Single Source Shortest Paths Problem: Dijkstra's Algorithm

REF.

E.W. Dijkstra. A note on two problems in connection with graphs. *Numerische Mathematik*, Volume 1, pp. 269-271, 1959.

- Greedy algorithm
- It works by maintaining a set S of ``special'' vertices whose shortest distance from the source is already known. At each step, a ``non-special'' vertex is absorbed into S.
- The absorption of an element of V - S into S is done by a greedy strategy.
- The following provides the steps of the algorithm.

Let

$$V = \{1, 2, \ldots, n\} \text{ and source} = 1 \tag{7.1}$$

$$C[i, j] = \text{Cost of the arc } (i, j) \text{ if the arc } (i, j) \text{ exists; otherwise } \infty$$

{

S= { 1 };

for (i = 2; i < n; i++)

D[i] = C[1,i];

for (i=1; i < = n-1; i++)

{

choose a vertex w $\in$ V-S such that D[w] is a minimum;

S = S $\cup$ {w };

for each vertex v $\in$ V-S

D[v] = min (D[v], D[w] + C[w, v])

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    }
}
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- The above algorithm gives the costs of the shortest paths from source vertex to every other vertex.
- The actual shortest paths can also be constructed by modifying the above algorithm.

Theorem: Dijkstra's algorithm finds the shortest paths from a single source to all other nodes of a weighted digraph with positive weights.

Proof: Let V = 1, 2, ..., n and 1 be the source vertex. We use mathematical induction to show that

(a)

If a node i ($\neq$ 1) $\in S$, then D[i] gives the length of the shortest path from the source to i.

(b)

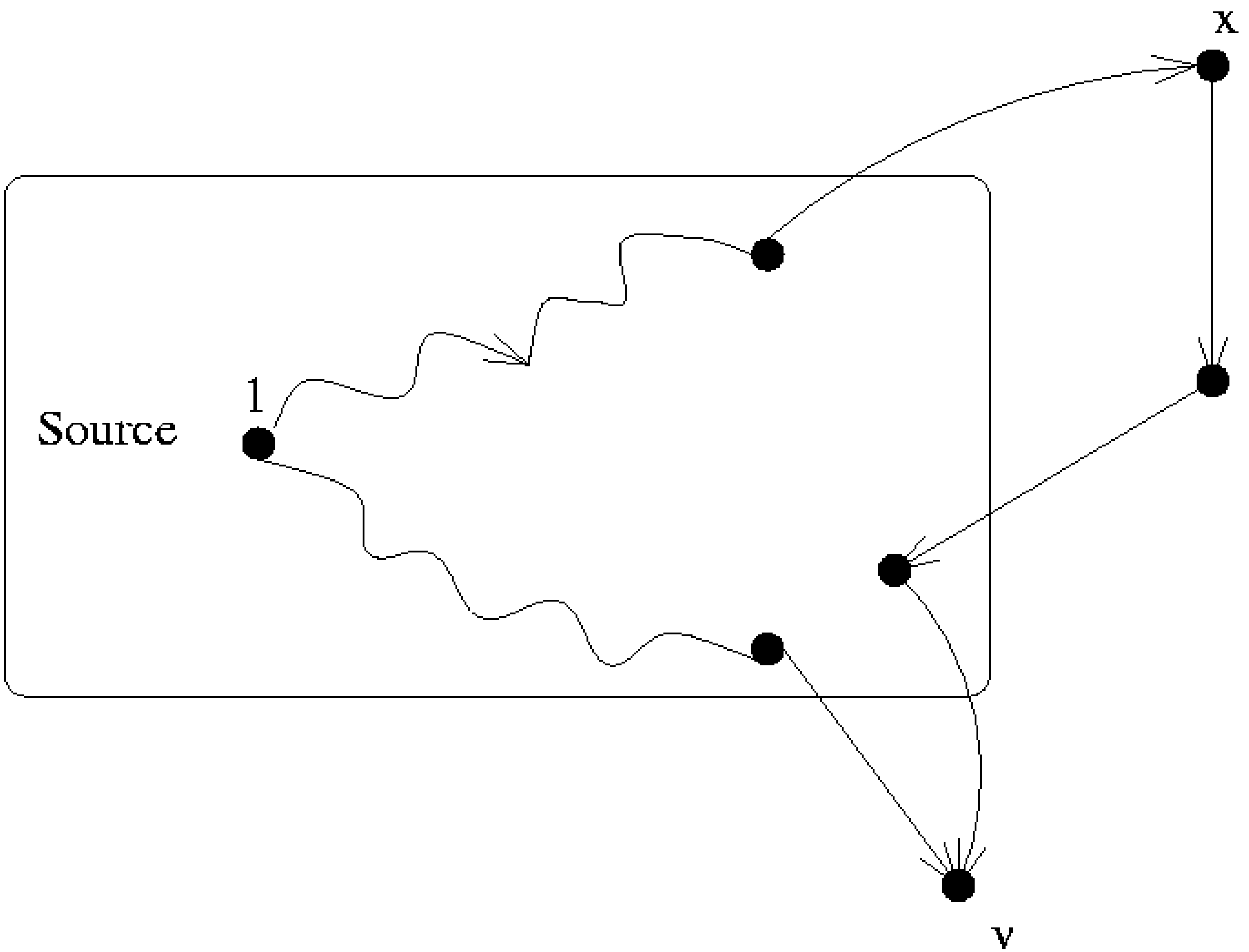
if a node $i \notin S$, then D[i] gives the length of the shortest special path from the source to i.

