159 $\quad a_1 = 2 \quad a_2 = 5$

ბევრი შემხვდა n-ზე n^2, n^3 ...

mod [illegible] ... [illegible] ... n-ზე [illegible] ... n^2, n^3 [illegible] modulo-ს [illegible] mod-ით [illegible].

$$a_{n+1} = (2-n^2)a_n + (2+n^2)a_{n-1}$$

p, q, r for which $\quad a_p \cdot a_q = a_r$.

$a_1 = 2$ | mod [illegible] (ინდუქციით a_k-ს [illegible])

$a_2 = 5$

$a_1 = 2 \equiv 2 \bmod 3$ | a_{n-1} არის

$a_2 = 5 \equiv 2 \bmod 3$ | a_n $2 \bmod 3$

a_{n+1}-ის [illegible]:

$$a_{n+1} = (2-n^2)a_n + (2+n^2)a_{n-1}$$

n^2-ი არის 0 ან 1 mod 3-ში

1) ~~n^2 ...~~ if $n^2 \equiv 0 \bmod 3$

$(2-n^2) \equiv 2 \bmod 3$

$2+n^2 \equiv 2 \bmod 3$

~~$a_{n+1} \equiv 2 \bmod 3 \cdot (2+n^2) a_{n-1}$~~

$$a_{n+1} \equiv 2\cdot 2 + 2\cdot 2 \equiv 4+4 \equiv 1+1 \equiv 2 \bmod 3$$

if $n^2 \equiv 1 \bmod 3$

$(2-n^2) \equiv 1 \bmod 3$

$(2+n^2) \equiv 0 \bmod 3$

$$a_{n+1} \equiv 1\cdot a_n + 0\cdot a_{n-1} \equiv 1\cdot 2 + 0\cdot 2 \equiv 2 \bmod 3$$

ანუ ვაჩვენეთ $a_{n+1} \equiv 2 \bmod 3$. ე.ი. ყველა a_k არის $2 \bmod 3$.

$$\underbrace{a_r}_{\text{უნდა } 2 \bmod 3 \text{ ყოფილიყო}} = \underbrace{a_p \cdot a_q}_{\text{არის } 2 \bmod 3} \equiv 2\cdot 2 \bmod 3 = 4 \equiv 1 \bmod 3$$

არის ინდუქციით ყველა წევრი $2 \bmod 3$ უნდა ყოფილიყო.

159 $\quad a_n + \ldots \quad k^{th}$ power.

~~755~~ $n > \cancel{1000}$ $1, 2 \cdots n$

2^n ~~is modulo~~

$\cancel{S_n > \frac{n}{2}(\log_2 n - 4)}$

~~a~~ ~~sum~~ of 2^n mod odd number > 1 is non-zero.

758 n at least 12

n-gon θ eq

$\epsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ | $r = 0, 1, \ldots$ ϵ the n^{th} root of unity.

equiangular polygon (given θ and side steps)

should have vertices coordinates $\sum_{i=0}^{k} \underbrace{\sigma(i)}_{\text{length}} \epsilon^i$ direction

sides are $\parallel$ to rays

σ is permutation of $1, 2, 3 \cdots n$

man vertices $\frac{2\pi}{n}$

for two sides

$\sum_{i=0}^{n-1} \sigma(i)\epsilon^i = 0$ (closed polygon $\Sigma 0$)

$n = ab$ i.e. a and b coprime

all n^{th} root of unity are $\epsilon^{aj + bk}$

for $j = 0, 1 \ldots b-1$, $k = 0, 1 \ldots a-1$

If $aj + bk \equiv aj' + bk' \pmod n$

$a(j - j') \equiv b(k' - k) \pmod n$

$(\Rightarrow)$ $j - j'$ divisible by b.

$k - k'$ divisible by a.

$(\Rightarrow)$ $j = j'$ and $k = k'$ $(\Rightarrow)$ we listed all n^{th} roots of unity

$n^p - n \equiv 0 \bmod p$ פרמה $n^p \equiv a \pmod p$

$n^{p-1} \equiv 1 \bmod p$; אם n co-prime

① $2^n + 3^n + 6^n - 1$ divisible by every p prime

$n = 1, n = 2$ נבדוק

$n = 2$ נבדוק $= 48$

$\Rightarrow p = 2, p = 3$ נכונים

$2^n + 3^n + 6^n - 1$

$2^2 + 3^2 + 6^2 - 1$

$2 = p-1$, $\boxed{p = 3}$

$2^{p-1} + 3^{p-1} + 6^{p-1}$

$(p-1 = 2)$ ✓

$p = 3$

now for $p > 3$

2^{p-1}, 3^{p-1} and 6^{p-1} are ... for prime > 3: $2, 3, 6$ aren't division by p.

$p = 2, 3$

$p > 3$

$2^{p-1} \equiv 1 \bmod p$

$3^{p-1} \equiv 1 \bmod p$

$6^{p-1} \equiv 1 \bmod p$

$3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 3 + 2 + 1 = 6 \bmod p$ when (6 ...)

$\Rightarrow 6 \cdot 2^{p-2} + 6 \cdot 3^{p-2} + 6 \cdot 6^{p-2} \equiv 6 \pmod p$

$\Rightarrow 2^{p-2} + 3^{p-2} + 6^{p-2} \equiv 1 \bmod p$

-1 (this expression) $\equiv 0 \bmod p$

② $x^6 + x^3 + x^3 y + y = 147^{157}$

$+\ x^3 + x^3 y + y^2 + y + z^9 = 157^{147}$

$(x^3 + y + 1)^2 + z^9 = 147^{157} + 157^{147} + 1$

$2 \cdot 9 + 1 = 19$

$a^{19-1} \equiv 1 \pmod{19}$ a^{18} נבדוק

$a^{18} \equiv 1 \pmod{19}$, a isn't multiple of 19

$\frac{18}{\gcd(18,2)} = \frac{18}{2} = 9$, ... 9 ... divisors: $1, 3, 9$

$(a^9)^2 = a^{18} \equiv 1 \bmod 19$

$\Rightarrow (a^9)^2 \equiv 1$

(760) n has $p-1$ digits 1, $p \neq 2, 3, 5$, n is div. by p

[illegible] 1 1 1 1 1

$p-1$ [illegible]

$$n = 1 + 1\cdot 10 + 1\cdot 100 + \dots + 1\cdot 10^{p-2} = \frac{10^{p-1}-1}{10-1} = \frac{10^{p-1}-1}{9}$$

$10^{p-1} \equiv 1 \bmod p$ [illegible] $n = 2 \cdot 5$ [illegible] $p \neq 2, 5$ [illegible]

$10^{p-1} - 1 \equiv 0 \bmod p$

$$n = \frac{p \cdot k}{9} \checkmark$$

(761) $p > 17$ [illegible] $p^{32}-1$ divisible by 16320 [illegible]

$16320 = 2^6 \cdot 3 \cdot 5 \cdot 17$ factoring [illegible] $p^m - 1$ [illegible]

$p^{98} - 1 = (p^9)^8 - 1$ [illegible] $(p^9 - 1)$ [illegible]

$\Rightarrow p^{32} - 1$ is divisible by $p^2 - 1$, $p^4 - 1$, $p^{16} - 1$

fermat [illegible]

$p^2 - 1 = p^{3-1} - 1$, $p^{3-1} \equiv 1 \bmod 3$, [illegible] $p > 17$ [illegible] p [illegible] 3 [illegible] co-prime

$p^{3-1} - 1 \equiv 0 \bmod p$ fermat [illegible]

3 [illegible]

$p^4 - 1 \equiv 0 \bmod 5$ [illegible] $p^{16} - 1 \equiv 0 \bmod 17$

[illegible] $p^{32} - 1$ [illegible] $(p^2 - 1)$, $(p^4 - 1)$ [illegible] $(p^{16} - 1)$ [illegible]

[illegible] $\vdots 3$ [illegible] $\vdots 5$ [illegible] 17 [illegible]

$16320 = 2^6 \cdot 3 \cdot 5 \cdot 17$ [illegible]

2^6 [illegible]

$$p^{32} - 1 = (2n+1)^{32} - 1 = (2n)^{32} + \binom{32}{1}(2n)^{31} + \dots + \binom{32}{2} 2n^2 + \binom{31}{1} 2n$$

last 5 term [illegible] contain power of ≥ 6.

p [illegible] odd $= 2n+1$

[illegible] prime, [illegible] $= 2$

[illegible] $p > 17$ [illegible]

$$\binom{32}{5}(2n)^5 + \binom{32}{4}(2n)^4 + \binom{32}{3} 2n^3 + \binom{32}{2} 2n^2 + \binom{31}{1} 2n$$

$\frac{32}{27!\,5!}$ [illegible] $\vdots 2$ [illegible]

$\vdots 2$ [illegible] $\vdots 2^2$ [illegible] 2^3 [illegible] 2^5 [illegible] 2^4 [illegible]

$\Rightarrow$ sum is divisible by 2^6

(7.2) ~~odd~~ p odd prime

$x^2 \equiv (a \bmod p)$ has solution

$a^{\frac{p-1}{2}} \equiv 1 \bmod p$

Fermat there are no any primes of form $4n-1$

x^2 [illegible] $p-1$) $\Rightarrow$ x [illegible] $p-1$).

[illegible]

$x \equiv 0 \bmod p$

$x^2 \equiv 0^2 \bmod p$

[illegible]

$x^2 \equiv a \bmod p$

[illegible]

~~$x^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} \bmod p$~~

Fermat [illegible]

$\boxed{x^{p-1} \text{ [illegible]}}$ $\frac{p-1}{2}$ [illegible]

$(x^2)^{\frac{p-1}{2}} \equiv a^{\frac{p-1}{2}} \bmod p$

Note p = prime p, x doesn't divide by p.

$x^{p-1} \equiv a^{\frac{p-1}{2}} \cancel{\bmod p} \equiv 1 \bmod p$

$x^{p-1} \equiv 1 \cancel{\bmod p} \equiv a^{\frac{p-1}{2}}$

p [illegible] prime [illegible]

x not divided by p, [illegible]

$\Rightarrow 1 \equiv a^{\frac{p-1}{2}} \bmod p$

$1 \equiv a^{\frac{p-1}{2}} \bmod p$

[illegible]

if $a = -1$

$a = -1$ [illegible]

$1 \equiv (-1)^{\frac{p-1}{2}} \bmod p$

$\frac{p-1}{2}$ = even

[illegible] $(-1)^{\text{even}} = 1$ $\Rightarrow$

$\frac{p-1}{2} = 2m$, $p = 4m+1$

[illegible]

$\boxed{p = 4m+1}$

$x^2 = a \bmod p$

$x^2 = -1 \bmod p$

[illegible] finite [illegible]

[illegible]

$x = (2p_1 p_2 \cdots p_n)^2$

$x^2 + 1 \equiv 0 \bmod p$

~~$p_1 \cdot p_2 \cdots p_n$ number~~

~~$(2 \cdot p_1 p_2 \cdots p_n)^2 + 1$~~

[illegible] $\rightarrow$ $(2p_1 p_2 \cdots p_n)^2 + 1$ has only

odd prime factor p, [illegible]

~~Must be in [illegible] form~~

[illegible] $4m+1$ [illegible]

[illegible] $+1 =$ [illegible]

[illegible] p [illegible]

[illegible]

[illegible] $y^2 - 1 - c$

(7.3) $x^2 = y^3 + 7$ no int solution

if y = even $y^3 + 7 \equiv 3 \bmod 4$

[illegible] $y = 2, 4, 6 \ldots$ $2^3 + 7$ [illegible] $2^4 + 7$ [illegible] $\ldots$

$y^3 + 7 \bmod 4 \equiv 0 + 3 \bmod 4$

x^2 [illegible] x^2 [illegible] $3 \bmod 4$. [illegible] x^2 [illegible]

[illegible] y = odd, $y = 2k+1$ $4n \bmod 4$ [illegible]

$x^2 = (2k+1)^3 + 7$, $x^2 + 1 = y^3 + 2^3 = (y+2)((y-1)^2 + 3)$ $(y+2)(y^2 \ldots 3)$

$x^2 + 1$: $4m + 3$ [illegible] $(4k^2 + \ldots)$ [illegible]

number of form.

$\Rightarrow$ must be divisor by prime number of this form which [illegible] 7.2

(704) $n \geq 2$ $\quad$ $(x+1)^n - x^n = ny$ $\quad$ [illegible]

p be smallest prime divides n $\Rightarrow$ n and $p-1$ are prime.

$\Rightarrow$ $(x+1)^n - x^n$ is divisible by p, but

x and $x+1$ both can't [illegible] p [illegible]

relatively prime

$$(x+1)^{p-1} \equiv 1 \equiv x^{p-1} \mod p$$

$\Rightarrow$ [illegible]

$$(x+1)^n \equiv x^n \pmod p$$

p is smallest that divides n. $\Rightarrow$ n and $p-1$ are prime

[illegible]

$$\Rightarrow \gcd(n, p-1) = 1$$

$$(x+1)^n = \left((x+1)^1\right)^n \qquad a(p-1) + bn = 1 \text{ [illegible]}$$

[illegible]

$$\cancel{x+1 = (x+1)^{a(p-1)+bn}}$$

$$(x+1)^1 = (x+1)^{a(p-1)+bn} \equiv x^{a(p-1)+bn} \equiv x^1 \mod p$$

100%.

$$x+1 \equiv x \mod p$$

$$1 \equiv 0 \mod p$$

$\Rightarrow$ 1 divides p
which is impossible

(705) $2^n - 3$ sequence.
infinite subsequence terms are pairwise relat prim

[illegible] x_k [illegible] x_{k-1} [illegible] prime

$2^n - 3 = x_n$. subsequence $(x_n)_n$

[illegible] prime factors of $x_1, x_2, x_3 \dots x_{k-1}$ [illegible] $p_1, p_2 \dots p_m$

$x_1, x_2 \dots$ are odd $2^n - 3$ [illegible] $\Rightarrow$ [illegible] $p \neq 2$

define $x_k = 2^{(p_1-1)(p_2-1)\cdots(p_m-1)} - 3$

$p_1 - 1, p_2 - 1 \dots$ [illegible]

[illegible] p [illegible] 2 [illegible] prime

fermat [illegible]

$$2^{p_1-1} \equiv 1 \mod (p_1)$$

$$2^{p_1-1} - 1 \equiv 0 \mod (p_1)$$

[illegible] 2^{p_2-1} [illegible]

subsequence (x_{n_k})

[illegible] $\equiv 0 \mod$

$$2^{(p_1-1)(p_2-1)\cdots(p_m-1)} - 1 \text{ are divisible}$$

by $p_1, p_2, \dots, p_m$

$$x_k = 2^{(p_1-1)\cdots} - 2$$

$$\Rightarrow x_k \equiv 0 - 2 \equiv -2 \mod p_i, \text{ but, } p_i \text{ is odd prime}$$

$\Rightarrow$ x_k ~~[illegible]~~ is relatively prime to $x_1, x_2 \dots x_{k-1}$. so can be added to sequence.

x_k [illegible] sequence.
x_{k-1} [illegible] $x_1, x_2 \dots x_{k-1}$
[illegible]
[illegible] x_k [illegible]
$x_1, x_2 \dots x_{k-1}$ [illegible]
[illegible] co-prime [illegible]
[illegible]
[illegible]

(766) $(x_n)_n$

$$x_{n+1} = 5x_n - 6x_{n-1}$$

[illegible] composite

recurrence [illegible] linear, x_n, x_{n-1}, x_{n+1} [illegible] $= 1$.

[illegible] ~~[illegible]~~ we look for $x_n = r^n$ [illegible]

$$r^{n+1} = 5r^n - 6r^{n-1} \quad (: r^{n-1})$$

$$r^2 = 5r - 6$$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0, \quad r=2, \; r=3$$

~~$x_n = A$~~

$$x_n = A\,2^n + B\,3^n$$

[illegible] x_n ~~x~~. $x_{n+1} = A\cdot 2^{n+1} + B\cdot 3^{n+1}$

$$x_n = A\cdot 2^n + B\cdot 3^n$$

$n=0$; $\boxed{x_0 = A + B}$

$n=1$; $\boxed{x_1 = A\cdot 2 + B\cdot 3 = 2A + 3B}$

[illegible] $A = 3x_0 - x_1$, $B = x_1 - 2x_0$

contradiction

assume all terms are prime of sequence x_n

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} ~~A\cdot 2^n + B\cdot 3^n~~ = \lim_{n\to\infty} 3^n\left(\frac{2^n}{3^n}A + B\right) = \infty$$

$$x_n = A\cdot 2^n + B\cdot 3^n \quad \infty\text{-[illegible]}$$

$\lim x_n = \infty$ [illegible] x_n [illegible]

$n\to\infty$

2 [illegible] 3 [illegible]

$p-1$ [illegible] multiple [illegible]

$$x_{n+k(p-1)} = A\cdot 2^{n+k(p-1)} + B\cdot 3^{n+k(p-1)}$$

$$= A\cdot 2^n\cdot 2^{k(p-1)} + B\cdot 3^n\cdot 3^{k(p-1)}$$

k [illegible] mod $\equiv 1$

Fermat [illegible]

$$2^{p-1} \equiv 1 \bmod p \qquad 3^{p-1} \equiv 1 \bmod p$$

[illegible] 2 & p are co-prime

$$2^{k(p-1)} \equiv 1^k \bmod(p) \equiv 1 \bmod p$$

k [illegible] x_n $\lim_{n\to\infty}$ k-[illegible]

$$1 \Rightarrow x_{n+k(p-1)} \equiv A\cdot 2^n\cdot 1 + B\cdot 3^n\cdot 1 = \underbrace{A\cdot 2^n + B\cdot 3^n}_{[illegible]} \bmod p$$

$$1 \Rightarrow x_{n+k(p-1)} \equiv x_n \bmod p$$

~~x_n~~ $x_n = \text{prime}$ [illegible]

$$x_{n+k(p-1)} \equiv p \bmod p$$

$$x_{n+k(p-1)} \equiv 0 \bmod p$$

$1 \Rightarrow$ ~~[illegible] x_{n+p-1}; $x_{n+k(p-1)}$ is divisible by p~~

~~[illegible] $x_n = p$~~ [illegible] $x_{n+k(p-1)}$ is greater than x_n [illegible] $1 \Rightarrow x_{n+k(p-1)} > p$ [illegible] p-[illegible] composite

[illegible]

a^k+1 divisible by 12321 for some prime pwr.

