

$$a_1 = 2 \quad a_2 = 5$$

$$a_{n+1} = (2-n^2)a_n + (2+n^2)a_{n-1}$$

Digir for which $a_p \cdot a_q = a_r$

$$a_p \cdot a_q = a_r$$

$$a_1 = 2 \quad | \quad \text{mod } 3 \text{ mod } 2 \text{ mod } n^2 \text{ mod } 3$$

$$a_2 = 5$$

$$a_1 = 2 \equiv 2 \pmod{3} \quad | \quad a_{n-2} \text{ mod } 3$$

$$a_2 = 5 \equiv 2 \pmod{3} \quad | \quad a_n \equiv 2 \pmod{3}$$

$a_{n+1} \pmod{2850}$

$$a_{n+1} = (2-n^2)a_n + (2+n^2)a_{n-1}$$

$$n^2 - 2 \rightarrow 3 \pmod{3-8}$$

$$\Rightarrow \cancel{n^2-1} \rightarrow 3 \pmod{3} \quad \text{if } n^2 \equiv 0 \pmod{3}$$

$$(2-n^2) \pmod{3} \equiv 2 \pmod{3}$$

$$2+n^2 \equiv 2 \pmod{3}$$

$$a_{n+1} = \cancel{2 \pmod{3}} \cdot (2+n^2)a_{n-1}$$

$$a_{n+1} = 2 \cdot 2 + 2 \cdot 2 = 4+4 = 8 \equiv 2 \pmod{3}$$

$$\text{if } \cancel{n^2 \equiv 1 \pmod{3}}$$

$$(2-n^2) \equiv 2 \pmod{3}$$

$$(2+n^2) \equiv 0 \pmod{3}$$

$$a_{n+1} = 0 \cdot a_n + 0 \cdot a_{n-1} \equiv 0 \cdot 2 + 0 \cdot 2 \equiv 0 \pmod{3}$$

$$\text{thus } a_{n+1} = 2 \pmod{3} \cdot 0 \pmod{3} \equiv 0 \pmod{3}.$$

$$\underbrace{a_p - a_p \cdot a_q}_{\text{mod } 3} \equiv 2 \cdot 2 \pmod{3} \equiv 4 \equiv 1 \pmod{3}$$

show induction
410P 673H

$$2 \pmod{3} \cdot 2 \pmod{3}$$

$$1 \cdot a_{n+1} \pmod{3}$$

$\pmod{3}$

$2 \pmod{3} \cdot 2 \pmod{3}$
15-15

(752) $n > 1000$ $1, 2, \dots, n$
 $2^n \pmod{n}$

~~2^n mod n~~

$$S_n \geq \frac{n}{2} (\log n - 1)$$

sum of $2^n \pmod{n}$ odd number $\geq S_n \geq \log n - 1$.

(753) n or first P_n
 $n = p_1 p_2 \dots p_k$

$$\epsilon = w \sqrt{\frac{2\pi}{n}} + i \sqrt{\frac{2\pi}{n}} |_{r=1, \theta} \in \text{set } n^{\text{th}} \text{ root of unity.}$$

equiangular polygon (you see more than stars)

side have $\sum_{i=0}^{k-1} \delta(i) \epsilon^i$ length
 vertices coordinates σ is permutation of $1, 2, 3, \dots, n$ direction

man vertices $\sum_{i=0}^{k-1} \delta(i) \epsilon^i = 0$ (closed polygon, $\Sigma 0$)

$n = ab$ i.e. a and b coprime

$$\text{If } \alpha_j + \beta k = \alpha_j' + \beta' k' \pmod{n}$$

$$\alpha(j - j') = \beta(k' - k) \pmod{n}$$

$\rightarrow j - j'$ divides β .

$k - k'$ divides α .

$\rightarrow j = j'$ and $k = k'$ \Rightarrow we listed all n^{th} roots of unity

$$\epsilon^j = e^{\frac{2\pi i}{n} (aj + bk)}$$

$$\epsilon^j = e^{\frac{2\pi i}{n} (aj + bk)}$$

$$j = 0, 1, \dots, n-1$$

$$k = 0, 1, \dots, n-1$$

$$A \cancel{\times} n^p - n \equiv 0 \pmod{p} \quad \text{since } n^p \equiv n \pmod{p}$$

~~for 2, 3, 5, 7~~

$$\textcircled{1} \quad 2^n + 3^n + 6^n \equiv 1 \pmod{p} \quad \text{for prime } p \\ \text{if } p=2 \quad n=2 \quad 2^n + 3^n + 6^n \equiv 1 \pmod{2}$$

$$1=2; 2=3 \quad \text{true} \quad n=2 \quad 2^n + 3^n + 6^n \equiv 1 \pmod{2}$$

$$\text{as } 2^n + 3^n + 6^n \equiv 1 \pmod{p} \quad \Rightarrow \quad p=3 \quad \text{true}$$

$$\begin{aligned} & 2^{p-1} + 3^{p-1} + 6^{p-1} \\ & 2^2 + 3^2 + 6^2 = 1 \\ & 2^2 + 3^2 + 6^2 \equiv 1 \pmod{p} \\ & (p-1) \text{ times} \\ & 2^{p-1} + 3^{p-1} + 6^{p-1} \\ & (p-1) = 2 \quad \text{true} \\ & p=3 \end{aligned}$$

now for $p > 3$

$$\begin{aligned} 2^{p-1} &\equiv 1 \pmod{p} \\ 3^{p-1} &\equiv 1 \pmod{p} \\ 6^{p-1} &\equiv 1 \pmod{p} \end{aligned}$$

and $6^{p-1} \equiv 1 \pmod{p}$ are ~~not~~ true for $p > 3$

$$\begin{aligned} p &= 7, 13 \\ p &= 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 149, 151, 157, 163, 173, 179, 181, 191, 197, 199, 209, 211, 223, 227, 233, 239, 241, 251, 257, 263, 271, 281, 287, 293, 299, 307, 311, 317, 323, 331, 347, 353, 359, 367, 373, 383, 397, 401, 407, 413, 427, 433, 443, 457, 463, 473, 487, 493, 503, 517, 523, 533, 547, 553, 563, 577, 583, 593, 607, 613, 623, 637, 643, 653, 667, 673, 683, 697, 703, 713, 727, 733, 743, 757, 763, 773, 787, 793, 803, 817, 823, 833, 847, 853, 863, 877, 883, 893, 907, 913, 923, 937, 943, 953, 967, 973, 983, 997, 1007, 1013, 1023, 1037, 1043, 1053, 1067, 1073, 1083, 1097, 1103, 1113, 1127, 1133, 1147, 1153, 1167, 1173, 1187, 1193, 1207, 1213, 1227, 1233, 1247, 1253, 1267, 1273, 1287, 1293, 1307, 1313, 1327, 1333, 1347, 1353, 1367, 1373, 1387, 1393, 1407, 1413, 1427, 1433, 1447, 1453, 1467, 1473, 1487, 1493, 1507, 1513, 1527, 1533, 1547, 1553, 1567, 1573, 1587, 1593, 1607, 1613, 1627, 1633, 1647, 1653, 1667, 1673, 1687, 1693, 1707, 1713, 1727, 1733, 1747, 1753, 1767, 1773, 1787, 1793, 1807, 1813, 1827, 1833, 1847, 1853, 1867, 1873, 1887, 1893, 1907, 1913, 1927, 1933, 1947, 1953, 1967, 1973, 1987, 1993, 2007, 2013, 2027, 2033, 2047, 2053, 2067, 2073, 2087, 2093, 2107, 2113, 2127, 2133, 2147, 2153, 2167, 2173, 2187, 2193, 2207, 2213, 2227, 2233, 2247, 2253, 2267, 2273, 2287, 2293, 2307, 2313, 2327, 2333, 2347, 2353, 2367, 2373, 2387, 2393, 2407, 2413, 2427, 2433, 2447, 2453, 2467, 2473, 2487, 2493, 2507, 2513, 2527, 2533, 2547, 2553, 2567, 2573, 2587, 2593, 2607, 2613, 2627, 2633, 2647, 2653, 2667, 2673, 2687, 2693, 2707, 2713, 2727, 2733, 2747, 2753, 2767, 2773, 2787, 2793, 2807, 2813, 2827, 2833, 2847, 2853, 2867, 2873, 2887, 2893, 2907, 2913, 2927, 2933, 2947, 2953, 2967, 2973, 2987, 2993, 3007, 3013, 3027, 3033, 3047, 3053, 3067, 3073, 3087, 3093, 3107, 3113, 3127, 3133, 3147, 3153, 3167, 3173, 3187, 3193, 3207, 3213, 3227, 3233, 3247, 3253, 3267, 3273, 3287, 3293, 3307, 3313, 3327, 3333, 3347, 3353, 3367, 3373, 3387, 3393, 3407, 3413, 3427, 3433, 3447, 3453, 3467, 3473, 3487, 3493, 3507, 3513, 3527, 3533, 3547, 3553, 3567, 3573, 3587, 3593, 3607, 3613, 3627, 3633, 3647, 3653, 3667, 3673, 3687, 3693, 3707, 3713, 3727, 3733, 3747, 3753, 3767, 3773, 3787, 3793, 3807, 3813, 3827, 3833, 3847, 3853, 3867, 3873, 3887, 3893, 3907, 3913, 3927, 3933, 3947, 3953, 3967, 3973, 3987, 3993, 4007, 4013, 4027, 4033, 4047, 4053, 4067, 4073, 4087, 4093, 4107, 4113, 4127, 4133, 4147, 4153, 4167, 4173, 4187, 4193, 4207, 4213, 4227, 4233, 4247, 4253, 4267, 4273, 4287, 4293, 4307, 4313, 4327, 4333, 4347, 4353, 4367, 4373, 4387, 4393, 4407, 4413, 4427, 4433, 4447, 4453, 4467, 4473, 4487, 4493, 4507, 4513, 4527, 4533, 4547, 4553, 4567, 4573, 4587, 4593, 4607, 4613, 4627, 4633, 4647, 4653, 4667, 4673, 4687, 4693, 4707, 4713, 4727, 4733, 4747, 4753, 4767, 4773, 4787, 4793, 4807, 4813, 4827, 4833, 4847, 4853, 4867, 4873, 4887, 4893, 4907, 4913, 4927, 4933, 4947, 4953, 4967, 4973, 4987, 4993, 5007, 5013, 5027, 5033, 5047, 5053, 5067, 5073, 5087, 5093, 5107, 5113, 5127, 5133, 5147, 5153, 5167, 5173, 5187, 5193, 5207, 5213, 5227, 5233, 5247, 5253, 5267, 5273, 5287, 5293, 5307, 5313, 5327, 5333, 5347, 5353, 5367, 5373, 5387, 5393, 5407, 5413, 5427, 5433, 5447, 5453, 5467, 5473, 5487, 5493, 5507, 5513, 5527, 5533, 5547, 5553, 5567, 5573, 5587, 5593, 5607, 5613, 5627, 5633, 5647, 5653, 5667, 5673, 5687, 5693, 5707, 5713, 5727, 5733, 5747, 5753, 5767, 5773, 5787, 5793, 5807, 5813, 5827, 5833, 5847, 5853, 5867, 5873, 5887, 5893, 5907, 5913, 5927, 5933, 5947, 5953, 5967, 5973, 5987, 5993, 6007, 6013, 6027, 6033, 6047, 6053, 6067, 6073, 6087, 6093, 6107, 6113, 6127, 6133, 6147, 6153, 6167, 6173, 6187, 6193, 6207, 6213, 6227, 6233, 6247, 6253, 6267, 6273, 6287, 6293, 6307, 6313, 6327, 6333, 6347, 6353, 6367, 6373, 6387, 6393, 6407, 6413, 6427, 6433, 6447, 6453, 6467, 6473, 6487, 6493, 6507, 6513, 6527, 6533, 6547, 6553, 6567, 6573, 6587, 6593, 6607, 6613, 6627, 6633, 6647, 6653, 6667, 6673, 6687, 6693, 6707, 6713, 6727, 6733, 6747, 6753, 6767, 6773, 6787, 6793, 6807, 6813, 6827, 6833, 6847, 6853, 6867, 6873, 6887, 6893, 6907, 6913, 6927, 6933, 6947, 6953, 6967, 6973, 6987, 6993, 7007, 7013, 7027, 7033, 7047, 7053, 7067, 7073, 7087, 7093, 7107, 7113, 7127, 7133, 7147, 7153, 7167, 7173, 7187, 7193, 7207, 7213, 7227, 7233, 7247, 7253, 7267, 7273, 7287, 7293, 7307, 7313, 7327, 7333, 7347, 7353, 7367, 7373, 7387, 7393, 7407, 7413, 7427, 7433, 7447, 7453, 7467, 7473, 7487, 7493, 7507, 7513, 7527, 7533, 7547, 7553, 7567, 7573, 7587, 7593, 7607, 7613, 7627, 7633, 7647, 7653, 7667, 7673, 7687, 7693, 7707, 7713, 7727, 7733, 7747, 7753, 7767, 7773, 7787, 7793, 7807, 7813, 7827, 7833, 7847, 7853, 7867, 7873, 7887, 7893, 7907, 7913, 7927, 7933, 7947, 7953, 7967, 7973, 7987, 7993, 8007, 8013, 8027, 8033, 8047, 8053, 8067, 8073, 8087, 8093, 8107, 8113, 8127, 8133, 8147, 8153, 8167, 8173, 8187, 8193, 8207, 8213, 8227, 8233, 8247, 8253, 8267, 8273, 8287, 8293, 8307, 8313, 8327, 8333, 8347, 8353, 8367, 8373, 8387, 8393, 8407, 8413, 8427, 8433, 8447, 8453, 8467, 8473, 8487, 8493, 8507, 8513, 8527, 8533, 8547, 8553, 8567, 8573, 8587, 8593, 8607, 8613, 8627, 8633, 8647, 8653, 8667, 8673, 8687, 8693, 8707, 8713, 8727, 8733, 8747, 8753, 8767, 8773, 8787, 8793, 8807, 8813, 8827, 8833, 8847, 8853, 8867, 8873, 8887, 8893, 8907, 8913, 8927, 8933, 8947, 8953, 8967, 8973, 8987, 8993, 9007, 9013, 9027, 9033, 9047, 9053, 9067, 9073, 9087, 9093, 9107, 9113, 9127, 9133, 9147, 9153, 9167, 9173, 9187, 9193, 9207, 9213, 9227, 9233, 9247, 9253, 9267, 9273, 9287, 9293, 9307, 9313, 9327, 9333, 9347, 9353, 9367, 9373, 9387, 9393, 9407, 9413, 9427, 9433, 9447, 9453, 9467, 9473, 9487, 9493, 9507, 9513, 9527, 9533, 9547, 9553, 9567, 9573, 9587, 9593, 9607, 9613, 9627, 9633, 9647, 9653, 9667, 9673, 9687, 9693, 9707, 9713, 9727, 9733, 9747, 9753, 9767, 9773, 9787, 9793, 9807, 9813, 9827, 9833, 9847, 9853, 9867, 9873, 9887, 9893, 9907, 9913, 9927, 9933, 9947, 9953, 9967, 9973, 9987, 9993, 10007, 10013, 10027, 10033, 10047, 10053, 10067, 10073, 10087, 10093, 10107, 10113, 10127, 10133, 10147, 10153, 10167, 10173, 10187, 10193, 10207, 10213, 10227, 10233, 10247, 10253, 10267, 10273, 10287, 10293, 10307, 10313, 10327, 10333, 10347, 10353, 10367, 10373, 10387, 10393, 10407, 10413, 10427, 10433, 10447, 10453, 10467, 10473, 10487, 10493, 10507, 10513, 10527, 10533, 10547, 10553, 10567, 10573, 10587, 10593, 10607, 10613, 10627, 10633, 10647, 10653, 10667, 10673, 10687, 10693, 10707, 10713, 10727, 10733, 10747, 10753, 10767, 10773, 10787, 10793, 10807, 10813, 10827, 10833, 10847, 10853, 10867, 10873, 10887, 10893, 10907, 10913, 10927, 10933, 10947, 10953, 10967, 10973, 10987, 10993, 11007, 11013, 11027, 11033, 11047, 11053, 11067, 11073, 11087, 11093, 11107, 11113, 11127, 11133, 11147, 11153, 11167, 11173, 11187, 11193, 11207, 11213, 11227, 11233, 11247, 11253, 11267, 11273, 11287, 11293, 11307, 11313, 11327, 11333, 11347, 11353, 11367, 11373, 11387, 11393, 11407, 11413, 11427, 11433, 11447, 11453, 11467, 11473, 11487, 11493, 11507, 11513, 11527, 11533, 11547, 11553, 11567, 11573, 11587, 11593, 11607, 11613, 11627, 11633, 11647, 11653, 11667, 11673, 11687, 11693, 11707, 11713, 11727, 11733, 11747, 11753, 11767, 11773, 11787, 11793, 11807, 11813, 11827, 11833, 11847, 11853, 11867, 11873, 11887, 11893, 11907, 11913, 11927, 11933, 11947, 11953, 11967, 11973, 11987, 11993, 12007, 12013, 12027, 12033, 12047, 12053, 12067, 12073, 12087, 12093, 12107, 12113, 12127, 12133, 12147, 12153, 12167, 12173, 12187, 12193, 12207, 12213, 12227, 12233, 12247, 12253, 12267, 12273, 12287, 12293, 12307, 12313, 12327, 12333, 12347, 12353, 12367, 12373, 12387, 12393, 12407, 12413, 12427, 12433, 12447, 12453, 12467, 12473, 12487, 12493, 12507, 12513, 12527, 12533, 12547, 12553, 12567, 12573, 12587, 12593, 12607, 12613, 12627, 12633, 12647, 12653, 12667, 12673, 12687, 12693, 12707, 12713, 12727, 12733, 12747, 12753, 12767, 12773, 12787, 12793, 12807, 12813, 12827, 12833, 12847, 12853, 12867, 12873, 12887, 12893, 12907, 12913, 12927, 12933, 12947, 12953, 12967, 12973, 12987, 12993, 13007, 13013, 13027, 13033, 13047, 13053, 13067, 13073, 13087, 13093, 13107, 13113, 13127, 13133, 13147, 13153, 13167, 13173, 13187, 13193, 13207, 13213, 13227, 13233, 13247, 13253, 13267, 13273, 13287, 13293, 13307, 13313, 13327, 13333, 13347, 13353, 13367, 13373, 13387, 13393, 13407, 13413, 13427, 13433, 13447, 13453, 13467, 13473, 13487, 13493, 13507, 13513, 13527, 13533, 13547, 13553, 13567, 13573, 13587, 13593, 13607, 13613, 13627, 13633, 13647, 13653, 13667, 13673, 13687, 13693, 13707, 13713, 13727, 13733, 13747, 13753, 13767, 13773, 13787, 13793, 13807, 13813, 13827, 13833, 13847, 13853, 13867, 13873, 13887, 13893, 13907, 13913, 13927, 13933, 13947, 13953, 13967, 13973, 13987, 13993, 14007, 14013, 14027, 14033, 14047, 14053, 14067, 14073, 14087, 14093, 14107, 14113, 14127, 14133, 14147, 14153, 14167, 14173, 14187, 14193, 14207, 14213, 14227, 14233, 14247, 14253, 14267, 14273, 14287, 14293, 14307, 14313, 14327, 14333, 14347, 14353, 14367, 14373, 14387, 14393, 14407, 14413, 14427, 14433, 14447, 14453, 14467, 14473, 14487, 14493, 14507, 14513, 14527, 14533, 14547, 14553, 14567, 14573, 14587, 14593, 14607, 14613, 14627, 14633, 14647, 14653, 14667, 14673, 14687, 14693, 14707, 14713, 14727, 14733, 14747, 14753, 14767, 14773, 14787, 14793, 14807, 14813, 14827, 14833, 14847, 14853, 14867, 14873, 14887, 14893, 14907, 14913, 14927, 14933, 14947, 14953, 14967, 14973, 14987, 14993, 15007, 15013, 15027, 15033, 15047, 15053, 15067, 15073, 15087, 15093, 15107, 15113, 15127, 15133, 15147, 15153, 15167, 15173, 15187, 15193, 15207, 15213, 15227, 15233, 15247, 15253, 15267, 15273, 15287, 15293, 15307, 15313, 15327, 15333, 15347, 15353, 15367, 15373, 15387, 15393, 15407, 15413, 15427, 15433, 15447, 15453, 15467, 15473, 15487, 15493, 15507, 15513, 15527, 15533, 15547, 15553, 15567, 15573, 15587, 15593, 15607, 15613, 15627, 15633, 15647, 15653, 15667, 15673, 15687, 15693, 15707, 15713, 15727, 15733, 15747, 15753, 15767, 15773, 15787, 15793, 15807, 15813, 15827, 15833, 15847, 15853, 15867, 15873, 15887, 15893, 15907, 15913, 15927, 15933, 15947, 15953, 15967, 15973, 15987, 15993, 16007, 16013, 16027, 16033, 16047, 16053, 16067, 16073, 16087, 16093, 16107, 16113, 16127, 16133, 16147, 16153, 16167, 16173, 16187, 16193, 16207, 16213, 16227, 16233, 16247, 16253, 16267, 16273, 16287, 16293, 16307, 16313, 16327, 16333, 16347, 16353, 16367, 16373, 16387, 16393, 16407, 16413, 16427, 16433, 16447, 16453, 16467, 16473, 16487, 16493, 16507, 16513, 16527, 16533, 16547, 16553, 16567, 16573, 16587, 16593, 16607, 16613, 16627, 16633, 16647, 16653, 16667, 16673, 16687, 16693, 16707, 16713, 16727, 16733, 16747, 16753, 16767, 16773, 16787, 16793$$

(56) n has $p-5$ digits \rightarrow
 $P \neq 2, 3, 5$, n is divisible by P

n is divisible by $\underbrace{11111}_{p-2} \cdot 10^5$

$$n = d + d \cdot 10 + d \cdot 100 + \dots + d \cdot 10^{p-2}$$

$$\frac{10^{p-1} - 1}{10 - 1} = \frac{10^{p-1} - 1}{9} = \frac{10^{p-2}}{9}$$

$$10^{p-2} \equiv 1 \pmod{P}$$

$$\frac{10^{p-2} - 1}{9} \equiv 0 \pmod{P}$$

$$n = \frac{d + \frac{P-1}{9} \cdot 10^{p-2}}{9}$$

(56) $P > 17 \Rightarrow P^{32} - 1$ is divisible by 16320
 $16320 = 2^6 \cdot 3 \cdot 5 \cdot 17$ factors

$$P^{98} - 2 = (P^9)^8 - 1 \quad \cancel{\text{is divisible by } P^8}$$

$\Rightarrow P^{32} - 1$ is divisible by P^8 and $P^{10} - 1$.

$$\frac{P^{32} - 1}{P^{2-2}} = P^{3-1} - 1 \Rightarrow P^{3-1} \equiv 1 \pmod{3}$$

$$P^{3-1} - 1 \equiv 0 \pmod{P}$$

$$P^{10} - 1 \equiv 0 \pmod{17}$$

$$P^{32} - 1 \equiv (P^2 - 1)(P^4 - 1) \dots (P^{16} - 1) \pmod{17}$$

$$16320 = 2^6 \cdot 3 \cdot 5 \cdot 17$$

2^6 factors.

$$P^{32} - 1 = (2m+1)^{32} - 1 = (2m)^{32} + \binom{32}{1}(2m)^{31} + \dots + \binom{32}{2}(2m)^2 + \binom{32}{1}(2m)$$

last 5 terms
possibly powers

P is odd $\Rightarrow 2m+1$

no other primes than 2

contains power of 2 ≥ 6 .

$$\frac{32}{27!5!} : 2-8) \rightarrow (2^5)^{(2^5)} + (2^5)(2^5)^4 + (2^5)(2^5)^3 +$$

$$(2^2)^{(2^2)} + (2^2)(2^2)^2 + (2^2)(2^2)$$

$$1 \rightarrow 5^n \text{ is divisible by } 2^6$$

(704) $n > 2$ ($\cancel{(x+1)^n - x^n = P}$) $x \neq 0 \Rightarrow x \neq 3^{\text{rd}}$
 P be smallest prime divisor of $n \Rightarrow n \mid P-1$ and $P-1$ are prime.
 $\Rightarrow (x+1)^n - x^n$ is divisible by P , but
 x and $x+1$ both cannot be or P is a prime.

$$(x+1)^{P-1} \equiv 1 \equiv x^{P-1} \pmod{P}$$

\Rightarrow ~~it shows~~ ~~gives~~ ~~implication~~

$$(x+1)^n \equiv x^n \pmod{P}$$

P is smallest prime divisor of $n \Rightarrow n \mid P-1$ and $P-1$ are prime

$$(x+1)^n = (x+1)^{P-1} \Rightarrow \gcd(n, P-1) = 1$$

$$a(P-1) + bn = 1 \text{ does not hold.}$$

Suppose
 \exists a, b such that
 $a(P-1) + bn = 1$
 $P \mid a(P-1)$
 $P \mid bn$

$$\begin{aligned} x+1 &= (x+1)^{a(P-1)+bn} \\ (x+1)^n &= (x+1)^{a(P-1)} \equiv X \pmod{P} \\ &\equiv X^{a(P-1)+bn} \end{aligned}$$

100%
~~so~~

$$\Rightarrow x+1 \equiv X \pmod{P}$$

$$\Rightarrow 0 \equiv 0 \pmod{P}$$

$\Rightarrow 0$ divides P

which is impossible, b/c

(705)

$2^n - 3$ congruent.

