Introduction to crisp set

Crisp set is a collection of **unordered distinct** elements, which are derived from a Universal set. A Universal set consists of all possible elements which take part in any experiment. A set is a quite useful and important way of representing data.

Let X represents a set of natural numbers, so

X = {1, 2, 3, 4, …}

Sets are always defined with respect to some universal set. Let us derive two sets A and B from this universal set X.

A = Set of even numbers = {2, 4, 6, …}

B = Set of odd number = {1, 3, 5, …}

Elements in the set are **unique**, i.e. A = {1, 1, 2, 2, 3, 3}, B = {1, 2, 3}, C = {1, 2, 2, 3, 3, 3} all are the same.

The order of elements in the set is not important, i.e. A = {1, 2, 3}, B = {2, 1, 3}, C = {3, 1, 2}, all correspond to identical set.

The element of the set is called a **member of the set. If any element is present in the set then it is considered a member of the set otherwise it is not a** member. In a crisp set, there is no concept of partial membership. Element is either fully present in the set or it is fully outside the set.

A crisp set is very important to model or represents many real-world entities, such as a set of boys, a set of books, a set of elements, a set of employees, a set of colours etc.

The membership function can be used to define a set A given by

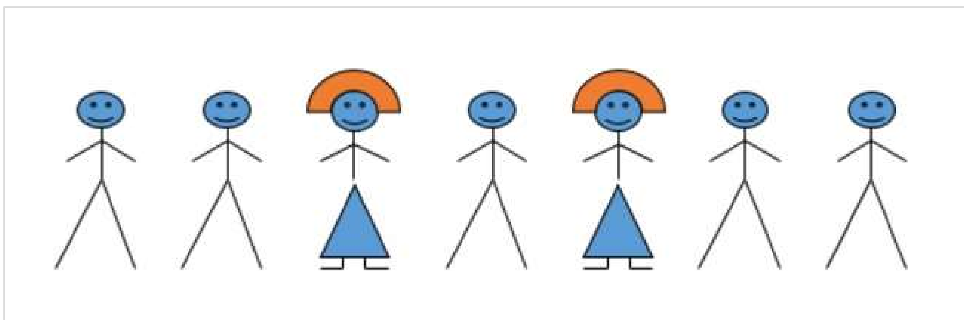
$$\chi_A(x) = \begin{cases} 1, & \text{if} x \in A \\ 0, & \text{if} x \notin A \end{cases}$$

The function χ (read as '*chi*') is known as the crisp **membership function**, which assigns membership value to the element of the universal set based on certain properties.

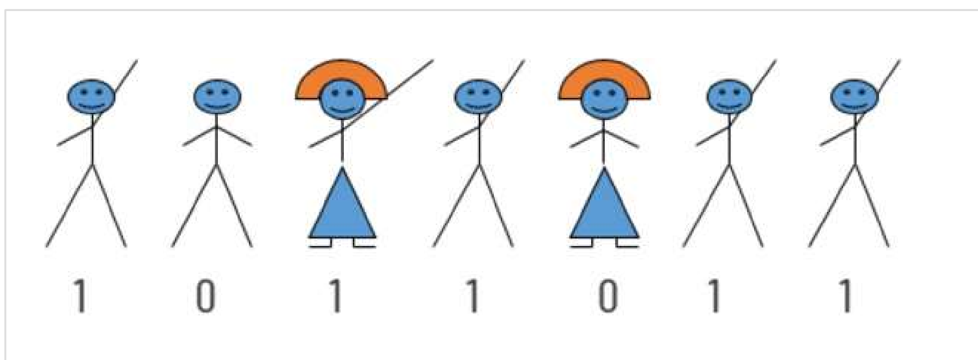
Examples of crisp set

Let us discuss an example of a crisp set. consider X represents a class of students which acts as the Universe of discourse. If you ask the question, *"who does have a driving license?"* Obviously, all students might not have a driving license. So those students who have a driving license will have a membership value of 1 for this particular set and the rest of them will have a membership value of zero.

We can define set A as equal to the set of students having driving licenses and A will be definitely a subset of universal set X



Universal set



Crisp Set

The best example of crisp set representation is the number system in mathematics, where,

- N: Set of natural numbers
- R: Set of real numbers
- Z: Set of integers
- Q: Set of rational numbers

Notations used in Crisp Set

We will discuss the various set notations with respect to the following sets:

X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
A = {2, 4, 6, 8, 10}
B = {1, 3, 5, 7, 9}
C = {4, 6, 8}
D = {x | x is perfect square and x > 10} = **Φ**

Various notations used in set theory are defined below:

- Φ: **Empty set** is represented by the symbol, Φ is a set which does not have any element in it. For given data, D = Φ
- x ∈ A represents element x is a **member** of set A. For given data, 2 ∈ A
- x ∉ A represents an element x that is **not a member** of set A. For given data, 3 ∉ A
- A ⊆ B represents every element of set A that is present in set B as well. In other words, A is a **subset** of B. For given data, A ⊆ X
- A ⊇ B represents every element of B is a member of set A as well. In other words, A is a **superset** of B. For given sets, A ⊇ C
- A ⊂ B represents every element of A in B as well as B has some additional element which is not in A. This notation says that A is a **proper subset** of B.
- A ⊃ B represents all the elements of B in set A as well as A has some additional element which is not in B. This notation says that A is a **proper superset** of B.
- if set A and B are identical then we can say A is a subset of B or B is a subset of A, but we cannot say that A is a proper superset of B or A is a proper subset of B
- A = B represents **Equal sets**, i.e. sets A and B have identical elements

- $A \neq B$ represents **Not equal sets**, i.e. sets A and B have different elements. For given sets, $A \neq B$
- $|A|$ represents the **Cardinality** of set A (i.e. a number of elements in set A). For given sets, $|A|= 5$
- p(A) represents the **Power set** of set A. For the given sets, $p(c) = \{ \Phi, \{4\}, \{6\}, \{8\}, \{4, 6\}, \{4, 8\}, \{6, 8\}, \{4, 6, 8\} \}$
- $|p(A)|$: $2^{|A|}$, i.e. power set of any set contains 2^n elements.

Let us understand various operations on set with the help of examples. We will consider the following data to execute various operations:

X = {1, 2, 3, 4, 5, 6, 7, 8, 9}

A = {1, 2, 3, 4, 5}

B = {3, 4, 5, 6}

C = {6, 7, 8, 9}

Union

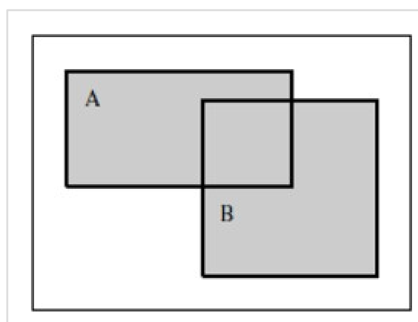
Union of sets is the collection of all the elements which are either in A **or** in B. Common elements from both sets are considered only once. Mathematically, we can represent union operation as follow:

$$A \cup B = \{ x \mid x \in A \textbf{ or } x \in B \}$$

For the given data

A ∪ B = {1, 2, 3, 4, 5, 6}

Graphically, we can describe union operation as shown below. The grey region represents the output of the operation.



Union of crisp sets

If there are *n* sets, called A_1, A_2, A_3, …, A_n, we can find the union of all by taking **unique elements** from each set, i.e. $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$

In shortened notation,

$$A = \bigcup_{i=1}^{n} A_i$$

Intersection

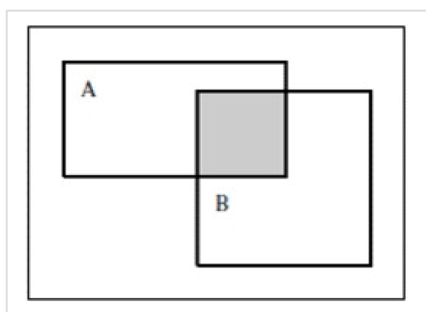
The intersection of sets is the collection of all the common elements from sets A **and** B. Mathematically, we can represent intersection operation as follow:

$$A \cap B = \{ x \mid x \in A \textbf{ and } x \in B \}$$

For the given data

$A \cap B = \{3, 4, 5\}$

Graphically, we can describe intersection operation as shown below. The grey region represents the output of the operation.



The intersection of crisp sets

If there are *n* sets, called $A_1, A_2, A_3, \ldots, A_n$, we can find the intersection of all by taking **common elements** from each set, i.e.
$A = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$

In shortened notation,

$$A = \bigcap_{i=1}^{n} A_i$$

Complement

Complement operation is always represented with respect to some set. If complement is performed with respect to a universal set, then it is called **absolute complement**.

The complement of set A is a collection of all the elements which are **not in** A but are in the universal set.

