n}{k}(e^{ix})^k = \left(2\cos\frac{x}{2}\right)^n\cdot\left(e^{i\frac{x}{2}}\right)^n = \left(2\cdot\cos\frac{x}{2}\right)^n\cdot\left(\cos\frac{nx}{2} + i\sin\frac{nx}{2}\right) - 1$

[illegible] $S_2 = \left(2^n\cdot\cos^n\frac{x}{2}\cdot\cos\frac{nx}{2} + i\sin\left(\frac{nx}{2}\right)\right)$

$S_1 = 2^n\cdot\cos^n\frac{x}{2}\cdot\cos\frac{nx}{2} - 1$

$1 + S_1 + S_2 = T$ [illegible] $\boxed{S_1 = T - S_2 - 1}$

S_1 [illegible] real part.

(669) taylor at 0

$f(t) = e^{x \cdot \cos\theta} \cdot \cos(x \cdot \sin\theta)$

(670) z_1, z_2, z_3 Same ~~[illegible]~~ [illegible]

$|z_1| = |z_2| = |z_3|$

Prove: if $z_1 + z_2 z_3,\ z_2 + z_3 z_1,\ z_3 + z_1 z_2$ are real

$z_1 z_2 \cdot z_3 = 1$

$\boxed{z_1 + z_2 z_3 = e^{it_1} + e^{i(t_2+t_3)}}$ are real, [illegible] $\sin$ [illegible] $=0$

$z_j = r(\cos t_j + i \sin t_j)$ $\quad r \neq 0 \quad t_j \in (0,\pi) \cup (\pi, 2\pi)$??

~~$\sin t_1 + r \cdot \sin(t_2+t_3) = 0$~~

[illegible] $\sin\ r \cdot \sin t_1 + r^2 \cdot \sin(t_2+t_3) = 0$

[illegible]

$$\begin{cases} \sin t_1 + r \cdot \sin(t_2+t_3) = 0 \\ \sin t_2 + r \cdot \sin(t_3+t_1) = 0 \\ \sin t_3 + r \cdot \sin(t_1+t_2) = 0 \end{cases}$$

[illegible] $t = \text{const}$ [illegible] $/ t$ $\quad t = t_1+t_2+t_3 \rightarrow t - t_3$

$\sin t_j = -r \cdot \sin(t - t_j)$ $\quad$ ~~$\sin t_3 =$~~

$\sin(t_j)$ [illegible]

$= -r \sin t \cdot \overset{\cos}{\cancel{\sin}} t_j - r \cdot \cos t \cdot \sin t_j$

~~$\cot t_j =$~~ $= -r \sin t \cot t_j \sin t_j$ ~~[illegible]~~ $- r \cos t \cdot \sin t_j = \sin t_j$

$\cot t_j = \frac{\cos t_j}{\sin t_j}$ $\quad$ $\cot t_j \cdot \sin t = \frac{1}{r} - \cos t$

$\cos t_j = \cot t_j \sin t$

(1) if $\sin t \neq 0$; $\cot t_1 = \cot_2 = \cot t_3$

$\cot t_{(1,2,3)} \underbrace{\sin t}_{\text{const}} = \underbrace{\frac{1}{r} - \cos t}_{\text{const}}$ [illegible] const [illegible]

$j \in (1,2,3)$ $\quad$ $\boxed{t = \text{const}}$ [illegible] $t_1 + t_2 + t_3$ [illegible]. $\quad$ $\boxed{\cot t_{1,2,3}}$

(2) [illegible] $\sin t = 0$

$\frac{1}{r} - \cos t = 0$

$1 - r\cos t = 0$

$r \cdot \cos t = \pm 1$

[illegible] $z_i = r \cdot \cos t = 1$

$\boxed{z_1 \cdot z_2 \cdot z_3 = r^3 \cdot \cos t = 1}$

[illegible] [illegible] $\sin t \neq 0$ $\quad \cot(t_1) = \cot(t_2) = \cot(t_3)$

$t_1 \neq t_2 \neq t_3$ [illegible], ~~[illegible]~~

$\cot$ [illegible] period π [illegible]

$\cot(t_1 + \pi) = \cot(t_2)$

[illegible] $(0, 2\pi)$ [illegible]

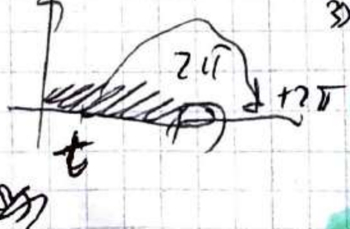
$t_1 + 2\pi$ [illegible]

[illegible] t_j

[illegible] $t_1 = t_j$ [illegible]

$t_2 = t_j$ ...

[illegible]



[illegible] $_3$ [illegible]

4.2.2 ~~2:26~~ 2:30

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \frac{z^n}{n!}$$

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

real's

$$e^{ix} = \cos x + i \sin x \qquad e^{nz} = (e^z)^n$$

$$e^{inx} = (e^{ix})^n$$

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n$$

Prove: given k and n, and also k and n ...

$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{2^n}{k} \sum_{j=1}^{k} \cos^n \frac{j\pi}{k} \cdot \cos \frac{nj\pi}{k}$$

Let $\epsilon_1, \epsilon_2 \ldots \epsilon_k$ be k^{th} root of unity

$$\epsilon_j = \cos \frac{2j\pi}{k} + i \sin \frac{2j\pi}{k}$$

$$\epsilon_1^s + \epsilon_2^s + \cdots \epsilon_k^s = k \quad \text{if } k \text{ divides } s.$$

$$= 0 \quad \text{if } k \text{ does not divide } s$$

$$\sum_{j=1}^{k} (1 + \epsilon_j)^n = \sum_{s=0}^{n} \binom{n}{s} \left(\sum_{j=1}^{k} \epsilon_j^s \right) = k \cdot \sum_{j=0}^{[n/k]} \binom{n}{jk}$$

if k divides s, only some k-th ...

$$1 + \epsilon_j = 2 \cos \frac{j\pi}{k} \left(\cos \frac{j\pi}{k} + i \sin \frac{j\pi}{k} \right)$$

$$1 + \cos \frac{2\pi j}{k} + i \sin \frac{2\pi j}{k} = 2 \cos \frac{\pi j}{k} \left(\cos \frac{\pi j}{k} + i \sin \frac{\pi j}{k} \right)$$

$$\boxed{1 + \cos x = 2 \cos^2 \frac{x}{2}}$$

de-moivre's form

$$\sum_{j=1}^{k} (1 + \epsilon_j)^n = \sum_{j=1}^{k} 2^n \cos^n \frac{j\pi}{k} \left(\cos \frac{nj\pi}{2} + i \sin \frac{nj\pi}{k} \right)$$

Proved

and k ... only divide $s = 0$ ...

$f(x) = a + b\cos 2x + c\sin 5x + d\cos 8x$ → complex

$f(t) = 4a$. Prove s for which $f(s) < 0$

Let $g(x) = b e^{2ix} - i c e^{5ix} + d e^{8ix}$, then

$f(x) = a + \mathrm{Re}\, g(x)$

$e^{ix} = (\cos x + i\sin x) \Rightarrow e^{5ix} = (\cos 5x + i\sin 5x)$

$e^{2ix}, e^{5ix}, e^{8ix}$ $\underbrace{\ }_{+3}$ $\underbrace{\ }_{+3}$

$i e^{5ix} = \underbrace{i(\cos 5x)}_{\mathrm{Im}} - \underbrace{1\cdot\sin 5x}_{\mathrm{Re}}$

$\Rightarrow \boxed{g(x) + g(x + \frac{2\pi}{3}) + g(x + \frac{4\pi}{3}) = 0}$

$g(x) + g(x + \frac{2\pi}{3}) + g(x + \frac{4\pi}{3}) = g(x)(1 + e^{\frac{2\pi i}{3}} + e^{\frac{4\pi i}{3}}) = 0$

$f(x) + f(x + \frac{2\pi}{3}) + f(x + \frac{4\pi}{3})$ ~~$+a+a+a$~~ $= 0$

$-a-a-a = 0$

$f(x) + f(\ldots) = 3a$, if $\boxed{a > 0}$

$f(t) = 4a$

$\boxed{s = t}$

$\underbrace{f(t)}_{4a} + f(t + \frac{2\pi}{3}) + f(x + \frac{4\pi}{3}) = 3a$

$-a$

if $a = 0$

negative

(Fourier series)