Infinite subsequence terms are pairwise ~~relatively prime~~ ~~pairwise relst prime~~

$$2^n - 3 = x_0, \text{ subsequence } (x_n)_n$$

3rdly, prime factors of $x_0, x_1, x_2, x_3, \dots, x_k, \dots$ are p_1, p_2, \dots, p_m

x_0, x_1, \dots are odd $\Rightarrow 2^n - 3$ is odd $\Rightarrow \text{odd } P \neq 2$

$$\left\{ \begin{array}{l} \text{define } x_k = 2^{\frac{(p_1-1)(p_2-1) \cdots (p_m-1)}{P_1-1}} - 3 \\ \text{so } x_k \text{ is odd prime} \\ \text{and } x_k \text{ is subsequence} \end{array} \right.$$

$P_1-1 \mid P_2-1 \mid \cdots \mid P_m-1$

from which $2 \mid (P_1-1)(P_2-1) \cdots (P_m-1)$

$$2^{P_1-1} \equiv 0 \pmod{P_1-1}$$

$x_k \in \text{one sequence.}$

x_{k-1}, x_k, \dots, x_m

$x_{k-1}, x_k, \dots, x_m</math$

760

(An)

$$\underbrace{x_{n+2}}_{\text{going to composite}} = \overline{5}x_n - 6x_{n-1}$$

Recurrence sch linear, $\Rightarrow x_n, x_{n-1}, x_{n-2}$ $\in \mathbb{Z}$

using PSLC

~~assume~~use look for $x_n = r^n$ & S18s

$$r^{n+2} = 5r^n - 6r^{n-2} \quad (\because r^{n-1})$$

$$r^2 = 5r - 6$$

$$r^2 - 5r + 6 = 0 \\ (r-2)(r-3) = 0, \quad r=2, r=3$$

~~$x_n = A \cdot 2^n + B \cdot 3^n$~~

~~$\lim_{n \rightarrow \infty} x_n = A \cdot 2^n + B \cdot 3^n \rightarrow \infty$~~

~~$x_n = A \cdot 2^n + B \cdot 3^n$~~

~~$n=0; x_0 = A + B$~~

~~$A = 3x_0 - x_1$~~

contradiction

~~$n=2; x_2 = A \cdot 2^2 + B \cdot 3^2 = 2A + 3B \quad B = x_1 - 2x_0$~~

assym, all terms are prime of sequence x_n

~~$x_n = \lim_{n \rightarrow \infty} A \cdot 2^n + B \cdot 3^n = \lim_{n \rightarrow \infty} 3^n \cdot \left(\frac{2^n}{3^n} \cdot A + B \right) = \infty$~~

2@3@3 steps

~~$x_n = A \cdot 2^n + B \cdot 3^n$ $\rightarrow \infty$ not possible.~~

~~$x_n = \infty$~~

as x_n are w-prime

~~$x_n + k(p-1) = A \cdot 2^{n+k(p-1)} + B \cdot 3^{n+k(p-1)}$~~

~~= A \cdot 2^n \cdot 2^{k(p-1)} + B \cdot 3^n \cdot 3^{k(p-1)}~~

~~$2^{k(p-1)} \equiv 1 \pmod p$~~

~~$2^{k(p-1)} \equiv 1 \pmod p \quad \text{as } 2 \not\equiv 0 \pmod p \quad \Rightarrow 3^{k(p-1)} \equiv 1 \pmod p$~~

~~$2^{k(p-1)} = 1 \pmod p$~~

$$\Rightarrow x_{n+k(p-1)} = A \cdot 2^n \cdot 1 + B \cdot 3^n \cdot 1 \pmod p$$

$$\Rightarrow x_{n+k(p-1)} = x_n \pmod p$$

Xn is prime, hence $p \mid x_n$

$$x_{n+k(p-1)} \equiv 0 \pmod p$$

$$x_{n+k(p-1)} \equiv 0 \pmod p$$

 ~~$x_{n+k(p-1)} \equiv 0 \pmod p$ $\Rightarrow x_{n+k(p-1)} \equiv 0 \pmod p$ is divisible by p~~
 ~~$x_n = p \Rightarrow x_{n+k(p-1)} = p \Rightarrow x_{n+k(p-1)} > p-2 \Rightarrow \text{not prime}$~~
 ~~$x_{n+k(p-1)} > p-2 \Rightarrow \text{not prime}$~~

For p prime
 $(p-1)! + 1$ divisor of p

$n! + 1$ and $(n+1)!$

I

If $(n+1)$ is composite, then each prime divisor of $(n+1)! < n+1$

~~$(n+1)!$~~
~~prime~~

$$\cancel{(n+1)!} = (n+1)(n) \dots$$

then $n+1$ (multiple of some $< n+1$)
 must divide $(n+1)!$

composite.

they also divide $n!$

but don't divide $n! + 1$

$(n! + 1) \neq (n+1)!$

gcd still 1 (so prime)

II If $(n+1)$ is prime then gcd can be

證明
方法
第二步

E.T. number n ; $\phi(n) = \text{number of prime numbers } p \leq n$ which are $\leq n$.

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right) \quad \phi(n) < n-1$$

if p_1 and n are prime

$$\phi(pn) = \phi(p) \phi(n)$$

n is prime & $\phi(n) = n-1$
 and if $n = p_1 p_2 \dots p_k$

$\phi(n) = (p_1-1)(p_2-1) \dots (p_k-1)$

then $\phi(n) = (p_1-1)(p_2-1) \dots (p_k-1)$

E.T. a is coprime to n

$$a \equiv 1 \pmod{n}$$

E.t. 1 n even; n^2-1 divides

$$\text{say } n = 2k-1, \text{ when } n = \text{odd}. \quad n^2-1 = (n-1)(n+1) = \frac{(n-1)}{2} \text{ odd}$$

$n!$ \rightarrow n is a prime relatively prime to n
 $n-2$ is a prime relatively prime to n
 \dots 2 is a prime relatively prime to n
 1 is a prime relatively prime to n

$\phi(n) < n-1 \Rightarrow \phi(n)$ divides $(n-1)!$
 All prime factors of $\phi(n)$ is in $(n-1)!$

$$(n-1)! \equiv 1 \pmod{\phi(n)}$$

$\phi(n)$ the n th positive integer less than n (\Rightarrow n is prime)
 prime number with n ($n-1$!)

$$\phi(n) \leq n-1$$

E.T. $a^{\phi(n)} \equiv 1 \pmod{n}$

$$2^{\phi(n)} \equiv 1 \pmod{n}$$

say n also divides $2^{(n-1)} - 1$

$$(n-1) \mid 2^{\phi(n)} - 1 \Rightarrow 2^{\phi(n)-1} \equiv 1 \pmod{n}$$

$n \leq n-2$ are relatively prime
 $\Rightarrow n(n-2)$ divides $2^{(n-1)} - 1$

$$(n-1) \mid 2^{\phi(n)} - 1 \Rightarrow 2^{\phi(n)-1} \equiv 1 \pmod{n}$$

Ex-2

$$S = \{ \phi^2 + a - d; \phi^3 + a^2 - d; \phi^4 + a^3 - d, \dots \}$$

(If n is product of elements of subset, then

Since elements of S are coprime to $a, \phi(a)$

$$S = \phi^n + a^{n-1} \quad \text{Product must be coprime to } n.$$

Since $\phi(n)$ is coprime to a
each s_i is coprime to a

a divides
both a^n and
 a^{n-1}

$$S = 0 + 0 - d \pmod{a}$$

Since S is
a d -cycle.
divide S .
 \Rightarrow write
 $a \equiv d \pmod{a}$

$$\text{E.T. } \phi(n) + 1 \quad \phi(n) + d \quad \phi(n) + d - 1 \equiv a + d - d \equiv d \pmod{n}$$

$\Rightarrow a^{\phi(n)+d} + a^{\phi(n)} - 1$ are coprime to n .

n , so can be added to S .

$\phi(n)$ is coprime to a
 $a \equiv d \pmod{n}$ if $n \pmod{a}$
are coprime.

Prop 5 $\sum_{k|n} \phi(k) = n$
if $k|n$

If k divides n \Rightarrow
 k is in form of $p^j m$, where $p \nmid m$

$$\sum_{j=0}^q \left(\sum_{m|q} \phi(p^j m) \right) = \sum_{j=0}^q \underbrace{\sum_{m|q} \phi(p^j) \phi(m)}_{\text{Property}} =$$

For prime p
 $\phi(p^j) = p^j - p^{j-1}$

$$= \left(d + \sum_{j=1}^{d-1} p^{j-1} (p-1) \right) q = p^d q = n$$

$n \neq 2$ (odd), $n \neq 6$

odd prime

$$\phi(n) \geq \sqrt{n}$$

If $n = 2^m$ then $n \geq 2$

$$\phi(n) = 2^m - 2^{m-1} = 2^{m-1} \geq \sqrt{2^m} = \sqrt{n}$$

If $n = p^m$, where $n \geq 2$ and p is odd prime

$$\phi(n) = p^{m-1} (p-1) \geq \sqrt{p^m} = \sqrt{n}$$

If $n = p^3$; $n \geq 2$ $p \geq 5$ $\phi(n) \geq \sqrt{2n}$

$$\boxed{1 \times 17 \times 2137} \quad \phi^2 + n^2 - 8n \cdot (3^{d-1} \cdot 5^{d-1} \cdot 8)^2 + (3^d \cdot 5^d)^2 = (3^{d-2} \cdot 5^{d-2} \cdot 8)^2$$

$$8^2 + 15^2 = 17^2 \quad (3^{d-1} \cdot 5^{d-1} \cdot 8)^2 + (3^{d-1} \cdot 5^{d-1} \cdot 3 \cdot 5)^2 = 8^{d-2} \cdot 5^{d-2} \cdot 17^2$$

$$32^2 + 37^2 = 41^2 \quad (10 \times \text{prime})$$

$(\phi(n))^2 + b^2$ graph with prime numbers ($3^d \cdot 5^d \approx 10000$)

$$\phi(15) = \phi(3 \cdot 5) = 2 \cdot 4 = 8$$

$\phi(pq) = (p-1)(q-1)$

prime numbers

$$n = 15 \times 11$$

$$(\phi(15))^2 + (15)^2 = 8^2 + 15^2 = 17^2, \quad \text{for } 3^d \cdot 5^d = 3^{d-1} \cdot 5^{d-1} \cdot (3-1) \cdot (5-1)$$

$$= 3^{d-1} \cdot 5^{d-1} \cdot 8. \quad \text{Ex-1}$$

$$\text{77} \rightarrow \psi(n) = m$$

$m = pmn$, $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{N}$: $m = 0$.

$$P(\lambda) = P_d^{d-2} \cdots P_k^{k-2} \underbrace{(P_2 - \lambda)}_{\lambda} \cdots \underbrace{(P_k - \lambda)}$$

$$\phi_{14} = \rho_1 \cdot \rho_2$$

Տարբերակ Փ2-ով
ուժի մեջ մտնելու
համար առաջ է մտնել

$$W_2 = \frac{d_2}{d_1 + d_2} \cdot P_1 + \frac{d_1}{d_1 + d_2} \cdot P_2$$

$$\text{If } \partial_1 = ? \quad J_0 = P^{\partial B}$$

$$q(p^{\theta}) = p^{\theta} \left(1 - \frac{1}{p}\right) \longrightarrow \phi(b) = p^{s-a} (p-s)$$

more than one factor of n
is P.

$$22wz?$$

$$\text{If } \phi = \omega \text{, then } \phi(n) = 2^{n-1} p^{2^n-1} (\phi - \omega)$$

~~• $\Delta V = P_1 - P_2$~~

$\phi = \perp$; $\vdash P = \exists (x_1 x_2) \varphi x_3$. So $\perp \vdash P$

2018 7.3-6

$$\text{अग्रिम रूप से } \beta, \eta \text{ का गुणात्मक } \phi(\eta) = p^{j-\delta} (p-j) \overset{\text{परिवर्तन}}{=} \frac{p^{j-\delta} \cdot 6}{p^{\text{पूरी}}} = \frac{7 \cdot 2 \cdot 3}{p^{\text{पूरी}}}$$

רֹאשֶׁת־הַבָּשָׂר : 2-3 > קְרֵבָה

vn : $\phi(y) \cdot \theta$

$$m = 2 \cdot 7^r n^{(m_3)l}$$

$$\begin{aligned} & \text{1. } \phi(4) = 0 \\ & \text{2. } \exists r \in \mathbb{N}_0 \text{ s.t. } \phi(r) = 1 \\ & \quad \xrightarrow{\text{from L}} \text{P(neg for all)} \\ & \quad \xrightarrow{\text{exists nR s.t.}} \text{M}_3 \text{ wrt L} \\ & \quad = \phi(b), \end{aligned}$$

$$a_m = \underbrace{a_n(a_n - k) + k}_{\text{1} \Rightarrow \quad a_n - k \text{ is divisible by } a_m \text{ now } j \leq m \leq n} = a_n \cdot a_{n-1} \cdot a_{n-2} \cdots a_1 + k$$

now

$$a_n \mid k$$

$$a_2 = 0 \cdot k + k \\ a_3 = a_2 \cdot a_1 + k$$

$(k+1, k)$ w.r.t. mers

$$\text{co-prime } \gcd(k+1, k) = 1$$

induction

$$a_m = a_{n-1} \cdot a_{n-2} \cdots a_1 + k$$

$d \rightarrow$ common divisor of a_n, k (yours t)

$$a_n - k = a_{n-1} \cdot a_{n-2} \cdots a_1$$

w.r.t. d, $a_{n-1}, a_{n-2}, \dots, a_1$ r.h.s.

in $a_{n-1} \cdot a_{n-2} \cdots a_1$ $k-1$ & a_{n-1}

$$a_n \mid a_1 \quad \boxed{d = 1} \quad Q$$

$$a_{n-1} \cdot a_{n-2} \cdots a_1 = \text{w.r.t.}$$

Proof \leq show prime & composite.

$$a+b+c+d+e+f \text{ polygonal number} \rightarrow \text{sum}$$

$$P(x) = abc + def \quad \text{so } abc + def + (a-d)(x-e)(x-f)$$

$$P(x) = (a+b+c+d+e+f)x^2 + (ab+bc+cd+de+ef+fd)x$$

$$+ abc + def = S \cdot x^2 + \text{w.r.t. } S \cdot x + \text{w.r.t. } S \cdot 1$$

$$= S(x^2 + \dots)$$

$$P(d) = (a+d)(b+d)(c+d) = \text{w.r.t. } S \mid P(x) \text{ also } n(t) \text{ for } S.$$

$a+d, b+d, c+d$

$y) \rightarrow S \rightarrow \text{so show } S \cdot \text{h.w.} = S$

$$S = ab + bc + cd + da + \dots + ac + bd + ca + db \quad \text{by now } S \text{ is composite.}$$

$$\text{7.8} \quad n; a, b > 0 \text{ Menge} \quad \text{gcd}(a, b) - 1$$

$$\underline{\text{gcd}(n^a - 1, n^b - 1) = n}$$

$$n^a \mid (n^a - 1) \quad n^b \mid (n^b - 1)$$

~~$\text{gcd}(a, b) \text{ ist der ggst von } n^a - 1 \text{ und } n^b - 1$~~ \Rightarrow
 ~~$n^a \mid d(a, b) \text{ und } n^b \mid d(a, b)$~~
~~it divides $\text{gcd}(n^a - 1, n^b - 1)$~~

$$ax - by = \text{gcd}(a, b)$$

$$ax - by = \text{gcd}(a, b)$$

$$\Rightarrow n^{ax - by} = n^{\text{gcd}(a, b)} =$$

$$\frac{n^{ax}}{n^{by}} = n^{\text{gcd}(a, b) - k - 1}$$
 ~~$n^{by} \mid n^{ax}$~~

greatest
common divisor

7.9

$$\text{7.9} \quad (q_n)_n \quad q_1 = k + j \quad q_{n+1} = q_n^2 - kq_n + k$$

anion are woprne

$$q_2 = (k + j)^2 - k(k + j) + k = (k + j) + k = q_1 + k$$

$$q_3 = q_2(q_2 - k) + k$$

$$q_3 = q_2(q_2 - k) + k = q_2q_1 + k$$

$$q_4 = q_3(q_3 - k) + k = q_3q_2q_1 + k$$

$$\text{thus } q_n = q_{n-1}q_{n-2} \dots q_2 + k \cdot q_1^{\text{gcd}}$$

Chinese remainder theorem

Mod 2... Mod pairwise prime so $(\mathbb{Z}/M_1\mathbb{Z}) \times (\mathbb{Z}/M_2\mathbb{Z})$ works
 $\Rightarrow \text{if } \gcd(M_i, M_j) = 1 \text{ then } (\mathbb{Z}/M_i\mathbb{Z}) \times (\mathbb{Z}/M_j\mathbb{Z})$

as $M_i \mid a_i \pmod{M_j}$ if $i \neq j$

$$x \equiv a_1 \pmod{M_2} \quad ; \quad x \equiv a_2 \pmod{M_1}$$

$$x \equiv a_2 \pmod{M_2}$$

solutions to $x \equiv a_2 \pmod{M_2}$ (so $x \equiv a_2 \pmod{M_1}$)

are congruent mod along $M = M_1 M_2 \dots M_k$

(2x1)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 8^5$$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 2^5$$

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 = t^3$$

have 00 pairs

solutions

\Rightarrow so x_1, x_2, x_3, x_4 have some symbols - x_1, x_2, x_3, x_4

$$\text{for } a = 1^2 + 2^2 + 3^2 + 4^2; b = 1^3 + 2^3 + 3^3 + 4^3$$

$$c = 1^5 + 2^5 + 3^5 + 4^5 \quad | \quad \text{look for solutions in form } x_i = a^n b^m c^p$$

$$x_1 = 1 a^n b^m c^p$$

$$x_2 = 2 a^n b^m c^p$$

$$x_3 = 3 a^n b^m c^p$$

$$x_4 = 4 a^n b^m c^p$$

$$2^x + 3 = \pi^3$$

? $\phi(7) = 6$ | $\phi(13) = 12$ here
 $\phi(9) = 6$ no of distinct cube residues
is $\frac{12}{\gcd(3, 12)} = \frac{12}{3} = 4$

$$(7^1) \phi(1) t_1^n + \phi(2) \left[\frac{n}{2} \right] + \dots = \frac{n(n+1)}{2}$$

1) Induction

2) Gauss's identity

$$\frac{n(n+1)}{2} = \sum_{m=1}^n m = \sum_{m=1}^n \sum_{k|m} \phi(k) = \sum_{k=1}^n \phi(k) \cdot \sum_{\substack{m=1 \\ k|m}}^n 1$$

$$\left[\sum_{k|n} \phi(k) = n \right] \times \text{LHS}$$

$\sum_{k|n} \phi(k)$ (for all k) for all positive n
 $\phi(k)$ covers for all positive n over all the positive divisors
of n .