Mathematically,

$$A' = A^c = X - A = \{ x \mid x \in X \textbf{ and } x \notin A \}$$

The complement of set A is often represented as A' or A^c or

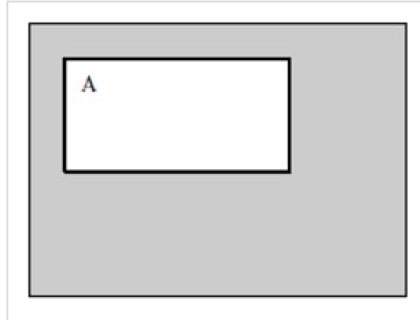
$$A$$

. We will be using any notation interchangeably in our discussion.

For the given data

A' = {6, 7, 8, 9}

Graphically, we can describe the complement operation as shown below. The grey region represents the output of the operation.



Complement of crisp set

Difference

The difference between set A with respect to set B is the collection of all the elements in A but not in B. It is also known as a **relative complement**.

Here the reference set is not the universal set, rather it is some set derived from the universe of discourse

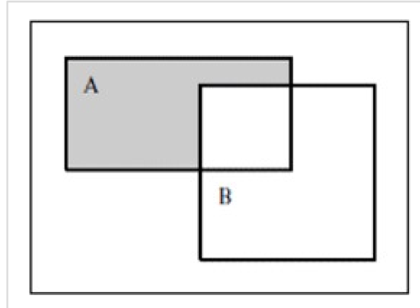
Mathematically,

$$A - B = \{ x \mid x \in A \textbf{ and } x \notin B \}$$

For the given data

$A - B = \{1, 2\}$

Graphically, difference operation can be represented as,



Difference of crisp sets

De Morgan's Law

De Morgan's law is very popular in set operations and it is quite useful in simplifying many complex computations. It is also useful in reducing the process of some proof techniques. De Morgan's law enjoys a special place in crisp set operations. There are two laws of De Morgan.

Law 1: $(A \cup B)' = A' \cap B'$

For the given data:

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$(A \cup B)' = \{7, 8, 9\} \rightarrow$ LHS

$A' = \{6, 7, 8, 9\}$

$B' = \{1, 2, 7, 8, 9\}$

$A' \cap B' = \{7, 8, 9\} \rightarrow$ RHS

Graphically,

Law 2: $(A \cap B)' = A' \cup B'$

For the given data:

$A \cap B=\{3, 4, 5\}$

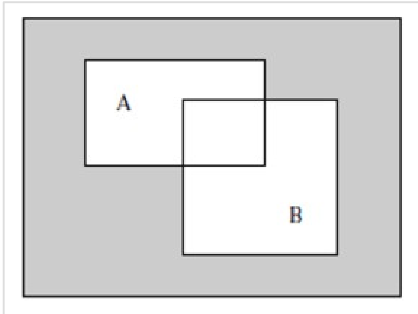
$(A \cap B)' = \{1, 2, 6, 7, 8, 9\} \rightarrow$ LHS

$A' = \{6, 7, 8, 9\}$

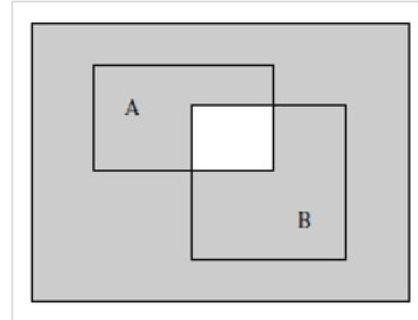
$B' = \{1, 2, 7, 8, 9\}$

$A' \cup B' = \{1, 2, 6, 7, 8, 9\} \rightarrow$ RHS

Graphically,



De Morgan's Law – 1



De Morgan's Law – 2

Fuzzy set is a natural way to deal with the imprecision. Many real world representation relies on significance rather than precision. Fuzzy logic is the best way to deal with them. Fuzzy set is an extension of crisp set.

Fuzzy Set – What wise men say?

Precision is not truth.

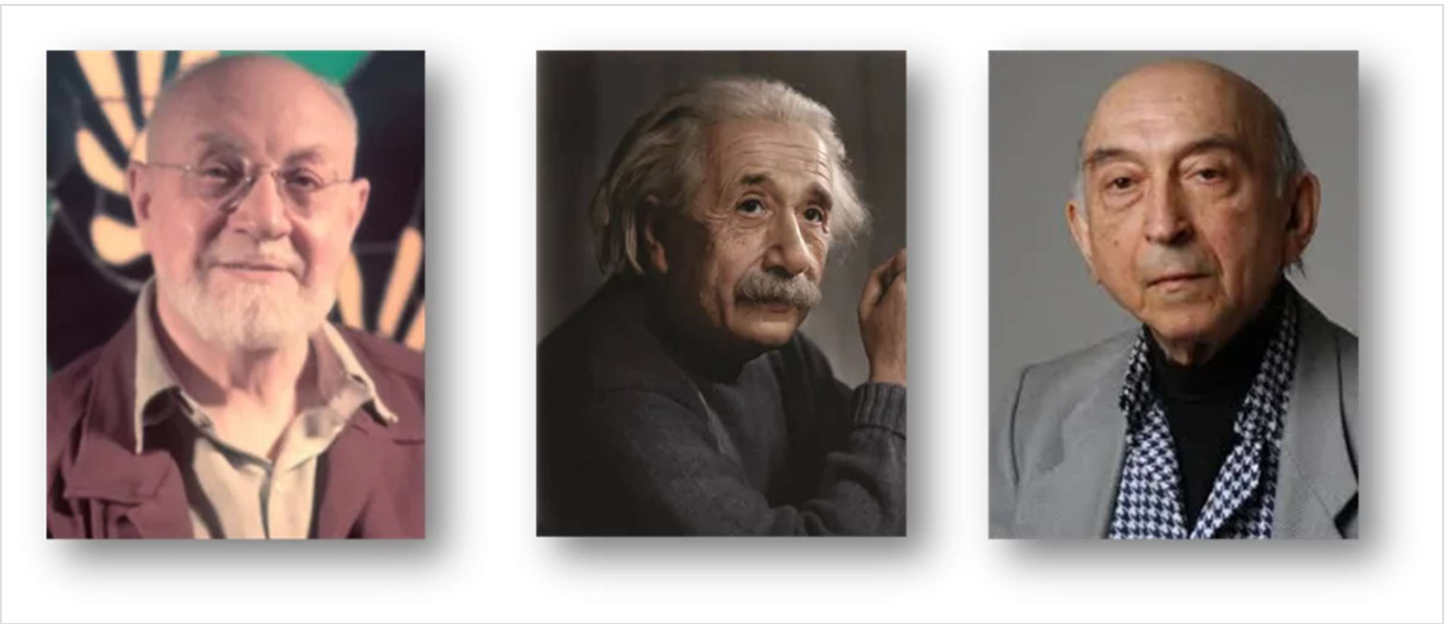
— Henri Matisse

So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.

— Albert Einstein

As complexity rises, precise statements lose meaning and meaningful statements lose precision.