If ~~$a > 0$~~ ~~$x = t$~~

$\boxed{\text{if } a < 0}$

(668)

$\left(\frac{1 + i\tan t}{1 - i\tan t}\right)^n = \frac{1 + i\tan nt}{1 - i\tan nt}$

$\left(\frac{(1 + i\tan t)\cdot\cos t}{(1 - i\tan t)\cdot\cos t}\right)^n = \frac{1 + i\tan nt}{1 - i\tan nt}\cdot\frac{\cos nt}{\cos nt} = \left(\frac{e^{it}}{e^{-it}}\right)^n = \frac{e^{int}}{e^{-int}}$

$e^{it} = \cos t + i\sin t$

$e^{-it} = \cos t - i\sin t$

$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{e^{it} - e^{-it}}{2i}}{\frac{e^{it} + e^{-it}}{2}} = \frac{e^{it} - e^{-it}}{i(e^{it} + e^{-it})}$

$1 + i\tan nt = 1 + i(\ldots) = 1 + \frac{e^{it} - e^{-it}}{e^{it} + e^{-it}} = \frac{2e^{it}}{e^{it} + e^{-it}}$

$1 - i\tan t = \frac{2e^{-it}}{e^{it} + e^{-it}}$ left hand side $= e^{2int}$, right hand side e^{2it}

e^{2it} raise to n power

637 ⇒ 645

① $2\sin x \cos 40^\circ$ $(0, 180)$

① $2\sin x \cos 40^\circ = \sin(x + 20^\circ)$

$$\boxed{\tan x} = \frac{\sin 20^\circ}{2\cos 40^\circ - \cos 20^\circ} = \tan 30^\circ$$

$x = 30^\circ$

$(0, 180)$

tan one-to-one function

② (a) $\cos \pi a = \frac{1}{3}$

$\cos \pi \cdot k = 0$

$\cos \pi \cdot a \cdot n = \pm 1$

integer

$\cos k a \pi = \frac{b_k}{3^k}$

b not divisible by 3^k

$k = 0$ and $k = 1$

$\cos(0 \cdot a\pi) = 1$

$$\frac{b_0}{3^0} = \frac{1}{1} = 1$$

$k = 1$ $\cos(a\pi) = \frac{b_1}{3^1}$

$\cos(k a\pi) = \frac{b_k}{3^k}$

$k = 0$

$\cos(k-1)a\pi = \frac{b_{k-1}}{3^{k-1}}$

$k = 1$

$$\cos(k+1)a\pi = 2\cos(a\pi)\cdot\cos(k a\pi) - \cos((k-1)a\pi)$$

$$= 2\cdot\frac{b_1}{3}\cdot\frac{b_k}{3^k} - \frac{b_{k-1}}{3^{k-1}} = \frac{2 b_1 b_k}{3^{k+1}} - \frac{b_{k-1}}{3^{k-1}}$$

$$= \frac{2b_1 b_k - 3b_{k-1}}{3^{k+1}} \qquad \frac{b_{k+1}}{3^{k+1}}$$

$b_{k+1} = 2b_1 b_k - 3b_{k-1}$

62

$x^3 + 3xy + y^3 = 1$ 3 non collinear A, B, C equidistant to each other find ABC area

$x^3 + 3xy \cdot (x+y) + y^3 = (x+y)^3$

let $x+y = s$; $x^3 + 3xy(s) + y^3 = s^3$

$\cancel{3xys - 3xy = s^3 - 1}$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = s(s^2 - 3xy)$

$s(s^2 - 3xy) + 3xy = 1$

$\ldots (s-1)(s^2 + s + 1 - 3xy) = 0$

$s = 1$ $(s^2) + s - 1 - 3xy = 0$

$x + y = 1$ $(x+y)^2 + x + y - 1 - 3xy = 0$

$\frac{1}{2}((x-y)^2 + (x+1)^2 + (y+1)^2) = 0$

$x = y$

or $x + y = 1$ IM and $(-1; -1)$ point A $x = y = -1$

B, C also on $x+y = 1$ symmetric to each other by D$(\frac{1}{2}; \frac{1}{2})$

$BC = \frac{\sqrt{3}}{2} AD$, $AD = \sqrt{(\frac{1}{2}+1)^2 + (\frac{1}{2}+1)^2} = \frac{3}{2}\sqrt{2}$

$BC = \sqrt{6}$ Area $= \frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$

692

$$u_n = \arcsin \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}\sqrt{n+1}}$$

[illegible]

Prove $S = u_0 + u_1 + \dots u_n + \dots$ [illegible]

~~$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}\sqrt{n+1}}$ [illegible] telescope [illegible]~~

$$u_n = \arcsin \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}\sqrt{n+1}} = \arcsin B_n - \arcsin B_{n+1}$$

[illegible] telescope [illegible] !

[illegible]

$$\sin(u_n) = \sin(\underbrace{\arcsin B_n}_{A} - \underbrace{\arcsin B_{n+1}}_{B}) =$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(\arcsin B_n) = \sqrt{1-\sin^2(\arcsin B_n)} = \sqrt{1-B_n^2}$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}\sqrt{n+1}} = \sin u_n = B_n \cdot \sqrt{1-B_{n+1}^2} - B_{n+1}\sqrt{1-B_n^2}$$

$B_n = \frac{1}{\sqrt{n+1}}$ [illegible].

$n = 0, 1, 2, \dots$ [illegible]

[illegible]

$u_n = \arcsin B_n - \arcsin B_{n+1}$

$$S = \lim_{n\to\infty} \sum_{n=0}^{n} \arcsin\left(\frac{1}{\sqrt{n+1}}\right) - \arcsin \frac{1}{\sqrt{n+2}}$$

$$= \arcsin 1 - \lim_{n\to\infty} \arcsin \frac{1}{\sqrt{n+2}} = \frac{\pi}{2}$$

677) $s = (1-x_1)(1-y_1) + (1-x_2)(1-y_2)$ max value

if $x_1^2 + x_2^2 = y_1^2 + y_2^2 = c^2$

x i y lie on circle radius $= c$ at (0,0)

$\boxed{x_1^2 + x_2^2 = R^2}$

$x_1 = c\cdot\cos\phi \; ; \; x_2 = c\cdot\sin\phi$

$y_1 = c\cdot\cos\psi \quad y_2 = c\cdot\sin\psi$

$s = 2 - c(\cos\phi + \sin\phi + \cos\psi + \sin\psi) + c^2(\cos\phi\cos\psi + \sin\phi\cdot\sin\psi)$

$= 2 + c\sqrt{2}\left(-\sin\left(\phi + \frac{\pi}{4}\right) - \sin\left(\psi + \frac{\pi}{4}\right)\right) + c^2\cos(\phi - \psi)$

$\cos\phi + \sin\phi + \cos\psi + \sin\psi$; $\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$(\sin\phi + \sin\psi)\left(\cos\frac{\pi}{4}\right)$

$= \sin\phi\cdot\cos\frac{\pi}{4} + \sin\psi\cdot\cos\frac{\pi}{4}$

max $\Rightarrow$ $2 + 2c\sqrt{2} + c^2 = (c + \sqrt{2})^2$ max

$\frac{5\pi}{4}$

$\sin\left(\phi + \frac{\pi}{4}\right) = -1$

$\sin\left(\psi + \frac{\pi}{4}\right) = -1$

$\cos(\phi - \psi) = 1$

$\cos(0) = 1$, $\phi - \psi = 0$

$\phi = \psi$

$\sin\left(\phi + \frac{\pi}{4}\right)$ $\sin\left(\psi + \frac{\pi}{4}\right)$

$\boxed{\phi = \psi}$

678) $\dfrac{|a-b|}{\sqrt{1+a^2}\sqrt{1+b^2}} \le \dfrac{|a-c|}{\sqrt{1+a^2}\sqrt{1+c^2}} + \dfrac{|b-c|}{\sqrt{1+b^2}\sqrt{1+c^2}}$

$a = \tan\alpha$; $b = \tan\beta$; $c = \tan\gamma$ for $\alpha, \beta, \gamma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$a^2 + 1 = \sec^2\alpha$