$$\sum_{k|n} \phi(k) = \phi(n) \cdot \left[\frac{n}{\phi(n)} \right], \text{ here since } \sum_{k|n} \text{ is sum of divisors of } n, \text{ and } \phi(n) \text{ is number of divisors of } n.$$

$$(1) \quad \phi \sum_{k|n} \phi(k) \text{ will give}$$

$$\phi(1) + \phi(2) + \dots + \phi\left(\frac{n}{\phi(n)}\right)$$

$\phi(d)$ if $d \mid n$. When
 $d \nmid n$ then $\phi(d) = 0$.

$$(2) \quad a_1, a_1 + ad_1, a_1 + 2ad_1, \dots, a_1 + kd_1 \cdot \phi(d) = d(\text{mod } d)$$

assume $a_1 \equiv 1 \pmod{d}$
 $a_1 + kd_1 \equiv 1 + kd_1 \equiv 1 \pmod{d}$

Part
Prime factors of

$a_1 + kd_1$
is same

$a_1 \not\equiv 1 \pmod{d}$

$$a_1^{k\phi(d)} = 1 + \frac{a_1 n_k \cdot d}{d} \pmod{d}$$

$$d^{k\phi(d)} = a_1 + n_k \cdot d \pmod{d}$$

(795)

$$ad - bc \neq \pm 1$$

$$\begin{cases} ax + by = m \\ cx + dy = n \end{cases}$$

If $a=0$ $by = m$ always has solutions $\Rightarrow b \neq 0 \Rightarrow b = \pm 1$ $mx = \pm m$ - one drop hence m

$$ax + by = m$$

in m ~~is~~ ~~not~~ ~~not~~solution possible $\Rightarrow m$ not \Rightarrow notdivide all m .now ~~not~~ ~~not~~ ~~not~~ m not \Rightarrow m not \Rightarrow m

~~$cx = n - dm$~~

now $n - dm$ integer solutions

$$c = \pm 1$$

$$\Rightarrow ad - bc = bc = \pm 1$$

and $a=0$.argument if b or c odd \Rightarrow LSF.If $a, b, c, d \neq 0$

$$\text{let } \Delta = ad - bc$$

$$\text{if } \Delta = 0, m \left| \frac{a}{c} = \frac{b}{d} = \lambda \right.$$

$$m = ax + by = \lambda(cx + dy) =$$

$$\text{if } \Delta \neq 0 \quad \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} m \\ n \end{array} \right); \text{ so } x, y \text{ integers}$$

$$m = x \cdot n, n \text{ min. no. of } \Rightarrow \Delta \neq 0$$

case

$$\text{if } \Delta \neq 0 \quad \rightarrow x = \frac{dm - bn}{\Delta} \quad \text{so } g = \frac{an - cm}{\Delta}$$

$$\text{det } \Delta \text{ is integer } \frac{an - cm}{\Delta} \text{ is integer}$$

but m, n , d, b , c, a

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) (m, n) \text{ is a gp. in } \mathbb{Z} \text{ from } \frac{d}{\Delta} \text{ to } \frac{a}{\Delta}$$

$$(m, n) = (\frac{d}{\Delta}, \frac{a}{\Delta}) \text{ since } x_1 = \frac{d}{\Delta}, y_1 = -\frac{c}{\Delta} \text{ soln.}$$

$$(m, n) = (0, 1) \text{ since } ; x_2 = -\frac{a}{\Delta}, y_2 = \frac{c}{\Delta}$$

$$\Rightarrow x_1 y_2 - x_2 y_1 = \frac{ad - bc}{\Delta^2} = \pm 1$$

$$x_1 y_2 + x_2 y_1 = \pm 1$$

$$\text{gcd}(a, b) = d \quad \left\{ \begin{array}{l} \text{no solution} \\ ax - by = 0 \\ cx - dy = 0 \end{array} \right. \Rightarrow a \cdot u - b \cdot v = 1$$

$$u, v \text{ no solns.}$$

$$u = g \text{ mod } b$$

$$v = g \text{ mod } a$$

$$u = t \text{ mod } c$$

$$v = z \text{ mod } d$$

$$\left(\begin{array}{cc} a & -z \\ b & -t \end{array} \right) \left(\begin{array}{c} x \\ y \end{array} \right) = \left(\begin{array}{c} c \\ -d \end{array} \right)$$

$$\det = d(-t) - (-z)b = -at + bz = -(at - bz) = -1$$

$$\text{rule-ny} \quad x = \frac{\det(c \ -z)}{-1} \quad y = \frac{\det(a \ -d)}{-1} \quad \left| \begin{array}{l} x = at + zd \\ y = ad + bc \end{array} \right. \text{ solns.}$$

$$(at + zd, ad + bc) \text{ gcd } (at + zd, ad + bc) \text{ solns.}$$

on $ax - by = c$ has solutions only if

$\gcd(a, b)$ divides c

If (x_0, y_0) are solutions, all others are of form

$$x = x_0 + \frac{b}{\gcd(a, b)} t$$

$$y = y_0 + \frac{a}{\gcd(a, b)} t.$$

$$ax - by = 1$$

$$\det \begin{pmatrix} a & y \\ b & x \end{pmatrix} = ax - by = 1$$

If $\det = \pm 1$ for 2×2 matrix, they form S.L. -> which are generated by

$$\begin{pmatrix} a & y \\ b & x \end{pmatrix} = ST^{\det} ST^{\det} S \dots ST^{\det} S$$

matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on Lobachevskii plane by

$$z \rightarrow \frac{az+b}{cz+d} \text{ with } z \in \mathbb{H}^2$$

$$\det \begin{pmatrix} 1 & 2 & 5 \\ 2 & 3 \end{pmatrix} = 12 \cdot 3 - 7 \cdot 5 = 1 \Rightarrow$$

~~12~~ $x + 2y = 1$ $(3, 5)$ is solution.
 $S \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a \\ p \end{pmatrix} + \begin{pmatrix} b \\ q \end{pmatrix}$

$$\frac{1}{7} = 1 + \cancel{\frac{5}{7}} \text{ frak. part. } \frac{1}{5} = 1 + \frac{-5}{5} = 1 + \frac{2}{5} \Rightarrow$$

$$\frac{5}{2} = 2 + \frac{5-2 \cdot 2}{2} = 2 + \frac{1}{2}$$

~~$\frac{1}{7} = 1 + \frac{5}{7}$~~

$$2. 1 \Rightarrow \frac{1}{5} = 1 + \frac{2}{5} \Rightarrow \frac{5}{7} = 1 + \frac{2}{5}$$

$$\frac{12}{7} = 1 + \frac{5}{7} = 1 + \frac{1}{1 + \frac{2}{5}} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}} = T \cdot S \cdot T \cdot S \cdot T^2 \cdot S \cdot T^2$$

dark arrow 2. T

$$2 \Rightarrow \text{dark arrow } 2 + \frac{1}{2} \mid S(2) = \frac{1}{2} \text{ so } T^2 \left(\frac{1}{2} \right) = \frac{1}{2} + 2 = \frac{5}{2}$$

$$3 \Rightarrow S\left(\frac{5}{2}\right) = \frac{2}{5} \mid T\left(\frac{2}{5}\right) = \frac{2}{5} + 1 = \frac{7}{5}$$

$$S\left(\frac{7}{5}\right) = \frac{5}{7} \mid T\left(\frac{5}{7}\right) = \frac{5}{7} + 1 = \frac{12}{7}$$

$$ax + by = c \cdot \text{num} \quad (\text{GCD}) \text{ ist werte.}$$

$a, b - a - b$ ist largest positive integer c

7.9.7

~~$7x + 11y = 60$~~

| | | |
|---|---|---|
| : | : | : |
| : | : | : |
| : | : | : |
| : | : | : |

num 1 syntax. \rightarrow 8.

+ 7

or num 2 syntax. \rightarrow 12

+ 8.

\Rightarrow a X times 8, b times 12. $7x + 11y + 8$, c & obvious

$$7x + 11y = n \in \mathbb{N}_0$$

possible.

gg. $\neq 7$ the coprime. $\Rightarrow 7 \nmid n$ \Rightarrow $n \equiv 0, 1, 2, \dots, 6 \pmod{7}$.
 \Rightarrow gg. $\neq 7$ worse $\Rightarrow \text{gcd}(7, 11) = 1$. \Rightarrow $n \equiv 0, 1, 2, \dots, 6 \pmod{7}$.

11. 0; 11. 1; 11. 2. . . 11. 6

form possible x $\pmod{7}$ mod $7 - 8$ \Rightarrow $0, 1, 2, \dots, 6$

$$7x + 11y = n ; 7x = n - 11y. \quad 0 \equiv n - 11y \pmod{7}$$

$$n \equiv 0 \pmod{7} \quad n \equiv 11y \pmod{7}$$

$$x = \frac{n - 11y}{7} \quad \text{solution}$$

for $0 \leq y \leq 6$,

which can y

utg $(60, 60)$ - 23

~~$y \rightarrow$ 23 solutions~~

~~$y = 0, 1, 2, \dots, 6$~~

$$60 = 7 \cdot 7 + 11 \cdot 2$$

$$61 = 7 \cdot 4 + 11 \cdot 3$$

$$62 = 7 \cdot 3 + 11 \cdot 5$$

$$63 = 7 \cdot 9 + 11 \cdot 0$$

$$64 = 7 \cdot 6 + 11 \cdot 2$$

$$65 = 7 \cdot 3 + 11 \cdot 4$$

$$66 = 7 \cdot 0 + 11 \cdot 6$$

$$0 \leq y \leq 6$$

$$7x \equiv 4 \pmod{11}$$

$$8 \cdot 7x \equiv 8 \cdot 4 \pmod{11}$$

$$x \equiv 32 \pmod{11}$$

$$x \equiv 10 \pmod{11}$$

$$x = 10 + k \cdot 11$$

$$7x + 11y = 59 + k \cdot 70$$

$$70 > 59$$

$$k = 0, 1, 2, \dots, 8$$

$$7x + 11y = 59 + 70 \cdot k$$

$$70 > 59$$

$$k = 0, 1, 2, \dots, 8$$

$$7x + 11y = 59 + 70 \cdot k$$

$$70 > 59$$

$$k = 0, 1, 2, \dots, 8$$

$$7x + 11y = 59 + 70 \cdot k$$

$$70 > 59$$

$$k = 0, 1, 2, \dots, 8$$

$$7x + 11y = 59 + 70 \cdot k$$

$$70 > 59$$

7 divisors how many y \rightarrow 23 solutions. $47N$ in N \Rightarrow $7x + 11y$.

orange y \rightarrow multiple y \rightarrow $x \rightarrow$ multiple x \rightarrow multiple N

multiple, so y it divisors \rightarrow multiple N \rightarrow multiple N

so form n . $+ 7 - 8$, \rightarrow x \rightarrow y \rightarrow N

so $n \geq 60$ proof.

proof: $n \leftarrow 59 - 8$ $n = 59 - 8 \rightarrow$ $y \rightarrow$

$$\rightarrow x + 11y = 59 \rightarrow 7x + 11(8 - 5) = 6$$

$$7x \equiv 4 \pmod{11}$$

$$(x, 11) \text{ s.t.}$$

$$\text{co-prime s.t.}$$

$$\text{multiple}$$

798 $\alpha x + \beta y = n$ f(x) ist orthogonal zu f(y).

$$(1) \frac{1}{1-x^q} = 1 + x^q + x^{2q} + \dots \quad (2) \frac{1}{1-x^8} = 1 + x^8 + x^{16} + \dots$$

(1) - (2) - 3) coeff of x^n is μ & no. of ways to fill k boxes with non-negative integers such that sum of all numbers is n .
 Number of ways = $\binom{n+k-1}{k-1}$

(Ans) * $x+y = n$, or $n \geq 6$
 x, y coprime \rightarrow $y_1 - n+8$ is also coprime.

For $n \geq 6$,
 n can be represented as $\frac{4m+4n+1}{4m+2}$
 \Rightarrow $4m+4$ is odd. \Rightarrow $4m+2$ is even.
 \Rightarrow $4m+1$ is even. \Rightarrow $4m+2$ is odd.
 \Rightarrow $4m+3$ is odd.

$4m = (2m-1) + (2m+1)$

$4m+1 = 4(2m) + (2m+1)$

$4m+2 = (2m-1) + (2m+3)$

$4m+3 = (2m+1) + (2m+2)$

$$\frac{4n^2 + 1}{4n+3} = ((2n+1) \cdot (2n+2))$$

↓
2 nos.

↳ 4 nos.

↳ 2 nos.

↳ 1 no.

↳ 6 nos.

↳ 16 nos.

↳ 1) consecutive (odd from even)
2) odd nos. (2n+1, 2n+3, ..., 2n+9)

800 ~~now~~ ~~dimensions~~
by 8 meter ~~dimensions~~ cut into ~~pieces~~
~~(A 1) 8 ft 100's (18)~~

$$(a^d - 1)x + (b^d - 1)y \equiv 1 \pmod{p}$$

for II mod

fix coprime
 α, β
 $\alpha \cdot x + \beta \cdot y = n$

for I mod

$\begin{cases} d-1 & \text{if } d \text{ is prime} \\ d-1 & \text{if } d \text{ is composite} \end{cases}$

$x = a^{d-1}$
 $y = b^{d-1}$
 $\alpha = a^{d-1}$
 $\beta = b^{d-1}$

$$e^{k-2} \xrightarrow{6} \{1, 2, \dots, n\}$$

σ_1, σ_2 are σ_2 permutations

$$\sigma_1^{-1} \circ \sigma_2^{-1} = \text{identity perm. (P)}$$

$$\sigma = \sigma_1 \circ \sigma_2$$

if σ has n cycles then σ .

$$\sigma - 1 \leq \text{length} = n. \text{ If } n = 1 \text{ or } 2$$

$$\sigma = (j_1 j_2) \quad \cancel{\sigma = (j_1 j_2)}$$

$$\sigma = \sigma_1 \circ I$$

identity
permutation

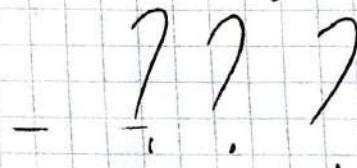
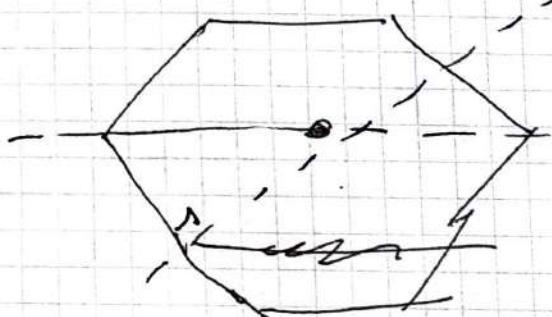
$$\sigma = \sigma_1 \circ I \quad \cancel{\sigma = (1, 2)}$$

identity perm.

otherwise, $\sigma = \text{rotation of regular } n\text{-gon}$

As, $\alpha_1, \alpha_2, \dots, \alpha_n$ by angle $\frac{2\pi}{n}$ around centre.

so $\alpha_1 + \alpha_2 + \dots + \alpha_n$



b2g

$$\alpha_1, \alpha_2, \dots, \alpha_{10} \text{ are integers } 1, 2, 3, \dots, 10$$

$$(|\alpha_1 - \alpha_2| + |\alpha_3 - \alpha_4| + |\alpha_5 - \alpha_6| + |\alpha_7 - \alpha_8| + |\alpha_9 - \alpha_{10}|)$$

$$\sum_{k=1}^n |\alpha_{2k-1} - \alpha_{2k}|$$

$$= n \cdot \text{average } (\alpha_1 - \alpha_2), \text{ i.e. } (\alpha_{2i-1} - \alpha_{2i})$$

is same for all $i = 1, 2, \dots, n$. when $\alpha_1 = k$

$$|\alpha_{2k-1} - \alpha_{2k}| = \frac{1+2+\dots+(k-1)}{k} = \frac{k(k-1)}{2k} \text{ now. } (\alpha_1 - \alpha_2)$$

$$\frac{(k-1) + (k-2) + \dots + 1 + 1 + 2 + (2n-k)}{2n-1} =$$

$$= \frac{1}{2n-1} \left(\frac{k(k-1)}{2} + \frac{(2n-k)(2n-k+1)}{2} \right) = \frac{k^2 - (2n+1)k + 2n(n+1)}{2n-1}$$

$$\text{when } k > \alpha_{2k}, \Rightarrow \text{sym} = \frac{k(k-1)}{2}, \text{ but } |\alpha_{2k-1} - \alpha_{2k}| = k - \alpha_{2k}$$

when $\alpha_{2k} > k$

$$\geq |k-j| = j - k = (1+2+\dots+2n-k) = \frac{(2n-k)(2n-k+1)}{2}$$

$$\text{average sym} = n \cdot \text{sym sym} = \frac{n(n+1)}{2}$$

$$G_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & l & m \\ 0_1 & 0_2 & - & \dots & 0_{l_1} & 0_{l_2} \end{pmatrix}$$

$$G_2 = \begin{pmatrix} 1 & 2 & 3 & \dots & l & m \\ a_{11} & a_{21} & a_{31} & \dots & a_l & a_m \end{pmatrix}$$

① If i, j same parity (i, j odd or even)

$i < j$ $a_{ii}(a_i, a_j)$ is inversion ($s > a_i > a_j$)

$a_2 - a_1$ ~~is inv. of $a_1, a_2, \dots, a_l, a_m$~~

$$\text{e.g. } (a_{11} \quad a_{12}) \text{ not}$$

$$a_{12} \dots$$

(a_{11} swap)

e.g. $a_{11} a_{12}, \dots, a_{1l}, a_{1m}$ and a_{1j} are swapped.

② If $i = \text{even}$ & $j = \text{odd}$,

$$\text{e.g. } a_{ii} a_j \text{ by } a_i > a_j$$

$G_2 - G_1$ position of a_i not a_j (swapped)

more swaps

③ If $i = \text{odd}$, $j = \text{even}$.. if $j > i + 1$ then (e.g.)

then a_{ij} and a_{ij} are interchanged, a_{ij} and a_{ij} swapped places.

more swaps

$G_2 - G_1$ $2 - n \rightarrow 2n$ swaps and $3n$ inversions

$\Rightarrow 2n^2$ swaps. By $a_{ij} = (\dots a_{ij} \dots)$

$$G_2 = (a_{11} \quad a_{21} \quad \dots)$$

not swap of a_{ij}

thus a_{ij} is preserved.

if $i = \text{odd}$, $j = \text{even}$ $a_{ij} \dots$

100 swaps

so 100 swaps of a_{ij} at most 100

swap inversions of G_2 is close to n^2 (no longer m number of pairs)

$(a_{ii} a_j)$ in G_2 pair swap?

$\binom{20}{2} = 190$ pairs in G_2 that are at least 90 swaps

190 - 40 = 150.

618

$$7^x - 3^x = 4$$

for $y=0$ $7^x - 1 = 4 \Rightarrow 3^y \text{ is a soln.}$

for $y=1 = x$ is a solution (3 is a divisor, so $10 \rightarrow 5$
 $8 \rightarrow 3 \rightarrow 1 = 0$)

for $y=2 \rightarrow 8$ $3^2 \text{ mod } 3^x \text{ is a soln.}$ \Rightarrow $x \rightarrow 3 \rightarrow 1 = 0$

~~$3^2 \text{ mod } 3^x \text{ is a soln.}$~~
 ~~$3^2 \equiv 9 \equiv 4 \text{ mod } 9$~~
 ~~$9 \equiv 0 \text{ mod } 9$~~

$7^x = 7 \Rightarrow x = 1$ top!

~~$2^x = 4 \text{ mod } 9$~~

$3^y \text{ mod } 9 - 8 \rightarrow 0 \Leftrightarrow$

$3^y \text{ mod } 9 = 0 \text{ always.}$

~~$7^x \equiv 4 \text{ mod } 9 - 8 \rightarrow 0$~~

$7^x \equiv 4 \text{ mod } 9$

$\Rightarrow x = \text{odd}$ so $2 \rightarrow 1 \rightarrow 0$

$x = \text{even}$

$7^2 \equiv 49 \equiv 4 \text{ mod } 9 \Rightarrow 7^2 \equiv 1 \text{ mod } 9$

so $x = \text{even.}$

$x = 2^n$

$3^x = 7^{2^n} - 4 = (7^n + 2)(7^n - 2)$

$y > 2 - 8$

P

else $x = 2 - 2$

~~$7^x \equiv 21 \text{ mod } 9 = 3$~~

~~$7^x \equiv 3 \text{ mod } 9$~~

619

$3^{2^x} = 2^{3^x} + 1$; $x = j = \text{solution}$

for $x > j - 1 \Rightarrow (j \geq 2 - 8)$

$x! = 1 \cdot 2 \cdots$

$x! = \text{even} \Leftrightarrow = 2 \cdot n$

By $3^{2^n} \equiv j \text{ mod } 4$ by

~~$3^{2^n} \equiv 2 \text{ mod } 4 \Rightarrow 2 \text{ mod } 4$~~

~~$2^{3^{2^n}} \equiv 2 \text{ mod } 4 \Rightarrow 2 \text{ mod } 4$~~

$2^1 = 2$
 $2^2 = 4$
 $2^3 = 8$
 $2^4 = 16$
 $2^5 = 32$

$2^{2^n} \equiv 2 \text{ mod } 4$

$\Rightarrow 2^{3^{2^n}} + 1 \equiv 2 + 1 \equiv 3 \text{ mod } 4$

$\text{mod } 4 \text{ mod } 7 \text{ mod } 3 \text{ mod } 2 \text{ mod } 1$

$2 \equiv 2 \text{ mod } 7$
 $4 \equiv 0 \text{ mod } 7$
 $8 \equiv 0 \text{ mod } 7$
 $16 \equiv 0 \text{ mod } 7$

$2^{3^2} = 8 \equiv 2 \text{ mod } 7$

(TOP)

1, 2, 3, 4, 5, 6, 7

P_1
 P_2
 P_3 \rightarrow minute

Q) $3 \rightarrow 1323$ stable P_1, P_2, P_3 on time | consecutive numbers
8 hours \leftarrow 3 days \uparrow

(1) $1213 \dots \overbrace{1980}^{77052}$
 $\overbrace{1980 \text{ sec}}^{72900 \text{ sec}}$

or

(2) min = 2 arithmetic time 3 hours.