— Lotfi Zadeh

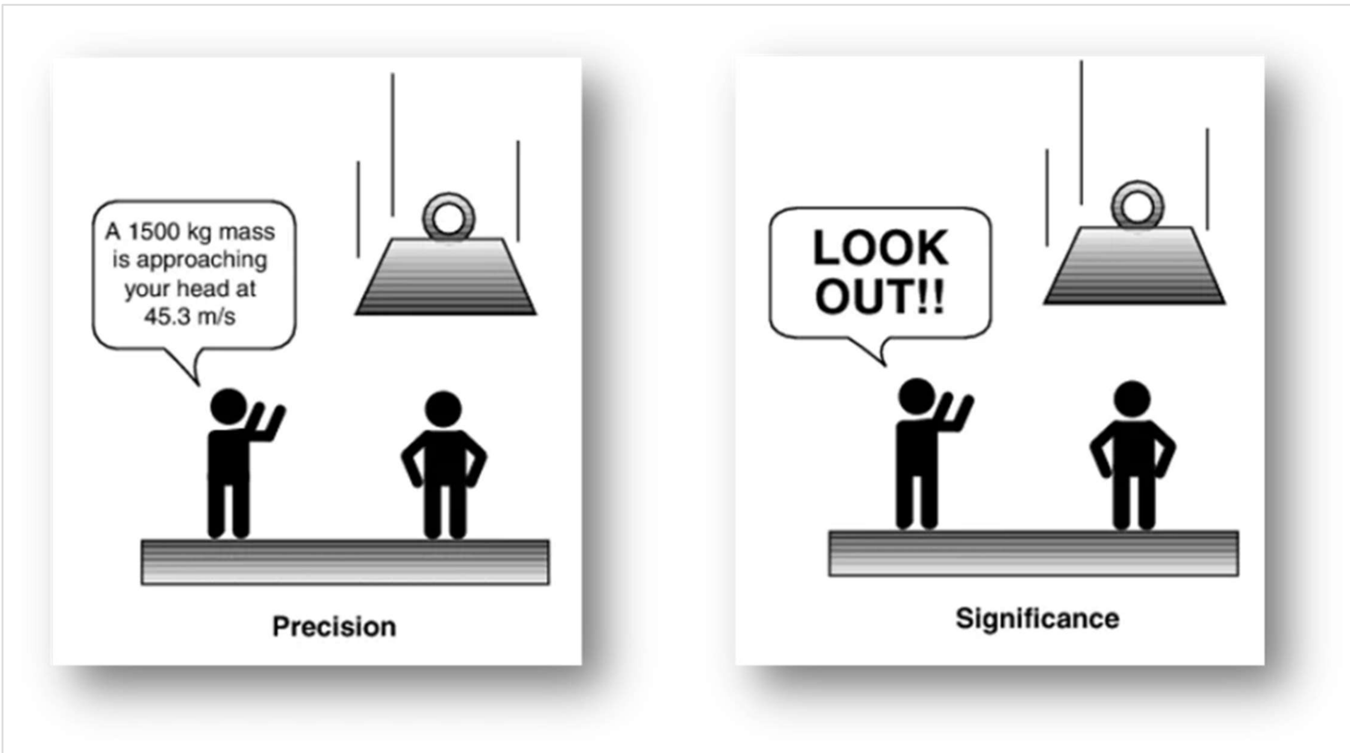


Henri Matisse (Left), Albert Einstein (Center) and Lotfi Zadeh (Right)

According to all the above greats, precision is not the everything. Most of the times, we are looking for the significance rather than precision.

For example if some heavy object is approaching to your friend from the top and you say that "hey friend, 1500 kg mass is approaching to you at the speed of 45.3 m/s". This statement is quite precise but before your friend Interpret it, the object might have fallen on him.

But if you simply shout "hello friend, **lookout**" then probably statement is not so precise but it convey quick information to your friend and he may move away from his place. In real world, we may be looking for significance rather than precision. Fuzzy logic helps us to model imprecision into data for many real world problems



Precision vs. Significance

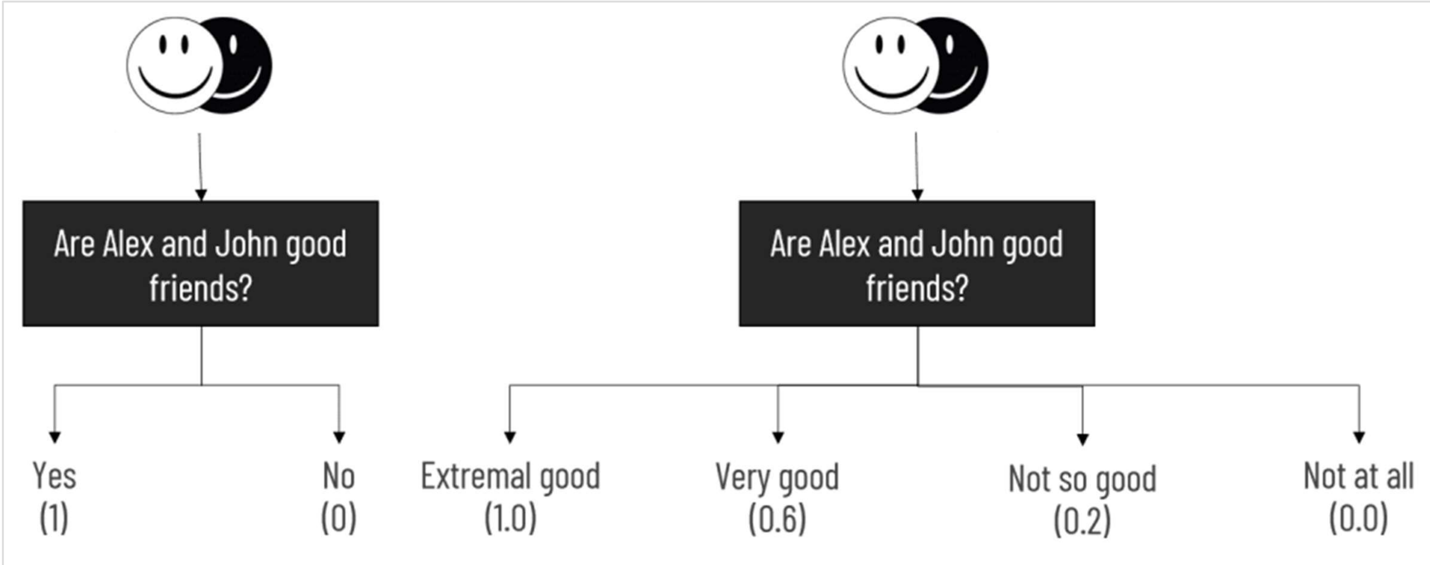
History of fuzzy logic

Though fuzzy logic gain popularity after the seminal work of Lofti Zadeh in 1965, the roots of fuzzy logic are very old. The concept of fuzzy logic dates back to the era of Aristotle.

- **Aristotle** supported the law of the Excluded Middle, i.e. there is no concept of partial truth. According to him, the proposition can be either true or false, it can not take middle value. This fact can be modelled by crisp set membership, where each element take membership value 0 or 1. The claim of Aristotle was strong base for binary valued logic.
- Later **Heraclitus** challenged the claim of Aristotle and claimed that things could be simultaneously True and not True. If we ask someone about Mr. A – if he is good or not? The answer is subjective. He might be good for some people, and he may not be good for others. So the predicate takes both the values simultaneously.
- **Plato** – the pupil of Aristotle – laid the foundation for what would become fuzzy logic. He said that the proposition could take values other then the extrema's, i.e. true (1) and false(0). According to him, predicate could take true and not true value simultaneously as well as the truthiness of statement not necessarily be 0 or 1. This logic suggest that the goodness of Mr. A can very between 0 and 1, rather then 0 or 1. This could be considered as pioneer thought in the direction of multivalued logic.
- Later, **Łukasiewicz** described a three-valued logic (True, False, Possible), along with the mathematics to accompany it. The possibility in fact can be correlated with probability of statement being true, which can take any real number between 0 and 1. Hence, it supports the claim of Plato of multi valued logic
- **Lofti Zadeh** introduced the notion of an infinite-valued logic in his seminal work "Fuzzy Sets" where he described the mathematics of fuzzy set theory, and by extension fuzzy logic. Rightly he is being referred as the **father of fuzzy logic**. He has proposed systematic way of modeling imprecision of the data using fuzzy sets. He also proposed strong mathematical background to support his claim.

What is fuzzy logic

Fuzzy logic is an approach to computing based on "degree of truth" rather than the usual "true or false value" (0 or 1) Boolean logic on which the modern computer is based.

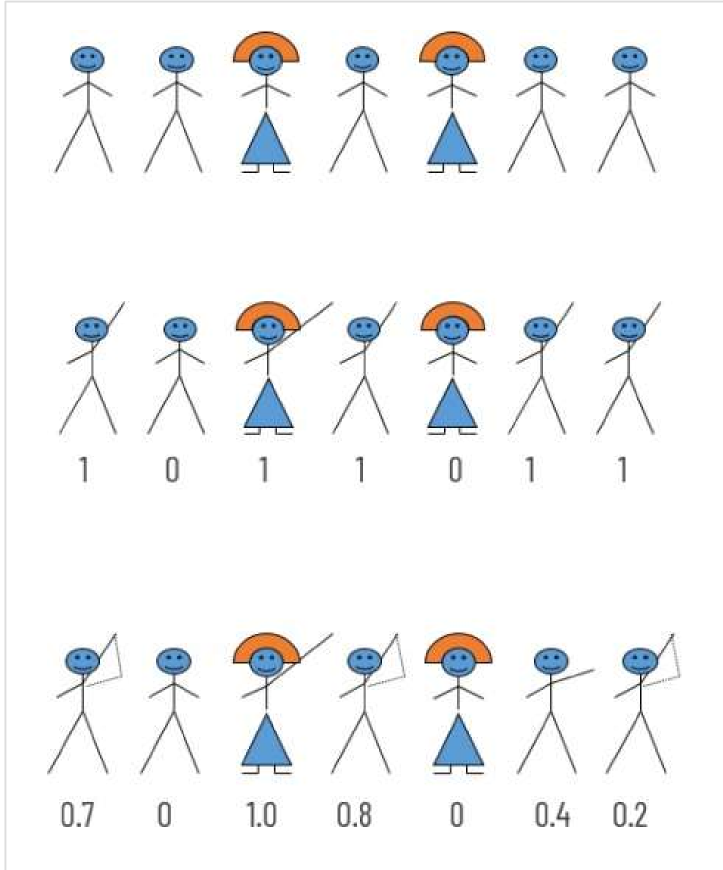


Crisp representation (left) vs. Fuzzy representation (right)

Crisp logic is binary valued logic. Any element in set has membership value either 0 or 1. Whereas, Fuzzy Logic which is also known as multi-valued has multiple membership values possible. For multi-valued logic, element can take any real value between 0 and 1.