$\underline{12321} = (111)^2 = \underline{3^2 \cdot 37^2}$ Prime power factoriz [illegible]

[illegible]

a^k+1 ~~also has factor 3~~

~~also has factor 3^2~~

$a^{k}+1 \equiv 0 \mod 12321$

$a^k \equiv -1 \mod 12321$

I

1=> a^k+1 has factor 3.

If a ~~divided by 3~~ isn't divided by 3 [illegible]

$a^{3-1} = a^2 \equiv 1 \mod (3)$

in mod 3 $a \equiv 1$ or $a = -1$

if k = even

$\boxed{\text{if } k = \text{even} \quad (-1)^k = (1)^k}$ $\Leftarrow$ $a^k = (a^2)^{\frac{k}{2}} \equiv 1^{\frac{k}{2}} \equiv 1 \mod 3$; V

if k is odd

$a^k = a\cdot(a^2)^{\frac{k-1}{2}} \equiv a\cdot 1^{\frac{k-1}{2}} \equiv a \mod 3$

$a^k \equiv a \mod 3$

mod 3 → $a \equiv 1$ or $a = -1$

[illegible] if k = odd [illegible] $a = -1$

[illegible] k is odd [illegible]

[illegible] a −1 or 1 [illegible]

[illegible] k = odd.

odd − 1 = even

$\frac{\text{even}}{2}$ [illegible] even.

$(a^2)^{\frac{k-1}{2}}$ a

[illegible] a [illegible] 3 [illegible]

$(a^2)^{\frac{k-1}{2}}$ [illegible]

1 [illegible] −1.

[illegible]

[illegible] a [illegible]

[illegible]

II

[illegible] mod 37 [illegible]

$a^{36} \equiv 1 \mod 37$

~~$a^k \equiv -1 \mod 37$~~

[illegible] $a^k \equiv -1 \mod 37$

($a^k \equiv -1 \mod 12321$ [illegible])

we wanna find d for which natural candidate $a^d \equiv -1 \mod (37)$ is $d = \gcd(k,36)$

because gcd shows largest shared factor in exponents k and 36

$\gcd(k,36) \to kx+36y = \gcd(k,36)$

$a^{\gcd(k,36)} = a^{kx+36y} =$

$= a^{kx}\cdot a^{36y} \equiv (-1)^x \cdot 1 \mod 37$

[illegible]

$a^k \equiv -1 \mod 37$

[illegible] $a^{36} \equiv 1 \mod 37$

x = odd, [illegible] −1 [illegible]

Q(3) For any prime p

$(p-1)!+1$ divisible by p

Q(4) $n!+1$ and $(n+1)!$

I. If $(n+1)$ is composite, then each prime divisor of $(n+1)! < n-1$

$(n+1)! = (n+1)(n)\cdots 1$ composite.

they also divide $n!$

but don't divide $n!+1$

$n+1$ composite

$(n!+1)$ and $(n+1)!$

gcd = 1 (co-prime)

II. If $(n+1)$ is prime then gcd can be

ETF

number n ; $\phi(n)$ = number coprime with n & which are $\leq n$.

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right)$$

$\phi(n) < n-1$

If m and n are prime

$\phi(mn) = \phi(m)\,\phi(n)$

n is prime if $\phi(n) = n-1$

and if $n = p_1 p_2 \cdots p_k$ distinct primes

then $\phi(n) = (p_1-1)(p_2-1)\cdots(p_n-1)$

E.T. a is coprime to n

$a^{\phi(n)} \equiv 1 \bmod n$

E(1) n-even ; n^2-1 divides $2^{n!}-1$

$n = m-1$, when m = odd. $n^2-1 = (n-1)(n+1) = (m-2)\,m$ divides $2^{n!}-1$

$m-2$ and m relatively prime, co-prime

$\phi(m) < m \Rightarrow \phi(m)$ divides $2^{(m-1)!}-1$

All prime factors of $\phi(m)$ is in $(m-2)!$

$(m-1)!$, $\Rightarrow 2^{\phi(m)}-1$ divides $2^{(m-1)!}-1$

if d divides n for any number a^d-1 divides a^n-1

$\phi(m) < m-1$

$\phi(m) \leq m-1-1$

E.T $a^{\phi(n)} \equiv 1 \bmod n$

$2^{\phi(m)}-1 \equiv 0 \bmod m$

$\phi(m)$ positive integer less than m $\Rightarrow$ prime factor will be in $(m-1)!$

m divides $2^{\phi(m)}-1$

m also divides $2^{(m-1)!}-1$.

$2^{\phi(m-2)}-1 \equiv 1 \bmod (m-2)$

m & $m-2$ are relatively prime $\Rightarrow m(m-2)$ divides $2^{(m-1)!}-1$

subset

Ex-2 $S = \{a^2+a-1;\ a^3+a^2-1;\ a^4+a^3-1;\ \dots\}$

If N is product of elements of subset, then since elements of S are coprime to a, also N must be coprime to N. $s_i \in S$ coprime to a

Product of coprimes also coprime

$N = s_1 \cdot s_2 \cdots s_n$ each s_i is coprime to a

$S = a^n + a^{n-1} - 1$

a divides both a^n and a^{n-1}

$S \equiv 0 + 0 - 1 \pmod a$

so a doesn't divide S.

$\Rightarrow$ coprime gcd $= 1$

E.T $a^{\phi(N)+1} + a^{\phi(N)} - 1 \equiv a + 1 - 1 \equiv a \pmod N$

$\Rightarrow a^{\phi(N)+1} + a^{\phi(N)} - 1$ are coprime to N, so can be added to S.

N also coprime to a

$a^{\phi(N)} \equiv 1 \pmod N$ Euler E.T. if N and a are co-prime.

Ex 5 $\sum_{\text{if } k \mid n} \phi(k) = n$

induction. $n = 1 - 2x > x$.

$n = p^a q$, p prime, q coprime to p

If k divides $n \Rightarrow$ k is in form of $p^j m$, where $m \mid q$

$$\sum_{j=0}^{a} \left(\sum_{m \mid q} \phi(p^j m) \right) = \sum_{j=0}^{a} \sum \phi(p^j)\phi(m) =$$

For prime p

$\phi(p^j) = p^j - p^{j-1}$

property

$$= \left(1 + \sum_{j=1}^{a} p^{j-1}(p-1)\right) q = p^a q = n$$

Ex 6 $n \neq 2$, $n \neq 6$

$\phi(n) \geq \sqrt{n}$. If $n = 2^m$ for $m \geq 2$

$\phi(n) = 2^m - 2^{m-1} = 2^{m-1} \geq \sqrt{2^m} = \sqrt{n}$

If $n = p^m$, when $m \geq 2$ and p is odd prime

$\phi(n) = p^{m-1}(p-1) \geq \sqrt{p^m} = \sqrt{n}$

If $n = p^m$; $m \geq 2$ $p \geq 5$ $\phi(n) \geq \sqrt{2n}$

Ex 1 $x^2 + n^2 = z^2$. $(3^{a-1}\cdot 5^{b-1}\cdot 8)^2 + (3^a\cdot 5^b)^2 = (3^{a-1}\cdot 5^{b-1}\cdot 17)^2$

$8^2 + 15^2 = 17^2$ $(3^{a-1}\cdot 5^{b-1}\cdot 8)^2 + (3^{a-1}\cdot 5^{b-1}\cdot 3\cdot 5)^2 = (3^{a-1}\cdot 5^{b-1}\cdot 17)^2$

(coprime)

Ex 7 $(\phi(n))^2 + n^2$

$\phi(pq) = (p-1)(q-1)$

$\phi(15) = \phi(3\cdot 5) = 2\cdot 4 = 8$

$n = 15$

$(\phi(15))^2 + (15)^2 = 8^2 + 15^2 = 17^2$ $\phi(3^a\cdot 5^b) = 3^{a-1}\cdot 5^{b-1}(3-1)(5-1) = 3^{a-1}\cdot 5^{b-1}\cdot 8$. Ex 1

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$\phi(n) = m$ $\quad$ $m = p \ldots$ [illegible]

$\bullet \; \phi(n) = p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1} (p_1 - 1) \cdots (p_k - 1)$

$\phi(n) = p_1^{\alpha} \cdot p_2^{\beta}$

[illegible] index [illegible]

$\cancel{n = 2^{\alpha_2}} = p_1^{\alpha - 1} \cdot p_2^{\beta - 1} (p_1 - 1)(p_2 - 1)$

if $k \geq 2$ [illegible] p_i [illegible] odd, $\phi(n) : 4$ [illegible]

$\Rightarrow \neq m$.

$n = 2^{\alpha} p^{\beta}$ [illegible]

$\bullet$ if $n = 2^{\alpha}$ or $2^{\alpha} p^{\beta}$ [illegible] for $\alpha > 2$ [illegible] $\phi(n) : 4$ [illegible] $\neq$ [illegible]

other if $n = 2^{\alpha} p^{\beta}$ if $\alpha = 0, \alpha = 1, \alpha = 2$

if $\alpha = 0$ $\quad \boxed{n = p^{\beta}}$

$\phi(p^{\beta}) = p^{\beta}\left(1 - \frac{1}{p}\right) \longrightarrow \phi(n) = p^{\beta - 1}(p - 1)$

where any prime factor of n is p.

m - [illegible] 6

$p - 1 = 6$

[illegible] $\boxed{p = 7}$

[illegible]?

$\phi(7^{\beta}) = 7^{\beta - 1} \cdot 6$

$\phi(n) : 2$ [illegible], [illegible] : 4 [illegible]

[illegible] 6.

$\bullet \; p - 1 = 6 ; p = 7$

$p - 1 = 2$ [illegible]

if $\alpha = 1$

$\phi(n) = 2^{\alpha - 1} p^{\beta - 1}(p - 1)$

[illegible] $p - 1$ [illegible] $p^{\beta - 1}$ [illegible] $p = $ [illegible]

$\alpha = 1 ; \Rightarrow p = 7$ [illegible]

[illegible] $7^{\beta - 1} \cdot 6$

[illegible] $\phi(n) = p^{\beta - 1}(p - 1) = \cancel{7^{\beta - 1} \cdot 6} = \underbrace{7^{\beta - 1} \cdot 2 \cdot 3}_{\text{prime factorization}}$

[illegible] : 2 [illegible] : $\phi(n)$ [illegible]

$m = \underline{2 \cdot 7^{r}}$ [illegible]. $\quad \phi(n)$ [illegible] 3 [illegible] prime factor.

[illegible] $= \phi(n)$.

(49) S, n [illegible] ?

sum of digits. (2)

$S = 2^{\alpha} \cdot 5^{\beta} \cdot t$, where t is coprime to 10 [illegible]

S [illegible] $(2 \cdot 5)$ [illegible] 10 [illegible]

[illegible] 2 [illegible] 5 [illegible]

define

$$n = 10^{\alpha+\beta}\left(10^{\phi(t)} + 10^{2\phi(t)} + \cdots 10^{S\phi(t)}\right)$$

sum of digits of n is S.

$2^{\alpha+\beta}$ [illegible] $5^{\beta+\alpha}$ [illegible]

(3) $10^{\phi(t)} \equiv 1 \bmod (t)$

so $10^{k\phi(t)} \equiv 1 \bmod t$ for $\boxed{k = 1, 2 \cdots S}$

$$n \equiv 10^{\alpha+\beta}(\underbrace{1+1+1 \cdots 1}_{S\text{-}[illegible]}) \equiv 10^{\alpha+\beta} \cdot S \pmod t$$

$S = 2^{\alpha} 5^{\beta} t$

(4) $10^{\alpha+\beta}$ is just 10 power doesn't effect divisibility.

$1 \Rightarrow n \equiv 10^{\alpha+\beta} \cdot t \cdot 2^{\alpha} \cdot 5^{\beta} \equiv$

$\equiv t \cdot \left(10^{\alpha+\beta} \cdot 2^{\alpha} \cdot 5^{\beta}\right) \bmod t$

t [illegible].

t [illegible] mod t [illegible] $k \cdot t$ [illegible] $t + t \cdot t$ [illegible] t [illegible].

$\boxed{n \equiv 0 \bmod t}$

n [illegible] $1 \bmod n$ [illegible]

[illegible] S [illegible]

[illegible] S [illegible] $10^{\phi(t)} + \cdots 10^{S\phi(t)}$ [illegible] S-[illegible]?

[illegible] S 2 4 6 [illegible]

[illegible] digit [illegible]

$\underline{1+1+1+1+1+1+1+1+1+1+1+1+1}$

[illegible] n [illegible] 0 [illegible] $S = 7$ [illegible] sum of digits of n [illegible] $= 1 + 1 + 1 \cdots 1$ 7 [illegible] S-[illegible]

$10^{\alpha+\beta}$, [illegible] 2^{α} [illegible] 5^{β} [illegible]

$n : S$-[illegible]

[illegible] $10^{\phi(t)}$ [illegible]

$\underbrace{10^{\phi(t)}}$ [illegible] t-[illegible]

$10^{\phi(t)} \equiv 1 \bmod (t)$ [illegible] $10^{\phi(t)}$ t [illegible] $= 1$.

[illegible] Σ [illegible] $= $ [illegible] X [illegible]

$$X \text{[illegible]} \equiv (\underbrace{1+1+1+1+1}_{S\text{-}[illegible]})^{\bmod t} \equiv S \bmod t \quad S = 2^{\alpha} \cdot 5^{\beta} \cdot t \quad (2^{\alpha} \cdot 5^{\beta}) t \bmod t$$

$a_{n+1} = a_n(a_n - k) + \cancel{k} = a_n \cdot a_{n-1} \cdot a_{n-2} \cdots a_1 + \cancel{k}$

$1 \Rightarrow$ $a_n - k$ is divisor of a_m [illegible] $1 \le m \le n$

$a_n - k = a_m \cdot a_{n-1} \cdot a_{n-2} \cdots$

[illegible]

$\cancel{k = a_n \cdot a_{n-1} \cdot a_{n-2}}$

[illegible] ~~$a_n + k$~~ ~~are pairwise prime~~

$a_2 = a_1 + k$

$a_3 = a_2 a_1 + k$

a_m [illegible] a_{k-1} rel. primes

$a_{n+1} = (a_n - k) + k$

a_m [illegible] $\cancel{(a_n - k)} + k$ also

rel. t. prime. const. a_m

$\boxed{m = 1 \text{ [illegible]}}$

$a_1 = k + 1$

$(k + 1, k)$ ← const. integers

coprime $\gcd(k, k+1) = 1$

λ induction

induction

$a_m = a_{m-1} \cdot a_{m-2} \cdots a_1 + k$

[illegible] rel. prime to k

$d \to$ common divisor of a_m, k ([illegible] 1)

$a_m - k = a_{m-1} \cdot a_{m-2} \cdots a_1$

const. d, [illegible] $a_{m-1}, a_{m-2}, \ldots, a_1$ [illegible]

[illegible] k [illegible] prime

so $\gcd(a_m, k) = 1$

[illegible] $\boxed{d = 1}$

$a_{n-1} \cdot a_{n-2} \cdots a_1 = $ const.

[illegible]

Proof | S [illegible] prime

[illegible] composite.

Also

$S = a + b + c + d + e + f$ [illegible]

$: abc + def$ [illegible] $ab + bc + ca - de - ef - fd$

$P(x) = (x+a)(x+b)(x+c) - (x-d)(x-e)(x-f)$

$P(x) = (a+b+c+d+e+f)x^2 + (ab + bc + ca - de - ef - fd)x$
$+ abc + def = S \cdot x^2 + \text{const}_1 \cdot S \cdot x + \text{const}_2 \cdot S \cdot$
$= S(x^2 + \cdots)$

~~$P(x) = $ const~~

$P(d) = (a+d)(b+d)(c+d) = \text{const} \cdot S$ | $P(x)$ also multiple of S.

$P(d)$

$a+d, b+d, c+d$

[illegible] S - [illegible] S. [illegible]

$S = a+b+c+d+e+f > a+d$
$> b+d$
$> c+d$

[illegible] S is composite.

717 $n, a, b > 0$ integer

$$\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1$$

~~$\gcd(a,b)$: both a, b~~

$n^{\gcd(a,b)} - 1$ divides $n^a - 1$ & $n^b - 1$ $\Rightarrow$ it divides $\gcd(n^a - 1, n^b - 1)$

greatest ~~common~~ divisor

~~$ax - by = \gcd(a,b)$~~

$$ax - by = \gcd(a,b)$$

$$\Rightarrow n^{ax-by} = n^{\gcd(a,b)} =$$

$$\frac{n^{ax}}{n^{by}} = n^{\gcd(a,b)} - 1$$

720

724 $(a_n)_n$ $\quad a_1 = k+1 \quad a_{n+1} = a_n^2 - k a_n + k$

min a_n i on the wprne

$$a_2 = (k+1)^2 - k(k+1) + k = (k+1) + k = a_1 + k$$

$a_n = a_{n-1} + k$

~~$a_3 = a_2(a_2 - k) + k$~~

$$a_3 = a_2(a_2 - k) + k = a_2 a_1 + k$$

$$a_4 = a_3(a_3 - k) + k = a_3 a_2 a_1 + k$$

so $a_n = a_{n-1} a_{n-2} \cdots a_1 + k$ (by ind)

(#3) $C(x,y)$ if x, y are coprime

for n, $\exists (a,b)$ s.t. no visible point $> n$.

(a,b) center in $(2n+1)\times(2n+1)$ square.
All lattice points in its interior and on sides aren't visible from origin.

[illegible] not visible [illegible] $\equiv 0 \bmod p_{ij}$

choose $(2n+1)^2$ distinct primes p_{ij}, $\boxed{-n \le i, j \le n}$

Shift from central point. max n to $-n$ [illegible]

$$\begin{cases} a+i \equiv 0 \bmod (p_{ij}) & \text{for all } i,j \\ b+j \equiv 0 \bmod (p_{ij}) & \text{for all } i,j \end{cases}$$

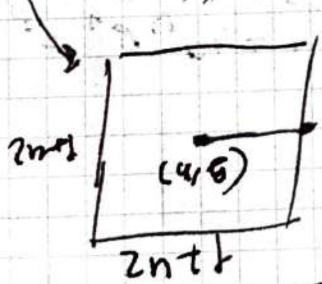
$\Rightarrow a+i$ & $b+j$ are $\dfrac{\text{divisible by}}{p_{ij}}$ p

$\Rightarrow$ none of $(a+i, b+i)$ is visible, since $a+i$ & $b+i$ [illegible] p_{ij} [illegible] coprime [illegible]

so $a+i$ & $b+j$ [illegible] for $\boxed{-n \le i, j \le n}$

[illegible] not visible from origin is outside of ~~2n+1~~ $(2n+1)\times(2n+1)$ square.