(3) hours = 3 days = less no. of hours

(1) $9 \rightarrow 13$ 2 time 4 days $\underbrace{1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120, 122, 124, 126, 128, 130, 132, 134, 136, 138, 140, 142, 144, 146, 148, 150, 152, 154, 156, 158, 160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188, 190, 192, 194, 196, 198, 200, 202, 204, 206, 208, 210, 212, 214, 216, 218, 220, 222, 224, 226, 228, 230, 232, 234, 236, 238, 240, 242, 244, 246, 248, 250, 252, 254, 256, 258, 260, 262, 264, 266, 268, 270, 272, 274, 276, 278, 280, 282, 284, 286, 288, 290, 292, 294, 296, 298, 300, 302, 304, 306, 308, 310, 312, 314, 316, 318, 320, 322, 324, 326, 328, 330, 332, 334, 336, 338, 340, 342, 344, 346, 348, 350, 352, 354, 356, 358, 360, 362, 364, 366, 368, 370, 372, 374, 376, 378, 380, 382, 384, 386, 388, 390, 392, 394, 396, 398, 400, 402, 404, 406, 408, 410, 412, 414, 416, 418, 420, 422, 424, 426, 428, 430, 432, 434, 436, 438, 440, 442, 444, 446, 448, 450, 452, 454, 456, 458, 460, 462, 464, 466, 468, 470, 472, 474, 476, 478, 480, 482, 484, 486, 488, 490, 492, 494, 496, 498, 500, 502, 504, 506, 508, 510, 512, 514, 516, 518, 520, 522, 524, 526, 528, 530, 532, 534, 536, 538, 540, 542, 544, 546, 548, 550, 552, 554, 556, 558, 560, 562, 564, 566, 568, 570, 572, 574, 576, 578, 580, 582, 584, 586, 588, 590, 592, 594, 596, 598, 600, 602, 604, 606, 608, 610, 612, 614, 616, 618, 620, 622, 624, 626, 628, 630, 632, 634, 636, 638, 640, 642, 644, 646, 648, 650, 652, 654, 656, 658, 660, 662, 664, 666, 668, 670, 672, 674, 676, 678, 680, 682, 684, 686, 688, 690, 692, 694, 696, 698, 700, 702, 704, 706, 708, 710, 712, 714, 716, 718, 720, 722, 724, 726, 728, 730, 732, 734, 736, 738, 740, 742, 744, 746, 748, 750, 752, 754, 756, 758, 760, 762, 764, 766, 768, 770, 772, 774, 776, 778, 780, 782, 784, 786, 788, 790, 792, 794, 796, 798, 800, 802, 804, 806, 808, 810, 812, 814, 816, 818, 820, 822, 824, 826, 828, 830, 832, 834, 836, 838, 840, 842, 844, 846, 848, 850, 852, 854, 856, 858, 860, 862, 864, 866, 868, 870, 872, 874, 876, 878, 880, 882, 884, 886, 888, 890, 892, 894, 896, 898, 900, 902, 904, 906, 908, 910, 912, 914, 916, 918, 920, 922, 924, 926, 928, 930, 932, 934, 936, 938, 940, 942, 944, 946, 948, 950, 952, 954, 956, 958, 960, 962, 964, 966, 968, 970, 972, 974, 976, 978, 980, 982, 984, 986, 988, 990, 992, 994, 996, 998, 1000, 1002, 1004, 1006, 1008, 1010, 1012, 1014, 1016, 1018, 1020, 1022, 1024, 1026, 1028, 1030, 1032, 1034, 1036, 1038, 1040, 1042, 1044, 1046, 1048, 1050, 1052, 1054, 1056, 1058, 1060, 1062, 1064, 1066, 1068, 1070, 1072, 1074, 1076, 1078, 1080, 1082, 1084, 1086, 1088, 1090, 1092, 1094, 1096, 1098, 1100, 1102, 1104, 1106, 1108, 1110, 1112, 1114, 1116, 1118, 1120, 1122, 1124, 1126, 1128, 1130, 1132, 1134, 1136, 1138, 1140, 1142, 1144, 1146, 1148, 1150, 1152, 1154, 1156, 1158, 1160, 1162, 1164, 1166, 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1500, 1502, 1504, 1506, 1508, 1510, 1512, 1514, 1516, 1518, 1520, 1522, 1524, 1526, 1528, 1530, 1532, 1534, 1536, 1538, 1540, 1542, 1544, 1546, 1548, 1550, 1552, 1554, 1556, 1558, 1560, 1562, 1564, 1566, 1568, 1570, 1572, 1574, 1576, 1578, 1580, 1582, 1584, 1586, 1588, 1590, 1592, 1594, 1596, 1598, 1600, 1602, 1604, 1606, 1608, 1610, 1612, 1614, 1616, 1618, 1620, 1622, 1624, 1626, 1628, 1630, 1632, 1634, 1636, 1638, 1640, 1642, 1644, 1646, 1648, 1650, 1652, 1654, 1656, 1658, 1660, 1662, 1664, 1666, 1668, 1670, 1672, 1674, 1676, 1678, 1680, 1682, 1684, 1686, 1688, 1690, 1692, 1694, 1696, 1698, 1700, 1702, 1704, 1706, 1708, 1710, 1712, 1714, 1716, 1718, 1720, 1722, 1724, 1726, 1728, 1730, 1732, 1734, 1736, 1738, 1740, 1742, 1744, 1746, 1748, 1750, 1752, 1754, 1756, 1758, 1760, 1762, 1764, 1766, 1768, 1770, 1772, 1774, 1776, 1778, 1780, 1782, 1784, 1786, 1788, 1790, 1792, 1794, 1796, 1798, 1800, 1802, 1804, 1806, 1808, 1810, 1812, 1814, 1816, 1818, 1820, 1822, 1824, 1826, 1828, 1830, 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2828, 2830, 2832, 2834, 2836, 2838, 2840, 2842, 2844, 2846, 2848, 2850, 2852, 2854, 2856, 2858, 2860, 2862, 2864, 2866, 2868, 2870, 2872, 2874, 2876, 2878, 2880, 2882, 2884, 2886, 2888, 2890, 2892, 2894, 2896, 2898, 2900, 2902, 2904, 2906, 2908, 2910, 2912, 2914, 2916, 2918, 2920, 2922, 2924, 2926, 2928, 2930, 2932, 2934, 2936, 2938, 2940, 2942, 2944, 2946, 2948, 2950, 2952, 2954, 2956, 2958, 2960, 2962, 2964, 2966, 2968, 2970, 2972, 2974, 2976, 2978, 2980, 2982, 2984, 2986, 2988, 2990, 2992, 2994, 2996, 2998, 3000, 3002, 3004, 3006, 3008, 3010, 3012, 3014, 3016, 3018, 3020, 3022, 3024, 3026, 3028, 3030, 3032, 3034, 3036, 3038, 3040, 3042, 3044, 3046, 3048, 3050, 3052, 3054, 3056, 3058, 3060, 3062, 3064, 3066, 3068, 3070, 3072, 3074, 3076, 3078, 3080, 3082, 3084, 3086, 3088, 3090, 3092, 3094, 3096, 3098, 3100, 3102, 3104, 3106, 3108, 3110, 3112, 3114, 3116, 3118, 3120, 3122, 3124, 3126, 3128, 3130, 3132, 3134, 3136, 3138, 3140, 3142, 3144, 3146, 3148, 3150, 3152, 3154, 3156, 3158, 3160, 3162, 3164, 3166, 3168, 3170, 3172, 3174, 3176, 3178, 3180, 3182, 3184, 3186, 3188, 3190, 3192, 3194, 3196, 3198, 3200, 3202, 3204, 3206, 3208, 3210, 3212, 3214, 3216, 3218, 3220, 3222, 3224, 3226, 3228, 3230, 3232, 3234, 3236, 3238, 3240, 3242, 3244, 3246, 3248, 3250, 3252, 3254, 3256, 3258, 3260, 3262, 3264, 3266, 3268, 3270, 3272, 3274, 3276, 3278, 3280, 3282, 3284, 3286, 3288, 3290, 3292, 3294, 3296, 3298, 3300, 3302, 3304, 3306, 3308, 3310, 3312, 3314, 3316, 3318, 3320, 3322, 3324, 3326, 3328, 3330, 3332, 3334, 3336, 3338, 3340, 3342, 3344, 3346, 3348, 3350, 3352, 3354, 3356, 3358, 3360, 3362, 3364, 3366, 3368, 3370, 3372, 3374, 3376, 3378, 3380, 3382, 3384, 3386, 3388, 3390, 3392, 3394, 3396, 3398, 3400, 3402, 3404, 3406, 3408, 3410, 3412, 3414, 3416, 3418, 3420, 3422, 3424, 3426, 3428, 3430, 3432, 3434, 3436, 3438, 3440, 3442, 3444, 3446, 3448, 3450, 3452, 3454, 3456, 3458, 3460, 3462, 3464, 3466, 3468, 3470, 3472, 3474, 3476, 3478, 3480, 3482, 3484, 3486, 3488, 3490, 3492, 3494, 3496, 3498, 3500, 3502, 3504, 3506, 3508, 3510, 3512, 3514, 3516, 3518, 3520, 3522, 3524, 3526, 3528, 3530, 3532, 3534, 3536, 3538, 3540, 3542, 3544, 3546, 3548, 3550, 3552, 3554, 3556, 3558, 3560, 3562, 3564, 3566, 3568, 3570, 3572, 3574, 3576, 3578, 3580, 3582, 3584, 3586, 3588, 3590, 3592, 3594, 3596, 3598, 3600, 3602, 3604, 3606, 3608, 3610, 3612, 3614, 3616, 3618, 3620, 3622, 3624, 3626, 3628, 3630, 3632, 3634, 3636, 3638, 3640, 3642, 3644, 3646, 3648, 3650, 3652, 3654, 3656, 3658, 3660, 3662, 3664, 3666, 3668, 3670, 3672, 3674, 3676, 3678, 3680, 3682, 3684, 3686, 3688, 3690, 3692, 3694, 3696, 3698, 3700, 3702, 3704, 3706, 3708, 3710, 3712, 3714, 3716, 3718, 3720, 3722, 3724, 3726, 3728, 3730, 3732, 3734, 3736, 3738, 3740, 3742, 3744, 3746, 3748, 3750, 3752, 3754, 3756, 3758, 3760, 3762, 3764, 3766, 3768, 3770, 3772, 3774, 3776, 3778, 3780, 3782, 3784, 3786, 3788, 3790, 3792, 3794, 3796, 3798, 3800, 3802, 3804, 3806, 3808, 3810, 3812, 3814, 3816, 3818, 3820, 3822, 3824, 3826, 3828, 3830, 3832, 3834, 3836, 3838, 3840, 3842, 3844, 3846, 3848, 3850, 3852, 3854, 3856, 3858, 3860, 3862, 3864, 3866, 3868, 3870, 3872, 3874, 3876, 3878, 3880, 3882, 3884, 3886, 3888, 3890, 3892, 3894, 3896, 3898, 3900, 3902, 3904, 3906, 3908, 3910, 3912, 3914, 3916, 3918, 3920, 3922, 3924, 3926, 3928, 3930, 3932, 3934, 3936, 3938, 3940, 3942, 3944, 3946, 3948, 395$

(1) $f(n) + 2f(f(n)) = 3n + 3$

$f(1) + 2f(f(1)) = 8 \Rightarrow f(1)$ is even number
 $\Rightarrow 2x$ even number \Rightarrow even number.

$f(1) \Rightarrow$ even number $\frac{(1+8)-2}{2}$

$f(1) = 2, 4, 6$

(a) $\left\{ \begin{array}{l} \text{if } f(1) = 2 \\ 2+2f(f(2)) = 11 \\ 2+2f(f(3)) = 11 \end{array} \right.$

$f(4) = n+d$ odd no. $f(3) = 4$.

Proof by induction
 $n+d + 2f(n+d) = 3n+3 ; f(n+1) = n+1$

(i) if $f(1) = 4$ $4+2f(4) = 8 ; f(4) = ?$...
 ~~$2+2f(f(4)) = 17$~~ \Rightarrow ~~not possible~~ \Rightarrow ~~not possible~~

(ii) if $f(1) = 6$ $6+2f(6) = 8$
 ~~$f(6) = 1$~~

$f(8) + 2f(f(8)) = 1+2 \cdot 6$
 $1+2 \cdot 6 = 1+2 \cdot 6$

(a) $f(4) = n+1$

$f(4) = n+1$

$\rightarrow 2f(f(x)) - 2f(f(x)) - f(x) + x = 0$

(2) $2f(f(x)) - 3f(x) + x = 0$
 $s(x) = f(x) - x$ $\Rightarrow s(x) = f(x)$

$2f(f(x)) - 2f(x) - f(x) + x = 0$
 $2g(f(x)) - g(x) = 0$

~~$2^k f^{(k)}(x) = g(x)$~~

for every x

~~$g(x) \neq 0$~~

$f^{(n)}(x)$ composed n times $1 \leq p(f)$

~~$g(x) \neq 0$~~
 $\Rightarrow g(x) \neq 0$
 $\Rightarrow g(x) \neq 0$

$g(500) \neq 0$
 $\Rightarrow g(250) \neq 0$
 $\Rightarrow g(125) \neq 0$
 $\Rightarrow g(62.5) \neq 0$
 $\Rightarrow g(31.25) \neq 0$

$$\Rightarrow f(4) = 1 \quad \text{and} \quad$$

no bijection

(703) $f(mn) = f(m) + f(n) + 3f(\alpha)f(\beta)$

α, β ~~are not integers~~ α, β ~~are not integers~~ $R(3, 1)$

g is bijective $g(xy) = g(x) \cdot g(y)$

$\Rightarrow g(1) = 1$

If $g(x \cdot y) = n$ then $n = \underbrace{p_1 p_2 \dots p_k}$ powers congruent to 2 mod 3

$\Leftrightarrow x, y$ prime.

* If p_1, p_2, \dots, p_k are numbers congruent to 2 mod 3
 $(\text{e.g. } 2, 5)$ then p_1, p_2, \dots, p_k are ~~not~~ factors of n .

$$g(q) = p^2 \quad ; \quad g(8) = q^2, \quad \text{so } q = p^2$$

$$g(p^2q^2) = g(p)^2 \cdot g(q)^2 = p^2q^2 = g(q)g(8) = g(16)$$

$$p^2 = q^2 \quad \text{with } p, q, \in \mathbb{Z}$$

~~hence, prime numbers~~

(704)

$$a_{n+1} \leq (a_{n+1} - a_n)^2 \leq a_n \quad (n \geq 2)$$

$1, 2, 3, 4, 5$: Non $n \leq 5$

$$b_n = a_{n+1} - a_n \quad \text{for } n \in \{1, 2, \dots, 5\}$$

$$\Rightarrow b_n \geq 0_{n-1} \quad \text{for } (n \geq 2) \quad 8$$

$$\Rightarrow b_n \geq 0_{n-1} \quad \text{so } b_{n+1} \geq b_n$$

$$n = 2, 3, 4, \dots$$

$$-a_n \leq b_n^2 \leq a_n - a_{n-1}$$

$$a_n \leq b_n^2 + a_n \leq a_{n+1}$$

$$-a_n \leq b_{n+1}^2 \leq a_{n+2}$$

$$0 < b_{n+1}^2 - b_n^2 \leq b_n + b_{n-1}$$

$$0 \leq (b_{n+1} - b_n)(b_{n+1} + b_n) \leq b_n + b_{n-1}$$

$$\text{so } b_{n+1} \geq b_n - b_{n-1} \quad \text{for } n = 2, 3, 4$$

$$0 < b_{n+1}^2 - b_n^2 \leq b_n + b_{n-1} \leq b_{n+1} + b_n$$

$$0 < b_{n+1} - b_n \leq \text{some value.} \quad \text{if } b_3 = b_4 = b_5$$

$$m \quad 8_3^2 \leq q_3 \leq 8_7^2 \leq q_7 \leq 8_5^2$$

$$a_3 = a_4 \quad B_4 + B_3 = 0 \text{ or } 2$$

July 20. 1974. No set.

~~No set of
instru numbers
getting~~

703

2

~~$x^2 + 10y^2 = 32$~~ $\frac{1}{2}$ odds with 0/1/1/4 P. 189

0, 1, 2, 3, (5) 7, 8. say 3 M 30 80 & 82 51
m 7 t p h r o 22

$$x = 5x_0 \quad z = 5z_0$$

$$25x_0^2 + 10y^2 = 3 \cdot 25z^2$$

121 500 ~~500~~
122 500 500 Pihla 200
123 500 500 200 70

8) $x = 5 \text{ m}$ ბეტა უსტა გვიშავთ
ეს როგორ.

2003-1, hna y nr sh, 1-63

$$y = 5 \text{ g} \text{d}$$

$$41 - 63 \quad n(s)_{10}^2 + 10 \cdot (s)_{10}^2 = 3(s)_{10}^2$$

$y_0, y_0, z_0 - w$ blank

$$\frac{1000}{\text{kg}} \cdot \frac{\text{Pa}}{\text{N/m}^2} = \text{J/m}^3 \text{ (Joules per cubic meter)}$$

٢٠٧

$$x^2 + 5y^2 = z^2$$

$$\frac{5x^2 + y^2 = t^2}{3}$$

$$\frac{6x^2}{6(x^2+y^2)} = \frac{x^2-y^2}{x^2+y^2} \cdot \frac{3-y}{3+y}$$

$$x^2 + c^2 = 3m$$

Jump ~~ways~~ to ~~out~~ ~~out~~

$$\phi\left(\underbrace{x^2+y^2}_{1}\right) = \phi(x^2+y^2)$$

$$t = 3 \text{ sec}$$

$$x^4 + y^2 = 3$$

$$\begin{aligned}x &= x_0 \\y &= 3y_0\end{aligned}$$

1) $y > i y_0, t \in [0, \infty)$
2) $y(0) = y_0$

$$\Delta \quad b_3^2 \leq a_3 \leq b_4^2 \quad a_3 = a_4 \quad b_4 + b_3 = a_4 - 0.1 \approx$$

705?

705, 200. 6th row 6th row
no set of

prime numbers
94th row

$$705 \quad x^2 + 10y^2 = 3z^2 \quad \text{pds with } 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500, 2609, 2716, 2824, 2936, 3056, 3176, 3296, 3416, 3536, 3656, 3776, 3896, 3916, 4036, 4156, 4276, 4396, 4516, 4636, 4756, 4876, 4996, 5116, 5236, 5356, 5476, 5596, 5716, 5836, 5956, 6076, 6196, 6316, 6436, 6556, 6676, 6796, 6916, 7036, 7156, 7276, 7396, 7516, 7636, 7756, 7876, 7996, 8116, 8236, 8356, 8476, 8596, 8716, 8836, 8956, 9076, 9196, 9316, 9436, 9556, 9676, 9796, 9916, 10036, 10156, 10276, 10396, 10516, 10636, 10756, 10876, 10996, 11116, 11236, 11356, 11476, 11596, 11716, 11836, 11956, 12076, 12196, 12316, 12436, 12556, 12676, 12796, 12916, 13036, 13156, 13276, 13396, 13516, 13636, 13756, 13876, 13996, 14116, 14236, 14356, 14476, 14596, 14716, 14836, 14956, 15076, 15196, 15316, 15436, 15556, 15676, 15796, 15916, 16036, 16156, 16276, 16396, 16516, 16636, 16756, 16876, 16996, 17116, 17236, 17356, 17476, 17596, 17716, 17836, 17956, 18076, 18196, 18316, 18436, 18556, 18676, 18796, 18916, 19036, 19156, 19276, 19396, 19516, 19636, 19756, 19876, 19996, 20116, 20236, 20356, 20476, 20596, 20716, 20836, 20956, 21076, 21196, 21316, 21436, 21556, 21676, 21796, 21916, 22036, 22156, 22276, 22396, 22516, 22636, 22756, 22876, 22996, 23116, 23236, 23356, 23476, 23596, 23716, 23836, 23956, 24076, 24196, 24316, 24436, 24556, 24676, 24796, 24916, 25036, 25156, 25276, 25396, 25516, 25636, 25756, 25876, 25996, 26116, 26236, 26356, 26476, 26596, 26716, 26836, 26956, 27076, 27196, 27316, 27436, 27556, 27676, 27796, 27916, 28036, 28156, 28276, 28396, 28516, 28636, 28756, 28876, 28996, 29116, 29236, 29356, 29476, 29596, 29716, 29836, 29956, 30076, 30196, 30316, 30436, 30556, 30676, 30796, 30916, 31036, 31156, 31276, 31396, 31516, 31636, 31756, 31876, 31996, 32116, 32236, 32356, 32476, 32596, 32716, 32836, 32956, 33076, 33196, 33316, 33436, 33556, 33676, 33796, 33916, 34036, 34156, 34276, 34396, 34516, 34636, 34756, 34876, 34996, 35116, 35236, 35356, 35476, 35596, 35716, 35836, 35956, 36076, 36196, 36316, 36436, 36556, 36676, 36796, 36916, 37036, 37156, 37276, 37396, 37516, 37636, 37756, 37876, 37996, 38116, 38236, 38356, 38476, 38596, 38716, 38836, 38956, 39076, 39196, 39316, 39436, 39556, 39676, 39796, 39916, 40036, 40156, 40276, 40396, 40516, 40636, 40756, 40876, 40996, 41116, 41236, 41356, 41476, 41596, 41716, 41836, 41956, 42076, 42196, 42316, 42436, 42556, 42676, 42796, 42916, 43036, 43156, 43276, 43396, 43516, 43636, 43756, 43876, 43996, 44116, 44236, 44356, 44476, 44596, 44716, 44836, 44956, 45076, 45196, 45316, 45436, 45556, 45676, 45796, 45916, 46036, 46156, 46276, 46396, 46516, 46636, 46756, 46876, 46996, 47116, 47236, 47356, 47476, 47596, 47716, 47836, 47956, 48076, 48196, 48316, 48436, 48556, 48676, 48796, 48916, 49036, 49156, 49276, 49396, 49516, 49636, 49756, 49876, 49996, 50116, 50236, 50356, 50476, 50596, 50716, 50836, 50956, 51076, 51196, 51316, 51436, 51556, 51676, 51796, 51916, 52036, 52156, 52276, 52396, 52516, 52636, 52756, 52876, 52996, 53116, 53236, 53356, 53476, 53596, 53716, 53836, 53956, 54076, 54196, 54316, 54436, 54556, 54676, 54796, 54916, 55036, 55156, 55276, 55396, 55516, 55636, 55756, 55876, 55996, 56116, 56236, 56356, 56476, 56596, 56716, 56836, 56956, 57076, 57196, 57316, 57436, 57556, 57676, 57796, 57916, 58036, 58156, 58276, 58396, 58516, 58636, 58756, 58876, 58996, 59116, 59236, 59356, 59476, 59596, 59716, 59836, 59956, 60076, 60196, 60316, 60436, 60556, 60676, 60796, 60916, 61036, 61156, 61276, 61396, 61516, 61636, 61756, 61876, 61996, 62116, 62236, 62356, 62476, 62596, 62716, 62836, 62956, 63076, 63196, 63316, 63436, 63556, 63676, 63796, 63916, 64036, 64156, 64276, 64396, 64516, 64636, 64756, 64876, 64996, 65116, 65236, 65356, 65476, 65596, 65716, 65836, 65956, 66076, 66196, 66316, 66436, 66556, 66676, 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x, y, z prime

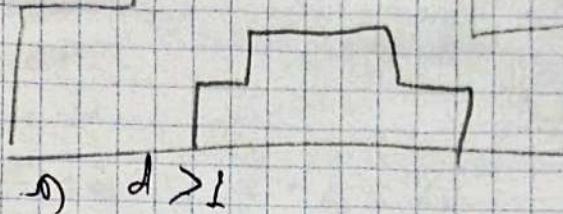
$$\begin{aligned} x^2 - y^2 &= 2xy \\ d^2 - y^2 &\equiv 0 \pmod{d} \\ d - y &\equiv 0 \pmod{d} \\ d &> 2y \end{aligned}$$

$$d = xy$$

$$\text{if } d = j \Rightarrow x = y = j$$

$$j^2 - j^2 = 2 \cdot d \cdot z$$

~~steps~~ rigid
rigid curve
 $y > d - z$



$$\text{① } d > 1$$

Let p be prime divisor of d

$$(x+y)(x-y) = x^2 - y^2 = 2xy \equiv 0 \pmod{p}$$

$$x+y \equiv 0 \pmod{p} \quad x \equiv y \pmod{p} \quad \text{or} \quad x \equiv -y \pmod{p}$$

$$\text{② } x-y \equiv 0 \pmod{p} \quad \xrightarrow{\text{Pys}} \quad x \equiv y \pmod{p}$$

$$\text{③ } \xleftarrow{\text{Pys}} \quad x = \frac{y}{p} \quad ; \quad y_1 = \frac{x}{p} - 1$$

thus position of y_1 .