$\dfrac{|\tan\alpha - \tan\beta|}{\sec^2\alpha\,\sec^2\beta} \le \dfrac{|\tan\alpha - \tan\gamma|}{\sec\alpha\sec\gamma} + \dfrac{|\tan\beta - \tan\gamma|}{\sec\beta\sec\gamma}$

$\boxed{\tan\theta - \tan\vartheta = \dfrac{\sin(\theta - \vartheta)}{\cos\theta\cos\vartheta}}$

$\left|\dfrac{\dfrac{\sin(\theta - \vartheta)}{\cos\theta\cdot\cos\vartheta}}{\sec\theta\cdot\sec\vartheta}\right|$; $|\sin(\alpha - \beta)| \le |\sin(\alpha - \gamma)| + |\sin(\beta - \gamma)|$

$|\sin(\alpha - \gamma + \gamma - \beta)| =$

$= |\sin(\alpha - \gamma)\cos(\gamma - \beta) + \sin(\gamma - \beta)\cos(\alpha - \gamma)|$

$\le |\sin(\alpha - \gamma)|\,|\cos(\gamma - \beta)| + |\sin(\gamma - \beta)\cos(\alpha - \gamma)|$

$\le |\sin(\alpha - \gamma)| + |\sin(\gamma - \beta)|$

max $= 1$

(679) $(ab+bc+ca-1)^2 \le (a^2+1)(b^2+1)(c^2+1)$

a, b, c ... $K \le a, b, c \le K$

... $K \cdot \cos(t)$ ...

... a^2+1 ... $\tan$ ...

$a = \tan u$
$b = \tan v$
$c = \tan w$

$u, v, w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\le \sec^2 u \cdot \sec^2 v \sec^2 w$

$\tan u \cdot \tan v + \dots$ ...

$(\tan u \tan v + \tan u \cdot \tan w + \tan v \tan w - 1) \cos u \cdot \cos v \cdot \cos w =$

$= \sin u \cdot \sin v \cdot \cos w + \sin u \cdot \cos v \cdot \sin w + \cos u \cdot \sin v \cdot \sin w -$

$- \cos u \cdot \cos v \cdot \cos w = \sin u \sin(v+w) - \cos u \cdot \cos(v+w) =$

$= \underbrace{-\cos(u+v+w)}_{-1 \le} \le 1$ proved

$\sin u(\sin v \cdot \cos w + \sin w \cos v \cdot \sin w) =$
$= \sin u \sin(v+w)$

$-\cos u \cdot (\sin v \cdot \sin w + \cos v \cos w) =$
$= -\cos u \cos(v+w)$

(680) $\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \le \frac{3\sqrt{3}}{2}$ $\quad x+y+z = xyz$

AM–GM

$x+y+z = 3\sqrt[3]{xyz}$

... $\tan$ ...

A, B, C ...
... $A+B+C = \pi$
$\tan A \cdot \tan B \cdot \tan C = \tan A + \tan B + \tan C$

$x = \tan\theta$

$\frac{\tan\theta}{\sec\theta} = \sin\theta$; $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

$\sin$ for $f(x)$ concave functions

Jensen's inequality
$x \in \left(0, \frac{\pi}{2}\right) - 3$ $\quad f(x_1) + f(x_2) + f(x_3) \le 3f\left(\frac{x_1+x_2+x_3}{3}\right)$

$\sin \frac{d^2y}{dx^2}(\sin x) \overset{<0}{=}$ concave

$3\sin\left(\frac{A+B+C}{3}\right) \le 3\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$

683 $a_0 = \sqrt{2}$; $b_0 = 2$

$$a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}} \qquad b_{n+1} = \frac{2 b_n}{2 + \sqrt{4 + b_n^2}}$$

(a) $(a_n)_n$ and $(b_n)_n$ decreasing converge to 0

$a_{n+1} = \sqrt{2 - \underbrace{\sqrt{4 - a_n^2}}}$ $\quad a_n = 2\sin t_n$ so $a_n = 2\cos t_n$, $t_n \in [0; \frac{\pi}{2}]$

[illegible] $x = \sqrt{4 - a_n^2}$

~~$x^2 = 4 - a_n^2$ | $x^2 + a_n^2 = 4$~~

$-2 \le a_n \le 2$

$$2\sin t_{n+1} = a_{n+1} = \sqrt{2 - \sqrt{4 - 4\sin^2 t_n}} = \sqrt{2 - 2\cos t_n} = 2\sin\frac{t_n}{2}$$

$\Leftrightarrow \quad 2\sin t_{n+1} = 2\sin\frac{t_n}{2} \Rightarrow t_{n+1} = \frac{t_n}{2}$

$a_0 = \sqrt{2} \Rightarrow t_0 = \frac{\pi}{4}$

$\boxed{t_n = \frac{\pi}{2^{n+2}}}$

[illegible] $\Rightarrow a_n = 2\sin\frac{\pi}{2^{n+2}}$

tan $-n$ subst [illegible]

$b_n = 2\tan u_n$; b_n is positive $\Rightarrow u_n \in (0; \frac{\pi}{2})$ [illegible] $\sqrt{4 + b_n^2}$ [illegible]

$$b_{n+1} = \frac{2 b_n}{2 + \sqrt{4 + b_n^2}}$$

$$2\tan u_{n+1} = b_{n+1} = \frac{2\tan 2u_n}{2 + \sqrt{4 + 4\tan^2 u_n}}$$

$$= \frac{4\tan u_n}{2 + \frac{2}{\cos u_n}} = 2\cdot\frac{\sin u_n}{1 + \cos u_n} = 2\tan\frac{u_n}{2}$$

$\Rightarrow u_{n+1} = \frac{u_n}{2}$

$b_0 = 2 \Leftrightarrow u_0 = \frac{\pi}{4} \quad u_n = \frac{\pi}{2^{n+2}}$; $b_n = 2\tan\frac{\pi}{2^{n+2}}$

b_n & a_n. sin & tan decreasing on $(0; \frac{\pi}{2})$ and as $x \to \frac{\pi}{2}$

$n \to \infty$; $2\tan(\frac{\pi}{\infty}) = 0$

$2\sin(\frac{\pi}{\infty}) = 0$

(b) Function $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$

increasing, decreasing

$2^n a_n = 2^n \cdot$ ~~$\sin\frac{\pi}{2^{n+2}}$~~

[illegible] 2^{n+2} [illegible] $\qquad a_n \cdot 2^n = \frac{\frac{\pi}{2} \sin(\frac{\pi}{2^{n+2}})}{\frac{\pi}{2^{n+2}}}$ $\qquad \frac{\sin x}{x}$ [illegible]

(621) $\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2} \geq \frac{3\sqrt{3}}{2}$

$0<x, y, z<1$ $\quad xy+yz+xz=1$

tan-ის გაორ. $\cdot 2$

$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2} \geq 3\sqrt{3}$

$\boxed{x=\tan\frac{A}{2},\ y=\tan\frac{B}{2},\ z=\tan\frac{C}{2}}$ then $0<x,y,z<1$ ~~ამიტომ~~ $A,B,C\in(0,\frac{\pi}{2})$

$xy+yz+xz=1$ $\quad \frac{1}{\tan x}=\cot$ $\quad \cot(\frac{\pi}{2}-x)=\tan(x)$

$\frac{1}{z}=\frac{x+y}{1-xy}$

$\cot\frac{C}{2}=\tan\frac{A+B}{2}$ იძ გამოდგება

$\cot(\frac{\pi}{2}-x)=\tan x$

$\frac{\pi}{2}-\frac{C}{2}=\frac{A+B}{2}$ or $A+B+C=\pi$

tanA tanB ჯამი. $\tan A+\tan B+\tan C \geq 3\sqrt{3}$ | Jensen's ineq. ...