$\Longleftrightarrow$ centered at a, b $\Rightarrow$ from (a,b) distance greater than n.



(#2) $a^n + n$ divides $b^n + n$. Prove $a = b$

Assume on contradict.

prime p doesn't divide $b-a$

CRT - [illegible] $\rightarrow n \equiv c_1 \bmod (m_1)$, $n \equiv c_2 \bmod (m_2)$ if m_1, m_2 are coprime.

p & $p-1$ are coprime [illegible]

$n \equiv 1 \pmod{p-1}$

$\boxed{n \equiv -a \pmod{p}}$ mod p - 1, $-a = n$

Fermat $a^n + n \equiv a + n \equiv a - a \equiv 0 \bmod p$

$b^n + n \equiv b + n \equiv b - a$ ~~[illegible]~~ mod p

so p divides $a^n + n$, but doesn't $b^n + n$. $\boxed{\text{so } a = b.}$

$n \equiv 1 \bmod (p-1)$

$\Rightarrow \boxed{n = 1 + k(p-1)}$

$a^n = a^{1+k(p-1)}$

$= a \cdot \underbrace{(a^{p-1})^k}_{\text{Fermat's}}$

$\equiv 1^k \equiv 1 \bmod p.$

Chinese remainder theorem

$m_1, m_2, \dots, m_k$ pairwise coprime [illegible] (m_i, m_j) coprime (m_i, m_j) coprime

i.e. $\gcd(m_i, m_j) = 1$, $i \neq j$

any integers $a_1, a_2, \dots, a_k$

$x \equiv a_1 \pmod{m_1}$, $x \equiv a_2 \pmod{m_2}$

$x \equiv a_k \pmod{m_k}$

solutions [illegible] (so x [illegible]) ~~[illegible]~~ any two are congruent modulo $m = m_1 m_2 \dots m_k$

Ex. 1

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = y^5$$
$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = z^2$$
$$x_1^5 + x_2^5 + x_3^5 + x_4^5 = t^3$$

have ∞ many solutions

1.) [illegible] $x_1, x_2, \dots$ [illegible]

$a = 1^2 + 2^2 + 3^2 + 4^2$, $b = 1^3 + 2^3 + 3^3 + 4^3$

$c = 1^5 + 2^5 + 3^5 + 4^5$ | look for solutions in form

$x_1 = a^m b^n c^p$

$x_2 = 2a^m b^n c^p$

$x_3 = 3a^m b^n c^p$

$x_4 = 4a^m b^n c^p$

(130) $2^x + 3 = z^3$

? $\phi(7) = 6$, $\phi(9) = 6$ | $\phi(13) = 12$ but here no of distinct cube residues is $\frac{12}{\gcd(3,12)} = \frac{12}{3} = 4$?

(131) $\phi(1)\left[\frac{n}{1}\right] + \phi(2)\left[\frac{n}{2}\right] + \cdots = \frac{n(n+1)}{2}$

1) induction

2) Gauss identity Euler's.

$$\frac{n(n+1)}{2} = \underbrace{\sum_{m=1}^{n} m}_{1+2+3\cdots} = \sum_{m=1}^{n} \sum_{k|m} \phi(k) = \sum_{k=1}^{n} \phi(k) \cdot \sum_{m=1}^{\left[\frac{n}{k}\right]} 1 \cdot \left[\frac{n}{k}\right]$$

$\boxed{\sum_{k|m} \phi(k) = m}$ $k|m$

if you sum $\phi(k)$ over ~~for any positive m~~ for any positive m over all positive divisors k of m $= m$.

$\sum_{k|m}^{n} \phi(k) = \phi(k) \cdot \left[\frac{n}{k}\right]$, how sums Σ sum up to all, for numbers $k|m$.

in $\sum_{k=1}^{n} \phi(k)$ all sum $\phi(1) + \phi(2) \cdots$ and $\cdot \left[\frac{n}{k}\right]$ - on

on

$a^{k\phi(d)}$ so a-th term always and in sequence, so same prime divisors.

of which divisible by a

(132) $a,\ a+d,\ a+2d,\ \cdots\ a+nd$.

same for assume $\gcd(a,d) = 1 \Rightarrow a^{\phi(d)} \equiv 1 \pmod d$ hence infinitely many divisible.

$a^{k\phi(d)} \equiv 1 \pmod d$ so on. → infinitely many $a+(\ldots)$ on

$a^{k\phi(d)} = 1 + m_k \cdot d$ if $\boxed{m_k = a n_k}$

~~Print~~ Prime factors of $a+n_k d$ is same as ~~...~~ $\Leftarrow$

$a^{k\phi(d)} = 1 + \frac{a n_k}{\cancel{m_k}\, a} \cdot d$ (-a) - on

$\boxed{a^{k\phi(d)+1} = a + n_k \cdot d}$ and all same prime - and, how $a^{k\phi(d)+1}$, how $=$.

(178) $ad - bc = \pm 1$ [illegible]

$ax + by = m$
$cx + dy = n$

if $a = 0$
$by = m$ always has solutions
$\Rightarrow$ ~~[illegible]~~ $b = \pm 1$.
so $y = \pm m$. [illegible]
$0 \cdot x + by = m$ [illegible] solution [illegible] must divide all m.

$\cancel{bx}$ $cx = \underbrace{n \mp dm}$ [illegible] integer solution

$c = \pm 1$

so $ad - bc = bc = \pm 1$ [illegible] $a = 0$.

[illegible] argument if b or c or $d = 0$ [illegible]

If $a, b, c, d \neq 0_{bc}$ $\left\{ \begin{array}{l} ax + by = m \\ cx + dy = n \end{array} \right.$
let $\Delta = ad - bc$

If $\Delta = 0$, then $\boxed{\frac{a}{c} = \frac{b}{d} = \lambda}$ [illegible] $m = ax + by = \lambda(cx + dy) = \lambda \cdot n$
$\boxed{m = \lambda \cdot n}$ [illegible] $\Rightarrow \Delta \neq 0$

If $\Delta \neq 0$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$, [illegible]
rule-[illegible]
$x = \frac{dm - bn}{\Delta}$ [illegible] $y = \frac{an - cm}{\Delta}$

if $\Delta \neq 0$

det [illegible] integer [illegible]
$dm - bn$ [illegible] $=$ int, [illegible].

[illegible] $\Delta = 1$ [illegible]

(m, n) [illegible]
$(m, n) = (1, 0)$ [illegible] $x_1 = \frac{d}{\Delta}$, $y_1 = -\frac{c}{\Delta}$ solution [illegible]
$(m, n) = (0, 1)$ [illegible] ; $x_2 = \frac{-b}{\Delta}$, $y_2 = \frac{a}{\Delta}$

$\Rightarrow x_1 y_2 - x_2 y_1 = \frac{ad - bc}{\Delta^2} = \frac{1}{\Delta}$ [illegible] $\Delta = 1$
x, y solution [illegible] $\Rightarrow$ [illegible]

(790) $\gcd(a, b) = 1$
$\left\{ \begin{array}{l} ax - yz - c = 0 \\ bx - yt + d = 0 \end{array} \right.$

$ax - yz = c$
$\underbrace{bx - ty = -d}$

$\begin{pmatrix} a & -z \\ b & -t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ -d \end{pmatrix}$

∞ many solutions [illegible] $\Rightarrow a \cdot u - b \cdot v = 1$ has ∞ many (u, v) positive solutions.
so $u = $ [illegible]
$v = $ [illegible]
$u = t$
$v = z$ [illegible]

$\det = a(-t) - (-z)b = -at + bz = -(at - bz) = -1$

rule-[illegible] $x = \frac{\det\begin{pmatrix} c & -z \\ -d & -t \end{pmatrix}}{-1 \leftarrow \det\Delta}$; $y = \frac{\det\begin{pmatrix} a & c \\ b & -d \end{pmatrix}}{-1 \leftarrow \Delta}$

$x = ct + zd$
$y = ad + bc$
$(ct + zd, ad + bc, z, t)$ [illegible] solution

a) $ax - by = c$ has solutions only if $\gcd(a,b)$ divides c

if (x_0, y_0) are solutions all others are the form

$$x = x_0 + \frac{b}{\gcd(a,b)} t$$

$$y_0 = y_0 + \frac{a}{\gcd(a,b)} t.$$

$ax - by = 1$

$\det \begin{pmatrix} a & y \\ b & x \end{pmatrix} = ax - by = 1$

if det $= 1$ for 2×2 matrix they form S.L.-s which are generated by

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} a & y \\ b & x \end{pmatrix} = ST^{d_1} ST^{d_2} S \dots ST^{d_n} \cdot S$$

matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on Lobachevski plane by

$z \to \dfrac{az+b}{cz+d}$ [illegible]

[illegible] z [illegible]

7g4

$\det \begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix} = 12\cdot 3 - 7\cdot 5 = 1$ (=>

$x \quad y \quad 12x - 7y = 1$ $(3,5)$ is solution.

$S\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}$ $\quad T^{n}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+nq \\ q \end{pmatrix}$

$\frac{12}{7} = 1 + \frac{5}{7}$ [illegible]

frac. part. [illegible] $\frac{7}{5} = 1 + \frac{7-5}{5} = 1 + \frac{2}{5}$ =>

[illegible] $\frac{5}{2} = 2 + \frac{5 - 2\cdot 2}{2} = 2 + \frac{1}{2}$

[illegible] ~~$\frac{12}{7} = 1 + \frac{5}{7}$~~ [illegible] 2. 1 => $\frac{7}{5} = 1 + \frac{2}{5}$ | so $\frac{5}{7} = \frac{1}{1+\frac{2}{5}}$

[illegible] $\frac{5}{2} = 2 + \frac{1}{2}$; so $\frac{2}{5} = \frac{1}{2+\frac{1}{2}}$

$$\frac{12}{7} = 1 + \frac{5}{7} = 1 + \frac{1}{1+\frac{2}{5}} = 1 + \frac{1}{1 + \frac{1}{2+\frac{1}{2}}} = T\cdot S\cdot T\cdot S\cdot T^2\cdot S\cdot T^2$$

[illegible] 2. | T

[illegible] $\frac{12}{7}$ [illegible]

[illegible] $2+\frac{1}{2}$ | $S(2) = \frac{1}{2}$ so $T^2(\frac{1}{2}) = \frac{1}{2} + 2 = \frac{5}{2}$

[illegible] $S(\frac{5}{2}) = \frac{2}{5}$ | $T^1(\frac{2}{5}) = \frac{2}{5} + 1 = \frac{7}{5}$

$S(\frac{7}{5}) = \frac{5}{7}$ | $T^1(\frac{5}{7}) = \frac{5}{7} + 1 = \frac{12}{7}$

$ax + by = c$. სადაც (a, b) არის coprime.

$ab - a - b$ is largest positive integer c

7ქ7

~~$7x + 12y = 60$~~

ყოველ 1 სვეტის ბოლო 8.

$+7$

ან ყოველ 1 სვეტის ბოლო 12

$+12$.

$\Rightarrow$ a x times 8, y times 12. $7x + 11y + 1$ არის 1 ოვლისთვის

$\square$ ერთი

მოთავსება.

$7x + 11y = n$ $\in$ სახის

ფორმის.

11 და 7 არის coprime. | :7-ზე ნაშთები შეიძლება < 7-ზე $0, 1, 2, \ldots 6$ სადაც 11 და 7 coprime $\gcd(7, 11) = 1$. არ მეორდება ნაშთები.

$11 \cdot 0, 11 \cdot 1, 11 \cdot 2 \ldots 11 \cdot 6$

form ყველა possible ნაშთს mod 7-ზე გაყოფისას. $(0, 1, 2 \ldots 6)$

$7x + 11y = n$; $7x = n - 11y$. $0 \equiv n - 11y \pmod 7$

ანუ :7-ზე

$n \equiv 11y \pmod 7$

$x = \dfrac{n - 11y}{7}$ არის solution

for $0 \le y \le 6$.

არსებობს ასეთი y

ანუ $(60, 66)$-ზე

y-ების solution:

~~...~~ $y = 6$

$60 = 7 \cdot 7 + 11 \cdot 1$

$61 = 7 \cdot 4 + 11 \cdot 3$

$62 = 7 \cdot 1 + 11 \cdot 5$

$63 = 7 \cdot 9 + 11 \cdot 0$

$64 = 7 \cdot 6 + 11 \cdot 2$

$65 = 7 \cdot 3 + 11 \cdot 4$

$66 = 7 \cdot 0 + 11 \cdot 6$

არის $0 \le y \le 6$

ამიტომ x-ები არის

7 მიმდევრული რიცხვის ყველა ის არის $7x + 11y$.

ანუ + 7-ის multiple-ების (x-ის გაზრდით)

მერე, ეს ის induction-ით მიიღება ყველა.

ამ form და. +7-ით x-ის გაზრდით.

ანუ $n \ge 60$ proved. y არის fixed $0 \le y \le 6$

Proof: ~~$n < 59$~~ $n = 59$-ზე თუ

$7x + 11y = 59 \longrightarrow 7x + 11(y-5) = 4$

მარჯვენა 11-ის multiple-ია

მარცხ. მხარე ... 11-ზე არის (mod 11-ში ...)

ამიტომ $7x \equiv 4 \bmod 11$

$8 \cdot 7x \equiv 8 \cdot 4 \bmod 11$

$x \equiv 32 \bmod 11$

$x \equiv 10 \bmod 11$

$32 > 11$

$x = 10 + 11k$ სადაც $k = 0$

$7x + 11y = 59$

$7 \cdot 10 = 70$

$70 > 59$

x-ის

$7x \equiv 4 \bmod 11$ $(7, 11)$ არის co-prime ამიტომ

$7 \cdot x \equiv 1 \bmod 11$

$7 \cdot 8$-ზე ...

(98) $\boxed{ax+by=n}$ f(x) ...

(1) $\frac{1}{1-x^a} = 1 + x^a + x^{2a} + \dots$ (2) $\frac{1}{1-x^b} = 1 + x^b + x^{2b} + \dots$

~~coeff of x^n~~

(1)·(2) -> coeff of x^n ...

Number of ways exponents of ... from ka and mb add up to n

... $x^a \cdot x^{2b} \to = x^{a+2b}$; $y=2$; $x=1$ ...

(99) $x+y=n$, for $n>6$

x, y coprime. -> ... coprime.

for ...

n can be represented as $4m, 4m+1, 4m+2, 4m+3$

... 4. odd -> odd. ... +2 ... odd ... >2 -> ... coprime ...

$4m = (2m-1) + (2m+1)$ +1

$4m+1 = 2m + (2m+1)$ +1

$4m+2 = (2m-1) + (2m+3)$ +1

$4m+3 = (2m+1) + (2m+2)$

...

$4m$ ... ?

Because any 2 number to be coprime ... 1) consecutive ... 2) odd ... 2

... 2 ... 4 ...

$4m+1$ ($2m$; $2m+1$)

$4m+3 = ((2m+1), (2m+2))$

... $b = x$ ... coprime ... x^d ... y ...

(800) ... d dimension ... d dimensional cube into k^d pieces

for any integer k ... k ... k^d ...

$(a^d-1)x + (b^d-1)y$ ... for ... (a,b) coprime ...

for coprime a, b : $a\cdot x + b\cdot y = n$ ... a^d-1 ... b^d-1 are coprime

for I cube ... -1 ...

for II cube ...

$a = a^d-1$ ... $b = b^d-1$ ...

$x = a^d-1$, $b = x$ ... ; $y = b^d-1 = x^{d-1}$ ... $x^d = (a^d-1)^d$; $x^d - y = 1$ ... $= (a^d-1)^{d}-1$ ... coprime ...

... cube -1 ... coprime ...

Ek-2 $\sigma \rightarrow \{1, 2 \cdots n\}$

σ_1 and σ_2 permutations $\sigma = \sigma_1 \sigma_2$

σ_1^2 and σ_2^2 = identity perm. (I)

sign of two elements.

σ - of length $= n$. if $n = 1$ or 2

$\sigma = (1)$ ~~or $\sigma = (1, 2)$~~

$\sigma = (1, 2)$

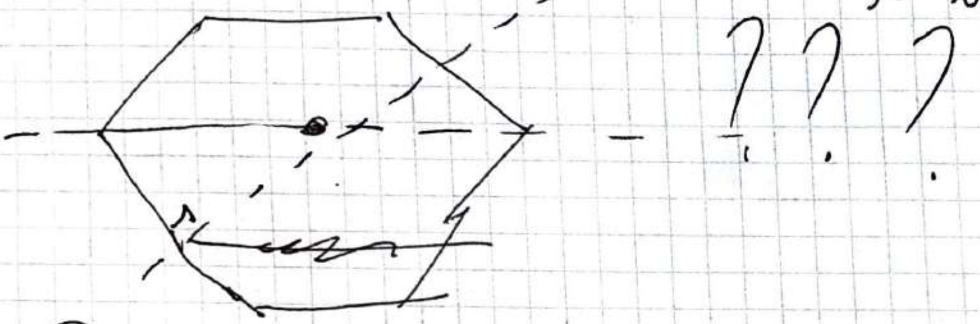
$\sigma = \sigma_1 \cdot I$

identity permutation

$\sigma = \sigma_1 \cdot I$ identity perm.

otherwise, σ = rotation of regular n-gon $A_1, A_2 \ldots A_n$ by angle $\frac{2\pi}{n}$ around centre.

or respectively



ex4 $a_1, a_2 \ldots a_{10}$ of integers $1, 2, 3 \cdots 10$

$|a_1 - a_2| + |a_3 - a_4| + |a_5 - a_6| + |a_7 - a_8| + |a_9 - a_{10}|$

$\sum_{k=1}^{n} |a_{2k-1} - a_{2k}|$ average

$= n \cdot$ average $|a_1 - a_2|$ which $|a_{2i-1} - a_{2i}|$

is same for all $i = 1, 2 \ldots n$. when $a_1 = k$

$\sum|k - i| = 1 + 2 \ldots (k-1) = \frac{k(k-1)}{2}$ and average $|a_1 - a_2|$

$$\frac{(k-1) + (k-2) + \ldots 1 + 1 + 2 + (2n-k)}{2n-1} =$$

$$= \frac{1}{2n-1}\left(\frac{k(k-1)}{2} + \frac{(2n-k)(2n-k+1)}{2}\right) = k^2 \frac{-(2n+1)k + n(2n+1)}{2n-1}$$

when $k > a_{2k}$. the $\sum$ sum $= \frac{k(k-1)}{2}$, where $|k - a_{2k}| = k - a_{2k}$

when $a_{2k} > k$

$$\sum |k - j| = j - k = (1 + 2 + \cdots 2n - k) = \frac{(2n-k)(2n-k+1)}{2}$$

average sum $= n \cdot$ average sum $= \frac{n(2n+1)}{3}$

[illegible]

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & 19 & 20 \\ a_1 & a_2 & \cdots & \cdots & a_{19} & a_{20} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & 19 & 20 \\ a_{19} & a_{20} & a_{17} & \cdots & a_1 & a_2 \end{pmatrix}$$

① $\Rightarrow$ [illegible] i, j same parity ([illegible] odd or even)

$i < j$ (a_i, a_j) is inversion ($\Rightarrow a_i > a_j$)

σ_2 - [illegible] [illegible] ~~[illegible]~~

[illegible] $(a_{17}$ $a_{19})$ [illegible]

[illegible]

$a_{19} \cdots\cdots$ (a_{17} [illegible])

[illegible] a_i, a_j [illegible] a_{19} [illegible]

a_i, a_j

② if i = even & j = odd.