705 ① If (q_{k+1}) is infinite and q_1

$$q_{k+1}^2 - q_k^2 = q_k^2 - q_{k-1}^2$$

must be increasing.

$$q_{k+1} + q_k \geq q_k + q_{k-1}$$

$$(q_{k+1} - q_k)(q_{k+1} + q_k) = (q_k - q_{k-1})(q_k + q_{k-1})$$

$$q_{k+1} - q_k \leq q_k - q_{k-1}$$

$$q_2 - q_1 > q_3 - q_2 > q_4 - q_3 \dots$$

$$a < q+d < a+rd$$

$f(q) < f(q+d) < f(q+rd)$ on $\mathbb{F}_{\text{finite field}}$ real numbers are distinct

$$f(a+nr) > f(a)$$

$f(a+2^{n+1}r) < f(a+2^n r)$, for all n , otherwise since a and $d = 2^n r$ satisfy

infinitely

$x^2 + y^2 = z^2$. If x, y, z are same parity, it divides $y + z$ evenly
 $\Rightarrow z^2 - x^2 = (z-x)(z+x)$
 So x & y are odd & z is even.

z is gcd, $z-x \mid z+x$
 $\Rightarrow z-x = k$, $z+x = 2k$
 $\Rightarrow z = k + \frac{1}{2}x$, $x = 2k - z$

$\Rightarrow z = k + \frac{1}{2}(2k - z) \Rightarrow z = \frac{3}{2}k$
 $\Rightarrow k = \frac{2}{3}z$

$\Rightarrow z = \frac{2}{3}z$ (Contradiction)

$\Rightarrow z^2 = u^2 + v^2$, $z = 2uv$
 $\Rightarrow x = u^2 - v^2$, $y = 2uv$, $z = u^2 + v^2$

$\Rightarrow x^2 + y^2 = z^2$, $y = \sqrt{u^2 - v^2}$, $z = \sqrt{u^2 + v^2}$

$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1$ (circle / conic sections)

$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ on B.C. Arcs

$t = \tan \theta$ (angle with radius / theta (B.C.))

$t = \text{some } \theta \Rightarrow z = r \sin \theta, y = r \cos \theta$

$t = \frac{m}{n}$ for some co-prime m, n

$\frac{x}{z} = 1 - \frac{(m/n)^2}{1 + (m/n)^2}; \frac{y}{z} = \frac{2(m/n)}{1 + (m/n)^2}$

x, y, z are odds
 $y = \text{even}$ $\Rightarrow \frac{y}{z} = \frac{2uv}{u^2 + v^2}$
 $\Rightarrow 2uv \text{ is irreducible.}$

$\Rightarrow u^2 + v^2$ is prime

$\Rightarrow 337000$

$\Rightarrow y = 2uv$
 $\Rightarrow z = u^2 + v^2$

$\Rightarrow x = u^2 - v^2$

by hypotenuse triples (h) Fibonacci

$$1, 1, 2, 3, 5, 8, \dots \quad \text{double}$$

$$\begin{aligned} f_4 = 3 & \quad f_5 = 5 \quad 3 \cdot 5 = 15 - \cancel{15} \quad 15^2 = 30 \\ 4+5=9 & \quad \text{not made from } 2 \times 2 \\ f_3 = 2 & \quad f_6 = 8 \quad \cancel{2 \cdot 8 = 16} \end{aligned}$$

$$\begin{aligned} \text{we get } 30/10. & \\ 15^2 + 16^2 = 1156 & \quad \text{square of } 39 \text{ is } 1525 \\ (\text{Fib} = 39) & \end{aligned}$$

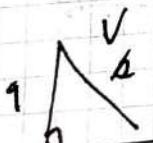
$$4+5=9=1+6$$

(Fib.) numbers $2f_n f_{n+1}, f_{n-1} f_{n+2}$, ~~from~~ triple
from Pythagorean triple

$$\begin{aligned} u = f_{n+1} & \quad u^2 - v^2 = (u-v)(u+v) = (f_{n+1} - f_n)(f_{n+1} + f_n) = \\ v = f_n & \quad = f_{n-1} \cdot f_{n+2} \end{aligned}$$

$$\begin{aligned} \text{11237 square, hd} & \\ f_{2n+1} = u^2 + v^2 = f_{n+1}^2 + f_n^2 & \end{aligned}$$

(60)



$$u^2 + v^2 = \text{perimeter } 54 \text{ square.}$$

(1, 2, 3) @ 0 pmc

as seen at \angle vertex (63)

$$u^3 + v^3 \text{ work}$$

(u, v)

$$d = u^2 - v^2 ; b = 2 \cdot u \cdot v$$

$$u^2 + v^2 = k^2$$

$$u^3 + v^3 = (u+v)(u^2 - uv + v^2)$$

$$= k^2(u^2 - v^2)^2 - 2uv(2uv(u^2 - v^2) + 4u^2v^2)$$

$$= k^2(u^2 + v^2 - 2uv^2 + 2u^2v^2) =$$

$$= (k(u^2 - uv))^2 + (k(v^2 + uv))^2$$

$$\begin{aligned} k &= u^2 - v^2 \\ r &= u^2 + v^2 \\ g &= 2uv \end{aligned}$$

13 min.

$$3x + y^2 = 5^2 \quad \text{in yr 53}$$

Stringent rules have been imposed by the Ministry

$$y^2 = \underbrace{5^2 - 2^2}_{\text{PWS}} \quad \text{PWS: 4-80, 732W, 3487L}$$

$$3x + y^2 \equiv 5 \pmod{8}$$

$$3^x + y^2 \equiv 1 \pmod{9} \equiv 3^x + 1 \pmod{9}$$

$$3 \times = \text{odd!} \quad 3 \cdot 3 \cdot 3 \cdot 3 = 27$$

~~✓ - 3 days - 100% / 1500 m³~~ 5500 m³

$$3 + 2 \cdot 8 = 17$$

$\gamma x = 0 \text{ or } 2\pi, x = 1$

$$\cancel{3} \cancel{x} = 3$$

$$X = \theta @ n = 2 \text{m}$$

$$\text{not } 3 - 2x \geq 2$$

$$x^2 + y^2 \equiv 15 \pmod{3} \quad \left. \begin{array}{l} x \not\equiv 0 \pmod{3} \\ y \not\equiv 0 \pmod{3} \end{array} \right\} \quad \text{矛盾}$$

~~5a) 1303~~

$$z = \operatorname{e}^{i\theta} y = w$$

$$\therefore (3^m)^2 + y^2 = (5^n)^2 \quad \therefore y = \text{even}$$

$$\frac{1}{1 - \alpha} \ln(1 - \frac{\alpha}{1-\alpha}) =$$

$$t \sqrt{u^2 - v^2} = 3 \quad | \quad u^2 - v^2 = (u-v)(u+v) =$$

$$S^2 = n^2 + v^2$$

$$\left(\frac{m^2}{n} - 4(n+1) \right) = 3^{13}$$

$m - v$ or $m + v$ sign powers of -

$$u = \frac{u - v + u + v}{2} \quad \text{bzw. } 1)$$

$$v - \frac{2v}{m+v} = \frac{m+v-2v}{m+v}$$

Aug 32 N. 100

$$M-V = 1 \Rightarrow M \approx 25^{\frac{1}{n}}$$

can see
when $n=5$

$$2 \cdot 5^n = g^5 + 1$$

~~8560031, 12~~ $\exists n=2$ ~~25~~

from eq. we have $M^2 + V^2 = 5^n$

$$M^2 + V^2 = 5^n$$

$n=$

- 25 - 25

$$g^2 = 5 - 1$$

25 - 25

$n=2$ sru.

$$\frac{25}{5} \mod 25 \equiv 1 \rightarrow$$

$$\text{and } 25 \equiv 1 \mod 25$$

$$(g^m \cdot n) \rightarrow 35^n \in R$$

$$g^2 \cdot 25 \equiv \text{nd } 25$$

by $n \neq 0$ normal argument

$$2 \cdot 5^n \equiv 0 \mod 25 \Rightarrow g^4 \equiv 0 \mod 25$$

$$\mod 25$$

- 1 mod

$$g^m \equiv -1 \mod 25$$

$$\frac{g^m}{g^2} \equiv -1 \mod 25$$

$$g^2 \equiv 1 \mod 25$$

$$g^2 \equiv g \cdot g \mod 25 =$$

$$g^2 \equiv 8 \equiv 6 \mod 25$$

$$g^3 \equiv g^2 \cdot g \equiv 6 \cdot g = 54 \equiv 4 \mod 25$$

$$g^5 = g^4 \cdot g \equiv 1 \cdot g \equiv g \equiv -1 \mod 25$$

$$g^5 \equiv -1 \mod 25$$

for $g^m \equiv -1 \mod 25$ to hold

If m is even multiple of 5 then $\frac{m}{5}$

$$m = 2k+5 \text{ even multiple of 5} \Rightarrow g^{2k+5} = (-1)^5 = 1$$

if $5 \cdot m$ is even $\Rightarrow 5 \cdot k$

$g^{5m} \rightarrow 1 \text{ mod } 25$

so $M \rightarrow \text{sru.}$

~~17~~ ~~odd and multiple of 5~~

$$n = 5(2k+1)$$

$$2 \cdot 5^n = \underbrace{(g^5)^{2k+1}}_{\text{multiple of } 5} + 1 = (g^5+1) \left((g^5)^{2k} - (g^5) \cdot 1 + \dots + 1 \right)$$

$$(g^5)^{2k+1} + 1 = \overline{5(g^5)^{2k+1} + 1}$$

~~so~~
~~2 · 5^n is not divisible by 5~~

$$\text{multiple of } g^5 + 1 = 2 \cdot 5^2 \cdot 1181 \quad \leftarrow$$

$$\frac{g \cdot g \cdot g \cdot g \cdot g + 1}{2 \cdot 5^n - \text{sp}} = \frac{59050}{1181} \quad \text{multiple of } 1181$$

$$\begin{array}{r|rr} 59050 & 2 \\ 29525 & 5 \\ 5905 & 5 \\ 1181 & 1181 \end{array} \Rightarrow 2 \cdot 5^2 \cdot 1181$$

$$2 \cdot 5^n = 2 \cdot 5^2 \cdot 1181 \quad \text{multiple of } 1181$$

$n = \text{multiple}$
of 3

$$2 \cdot 5^2 \cdot 5^{n-2}$$

$5^{n-2} \neq 1181$
multiple

~~multiple of 5~~ $n=1 \rightarrow$ 2 is multiple of 3

~~multiple of 3~~ $x=2$; $2=2-63$

$$(103) 2^x + 5^y = \text{perfect square}$$

3. a. a. 18392c 83L

fact: last digit of perfect squares
never ends in 6 or 2, 3, 7, 8.

look perfect squares at end 10 (last digit)

$$\text{so, } x = \text{even. } x = ? \times 1$$

$$(2^{x_1})^2 + (5^y)^2 = z^2$$

if x_1 is even, then z^2 ends in 5

$$(9) \quad 5^2 \equiv 1 \pmod{4} \quad z^2 \equiv 1 \pmod{4}$$

$$(10) \quad 2^{x_1} \equiv 0 \pmod{4} \quad z^2 \equiv 1 \pmod{4}$$

by looking
at p. parity
(odd/even)

if z even, z^2

$$z^2 = 2m^2$$

$$z^2 - 1 = 2m^2 - 1$$

then $z^2 - 1$ is divisible by 2

(a) $2^x - 5^y \neq 0 \pmod{2}$

$$z^2 - 1 = (z-1)(z+1)$$

$z \equiv 1 \pmod{2}$ or $z \equiv -1 \pmod{2}$

$$z = u+v+4-u \quad z = 2m = u+v+4-u \quad \text{if } u=5 \text{ is multiple}$$

$$v = \frac{u+v+4-u}{2} = m \quad v \equiv 0 \pmod{2}$$

$2v = 5 \text{ is multiple of } 4$

$$(10) \quad 2^x - 5^y \equiv 0 \pmod{4}$$

$$y \leq 1 \quad u = 2^{x_1-1} \text{ and}$$

so $u \equiv 1 \pmod{4}$ from 2nd if $p \neq 0$

$$u = 2^{x_1-1} \quad 2^{x_1-1} \not\equiv 5 \pmod{4} \quad \text{why?}$$

$$\text{so } u = 1 \quad u = 2^{x_1-1}$$

$$(11) \quad 2^x - 5^y \equiv 0 \pmod{4}$$

$$5^y = 2^{x_1-2} - 1 = (2^{x_1-1}-1)(2^{x_1-1}+1)$$

so x_1 differs by 2.
so both solutions don't exist

$$5^n \cdot 2^m \text{ परिवर्तन, हजार } 5^x = (2^{x-1} - 1) (2^{x'-1} + 1)$$

$$5^a \cdot 5^b = \underbrace{5^a}_{\text{मूल}} \cdot \underbrace{5^b}_{\text{पूर्ण घटक}} = 2^{x-1} - 1 \quad ; b = 2^{x'-1} + 1$$

यहाँ से $5^a = 2^{x-1} - 1$ का अनुप्रयोग नहीं होता।

$$5^a \cdot 5^b = 2^{x-1} - 1 \cdot 2^{x'-1} + 1 = 5 - 5^2$$

~~$$5^a \cdot 5^b = 2^{x-1} - 1 \cdot 2^{x'-1} + 1 = 5 - 5^2$$~~

इसी तरह $5^a \cdot 5^b = 2^{x-1} - 1 \cdot 2^{x'-1} + 1 = 5 - 5^2$

इनका लिखें।

$$x^2 - (n^2 + 1) y^2 = 1$$

~~$$x^2 - D y^2 = 1$$~~

$$D \text{ का रूप } D = n^2 + 1$$

इसका अर्थ यह है कि दोनों ओर विभाजन करके दो भागों में विभाजित कर सकते हैं।

$$x = n^2 + 1 \quad ; \quad y = n - 1 \text{ यहाँ से}$$

$$1 + 2x^2 + 2y^2 \text{ का अर्थ } 1 + 2(n^2 + 1)^2 + 2(n - 1)^2 = ?$$

$$1 + 2(n^2 + 1)^2 + 2(n - 1)^2 = (n + 1)^2 + (n - 1)^2 = 2x^2 = a^2$$

$$\frac{2 \cdot 2 \cdot 22}{2} = a \cdot q$$

अब अपना लिखें।

अब हम यह बताएँ कि यह कैसे होता है।

$$1 + x^2 = 2n$$

$$1 + x^2 = 22$$

~~$$a = 2, b = 22$$~~

यहाँ $n, n+1, n+2$ का अर्थ है कि यह दो सम्पूर्ण विभाजन होता है।

$$x^2 - 2y^2 = 1$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

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$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

$$1 + n^2 + 2n + 1 = n^2 + 2n + 2$$

so n can't have form $7 \cdot 2^k + 2$ for $k \geq 1$.

$$\text{or } x^2 - 2 \equiv 7 \pmod{8} \text{ can't happen.}$$

~~$\Rightarrow n \equiv 7 \pmod{8}$~~

~~$7 \equiv -1 \pmod{8}$~~

$$7^y \equiv (-1) \pmod{8}$$

if $y = \text{even}$:

$$7^y \equiv 1 \pmod{8}$$

if $y = \text{odd}$:

$$7^y \equiv -1 \pmod{8}$$

squares mod 8 can be only

$$0, 1, 4.$$

~~If $x^2 \equiv 0 \pmod{8}$ then~~

then

then

~~$x^2 \equiv 0 \pmod{8} \Rightarrow x^2 - 2 \equiv -2 \pmod{8} \Rightarrow 7 \equiv 7 \pmod{8}$~~

$$\text{if } x^2 \equiv 1 \pmod{8} \quad | \quad x^2 - 2 \equiv -1 \pmod{8} \quad | \quad 7 \equiv 7 \pmod{8}$$

$$x^2 \equiv 4 \pmod{8} \quad | \quad 7 \equiv 7 \pmod{8}$$

$$7 \equiv -1 \pmod{8}$$

so $y = 2k$

$$7^y \equiv 1 \pmod{8}$$

graph

$y = 7^x$
with x odd
odd \Rightarrow even
 \Rightarrow even
using $7^{2k+1} = 7 \cdot 7^{2k}$
and $7^{2k+1} = 7^2 \cdot 7^{2k-1}$
so $y = 7^{2k+1}$

$$\Rightarrow y = \text{odd} - b$$

$$y = 2z - 1$$

$$x^2 - 2 = 7^{2k+1} - 2 = 7 \cdot (7^{2k})^2 + 1 = 7^{2k+2} + 1$$

$$x^2 = 7(7^2)^k + 1 = 7^{2k+2} + 1$$

$$x^2 = 7^{2k+2} + 1 = 7^{2k+2} + 2 \cdot 7^{2k+1}$$

$$x^2 = 7^{2k+2} + 2 \cdot 7^{2k+1} + 1$$

$$x^2 = 7^{2k+2} + 2 \cdot 7^{2k+1} + 1$$

$$= (7^{2k+1} + 1)^2$$

$$(7^{2k+1} + 1)^2 - 7^{2k+2} = 1$$

$$7(7^{2k+1} + 1)^2 - 7 \cdot (7^{2k+1})^2 = 1$$

so $7^{2k+1} = 1 - 1 = 0$

and $7^{2k+1} = 1 - 1 = 0$

$x = -7$, $y^2 = 2$, $y_2 = 8$, $j, j_1 = 3$, is $s = 1/4$ true.
pell equation. by now general solution.

$$x_k + y_k \sqrt{7} = (8 + 3\sqrt{7})^k$$

$$\begin{aligned} x = 7^{2k+1}, y = 7^{2k+1} \\ \therefore x_k + y_k \sqrt{7} = 8^k + \binom{k}{2} 8^{k-2} \cdot 3^2 \cdot 7 + \end{aligned}$$

$$\begin{aligned} x_k + y_k \sqrt{7} = \binom{k}{1} 8^{k-1} \cdot 3 + \binom{k}{3} 8^{k-3} \cdot 3^3 \cdot 7 + \end{aligned}$$

$$\left(\frac{k}{m+1} \right) 7^m = \frac{m \cdot k(k-1) \cdots (k-m)}{1 \cdot 3 \cdot 5 \cdots k} \cdot 7^m$$

$$7^2 \text{ divides } k \\ 8^k > 7^2 > 7^{2k+1}$$

$$\begin{aligned} \binom{k}{m+1} &= \frac{k!}{m!(k-m)!} \\ \binom{k}{m+1} &= \frac{k!}{(k-m)!} \\ \therefore (k-m)! &= \frac{k!}{\binom{k}{m+1}} \end{aligned}$$

so now to iterate pell's eqn.
2800400.

$$\text{by } (x+j)^3 - x^3 = j^2$$

$$\text{express } 3x^2 + 3x + j = s^2$$

where x, s are integers

$$(2y)^2 - 3(2x+j)^2 = 1$$

$$u^2 - 3v^2 = -8 \text{ is pell's eqn.}$$

solutions

odd

$$u_n \pm v_n \sqrt{3} = (2 \pm \sqrt{3})^n \quad \text{if } n = 2m+1$$

$$u_n \pm (2x_n + 1)\sqrt{3} = (2 \pm \sqrt{3})(7 \times 1)^n \cdot (2 + \sqrt{3})^2 \cdot (2 + \sqrt{3})^{2m+1} = (2 + \sqrt{3})(7 + 4\sqrt{3})^{2m+1}$$

x_n, y_n odd numbers
p.s. $(h_{2k+1} + j) \sqrt{3}$
are odd
odd so y_n .

$$701$$

(Q6) $(x-y)^5 = x^3 - y^3$ ∴ L.S.O.R.H.S is
 relatively prime $x-y = x-y$

Let t be

gcd of x, y

$$t \stackrel{x}{=} \frac{x}{t} ; t \stackrel{y}{=} \frac{y}{t}$$

(x, y) are relatively prime

$$t^5 (u-v)^5 = t^3 (u^3 - v^3)$$

$$t^2 (u-v)^4 = u^3 - v^3$$

$$t^2 (u-v)^4 = \frac{u^3 - v^3}{u-v}$$

$$= u^2 + uv + v^2 = (u-v)^2 + 3uv$$

$$\therefore (u-v)^2 (t^2 (u-v)^2 - 1) = 3uv$$

$\Rightarrow (u-v)^2$ divides $3uv$ \Rightarrow u & v are relatively prime

~~$(u-v)(u+v) = 3 \cdot u \cdot v$~~

(u, v) relatively prime

~~$(u-v)(u+v) = \sqrt{3} \cdot u \cdot \sqrt{3} \cdot v$~~

$\Rightarrow u = v$ \Rightarrow $u = v$. $3uv$

~~$\therefore (u-v)^2 \cdot (t^2 (u-v)^2 - 1) = 3uv$~~

\therefore

~~$(u-v)^2 = 3uv$~~

$\Rightarrow u^2 - 2uv + v^2 = 3uv$

~~$t^2 (u-v)^2 - 1 = \frac{3uv}{(u-v)^2}$~~

$\Rightarrow 3(u-v)^2$ divides $3uv$

$(u-v)$ relatively prime

~~$\therefore \frac{u-v}{u-v} = \frac{3uv}{u-v}$~~

No 32.