(682) $\frac{3x-y}{x-3y}=x^2$ (1) $\quad \frac{3y-z}{y-3z}=y^2$ $\quad \frac{3z-x}{z-3x}=z^2$

(1)-დან $x=0$; ... $\frac{-y}{-3y}=0$ $\quad \frac{1}{3}\neq 0$

ამიტომ $y=0$ და $x=0$ არაა ამონახსნი

$3x-y=x^2(x-3y)=x^3-3x^2y$

$3x-x^3=y(1-3x^2)$

$y=\frac{x^3-3x}{3x^2-1}=\frac{3x-x^3}{1-3x^2}$

$z=\frac{3y-y^3}{1-3y^2}$; $x=\frac{3z-z^3}{1-3z^2}$

tan ჯ(3x) tan formula

$y=\frac{3\tan(x)-\tan^3 x}{1-3\tan^2(x)}=\tan(3x)$ ← y იძ ვიპოვე

$\boxed{x=\tan x}$

$z=\frac{3\tan 3x-\tan^3 3x}{1-3\tan^2 3x}=\tan 9x$

~~$x=\tan 27x$~~

$x=\tan 27x$ $\quad -\frac{\pi}{2}<x=\frac{k\pi}{26}<\frac{\pi}{2}$

k possible $\pm1, \pm2\ldots \pm12$

~~$\tan x=\tan 27x$~~ ~~$\tan(x+26\pi)$~~

$x=\tan\frac{k\pi}{26}$; $y=\tan\frac{3k\pi}{26}$, $z=\tan\frac{9k\pi}{26}$

3 ამონახსნი ...

$\tan(x+\pi)=\tan(x)$ ამიტომ.

[illegible]

$\boxed{x_1 = y_1 = \sqrt{3}}$ $\quad x_{n+1} = x_n + \sqrt{1+x_n^2}$

$$y_{n+1} = \frac{y_n}{1+\sqrt{1+y_n^2}}$$

Prove $2 < x_n y_n < 3$

$x_n = \tan a_n$ $\boxed{0^\circ \le a_n < 90^\circ}$

$$x_{n+1} = \tan\theta_n + \sec\theta_n = \frac{1+\sin\theta_n}{\cos\theta_n} = \tan\left(\frac{90^\circ+\theta_n}{2}\right)$$

$x_1 = \sqrt{3} \implies a_1 = 60^\circ;\ \theta_2 = 75^\circ;\ \theta_3 = 82.5^\circ$

$\boxed{a_n = 90^\circ - \frac{30^\circ}{2^{n-1}}}$ [illegible] recursive [illegible]

$\tan\theta_n$
$= x_n = \tan\left(\underbrace{90^\circ - \frac{30}{2^{n-1}}}_{n-1\ [illegible]}\right) = \cot\left(\frac{30^\circ}{2^{n-1}}\right) = \cot\theta_n$

$\theta_n = \frac{30^\circ}{2^{n-1}}$

[illegible] y_n [illegible] $y_n = \cancel{\tan 2\theta_n}\ 2\tan\theta_n$ [illegible] $2\tan\theta_n$.

$y_{n+1} = 2\tan\theta_n$

θ_n [illegible]

$y_n = \tan 2\theta_n = \frac{2\tan\theta_n}{1-\tan^2\theta_n}$

$\boxed{x_n \cdot y_n = \frac{2}{1-\tan^2\theta_n}}$ $\quad 0^\circ < \theta_n \le 45^\circ$

$\theta_n = \frac{30^\circ}{2^{n-1}}$

$0 < \tan^2\theta_n < 1$

$\implies x_n y_n > 2$. If $n > \cancel{1}$

$\theta_n < 30^\circ$

$\tan^2\theta_n < \frac{1}{3}$

$\boxed{x_n \cdot y_n = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} < 3}$

$x^2 - y^2 = 1$ $\quad x = \cosh t \quad y = \sinh t$

$\boxed{x = a\cosh t}$ $\cdot$ in $\sqrt{a^2-1}$

let $a_1 = a_2 = 97$

$a_{n+1} = a_n \cdot a_{n-1} + \sqrt{(a_n^2-1)(a_{n-1}^2-1)}$ $\Rightarrow \sqrt{a_n^2-1} \cdot \sqrt{a_{n-1}^2-1}$

(a) $2 + 2a_n$ is perfect square

(b) $2 + \sqrt{2+2a_n}$ is perfect square

$a_n = \cosh t_n$

$\cancel{\cosh t_{n+1}} = \cosh t_{n+1} = a_{n+1} = \cosh t_n \cdot \cosh t_{n-1} + \sinh t_n \sinh t_{n-1} =$

$= \cosh(t_n + t_{n-1})$

assume $\boxed{t_0 = t_1}$

$t_{n+1} = t_n + t_{n-1}$

Fibonacci

that $t_n = F_n(t_0)$

$a_n = \cosh t_n = \cosh(F_n t_0)$

$2(\cosh t)^2 - 1 = \cosh 2t \;\Rightarrow$

$2 + 2\cosh F_n t_0 = \left(2\cosh F_n \frac{t_0}{2}\right)^2$

$2\cosh(k+1)t = (2\cosh t)(2\cosh kt) - 2\cosh(k-1)t$

$2\cosh \frac{t_0}{2} = \sqrt{2+2 \cdot 97} = 14$. so $2\cosh k \frac{t_0}{2}$ is integer.

(b) $2 + \sqrt{2+2a_n} = \left(2\cosh F_n \frac{t_0}{4}\right)^2$

if $n=1$ is int, we get integer

$\sqrt{2+\sqrt{2+2 \cdot 97}} = \sqrt{2+14} = 4$

(636) $\int \frac{1}{x+\sqrt{x^2-1}}\, dx$

$x = \cosh t \;\Rightarrow\; \int \frac{1}{\sinh t \cdot \cosh t} \sinh t\, dt =$

$= \int \frac{e^t - e^{-t}}{2e^t} = \frac{1}{2}\int \frac{e^t}{e^t} - \frac{e^{-t}}{e^t} =$

$= \frac{1}{2}\int 1 - e^{-2t}\, dt =$

$= \frac{1}{2}t + \frac{e^{-2t}}{4} + C$

Rest is going to...

(683) $27\sin^3 9^\circ + 9\sin^3 27^\circ + 3\sin^3 81^\circ + \sin^3 243^\circ = 20\sin 9^\circ$

$\sin 3x = 3\sin x - 4\sin^3 x$ tele-scopic [illegible]

$27 \cdot \frac{3\sin 9^\circ - \sin 27^\circ}{4} + 9 \cdot \frac{3\sin 27^\circ - \sin 81^\circ}{4} + \dots$

$= \frac{81 \cdot \sin 9^\circ - \sin 729^\circ}{4} = \frac{81\sin 9^\circ - \sin 9^\circ}{4} = 20\sin 9^\circ$

(689) $\frac{1}{\cot 9^\circ - 3\tan 9^\circ} + \frac{3}{\cot 27^\circ - 3\tan 27^\circ} + \frac{9}{\cot 81^\circ - 3\tan 81^\circ} \dots$

$3\tan 3x = \frac{3(3\tan x - \tan^3 x)(-1)}{(1 - 3\tan^2 x)(-1)} = \frac{3\tan^3 x - 9\tan x}{3\tan^2 x - 1} = \frac{3\tan^3 x - 1 - 8\tan x}{3\tan^2 x - 1}$

$= \tan x - \frac{8\tan x}{3\tan^2 x - 1} = 3\tan 3x$

$\frac{1}{\cot x - 3\tan x} = \frac{\tan x}{1 - 3\tan^2 x}$

$\frac{1}{\tan x}$

$\frac{1}{8}(3\tan 3x - \tan x) = \frac{\tan x}{3\tan^2 x - 1} = \frac{1}{\cot x - 3\tan x}$

[illegible]

$= \frac{1}{8}(81\tan 9^\circ - \tan 9^\circ)$

~~tele[illegible]~~

$\tan 3x$ [illegible]

(690) $\frac{1}{\sin 45 \sin 46^\circ} + \frac{1}{\sin 47 \sin 48} + \dots \frac{1}{\sin 133^\circ \sin 134} = \frac{1}{\sin 1^\circ}$ $(\cdot \sin 1^\circ)$

$\frac{\sin 1^\circ}{\sin 45^\circ 46^\circ} + \frac{\sin 1^\circ}{\sin 47 \sin 48^\circ}$ [illegible] $\dots \frac{\sin((k+1^\circ) - k^\circ)}{\sin k^\circ \sin(k+1)^\circ} =$ [illegible]

[illegible]