[illegible] a_i, a_j [illegible] $a_i > a_j$

σ_2-[illegible] position [illegible]

[illegible]

③ if i = odd j = even . if $j > i+1$ [illegible]

[illegible] j & i are consecutive, [illegible] switch places.

[illegible]

σ_2 - [illegible] 2 - [illegible] $\Rightarrow$ [illegible]

[illegible] $\sigma_1 = (\cdots a_{19}\ a_{20})$

$\sigma_2 = (a_{19}\ a_{20} \cdots)$

inversion is preserved. [illegible]

i = odd, 1, 3, 5 ⋯ 19

10 [illegible]

$\boxed{\text{[illegible] } 100 \text{ inversion}}$ of σ_1 [illegible] at most . 10

[illegible] inversion of σ_2 . 90 is lost. (no longer inversion of σ_2)

(a_i, a_j) [illegible] pair [illegible] ?

$\binom{20}{2} = 190$ pairs in σ_2 [illegible] ~~[illegible]~~ at least 90 [illegible] inversions.

$190 - 90 = 100$.

(618) $7^x - 3^y = 4$

for $y=0$, $7^x - 1 = 4$ ~~$\emptyset$~~

for $y = 1 = x$ is solution

for $y \geq 2$ – אז 3^2 ... 3^y ... mod 9

~~$7^x \equiv 4 \mod 9$~~

$3^y \mod 9$ – אז $\equiv 0$

~~$7^x \equiv 4 \mod 9$~~ – אז $\boxed{7^x \equiv 4 \mod 9}$

$\Rightarrow$ אז $x = \text{odd}$ ...

אז $x = \text{even}$

$7^2 \equiv 49 \equiv 4 \mod 9 \Rightarrow 7^1 \equiv 7 \mod 9$

אז $x = \text{even}$.

$x = 2n$.

$3^y = 7^{2n} - 4 = (7^n + 2)(7^n - 2)$

אז $7^n + 2 - (7^n - 2) = 4$

$y \geq 2$ – אז $\emptyset$

$\boxed{x = 1 = 1}$

$7^x = 7^1$, $x = 1$!

(619) $3^{2^{x!}} = 2^{3^{x!}} + 1$, $x = 1$ = solution

for $x > 1$ – אז ($x \geq 2$ – אז)

$x! = 1 \cdot 2 \cdot \ldots$

$x! = \text{even}$

$= 2n$

$3^{2n} \equiv 1 \mod 4$

~~$2^{3^{2n}} \equiv 2 \mod 4 \equiv 2 \mod 4$~~

$2^{3^{2n}}$ ... $= 2$

$\Rightarrow 2^{3^{2n}} + 1 = 2 + 1 = 3$

mod 4 –

$2 \equiv 2 \mod 4$

$4 \equiv 0 \mod 4$

$8 \equiv 0 \mod 4$

$16 \equiv 0 \mod 4$

... $32 \equiv$ ... $2 \mod 4$

$3^{2^{x!}}$

$2^{3^{2n}} \equiv 2 \mod 4$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32$

$2^1 = 2$

$2^3 = 8$

~~$2^5 = 32$~~

(100) 1, 2, 3, 4, 5, 6, 7

B_1

B_2 → infinite

B_3 → 0

(1) 3 [illegible] B_1 B_2 B_3 [illegible] consecutive numbers (1, 2, 3 . . . 1980 [illegible])

8 [illegible] 3 [illegible]

[illegible] (1) [illegible] 2 arithmetic [illegible] 3 [illegible].

(1) [illegible] at least 4 of those numbers.

(1) [illegible] 4 [illegible] 2, 4, 6, 8 [illegible]

Positive [illegible] 1980 [illegible]

If 1 3 5 7 [illegible] 2 [illegible] 2, 4, 6, 8

[illegible] 2 consecutive even numbers [illegible]

[illegible] 2, 6 [illegible]

4, 8 [illegible]

1980 [illegible]

(2) 3 [illegible] 3 [illegible]

3 [illegible] 6 [illegible] 3 [illegible]

3 [illegible] 3, 5, 7, 3 [illegible] 2 [illegible]

6 [illegible]

4, 6, 8 [illegible]

1980 = 18 · 3 · 8 [illegible]

2, 4, 6, 8

1980 [illegible]

N01 [illegible]

$f(n) + 2f(f(n)) = 3n+3$ — $f(0)$; $f(f(0))$ [illegible]

$f(1)$ is even number

$f(1) + 2f(f(1)) = 8$ => [illegible]

$f(1)$ [illegible] even number $(1;6)$ [illegible]

$f(1) = 2; 4; 6$

(a) if $\boxed{\text{if } f(1) = 2}$

$2 + 2f(2) = 8$; $f(2) = 3$

$3 + 2f(3) = 11$

$3 + 2f(3) = 11$

$f(3) = 4$.

$f(n) = n+1$ [illegible].

Proof inductive

$n+1 + 2f(n+1) = 3n+3$; $f(n+1) = n+2$

Proven.

(b) if $f(1) = 4$ $\cancel{\varnothing}$ $4 + 2f(4) = 8$; $f(4) = 2$

$2 + 2f(f(4)) = 17$ [illegible]

(c) if $f(1) = 6$ $6 + 2f(6) = 8$

$f(6) = 1$

$\varnothing$

[illegible] $f(6) + 2f(f(6)) = 1 + 2\cdot 6$

$1 + 2\cdot f(1) = 1 + 2\cdot 6$

$1 + \cancel{2 \cdot 6 = 1 + 2\cdot 6}$

(a) [illegible]

$f(n) = n+1$

N02 $2f(f(x)) - 3f(x) + x = 0$

$\boxed{2f(t)} - \boxed{3\cdot t} + x = 0$

$\to 2f(f(x)) - 3f(x)$

$2f(x) - f(x) + x = 0$

$g(x) = f(x) - x$ input in $f(x)$. [illegible]

$$\underbrace{2f(f(x)) - 2f(x)}_{2g(f(x))} \underbrace{- f(x) + x}_{g(x)} = 0$$

$$2g(f(x)) = g(x)$$

induction

$$2^n f^{(n)}(x) = g(x)$$

($f^{(n)}(x)$ composed n times itself)

for every x

$g(x)$ $\boxed{g(x) \text{ [illegible]}}$

[illegible] 2^n [illegible] => $g(x) = 0$ only

g 500-[illegible] 502-[illegible] [illegible]

$\Rightarrow$ $f(x) = x$ ...

no bijection

(103) $f(mn) = f(m) + f(n) + 3f(m)f(n)$ Rl 3 1

$g(x) = 3f(x) + 1$

g is bijective $\quad$ ~~$g(xy) = g(x)\cdot g(y)$~~

$\Rightarrow$ $g(1) = 1$

If $g(x\cdot y) = n$ when $n = pq$ primes congruent to 2 mod 3

$\Rightarrow$ x is prime.

if $x = yz$ prime ...

$n = g(x) = g(y)\cdot g(z)$

... n can be factorised as product of 2 numbers which are prime. (Prime ...)

If p, q are numbers congruent to 2 mod 3 (eg. 2, 5)

then pq; $p\cdot p$; $q\cdot q$ are ~~image of~~

All in image of g.

$g(a) = p^2$; $g(b) = q^2$, $g(c) = pq$

$g(c^2) = g(c)^2 = p^2q^2 = g(a)g(b) = g(ab)$

$c^2 = ab$ with a, b, c distinct

$\emptyset$. , prime numbers ...

(104)

$a_{n-1} \le (a_{n+1} - a_n)^2 \le a_n$ $\quad \boxed{n \ge 2}$

$1, 2, 3, 4, 5$: Max $n \le 5$

if a_6 exists ...

$b_n = a_{n+1} - a_n$ for $n \in (1, 2 \dots 5)$

$a_n \ge a_{n-1}$ for $(n \ge 2)$

$\Rightarrow$ ~~b_n is ness~~ $a_{n+1} \ge a_n$

~~$b_n \ge 0$~~

b_n is nonnegative for $1, 2 \dots 5$

$n = 2, 3, 4 - b_n$

~~$-a_n \le b_n^2 \le a_n - a_{n-1}$~~

$a_n \le b_{n+1}^2 \le a_{n+1}$

$-a_n \le b_{n+1}^2 \le a_{n+1}$

$0 \le b_{n+1}^2 - b_n^2 \le b_n + b_{n-1}$

$0 \le (b_{n+1} - b_n)(b_{n+1} + b_n) \le b_n + b_{n-1}$

$b_{n+1} \ge b_n - b_0$ for $n = 2, 3, 4$

$0 < b_{n+1}^2 - b_n^2 \le b_n + b_{n-1} < b_{n+1} + b_n$

$0 < b_{n+1} - b_n < 1$ ... for $b_3 = b_4 = b_5$

$b_3^2 \le a_3 \le b_4^2 \le a_4 \le b_5^2$

$a_3 = a_4$ but $b_3 = a_4 - a_1 \le 0$

[illegible] no set of positive integers [illegible]

705)

706 $x^2 + 10y^2 = 3z^2$ ends with 0, 1, 4, 5, 6, 9

ends with 0, 2, 3, 5, 7, 8. [illegible]

$x = 5x_0 \quad z = 5z_0$

$25x_0^2 + 10y^2 = 3 \cdot 25z^2$

y [illegible] $10y^2$ [illegible] 5-ზე [illegible]

[illegible] $x = 5$-ზე [illegible]

[illegible] $\boxed{y = 5y_0}$

[illegible] $\cancel{25} x_0^2 + 10 \cdot \cancel{25} y_0^2 = 3 \cdot \cancel{25} z_0^2$

x_0, y_0, z_0 [illegible]

[illegible]

707 $x^2 + 5y^2 = z^2$

$+ \quad 5x^2 + y^2 = t^2$

$\cancel{6x^2}\ 6(x^2+y^2) = \underbrace{z^2 + t^2}_{[illegible]}$ 3-ზე [illegible]

$z^2 + t^2 = 3$ [illegible] : 3-ზე

[illegible] z და t [illegible] $z = 3z_0$

$t = 3t_0$

$\cancel{6}(x^2+y^2) = \cancel{6}(z^2 + t^2)$ ($6 \to 2$, $6 \to 3$)

$x^2 + y^2 : 3$-ზე $\quad \cancel{x = x_0}$ $x = 3x_0$ $y = 3y_0$

[illegible] x_0, y_0, t_0, z_0 [illegible]

ე. $b_3^2 \le a_3 \le b_4^2 \le a_4 \le b_5^2$

$a_3 = a_4 \qquad b_4 + b_3 = a_4 - a_1 \le 0$

[illegible] no set of positive integers [illegible]

705)

---

706 $x^2 + 10y^2 = 3z^2$ ends with 0, 1, 4, 5, 6, 9

ends with 0, 2, 3, 5, 7, 8. [illegible]

[illegible] x, y, z [illegible]

$x = 5x_0 \qquad z = 5z_0$

$25x_0^2 + 10y^2 = 3 \cdot 25z^2$

y [illegible] $10y^2$ [illegible]

ე. $x = 5n$ [illegible]

[illegible] y [illegible] $\boxed{y = 5y_0}$

[illegible] $\cancel{25}x_0^2 + 10 \cdot \cancel{25}y_0^2 = 3\cancel{25}z_0^2$

x_0, y_0, z_0 [illegible]

[illegible]

707 $x^2 + 5y^2 = z^2$

$+ \quad 5x^2 + y^2 = t^2$

$\cancel{6x^2} \quad 6(x^2+y^2) = \underbrace{z^2 + t^2}_{\text{[illegible]}}$ 3-ის [illegible]

$z^2 + t^2 = 3n$: 3-ის

[illegible] z და t-ს

$z = 3z_0$

$t = 3t_0$

$\overset{2}{\cancel{6}}(x^2+y^2) = \overset{3}{\cancel{9}}(z_0^2 + t_0^2))$

$x^2 + y^2 \vdots 3$-ის $\quad \cancel{x = x_0}$

$x = 3x_0$
$y = 3y_0$

[illegible] x_0, y_0, t_0, z_0 [illegible] x, y, z positive

(708) $x^2 - y^2 = 2xyz$ $\quad d = xy$

[illegible]

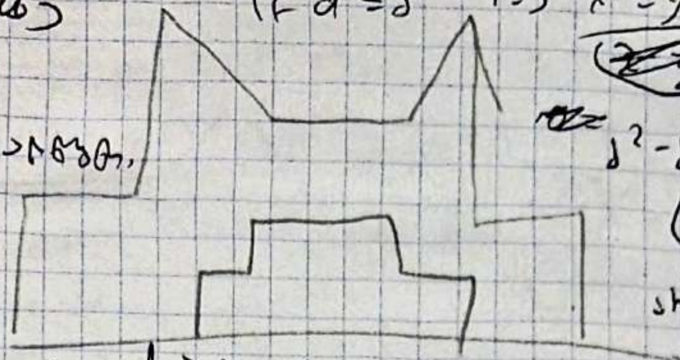
if $d = 1 \Rightarrow x = y = 1$

$1 - 1 = 0 - 2\cdot$ [illegible]

$1^2 - 1^2 = 2 \cdot d \cdot z$

$\boxed{z = 0}$

$d >$ [illegible]



since x, y, z positive

$\Rightarrow d > 1$

let p be prime divisor of d

$(x+y)(x-y) = x^2 - y^2 = 2xyz = 0 \bmod p$

$x + y = 0 \bmod p$ $\quad x = y \bmod p$ $\quad$ or $\quad x = -y \bmod p$

$x - y = 0 \bmod p$

p divides x or y - ...

[illegible] p [illegible]

$x_1 = \frac{x}{p}$, $y_1 = \frac{x}{p} - v$

[illegible] equation [illegible]

(709) If (a_n) is infinite [illegible]

$a_{k+1}^2 - a_k^2 = a_k^2 - a_{k-1}^2$ $\quad d$

[illegible]

must be increasing.

$a_{k+1} + a_k > a_k + a_{k-1}$

$(a_{k+1} - a_k)(a_{k+1} + a_k) = (a_k - a_{k-1})(a_k + a_{k-1})$

$a_{k+1} - a_k < a_k - a_{k-1}$

$a_2 - a_1 > a_3 - a_2 > a_4 - a_3 \dots$

(710) $a < a+d < a+2d$

$f(a) < f(a+d) < f(a+2d)$ on for $f(a+k)$ finitely many numbers are that are less than a.

fix a, [illegible] it.

$f(a+nr) > f(a)$

$f(a+2^{n+1}r) < f(a+2^n r)$, for all n,

otherwise a and $d = 2^n r$ would satisfy.

infinitely

$x^2+y^2=z^2$; ~~If x,z have same factor it divides y as well~~

$y^2 = z^2-x^2 = (z-x)(z+x)$

if (x,z) co-prime — [illegible] x [illegible] z [illegible]

$z-x$ [illegible] $z+x$ [illegible]

z is odd [illegible] ; $z-x+z+x = 2z$

$\Rightarrow y=$ even. [illegible] $z+x-(z-x) = 2x$

$y = 2\cdot u\cdot v$

$\Rightarrow z+x = 2u^2$; $z-x = 2v^2$

solutions

$x = k(u^2-v^2)$ [illegible] $y = 2kuv$, $z = k(u^2+v^2)$

~~$z = k(u^2+v^2)$~~ ~~$y = k(u^2-v^2)$~~

~~$x = z$~~

$x = 2kuv$, $y = k(u^2-v^2)$ [illegible] $z = k(u^2+v^2)$

$x^2+y^2 = z^2$ $\cdot z^2$ [illegible]

$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1$

circle (conic [illegible] parameterization)

$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ [illegible] t [illegible]

[illegible] rational [illegible] t = rational

$\boxed{t \text{ rational } \Rightarrow t = \frac{u}{v}}$ for some co-prime u,v

$\frac{x}{z} = \frac{1-\left(\frac{u}{v}\right)^2}{1+\left(\frac{u}{v}\right)^2}$, $\frac{y}{z} = \frac{2\frac{u}{v}}{1+\left(\frac{u}{v}\right)^2}$

when x,z are odd

odd y = even

$\frac{y}{z} = \frac{2uv}{u^2+v^2}$

$\Rightarrow \frac{2uv}{u^2+v^2}$ [illegible] irreducible.

[illegible]

$\Rightarrow y = 2uv$

$z = u^2+v^2$

$\Rightarrow x = u^2-v^2$

Pythagorean triples in Fibonacci

1, 1, 2, 3, 5, 8 . . . .

$F_4 = 3$ $F_5 = 5$. $3 \cdot 5 = 15$ ← double ~~15·2~~ $15 \cdot 2 = 30$

$4+5=3+6$ [illegible]

$F_3 = 2$ $F_6 = 8$ ~~[illegible]~~ $2 \cdot 8 = 16$.

we got 30, 16.

$30^2 + 16^2 = 1156$ & square of [illegible] $F_9 = 34$

$4+5=9=3+6$

Fib numbers $2F_nF_{n+3}$, $F_{n-1}F_{n+2}$, ~~[illegible]~~ F_{2n+3} form pythagorean triple

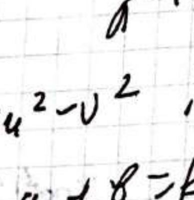
$u = F_{n+1}$, $v = F_n$ | $u^2 - v^2 = (u-v)(u+v) = (F_{n+1} - F_n)(F_{n+1} + F_n) =$ recursive

$= F_{n-1} \cdot F_{n+2}$

similarly

$F_{2n+1} = u^2 + v^2 = F_{n+1}^2 + F_n^2$

(b) $a + b =$ perfect square.



