$\tilde{R}$:

	1	2	3	4
1	1.0	0.5	0.0	0.0
2	0.5	1.0	0.5	0.0
3	0.0	0.5	1.0	0.5
4	0.0	0.0	0.5	1.0

Suppose that we want to perform the following fuzzy logic inference (approximate reasoning):

Premise	*a* is small
Implication	*a* and *b* are approximately equal
Conclusion	*b* is somewhat small

Then, we can apply the *max-min* composition of the fuzzy relations as follows:

(i) "*a* is small": $\mu_A(a)$ is available;

(ii) "*a* and *b* are approximately equal": $\mu_{\tilde{R}}(a,b)$ is given by the table;

(iii) let $\mu_B(b)$ be the membership function for the conclusion (a fuzzy modus ponens for this example):

$$\mu_B(b) = \max_{a\in A} \{ \min\{ \mu_A(a),\ \mu_{\tilde{R}}(a,b) \} \}, \qquad b\in B=A.$$

The result, for $b = 2$, say, is

$$\begin{aligned}\mu_B(2) &= \max_{a\in A} \{ \min\{ \mu_A(a),\ \mu_{\tilde{R}}(a,2) \} \} \\ &= \max\{ \min\{1.0,0.5\}, \min\{0.7,1.0\}, \min\{0.3,0.5\}, \\ &\qquad \min\{0.0,0.0\} \} \\ &= \max\{ 0.5, 0.7, 0.3, 0.0 \} \\ &= 0.7.\end{aligned}$$

Similarly, one can evaluate $\mu_B(1)$, $\mu_B(3)$, and $\mu_B(4)$. The final result is

$$\mu_B(b) = \begin{cases} 1.0 & \text{if } b=1, \\ 0.7 & \text{if } b=2, \\ 0.5 & \text{if } b=3, \\ 0.3 & \text{if } b=4. \end{cases}$$

VI. FUZZY LOGIC RULE BASE

A. Fuzzy IF-THEN Rules

In Section IV, we discussed in somewhat detail the fuzzy logic operations *and, or, not, implication,* and *equivalence*:

$$a \wedge b, \qquad a \vee b, \qquad \overline{a}\,, \qquad a \Rightarrow b, \qquad a \Leftrightarrow b$$

and their evaluations on a fuzzy set A with the membership function $\mu_A(\cdot)$:

$$\begin{aligned}
\mu_A(a \wedge b) &= \mu_A(a) \wedge \mu_A(b) = \mathit{min}\{\, \mu_A(a), \mu_A(b)\, \}; \\
\mu_A(a \vee b) &= \mu_A(a) \vee \mu_A(b) = \mathit{max}\{\, \mu_A(a), \mu_A(b)\, \}; \\
\mu_A(\,\overline{a}\,) &= \mu_{\overline{A}}(a) = 1 - \mu_A(a); \\
\mu_A(a \Rightarrow b) &= \mu_A(a) \Rightarrow \mu_A(b) = \min\{\, 1, 1 + \mu_A(\mathrm{b}) - \mu_A(a)\, \}; \\
\mu_A(a \Leftrightarrow b) &= \mu_A(a) \Leftrightarrow \mu_A(b) = 1 - |\, \mu_A(a) - \mu_A(b)\, |.
\end{aligned}$$

In Section V, we also discussed fuzzy relations between elements of two subsets A and B, on which a membership function $\mu_{A\times B}(a,b)$ is defined, with $a \in A$ and $b \in B$. It is clear that one can consider the above fuzzy logic operations as some special fuzzy relations, with $A = B$ and $\mu_{A\times A} = \mu_A$.

In this section, we take a closer look at the implication relation $a \Rightarrow b$ and its application in fuzzy logic rules.

The implication relation $a \Rightarrow b$ can be interpreted, in linguistic terms, as "IF a is true THEN b is true." Of course, this is valid for both the classical (two-valued) logic and the fuzzy (multi-valued) logic. For fuzzy logic performed on a fuzzy subset A, we have a membership function μ_A describing the truth values of $a \in A$ and $b \in A$. In this case, a more complete linguistic statement would be

"(IF $a \in A$ is true with a truth value $\mu_A(a)$ THEN $b \in A$ is true with a truth value $\mu_A(b)$) has a truth value $\mu_A(a \Rightarrow b) = \mathit{min}\{\, 1, 1 + \mu_A(b) - \mu_A(a)\, \}$."