For instance, if you ask Alex and John are good friends? Using crisp representation, the possible outcomes for this question is either 0 or 1. But with fuzzy logic, the possible outcomes could be such as they are extremely good friend, they are good friends, they are not so good friend, or they are not friends at all. This is subjective approach and the membership value of the element is assigned based on the perception or the confidence of the user. For particular case, we have assigned membership value 1 to extremely good, 0.6 to very good, 0.2 to not so good and 0.0 to not at all

Example



Universal set (top), crisp set (middle), fuzzy set (bottom)

As discussed in the article on introduction to crisp set, we can consider the class of students as a universal set. If we ask question *"who does have a driving license?"*, – student may or may not have driving license. Based on that, the membership value assigned to student will be either 0 or 1

But if we ask the question *"who can drive well?"* – answer to this question is quite subjective. Based on the skill of student, the membership value of student in particular set will vary from 0 to 1, where 0 indicates no driving skill and 1 indicate the highest level of driving skill. This is how fuzzy representation helps us to capture the uncertainty in the data.

From above examples, we can say that the crisp sets have crisp boundary (element is either inside set or outside set), and fuzzy set is having fuzzy boundary (element can be partially member of set)

Motivation for fuzzy set

There are many motivational factors for fuzzy set. Few of them are listed here.

- Knowledge in real world can be: inaccurate, unclear, imprecise, indecisive, probabilistic, approximate. Crisp set can not tolerate the imprecision in the data.
- Human thinking and reasoning include fuzzy nuances. Human mind compares/measure the things relatively, for example, while we touch something, instead of measuring exact value of temperature, we think whether the object is cold, warm, hot or very hot etc. Fuzzy set inherently represents the real data into such linguistic terms.
- Real world systems should function with vague information. Many times, data might be missing or may not be recorded. Fuzzy logic deals with the range, rather then individual elements, so its easy to handle such vague information too.
- Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive.
- Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

Advantages of fuzzy set

Fuzzy sets are quite useful in many industrial and hand hold device design due to its simplicity and interpretability. Some of the most prevailing advantages are listed here:

- Conceptually easy to understand as it is built upon “natural” mathematics
- Tolerant to imprecise data
- Universal approximation: It can model arbitrary nonlinear functions
- Intuitive
- Based on linguistic terms
- Convenient way to express expert and common sense knowledge

Limitations of fuzzy set

- How do we define the membership functions? For the same range of inputs, different person can use different membership function, which results into different result.
- There is no proper way of learning in fuzzy logic. Membership function is to be chosen from the experience of the domain expert.
- What if we have membership functions provided from two different people E.g. What a 6’11” Basketball player defines as tall will differ from a 4’10” Gymnast. A guy having height 6.11 may not be considered tall in case of Basketball, but a player with height 4.10 is considered as tall in Gymnast. Thus, there might be ambiguity in interpretation of membership function, that is, domain changes, interpretation also changes.
- Defuzzification can produce undesired results. Different defuzzification techniques can produced different crisp output for same fuzzy value. The produced crisp value might have large variations in them, this creates lots of ambiguity in selection of defuzzification method.
- Crisp/precise models can be more efficient and even convenient
- Membership values begin to move away from expectations when chains of logic are lengthy so this approach is not suitable for many KBS problems (e.g., medical diagnosis)

Probability vs. Fuzziness

People often miss-interpret probability with fuzziness.

Probability describes the uncertainty of an event occurrence.

- Probability defines frequency of likelihood that an element is in a class
- There is a 50% chance of an apple being in the refrigerator: If you open the refrigerator 100 times, 50 times you will find the apple in refrigerator

Fuzziness describes event ambiguity.

- Fuzziness defines similarity of an element to a class
- There is a half an apple in the refrigerator: If you open the refrigerator 100 times, all 100 times you will find half apple (50%) in refrigerator.

Applications of fuzzy logic

Fuzzy logic covers wide spectrum of applications. Few of them are listed here.

- Reasoning tool like Fuzzy Logic Controller
- Automation (Flight Control, Washing Machine, …)
- Environment Control (Air Conditioner)
- Clustering using fuzzy logic
- Fuzzy Mathematical Programming
- Fuzzy Graph Theory
- Hybrid systems (ANFIS)

Test your knowledge

1. What is the fundamental difference between fuzzy set and crisp set?

2. State the scenario where crisp set is preferred over fuzzy set?
3. State the scenario where fuzzy set is preferred over crisp set?
4. Who is known as the father of fuzzy logic?

Union:

In the case of the union of crisp _sets, we simply have to select repeated elements only once. In the case of fuzzy sets, when there are common elements in both fuzzy sets, we should select the element with the **maximum membership value**.

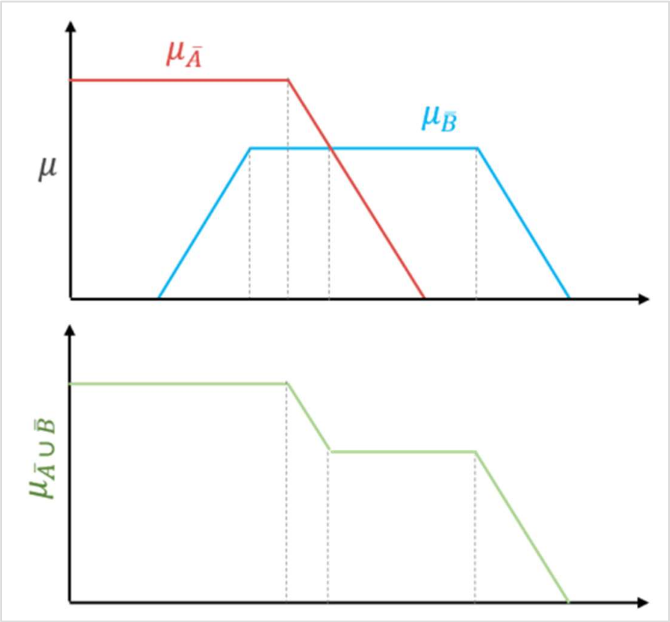
The **union** of two fuzzy sets $\underline{A}$ and $\underline{B}$ is a fuzzy set $\underline{C}$, written as $\underline{C} = \underline{A} \cup \underline{B}$

$$\underline{C} = \underline{A} \cup \underline{B} = \{(x, \mu_{\underline{A} \cup \underline{B}}(x)) \mid \forall x \in X\}$$

$$\mu_{\underline{C}}(x) = \mu_{\underline{A} \cup \underline{B}}(x) = \mu_{\underline{A}}(x) \vee \mu_{\underline{B}}(x)$$

$$= \max(\mu_{\underline{A}}(x), \mu_{\underline{B}}(x)), \forall x \in X$$

Graphically, we can represent union operations as follows: Red and Blue membership functions represent the fuzzy value for elements in sets A and B, respectively. Wherever these fuzzy functions overlap, we have to consider the point with the maximum membership value.



Fuzzy union

Example of Fuzzy Union:

$\underline{C} = \underline{A} \cup \underline{B} = \{(x, \mu_{\underline{A} \cup \underline{B}}(x)) \mid \forall x \in X\}$

$\underline{A} = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.6), (x_4, 0.8), (x_5, 1.0) \}$

$\underline{B} = \{ (x_1, 0.8), (x_2, 0.6), (x_3, 0.4), (x_4, 0.2), (x_5, 0.1) \}$

$\mu_{\underline{A} \cup \underline{B}}(x_1) = \max(\mu_{\underline{A}}(x_1), \mu_{\underline{B}}(x_1)) = \max \{ 0.2, 0.8 \} = 0.8$

$\mu_{\underline{A} \cup \underline{B}}(x_2) = \max(\mu_{\underline{A}}(x_2), \mu_{\underline{B}}(x_2)) = \max \{ 0.5, 0.6 \} = 0.6$

$\mu_{\underline{A} \cup \underline{B}}(x_3) = \max(\mu_{\underline{A}}(x_3), \mu_{\underline{B}}(x_3)) = \max \{ 0.6, 0.4 \} = 0.6$

$\mu_{\underline{A} \cup \underline{B}}(x_4) = \max(\mu_{\underline{A}}(x_4), \mu_{\underline{B}}(x_4)) = \max\{0.8, 0.2\} = 0.8$

$\mu_{\underline{A} \cup \underline{B}}(x_5) = \max(\mu_{\underline{A}}(x_5), \mu_{\underline{B}}(x_5)) = \max\{1.0, 0.1\} = 1.0$

So, $\underline{A} \cup \underline{B} = \{ (x_1, 0.8), (x_2, 0.6), (x_3, 0.6), (x_4, 0.8), (x_5, 1.0) \}$

Intersection:

In the case of the intersection of crisp_sets, we simply have to select common elements from both sets. In the case of fuzzy sets, when there are common elements in both fuzzy sets, we should select the element with **minimum membership value**.