(a, b) two prime [illegible] the sum of 2 perfect cubes

$a^3 + b^3$

(ii)

$a = u^2 - v^2$; $b = 2 \cdot u \cdot v$

$a + b = k^2$

[illegible]

$x = u^2 - v^2$

$z = u^2 + w^2$

$y = 2uv$

[illegible]

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$= k^2((u^2 - v^2)^2 - 2uv(u^2 - v^2) + 4u^2v^2) =$

$= k^2(u^4 + v^4 - 2u^3v + 2uv^3 + 2u^2v^2)$

$= (k(u^2 - uv))^2 + (k(v^2 + uv))^2$

002

$$3^x + y^2 = 5^z$$

$x = 2$ $z = 2$

$$2^2 + y^2 = 5^2$$

giving hint how to approach problem

$$y^2 = \underbrace{5^2 - 2^2}_{\text{even}}$$

so $\underline{y = \text{even}}$ $y^2 : 4 \cdot 8$

$$3^x + y^2 \equiv 5^z \equiv 1 \pmod 4$$

$$3^x + y^2 \equiv 1 \bmod 4 \equiv 3^x + 0 \bmod 4 = 1$$

$y = \text{even}$

$3^x = \text{odd}$ ~~3, 3·3, 27~~

$\equiv 1 \bmod 4$.

so x = odd, no. $x : 1$, $x = 3$

$27 \bmod 4 = \underline{3 \bmod 4}$

so $x = 2$ or $x = 4$

$$3^2 \equiv 1 \bmod 4$$

$x = \text{even} = 2n$

~~mod 3~~

$$3^x + y^2 \equiv 5^z \pmod 3$$

z = even = 2m

$$(3^n)^2 + y^2 = (5^m)^2 \quad ; y = \text{even}$$

$u^2 - v^2 = 3^n$

$5^m = u^2 + v^2$

$(u - v)(u + v) = 3^n$

so $u - v$ & $u + v$ are powers of 3

$u - v$ & $u + v$ odd

$$u = \frac{2u}{2} = \frac{u - v + u + v}{2}$$

$$v = \frac{2v}{2} = \frac{u + v - u + v}{2}$$

$u^2 + v^2 = 5^m$ ... $(5, 3)$

[illegible] $m - u = 1 \Rightarrow m^2 + u^2 = 3^n$ [illegible]

$m^2 + u^2 = 5^n$ [illegible]

$2 \cdot 5^n = 9^n + 1$

[illegible] $n = 1$ [illegible]

[illegible] $m^2 + u^2 = 5^n$ $n = 1$

[illegible] $7 = 5 - 8$ [illegible] $n = 1$ [illegible]

~~mod 25~~ [illegible]

$9^n \geq 1$ [illegible]

mod 25 - [illegible] 5^n [illegible]

[illegible] $n \geq 1$ [illegible] 5^2 [illegible] mod 25 [illegible]

[illegible]

$2 \cdot 5^n \equiv 0 \mod 25 \Rightarrow 9^n + 1 \equiv 0 \pmod{25}$

$-1 \mod 25$ [illegible]

-1 [illegible]

$9^m \equiv -1 \mod 25$

[illegible] $9^1 \equiv 9 \mod 25$

$9^2 \equiv 9 \cdot 9 \mod 25 = 81 \equiv 6 \mod 25$

$9^3 \equiv 9^2 \cdot 9 \equiv 6 \cdot 9 = 54 \equiv 4 \mod 25$

$\vdots$

$9^5 = 9^4 \cdot 9 \equiv 11 \cdot 9 = 99 \equiv -1 \mod 25$

$9^5 \equiv -1 \mod 25$

for $9^m \equiv -1 \mod 25$ to Hold

If m = even and [illegible] of 5 then

$m = 2k$ [illegible] $9^{2k} = (9^5)^{\frac{2k}{5}} \equiv (-1)^{\frac{2k}{5}} \equiv 1$

of 5 [illegible] $2k$ [illegible] 5 [illegible]

[illegible]

$\Rightarrow$ ~~n≡5 mod 10~~ n = odd multiple of 5

$$n = 5(2k+1)$$

$$2\cdot 5^n = (9^5)^{(2k+1)} + 1^{2k+1} = (9^5+1)\left((9^5)^{2k} - (9^5)^{2k-1}\cdot 1^1 + \dots + 1^{2k}\right)$$

[illegible]

$$(9^5)^{2k+1} + 1 = (9^5)^{2k+1} + 1^{2k+1}$$

[illegible] $2\cdot 5^n$ is not the multiple of $9^5+1 = 2\cdot 5^2\cdot 1181$

$$9\cdot 9\cdot 9\cdot 9\cdot 9 + 1 = 59050$$

[illegible]

[illegible] $2\cdot 5^n$ [illegible]

59050	2
29525	5
5905	5
1181	1181

$\Rightarrow 2\cdot 5^2\cdot 1181$

[illegible]

$$2\cdot 5^n = 2\cdot 5^2\cdot 1181$$ [illegible]

[illegible] n = [illegible]

$$2\cdot 5^2\cdot 5^{n-2}$$

[illegible] $5^{n-2} \neq 1181$ [illegible]

[illegible] $n=1$ [illegible]

$x=2;\ z=2$ [illegible]

(80 3) $2^x + 25^y$ = perfect square

fact: last digit of perfect square never can be 2, 3, 7, 8.

look perfect squares at mod 10 (last digit)

so, x = even. $x = 2x'$

$$(2^{x'})^2 + (5^y)^2 = z^2$$

pythagorean

(a) $5^y = u^2 - v^2$

(b) $2^{x'} = 2u\cdot v$

by looking at parity (odd, even) ... $5^y = 2uv$ ... $2^{x'} = u^2 - v^2$ ...

(a)

$$5^y = (u-v)(u+v)$$

5-ის multiple

$$u = \frac{(u+v)+(u-v)}{2}$$

$2u$ = 5-ის multiple only if u = 5's multiple

$$v = \frac{u+v+v-u}{2}$$

$2v$ = 5-ის multiple

$u - v$

(b) $2^{x'} = 2u\cdot v$

u და v 2-ის

$v = 1$ $\quad u = 2^{x'-1}$

(a) / (b) u და v

2-ის და 5-ის

$v = 1$; $\quad 2^{x'} = 2\cdot 1\cdot u \Rightarrow \boxed{u = 2^{x'-1}}$

(a)

$$5^y = 2^{2x'-2} - 1 = (2^{x'-1}-1)(2^{x'-1}+1)$$

differ by 2.

$5^y = (2^{x'-1}-1)(2^{x'-1}+1)$

$5^a \cdot 5^b =$

$5^a = 2^{x'-1}-1$ and $5^b = 2^{x'-1}+1$

but 5^a and 5^b ... differ by 2.

$b = a+1$

5^a and 5^{a+1} $= 5 \cdot 5^a$

$= 5^a(5-1) = 4 \cdot 5^a$

Solution

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$x^2 - (n^2+1)y^2 = 1$

$x^2 - Dy^2 = 1$

$D = n^2+1$

$x = 2n^2+1$ and $y = 2n$

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$1 + 2^{x^2} + 2^{y^2}$

$2^{x^2} = a^2$

and $2^{y^2} = 2a$

$1 + 2a + a^2 = (a+1)^2$

$1 + 2^{x^2} + 2^{y^2} = (a+1)^2$

$2^{x^2} = a^2$

$2 \cdot 2 \cdot 2 \cdot 2 = a \cdot a$

$\Rightarrow x^2 = 2m$

$\boxed{x = 2z}$

$a = 2^{2z^2} =$

$x^2 - 1 = n+1$

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$n, n+1, n+2$ each sum of perfect two squares.

$x^2 - 2y^2 = 1$

$n = x^2 - 1$

$n = y^2 + y^2$

and $n+2 = x^2 + 1^2$

$x^2 - 1 = 2y^2 = y^2 + y^2$

$n+1 = x^2 - 1 + 1 = x^2 + 0^2$

(108)

no n can be equal power 7 with exp > 1.

so $x^2 - 2 = 7^y$ can't happen.

$7^y \equiv (-1)^y \bmod 8$

if y = even, $7^y \equiv 1 \bmod 8$

if y = odd, $7^y \equiv -1 \bmod 8$

since $7 \equiv -1 \bmod 8$

squares in mod 8 can be only 0, 1, 4.

If $x^2 \equiv 0 \bmod 8$ if

then $x^2 - 2 \equiv -2 \bmod 8 \equiv 6 \bmod 8$

if $x^2 \equiv 1 \bmod 8$ | $x^2 - 2 \equiv -1 \equiv 7 \bmod 8$

$x^2 \equiv 4 \bmod 8$ | $\equiv 2 \bmod 8$

$7 \equiv -1 \equiv 7 \bmod 8$

$7^y \equiv 7 \bmod 8$

$\Rightarrow y$ = odd

$y = 2z+1$

$x^2 - 2 = 7^{2z+1} = 7 \cdot (7^{2z}) = 7(7^z)^2$

$x^2 = 7(7^z)^2 + 2 = 7^{2z+1} + 2$

$(7^{2z+1}) - 2$

$x^2 \cdot 7^{2z+1} = 7^{2z+1}(7^{2z+1} + 2)$

$= 7^{4z+2} + 2 \cdot 7^{2z+1}$

perfect square +1 both sides

$7^{2z+1} x^2 + 1 = 7^{4z+2} + 2 \cdot 7^{2z+1} + 1$

$= (7^{2z+1} + 1)^2$

$(7^{2z+1} + 1)^2 - 7^{2z+1} x^2 = 1$

$(7^{2z+1} + 1)^2 - 7 \cdot (7^z \cdot x)^2 = 1$

y = odd, by $7^{2z+1} = 7 \cdot (7^z)^2$

y = even $(7^z)^2$

pell equation

$a^2 - dy^2 = 1$

$x^2 - 7y^2 = 1$ $\quad x_1 = 8,\ y_1 = 3$ is solution.

pell equation. [illegible] general solution.

$$x_k + y_k\sqrt{7} = (8+3\sqrt{7})^k$$

$x = 7^{2z+1}+1$

$y = 7^z$

$$\iff 7^{2z+1}+1 = 8^k + \binom{k}{2}8^{k-2}\cdot 3^2\cdot 7 + \binom{k}{4}8^{k-4}3^4\cdot 7^2$$

$$7^z = \binom{k}{1}8^{k-1}\cdot 3 + \binom{k}{3}8^{k-3}\cdot 3^3\cdot 7 + \dots + \binom{k}{2m+1}7^m$$

$x>$ [illegible]

$$\binom{k}{2m+1}7^m = \frac{k(k-1)\dots(k-2m-1)}{1\cdot 2\cdot 3\cdot 4\cdots k}\cdot 7^m$$

7^{2z+1} [illegible]

7^z divides k.

$$8^k > 8^{7^z} > 7^{2z+1}$$

∞ [illegible] solutions pell [illegible]

[illegible]

$$(x+1)^3 - x^3 = y^2$$

expand

$$3x^2+3x+1 = y^2 \qquad (4-3)$$

$4y^2 = (2y)^2$ [illegible]

$$(2y)^2 - 3(2x+1)^2 = 1$$

$$u^2 - 3v^2 = 1 = \text{pell equation}$$

solutions

$$u_n \pm v_n\sqrt{3} = (2\pm\sqrt{3})^n. \quad \text{if } n = 2m+1 \text{ odd}$$

$$u+v\sqrt{3} = (2+\sqrt{3})^{2m+1} = (2+\sqrt{3})(2+\sqrt{3})^{2m} = (2+\sqrt{3})(7+4\sqrt{3})^m$$

$$2y_n \pm (2x_n+1)\sqrt{3} = (2\pm\sqrt{3})(7+4\sqrt{3})^n$$

x_n, y_n [illegible]

???

(10) $(x-y)^5 = x^3-y^3$. 1 solution is (n;n) $x-y=x-y$

let t be gcd of x,y

$u = \frac{x}{t}$; $v = \frac{y}{t}$ [illegible]

(u,v = relatively prime)

$t^5(u-v)^5 = t^3(u^3-v^3)$

$t^2(u-v)^{5} = u^3-v^3$

$t^2(u-v)^4 = \frac{u^3-v^3}{u-v} = u^2+uv+v^2 = (u-v)^2+3uv$

$(u-v)^2(t^2(u-v)^2-1) = 3uv$

$(u-v)^2$ divides $3uv$ [illegible]

u & v are relatively prime

~~$(u-v)(u-v) = 3 \cdot u \cdot v$ (u,v) rel. prime~~

~~$(u-v)(u-v) = \sqrt{3} \cdot u \cdot \sqrt{3} \cdot v$~~

[illegible] $3uv$

$u-v=1$.

$$\underbrace{(u-v)^2}_{\mathrm{I}} \cdot \underbrace{(t^2(u-v)^2-1)}_{\mathrm{II}} = 3uv$$

~~$(u-v)^2 = \frac{3uv}{t^2}\dots$~~

[illegible]

$t^2(u-v)^2-1 = \frac{3uv}{(u-v)^2}$

[illegible] $3(u-v)^2$ [illegible] $3uv$ [illegible] ; $(u-v)(u-v) = 3uv$

~~(u,v) rel-prime~~

$u-v = \frac{3uv}{u-v}$ | u,v rel-prime

$\boxed{u-v=1}$

$t^2 - 1 = 3w = 3u(u+1)$ →

$w - u = 1$.

~~$w = u+1$~~

$t^2 - 1 = 3u(u+1)$, $t^2 = \underbrace{3u(u+1)}_{} + 1$

[illegible] "3" [illegible]

$(u)^3$ [illegible] $(u+1)^3$ [illegible]

[illegible] olympiad [illegible]

[illegible]

$$t^2 = (u+1)^3 - u^3$$ [illegible]

$x+y = 20$

$z+t = -1$ [illegible]

(611) $x^3 + y^3 + z^3 + t^3 = 1999$

$x = 10;\ y = 10;\ z = -1;\ t = 0.$ is solution.

[illegible] → [illegible] simplicity

Find numbers n that $(10+n;\ 10-n;\ -\frac{1}{2}+u;\ -\frac{1}{2}-u)$ is solution.

[illegible]: $x = 10;\ y = 10$ [illegible]

$\frac{10+10}{2} = 10$ [illegible] $10 \pm n$

$z = -1;\ t = 0$

$\frac{-1+0}{2} = -\frac{1}{2}$ | $-\frac{1}{2} + u$, $-\frac{1}{2} - u$

u is half integer / [illegible] $\frac{1}{2} \pm u$, integer

[illegible]. $u = \frac{w}{2}$, [illegible] w = odd.

$$\boxed{(10+n)^3 + (10-n)^3} + \underline{\left(-\frac{1}{2}+u\right)^3 + \left(-\frac{1}{2}-u\right)^3} = 1999$$

$2000 + 60n^2$

$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$\boxed{a+b = -1}$

$ab = \frac{1}{4} - u^2$

$\boxed{u = \frac{w}{2}}$

$2000 + 60n^2$ ~~...~~ $= 1999$

$-\frac{1}{4} + 3u^2 = 1999$

~~$2000 + 60$~~ $2000 + n^2 - \frac{1}{4} + 3\left(\frac{w}{2}\right)^2 = 1999$

$(2000 + n^2) - \frac{1 + 3w^2}{4} = 1999$

$w^2 - 80n^2 = 1$

$(w_1; n_1) = (9; 1)$

$w_n + n_n\sqrt{80} = (w_1 + n_1\sqrt{80})^n$

$x^4 + y^4 + z^4 = (2002)^{w}$

$3^4 + 5^4 + 6^4 = 2002$

$2002 = \underbrace{3^4 + 5^4 + 6^4}_{\text{sum}}$ 4 [illegible]

consider $x_k = 3 \cdot 2002^k$ [illegible] x, y, z [illegible] 2002 [illegible]

x^4, y^4, z^4 [illegible] [illegible] 2002 [illegible]

2002 [illegible]

$x_k = 3 \cdot 2002^k$

$y_k = 5 \cdot 2002^k$

$z_k = 6 \cdot 2002^k$

$x_k = 3 \cdot 2002^k$ [illegible] $3 \cdot 2002^k$

$x_k^4 + y_k^4 + z_k^4 =$

$= (3^4 + 5^4 + 6^4)(2002)^{4k} =$

$= 2002^{4k+1}$

[illegible] x^4 [illegible] $3^4 + \dots = 2002$

2002 [illegible]

816 $\cancel{4^x + 4^y = 4^z}$

817 $x^2 + y^2 + z^2 + 3x + 3y + 3z + 5 = 0$

complete square

$x^2 + 3x = \left(x + \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$

[illegible] y, z [illegible]

$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 + \left(z + \frac{3}{2}\right)^2 = \frac{7}{4}$

$4\left(x + \frac{3}{2}\right)^2 + 4\left(y + \frac{3}{2}\right)^2 + \dots$

$4\left(x + \frac{3}{2}\right)^2 = (2x+3)^2$

$(2x+3)^2 + (2y+3)^2 + \dots = 7$

$u = 2k - 3$; $v = 2y + 3$ $w = 2z + 3$

$u^2 + v^2 + w^2 = 7$

$u = \frac{U}{T}$; $v = \frac{V}{T}$; $w = \frac{W}{T}$ [illegible]

$U^2 + V^2 + W^2 = 7T^2$ ვთქვათ U, V, W, T მთელი რიცხვებია

~~$1 + 1 + 7(W) + (T)$~~ [illegible]

$U^2 + V^2 + W^2 \equiv 7T^2 \pmod 4$

T-ც [illegible] mod-ები U^2, V^2, W^2 [illegible] square არის [illegible] mod 8, mod 4-ზე [illegible]

mod 4-ზე square არის მხოლოდ 0 ან 1

და U^2, V^2, W^2 [illegible] 0 ან 1

1) $7T^2$ უნდა იყოს 0, 1, 2, 3 mod 4-ზე

მაგრამ T^2 არის square არის (0 ან 1 [illegible])

ე.ი. $7T^2 \equiv 3T^2 \bmod 4$.

ე.ი. $T \equiv 0 \bmod 4$ ან $T =$ ~~[illegible]~~ even

$3T^2 \equiv 0 \bmod 4$ ან $3T^2 \equiv 3 \cdot 1 \bmod 4$

ე.ი. $T \equiv 1 \bmod 4$ ან

ან U, V, W, T ყველა odd ან ყველა even.