$= \cot k^\circ - \cot(k+1)^\circ$

A B

$\frac{\sin(A - B)}{\sin B \sin A} = \cot B - \cot A$

$\cot 45^\circ - \cot 46^\circ + \cot 47^\circ - \cot 48^\circ + \dots \cot 131^\circ - \cot 132^\circ + \cot 133^\circ - \cot 134^\circ =$

~~$\cot 45^\circ + \cot 47^\circ +$~~

$\cot 45^\circ - (\cot 46^\circ + \cot 134^\circ) + (\cot 47^\circ + \cot 133^\circ) \dots$

$\cos(\pi - x) = -\cos x$

$\sin(\pi - x) = \sin x$

$\cot(\pi - x) = -\cot(x)^\circ$

$0 - \cot 90^\circ =$

$= 1$

(69)

$\tan a = x$; $\tan\beta = y$ $\Longleftrightarrow$ $a=\arctan x$; $\beta=\arctan y$

(א) $\sum_{n=1}^{\infty} \arctan\frac{2}{n^2}$ | $\tan(a-\beta)=\frac{\tan a-\tan\beta}{1+\tan a\tan\beta}$

$\arctan\frac{x-y}{1+xy} = \arctan x - \arctan y$

[illegible] טלסקופי

$x\ge$ (x ו-y [illegible])

$\frac{2}{n^2}$ [illegible]

$x-y=2$ | $x=n+1$
$1+xy=n^2$ | $y=n-1$

$\tan(\arctan A-\arctan y)=\frac{x-y}{1+xy}$

$\arctan A-\arctan B=\arctan\frac{x-y}{1+xy}$

$=\arctan(n+1)-\arctan(n-1)$

$$\sum_{n=1}^{\infty}\frac{2}{n^2}=\lim_{N\to\infty}\sum_{n=1}^{N}\arctan\frac{2}{n^2}=\lim_{N\to\infty}\sum\arctan(n+1)-\arctan(n-1)$$

$$=\lim_{N\to\infty}\cancel{\arctan(N+1)}\ \big(\arctan(2)-\arctan(0)\big)+$$
$$+\arctan(3)-\arctan 1+\ldots$$
$$+\arctan(4)-\arctan(2)\ \ldots$$

$n=$ [illegible]

$$=\lim_{N\to\infty}\big(\arctan(N+1)+\arctan N-\arctan 1-\arctan 0\big)$$
$$=\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{4}=\frac{3\pi}{4}$$

(ב) $\sum_{n=1}^{\infty}\arctan\frac{8n}{n^4-2n^2+5}$ $\longrightarrow$ [illegible] $\left(\frac{x-y}{1+xy}\right)$

$$\frac{8n}{n^4-2n^2+5}=\frac{8n}{4+(n^2-1)^2}=\frac{8n}{4+(n+1)^2(n-1)^2}=\frac{\left(\frac{n+1}{\sqrt2}\right)^2-\left(\frac{n-1}{\sqrt2}\right)^2}{1+\left(\frac{n+1}{\sqrt2}\right)^2\left(\frac{n-1}{\sqrt2}\right)^2}$$

$(n^4=t^2)$ [illegible]

x y [illegible]

[illegible]

xy [illegible] | $\frac{x-y}{1+xy}$

[illegible]

$8n=\ldots 2\big((n+1)^2-(n-1)^2\big)$ [illegible]

$:4-8n$

[illegible]

$x=\left(\frac{n+1}{\sqrt2}\right)^2$, $y=\left(\frac{n-1}{\sqrt2}\right)^2$

$$=\lim_{N\to\infty}\left(\sum\arctan\left(\frac{n+1}{\sqrt2}\right)^2-\arctan\left(\frac{n-1}{\sqrt2}\right)^2\right)$$

[illegible] $N=4$

$$=\lim_{N\to\infty}\left(\arctan\left(\frac{N+1}{\sqrt2}\right)^2+\arctan\left(\frac{N}{\sqrt2}\right)^2-\arctan 0-\arctan\frac12\right)$$

$$=\pi-\arctan\frac12$$

742 $f, g : \{0, 1, 2 \dots\} \times \{0, 1, 2 \dots\} \to \{0, 1, 2 \dots\}$

$n = f(a,b) - g(a,b)$

n is composite odd positive numbers

each number can be written as

$(2a+3)(2b+3)$

a, b non-negative integers

is $a, b \geq 0$ integer or integers

$\Rightarrow$ n is composite

$(2a+3)(2b+3) = (a+b+3)^2 - (a-b)^2$

write as diff of 2 squares (integers)

$f(a,b) = (a+b+3)^2$

$g(a,b) = (a-b)^2$

n is composite

non-negative a, b such that $n = f(a,b) - g(a,b)$

$p = 3k+1$ [illegible]

$\frac{m}{n}$ [illegible] $m = n$ [illegible]

m, n coprime (p) [illegible]

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \dots \frac{1}{(2k-1)2k} = \frac{m}{n}$$

Prove p divides m

$$\frac{m}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{2k-1} - \frac{1}{2k}$$

[illegible]

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{2k} - 2\left(\frac{1}{2} + \frac{1}{4} \dots \frac{1}{2k}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} \dots \frac{1}{2k} - \left(1 + \frac{1}{2} + \dots \frac{1}{k}\right)$$

$\frac{1}{k}$ [illegible] $\frac{1}{2k}$ [illegible]

$$= \frac{1}{k+1} + \frac{1}{k+2} + \dots \frac{1}{2k-1} + \frac{1}{2k} =$$

$$= \left(\frac{1}{k+1} + \frac{1}{2k}\right) + \left(\frac{1}{k+2} + \frac{1}{2k-1}\right) + \dots$$

$$= \frac{3k+1}{(k+1)2k} + \frac{3k+1}{(k+2)(2k-1)} + \dots \frac{}{(k+1)(k+2)\cdot 2k\cdot(2k-1)}$$

[illegible] k [illegible] $2k$ [illegible]

$$\frac{m}{n} = \frac{(3k+1)\cdot q}{(2k!)(k+1)}$$

[illegible] q.

$m(2k)! = n(3k+1)q$

[illegible] $p = 3k+1$ divides

$p = 3k+1$

[illegible] p [illegible] $2k, 2k-1 \dots$ $\frac{1}{2k}, \frac{1}{2k-1} \dots$

$$x^{x+y} = y^{y-x}$$

[illegible] x, y [illegible]

$\cancel{x^{x+y}} = \cancel{y^{y-x}}$ $= x(\dots)$

[illegible] prime [illegible]

$$(x)^{x+y} = \left(\prod_{i=1}^{k} p_i^{a_i}\right)^{x+y} \qquad y = \prod_{i=1}^{k} p_i^{b_i}$$

$$y^{y-x} = \left(\prod_{i=1}^{k} p_i^{b_i}\right)^{b(y-x)}$$

x [illegible] y [illegible] $\prod p_i$ - prime factorization [illegible] prime [illegible] $\leq y$ [illegible] prime [illegible]

$$\prod_{i=1}^{k} p_i^{a_i(x+y)} = \prod_{i=1}^{k} p_i^{b_i(x+y-x)}$$

$$1 \Rightarrow a_i(x+y) = b_i(y-x) \qquad a_i < b_i \quad 1 \Rightarrow$$

x divides y

$x = \prod p_i^{a_i}$ [illegible] $\prod p_i^{b_i} = y$ $a_i < b_i$

[illegible] y [illegible] prime [illegible] $1^2 \cdot 3^2 \cdot 5^2$ [illegible] $1^3\cdot 3^3 \cdot 5^3$

$y = z \cdot x$

$$x^{x(z+1)} = (xz)^{x(z-1)}$$

$$x^2 = z^{z-1}$$

$$y^2 = (xz)^2 = z^{z+1}$$

x^2 [illegible] y^2 [illegible]

$$x^2 \cdot z^2 = z^{z+1}$$

$$(x\cdot z)^{x+1} = z$$

z^{z+1} [illegible] z^{z-1} perfect square [illegible]

[illegible] z [illegible] perfect square [illegible]

z [illegible] !

$z = t^2$; $x = t^{t^2-1}$ $y = t^{t^2+1}$ (1) solution (2)

$z - 1 = 2s$

$x = (2s+1)^s$ [illegible] $y = (2s+1)^{s+1}$

(130) $A - B = n$.

If n is even. ~~$A = 2n$, $B = n$~~

~~$A - B = 2n - n = n$~~

$A - B = 2n - n = n$

If n is odd.