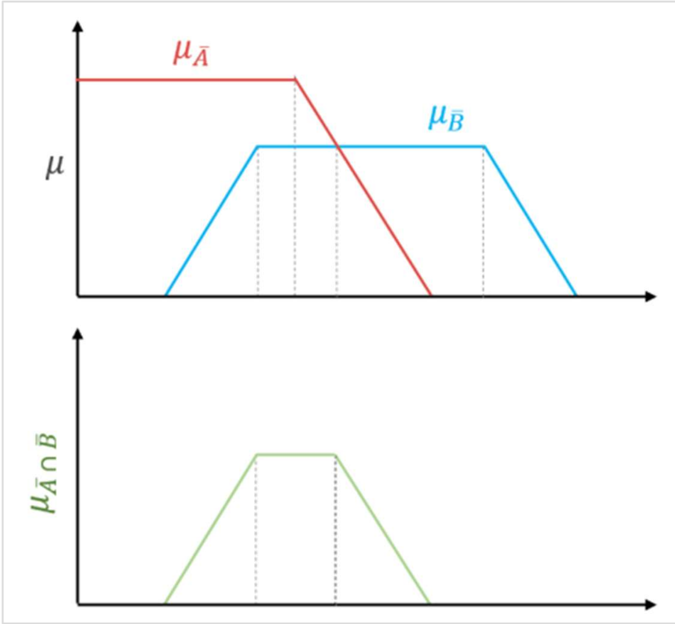
The **intersection** of two fuzzy sets $\underline{A}$ and $\underline{B}$ is a fuzzy set $\underline{C}$, written as $\underline{C} = \underline{A} \cap \underline{B}$

$\underline{C} = \underline{A} \cap \underline{B} = \{(x, \mu_{\underline{A} \cap \underline{B}}(x)) \mid \forall x \in X\}$

$\mu_{\underline{C}}(x) = \mu_{\underline{A} \cap \underline{B}}(x) = \mu_{\underline{A}}(x) \wedge \mu_{\underline{B}}(x)$

$= \min(\mu_{\underline{A}}(x), \mu_{\underline{B}}(x)), \forall x \in X$

Graphically, we can represent the intersection operation as follows: Red and blue membership functions represent the fuzzy value for elements in sets A and B, respectively. Wherever these fuzzy functions overlap, we have to consider the point with the minimum membership value.



Fuzzy intersection

Example of Fuzzy Intersection:

$\underline{C} = \underline{A} \cap \underline{B} = \{(x, \mu_{\underline{A} \cap \underline{B}}(x)) \mid \forall x \in X\}$

$\underline{A} = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.6), (x_4, 0.8), (x_5, 1.0) \}$

$\underline{B} = \{ (x_1, 0.8), (x_2, 0.6), (x_3, 0.4), (x_4, 0.2), (x_5, 0.1) \}$

$\mu_{\underline{A} \cap \underline{B}}(x_1) = \min(\mu_{\underline{A}}(x_1), \mu_{\underline{B}}(x_1)) = \max \{ 0.2, 0.8 \} = 0.2$

$\mu_{\underline{A} \cap \underline{B}}(x_2) = \min(\mu_{\underline{A}}(x_2), \mu_{\underline{B}}(x_2)) = \max \{ 0.5, 0.6 \} = 0.5$

$\mu_{\underline{A} \cap \underline{B}}(x_3) = \min(\mu_{\underline{A}}(x_3), \mu_{\underline{B}}(x_3)) = \max \{ 0.6, 0.4 \} = 0.4$

$\mu_{\underline{A} \cap \underline{B}}(x_4) = \min(\mu_{\underline{A}}(x_4), \mu_{\underline{B}}(x_4)) = \max\{0.8, 0.2\} = 0.2$

$\mu_{\underline{A} \cap \underline{B}}(x_5) = \min(\mu_{\underline{A}}(x_5), \mu_{\underline{B}}(x_5)) = \max\{1.0, 0.1\} = 0.1$

So, $\underline{A} \cap \underline{B} = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.4), (x_4, 0.2), (x_5, 0.1) \}$

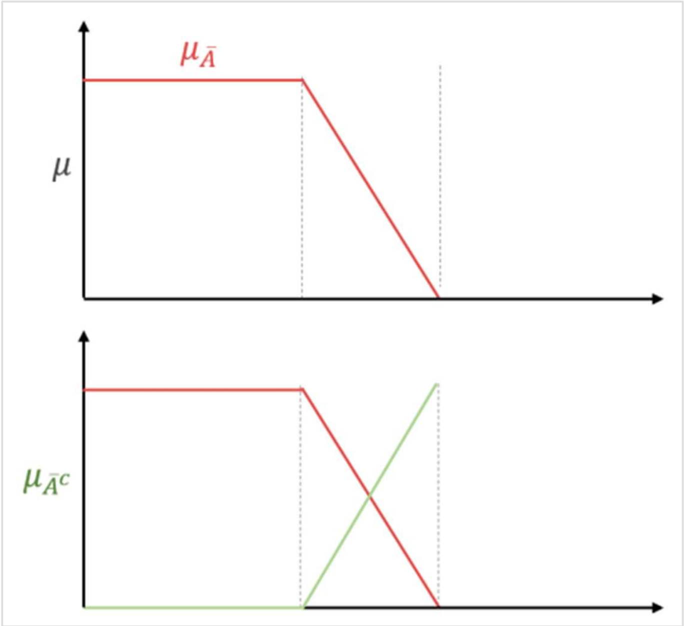
Complement:

Fuzzy complement is identical to crisp complement operation. The membership value of every element in the fuzzy set is complemented with respect to 1, i.e. it is subtracted from 1.

The **complement** of fuzzy set $\underline{A}$, denoted by $\underline{A}^C$, is defined as

$\underline{A}^C = \{(x, \mu_{A^c}(x)) \mid \forall x \in X\}$

$\underline{A}^C(x) = 1 - \mu_A(x)$



Fuzzy complement

Example of Fuzzy Complement:

$\underline{A}^C(x) = 1 - \mu_A(x)$

$\underline{A} = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.6), (x_4, 0.8), (x_5, 1.0) \}$

$\underline{A}^C = \{ (x_1, 0.8), (x_2, 0.5), (x_3, 0.4), (x_4, 0.2), (x_5, 0.0) \}$

$\underline{A} \cup \underline{A}^C = \{ (x_1, 0.8), (x_2, 0.5), (x_3, 0.6), (x_4, 0.8), (x_5, 1.0) \} \neq X$

$\underline{A} \cap \underline{A}^C = \{ (x_1, 0.2), (x_2, 0.5), (x_3, 0.4), (x_4, 0.2), (x_5, 0.0) \} \neq \Phi$

Unlike crisp sets, fuzzy sets do not hold the law of contradiction and the law of excluded middle.

	Crisp Set	Fuzzy Set
Law of contradiction	$A \cap A^c = \phi$	$\bar{A} \cap \bar{A}^c \neq \phi$
Law of excluded middle	$A \cup A^c = X$	$\bar{A} \cup \bar{A}^c \neq X$

Note: As stated earlier, to distinguish the fuzzy set from the crisp set, we will be using a **bar under the set letter**, i.e.

A : Crisp set

$\underline{A}$: Fuzzy set

The membership value of elements in the crisp set is defined by the characteristic function χ (chi), where as the membership value of elements in a fuzzy set is defined by the membership function μ (mu).