ე.ი. ყველა odd. (mod 8-ზე [illegible]) ვნახოთ.

mod 8 [illegible] x [illegible] x^2 [illegible]

ე.ი. $x =$ odd, $x^2 \equiv 1 \pmod 8$ [illegible]

$U^2 + V^2 + W^2 = 7T^2 \bmod 8$ $1 = 1$ ყველა

$3 \neq 7 \bmod 8$

ყველა = even.

ე.ი. $U = 2U'$; ... ე.ი. U, V, W, T არ [illegible] მთელი რიცხვები.

4:47 intro. n dimensional · n mutually perp. chords.

 so

Prove: Σ chord's lengths doesn't depend on direction.

so choose coordinates where chords are n orthogonal axis (perp. axis).
now assuming P fixed point not the origin

so

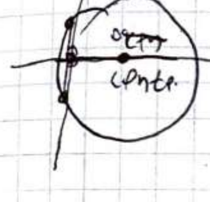
(a, b) $(0,0)$ (c,d)

~~other non-zero~~ endpoint.

~~endpoints~~

Endpoints of kth cord $(x_k, 0, 0, 0 \dots)$ and $(y_k, 0, 0, 0)$

Center $(0,0)$ ... cord ...



midpoint

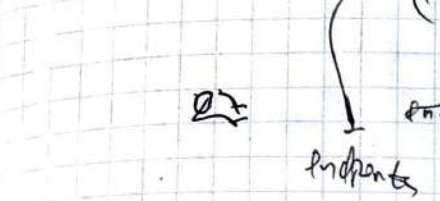
center

O-C coord $= \left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \dots \frac{x_n + y_n}{2} \right)$

x_k & y_k lie on same sphere

so $\left(x_k - \frac{x_k + y_k}{2} \right)^2 + \Sigma \left(\frac{x_j + y_j}{2} \right)^2 = R^2$

likewise $\left(y_k - \frac{x_k + y_k}{2} \right)^2 + \Sigma \left(\frac{x_i + y_i}{2} \right) = R^2$



x_1, y_1

edge

x_1, y_1

$x_2 y_2$

~~edge~~ edge's ~~coord~~ edge $= L = \sqrt{m}$ ...

(63) $(2n+1)$ dim. cube have integer $(2n+1)$ ... pythagorean theorem.

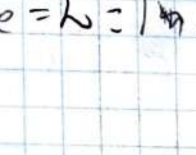
... n=1 ~~(n positive ... $n \geq 0$)~~

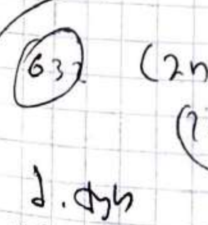
~~$2(1)+1 = 4$~~

n = 0

edge

square root of integer





cube's volume $= \left| \begin{matrix} \text{coordinates of} \\ \text{vertices} \end{matrix} \right| \Rightarrow$ integer

~~integer~~

$L = \sqrt{M}$

$\boxed{L^2 = M}$ = integer 100%

$\equiv$ ... L^{2n+1}

$\frac{L^{2n+1}}{L^2} = L$ = rat. number.

also L^2 = integer

so L is integer. so $\boxed{L^{2n+1} \equiv \text{integer}}$

(833) $\tau = (n, \ldots 2, 1)$

τ = same $(n, 1)$

A^n

$$\sum_{\sigma} \operatorname{sign}(\sigma)\, x_{\sigma(1)}\, x_{\tau(\sigma(2))} \cdots x_{\tau^{n-1}(\sigma(n))}$$

Prove this with the plane.

Leibniz form. of det

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(\prod_{i=1}^{n} a^{i}_{\sigma(i)} \right)$$

* $\tau(1) = n$

$X_{(1)} = n$

$X_{(2)} =$

$\tau(1) = n$

$\tau(2) = 1$

$$r_1 \to \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_n & x_1 & \cdots & x_{n-1} \\ \vdots & & & \\ x_2 & x_3 & \cdots & x_1 \end{vmatrix} = 0$$

det = 0

add rows to 1st

$$r_1 = (x_1 + x_2 + \cdots x_n)$$

$$\det(A) = (x_1 + x_2 + \cdots x_n) \cdot$$ fn of $x_1 x_2 \cdots x_n$

$\det(A) = 0$ two $= 0$.

(834) $\tau = (n, n-1 \cdots 2, 1)$

cycle permutation shifts at left position

$n = 4$ $T = (5, 4, 3, 2, 1)$

$T(1) = 5$

$\tau(\sigma(2))$? $T(2) = 1$ → $T = (4, 3, 2, 1, 5)$

$\sigma(2)$ $(2, 3, 1)$

$\sigma(2) = 3$

$$\sum \operatorname{sign}(\sigma)\, x_{\sigma(1)}\, x_{\tau(\sigma(2))} \cdots x_{\tau^{n-1}(\sigma(n))}$$

(834)

n-dim. cube centered at origin
edges parallel to coord. axis
with planes of vectors

$$\vec{a} = \left(\cos\frac{2\pi}{n}, \cos\frac{4\pi}{n} \dots \cos\frac{2n\pi}{n}\right)$$

$$\vec{b} = \left(\sin\frac{2\pi}{n}, \sin\frac{4\pi}{n} \dots \sin\frac{2n\pi}{n}\right)$$

is regular 2n-gon

(835)

R^n - on sphere $R = 1$
distance of 2 points $> \sqrt{2}$

unit sphere

$$S^{n-1} = \left\{ (x_1, x_2 \dots x_n) \in R^n \mid \sum_{k=1}^{n} x_k^2 = 1 \right\} = 1$$

any 2 points

$X = (x_1, x_2 \dots x_n)$ $\quad Y = (y_1, y_2 \dots y_n)$

$$d(X, Y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2\right)^{\frac{1}{2}} \geq \sqrt{2} - \delta$$

$\sqrt{1 + \delta + \dots}$

$\Sigma(x^2 - 2xy + y^2)$

$$d^2(X, Y) = \sum_{k=1}^{n} x_k^2 + \sum_{k=1}^{n} y_k^2 - 2\sum_{k=1}^{n} x_k y_k > 2$$

$$1 \Rightarrow \sum_{k=1}^{n} x_k y_k < 0$$

$A_1, A_2 \dots A_{m}$

A_m points of sphere $1 \Rightarrow A_1 = (-1, 0 \dots 0)$

$A_i = (x_1, x_2 \dots x_n)$ $\quad A_j = (y_1, y_2 \dots y_n)$

$d(A_i, A_j) > \sqrt{2}$

$\sum_{k=1}^{n} x_k y_k < 0$ $\quad x_1 y_1 = $ positive

$\Rightarrow \sum_{k=2}^{n} < 0$

~~$d(A_i, A_j)$~~ $d(A_1, A_i)$ Both ~~…~~ $> \sqrt{2}$
$d(A_1, A_j)$

$d(A_1, A_i) \Rightarrow \sum x_k x_k < 0$

$(-1) x_1 y_1 + x_2 y_2 \dots < 0$

A_1 - $= (-1, 0, 0, 0)$

$d(A_1, A_j)$ ~~$- \delta$~~

$\boxed{y_1 > 0}$ $\quad \boxed{x_1 > 0}$

$$\sum_{k=2}^{n} x_k y_k < 0$$

normalize — last $n-1$ coord. of [illegible] A_i

$$\text{by } x'_k = \frac{x_k}{\sqrt{\sum_{k=1}^{n-1} x_k^2}} \qquad k = 1, 2 \ldots (n-1)$$

$\longrightarrow x_k =$ [illegible] 3D/4D…

x_k

[illegible] normalized coordinates

[illegible]

$x-b_1$

[illegible] C [illegible]

[illegible]

[illegible]

$$x'_k = \frac{x_k}{\sqrt{x_1^2 + x_2^2 \ldots x_k^2}}$$

[illegible]

[illegible]

$A_1 \ldots A_m$

$A_1, A_2 \ldots A_{m_n}$

coordinates $(x_1', x_2' \ldots x_{n-1}')$

[illegible]

a coord [illegible]

$m_n \leq 1 + m_{n-1}$ — [illegible] B_i [illegible]

S^{n-2} [illegible] $\mathbb{R}^{n-1}$ [illegible]

$\boxed{m_1 = 2}$ $m_n \leq n+1$

[illegible]

$m_n = n+1$ [illegible] simplex [illegible] n dimensional [illegible] unit sphere [illegible].

$$\boxed{\sum_{k=1}^{n-1} x'^2_k = 1}$$

$B_2, B_3 \ldots B_m$ unit sphere [illegible].

Distance from cent. of sphere to hyperface of n-dimensional simplex is $\frac{1}{n}$. [illegible] $A_1 = (-1, 0, 0 \ldots 0)$

$$A_2 = \left(\frac{1}{n}; -c_1; 0; 0 \ldots\right)$$

$$A_3 = \left(\frac{1}{n}; \frac{1}{n-1} \cdot c_1; -c_2; 0 \ldots 0\right)$$

$$A_{n+1} = \left(\frac{1}{n}; \frac{1}{n-1} c_1 \ldots\right)$$

$$A_n = \left(\frac{1}{n}; \frac{1}{n-1} \cdot c_1, \frac{1}{n-2} \cdot c_2; c_3 \ldots\right)$$

$$c_k = \sqrt{\left(1+\frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)}$$ [illegible] distance $= 2$

c_k [illegible] n dim. simplex [illegible] distance [illegible]

$$d(A_i A_j) = \sqrt{(x_1 - x_2)^2 + (y_1 + y_2)^2 \ldots} = \sqrt{2}\sqrt{1 + \frac{1}{n}}$$

$$c_k = \sqrt{2}\sqrt{1 + \frac{1}{n}} \to \sqrt{2}$$

(4.1.) R m rectangular strips of width 2
prove $m \geq R$

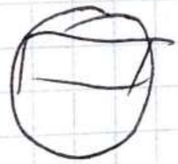
$$2\pi\int_a^b f(x)\cdot\sqrt{f'(x)^2+1}\,dx = 2\pi\int_a^b \sqrt{R^2-x^2}\,\frac{R}{\sqrt{R^2-x^2}}\,dx$$
$$= 2\pi\int_a^b R\,dx = 2\pi R d$$

Proof

Area cut from sphere of R radius by 2 parallel planes at distance d from each other
$= 2\pi R d$, if $\boxed{d=2}$ $4\pi R$

sphere's Area $4\pi R^2$ is covered by m surfaces.
each area $4\pi R$. $4\pi m R \geq 4\pi R^2$ so $m \geq R$.

3D
strips analogy Area ...
3D's surface



Q [illegible] $f: \{0,1,2\cdots\} \to \{0,1,2\cdots\}$

$$2f(m^2+n^2) = (f(m))^2 + (f(n))^2$$

$m=n=0$ [illegible] $2f(0^2+0^2) = f(0)^2 + f(0)^2$

$\Rightarrow f(0^2) = f(0)$

$f(0)=0$ [illegible] $f(0)=1$

[illegible] $f(0)=0$

$2f(1^2+0^2) = f(1)^2 + f(0)^2$

$2f(1) = f(1)^2 \Rightarrow f(1)=0$ [illegible] $f(1)=2$

[illegible] $f(1)=2$ $f(0)$ [illegible] 0 [illegible]

[illegible] $f(1)=f(0)=0$ [illegible]

~~$2f(2$~~

~~$2f(1^2+1^2) = 2f(1$~~

$2f(2) = 2f(1^2+1^2) = f(1)^2+f(1)^2 = 8$

[illegible] $f(2)=4$

$2\cdot f(4) = 2f(2^2+0^2) = f(2)^2+f(0)^2 = 16$

$2f(5) = 2f(2^2+1^2) = f(2)^2+f(1)^2 = 20$

[illegible] $2f(8) = 2f(2^2+2^2) = (f(2))^2 + (f(2))^2 = 32$

$f(4)=8$, $f(5)=$ ~~16~~ 10, $f(8)=16 \ldots$

$f(n)=2n$ ~~$n \le 10$~~ [illegible] $n=10$ [illegible]

~~100=~~ [illegible] $n=10$ [illegible]

$(f(5))^2 = (10)^2 = 100$

$(f(5))^2 = f(5)^2 + f(0)^2 =$ ~~$2f(100)$~~ $= 2\cdot f(5)^2 + f(0)^2 =$

$= 2\cdot f(5^2) = 2f(3^2+4^2) = f(3)^2 + f(4)^2 = f(3)^2 + 64$

[illegible] $f(0)=0$

$100 = f(3)^2 + 64$ [illegible]

$\Rightarrow f(3)=6$

[illegible]

$\Rightarrow 2f(9) = 2f(3^2+0^2) = f(3)^2+f(0)^2 = 36$

$2f(10) = 2f(3^2+1^2) = f(3)^2+f(1)^2 = 40$

~~400=~~

$(f(10))^2 = f(10)^2 + f(0)^2 =$

$= 2f(10^2) = 2f(6^2+8^2) = f(6)^2 + f(8)^2 =$

$400 = f(6)^2 + 256$

$f(6)=12$. $f(7)$ [illegible]

$\boxed{7^2+1^2 = 5^2+5^2}$

$f(7)^2 + f(1)^2 = f(5)^2 + f(5)^2$

$f(7)=14$

$n \le 10$ [illegible]

[illegible] integer [illegible]

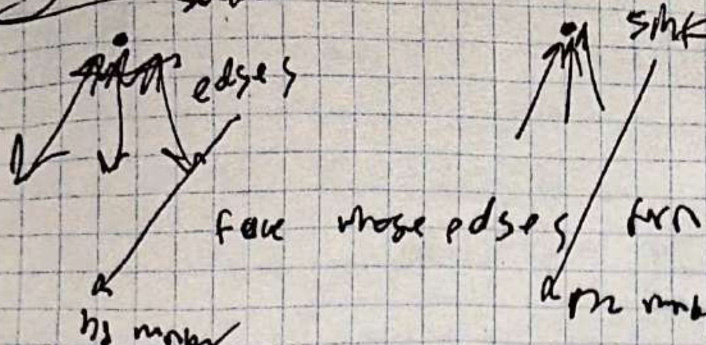
induction [illegible]

$n \ge 10$ [illegible]

[illegible] $f(5)$ [illegible] $f(10) \ldots$ [illegible]

[illegible] $f(5 \ldots)$ [illegible]

p 53



face whose edges form cycle ~~circulation~~ circulations

n_3 = number of circulations

n_4 = sum of all | of vertex is source, it has 0 incoming edges.

incoming edges ~~total~~ counted at vertices.

$E - V - n_1$ → number of sources. / → number of vertices (is not a source, it has min 1 incoming edge / ...)

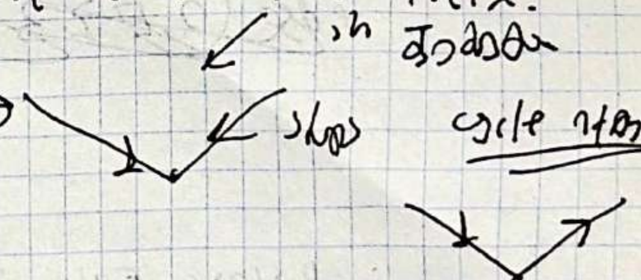
~~number of edges~~

Each non circulation has 2 edges towards same vertex

$$E = \underbrace{V - n_1}_{\text{incoming edge on}} + f - n_3$$

→ ... $E \geq V - n_1$ ... every vertex ... incoming lines ...

~~face~~ non-cycle face ... edge ..., ... two edges point toward same vertex.



... for non-circular ... face $f - c$ ...

$E - C$ overcounting ...

... E, ... vertex ... pointer ..., ... $\boxed{E = V - n_1}$

...

$E = V - n_1 + f - n_3$.

... sinks ... E ... incoming edges ~~...~~

$E = V - n_1 + f - n_3 - n_2 - n_4$

Euler formula ... $n_1 + n_2 + n_3 + n_4 = V - E + f = 2$

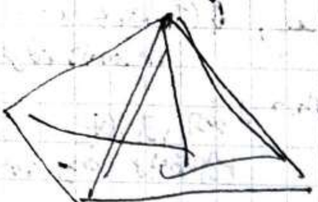
$n_4 \geq 0$ ...

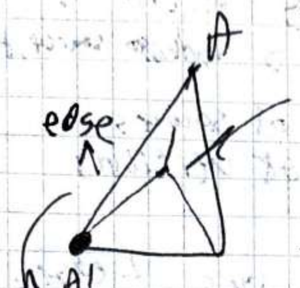
$n_1 + n_2 + n_3 \leq 2$

...

(382) (4) $V - E + F = 2$

~~[illegible]~~ [illegible] 3 face Polyhedron [illegible]





[illegible] 2 [illegible] 3 [illegible] face [illegible]

$2 \cdot 3 = 6$ (at least) edges.

Also, each edge has 2 vertices, [illegible]

~~[illegible]~~ ~~[illegible]~~

[illegible] 6 [illegible] overcount (A [illegible] A' [illegible])

[illegible] each edge $= 2E$. [illegible] edge $\geq 6V$

$\Rightarrow$ $2E \geq 6V$

[illegible] [illegible] E [illegible]. each face at least 3 edge.

~~edge $2E \geq 3F$~~

edge $\geq 3F$ face [illegible].

edge $= 2E$

edge $-3-$ [illegible] [illegible]

$2E \geq 3F$

[illegible] Euler-3 $V - E + F = 2$

$2 \leq V$ $2 \leq \frac{1}{3}E - E + \frac{2}{3}E = 0$

[illegible] [illegible].

[illegible] at least 6 edge [illegible] [illegible]

[illegible] at MAX 5 edge

(351) $V - E + F = 2$ $\quad 2\pi V - \Sigma = 4\pi$ [illegible]

$\cdot(2\pi)$

$2\pi V - 2\pi \cdot E + 2\pi F = 4\pi$

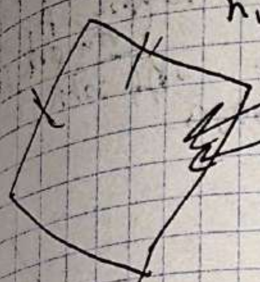
n_k = number of faces that are k-gons.

$F = n_3 + n_4 + n_5 + \dots$

$k \geq 3$ of course

each edge belongs to 2 faces.

