```
Algorithm Greedy(a, n)
// a[1 : n] contains the n inputs.
{
    solution := ∅; // Initialize the solution.
    for i := 1 to n do
    {
        x := Select(a);
        if Feasible(solution, x) then
            solution := Union(solution, x);
    }
    return solution;
}
```

Algorithm 4.1 Greedy method control abstraction for the subset paradigm

function Greedy describes the essential way that a greedy algorithm will look, once a particular problem is chosen and the functions Select, Feasible, and Union are properly implemented.

For problems that do not call for the selection of an optimal subset, in the greedy method we make decisions by considering the inputs in some order. Each decision is made using an optimization criterion that can be computed using decisions already made. Call this version of the greedy method the *ordering paradigm*. Sections 4.2, 4.3, 4.4, and 4.5 consider problems that fit the subset paradigm, and Sections 4.6, 4.7, and 4.8 consider problems that fit the ordering paradigm.

EXERCISE

1. Write a control abstraction for the ordering paradigm.

4.2 KNAPSACK PROBLEM

Let us try to apply the greedy method to solve the knapsack problem. We are given n objects and a knapsack or bag. Object i has a weight w_i and the knapsack has a capacity m. If a fraction x_i, $0 \leq x_i \leq 1$, of object i is placed into the knapsack, then a profit of $p_i x_i$ is earned. The objective is to obtain a filling of the knapsack that maximizes the total profit earned. Since the knapsack capacity is m, we require the total weight of all chosen objects to be at most m. Formally, the problem can be stated as

$$\text{maximize} \sum_{1 \le i \le n} p_i x_i \tag{4.1}$$

$$\text{subject to} \sum_{1 \le i \le n} w_i x_i \le m \tag{4.2}$$

$$\text{and } 0 \le x_i \le 1, \quad 1 \le i \le n \tag{4.3}$$

The profits and weights are positive numbers.

A feasible solution (or filling) is any set $(x_1, \ldots, x_n)$ satisfying (4.2) and (4.3) above. An optimal solution is a feasible solution for which (4.1) is maximized.

Example 4.1 Consider the following instance of the knapsack problem: $n = 3, m = 20, (p_1, p_2, p_3) = (25, 24, 15)$, and $(w_1, w_2, w_3) = (18, 15, 10)$. Four feasible solutions are:

	(x_1, x_2, x_3)	$\sum w_i x_i$	$\sum p_i x_i$
1.	(1/2, 1/3, 1/4)	16.5	24.25
2.	(1, 2/15, 0)	20	28.2
3.	(0, 2/3, 1)	20	31
4.	(0, 1, 1/2)	20	31.5

Of these four feasible solutions, solution 4 yields the maximum profit. As we shall soon see, this solution is optimal for the given problem instance. □

Lemma 4.1 In case the sum of all the weights is $\le m$, then $x_i = 1$, $1 \le i \le n$ is an optimal solution. □

So let us assume the sum of weights exceeds m. Now all the x_i's cannot be 1. Another observation to make is:

Lemma 4.2 All optimal solutions will fill the knapsack exactly. □

Lemma 4.2 is true because we can always increase the contribution of some object i by a fractional amount until the total weight is exactly m.

Note that the knapsack problem calls for selecting a subset of the objects and hence fits the subset paradigm. In addition to selecting a subset, the knapsack problem also involves the selection of an x_i for each object. Several simple greedy strategies to obtain feasible solutions whose sums are identically m suggest themselves. First, we can try to fill the knapsack by including next the object with largest profit. If an object under consideration doesn't fit, then a fraction of it is included to fill the knapsack. Thus each time an object is included (except possibly when the last object is included) into the knapsack, we obtain the largest possible increase in profit value. Note that if only a fraction of the last object is included, then it may be possible to get a bigger increase by using a different object. For example, if we have two units of space left and two objects with ($p_i = 4$, $w_i = 4$) and ($p_j = 3, w_j = 2$) remaining, then using j is better than using half of i. Let us use this selection strategy on the data of Example 4.1.