In the above, both a and b belong to the same fuzzy subset A and share the same membership function μ_A. If they belong to different fuzzy subsets A and B with different membership functions μ_A and μ_B, then we have a nontrivial fuzzy relation, which can be quite complicated. In most cases, however, the implication relation $a \Rightarrow b$, performed on fuzzy subsets A and B, where $a \in A$ and $b \in B$, is simply defined in linguistic terms as

"IF $a \in A$ is true with a truth value $\mu_A(a)$ THEN $b \in B$ is true with a truth value $\mu_B(b)$."

Throughout this book, we often consider this kind of implication. Because such statements have a standard format and their meaning is clear, it is common to write them in the following simple form:

"IF a is A THEN b is B."

A fuzzy logic implication statement of this form is usually called a *fuzzy IF-THEN rule*.

To be more general, let $A_1,\ldots,A_n$, and B be fuzzy subsets with membership functions $\mu_{A_1},\ldots,\mu_{A_n}$, and μ_B, respectively.

Definition 2.1. A *General Fuzzy IF-THEN Rule* has the form

"IF a_1 is A_1 AND ... AND a_n is A_n THEN b is B."

Using the fuzzy logic AND operation, this rule is implemented by the following evaluation formula:

$$\mu_{A_1}(a_1) \wedge \ldots \wedge \mu_{A_n}(a_n) \Rightarrow \mu_B(b),$$

where

$$\mu_{A_i}(a_i) \wedge \mu_{A_j}(a_j) = min\{\ \mu_{A_i}(a_i),\ \mu_{A_j}(a_j)\ \},$$

$1 \le i, j \le n$, and, therefore,

$$\mu_{A_1}(a_1) \wedge \ldots \wedge \mu_{A_n}(a_n) = min\{\ \mu_{A_1}(a_1), \ldots, \mu_{A_n}(a_n)\ \}.$$

About this general fuzzy IF-THEN rule and its evaluation, a few issues have to be clarified:

(i) There is no fuzzy logic OR operation in a general fuzzy IF-THEN rule. What should we do if a fuzzy logic implication statement involves the OR operation?

(ii) There is no fuzzy logic NOT operation in a general fuzzy IF-THEN rule. What should we do if a fuzzy logic implication statement involves the NOT operation?

(iii) How do we interpret a fuzzy IF-THEN rule in a particular application? Is this interpretation unique?

We provide answers to these questions in the next two subsections.

B. Fuzzy Logic Rule Base

We first consider questions (i) and (ii). Let us first discuss, for example, the following fuzzy IF-THEN rule containing an OR operation:

"IF a_1 is A_1 AND a_2 is A_2 OR a_3 is A_3 AND a_4 is A_4 THEN b is B."

By convention, this is understood in logic as

"(IF a_1 is A_1 AND a_2 is A_2) OR (IF a_3 is A_3 AND a_4 is A_4)
THEN (b is B)."

With this convention and understanding, it is clear that this statement is equivalent to the combination of the following two fuzzy IF-THEN rules:

(1) "IF a_1 is A_1 AND a_2 is A_2 THEN b is B."
(2) "IF a_3 is A_3 AND a_4 is A_4 THEN b is B."

Hence, the fuzzy logic OR operation is not necessary to use: it may shorten a statement of a fuzzy IF-THEN rule, but it increases the format complexity of the rules.