For crisp set: $\chi \in \{ 0, 1 \}$

For fuzzy set: $\mu \in [0, 1]$

Fuzzy terminologies

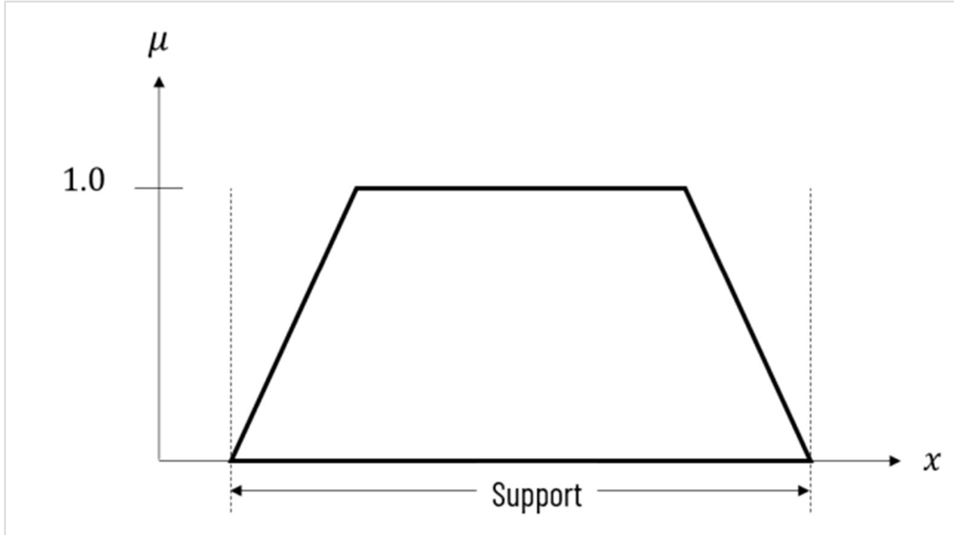
Fuzzy terminologies define the properties of fuzzy sets. A complete set of fuzzy terminologies is discussed here.

Support:

The **support** of a fuzzy set $\underline{A}$ is the set of all points $x \in X$ such that $\mu_A(x) > 0$

Support($\underline{A}$) = $\{ x \mid \mu_A(x) > 0, x \in X \}$

Graphically, we can define support of fuzzy set as,



Support of fuzzy set

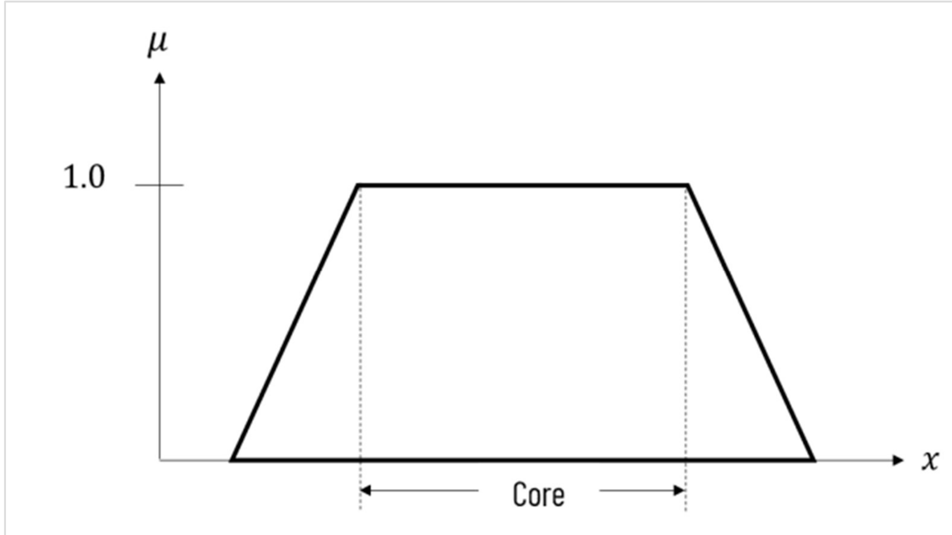
Note: Support of fuzzy set is its Strong 0-cut (Discussed in the later part of this article)

Core:

The **core** of a fuzzy set $\underline{A}$ is the set of all points $x \in X$ such that $\mu_A(x) = 1$

Core($\underline{A}$) = $\{ x \mid \mu_A(x) = 1, x \in X \}$

All fuzzy sets might not have a core present in them.



The core of fuzzy set

Height of Fuzzy Set: It is defined as the largest membership value of the elements contained in that set. It may not be 1 always. If the core of the fuzzy set is non-empty, then the height of the fuzzy set is 1.

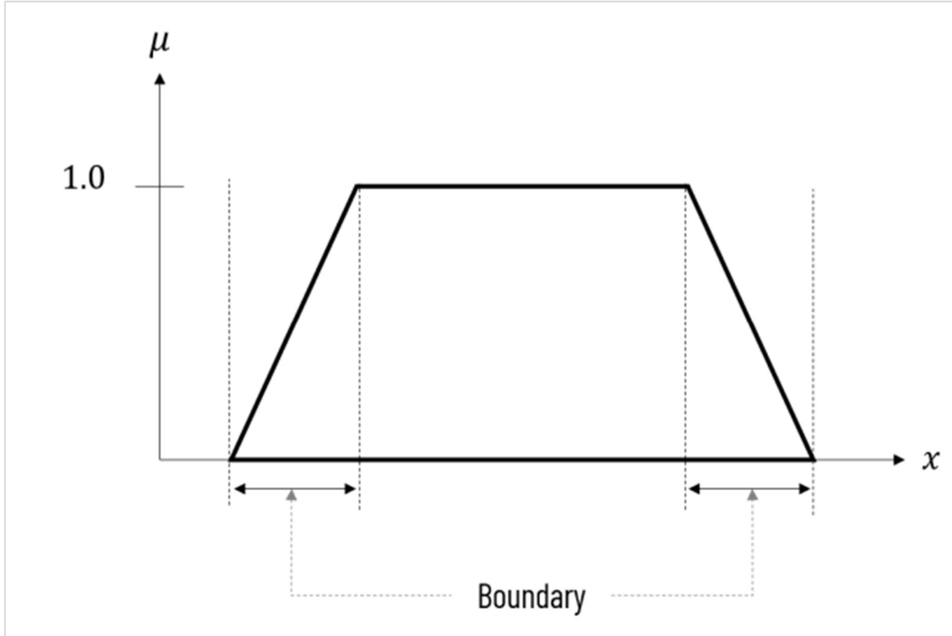
Boundary:

Boundary comprises those elements x of the universe such that $0 < \mu_A(x) < 1$

$$\text{Boundary}(\underline{A}) = \{ x \mid 0 < \mu_A(x) < 1 , x \in X \}$$

We can treat boundary as the difference between support and core.

Graphically, it is represented as



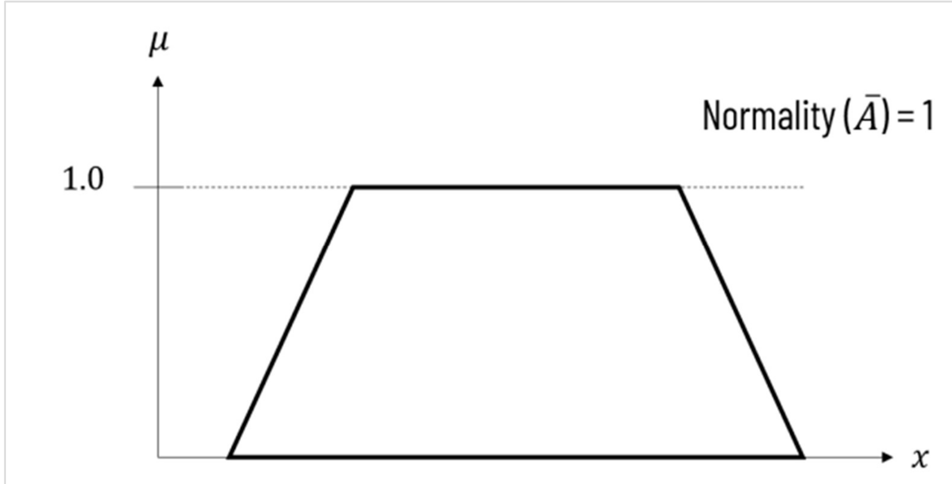
The boundary of fuzzy set

Normality:

A fuzzy set A is **normal** if its core is non-empty.

In other words, a fuzzy set is normal if its height is 1

Sub-normal Fuzzy set: For a sub-normal fuzzy set, h(A) < 1, where h(A) represents the height of the fuzzy set / highest membership value in the fuzzy set.