~~Let p be smallest odd prime that doesn't divide n~~

$A = p \cdot n$ & $B = (p-1) \cdot n$

$\Rightarrow p \cdot n - (p-1) \cdot n = n$

smallest odd prime that doesn't divide n,

1) Prime A ... primes p ... primes. $(p-1)$.

2) odd B ... ~~primes p~~

$(p-1) - 1$.

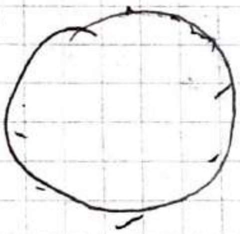
even ... $B - 1$ ... prime

... +2 ... prime

$\Rightarrow$ $B-1$ ... prime, ... $n-1$.

... $pn - 1$.

(131)



$a_0, a_1, a_2 \ldots a_{1995}$ integers.

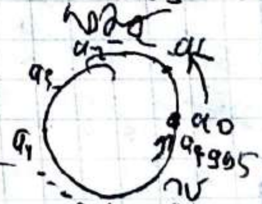
ratio $\frac{a_{k-1}}{a_k}$ is prime (1)

or $\frac{1}{\text{prime}}$ (2)

(1) ... m ... (prime-1)

(2) $1995 - m$ ... $\left(\frac{1}{\text{primes}}\right)$

$a_0 = a_{1995}$



$$\underbrace{\frac{a_0}{a_1} \cdot \frac{a_1}{a_2} \cdot \frac{a_2}{a_3} \cdots \frac{a_{1994}}{a_{1995}}} = \frac{a_0}{a_{1995}} = 1$$

$\Rightarrow$

$m \cdot \text{primes} \cdot (1995 - m) \cdot \frac{1}{\text{primes}} = 1$

... prime ...

$1 = m = 1995 - m$

$\varnothing$.

$$\frac{p_1 \cdot p_2 \cdots p_m}{q_1 \cdot q_2 \cdots q_{1995-m}} = 1$$

$p_1 \cdot p_2 \cdots = q_1 \cdot q_2 \cdots$

m ... prime

prime -1 ...

$(1995 - m)$ ...

... prime.

(439) $N = k \cdot p$ [illegible] !

10-ით

$p = 2$ ან $p = 5$-ის [illegible].

[illegible] 10-ის

$0,1,2,3, \dots 9$

10 [illegible]

$p = 2$ [illegible] $2 \cdot k = N$, N [illegible] 2-ის [illegible]

$~~2 \cdot 1023456789~~ =$

$~~= 2046810~~$

[illegible] 10 [illegible].

[illegible] $N = 10234567\cancel{9}8$ [illegible]

$p = 2$ [illegible].

$p = \underline{5\text{-ის}}$ [illegible]

$$N = 1023456785$$

If $p \neq 2$ და $p \neq 5$, ის p is relatively prime to 10

10-ის [illegible] 10-ს [illegible].

Fermat theorem $10^{p-1} \equiv 1 \pmod p$ ($p \neq 10$-ის)

$10^{k(p-1)} \equiv 1 \pmod p$ [illegible] p-ის [illegible]

let $\boxed{a \equiv n \pmod p}$, where $0 \leq n \leq p-1$ [illegible] $p \geq 3$-ის

a be number 10 digit [illegible] p-ის [illegible] n-ის [illegible] $k = 6(p-1)$ [illegible]?

define $N_a = 10^{(p-n+5)(p-1)} + 10^{(p-n+4)(p-1)} \dots + 10^{6(p-1)} + a$

[illegible] 10-ის [illegible] $10^{6(p-1)}, 10^{7(p-1)} \dots 10^{(p-n+5)(p-1)}$

~~form arithmetic progression.~~

[illegible] $(p-n+5) - 6 + 1 = p - n$ [illegible]

[illegible] n [illegible].

each term is congruent

$10^{k(p-1)} \equiv 1 \bmod (p)$ / [illegible] $a \equiv n \pmod p$ [illegible]

$\Rightarrow N_a = \underbrace{1 + 1 + 1 \dots + n}_{p-n\text{-ჯერ}} \equiv 0 \pmod p$

$\Rightarrow$

$10^{10} k p + N_a$

[illegible] LoL

a [illegible] 10 digit [illegible].

10^{10} [illegible]

$1 + 10000000000$ [illegible] $+ N_a$ [illegible] N_a-ს [illegible] 10 [illegible].

[illegible] for $p \geq 3$ $10^{6(p-1)} > 10^{10}$

[illegible] N [illegible] $p \neq 5$ და $p \neq 2$ [illegible] $p \nmid 10$-ის

Fermat-ის 10-ის [illegible] (p-ს) [illegible]. a [illegible] n [illegible] p-ის [illegible]

$10^{6(p-1)}$ $a = 1023456789$ [illegible] 100000000 [illegible]

$p \geq 3$ [illegible] 10 [illegible]

(740) $A = a^2 + 2b^2 \quad a \neq 0$

A set. If $p^2 \in A$, p prime then $p \in A$.

$p = 2$ easy. $2^2 = 4$

$a = 0$

$b =$

p is odd

$p^2 = a^2 + 2b^2$; $\underbrace{2b^2}_{\text{even}} = (p-a)(p+a)$

p odd. odd - odd is even

a = odd

$\gcd(p-a, p+a) = 2$

p-a, p+a even

Any common divisor d ... what?

$(p+a) - (p-a) = 2a$

a is odd

so only even divisor they share is 2.

Any common divisor d must divide their difference $(p+a) - (p-a) = 2a$ ... d divides it; a is odd

$p - a = 2X$
$p + a = 2Y$ | 2 even

$2b^2 = 2X \cdot 2Y$

$b^2 = X \cdot 2Y$

$\gcd(p-a, p+a) = \gcd(2X, 2Y) = 2$

$\Rightarrow \gcd(X, Y) = 1$

co-prime X, Y

$X = n^2$ so $Y = 2m^2$

$X = 2n^2 \quad Y = m^2$

???

(741) $n > 1$ and $1 = d_1 < d_2 < \dots < d_k = n$

$s = d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$. Prove $s < n^2$

$s = \sum_{i=1}^{k-1} d_i d_{i+1} = n^2 \sum_{i=1}^{k-1} \frac{1}{d_i d_{i+1}} \leq n^2 \sum \left(\frac{1}{d_i} - \frac{1}{d_{i+1}}\right) < n^2 \cdot \frac{1}{d} = n$

for pos. number

If d is divisor of n, so is $\frac{n}{d}$ (d_i, d_{i+1})

d_i, d_{i+1}

5 : 1, 15, ... $\frac{15}{5} = 3$

$\frac{15}{3} = 5$

$d \Rightarrow \boxed{t = \frac{n}{d}}$

(1) ???!

$\left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] \to \underbrace{1 \cdot 2 \cdot 3 \cdots m}_{-ში}$ ამდენი p-ს ფაქტორი

(ა) $m > 1$

$\underbrace{1 \cdot 2 \cdots m}_{m \mid r}$

$a \quad a+r \cdots$

$p : n!$

$a(a+r)(a+2r) \cdots (a+(n-1)r)$

$\gcd(r, n) = 1$

744 n, $n!$ ends with 1000 zeros.