As to the fuzzy logic NOT operation, although we may have a negative statement like "IF a is not A," we can always interpret this negative statement by a positive one "IF $\bar{a}$ is A" or "IF a is $\bar{A}$," where $\bar{A}$ means "not A" in logic and "complement of A" in set theory. Moreover, the statement "$\bar{a}$ is A" or "a is $\bar{A}$" can be evaluated by

$$\mu_A(\ \bar{a}\ ) = \mu_{\bar{A}}(a) = 1 - \mu_A(a).$$

Example 2.6. Given a fuzzy logic implication statement

"IF a_1 is A_1 AND a_2 is not A_2 OR a_3 is not A_3 THEN b is B,"

how can we rewrite it as a set of equivalent general fuzzy IF-THEN rules in the unified form?

We may first drop the fuzzy logic OR operation by rewriting the given statement as

(1) "IF a_1 is A_1 AND a_2 is not A_2 THEN b is B."
(2) "IF a_3 is not A_3 THEN b is B."

We may then drop the fuzzy logic NOT operation by rewriting them as

(1') "IF a_1 is A_1 AND $\bar{a}_2$ is A_2 THEN b is B."

(2') "IF $\bar{a}_3$ is A_3 THEN b is B."

Finally, these two general fuzzy IF-THEN rules can be evaluated as follows:

$$\mu_{A_1}(a_1) \wedge \mu_{A_2}(\bar{a}_2) \Rightarrow \mu_B(b),$$

where $\mu_{A_2}(\bar{a}_2) = 1 - \mu_{A_2}(a_2)$, and

$$\mu_{A_3}(\bar{a}_3) \Rightarrow \mu_B(b),$$

where $\mu_{A_3}(\bar{a}_3) = 1 - \mu_{A_3}(a_3)$. Therefore, we only need two general fuzzy IF-THEN rules (1') and (2') and three membership values $\mu_{A_2}(a_1)$, $\mu_{A_2}(a_2)$, and $\mu_{A_3}(a_3)$ to infer the conclusion "b is B," namely, $b \in B$ with the truth value $\mu_B(b)$.

All the other fuzzy logic operations can be simply defined and expressed by the AND and OR operations. They can be evaluated via the min and max operations as follows:

$$\mu_{A_1}(a_1) \wedge \mu_{A_2}(a_2) = min\{ \mu_{A_1}(a_1), \mu_{A_2}(a_2) \};$$

$$\mu_{A_1}(a_1) \vee \mu_{A_2}(a_2) = max\{ \mu_{A_1}(a_1), \mu_{A_2}(a_2) \};$$

$$\mu_A(\bar{a}) = \mu_{\bar{A}}(a) = 1 - \mu_A(a);$$

$$\mu_A(a \Rightarrow \tilde{a}) = \mu_A(a) \Rightarrow \mu_A(\tilde{a}) = min\{ 1, 1+\mu_A(\tilde{a}) - \mu_A(a) \};$$

$$\mu_A(a \Leftrightarrow \tilde{a}) = \mu_A(a) \Leftrightarrow \mu_A(\tilde{a}) = 1 - | \mu_A(a) - \mu_A(\tilde{a}) |.$$

Thus, all finite combinations of these fuzzy logic operations can also be expressed by the AND and OR operations, so that in any finite fuzzy logic inference statement

IF ... THEN ...,

the condition part "IF ..." can be expressed only by the AND and OR operations. This eventually leads to using only the AND operation, as discussed above.

In summary, a finite fuzzy logic implication statement can always be described by a set of general fuzzy IF-THEN rules containing only the fuzzy logical AND operation, in the form

(1) "IF a_{11} is A_{11} AND ... AND a_{1n} is A_{1n} THEN b_1 is B_1."

(2) "IF a_{21} is A_{21} AND ... AND a_{2n} is A_{2n} THEN b_2 is B_2."

⋮

(m) "IF a_{m1} is A_{m1} AND ... AND a_{mn} is A_{mn} THEN b_m is B_m."

This family of general fuzzy IF-THEN rules is usually called a *fuzzy logic rule base*.

We remark that the number of components in each rule above needs not to be the same. If $n = 2$ but a rule has only one component in the condition part, say

"IF a_{11} is A_{11} THEN b_1 is B_1,"

we can formally rewrite it as

"IF a_{11} is A_{11} AND a_{12} is I_{12} THEN b_1 is B_1,"

where I_{12} is a fuzzy subset with $\mu_{I_{12}}(a) = 1$ for all $a \in I_{12}$. Here, we actually insert a "always true" (redundant) condition into the "IF ... AND ..." part to fill in the gap of the statement. In so doing, we can keep the format of a fuzzy logic rule base simple in all the general discussions throughout the book.