Basis: Initially S = 1 and hence (a) is vacuously true. For i $\in$ S, the only special path from the source is the direct edge if present from source to i and D is initialized accordingly. Hence (b) is also true.

Induction for condition (a)

- The induction hypothesis is that both (a) and (b) hold just before we add a new vertex v to S.
- For every node already in S before adding v, nothing changes, so condition (a) is still true.
- We have to only show (a) for v which is just added to S.

Figure 7.4: The shortest path to v cannot visit x



- Before adding it to S, we must check that D[v] gives the length of the shortest path from source to v. By the induction hypothesis, D[v] certainly gives the length of the shortest special path. We therefore have to verify that the shortest path from the source to v does not pass through any nodes that do not belong to S.
- Suppose to the contrary. That is, suppose that when we follow the shortest path from source to v, we encounter nodes not belonging to S. Let x be the first such node encountered (see Figure 7.4). The initial segment of the path from source to x is a special path and by part (b) of the induction hypothesis, the length of this path is D[x]. Since edge weights are no-negative, the total distance to v via x is greater than or equal to D[x]. However since the algorithm has chosen v ahead of x, D[x] $\geq$ D[v]. Thus the path via x cannot be shorter than the shortest special path leading to v.

Induction step for condition (b): Let $\omega \neq$ v and $\omega \in$ S. When v is added to S, these are two possibilities for the shortest special path from source to w:

1. It remains as before
2. It now passes through v (and possibly other nodes in S)

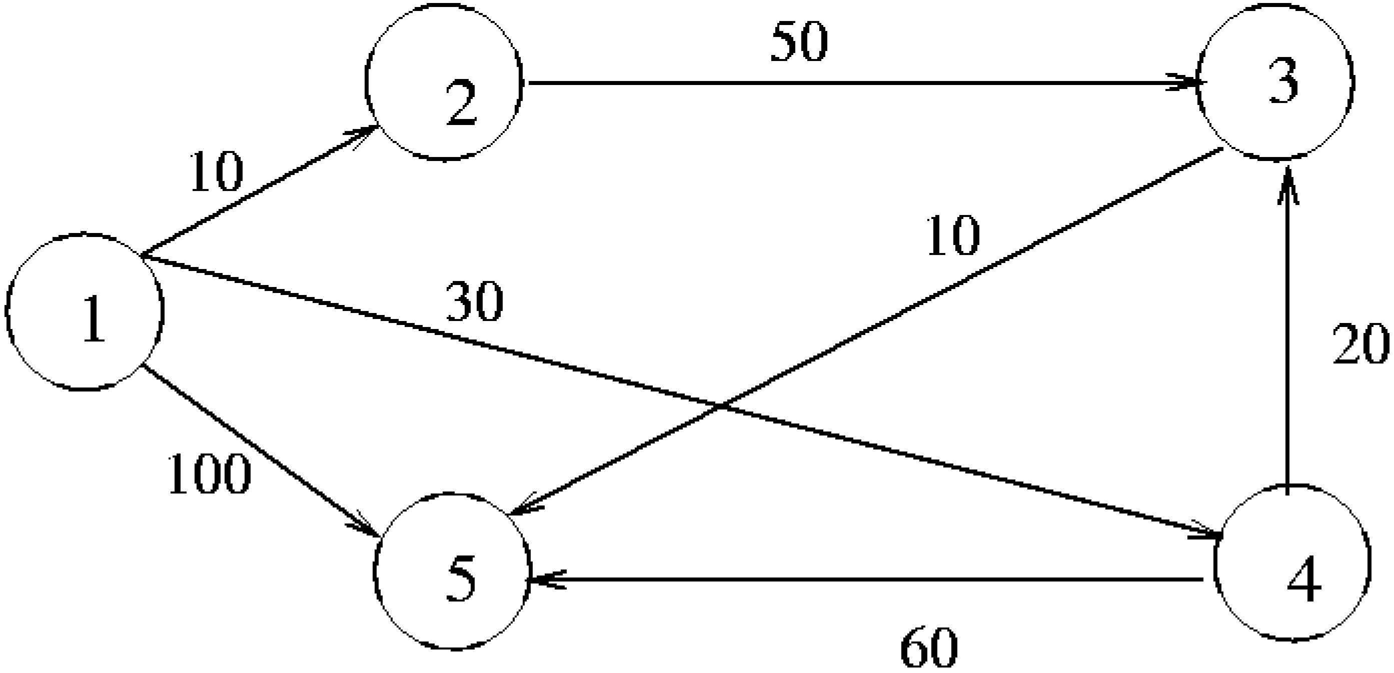
In the first case, there is nothing to prove. In the second case, let y be the last node of S visited before arriving at w. The length of such a path is D[y] + C[y,w].

- At first glance, to compute the new value of d[w], it looks as if we should compare the old value of D[w] with D[y] + C[y,w] for every $y \in S$ (including v)
- This comparison was however made for all $y \in S$ except v, when y was added to S in the algorithm. Thus the new value of D[w] can be computed simply by comparing the old value with D[v] + C[v,w]. This the algorithm does.

When the algorithm stops, all the nodes but one are in S and it is clear that the vector D[1], D[2], ..., D[n]) contains the lengths of the shortest paths from source to respective vertices.

Example: Consider the digraph in Figure 7.5.

Figure 7.5: A digraph example for Dijkstra's algorithm



Initially:

$S = \{1\}$ $D[2] = 10$ $D[3] = \infty$ $D[4] = 30$ $D[5] = 100$

Iteration 1

Select w = 2, so that S = {1, 2}

$$D[3] = \min(\infty, D[2] + C[2, 3]) = 60 \tag{7.2}$$
$$D[4] = \min(30, D[2] + C[2, 4]) = 30 \tag{7.3}$$
$$D[5] = \min(100, D[2] + C[2, 5]) = 100$$

Iteration 2

Select w = 4, so that S = {1, 2, 4}

$$D[3] = \min(60, D[4] + C[4, 3]) = 50 \tag{7.4}$$
$$D[5] = \min(100, D[4] + C[4, 5]) = 90$$

Iteration 3

Select w = 3, so that S = {1, 2, 4, 3}

$$D[5] = \min(90, D[3] + C[3, 5]) = 60$$

Iteration 4

Select w = 5, so that S = {1, 2, 4, 3, 5}

$$D[2] = 10 \quad (7.5)$$
$$D[3] = 50 \quad (7.6)$$
$$D[4] = 30 \quad (7.7)$$
$$D[5] = 60$$

Complexity of Dijkstra's Algorithm

With adjacency matrix representation, the running time is O(n2) By using an adjacency list representation and a partially ordered tree data structure for organizing the set V - S, the complexity can be shown to be

O(elog n)

where e is the number of edges and n is the number of vertices in the digraph.