Object one has the largest profit value ($p_1 = 25$). So it is placed into the knapsack first. Then $x_1 = 1$ and a profit of 25 is earned. Only 2 units of knapsack capacity are left. Object two has the next largest profit ($p_2 = 24$). However, $w_2 = 15$ and it doesn't fit into the knapsack. Using $x_2 = 2/15$ fills the knapsack exactly with part of object 2 and the value of the resulting solution is 28.2. This is solution 2 and it is readily seen to be suboptimal. The method used to obtain this solution is termed a greedy method because at each step (except possibly the last one) we chose to introduce that object which would increase the objective function value the most. However, this greedy method did not yield an optimal solution. Note that even if we change the above strategy so that in the last step the objective function increases by as much as possible, an optimal solution is not obtained for Example 4.1.

We can formulate at least two other greedy approaches attempting to obtain optimal solutions. From the preceding example, we note that considering objects in order of nonincreasing profit values does not yield an optimal solution because even though the objective function value takes on large increases at each step, the number of steps is few as the knapsack capacity is used up at a rapid rate. So, let us try to be greedy with capacity and use it up as slowly as possible. This requires us to consider the objects in order of nondecreasing weights w_i. Using Example 4.1, solution 3 results. This too is suboptimal. This time, even though capacity is used slowly, profits aren't coming in rapidly enough.

Thus, our next attempt is an algorithm that strives to achieve a balance between the rate at which profit increases and the rate at which capacity is used. At each step we include that object which has the maximum profit per unit of capacity used. This means that objects are considered in order of the ratio p_i/w_i. Solution 4 of Example 4.1 is produced by this strategy. If the objects have already been sorted into nonincreasing order of p_i/w_i, then function GreedyKnapsack (Algorithm 4.2) obtains solutions corresponding to this strategy. Note that solutions corresponding to the first two strategies can be obtained using this algorithm if the objects are initially in the appropriate order. Disregarding the time to initially sort the objects, each of the three strategies outlined above requires only $O(n)$ time.

We have seen that when one applies the greedy method to the solution of the knapsack problem, there are at least three different measures one can attempt to optimize when determining which object to include next. These measures are total profit, capacity used, and the ratio of accumulated profit to capacity used. Once an optimization measure has been chosen, the greedy

2. [0/1 Knapsack] Consider the knapsack problem discussed in this section. We add the requirement that $x_i = 1$ or $x_i = 0$, $1 \le i \le n$; that is, an object is either included or not included into the knapsack. We wish to solve the problem

$$\max \sum_1^n p_i x_i$$

$$\text{subject to} \sum_1^n w_i x_i \le m$$

$$\text{and} \quad x_i = 0 \text{ or } 1, \; 1 \le i \le n$$

One greedy strategy is to consider the objects in order of nonincreasing density p_i/w_i and add the object into the knapsack if it fits. Show that this strategy doesn't necessarily yield an optimal solution.

4.3 TREE VERTEX SPLITTING

Consider a directed binary tree each edge of which is labeled with a real number (called its *weight*). Trees with edge weights are called *weighted trees*. A weighted tree can be used, for example, to model a distribution network in which electric signals or commodities such as oil are transmitted. Nodes in the tree correspond to receiving stations and edges correspond to transmission lines. It is conceivable that in the process of transmission some loss occurs (drop in voltage in the case of electric signals or drop in pressure in the case of oil). Each edge in the tree is labeled with the loss that occurs in traversing that edge. The network may not be able to tolerate losses beyond a certain level. In places where the loss exceeds the tolerance level, boosters have to be placed. Given a network and a loss tolerance level, the *Tree Vertex Splitting Problem (TVSP)* is to determine an optimal placement of boosters. It is assumed that the boosters can only be placed in the nodes of the tree.

The TVSP can be specified more precisely as follows: Let $T = (V, E, w)$ be a weighted directed tree, where V is the vertex set, E is the edge set, and w is the weight function for the edges. In particular, $w(i,j)$ is the weight of the edge $\langle i,j \rangle \in E$. The weight $w(i,j)$ is undefined for any $\langle i,j \rangle \notin E$. A *source vertex* is a vertex with in-degree zero, and a *sink vertex* is a vertex with out-degree zero. For any path P in the tree, its *delay*, $d(P)$, is defined to be the sum of the weights on that path. The delay of the tree T, $d(T)$, is the maximum of all the path delays.

Let T/X be the forest that results when each vertex u in X is split into two nodes u^i and u^o such that all the edges $\langle u,j \rangle \in E$ ($\langle j,u \rangle \in E$) are