~~$\therefore u-v = 1$~~

$$c^2 - 1 = 3uv = 3u(v+1) \rightarrow$$

~~(cancel)~~ ~~3u(v+1)~~

$$c^2 - 1 = 3u(v+1), c^2 = 3u(v+1) + 1$$

Q 63A (2nd part)
SMS-3C "In
Jewel, no
(u) $\rho(u+d)^3$ is 100000.

~~u-v=1~~
~~u=v+1~~
~~u must be even~~

~~u > v~~
~~u > v+1~~
~~u > v+2~~
~~u > v+3~~
~~u > v+4~~
~~u > v+5~~
~~u > v+6~~
~~u > v+7~~
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~~u > v+99~~
~~u > v+100~~

61) $x^3 + y^3 + z^3 + t^3 = 1999$

$x=10, y=10, z=-1, t=0$ is solution.

~~77+1172~~ signs \rightarrow symmetry

Four numbers ~~not~~ $(10+n, 10-n, -\frac{1}{2}+u)$ \rightarrow ~~not~~ 210:

u is half integer/

WTS $\frac{1}{2} \pm u$, ~~is integer~~

\therefore ~~2000~~ $u = \frac{w}{2}$, ~~WTS~~ $w = \text{odd}$.

$$\left(10+n\right)^3 + \left(10-n\right)^3 + \left(-\frac{1}{2}+u\right)^3 + \left(\frac{1}{2}-u\right)^3 = 1999$$

$$\begin{aligned} x=10, y=10 &\text{ are solutions} \\ \frac{10+10}{2} = 10 &\text{ by } 10+n \\ z=-1, t=0 & \\ -\frac{1+0}{2} = -\frac{1}{2} & \text{ by } -\frac{1}{2}+u \\ -\frac{1-0}{2} = -\frac{1}{2} & \text{ by } -\frac{1}{2}-u \end{aligned}$$

$$\left(10+n\right)^3 + \left(10-n\right)^3 + \left(-\frac{1}{2}+u\right)^3 + \left(\frac{1}{2}-u\right)^3 = 1999$$

$$2000 + 60n^2 \neq$$

$$\begin{cases} u+b = 4 \\ u-b = \frac{1}{2} - u \end{cases} \quad -3ab(a+b)$$

$$\left(u = \frac{w}{2}\right)$$

$$2000 + 60n^2 \neq 1999$$

$$-\frac{1}{4} + 3u^2 = 1999$$

$$2000 + 60n^2 + \frac{1}{4} + \frac{3w^2}{4} = 1999$$

$$(2000+60n^2) - 1 + \frac{3w^2}{4} = 1999$$

$$w^2 - 80n^2 = 1 \quad (w, n) = (9, 1)$$

$$w + \sqrt{80} \quad (w + \sqrt{80})^2$$

$$x^4 + y^4 + z^4 = (2002)$$

$$2002 = \underbrace{3^4 + 5^4 + 46^4}_{\text{max}} = 1002 \text{ sush.}$$

$$\text{A max, der } x_F = 2002 - 3^4$$

62. y, y_1, z, z_1 max
 $2002 - 3^4$ max
 ~~$2002 - 3^4$~~ max

$$x^4, y^4, z^4 \text{ max}$$

~~x^4, y^4, z^4~~ max (max)
 ~~x^4, y^4, z^4~~ max
 ~~x^4, y^4, z^4~~ max

$$x_F > x_F = 3$$

$$x_F = 3 \cdot 2002^k$$

$$y_F = \underbrace{5 \cdot 2002^k}_{\text{min}}$$

$$z_F = 4 \cdot 2002^k$$

$$2002^k$$

$$\text{max } y_F = 4 \cdot 2002^k$$

$$\text{min } x_F = 3 \cdot 2002^k$$

$$\leftarrow \text{je RZL } \rightarrow \text{RZL}$$

$$3^4 \cdot 2002^k \leq 3^4$$

$$1000 \cdot 2002^k \leq 2002^k$$

$$\rightarrow \text{RZL } \rightarrow \text{RZL}$$

$$2002^k \leq 2002^k$$

$$\begin{aligned} x_F^4 + y_F^4 + z_F^4 &\geq 3 \\ \Rightarrow 3(3^4 + 5^4 + 46^4) (2002)^{4k} &= \\ &= 2002^{4k+4} \end{aligned}$$

$$\textcircled{816} \quad \cancel{x^4 + y^4 = 4^2 = 16}$$

$$\textcircled{817} \quad x^2 + y^2 + z^2 + \cancel{3x + 3y + 3z + 5} = 0$$

$$\cancel{x^3 + 3x} = \left(x + \frac{3}{2}\right) - \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 + \left(z + \frac{3}{2}\right)^2 = \frac{2}{9}$$

$$4\left(x + \frac{3}{2}\right)^2 + 4\left(y + \frac{3}{2}\right)^2 + \dots$$

$$4\left(x + \frac{3}{2}\right)^2 + 4\left(y + \frac{3}{2}\right)^2 = (2x + 3)^2$$

$$(2x + 3)^2 + (2y + 3)^2 + \dots = 71$$

$$u = 2v - 3 \quad ; \quad 6v = 7v + 3 \quad w = 2v + 3$$

$$u^2 + v^2 + w^2 = 7 \quad ; \quad u = \frac{6v}{T} \quad ; \quad w = \frac{w}{T}$$

$u^2 + v^2 + w^2 = 7T^2$ since $u, v, w \in \mathbb{Z}$
 ~~$14T^2 + 10v^2 + 7w^2$~~ is not
 $u^2 + v^2 + w^2 = 7T^2 \pmod{4}$

$T \in \mathbb{Z}$
 $\text{for } u, v, w$
 $\text{mod } 4 \Rightarrow u^2, v^2, w^2$
 $\text{quadratic residue mod } 4$
 $14T^2 \pmod{4}$
 $14 \pmod{4} \text{ and } 14 \equiv 2 \pmod{4}$
 $2 \pmod{4}$

$\pmod{4} \Rightarrow$ square $\forall T$
 $\text{mod } 4 \times 0, 1, 2, 3 \pmod{4}$

$14T^2 \pmod{4}$
 $14 \pmod{4} \Rightarrow$ square $\forall T$ ($\pmod{4}$ 0 or 2)

$$3 \pmod{4} \quad ; \quad 7T^2 = 3T^2 \pmod{4}$$

$$3T^2 \equiv 0 \pmod{4} \Rightarrow T = \sqrt[4]{0}$$

$$3T^2 \equiv 3 \pmod{4} \Rightarrow T = \sqrt[4]{3}$$

$$3T^2 \equiv 0 \pmod{4}$$

$$3T^2 \equiv 3 \pmod{4}$$

$g \cdot u, v, w, T$ is even

$g \cdot u, v, w, T$ is odd

$g \cdot u, v, w, T$ is even

$$\text{mod } 8 \quad \text{by } x^2 \pmod{8} \quad \Rightarrow \quad x^2 \equiv 0 \pmod{8} \quad \text{or} \quad x^2 \equiv 4 \pmod{8}$$

$$\text{mod } 8 \quad x^2 \pmod{8} \quad \Rightarrow \quad x = \text{odd} \quad ; \quad x^2 \equiv 4 \pmod{8}$$

$$u^2 + v^2 + w^2 = 7T^2 \pmod{8}$$

$$3 \pmod{8} \quad ; \quad 7 \pmod{8}$$

$$g \cdot u, v, w, T$$

$$g \cdot u, v, w, T$$

u:47 info. n dimensions. In matrix rep. chords.

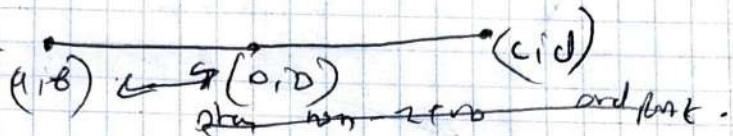
new. so chord's length doesn't depend on direction.



so (hook) word casting where chords are in orthogonal axis (perp. of B).

Now consider suppose P fixed point not at origin

③

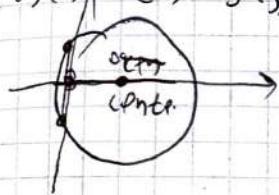


④

endpoints (right) & (left)

endpoints of k-th word $(x_k, (0, 0, 0, \dots))$ and $(y_k, (0, 0, 0, \dots))$

(center $(0,0)$)



intersection of word chord - k

midpoint =

$$0\text{-L coord} = \left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, \dots, \frac{x_n + y_n}{2} \right)$$

$x_k = 0$ & $y_k = 1$, P is some sphere

$$\left(x_k - \frac{x_1 + y_1}{2} \right)^2 + \sum \left(\frac{x_i + y_i}{2} \right)^2 = R^2$$

$$\left(y_k - \frac{x_1 + y_1}{2} \right)^2 + \sum \left(\frac{x_i + y_i}{2} \right)^2 = R^2$$

a 3D cube,



width

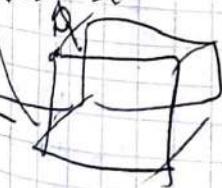
$$\text{edge} = h = \sqrt{5 \cdot 1 \cdot a}$$

63.

$(2n+3)$ dim. cube have integer

$(n+1)$ -th 3D face proportion there.

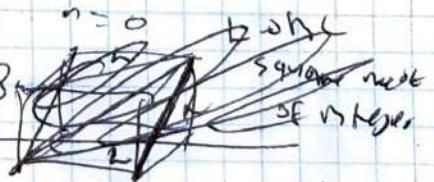
1. dim
edge 2D.



2. 2D. 2G3D. 3D 3D. 4D 4D

$$n=1 \quad (n \geq 0) \quad n \geq 0$$

$$2(1)+3=4$$



cube's volume = $| \text{word entries of } f | \Rightarrow$
vertices

U2

$$| L^2 = M | \text{ integer } 100\%$$

$$\frac{N}{L^2} = k = \text{int. number.}$$

$$k^2 = M \text{ int.}$$

$$\Rightarrow L \text{ is int. } \Rightarrow$$

$$\equiv \text{int. } \frac{M}{L^2} \text{ int.}$$

$$\left| \frac{L}{L^2} = \text{int.} \right|$$

(635) $\gamma = (n; \dots; 2, 1)$

$\gamma = \text{range } (\bar{\sigma}; \delta) \rightarrow \mathbb{N}^*$

$\mathbb{N}^n - \emptyset$

$$\sum_{\sigma} \text{sign}(\sigma) (\rightarrow x_{\sigma(1)} \times x_{\sigma(2)} \cdots x_{\sigma(n)})$$

From this we have plane.

$$*\quad \gamma(1) = n$$

Liebniz form. of det

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \left(\prod_{i=1}^n a_{\sigma(i)}^i \right)$$

$\sigma = \sigma_1 \sigma_2 \dots \sigma_n$

and matrix

permute rows
($\sigma \mapsto \text{row } i$)
as follows.

$$\gamma(1) = 1$$

$$\gamma(2) = 1$$

$$\begin{array}{c|cccc|c} & x_1 & x_2 & \dots & x_n \\ r_\sigma & x_n & x_1 & \dots & x_{n-1} \\ & \vdots & & & \\ & x_2 & x_3 & \dots & x_n \end{array} \quad \det = 0$$

$$x_{(1)} =$$

$$x_{(1)} = n$$

$$x_{(2)} =$$

7.

a plane-line diagram

add paths to dist

$$r_\sigma = (x_1 + x_2 + \dots + x_n)$$

$$\det(n) = (x_1 + x_2 + \dots + x_n) \parallel \cdot \text{fun}(\text{fun of } x_1 x_2 \dots x_n)$$

$$\det(n) = 0 \quad \text{fun} = 0.$$

(636)

$$\gamma' = (n; n-1; \dots; 2, 1)$$

cycle permutations shifts at left position

~~numbering~~.

$$\text{e.g. } n = 4, 1, 3, 2 \quad T = (5, 4, 1, 3; 1, 2, 3, 4)$$

$$T(2, 1) = 5 \quad \text{(-> 5) 1 cycle in 4}$$

from 2nd

5 -> 1

new 1st

1 -> 2

$\gamma(\sigma(2)) \text{ 1st?}$

$$\tau(2) = 1$$

<-> 1st? { 5 sequences per

$\sigma(2)$ sequence $(2, 1, 3, 4)$

etc.

$$\sigma(2) = 3$$

$\sum \text{sign}(\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}$ in sequences shifting

b2m
 n -dim. Raum mit
 endlich Parallelen zu einer Apfelpi

mit Planes als Winkel

$$\vec{a} = \left(\cos \frac{2\pi}{n}, \cos \frac{4\pi}{n}, \dots, \cos \frac{2(n-1)\pi}{n} \right)$$

$$\vec{b} = \left(\sin \frac{2\pi}{n}, \sin \frac{4\pi}{n}, \dots, \sin \frac{2(n-1)\pi}{n} \right)$$

3 regular 2D-Subs

b3b n -dim. Raum $R=1$

distance of 2 points $> \sqrt{2}$

b3m unit sphere
 $S^{n-2} = \{ (x_1, x_2, \dots, x_n) \in A^n \mid \sum_{k=1}^n x_k^2 = 1 \} = \{ \text{points on } S^{n-1} \}$

a2
 2 points
 $X = (x_1, x_2, \dots, x_n) \quad Y = (y_1, y_2, \dots, y_n)$
 $d(X, Y) = \sqrt{\sum_{k=1}^n (x_k - y_k)^2} \geq \sqrt{2} - 8$

$$\sum (x_k^2 - 2x_k y_k + y_k^2) \quad d^2(X, Y) = \sum_{k=1}^n x_k^2 + \sum_{k=1}^n y_k^2 - 2 \sum_{k=1}^n x_k y_k \geq 2$$

$$\Rightarrow \sum_{k=1}^n x_k y_k \leq 0$$

b3m A_1, A_2, \dots, A_m \rightarrow 2D-Subs
 so \exists 2 points A_i, A_j with $d(A_i, A_j) > \sqrt{2}$

A_i, A_j Sums of $1 \Rightarrow A_i = (-1, 0, \dots, 0)$

$$A_i = (x_1, x_2, \dots, x_n) \quad A_j = (y_1, y_2, \dots, y_n)$$

$$d(A_i, A_j) > \sqrt{2}$$

$$\sum_{k=1}^n x_k y_k < 0 \quad x_k, y_k = \text{positive}$$

$$\Rightarrow \sum_{k=1}^n < 0$$

~~$d(A_i, A_j)$~~ $d(A_i, A_j)$ Both ~~$\geq \sqrt{2}$~~
 $d(A_i, A_j) > \sqrt{2}$

$$\text{by } d(A_i, A_j) \geq \sum_{k=1}^n x_k y_k < 0$$

$$\text{as } d(A_i, A_j) = \sqrt{(y_1 + 1)^2 + y_2^2 + \dots + y_n^2} > 0$$

$$(y_1 > 0) \quad | \quad x_1 > 0$$

$$\sum_{k=1}^n x_k y_k \leq 0$$

$k=2$

normalize

last $n-1$ coord. of mts A_i

$$\text{by } x'_k = \frac{x_k}{\sqrt{\sum_{k=1}^{n-1} x_k^2}} \quad k = 1, 2, \dots, n-1$$

loop, normalized (order dist)

x_k

~~x_1, x_2, \dots, x_n~~
 ~~x_1, x_2, \dots, x_{n-1}~~
 ~~C_{2M36}~~

~~y_1, y_2, \dots, y_n~~
 ~~D_{2M36}~~

$$A = \begin{pmatrix} M_{11} & & \\ M_{12}, M_{13}, \dots, M_{1n} \\ \vdots \\ M_{n1}, M_{n2}, \dots, M_{nn} \end{pmatrix}$$

P_{n1}

$$M_n \geq d + M_{n-1}$$

$$(M_2 = 2) \quad M_n \leq n+1$$

$M_n = n+1$ by simplex

~~size of n dimension~~

~~length of $n+1$ side of unit simplex~~

$$\text{coordinates } (x'_1, x'_2, \dots, x'_{n-1})$$

or word dim $n-1$

$(0, 0, \dots, 0, 1) \rightarrow e_n$, e_n

$S^{n-2} - \text{set of } S^{n-1} - e_n$

~~unit sphere~~ ~~unit simplex~~

$$\text{unit sphere} \quad \sqrt{\sum_{k=1}^{n-1} x_k'^2} = 1$$

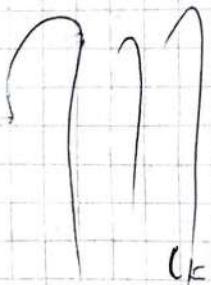
unit simplex \rightarrow set.

Distance from center of sphere to hyperspace of n -dim 3,000

simplex is $\frac{1}{n}$. $\Rightarrow A_1 = (-1, 0, 0, \dots, 0)$

$$A_2 = \left(\frac{1}{n}; -1, 0, 0, \dots, 0 \right)$$

$$A_3 = \left(\frac{1}{n}; \frac{1}{n-1}, -1, 0, \dots, 0 \right)$$



$$A_{n+1} = \left(\frac{1}{n}; \frac{1}{n-1}, \dots, \frac{1}{2}, 1 \right)$$

$$A_n = \left(\frac{1}{n}; \frac{1}{n-1}, \dots, 1, \frac{1}{n-2}, 2, \dots, 3, \dots \right)$$

$$c_k = \sqrt{\left(1 + \frac{1}{n}\right)\left(1 - \frac{1}{n-k+1}\right)} = 2$$

c_k where n dim simplex \rightarrow distance between points

$$d(A_i, A_j) =$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots} = \sqrt{2 \cdot \frac{1}{n}}$$

$$c = \sqrt{2 \cdot \frac{1}{n}}$$

for n rectangular strips of width 2

where $n \geq m$

$$2\pi \int_a^B f(x) \cdot \sqrt{f'(x)^2 + 1} dx = 2\pi \int_a^B \sqrt{R^2 - x^2} \frac{R}{\sqrt{R^2 - x^2}} dx$$
$$= 2\pi \int_a^B R dx = 2\pi R d$$

Arch cut for sphere of radius R
2 parallel planes at distance d from each other
 $= 2\pi R d$ if $d = 2R$

sphere's area $4\pi R^2$ is covered by m surfaces.

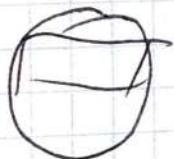
each area $4\pi R^2 / m^2$.

3D

strips \rightarrow Area of strips

\rightarrow 2D, 3D problems.