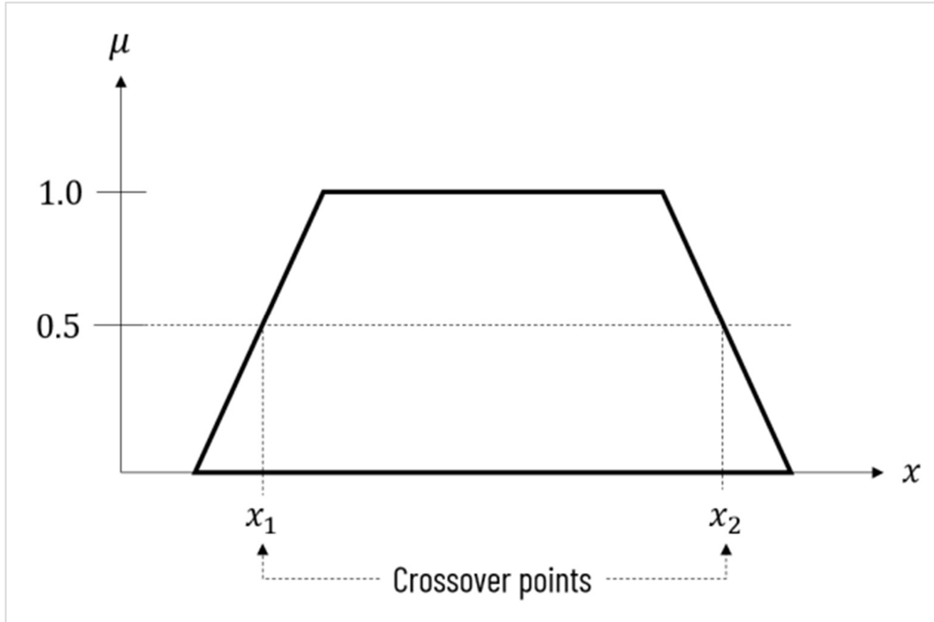
Normality of fuzzy set

Crossover points:

A crossover point of a fuzzy set A is a point x ∈ X at which $\mu_A(x) = 0.5$

Crossover(A) = $\{ x \mid \mu_A(x) = 0.5 \}$

Graphically, we can represent it as



Cross-over points of fuzzy set

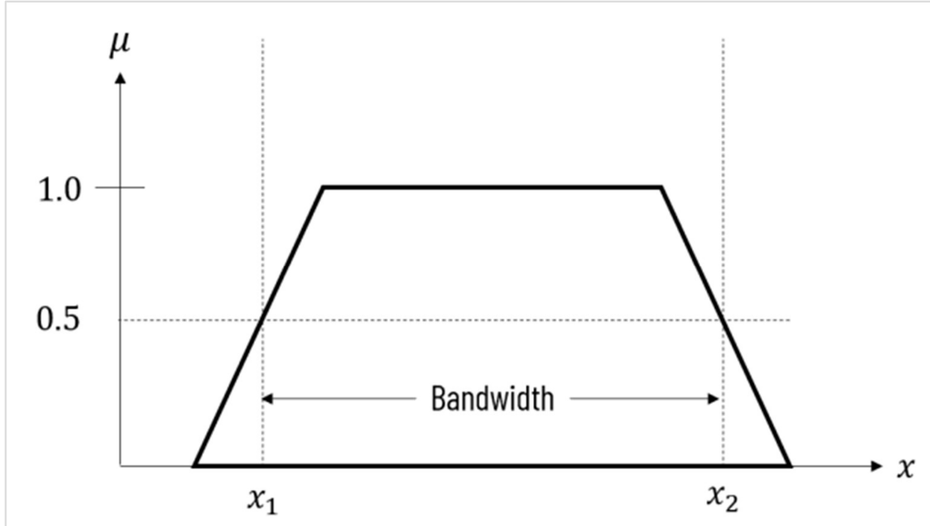
Bandwidth:

For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points.

Bandwidth($\underline{A}$) = $|x_1 - x_2|$

Where, $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Graphically,



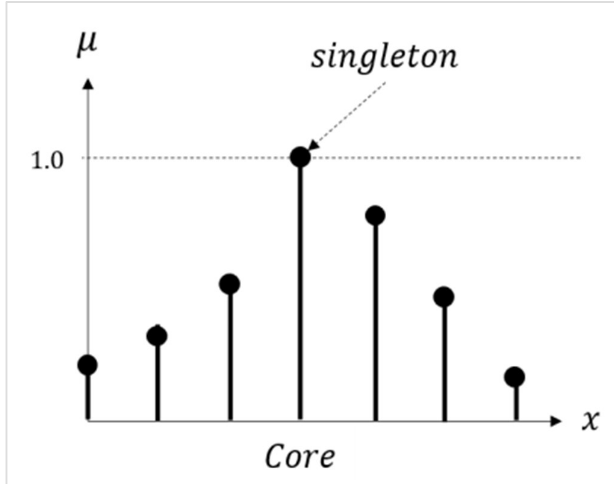
The bandwidth of fuzzy set

Fuzzy singleton:

A fuzzy set whose core is a single point in X with $\mu_A(x) = 1$, is called a fuzzy singleton. In other words, if the fuzzy set is having only one element with a membership value of 1, then it is called a fuzzy singleton.

$|A| = \{ \mu_A(x) = 1 \}$

Graphically,



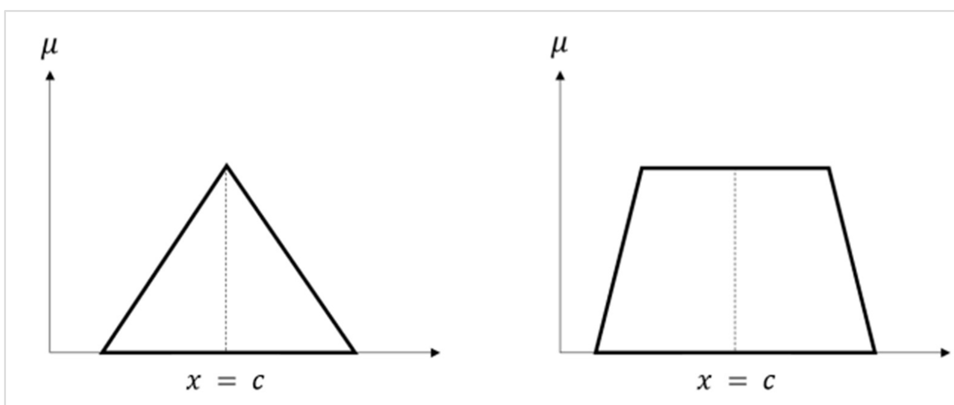
Fuzzy singleton

Symmetry:

Fuzzy set A is symmetric if its membership function around a centre point x = c is symmetric

i.e. $\mu_A(x + c) = \mu_A(x - c), \forall x \in X$

Triangular, Trapezoidal, Gaussian etc. are mostly symmetric. This is more natural to represent the membership than a non-symmetric shape.

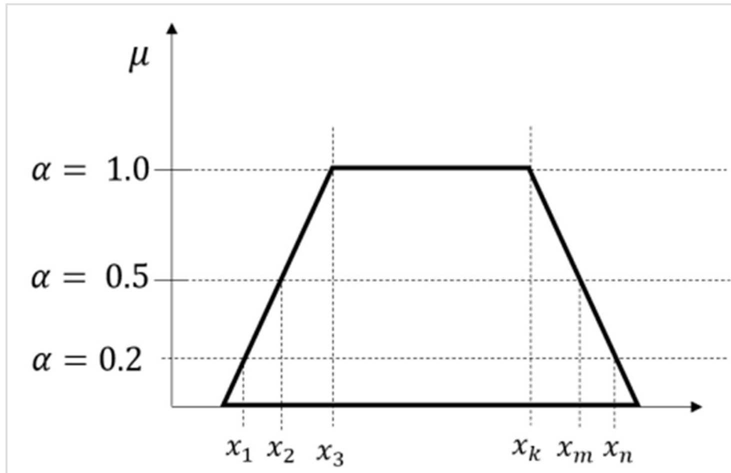


Symmetric fuzzy sets

Alpha cut:

The **α-cut** of a fuzzy set $\underline{A}$ is a crisp set defined by $A_\alpha = \{ x \mid \mu_A(x) \geq \alpha \}$

Strong α-cut of a fuzzy set $\underline{A}$ is a crisp set defined by $A_\alpha + = \{ x \mid \mu_A(x) > \alpha \}$

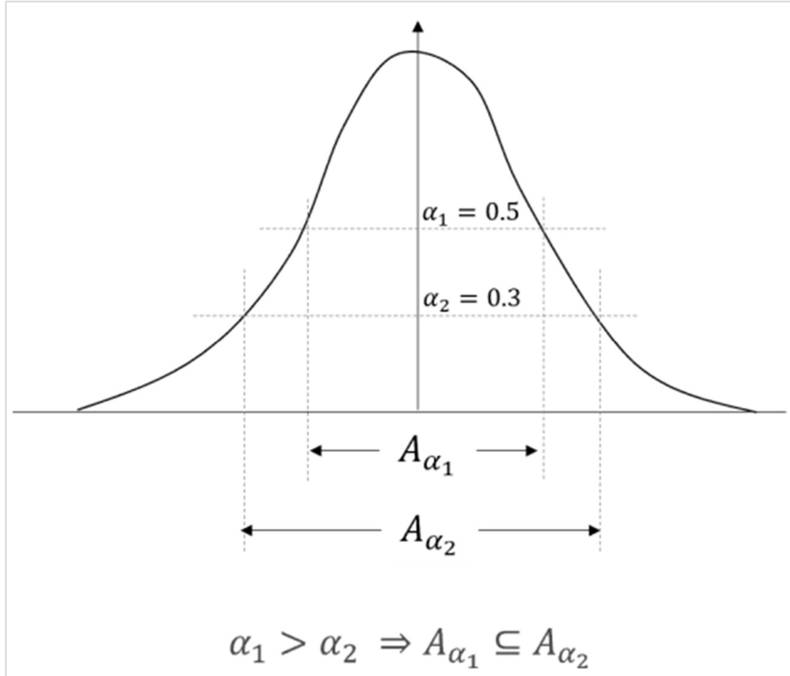


Alpha cut in fuzzy set

For the above diagram,

- The set $A_{\alpha=0.2}$ contains all the elements from x_1 to x_n, including both end values
- The set $A_{\alpha=0.5}$ contains all the elements from x_2 to x_m, including both end values
- The set $A_{\alpha=1.0}$ contains all the elements from x_3 to x_k, including both end values