[illegible] n_4 [illegible] 4-gon [illegible] $4n_4$ edge.



n_4 · 4-gon

n_4 = face. 4-gon [illegible]

1 face [illegible] 1 face → 4 edge

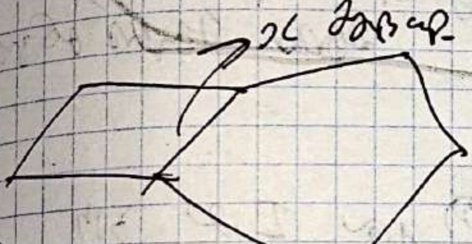
⇒ [illegible] k-gon [illegible]

$\Rightarrow 2E = 3n_3 + 4n_4 + 5n_5 + \dots$

here each edge belongs to 2 faces

[illegible] polyhedron [illegible]

1 edge belongs to 2 faces. [illegible]



~~$2\pi V - 2\pi \cdot E + 2$~~

[illegible] $2\pi V - 2\pi(n_3 + 2n_4 + 3n_5 + \dots) = 4\pi$

sum of all angles A of k-gon $= (k-2)\pi$ $= \Sigma$

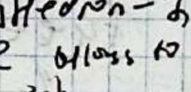
of faces $= \Sigma$ [illegible].

?

→ $k = 4$ [illegible] $\quad \pi \cdot (4-2)$

$= \pi \cdot 2$

(350) $F \geq 5$ and $E = \frac{3V}{2}$ 3 edges emerge from each vertex.



[illegible] face of polyhedron [illegible]

triangle [illegible] $E = \frac{3F}{2}$ $\quad 1 \to 2.$ $F - \frac{3F}{2} + F = 2,$

$F = 4$ [illegible]

show $F \geq 5$ [illegible].

[illegible] face-[illegible] show triangle

$(A_1 A_2 \dots A_n - B)$; $n \geq 4$ [illegible] face-[illegible]

[illegible] F [illegible] common-vertex-[illegible] polygon [illegible]

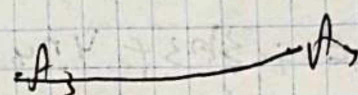
ე.ი. II-მ ქონდა [illegible] face-ი, რაც
~~საერთო I-ის ქვემოთ~~ A_1A_2-ს, (შიგ არის A_1A_2-ით [illegible])

$\boxed{\text{(ყველა edge = 2 [illegible] faces} = A_1A_2\text{, რაც face-ის [illegible] ყველა edge "შიგნით".)}}$

I არის შიგა face, რაც ~~[illegible]~~ შეიცავს ~~[illegible]~~
A_3A_4-ს.

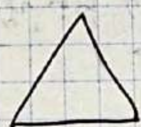
II [illegible]-მ [illegible] face ~~[illegible]~~ [illegible] A_3 ~~[illegible]~~
~~[illegible] A_4 [illegible] [illegible]~~
მაგრამ [illegible] we know that each edge contains 2 vertices, ანუ II-მ [illegible] A_3 და A_4 [illegible]
I [illegible] A_3-ის [illegible], I-მ A_4-ის [illegible]
[illegible], მაგრამ.

მ [illegible].

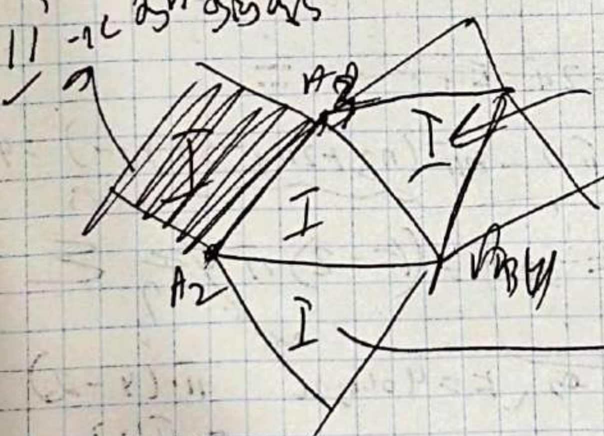


[illegible] [illegible] [illegible],
[illegible] [illegible] [illegible].

face-ებს



"II-ის [illegible]"



I [illegible] [illegible] I, II [illegible]
[illegible], რაც
A_3 [illegible] II-ის
[illegible].

[illegible].

(ვაჰ)

$$v - E + F = 2$$
$$E - F = n+2$$

$\boxed{v = n}$ ⇒ $E = 3n-6$
$E \leq 3n-6$.

every edge belongs to 2 faces,
& every face has 3 edge. $(2E = 3F)$

① $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ [illegible] $\binom{n}{k} = \binom{n}{n-k}$ [illegible]

[illegible]

(ק.1) $n \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} =$

$1 \le m \le n$

$= n \sum_{k=0}^{m-1} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right)$

$\binom{n}{m} = n \cdot \binom{n-1}{m}$ [illegible]

$= n \sum_{k=0}^{m-1} (-1)^k \binom{n-1}{k} - n \sum_{k=0}^{m-2} (-1)^k \binom{n-1}{k}$

$\binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1}$

$= n(-1)^{m-1} \binom{n-1}{m-1} = m(-1)^{m-1} \binom{n}{m}$

[illegible]

$\Rightarrow n\binom{n-1}{m-1} = m\binom{n}{m}$

[illegible]

~~$(x^{k+m}-1)(x^{k+m-1}-1)\cdots(x^{k+1}-1)$~~

~~[illegible]~~

$$\frac{(x^{k+m}-1)(x^{k+m-1}-1)\cdots(x^{k+1}-1)}{(x^m-1)(x^{m-1}-1)\cdots(x-1)} =$$

[illegible] $x=1$

$= \binom{k+m}{m}$ [illegible]

[illegible]

$\binom{k+m+1}{m} = \binom{k+m}{m} + \binom{k+m}{m-1}$

$$= \frac{(x^{k+m}-1)\cdots(x^{k+1}-1)}{(x^m-1)\cdots(x-1)} + x^{k+1}\frac{(x^{k+m}-1)\cdots(x^{k+2}-1)}{(x^{m-1}-1)\cdots(x-1)}$$

[illegible]

$m=0$ [illegible]

$\frac{x^{k+1}-1}{x-1} = x^k + x^{k-1} + \cdots + 1$

$(n+k)$ [illegible]

$$[x+y]_n = \sum_{k=0}^{n} \binom{n}{k} [x]_k \, [y]_{n-k}$$

$$[x]_n = x(x-1)\cdot \ldots (x-n+1)$$

[illegible] ?

$$\frac{d}{dt} t^{x} = x\, t^{x-1}$$

$$\frac{d^2}{dt^2} t^{x+1} = (x-1)(x)\, t^{x-2}$$

$$[x+y]_n = \underbrace{(x+y)(x+y-1) \cdots (x+y-n+1)}$$

$$\frac{d}{dt} t^{x+y} = (x+y)\, t^{x+y-1}$$

$$\left(\frac{d}{dt}\right)^n t^{x+y} = (x+y)(x+y-1)\cdots(x+y-n+1)\, t^{x+y-n}$$

$$\left(\frac{d}{dx}\right)^n t^{x+y} = [x+y]_n\, t^{x+y-n} \quad (1)$$

[illegible] $$\left(\frac{d}{dt}\right)^n (t^x \cdot t^y) = \sum_{k=0}^{n} \binom{n}{k} \left(\left(\frac{d}{dt}\right)^k t^x\right) \cdot \left(\frac{d}{dt}\right)^{n-k} t^y$$

leibnitz rule

$$\left(\frac{d}{dt}\right)^n (f(t)\cdot g(t)) = \sum_{k=0}^{n} \binom{n}{k} \left(\underbrace{\left(\frac{d}{dt}\right)^k f(t)}\right) \cdot \left(\frac{d}{dt}\right)^{n-k} g(t)$$

$$= \sum_{k=0}^{n} \binom{n}{k} [x]_k \, [y]_{n-k} \cdot t^{x+y-n} \quad (2)$$

(1) = (2) [illegible] $\left(\frac{d}{dt}\right)^n (t^{x+y})$ [illegible]

(807)

$$a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{4} = \underbrace{(1+\sqrt[3]{2}+\sqrt[3]{4})}^{n}$$

$$\sqrt[3]{2} \cdot (1+\sqrt[3]{2}+\sqrt[3]{4}) =$$

p.s.

$$\boxed{\sqrt[3]{4} = (\sqrt[3]{2})^2}$$

$$= \sqrt[3]{2} + \sqrt[3]{2}\cdot\sqrt[3]{2} + \sqrt[3]{8} \quad = 2$$

$$= 2 + \sqrt[3]{2} + \sqrt[3]{4}$$

n ...

$$(\sqrt[3]{2})^n (1+\sqrt[3]{2}+\sqrt[3]{4})^n = (2+\sqrt[3]{2}+\sqrt[3]{4})^n$$

$$(1+\sqrt[3]{2}+\sqrt[3]{4})^n = \frac{(2+\sqrt[3]{2}+\sqrt[3]{4})^n}{(\sqrt[3]{2})^n} \; ?$$

(803)

$$S_n = \binom{2n+1}{0} 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2}\, 3 + \dots \binom{2n+1}{2n} \cdot 3^n$$

is sum of 2 consecutive perfect squares.

$$S_n = \frac{1}{4}\left((2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1}\right)$$

$$S_n = (k-1)^2 + k^2$$

$$2k^2 - 2k + 1 - S_n = 0$$

$$D = 4(2S_n - 1) =$$

$a + b\sqrt{3}$ ... ?

$$= 2\left[(2+\sqrt{3})^{2n+1} + (2-\sqrt{3})^{2n+1} - 2\right)$$

$2+\sqrt{3}$ ... $2-\sqrt{3}$ are reciprocal.

$$(1+1\sqrt{3})^2 = 4+2\sqrt{3}$$

$(4 \pm 2\sqrt{3})$ are squares of $(1 \pm \sqrt{3})$

$$(2+\sqrt{3})(2-\sqrt{3}) = 1$$

$$= \left(\frac{(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}}{2^n}\right)^2$$

$$k = \frac{-b \pm \sqrt{4ac}}{}$$

$$k = \frac{1}{2} + \frac{(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}}{2^{n+2}} = \frac{1}{2} + \frac{1}{4}\left((1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n\right)$$

$$= \frac{1}{2} + (1+\sqrt{3})^{2}\,\underbrace{(1+\sqrt{3})^{2n}}_{4+2\sqrt{3} = 2(2+\sqrt{3})} + (1-\sqrt{3})(1-\sqrt{3})^{2n}$$

$2+\sqrt{3}$ და $2-\sqrt{3}$ ამოხსნებია n ფესვია.

ეს ორი ...

viettas-ით $x_1 \cdot x_2$ ...
$x_1 + x_2$ ...

$\lambda^2 - 4\lambda + 1 = 0$

რეკურსიული ...

recursive ...

$\lambda^2 - 4\lambda + 1 = 0$

$\lambda = 2+\sqrt{3}$, $\lambda = 2-\sqrt{3}$

$x_{n+1} = \lambda^2$

$x_n = \lambda$

$x_{n-1} = 1$

$x_{n+1} - 4x_n + x_{n-1} = 0$

$x_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$

A და B ...

$1+\sqrt{3}$ და $1-\sqrt{3}$

$(1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n = x_n$

$n=0$, $n=1$ ... $x_0 = 2$, $x_1 = 10$

$x_0 = A + B = 2$

$\sqrt{3} = A(2+\sqrt{3}) + B(2-\sqrt{3})$

induction ... $x_n : 2 - 8$

... $4-8$.

... $k = $ integer

$k = \frac{1}{2} + \frac{1}{4}(m)$ ← $m : 2-8$

$: 6$

$k = \frac{1}{4}\left(\frac{2+2\sqrt{3}}{4}\right)$

$: 4-8$

$: 2-8$

m ... $(m-6)$

$+2$

integer.

(2002) $a_1, a_2 \dots a_n$

$S_n = a_1 + a_2 + \dots a_n$

$$\sum_{k=0}^{n} \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} S_{n+1}$$

$a_2 = a_1 + d$

$a_3 = a_1 + 2d$

so $a_i = a_1 + (i-1)d$

$a_{k+1} = a_1 + kd$ & $a_{n-k+1} = a_1 + (n-k)d$

$a_{k+1} + a_{n-k+1} = 2a_1 + (k+n-k)d =$
$= 2a_1 + nd$

$$S_{n+1} = \frac{n+1}{2}(a_1 + a_{n+1}) = \frac{n+1}{2}(a_1 + a_1 + nd) = \frac{n+1}{2}(2a_1 + nd)$$

$$2S_{n+1} = (n+1)(2a_1 + nd)$$

$$\sum \binom{n}{k}(a_{k+1} + a_{n-k+1}) = \frac{2S_{n+1}}{n+1} \sum_{k=0}^{n} \binom{n}{k} \quad \frac{2^{n+1}}{n+1} S_{n+1}$$

$\sum \binom{n}{k} a_{k+1}$ & $\sum \binom{n}{}$

(3.01) $f_1 = f_2 = 1$

$f_{n+1} = f_n + f_{n-1}$.

Prove: $f_1\binom{n}{1} + f_2\binom{n}{2} + \dots f_n\binom{n}{n} = f_{2n}$

$$f_n = \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right]$$

f_n [illegible] $(1+x)^n$ [illegible]

$f_0 = 0$; $\sum f_i\binom{n}{i} = \frac{1}{\sqrt5}\left[\sum_{i=0}^{n}\binom{n}{i}\left(\frac{1+\sqrt5}{2}\right)^i - \sum\binom{n}{i}\left(\frac{1-\sqrt5}{2}\right)^i\right]$

$$= \frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^{2n} - \left(\frac{1-\sqrt5}{2}\right)^{2n}\right) = f_{2n}$$

$$= \frac{1}{\sqrt5}\left(\left(\frac{3+\sqrt5}{2}\right)^n - \left(\frac{3-\sqrt5}{2}\right)^n\right)$$

[illegible] $\frac{3\pm\sqrt5}{2} = \left(\frac{1\pm\sqrt5}{2}\right)^{2}$

f_n - [illegible] golden ratio-[illegible] [illegible]

$$\sum_{i=0}^{n}\binom{n}{i}x^i = (1+x)^n$$ [illegible] $\left(1+\frac{1+\sqrt5}{2}\right)^n = \left(\frac{3+\sqrt5}{2}\right)^n$

$x = \frac{1+\sqrt5}{2}$ [illegible]

(3.5) [illegible] k [illegible] [illegible] [illegible]

I_k [illegible] I_{k-1} [illegible]

$= (k(k-1)!)^2 = (k-1)!^2$

$I_k = \frac{4k-2}{k} I_{k-1}$ [illegible] integral-[illegible] [illegible]

$$\binom{2k}{k} = \frac{(2k)(2k-1)\cdots(2k-2)!}{k!\,k!} = \frac{4k-2}{k}\binom{2k-2}{k}$$

$(k!)^2$ $\cancel{2(\ )^2}$

(860) $P = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & & 1 \\ 0 & 0 & 1 & & 1 \\ 0 & & & \ddots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \end{pmatrix}$

A^k [illegible]

A

$n \times n$

[illegible] [illegible] [illegible]

[illegible]

$A^2 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & & n-1 \\ 0 & 0 & 1 & \cdots & n-2 \\ & & & & \vdots \\ 0 & 0 & 0 & & 1 \end{pmatrix} = \begin{pmatrix} \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} \\ 0 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{n-1}{1} \\ 0 & 0 & 0 & \cdots & \binom{1}{1} \end{pmatrix}$

$A^3 = \begin{pmatrix} \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \cdots & \binom{n+1}{2} \\ 0 & \binom{2}{2} & \binom{3}{2} & & \binom{n}{2} \\ & & & & \binom{n-2}{2} \end{pmatrix}$

$A^k = \Big(\cancel{\;} \quad \binom{k-1}{k-1} \binom{k}{k-1} \binom{k+1}{k-1} \cdots \binom{k+n-2}{k-1}$

$\binom{k-1}{k-1}$

[illegible]

[illegible]

$\binom{n}{n} + \binom{n+1}{n} + \cdots + \binom{n+p}{n} = \binom{n+p+1}{n+1}$

(639)

(a_n) ...

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$

If ... 3 ...

$$a_{n+1} + u \cdot a_n + v \cdot a_{n-1} = 0$$

$$G(x) = \frac{a_0 + (u a_0 + a_1)x}{1 + ux + vx^2}$$

If r_1 & r_2 are roots of characteristic equation $\lambda^2 - u\lambda + v = 0$

then

$$G(x) = \frac{a_0 + (u a_0 + a_1)x}{(1 - r_1 x)(1 - r_2 x)} = \frac{\alpha}{1 - r_1 x} + \frac{\beta}{1 - r_2 x} = \sum_{n=0}^{\infty} \underbrace{(\alpha r_1^n + \beta r_2^n)}_{\ldots} x^n$$

... term ...

$$a_n = \alpha r_1^n + \beta r_2^n$$

C_n ~~C_n~~ C_n ...

$$x_0 \cdot x_1 \cdot x_2 \cdots x_n$$

... multiplication ...

... x_k & x_{k+1} ...

... $\underbrace{x_0, \dots x_k}_{k+1 \text{ ... element}}$ C_k ...

$$\underline{x_{k+1}, \dots\dots x_n} \quad C_{n-k-1}$$

C_n ... ...

$$1 \Rightarrow C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots C_{n-1} C_0 \Rightarrow C_n = \sum_{k=0}^{n} C_k C_{n-k}$$

$$G(x) = \sum \underbrace{C_n x^n}_{\text{catalan numbers}} ; \quad (G(x))^2 = \left(\sum C_n x^n\right)^2 = \sum_{n \geq 0} \left(\sum_{k=0}^{n} C_k C_{n-k}\right) x^n$$

$n \geq 1$ ... catalan C_n

$$x \cdot (G(x))^2 = \sum_{n \geq 0} \sum^{n} (C_k C_{n-k}) x^{n+1} = \sum_{n \geq 1} \left(\sum^{n-1} C_k \cdot C_{n-1-k}\right) x$$

$G(x)$ ...

x ... x^{n+1} ... x & x^n ... C_n

$x \cdot U(x)^2 = \sum_{n \geq 1} c_n x^n$; with $c_0 = 1$, ...

~~$U(x) = c_0 + \sum_{n\geq 1}$~~

$U(x) = \sum c_n x^n = c_0 + \sum_{n \geq 1} c_n x^n = 1 + x(U(x))^2$

$U(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$; $\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n =$

$= \sum_{n=0}^{\infty} \dfrac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} (-4x)^n$

$\vdots$

$= 1 - 2 \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \dfrac{x^n}{n}$

$U(x) = \dfrac{1 - 1 + 2\sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{x^n}{n}}{2x} = \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \dfrac{x^{n-1}}{n}$, ...