$1 \cdot 2 \cdot 3 \cdots n$-ში

$~~\left[\frac{n}{5}\right] + \frac{n}{5}~~$

1000 [illegible]

2-ის 5-ზე, [illegible] [illegible]

[illegible]

$\left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \cdots = 1000$

[illegible]

n

$< \frac{n}{5} + \frac{n}{5^2} + \frac{n}{5^3} + \cdots = \frac{n}{5} \cdot \frac{1}{1 - \frac{1}{5}} = \frac{n}{4} > 1000$

$n > 4000$

use inequality $[a] > a - 1$ (2) [illegible] (1)

n [illegible] $\left(\frac{n}{5}\right)$

$1000 > \left(\frac{n}{5} - 1\right) + \left(\frac{n}{5^2} - 1\right) + \left(\frac{n}{5^3} - 1\right) + \left(\frac{n}{5^4} - 1\right) + \left(\frac{n}{5^5} - 1\right)$

$= \frac{n}{5}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}\right) - 5$

$= \frac{n}{5} \cdot \frac{1 - \left(\frac{1}{5}\right)^5}{1 - \frac{1}{5}} - 5$

n [illegible] > 4000-ზე

(1) [illegible]

$\left[\frac{n}{5}\right] + \left[\frac{n}{5^2}\right] + \left[\frac{n}{5^3}\right] + \left[\frac{n}{5^4}\right] + \left[\frac{n}{5^5}\right]$

$\left[\frac{n}{x}\right] = 0$, [illegible]

$x > 4000$-ზე

$5^1, 5^2, 5^3$,

$5^4 = 3125$

$\boxed{5^5 = ~~[illegible]~~}$

$\boxed{5^6 = 15625 > 4000}$

$1 \Rightarrow n < \frac{1005 \cdot 4 \cdot 3125}{3124} < 4022$

ე.ი. $n > 4000$-ზე და < 4022

$\underbrace{4001, 4002 \cdots 4021.}$

[illegible]

$\left[\frac{n}{5}\right] + \cdots$ [illegible]

[illegible]

$n = 4005, 4006$

$4007, 4008, 4009$

[illegible]

(746) $n!$ not divisible by 2^n — პოლინიაკის ფორმულა: ნებისმიერი prime p appears in prime factorization of $n!$

~~2^{n}~~

$$\rightarrow \quad = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \dots$$

2 არის prime და prime in $n!$) => პოლინიაკის ფორმულა

$$\underbrace{\frac{n}{2} + \frac{n}{2^2} + \frac{n}{2^3} + \dots}_{\text{notice}} = n \Rightarrow \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \dots = n$$

[illegible] integer ([illegible] $\lfloor\ \rfloor$ -ის whole number / integer-ით [illegible]).

|=> [illegible] -whole- $\frac{n}{2} + \frac{n}{2^2} + \dots$ [illegible]

[illegible].

-whole [illegible] $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \dots < n$

=> total number of factors in $2^{n!}$ is $<$ than n. and 2^n has $\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdots}_{n}$ times factors.

=> [illegible] factor [illegible] $n!$-ს

~~(748)~~

(746)

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\left(1, 2 \dots \left\lfloor \frac{n}{i} \right\rfloor\right)$$

p prime number. power of p in $\operatorname{lcm}\left(1, 2 \dots \left\lfloor \frac{n}{i} \right\rfloor\right) = k$ if and only if

$$\left\lfloor \frac{n}{p^{k+1}} \right\rfloor < i \le \left\lfloor \frac{n}{p^k} \right\rfloor$$

$\underbrace{\operatorname{lcm}(1, 2 \dots n)}$

lcm არის least common multiple [illegible]. [illegible] lcm -ის prime factorization

$$p^k \le n$$

=> exponent of p in $\operatorname{lcm}(1, 2 \dots n)$ is k only if $\underline{p^k \le n < p^{k+1}}$

[illegible] $\left\lfloor \frac{n}{i} \right\rfloor$

$$p^k \le \left\lfloor \frac{n}{i} \right\rfloor < p^{k+1}$$

$$\left\lfloor \frac{n}{p^{k+1}} \right\rfloor < i \le \left\lfloor \frac{n}{p^k} \right\rfloor$$

=> number of i for which factor $\operatorname{lcm}\left(1, 2 \dots \left\lfloor \frac{n}{i} \right\rfloor\right)$ [illegible] p^k is $\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor$ => for each prime p how many times p appears in prime factorization of $\operatorname{lcm}\left(1, 2 \dots \left\lfloor \frac{n}{i} \right\rfloor\right)$ [illegible]

$\prod \operatorname{lcm}\left(1, 2 \dots \left\lfloor \frac{n}{i} \right\rfloor\right)$ [illegible] [illegible] i-ს. => [illegible] $k \cdot \left(\left\lfloor \frac{n}{p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor \right)$

$$\sum_{k \ge 1} \left(\frac{n}{p^k} - \frac{n}{p^{k+1}} \right)$$

[illegible] k-[illegible] factors [illegible]

7.47 $\frac{\gcd(m,n)}{n}\binom{n}{m}$ ← gcd integer? ან integer? [illegible]

2 ვარიანტი

$\gcd(m,n)$ არის ან m ან n [illegible]

$\frac{\gcd(m,n)}{n}\binom{n}{m}$

1) ვთქვათ $\gcd(m,n)=m \iff \frac{m}{n}\binom{n}{m} = \binom{n-1}{m-1}$

2) $=n \iff \frac{n}{n}\binom{n}{m} = \binom{n}{m}$ integer არის [illegible]

1 ვარიანტი — $\frac{m}{n}\cdot\frac{n!}{m!(n-m)!}$

integer [illegible]

[illegible] $\iff$ [illegible] p [illegible]

$n \geq m$ — $\gcd(n,m)$ → $\alpha \geq \beta \geq 0$ [illegible]

ვთქვათ p არის power α in n. (p^α [illegible] p-[illegible]) $p^\alpha \cdot n_0$

power β in m. $\alpha \geq \beta \geq 0$

idea. $\gcd = n$

$\lfloor x\rfloor + \lfloor y\rfloor \leq \lfloor x+y\rfloor$

$\left\lfloor \frac{n}{p^k}\right\rfloor \geq \left\lfloor\frac{m}{p^k}\right\rfloor + \left\lfloor\frac{n-m}{p^k}\right\rfloor$ | [illegible] $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

$x = \frac{m}{p^k}$ $y = \frac{n-m}{p^k}$

$x+y = \frac{n}{p^k}$

$\gcd(m,n)$ [illegible] p^β-c

[illegible] $(v_p(\gcd(m,n)) = \beta$

$(1, 2 \ldots n)$ [illegible] how many divisible by p

use this inequality

$\left\lfloor\frac{n}{p}\right\rfloor$ counts how many divisible by p

$\left\lfloor\frac{n}{p^2}\right\rfloor$ counts by p^2 ...

$v_p(n!) = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \ldots$ (1)

$v_p(m!) = \left\lfloor\frac{m}{p}\right\rfloor + \left\lfloor\frac{m}{p^2}\right\rfloor +$ (2)

$v_p(n-m)! = \left\lfloor\frac{n-m}{p}\right\rfloor + \left\lfloor\frac{n-m}{p^2}\right\rfloor +$ (3)

~~(3) ≥ (1) + (2)~~ (1) − (2) − (3) $= v_p\binom{n}{m}$ [illegible]

$\sum\left\lfloor\frac{n}{p^k}\right\rfloor - \left\lfloor\frac{m}{p^k}\right\rfloor - \left\lfloor\frac{n-m}{p^k}\right\rfloor = v_p\binom{n}{m}$

$\alpha \geq k > \beta$

power of p is at least $\alpha - \beta$

[illegible] $v_p = \alpha$ ~~β~~ [illegible]

$\beta - \alpha + \alpha - \beta = 0$

so $p_{(\text{pow})}$ has non-negative ~~integer~~. $\Rightarrow$ power $\Rightarrow$ integer ordering [illegible] $2^{29} \equiv$ sum of digits $\bmod(9)$

① modulo 9 is congruent to sum of its digits

2^{29} has 9 distinct digits (0,1,2 ...) [illegible]

~~of ① [illegible] 0+1+2 ... +9 = 45 which : 9 - [illegible]~~

$0+1+2 \cdots 9 = 45$
which $\vdots$ 9 - [illegible].

~~$2^{29} = 2^2(-1)^9 = -4 \pmod 9$~~

~~$2^{29} = 2^2 \cdot (2^{27}) = 2 \cdot 2^{18} \cdot 2^9$~~

①

~~$\sum t_n \cdot 10^n \equiv \sum t_n \bmod 9$~~ [illegible]

$2^{29} = 2^{27} \cdot 4 = 8^9 \cdot 4 = ~~(9-1)^9~~ \cdot 4$

$2^{29} = 8n$

$8 \equiv -1 \bmod(9)$

$8^9 \equiv (-1)^9 \bmod(9)$

$\Rightarrow 2^{27} \equiv -1 \bmod 9$

$= -1 \cdot 4 = \underline{-4 \bmod 9}$ [illegible] 2^{29} [illegible] (-4 [illegible])

from multiple of 9.
$\Rightarrow 9-4$

$-4 \bmod 9$ [illegible]

$-4 \equiv 5 \bmod(9)$ (0 ; 8 - [illegible])

$9 - 5 = 4$ [illegible].