One can also verify that such a general form of a fuzzy logic rule base includes the nonfuzzy case and the unconditional (degenerate) case (with only "b is B") as special cases. Moreover, this general fuzzy logic rule base (with only the fuzzy logical AND operation in the condition part) also covers many unusual fuzzy logic implication statements, such as the one shown in the next example.

Example 2.7. Given a fuzzy logic implication statement

"b is B unless a_1 is A_1 AND ... AND a_n is A_n,"

which is understood in logic as

"(b is B) unless (a_1 is A_1 AND ... AND a_n is A_n),"

one can first convert it by using the fuzzy logic NOT and OR operations as follows:

"IF a_1 is $\overline{A}_1$ OR ... OR a_n is $\overline{A}_n$ THEN b is B,"

and then replace all the OR operations by a fuzzy logic rule base of the form:

(1) "IF a_1 is $\overline{A}_1$ THEN b is B,"

(2) "IF a_2 is $\overline{A}_2$ THEN b is B,"

$\vdots$

(n) "IF a_n is $\overline{A}_n$ THEN b is B."

This rule base is in the general format, indeed.

In this example, however, we should note that the given statement is not equivalent to the following:

"IF a_1 is A_1 AND ... AND a_n is A_n THEN b is NOT B,"

since the conclusion can be "b has no relation with B."

Finally, we remark that a fuzzy rule base has to satisfy some properties or requirements. For example, a fuzzy rule base has to be *complete* in the sense that no other possible conditions are left out. The following rule base is incomplete:

(1) IF $a > 0$ THEN $b > 0$,

(2) IF $a = 0$ THEN $b < 0$,

because the case of $a < 0$ is left out. Also, a fuzzy rule base has to be *consistent* in the sense that no conclusions are contradictive. The following rule base is inconsistent:

(1) IF $a > 0$ THEN $b > 0$,

(2) IF $a > 0$ THEN $b = 0$,

(3) IF $a = 0$ THEN $b < 0$,

(4) IF $a < 0$ THEN $b = 0$,

since the first two rules contradict each other. Yet this rule base is complete. Note that the following two rules are consistent:

(1) IF $a > 0$ THEN $b > 0$,

(2) IF $a = 0$ THEN $b > 0$,

where two different conditions give the same conclusion, which is not a conflict, and the following two rules are consistent, too:

(1) IF $a > 0$ THEN $b > 0$,
(2) IF $a > 0$ THEN $c > 0$,

which are equivalent to "IF $a > 0$ THEN $b > 0$ AND $c > 0$."

Some other requirements may need to be imposed as well for a fuzzy rule base in a particular application. For example, a rule base should be *concise* with less or no redundancy.

C. Interpretation of Fuzzy IF-THEN Rules

Now, we return to Definition 2.1 and consider question (iii) thereafter: how do we interpret a fuzzy IF-THEN rule in a particular application, and is such an interpretation unique?

In the classical two-valued logic, the IF-THEN rule can be easily interpreted, namely,

"IF a is A THEN b is B"

is itself clear: the condition "a is A" infers the conclusion "b is B." For example, the statement

"IF a is positive THEN b is negative"

is crisp, nonvague, and absolute. In fuzzy multi-valued logic, however, both A and B are fuzzy subsets associated with fuzzy membership functions μ_A and μ_B. Depending on the actual membership values, $\mu_A(a)$ and $\mu_B(b)$ for the actual member values $a \in A$ and $b \in B$, respectively, both the condition "a is A" and the conclusion "b is B" can have various interpretations. We explain this in more detail by the following example.