3D's surface



$$O^{M\#} f: \{o_1, o_2, \dots\} \rightarrow \{o_1, o_2, \dots\}$$

$$2f(n^2) = (f(n))^2 + (f(n))$$

$$n = n = 0 - B_2 \quad 2f(0^2 + 0^2) = f(0)^2 + f'(0)^2$$

$$\Rightarrow \underbrace{f(\alpha^2) = f(\alpha)}_{f(\alpha) = \alpha} \Rightarrow f(\alpha) = L$$

$$\text{Since } f(1) = 0 \text{ and } f(1)^2 + f(0)^2 = f(1)^2 + f(0)^2$$

$$2f(x^2 + 0) = 0 \quad (1)$$

$$2f(1) = f(j) \Rightarrow \frac{f(1) = 2}{f(j)} = 2$$

$\Rightarrow f(1) = 2$ $f(0)$ ی ۳۱ است و ۰ را نمایم

$$f(0) = f(0) = 0 > b$$

$af(2 + \epsilon)$

$$2f(1) = f(1)$$

$$2f(2) = 2f(1^2 + 1^2) = f(1)^2 + f(1)^2 = 8$$

$$m(f(2)) = 4$$

$$2 \cdot f(4) = 2f(2^2 + 0^2) = f(2)^2 + f(0)^2 = 18$$

$$\begin{aligned}2f(5) &= 2f(2^2 + 3^2) = f(2)^2 + f(3)^2 = 20 \\2f(8) &= 2f(2^2 + 2^2) = (f(2))^2 + (f(2))^2 = 32\end{aligned}$$

$$f(2) = 8 \quad ; \quad f(5) = 16 \quad ; \quad f(8) = 16 \quad \dots$$

$$f(n) = 2n \quad \cancel{f(n) \leq 2n}$$

$$100 = 80 \text{ (остаток)} + 20 \text{ (делитель)}? \quad n = 10 - 80 \text{ (остаток)} + 20 \text{ (делитель)} = 100$$

$$\frac{1}{100} \cdot \cancel{\left(\frac{1}{100} \right)^2} = 1 \cdot \left[(-5)^2 + (-) \right]^2 =$$

$$(f(5))^2 = f(5)^2 + f(12) \xrightarrow{f(5)=0} 0 = 2 \cdot f(5^2) = 2f(3^2+4^2) = f(3)^2 + f(4)^2$$

۷۸۳

$$\begin{aligned} (f(10))^2 &= f(10)^2 + f(0)^2 = \\ &= 2f(10)^2 = 2f(6^2 + 8^2) = f(6)^2 + f(8)^2 = \end{aligned}$$

$h \leq 10$, \exists $\varphi_{2289275}$

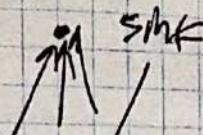
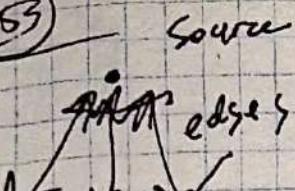
~~4400 1445er - B).~~

Introduction

5210-2

$f(5) < f(f(5)) < f(f(f(5)))$

63



face whose edges form cycle containing source and sink.

n_3 number

n_3 = number of cycles

$n_4 = \sum$ of all vertices is source, it has 0 incoming edges.
is sink, it has min + 2 edges (both arguments)

number of edges
number of vertices

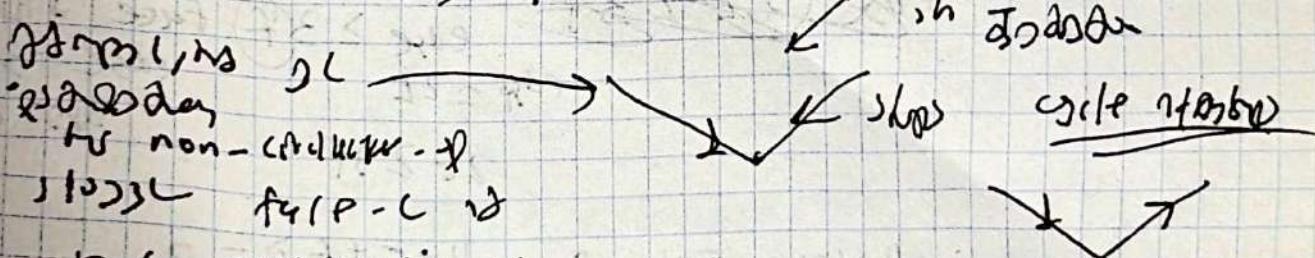
face which contains sink has 2 edges towards sink vertex
min edge on

$$E = v - n_1 + f - n_3$$

$E \geq v - n_1$, by how many vertices does it contain? n_3 vertices.

face = non-cycle faces edge incident to, n_3

two edges point toward sink vertex.



$$E - c \text{ over counting } n_3$$

now E, n_3 are given n_1, n_2

non-sink vertex - is away from n_1, n_2 $E = v - n_2$

so $v > n_2$ \Rightarrow non-sink vertex graph.

$$E = v - n_1 + f - n_3$$

but sink vs. non-sink $E > v$ having edges ~~incident~~

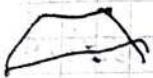
$$E = v - n_1 + f - n_3 - n_2 - n_3$$

$$v + n_1 + n_2 + n_3 + n_4 = v - E + f = 2$$

$$\frac{n_4 \geq 0, \text{ so } n_1 + n_2 + n_3 \geq 2}{\text{new strategy}}$$

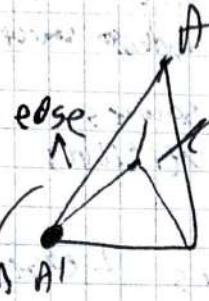
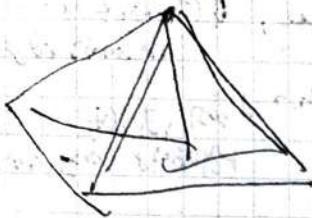
(682)

$$(a) v - e + f = 2$$



Given 2 faces & 3 edges

$$v = 2, e = 3$$



area of $\triangle \rightarrow 5$ sides (sides)
abspm, 3 edges (edges)
 $\Rightarrow 2 \cdot 3 = 6$ (at least edges.)

Also, each edge has 2 vertices, right?

any 3 edges
G - 2 edges doesn't cover all
(A B C D E F)
 $A \sim B \sim C \sim D$

~~another 3 edges~~

~~at least 3 edges, right?~~

$$\Rightarrow \text{each edge} = 2E \cdot \text{edges} \geq 6V$$

$\Rightarrow 2E \geq 6V$

~~$2E \geq 3F$~~

At least 3 edges.

~~$edge \geq 3f$~~ face & m.

~~$edge = 2E$~~

edge = 3 - 6f

~~$2E \geq 3F$~~

$$R/1224 E 46e - 3 \quad v - e + f = 2$$

$$v = n \quad 2 \leq \frac{1}{3}E - 6 + \frac{2}{3}E = 0$$

No unique solution.

So, at least 6 edges in total required.

So, max 5 edges

$$69) \quad V - E + F = 2 \quad \xrightarrow{(2\pi)} \quad \text{sum of exterior angles}$$

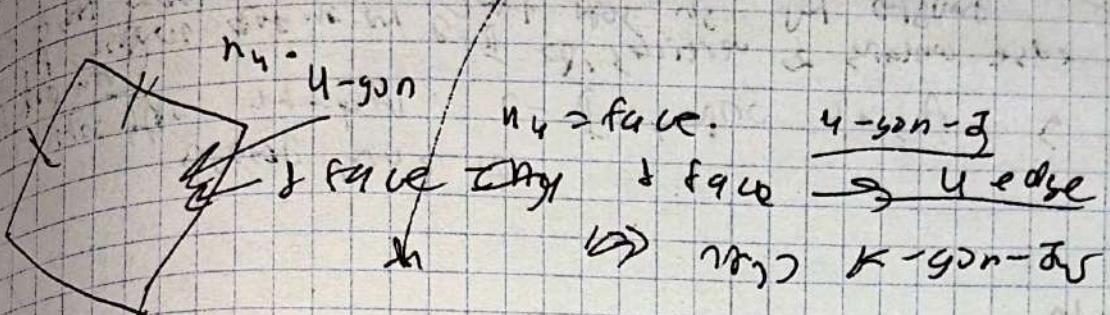
$$2\pi V - 2\pi \cdot E + 2\pi F = 4\pi$$

n_F - number of faces that are k -gons.

$$F = n_3 + n_4 + n_5 + \dots$$

each edge belongs to $\frac{1}{2}$ faces $\rightarrow 2$ faces.

\Rightarrow $2n_3 - 3$ 4-gon-edges meet at every edge.



$$\Rightarrow 2E = 3n_3 + 4n_4 + 5n_5 + \dots$$

loop each edge belongs to 2 faces

$$2\pi V - 2\pi \cdot E + 2$$

$$\Rightarrow 2\pi V - 2\pi (n_3 + n_4 + n_5 + \dots) = 4\pi$$

$$\left. \begin{aligned} \text{sum of all angles } A \text{ of } k\text{-gons} &= (k-2)\pi \cdot \frac{1}{2} \sum \\ \text{of faces} &= \sum 2\pi n_k \end{aligned} \right\} ?$$

$$\Rightarrow F = 4 \cdot \frac{n_3 + n_4 + \dots}{2} \cdot (k-2)$$

$$\approx 111 \cdot 2$$

$$70) \quad F \geq 5 \quad \text{and} \quad \Sigma = \sum n_k$$

3 edges

leave from each vertex.



\Rightarrow 3rd face of polygonal shell

$$\text{triangle} \Rightarrow E = \frac{3f}{2} \quad \Rightarrow \quad f - \frac{3f}{2} + f = 2,$$

$$\left. \begin{aligned} F &= 4 \\ \Sigma n_k &\geq 5 \end{aligned} \right\} \text{sum of exterior angles}$$

from 4th face by induction has triangle $(A_1, A_2, \dots, A_{n-3})$; hence 4th face - 6 has 3 edges in total, which is 7 edges. 3 edges system yields $F = 7$ for 4th face, which is wrong.

Ex. 11-2 ~~the~~ a word having no face - i., man

$A_1 A_2 - c$, (L2W2 W1 A1 A2 - w1)
 $A_1 A_2 - c$, (L2W2 W1 A1 A2 - w1)
 $A_1 A_2 - c$, (L2W2 W1 A1 A2 - w1)

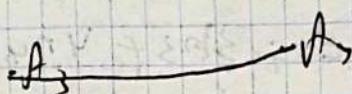
1) Listen - & sing 5 songs of your favorite artist. ^{free} ~~which you like~~

~~200m 0.5 day 200 short 30~~

- thoughts by John Thorpe & I know first each
edge contains 2 vertices, so if Δ is the total number of edges, $A_3 \leq$
 $\frac{\Delta}{2}$

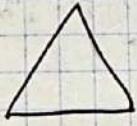
Σ is a A_3 -like surface, $I \rightarrow A_4$ -like surface.

J. 3010.



free - jh

“અનુભાવ”



as how the physcs,
and maths of grav.

~~11-12 years~~

148

A2

1

11

I 3) 10 I, 2 314
59 3 100 2012211

A₃ → 332 11 16
23 h द्वितीय.

$$V - E + F = 2$$

$F - E = n+2$

$$\vec{r} \cdot \vec{v} = n$$

$$E = 3n - 6$$

Every edge belongs to 2 faces

why edge belongs to 2 faces
eg: every face has 3 edges. ($2E = 3F$)

$$0 \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{and} \quad \binom{n}{k} = \binom{n}{n-k}$$

$$\text{Ex. 1. } \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} =$$

$$1 \leq m \leq n \quad = \sum_{k=0}^{m-1} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right)$$

$$\binom{n}{m} = n \cdot \binom{n-1}{m-1} = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} - \sum_{k=0}^{n-2} (-1)^k \binom{n-1}{k}$$

$$\binom{n}{m} = \frac{n}{m} \binom{n-1}{m-1} \quad n(-1)^{m-1} \binom{n-1}{m-1} = m(-1)^{m-1} \binom{n}{m}$$

$$\Rightarrow \binom{n}{m-1} = m \binom{n}{m}$$

$$\text{Polynomial representation}$$

$$\text{Ex. } (x^{k+1}-1)(x^{m-j}-1) \cdots (x^{l+1}-1)$$

$$\frac{(x^{k+1}-1)(x^{m-j}-1) \cdots (x^{l+1}-1)}{(x^m-1)(x^{m-1}-1) \cdots (x-1)}$$

$$x = \beta \text{ で } \binom{k+m}{m} = \binom{k+m}{m} \times \binom{k+m}{m-1}$$

$$\text{A. } \binom{k+m+1}{m} = \binom{k+m}{m} + \binom{k+m}{m-1}$$

\Rightarrow Polynomials.

$$= \frac{(x^{k+1}-1) + \cdots + (x^{l+1}-1)}{(x^m-1) \cdots (x-1)}$$

$m=0, 1, 2, \dots$

\Rightarrow x^k-1

$$\frac{x^{k+1}-1}{x-1} = x^k + x^{k-1} + \cdots + 1$$

$$+ x^{k+1} \frac{(x^{m-k}-1) - \cdots - (x^{l+1}-1)}{(x^{m-1}-1) \cdots (x-1)}$$

$$[x+y]_n = \sum_{k=0}^n \binom{n}{k} [x]_k [y]_{n-k}$$

$$[x]_n = x(x-1) \cdots (x-n+1)$$

↗ (wie kann man dies in der Form schreiben?)

$$\frac{d}{dx} t^{x+j} = (x+\cancel{j}) t^{x-1}$$

$$\frac{d^2}{dx^2} t^{x+j} = \cancel{x(x-1)} (x-1)(x) t^{x-2}$$

$$[x+y]_n = \underline{(x+y)(x+j-1)} : \dots : \underline{(x+y)-n+j} t^n$$

$$\frac{d}{dx} t^{x+j} = (x+y) t^{x+y-1}$$

$$\left(\frac{d}{dx}\right)^n t^{x+j} = (x+j)(x+j-1) \cdots (x+y-n+j) t^{x+y-n}$$

$$\left(\frac{d}{dx}\right)^n t^{x+j} = [x+y]_n t^{x+y-n} \quad \text{Kl} \quad \text{①}$$

$$\rightarrow \left(\frac{d}{dt}\right)^n (t^x \cdot t^y) = \sum_{k=0}^n \binom{n}{k} \left(\frac{d}{dt}\right)^k t^x \cdot \left(\frac{d}{dt}\right)^{n-k} t^y$$

leider

$$\left(\frac{d}{dt}\right)^n (f(t) \cdot g(t)) = \sum_{k=0}^n \binom{n}{k} \underbrace{\left(\frac{d}{dt}\right)^k f(t)}_{f'(t) \cdot g(t)} \cdot \underbrace{\left(\frac{d}{dt}\right)^{n-k} g(t)}_{g'(t)}$$

$$= \sum_{k=0}^n \binom{n}{k} [x]_k [y]_{n-k} \cdot t^{x+y-n}$$

$$\textcircled{1} = \textcircled{1}, \text{ aber nach } \left(\frac{d}{dt}\right)^n (t^{x+j}) - j! -$$

$$\textcircled{607} \quad a_n + b_n \sqrt[3]{2} + c_n \sqrt[3]{7} = (\underbrace{1 + \sqrt[3]{2} + \sqrt[3]{4}}_n)^n$$

$$3\sqrt{2} = \cancel{(\cancel{3\sqrt{2}} + 3\sqrt{2} + 3\sqrt{4})} = \\ = 3\sqrt{2} + 3\sqrt{2} \cdot 3\sqrt{2} + \cancel{3\sqrt{8}} = \\ = 3\sqrt{2} + 2 + 3\sqrt{2} + 3\sqrt{4}$$

$$(3\sqrt{2})^n (1 + 3\sqrt{2} + 3\sqrt{4})^n = (2 + 3\sqrt{2} + 3\sqrt{4})^n$$

$$\Rightarrow (1 + 3\sqrt{2} + 3\sqrt{4})^n = \frac{(2 + 3\sqrt{2} + 3\sqrt{4})^n}{(3\sqrt{2})^n}$$

$$S_n = \binom{2n+1}{0} 2^{2n} + \binom{2n+1}{2} \cdot 2^{2n-2} 3^2 + \dots + \binom{2n+1}{2n} 3^n$$

~~is sum of 2 consecutive squares~~ perfect squares.

$$\Rightarrow S_n = \frac{1}{4} (2 + \sqrt{3})^{2n+2} + (2 - \sqrt{3})^{2n+2}$$

$$S_n = (k-1)^2 + k^2 - S_n = 0$$

$$\begin{aligned}
 & \text{If } \theta \neq 3^\circ \text{ min? } \\
 & y_{3n+3} = 2[(1+\sqrt{3})^{2n+2} + (1-\sqrt{3})^{2n+2}] - 2 \\
 & (1+\sqrt{3})^2 = 2 + 2\sqrt{3} \text{ and } 2 - 2\sqrt{3} \text{ are reciprocals.} \\
 & \text{Now solving for } (1 \pm 2\sqrt{3}) \text{ gives solutions of } (1 \pm \sqrt{3})^{2n+2} \\
 & (1+2\sqrt{3})(2-2\sqrt{3}) = 2^{2n+2} = (1+2\sqrt{3})^{2n+2} + (1-2\sqrt{3})^{2n+2} \\
 & k = \frac{1+2\sqrt{3}}{2} = 1 + \sqrt{4n+1}
 \end{aligned}$$

$$k = \frac{1}{2} + \frac{(1-\sqrt{3})^{2n+1} e^{(1-\sqrt{3})^{2n+1}}}{2^{2n+2}} = \frac{1}{2} \sqrt{3} e^{\sqrt{3}} = \frac{1}{2} + \frac{(1+\sqrt{3})(2+\sqrt{3})}{2^{2n+2}} e^{(1+\sqrt{3})(2+\sqrt{3})}$$

$2 + \sqrt{3}$ ou $2 - \sqrt{3}$ > 3000, 10000 n' est pas.

$$\begin{array}{l} 2 + \sqrt{3} \\ 2 - \sqrt{3} \\ \hline \end{array} \quad \frac{\Delta}{\Delta} \quad \frac{x^2 - 4x + 1 = 0}{x^2 - 4x + 3 = 0}$$

valeurs x_1, x_2
 $x_1 + x_2$

$$\lambda^2 - 4\lambda + 1 = 0$$

$$\lambda = 2 + \sqrt{3}, \lambda = 2 - \sqrt{3}$$

$$\Rightarrow x_n = A(2 + \sqrt{3})^n + B(2 - \sqrt{3})^n$$

$\begin{cases} x_0 = 2 \\ x_1 = 2 + \sqrt{3} \end{cases}$



$$\begin{cases} x_0 = 2 \\ x_1 = 2 + \sqrt{3} \end{cases} \quad ? + \text{racine de } 3$$

soit $x_0 = 2$ et $x_1 = 2 + \sqrt{3}$

$$(1 + \sqrt{3})(2 + \sqrt{3})^n + (1 - \sqrt{3})(2 - \sqrt{3})^n = x_n$$

$$n=0, n=1, n=2, \dots, x_0 = 2, x_1 = 2 + \sqrt{3}$$

$$x_0 = A + B = 2$$

$$A(2 + \sqrt{3}) + B(2 - \sqrt{3}) = 1$$

et $x_1 = 2 + \sqrt{3}$

$A = 1, B = 0$

$$x_n = A(2 + \sqrt{3})^n + B(2 - \sqrt{3})^n$$

$$k = \frac{1}{2} + \frac{1}{\sqrt{3}} \left(\frac{M}{N} \right)$$

entiers.

$$2 + \sqrt{3}$$

$$2 + \sqrt{3} = \frac{2 + \sqrt{3}}{1}$$

$$2 + \sqrt{3} = 2 + \sqrt{3}$$

$$2 + \sqrt{3} = 2 + \sqrt{3}$$

$$M = 2 + \sqrt{3}$$

$$+ 2$$

$\text{Q2) } a_1, a_2, \dots, a_n$
 $s_n = a_1 + a_2 + \dots + a_n \quad \frac{2^n}{n+1} s_{n+1}$
 also $\sum_{k=0}^{n-1} \binom{n}{k} a_{k+1} = \frac{2^n}{n+1} s_{n+1}$

$a_2 = a_1 + d \quad | \cdot 2d$
 $a_3 = a_1 + 2d \quad | \cdot (i-d)d$
 so $a_i = a_1 + (i-1)d = a_1 + (n-i)d$
 $a_{k+1} = a_1 + kd \quad \Rightarrow \quad a_{n-k+1} = a_1 + (n-k)d =$
 $a_{k+1} - a_{n-k+1} = \cancel{a_1} - 2a_1 + (k+n-k)d =$
 $= 2a_1 + nd \quad \rightarrow$
 $s_{n+1} = \frac{n+1}{2} (a_1 + a_{n+1}) = \frac{n+1}{2} (a_1 + a_1 + nd) = \frac{n+1}{2} (2a_1 + nd)$
 $\therefore s_{n+1} = (n+1)(2a_1 + nd)$

$\sum_{k=0}^{n-1} \binom{n}{k} (a_{k+1} + a_{n-k+1}) = \frac{2s_{n+1}}{n+1} \sum_{k=0}^{n-1} \binom{n}{k} = \frac{2^{n+1}}{n+1} s_{n+1}$

$\sum_{k=0}^{n-1} \binom{n}{k} a_{k+1} \quad \Rightarrow \quad \sum_{k=0}^{n-1} \binom{n}{k} a_{k+1}$

(B01)

$$f_1 = f_2 = 3$$

$$f_{n+2} = f_n + f_{n-2}$$

Prove: $f_1 \binom{n}{1} + f_2 \binom{n}{2} + \dots + f_n \binom{n}{n} = f_{2n}$

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \right]$$

$$f_0 = 0, \sum f_0 \binom{n}{i} = \frac{1}{\sqrt{5}} \sum_{i=0}^n \binom{n}{i} \left(\frac{1+\sqrt{5}}{2} \right)^i - \sum_{i=0}^n \binom{n}{i} \left(\frac{1-\sqrt{5}}{2} \right)^i$$
$$= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{2n} - \left(\frac{1-\sqrt{5}}{2} \right)^{2n} \right) = f_{2n}$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{3+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right)^n \right)$$

$$\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2} \right)^2$$

$$f_n = \text{golden ratio } \varphi^{2n} \text{ by induction}$$

$$\sum_{i=0}^n \binom{n}{i} x^i = (1+x)^n \Rightarrow \left(1 + \frac{1+\sqrt{5}}{2} \right)^n = \left(\frac{3+\sqrt{5}}{2} \right)^n$$
$$x = \frac{1+\sqrt{5}}{2} \text{ by def.}$$

(B5)

to prove: $\frac{1}{k!} < \frac{1}{(k-1)!}$

$I_k > I_{k-1} - \alpha$

$$I_k = \frac{4k-2}{k} I_{k-2}$$

$\alpha = \frac{5}{4} \cdot \frac{1}{(k-1)!}$

$$\binom{2k}{k} = (2k)(2k-1) \cdots (2k-1)!$$

$$= \frac{4k-2}{k} (2k-2)!$$

$$= \frac{(2k-1)!!}{(k!)^2} \times (2k-2)!$$

$$\textcircled{36} P = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}$$

$n \times n$
matrix

$$A^2 = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 1 & \dots & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

81 hours - 20 flights
20 flights
62 hours
average 3 hours

$$A^3 = \begin{pmatrix} (2) & (3) & (4) & \dots & (n+1) \\ 0 & (2) & (3) & \dots & (n+1) \\ 0 & 0 & (2) & \dots & (n+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (2) \end{pmatrix}$$

$$A^k = \begin{pmatrix} (2) & (3) & (4) & \dots & (k+1) \\ 0 & (2) & (3) & \dots & (k+1) \\ 0 & 0 & (2) & \dots & (k+1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (2) \end{pmatrix}$$

$$(k-m-2)$$

$$(k-1)$$

71 hours

$$(B) + (B^{-1}) + \dots + (A+P) \leftarrow k \dots$$

$$= 2 \frac{(B+P)^{k-1}}{k+1}$$

63

average 16 hours (20 flights)