$= \sum_{n=0}^{\infty} \binom{2n-2}{n-1} \dfrac{x^n}{n} = \sum_{n=0}^{\infty} \dfrac{1}{n+1} \binom{2n}{n} x^n$

Binomial coeffs $\binom{n}{k}$ are generated by $\underbrace{(x+1)^n}$, so generating ...

Ex: 1

$\sum_{j=0}^{n} \binom{n}{j} 2^{n-j} \binom{j}{\lfloor \frac{j}{2} \rfloor} = \binom{2n+1}{n}$

(we want coeff. of ... x^{-2} ... x^{-1} ... $j-3k=0$ ...)

$(x^{-1} + x)^j = \sum_{k=0}^{j} \binom{j}{k} x^{-k} x^{j-k} = \sum_{k=0}^{j} \binom{j}{k} x^{j-2k}$

$\cdot (x+1) \to x \cdot \underbrace{\sum_{k=0}^{j} \binom{j}{k} x^{j-2k}} + 1 \cdot \sum_{k=0}^{j} \binom{j}{k} x^{j-2k}$

$= \underbrace{\sum_{k=0}^{j} \binom{j}{k} x^{j-2k+1}}_{\text{term 1}} + \underbrace{\sum_{k=0}^{j} \binom{j}{k} x^{j-2k}}_{\text{term 2}}$

2 sum. ... term 1 ... term 2

the term 2 ... = 0 (... const)

$\binom{n}{k} = \binom{n}{n-k}$; $j - 2k + 1 = 0$; $k = \dfrac{j+1}{2}$; $j - 2k = 0$; $k = \dfrac{j}{2}$

binom. coeff ...

$\binom{j}{\frac{j+1}{2}} = \binom{j}{j - \frac{j+1}{2}} = \binom{j}{\frac{j-1}{2}}$

if j = even const term is $\binom{j}{\frac{j}{2}} = \lfloor \frac{j}{2} \rfloor$

if j = odd $\lfloor \frac{j}{2} \rfloor = \frac{j-1}{2}$

$\binom{j}{\frac{(j+1)}{2}}$ [illegible] $\binom{j}{\frac{(j-1)}{2}} \to ?$

[illegible] $\binom{j}{\lfloor \frac{j}{2} \rfloor}$ [illegible] $\lfloor \frac{j}{2} \rfloor = \frac{(j-1)}{2}$, [illegible] j = odd

[illegible] $\lfloor \frac{j}{2} \rfloor = \left(\frac{j}{2}\right)$, [illegible] j = even

$$\sum_{j=0}^{n} \binom{n}{j} 2^{n-j} (1+x)(x^{-1}+x)^j = (1+x) \sum_{j=0}^{n} \binom{n}{j} (x^{-1}+x)^j 2^{n-j} =$$

$$= (1+x)(2+x^{-1}+x)^n \quad \frac{\cdot x^n}{x^n}$$

[illegible] theorem [illegible] $A = x^{-1}+x$; $B = 2^n$

$(x^{-1})^n \cdot x^n = 1$

$$\sum \binom{n}{j} A^j B^{n-j} = (A+B)^n$$

$$= (1+x)(2x+1+x^2)^n \cdot \frac{1}{x^n} = \frac{1}{x^n}(1+x)^{2n+1}$$

x^n [illegible]

[illegible] $x^n (a+b+c)^n$

[illegible]

$B+B$ $A_1; A_2 \cdots A_n$ $B_1; B_2 \cdots B_n \ldots$

$A_1 = \emptyset$; $B_1 = \{0\}$; $A_{n+1} = \{x+1 \mid x \in B_n\}$;

$B_{n+1} = (A_n \cup B_n) \setminus A_n \cap B_n)$

which n is $B_n = \{0\}$.

A_n $a_n x = \sum_{a=1}^{\infty} c_a x^a$ | $b_n x$ $B_n = \sum_{b=1}^{\infty} c_b x^b$

$c_a = 1$ if $a \in A_n$

$c_a = 0$ if $a \notin A_n$

$a_1(x) = 0$

$b_1(x) = 1$

$b_{n+1}(x) = x\, b_n(x)$

ეს set, რომ non-negative ელემენტ[illegible] [illegible] [illegible]

$$f_S(x) = \sum_{s \in S} c_s \cdot x^s \quad \cdot \; c_s = 1 \text{ თუ } s \text{ [illegible]}$$

set-ში [illegible] $= 0$.

$S = \{0, 2, 3\}$ [illegible]

$$f_S(x) = \underline{x^0 + x^2 + x^3} = 1 + x^2 + x^3$$

x [illegible]

$$a_n(x) = \sum c_a x^a \; , \quad b_n = \sum_{b=1} c_b x$$

B_{n-1} [illegible]

$a_1(x)$

A_1 [illegible], $A_1 = \emptyset$ $\quad B(x) = \{0\}$.

$1 \Rightarrow a_1(x) = 0$, [illegible] set A-ს [illegible] "1" [illegible] $\emptyset$.

~~$B_1(x)$~~ $A_{n+1} = \{x+1, \; x \in B_n\}$

$\Rightarrow$ x [illegible] B_n-[illegible], [illegible] $x+1$ [illegible] A_{n+1}-[illegible].

B_n [illegible] $(c_b = 1)$ [illegible] $(b = x)$

$A_{n+1} =$ [illegible]

$$B_n = \sum x^x$$

$$x^{(x+1)} = x \cdot x^x \quad 1 \Rightarrow$$

$c_a \cdot x \rightarrow$ [illegible]

$$\text{I.} \quad a_{n+1}(x) = x \cdot B_n(x)$$

$$\text{II.} \quad B_{n+1} = (A_n \cup B_n) \setminus (A_n \cap B_n)$$

[illegible] $A_n(2; 3; 5)$

$B_n(4; 3; 7)$

$A_n \cup B_n = (2; 3; 4; 5; 7)$

$A_n \cap B_n = (3)$.

$(A_n \cup B_n) \setminus (A_n \cap B_n) = (2; 4; 5; 7)$

[illegible]

$B_{n+1} = a_n(x) + B_n(x) \pmod 2$

~~$B_1(x) = B_2(x) = 1$~~

$a_2(x) = x \cdot B_1(x) = x \cdot 1 = x$

$B_2(x) = \underbrace{a_1(x)}_{a_1(x) = 0} + \underbrace{B_1(x)}_{B_1(x) = 1} = 0 + 1 = 1 \bmod 2.$

[illegible]

$B_1 = B_2 = 1$

the second order recurrence

[illegible] $B_n(x) = p(x)^n$

$$p(x)^{n+1} = p(x)^n + x \cdot p(x)^{n-1}$$

$: p(x)^{n-1}$

$$p(x)^2 = p(x) + x$$

$p(x) = ($ ~~$p(x)(p(x)-1) = x$~~

B_n [illegible] $c_a = 1$ [illegible] B_n-[illegible].

[illegible]

$a_n(x)$-[illegible] $+ B_n(x)$ [illegible]

[illegible] (20.3)

c_a [illegible] $c_b = 1$ [illegible]

[illegible] mod 2-ს [illegible]

[illegible] $= 0$.

$P(x)^2 - P(x) - x = 0$, [illegible] mod 2 [illegible]

$2 \equiv 0 \mod (2)$

$-x \equiv$ [illegible] $\boxed{1 \equiv -1 \mod 2}$

$P(x)^2 - P(x) - x \equiv$

$(1 \cdot P(x))^2$ [illegible] $+ (-1 \cdot P(x)) + (-1 \cdot x)$

[illegible] mod 2 [illegible]

$\equiv P(x)^2 + P(x) + x \equiv 0$

$P(x)(P(x)+1) = x$

[illegible] mod 2

$1 \cdot P(x)^2 \equiv -1 (P(x))^2$ [illegible]

[illegible] mod 2 [illegible]

$-P(x) \equiv -1 - P(x) \equiv P(x) \mod 2$ [illegible]

solutions $P_1(x)$ [illegible] $P_2(x)$

$P_1(x) \cdot (P_1(x)+1) = x$

$P_2(x) = 1 + P_1(x)$

~~$P_1 + P_2$~~ $P_1(x) + P_2(x) = P_1(x) + (1 + P_1(x)) = 1$

$\cancel{P_1(x) = \sum_{a=1} d_a X^a}$

$P_1(x) =$
$d_1 x + d_2 x^2 \dots$ $\quad P_1(x) = \sum \underbrace{d_a}_{\text{coeff of } X^a} \cdot X^a$

$\boxed{P_1(x)(P_1(x)+1) \equiv X}$ [illegible] mod 2 [illegible]

$\cancel{X^1}$ [illegible]

$P_1(x)$

$P_1(x) + 1 = 1 + d_1 X + d_2 X^2 + \dots$

[illegible] constant term [illegible]

$P_1(x)$ [illegible] const term. [illegible] $= 0$
([illegible] X^1 [illegible])

[illegible]

$(d_1 X + \dots)(1 + d_1 X \dots)$

X [illegible]

$d_1 X \cdot 1 = d_1 \cdot X$.

[illegible] $\equiv X$ [illegible] $\boxed{d_1 = 1}$

X^2 [illegible]

[illegible] $\cancel{d_1 d_1 \cdot d_1 \cdot X^2}$ $d_1 X \cdot d_1 \cdot X = d_1^2 \cdot X^2$

$d_2 X^2 \cdot 1 = d_2 X^2$. [illegible] total coeff of X^2

$d_1^2 + d_2$. mod 2 [illegible] $d_1 = 1$

ג) $d_2 = 1$ ... x^3 coeff of

$$a_1 x \cdot d_2 x^2 + a_2 x^2 \cdot a_1 x + a_3 x^3 \cdot 1$$

$$a_1 = d_2 = 1 \pmod 2$$

$$x^3 = 1\cdot 1 + 1\cdot 1 + a_3 =$$
$$= 2 + a_3 = 0 \pmod 2$$

$$d_3 = 0.$$

pattern

$$x^4 ; x^5 \dots \quad d_{odd} = 0$$

$$d_{2k+1} = 0 \mod 2$$

$$d_1 = 1 ; d_2 = 1 ; d_3 = 0 ; d_4 = 1 ; d_5 = 0$$

$$P_1(x) = x + x^2 + x^4 + x^8 + \dots = \sum_{k=0} x^{2^k}$$

$$(1 + d_1 x + d_2 x^2)\cdot(a_1 x + a_2 x^2) + \dots$$

x^2

$$a_1 \cdot x \cdot d_1 x \cdot$$
$$a_2 \cdot x^2 \cdot 1$$

~~$P_1(x) + P_2(x) = P_1(x)^2 + P_2(x^2) = 1$~~

$$Q_n(x) = P_1(x)^n + (1 + P_1(x))^n = P_1(x)^n + P_1(x)^n + 1^n \equiv 1 \mod 2$$

$2P_1(x)$ mod 2 $= 0$

$$2 \cdot P_1(x) = 0$$

$$(1+P_1(x))^n = \sum_{j=0}^{n} \binom{n}{j} P_1(x)^j$$

~~$(1+P_1(x)) = \sum_{j=0} c_j P_1(x)^j$~~ ~~$c_j = 0$ or 1 when $\binom{n}{j}$ is even or odd~~

$n = 2$ mod 2

$$(1+P_1(x))^2 = 1 + \underbrace{2P_1(x)}_{\text{mod } 2 = 0} + P_1(x)^2$$

$$(1+P_1(x))^2 \equiv 1 + P_1(x)^2 \mod 2.$$

$n = 3$

$$(1+P_1(x))^3 = 1 + \underbrace{3P_1(x) + 3P_1(x)^2}_{\text{mod } 2} + P_1(x^3)$$

$$= 1 + \cancel{P_1(x)} \dots$$

$$(1+P_1(x))^n \equiv 1 + P_1(x)^n \mod 2$$

$d_3 = 0$, $d_{odd} = 0$

$$3P_1(x) = 1 P_1(x) \mod 2$$

$$P_1(x) + P_1(x^2) =$$

$$P_1(x) = x\left(\sum_{k=0}^{n} x^{2^k}\right)^n + \dots$$

$$P_1(x) + P_1(x)^2 = x$$

$$x + (x^2 + x^2) + (x^4 + x^4) + x^8 + \dots$$

$(x^2 + x^4 \dots)$

mod 2 $= 0$

$$s_n(x) = p_1(x^n) + p_2(x)^n = \left(\sum_{k=0}^{\infty} x^{2^k}\right)^n + \left(1 + \sum_{k=0}^{\infty} x^{2^k}\right)^n =$$

$b_n \equiv 1 \mod 2$

$$\left(\sum_{k=0}^{\infty} (x^{2^k}) + 1\right)^n \equiv \underbrace{\sum_{k=0}^{\infty} x^{2^k} + 1}_{} \mod 2$$

mod 2 ... $-1 \equiv 1$ mod 2 ...

... power of 2.

if n is not power of 2, ... $n = 2^i(2j+1)$

then smallest m for which $\binom{n}{m}$ is odd is

$2^j \leftarrow$

Continuation

$a_1, a_2 \ldots a_n$; $b_1 \ldots b_n \ldots$

$\binom{n}{2}$ sums $a_i + a_j$ are same as $\binom{n}{2}$ sums $b_i + b_j$

... n is not power of 2.

$f(x) = x^{a_1} + x^{a_2} + \ldots x^{a_n}$, $\underbrace{g(x)}_{} = x^{b_1} + x^{b_2} + \ldots$

$a_1, a_2 \ldots$ set (A) ; $b_1, b_2 \ldots$ set (B)

$$f(x)^2 = (x^{a_1} + x^{a_2} \ldots)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \underbrace{x^{a_i} \cdot x^{a_j}}_{= x^{a_i + a_j}} = f(x)^2$$

$\boxed{i = j \quad x^{2a_i}}$ ...

$$\sum_{i=1}^{n} x^{2a_i} = x^{2a_1} + x^{2a_2} + \dots x^{2a_n}$$

$$f(x)^2 - f(x^2) = \sum_{i\neq j} x^{\underbrace{a_i+a_j}}$$ without ~~$i=j$~~

then $a_i + a_j$ each pair (i,j) are counted twice (i,j) or (j,i).

$$\sum_{i\neq j} x^{a_i+a_j} = 2\sum_{i<j} x^{a_i+a_j}$$

$$= 2\sum_{i<j} x^{a_i+a_j} = 2\sum_{i<j} x^{b_i+b_j} = g^2(x) - g(x^2)$$

so

$$\underbrace{(f(x) - g(x))}_{h(x)}\underbrace{(f(x)+g(x))}_{p(x)} = f(x^2) - g(x)^2$$

if n isn't power of 2, $h = 0$

$h(1) = 0.$

if all derivatives of h is 0, $h \sim 0$

Taylor

$h(x)\cdot p(x) = h(x^2)$ $\xrightarrow{\text{derivative}}$ $h'(x)\cdot p(x) + h(x)\,p'(x) = 2x\,h'(x^2)$

$h'(1)\,p(1) = 2h'(1)$, $p(1) = f(1)+g(1) = 2n$ isn't power of 2.

$\Rightarrow h'(1) = 0$

$h^{(k)}(1)\,(p(1)) = 2^k\, h^{(k)}(1)$

$h^{(k)}(1) = 0$

so $h(x) = 0$ for all x.

so $f(x) = g(x)$ for all x.

so $A = B$.

878 $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0. = S(n)$

$$S(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cdots$$

$\cancel{\pm 1 ; \pm 2}$ [illegible] $\underbrace{\pm 1 ; \pm 2 \ldots}_{\text{[illegible]}}$

[illegible] $j \in (1, n)$

$\pm 1 \to (x^{1} + x^{-1})$

$\pm 2 \to (x^{2} + x^{-2})$

$\pm n \to (x^{n} + x^{-n})$

$$G_n(x) = \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) \cdots \left(x^n + \frac{1}{x^n}\right)$$

~~S(n) - [illegible]~~ [illegible]

[illegible]

[illegible]

$\cancel{x^{\epsilon_1 \cdot 1 + \epsilon_2 2 + \cdots \epsilon_n}}$

$x^{\pm 1 \pm 2 \pm 3 \cdots \pm n}$

$S(n) - [illegible]\ x^0 = 1$ = [illegible]

$S(n) = \pm 1 \pm 2 \pm \cdots = 0$

$$\left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) \cdots \left(x^n + \frac{1}{x^n}\right) = S(n) + \sum_{k \neq 0} c_k x^k$$

$x = e^{it}$ - [illegible]

$z + \frac{1}{z} = 2\cos t$ [illegible]

$\int_0^{2\pi}$

$$\int_0^{2\pi} (2\cos t)(2\cos 2t) \cdots (2\cos nt)\, dt = = 2\pi S_n + 0$$

[illegible]

875 p = odd, prime.

$\{1, 2 \cdots p-1\}$ [illegible] $\cdots p$.

for each number k in set.

either include k or not in subset

don't include? [illegible] x^0 [illegible] sum subset [illegible] $+0$

include? x^k (sum subset [illegible] $+k$)

[illegible] [illegible] [illegible] [illegible] $(1+x)(1+x^2 \cdots)$

$x^0 + x^k = 1 + x^k$ [illegible]

$G(x) = (1+x)(1+x^2) \cdots (1 + x^p)$

[illegible] z [illegible] p^{th} root of unity

$$\sum_{k=1}^{p} z^k = 0, \text{[illegible]} \quad z = 1$$

sum of coeff of x^n for n divisible by p.

$$\sum_{k=1}^{p} G(z^k) = p\,s(p)$$

$- (-1-\varepsilon)$, [illegible] p [illegible] $(-1)^p$

$$G(z^k) = \prod_{j=1}^{p} (1+\varepsilon^{kj}) = \prod_{j=1}^{p} (1+\varepsilon^{j}) = (-1)^p \prod_{j=1}^{p} ((-1)-\varepsilon^{j}) =$$

$$= (-1)^p \left((-1)^p - 1\right) = 2.$$

If $\varepsilon_1, \varepsilon_2 \dots \varepsilon_p$ are p^{th} root of unity

[illegible] $\quad x^p - 1 = \prod_{j=1}^{p} (x - \varepsilon_j)$

[illegible] $x = -1$

$$(-1)^p - 1 = \prod_{j=1}^{p} (-1 - \varepsilon_j)$$

[illegible] $p\,s(p) = 2^p + 2(p-1) =$

$= 2^p + 2p - 2$

$$\boxed{s(p) = \frac{2^p - 2}{p} + \dots}$$

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