$\frac{r}{ps} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{p}$ harmonic series

p^3 divides $r - s$

sum of first p terms

$\frac{r}{p \cdot s} = \frac{p!}{1} + \frac{p!}{2} + \cdots \frac{p!}{p}$ over $p!$

[illegible] p - [illegible] common prime divisor $< p$ - [illegible]

$s = (p-1)!$ [illegible] r = [illegible]

$r - s = p\left(\frac{(p-1)!}{1} + \frac{(p-1)!}{2} + \cdots \frac{(p-1)!}{p-1}\right)$

$s = (p-1)!$

[illegible]

$p!$ ~~[illegible]~~ [illegible] p [illegible]

[illegible] p^3 - [illegible]

p [illegible]

p^2 - [illegible]

[illegible] Σ [illegible]

~~$\frac{(p-1)!}{1} + \frac{(p-1)!}{2} = \frac{2(p-1)! + (p-1)!}{2} = \frac{(p-1)!(2+1)}{2}$~~

~~$\frac{(p-1)!}{1} + \frac{(p-1)!}{2} + \frac{(p-1)!}{3} = \frac{(p-1)!(6+3+2)}{6}$~~

If k^{-1} = inverse of k modulo P, $\left(k \cdot k^{-1} \equiv 1 \bmod P\right)$

$\boxed{\text{mod } P\text{-ში}}$ then $P - k^{-1}$ is inverse of $p - k$ mod P

რადგან ეს p-ზე იყოფა [illegible] $\equiv 0$ [illegible]

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{(p-1)!}{k(p-k)} \equiv (p-1)! \sum_{k=1}^{\frac{p-1}{2}} \left[k(p-1)\right]^{-1} \equiv$$

Σ-ს [illegible]

$$\equiv (p-1)! \sum_{k=1}^{\frac{p-1}{2}} \underbrace{k(p-k)}_{kp-k^2} \equiv -(p-1)! \sum k^2 = (-p-1)\frac{\frac{p-1}{2} \cdot \frac{p+1}{2} \cdot p}{6} =$$

$$\equiv \boxed{0 \bmod P}$$

(749) a, b, c

$a^3b - ab^3$ is multiple of 10.

$\underbrace{a^3b - ab^3}_{} = ab(a^2 - b^2) = ab(a-b)(a+b) = E$

[illegible] factor [illegible]

a, b are both odd $\Rightarrow a + b =$ even

$\Rightarrow$ ~~$ab(a-$~~ $E : 2$-ზე [illegible].

[illegible] $: 5$-ზე [illegible]

if ~~$a-b$~~ ab, $a-b$ or $a+b$ is multiple of 5 ✓

$\Rightarrow a$ or $b \equiv 0 \bmod 5$

If not

mod 5-ის [illegible] $(1; 2; 3; 4)$ [illegible].

a, b

[illegible]

$\{1; 4\}$ [illegible] 1

$\{2; 3\}$ [illegible] 2

$ab(a-b)(a+b)$

3 [illegible]

2 [illegible]

[illegible] 2 [illegible] [illegible]-ში [illegible]

mod 9-ის [illegible]

$(2+0+0+2)$

(mod 9-ის [illegible])

(mod 9-ზე [illegible])

$2002 \equiv 4 \bmod 9$

$(2002)^{2002} \equiv 4^{2002} \bmod 9$

(750) 2002^{2002}

$\underbrace{2002 = 10^3 + 10^3 + 1^3 + 1^3}_{}$

[illegible]

$2002^{2002} = 2002^{2001} \cdot 2002 =$

[illegible]

$= \left((2002)^{667}\right)^3 (10^3 + 10^3 + 1^3 + 1^3)$

$= \left(10 \cdot 2002^{667}\right)^3 + \left(10 \cdot 2002^{667}\right)^3 + \left(2002^{667}\right)^3 + \left(2002^{667}\right)^3$

[illegible] n^3-ის [illegible] remainder : 9-ზე

$= -1; 0; 1$

[illegible] $(0; 1 \dots 9)$ integer [illegible]

10 [illegible] $(8)^k \equiv (-1)^k \bmod 9$ [illegible]

3 [illegible] n^3-ის [illegible]

mod 9-[illegible] $(-3; -2; -1; 0; 1; 2; 3)$

[illegible]

$2002^{2002} \equiv 4^{2002} \equiv (4^3)^{667} \equiv 1 \cdot 4 \equiv 4 \pmod 9$

(#51) k^2 ~~[illegible]~~ ბოლო 4 position-ზე digit ერთი a

~~$k^2 = a \cdot 1111 \pmod{10\,000}$~~

ეგ ~~$aaaa$~~

$\underbrace{aaaa}_{\text{ამ 4 ციფრი}} = a \cdot 1000 + a \cdot 100 + a \cdot 10 + a \cdot 1$
$= a \cdot (1000 + 100 + 10 + 1) = a \cdot 1111$

$k^2 \equiv a \times 1111 \pmod{10\,000}$ last 4 digit [illegible]

ეს [illegible] ბოლო 4 digit (k^2-ის) არის $a \cdot 1111$ [illegible] k^2-ის.

Perfect squares end in 0, 1, 4, 5, 6, 9

$a \cdot 1111$-ზე [illegible] perfect square [illegible] a [illegible], არის, ~~mod 8~~ [illegible] mod 8, mod 16-ზე [illegible]

If $a = 0$. we're done

~~[illegible]~~ If $a = 1$ ან 5, ან 9

$k^2 = a \cdot 1111.$

~~k^2~~

mod 8-ის [illegible]

Perfect square [illegible] mod 8-ზე [illegible] 0, 1, 4

$1111 \equiv 7 \bmod 8$ ([illegible]

$a \cdot 1111 \equiv a \cdot 7 \bmod 8$
$\equiv 1 \cdot 7 \bmod 8$
7 არ არის (0, 1, 4-ში)

$(5 \cdot 7) \bmod 8 = 35 \bmod 8 =$
$= 8 \cdot 4 + 3 = 3 \bmod 8$
3 არ [illegible]

[illegible] perfect-ის [illegible] mod 8-ზე 0, 1, 4 [illegible] mod-ზე [illegible]

$1111 = 8 \cdot 138 + 7$
$1111 \equiv 7 \bmod 8$

$a = 2$ და $a = 4$

mod 16-ის [illegible] 8-ზე

Perfect square-ის [illegible]

0, 1, 4, 9.

$9 = 9 - 8, 0$ [illegible].

$1111 = 8 \cdot 138 + 7$ [illegible] $=$ [illegible]
mod 8
$= 16 \cdot 69 + 7$
[illegible] mod 16

$1111 = 16 \cdot 69 + 7$ [illegible]

$1111 \equiv 7 \pmod{16}$ [illegible]

$2 \cdot 7 \equiv 14 \bmod 16$ (14 არ [illegible] 0, 1, 4, 9-ში)

(#52)

$2^x \cdot 3^y = 1 + 5^z$

mod 4-ის [illegible]

$2^x \cdot 3^y \equiv ~~1~~(1 + 5^z) \bmod 4 = (1 + 1) \bmod 4$

[illegible] $5 \bmod 4 = 1 \bmod 4$ [illegible]
$(5)^z \equiv (1)^z \bmod 4$ [illegible] 4-ზე
$1 \cdot 5 \cdot 1 \bmod 4 = -3 \bmod 4$

1) $2^x \cdot 3^y \equiv 2 \bmod 4$
$\Rightarrow$ [illegible] :4-ზე $2^x \cdot 3^y$ [illegible] :4-ზე $- 1 \Rightarrow 2^x$-ი [illegible] :4-ზე $x = 1$.

2) If y [illegible] $y \geq 2$-ზე $3^2 = 9$. mod 9-ის [illegible]

$0 \equiv 1 + 5^z \bmod 9 \iff 5^z \equiv 1 \bmod 9$ [illegible] 5-ის [illegible] 1-?

[illegible] $y \geq 2$-ზე $2^x 3^y \equiv 0 \bmod 9$ [illegible] mod 6-ზე 5^6 [illegible] z has to divisible by 6. $(2 \cdot 3)^{x+y}$ [illegible] $1 + (-1)^z = 2$