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

then if there is 3 roots r_1, r_2, r_3

$$a_{n+1} + u \cdot a_n + v \cdot a_{n-1} = 0$$

$$G(x) = \frac{a_0 + (u a_0 + a_1)x}{(1 - r_1 x)(1 - r_2 x)}$$

If $r_1 \neq r_2$ are roots of characteristic equation $\lambda^2 - u\lambda + v = 0$

then

$$G(x) = \frac{a_0 + (u a_0 + a_1)x}{(1 - r_1 x)(1 - r_2 x)} = \frac{1}{1 - r_1 x} + \frac{1}{1 - r_2 x} = \sum_{n=0}^{\infty} (\alpha r_1^n + \beta r_2^n) x^n$$

terms ≤ 1
 $a_n = \alpha r_1^n + \beta r_2^n$

$c_n = c_0 + c_1 n + \dots$

$x_0 \cdot x_1 \cdot x_2 \cdots x_n$
 from multiplying terms in groups (sharry)
 notably x_k is $k+1$ -th term

for each x_0, \dots, x_k
 x_{k+1}, \dots, x_n we get c_{n-k} (since x^{n-k})

$c_n = c_0 x_0 + c_1 x_1 + \dots + c_{n-k} x_{n-k}$
 $c_n = c_0 x_0 + c_1 x_1 + \dots + c_{n-k} x_{n-k}$

$$\therefore c_n = c_0 c_{n-1} + c_1 c_{n-2} + \dots + c_{n-1} c_0 \quad (0 \neq 1) \quad c_n = \sum_{k=0}^n (c_k c_{n-k})$$

$$G(x) = \sum c_n x^n; (G(x))^2 = \left(\sum c_n x^n\right)^2 = \sum_{n \geq 0} \left(\sum_{k=0}^n (c_k c_{n-k}) x^n\right)$$

c_n numbers
 c_n ways to choose k objects from n objects

$$x \cdot G(x) = \sum_{n \geq 0} \sum_{k=0}^n (c_k c_{n-k}) x^{n+1} = \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} c_k c_{n-1-k}\right) x^n$$

$$1 \cdot U(t)^2 = \sum_{n \geq 0} c_n x^n ; \text{ with } c_0 = 1, \text{ others } 0$$

$$U(x) = (c_0 + \sum_{n \geq 1} c_n x^n)$$

$$U(x) = \sum (c_n x^n) \div (c_0 + \sum_{n \geq 1} c_n x^n) = 1 + X \cdot U(x)$$

$$\begin{aligned} U(x) &= 1 - \frac{\sqrt{1-4x}}{2x} \quad | \quad \sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4x)^n = \\ &= \sum_{n=0}^{\infty} \frac{\binom{\frac{1}{2}}{n} (\frac{1}{2}-1) \cdots (\frac{1}{2}-n+1)}{(n+1)!} (-4x)^n \\ &= (1-2 \cdot \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{x^n}{n}) \end{aligned}$$

$$\begin{aligned} U(x) &= x^{-1} + 2 \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{x^n}{n} = \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{x^n}{n}, \text{ with} \\ &\quad x = \sum_{n=0}^{\infty} \binom{2n-2}{n-1} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n \end{aligned}$$

Binomial coefficients are generated by $(x-a)^n$, with previous terms.

$$\binom{n}{k}$$

$$\sum_{j=0}^n \binom{n}{j} 2^{n-j} \left(\frac{j}{x+a} \right) = \binom{n+1}{n}$$

$$\begin{aligned} \text{with } n &= j+k \\ x^{-2} - a &= 3x^2 + 3x \\ x^{-1} - a &= 2x^3 + 3x^2 \\ x^0 - a &= x^4 + 3x^3 + 3x^2 + x \\ x^1 - a &= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 \end{aligned}$$

$$(x-a)^j = \sum_{k=0}^j \binom{j}{k} x^{-k} a^k = \sum_{k=0}^j \binom{j}{k} x^{j-k}$$

$$\begin{aligned} (x-a)-b &\Rightarrow x \cdot \underbrace{\sum_{k=0}^j \binom{j}{k} x^{j-k}}_{\text{term 1.}} + b \cdot \underbrace{\sum_{k=0}^j \binom{j}{k} x^{j-k}}_{\text{term 2.}} \\ &= \underbrace{\sum_{k=0}^j \binom{j}{k} x^{j-k+1}}_{\text{term 1.}} + \underbrace{\sum_{k=0}^j \binom{j}{k} x^{j-k}}_{\text{term 2.}} \end{aligned}$$

now $a = b$

$$\binom{n}{k} = \binom{n}{n-k} \quad \binom{j}{k} \quad j-2k+1=0 ; k=\frac{j+1}{2}$$

binom. coeff. same

$$\binom{j}{\frac{j+1}{2}} = \binom{j}{j-\frac{j+1}{2}} \binom{j}{j-\frac{j+1}{2}} = \binom{j}{\frac{j-1}{2}}$$

$$\text{now term 2 in binom.} = 0 \quad (\text{not worst})$$

$$* j-2k=0 ; k=\frac{j}{2}$$

$$\begin{aligned} j &= even \\ \text{const term is} & \binom{j}{\frac{j}{2}} = \binom{j}{\frac{j-1}{2}} \\ j &= odd \\ \left[\frac{j}{2} \right] &= \frac{j-1}{2} \end{aligned}$$

$$\begin{aligned}
 & \left(\frac{j}{2} \right) \text{ 2nd term. Now } \left(\frac{j}{2} \right) \rightarrow ? \\
 & \text{Now } \left(\frac{j}{2} \right) \text{ has } \left[\frac{j}{2} \right] = \left(\frac{j-1}{2} \right), \text{ but } j \text{ is odd} \\
 & \Rightarrow \left[\frac{j}{2} \right] \text{ is also } \left(\frac{j-1}{2} \right)^2 \text{, but } j \text{ is even} \\
 & \sum_{j=0}^n \binom{n}{j} 2^{n-j} (1+x)(x^{-1}+x)^j = (1+x) \sum_{j=0}^n \binom{n}{j} (x^{-1}+x)^j 2^{n-j} = \\
 & = (1+x)(2+x^{-1}+x)^n \cdot \frac{x^n}{x^n - 1}
 \end{aligned}$$

$$\begin{aligned}
 & (x^{-1})^n \cdot x^n = 1 \quad \text{so} \\
 & \leq \binom{n}{j} n^j \theta^{n-j} = (\theta \cdot \omega)^n \\
 & = (1+x)(2x + j + x^2)^n \cdot \frac{1}{x^n} = \frac{1}{x^n} (1+x)^{2n+1} \\
 & \text{so } \theta = 2 \text{ always} \\
 & \text{so } \theta = 2 \text{ always. } x^n (2 + j + x^2)^n \\
 & \text{so } \theta = 2 \text{ always. } n \text{ unknown} \rightarrow 31.
 \end{aligned}$$

~~but~~ $\theta_1, \theta_2, \dots, \theta_n$

$$\begin{aligned}
 A \neq A_1 = \emptyset; \theta_1 = \{0\}; A_{n+1} = \{x + 1 \mid x \in \theta_n\}; \\
 \theta_{n+1} = (A_n \cup \theta_n) \setminus \theta_n \cap \theta_n
 \end{aligned}$$

so when $n \geq 3$ $\theta_n = \{0\}$.

$$\begin{aligned}
 & A \rightarrow a_n x = \sum_{i=0}^n a_i x^i \\
 & (a_i = 0 \mid i \in \theta_n) \\
 & (a_i = 0 \mid i \notin \theta_n) \\
 & a_{n+1}(x) = 0 \\
 & a_{n+1} x \theta_n = \sum_{i=0}^n b_i x^i \quad (b_i \in \theta_n) \\
 & b_1(x) = k \cdot a_{n+1}(x)
 \end{aligned}$$

BN spez. kann man nicht für Polynome > 1x anfordern, da es keine.

$$f_S(x) = \sum_{S \in S} c_S \cdot x^S \quad \text{mit } c_S = 1 \Leftrightarrow S \text{ ist ein } S \text{ Schleife.}$$
$$\sum_{S \in S} c_S \cdot x^S = 0 \quad \text{für alle } S \neq \emptyset.$$

$$B_3 \cdot S \subset f_S(t) = t^0 + t^2 + t^3 = 1 + t^2 + t^3$$

$$a_n(t) = \sum_{A_n(t)} c_A t^A \quad , \quad B_n = \sum_{B \in B_n} c_B t^B \quad \begin{array}{l} \text{X beschreibt } \\ \text{Basis elemente} \\ \text{in } B_n \end{array}$$

Aus Gründen, $A_n = \emptyset$ ist $\{0\}$.

$$\Rightarrow A_1(t) = \emptyset \quad \text{d.h. } B_1 = \emptyset \quad \text{und } c_0 = 1.$$

$$\text{a. } A_{n+1} = \{k+d \mid k \in B_n\}$$

X ist in $B_n - \emptyset$, also X ist die Basis von $A_{n+1} - \emptyset$.

$$A_{n+1} = \underbrace{\text{Basis von } B_n}_{\text{Basis von } A_{n+1}}$$

$$B_n = \sum_{B \in B_n} X$$

$$(q \cdot X) \rightarrow \text{Basis von } B_n$$

$$I \cdot a_{n+1}(x) = x \cdot B_n(x)$$

$$\text{II. } B_{n+1} = (A_n \cup B_n) \setminus (A_n \cap B_n), \text{ d.h. } \text{ aus } A_n(2; 3; 5)$$

ausser

$$R_{B_n}(4; 3; 7)$$

$$B_{n+1} \approx A_n(t) + B_n(t) \text{ mod 2}$$

$$B_n \cup B_n = (2; 3; 4; 5; 7)$$

$$B_n \cap B_n = (3).$$

$$(A_n \cup B_n) \setminus (A_n \cap B_n) = (2; 4; 5; 7)$$

x soll kein passen.

Consequently, we get
Basis 2nd. order & $c_0 = 1$
mod 2.

$$Q_1(t) = x \cdot Q_1(t) = x \cdot 1 = x$$

$$Q_2(t) = Q_1(t) + Q_1(t) = 0 + 0 = 0 \text{ mod 2.}$$

$$Q_1(t) = 0 \quad Q_1(t) = 0 \text{ mod 2.}$$

$$Q_1 = Q_2 = 1$$

1st. order recurrence
no basis \Rightarrow $B_n(t) = P(t)$

$$P(t)^{n+1} = P(t)^n + t \cdot P(t)^{n-1}$$

$$P(t)^2 = P(t) + t \quad : P(t)^{n-1}$$

$$P(t) = t \quad \cancel{P(t)^{n-1} = t}$$

$$Q_1(t) = t + t^2 + t^3 + \dots$$

$$Q_1(t) = t + t^2 + t^3 + \dots \text{ mod 2.}$$

$$Q_1(t) = 1 \quad \text{mod 2.}$$

$$Q_1(t) = 1 \quad \text{mod 2.}$$

$$Q_1(t) = 0.$$

$$P(X)^2 - P(H-X) = 0 \quad \text{mod } 2 \quad (\text{since } P(H) \text{ is even})$$

$$2 \equiv 0 \pmod{2}$$

$$-X \equiv 1 \pmod{2} \quad (X \equiv -1 \pmod{2})$$

$$P(X)^2 - P(H) - X =$$

$$\equiv P(X)^2 + P(H) + X = 0$$

$$P(X)(P(X) + 1) = X$$

$$\begin{cases} P(X) \equiv 1 \\ P(H) \equiv 0 \end{cases} \quad \text{mod } 2$$

$$480 \cdot 2 \equiv 32 \pmod{32}$$

$$1 \cdot (P(X))^2 \equiv -1 \cdot (P(H)) \pmod{2}$$

$$(2020 \cdot 2 \pmod{32}) \pmod{2-2}$$

$$-P(H) \equiv -1 - P(X) \equiv P(X) \pmod{2}$$

$$P_1(X) \in \mathbb{F}_2[X]$$

$$P_1(X) \cdot (P_1(X) + 1) = X$$

$$P_2(X) = 1 + P_1(X)$$

$$R + P_2 \cdot P_1(X) + P_1(X) = P_1(X) + (1 + P_1(X)) = 1$$

$$P(X) = \sum_{a \geq 1} d_a \cdot X^a$$

$$P_1(X) = d_3 X + d_2 X^2 + \dots \quad P_1(X) = \sum_{a \geq 1} d_a \cdot X^a$$

$$P_1(X) \cdot (P_1(X) + 1) \equiv X \pmod{2}$$

$$d_3 = 1 \quad d_2 = 1 \quad d_1 = 0 \quad d_0 = 1$$

$$P_1(X)$$

$$P_1(X) + 1 = 1 + d_3 X + d_2 X^2 + \dots$$

$$\text{solution: } (d_3, d_2, \dots) (1 + d_3 X + \dots)$$

$$d_1 \cdot X \cdot 1 = d_1 \cdot X \quad d_1 = 1$$

$$d_2 \cdot X^2 \cdot 1 = d_2 \cdot X^2 \quad \text{constant term} = 1$$

$$P_1(X) + 1 - 1 \cdot d_2 \cdot X^2 - d_3 \cdot X \cdot 1 = 0$$

$$(d_3, d_2, d_1, d_0)$$

$$d_3 \cdot X \cdot d_2 \cdot X^2 + d_2 \cdot X^2 = d_2 \cdot X^2$$

$$d_2 \cdot X^2 \cdot 1 = d_2 \cdot X^2$$

$$d_1^2 + d_2 \cdot 1 \pmod{2} \quad d_1 = 1$$

$$\text{coeff } d_2 = 1$$

$$\frac{a_1 x \cdot d_2 x^2 + a_2 x^2 \cdot a_1 x + a_3 x^3 \cdot 2}{d_1 = d_2 = 2 \pmod{2} - 3}$$

$$x^3 - 5x^2 + 1 \cdot 1 + 1 \cdot 1 + 0_3 =$$

$$= 2 + 0_3 = 0 \pmod{2}$$

$$\underline{d_3 = 0}$$

Hence we have a_1, a_2, a_3

$$\begin{aligned} d_1 &= 3; d_2 = 1; d_3 = 0; d_4 = 1; d_5 = 0 \\ \therefore p(x) &= x + x^2 + x^4 + x^6 + \sum_{k=0}^{\infty} x^{n_k} \end{aligned}$$

$$\begin{aligned} P_1(x) + P_2(x) &= P_1(x)^2 + P_2(x^2) \geq 1 \\ P_1(x) &= P_1(x)^2 + (1+P_1(x))^2 = P_1(x)^2 + P_1(x)^2 + 1 \geq 1 \text{ mod } 2 \end{aligned}$$

$$(1 + p_1(x))^n = \sum_{j=0}^n \binom{n}{j} p_1^j(x)$$

$$(1 + P_1(x)) - \sum_{j=0}^{\infty} c_j P_1(x)^j$$

$c_j = 0 \text{ or } 1$ where c_j is power odd.

$$\eta = \rho_1 \delta n \quad \Rightarrow \quad n=2 \quad L_0 \beta \bar{x} \cdot \frac{(\text{mod } ? - \bar{x})}{(1+\rho_1(t))^2} = 1 + 2\rho_1(t) + \rho_1(t) \cancel{x} \stackrel{\text{mod } 2 \text{ mod } x}{=} 0$$

$$\begin{aligned} & \text{Let } n = 3 - 2t \text{ since } (1 + p_1(x))^{3-t} \equiv 1 + p_1(x)^2 \pmod{2}. \\ & (1 + p_1(x))^{3-t} = (1 + 3p_1(x) + 3p_1(x)^2 + p_1(x)^3) \pmod{2} \end{aligned}$$

$$\rightarrow n = \text{span } y = 1 + P_1(F) \quad \text{span } h \text{ of } \omega \quad (1 + P_1(F)) = 1 + P_1(F)$$

$$P_1(x) = \sqrt{\sum x^2} + (\bar{x})^2$$

$$\Leftrightarrow p_1(t) + p_1(x)^2 = X$$

$$x^2 + x^4 = 0$$

$$e_n(x) = p_1(x^n) + p_2(x^k) = \left(\sum_{k=0}^{\infty} x^{2^k k} \right)^n + \left(1 + \sum_{k=0}^{\infty} x^{2^k k} \right)^n =$$

~~$\left(\sum_{k=0}^{\infty} x^{2^k k} + 1 \right)^n$~~

mod 2 $\Leftrightarrow -1 \equiv 1$
mod 2 $\Leftrightarrow 2 \equiv 0$

if n is power of 2,

$\Rightarrow n$ is power of 2.

If n is not power of 2, then smallest m for which $(\frac{n}{2})^m$ is odd is

~~$\sum_{i=0}^j$~~

Corollary

~~Ex~~ $\underbrace{a_0 a_1 \dots a_j}_{(\frac{n}{2}) \text{ bits}} \underbrace{b_1 \dots b_m}_{a_i + a_j \text{ are same as } (\frac{n}{2}) \text{ bits}}$

wanted. n isn't power of 2.

$$f(x) = \underbrace{x^{a_1} + x^{a_2} + \dots + x^{a_n}}_{a_1, a_2, \dots, a_n \in \text{Set } A} / \underbrace{y(y) = x^{b_1} + x^{b_2} + \dots + x^{b_m}}_{b_1, b_2, \dots, b_m \in \text{Set } B}$$

$$\begin{aligned} f^2(x) - f(x^2) &= \\ f(x)^2 &= (x^{a_1} + x^{a_2} + \dots)^2 = \sum_{i=1}^n \sum_{j=1}^n x^{\underbrace{a_i + a_j}_{i=j \text{ (odd)}}} = x^{a_i + a_j} = f(x)^2 \end{aligned}$$

if $i > j$ then $x^{a_i + a_j} = x^{a_i + a_{i-j}}$.

$$\sum_{i=1}^n x^{q_1 q_2 \dots q_n} = x^{q_1} + x^{q_2} + \dots + x^{q_n}$$

$f(x^2) - g(x^2) = \sum_{i+j} x^{q_i+q_j}$ with $i+j$ even $\cancel{i+j \neq 0}$

then (i,j) eqn pair (i,j) are counted twice (i,j)

$$\sum_{i+j} x^{q_i+q_j} = 2 \sum_{i < j} x^{q_i+q_j}$$

$\Rightarrow q_i+q_j$ even, q_i+q_j "odd"

 $p = 2 \sum_{i < j} x^{q_i+q_j} = 2 \sum_{i < j} x^{q_i+q_j} = g'(x) - g(x^2)$

$$\frac{(f(x) - g(x))}{h(y)} \frac{(f(y) + g(y))}{p(y)} = f(x^2) - g(y^2)$$

if n is not power of 2, $h=0$

$h(1) = 0$, if all derivatives of h is 0,

$$h(x) \cdot p(y) = n'(x) \cdot p(x) + h(x) p'(x) = 2x h'(x^2)$$

$$h(x) \cdot p(y) = h(1) p'(1) = 2h'(1)$$

$$h'(1) p'(1) = 2h'(1)$$

$$h'(1) = 0$$

$$h''(1) = 0$$

$$h'''(1) = 0$$

$$h''''(1) = 0$$

$\therefore h'(1) = 0$ for all x .

$\therefore f(x) = g(x)$ for all x .

$\therefore A = \emptyset$. Since $A \cap B \neq \emptyset$

$$820 \quad 1 + 2 + 3 + \dots + n = 0. = f(n)$$

$$S(n) = \sum_{k=1}^{2^n-1} \int_0^{2\pi}$$

$$\pm 1 + \pm 2 + \dots + \underbrace{\pm j + \pm 2 + \dots}_{\text{only } k \text{ terms}}$$

\star If j is even then $j \in (j, n)$

$$\pm j \rightarrow (x^1 + \dots + x^{-j})$$

$$\pm 2 \rightarrow (x^2 + \dots + x^{-2}) \quad b_n(x) = (x + \frac{1}{x})(x^2 + \frac{1}{x^2}) \dots (x^n + \frac{1}{x^n})$$

$$\pm n \rightarrow (x^n + x^{-n})$$

$S(n) \rightarrow$ how many terms

are there?

odd terms

even terms

$$\frac{e_1 + e_2 + \dots + e_n}{2}$$

$$S(n) - \text{rest } X =$$

$$S(n) = \pm 1 \pm 2 \pm \dots = 0$$

$$(x + \frac{1}{x})(x^2 + \frac{1}{x^2}) \dots (x^n + \frac{1}{x^n}) = S(n) + \sum_{k \neq 0} x^k$$

$$x = e^{it} - 1$$

$$2 + \frac{1}{2} = 2 \text{ rest}$$

at 2pm

$$x^k \text{ is rest } \int_0^{2\pi}$$

$$\int_0^{2\pi} (2 - \cos t)(2 - \cos 2t) \dots (2 - \cos nt) dt =$$

$$= 2^n S_n + 0$$

675 $P = \text{odd, prime.}$

$$\{1, 2, \dots, P-1\} \rightarrow \dots P.$$

either each number \in 1st set or not

don't include? $\leftarrow x^0 \text{ (is } \sum \text{ subset of } + 0\right)$

include? $x^k \leftarrow (\text{sum subset } \leftarrow + k)$

$$4. x^0 + x^k = \frac{1 + x^k}{(1-x)^P} \text{ (LHS = 0)}$$

$$LHS = (1+x)(1+x^2)(1+\dots)(1+x^P)$$

but $x^P = 1$ (root of unity)

$$\sum_{k=1}^P x^k = 0, \text{ but } x^0 = 1$$

$$x = 1$$

- $(-1 - \epsilon)$, non-prime powers
Now P
 (-1)

auto sum of roots of x^n or dividing by p .

$$\sum_{k=1}^p \zeta(z^k) = ps(p)$$

$$s(z^p) = \prod_{j=1}^p (1 + \epsilon^{kj}) = \prod_{j=1}^p (1 + \epsilon^j) = (-1)^p \prod_{j=1}^p ((-1) - \epsilon^j) = (-1)^p ((-1)^p - 1) = 2.$$

If $\epsilon_1, \epsilon_2, \dots, \epsilon_p$
 or p^{th} root of unity
 $x^p - 1 = \prod_{j=1}^p (x - \epsilon_j)$
 $x = -1$ $(-1)^p - 1 = \prod_{j=1}^p (-1 - \epsilon_j)$

ps $ps(p) = 2^p + 2(p-1) =$
 $= 2^p + 2p - 2$
 $\boxed{s(p) = \frac{2^p - 2}{p} + }$

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