Example 2.8. Let

$y = f(x)$

be a real-variable real-valued and invertible function defined on $X = [0,4]$ with range $Y = [-4,0]$ as shown in Figure 2.4 (a). If a crisp value x is given then $y = f(x)$ and if a crisp value y is given then $x = f^{-1}(y)$. Suppose that we don't actually know the exact formula of f. We let $\mu_S(\cdot)$, $\mu_M(\cdot)$, and $\mu_L(\cdot)$ be membership functions defined on X and Y, describing "small," "medium," and "large" in absolute values, respectively, as shown in Figure 2.4 (b). Thus, we may approximate the real function $y = f(x)$ by the following fuzzy rule base as shown in Figure 2.4 (c):

(1) "IF x is positive small THEN y is negative small."
(2) "IF x is positive medium THEN y is negative medium."
(3) "IF x is positive large THEN y is negative large."

Using the brief notation "a is A" to mean "$a \in A$ has a membership value $\mu_A(a)$" as we did before, one may now rewrite the above three implication statements as follows:

(1') "IF x is PS THEN y is NS."
(2') "IF x is PM THEN y is NM."
(3') "IF x is PL THEN y is NL."

Comparing it to the classical two-valued logic inference,

"IF x is positive THEN y is negative"

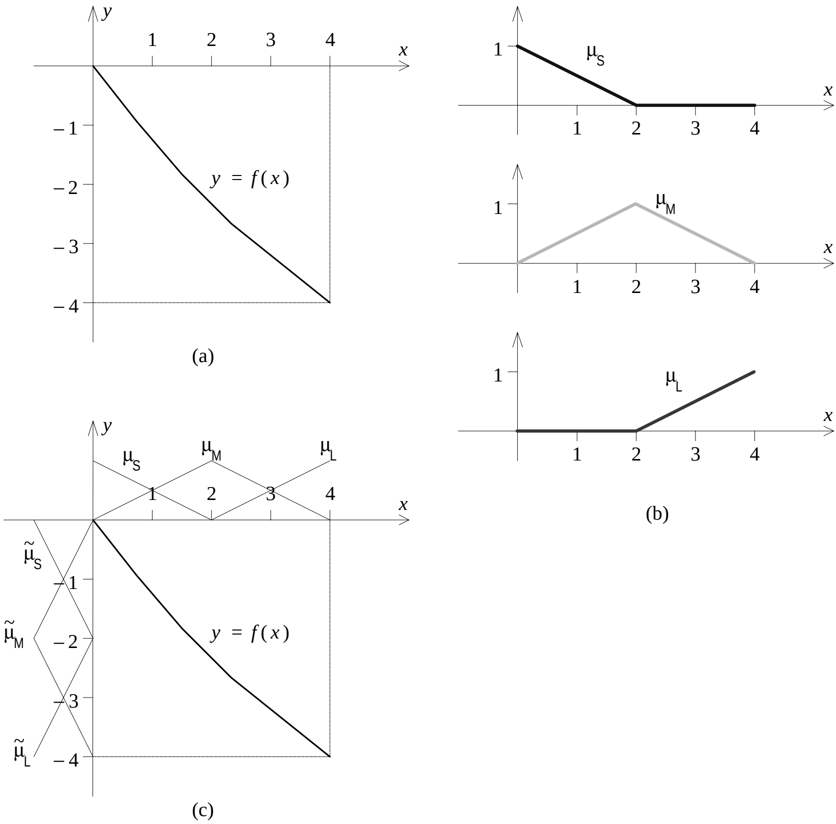

Figure 2.4 An example of approximating a real function by a fuzzy rule base.

or

"IF x is small THEN y is small,"

we see that we need the following long statement from the classical logic to express the same meaning of the fzzy logic inference (1):

"IF x is positive AND x is small THEN y is negative AND y is small."

Even if so, the classical logic can only be used to determine "x is small" or "x is not small," while the fuzzy membership function μ_S gives infinitely many different truth values to describe how small x is.

We note that in this example we only use one membership function for one evaluation, namely, if we are interested in the fuzzy implication statement (1), then we only apply the membership function μ_S to both x and y, but the other two membership functions μ_M and μ_L are not used. Later, we will discuss the cases where we apply more than one membership functions to one evaluation, to describe the situations like "x is small as well as medium" with different