For different values of α, we get different crisp sets. In general, if $\alpha_1 > \alpha_2$ then $A_{\alpha 1} \subseteq A_{\alpha 2}$



Relation between different α values

Cardinality:

Scalar cardinality:

Scalar cardinality is defined by the summation of membership values of all elements in the set. For the data given in the table,

Age	Infant	Young	Adult	Senior
5	0	0	0	0
15	0	0.2	0	0
25	0	0.8	0.8	0
35	0	1.0	0.9	0
45	0	0.6	1	0
55	0	0.5	1	0.3
65	0	0.1	1	0.9
75	0	0.0	1	1
85	0	0.0	1	1

$| \underline{A} | = \Sigma_{x \in X} \{ \mu_A(x) \}$

$|Senior| = 0.3 + 0.9 + 1 + 1 = 3.2$

Relative cardinality:

$|| \underline{A} || = | \underline{A} | / | X |$

$|| Senior || = 3.2 / 9 = 0.356$

Fuzzy cardinality:

$\alpha = \mu_{senior}$	$\|A_\alpha\|$
0.3	4
0.9	3
1	2

$| \underline{A} |_F = \{ (\alpha , \mu_{A\alpha}(x)) \}$

$| Senior |_F = \{ (4, 0.3), (3, 0.9), (2, 1.0) \}$

Open and Closed fuzzy sets:

Open left: As the name suggests, open left fuzzy sets have all the elements on left after a certain point have a membership value of 1, and all the elements on the right side after a certain point have a membership value of 0.

Open left: if $\lim_{x \to -\infty} \mu_{\bar{A}}(x) = 1$ and $\lim_{x \to +\infty} \mu_{\bar{A}}(x) = 0$

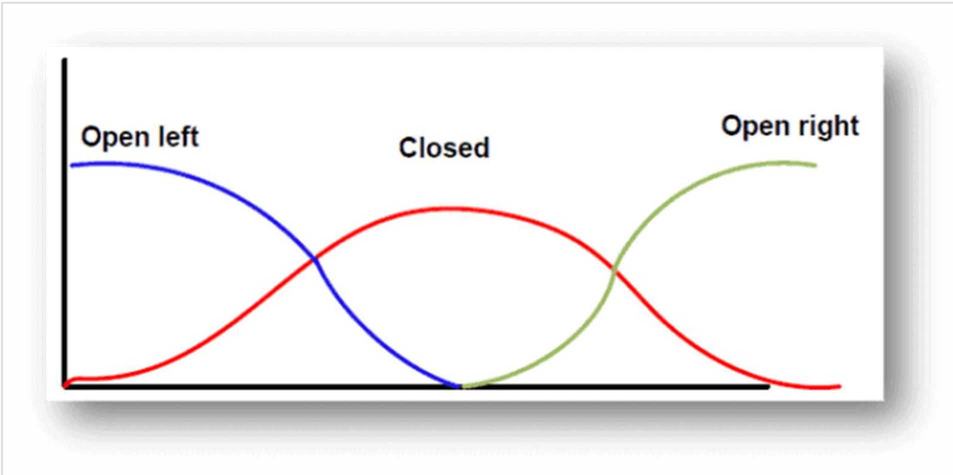
Open right: Open right fuzzy sets have all the elements on left after a certain point have a membership value of 0, and all the elements on the right side after a certain point have a membership value of 1.

Open right: if $\lim_{x \to -\infty} \mu_{\bar{A}}(x) = 0$ and $\lim_{x \to +\infty} \mu_{\bar{A}}(x) = 1$

Closed: Closed fuzzy sets have all the elements on the left or right side after a certain point have a membership value of 0.

Closed: if $\lim_{x \to -\infty} \mu_{\bar{A}}(x) = 0$ and $\lim_{x \to +\infty} \mu_{\bar{A}}(x) = 0$

The following diagram graphically demonstrates all three kinds of fuzzy sets.



Open and closed fuzzy sets

Convexity:

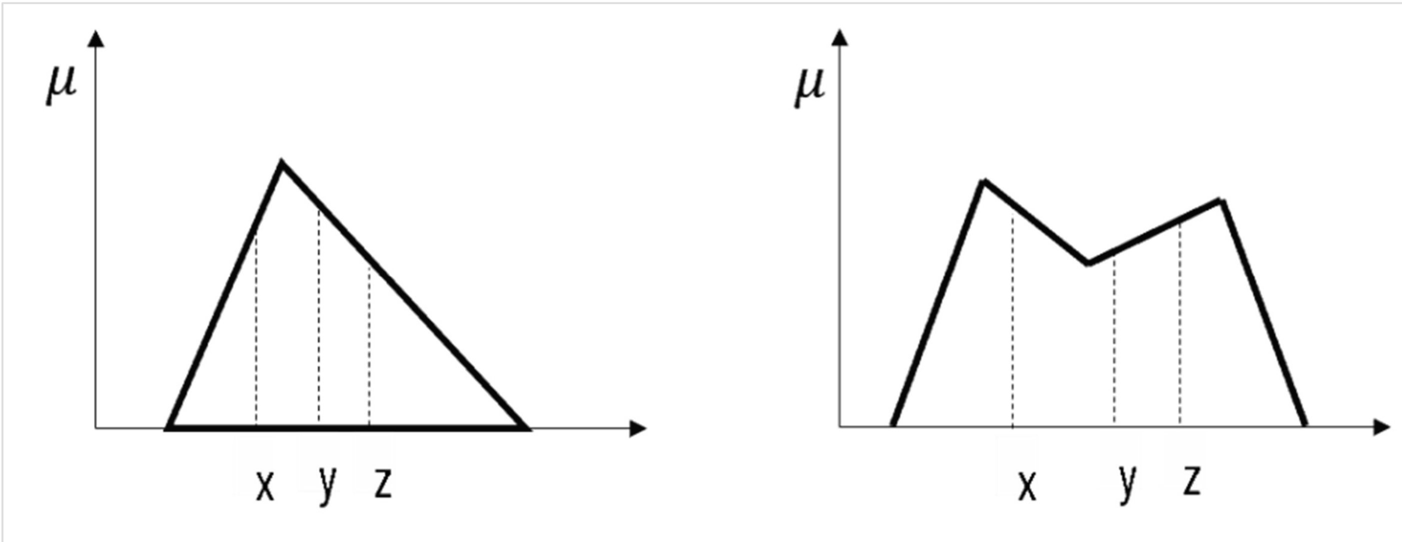
Crisp Set A is convex if $(\lambda x_1 + (1 - \lambda) x_2)$ in A, where $\lambda \in [0, 1]$

Fuzzy Set $\underline{A}$ is convex if $\mu_A(\lambda x_1 + (1 - \lambda) x_2)) \geq \min(\mu_A(x_1), \mu_A(x_2))$, where $x_1, x_2 \in X$

In other words, for any elements x, y and z in a fuzzy set A, the relation $x < y < z$ implies that: $\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$. If this condition holds for all points, the fuzzy set is called a convex fuzzy set.

Convex fuzzy sets are strictly increasing and then strictly decreasing

A is convex if all its α-level sets are convex



Convex fuzzy set (left), Non-convex fuzzy set (right)

Cartesian product

Fuzzy relation defines the mapping of variables from one fuzzy_set to another. Like crisp_relation, we can also define the relation over fuzzy sets.

Let $\underline{A}$ be a fuzzy set on universe X and $\underline{B}$ be a fuzzy set on universe Y, then the Cartesian product between fuzzy sets $\underline{A}$ and $\underline{B}$ will result in a fuzzy relation $\underline{R}$ which is contained with the full Cartesian product space or it is a subset of the cartesian product of fuzzy subsets. Formally, we can define fuzzy relation as,

$$\underline{R} = \underline{A} \times \underline{B}$$

and

$$\underline{R} \subset (X \times Y)$$

where the relation $\underline{R}$ has a membership function,

$$\mu_{\underline{R}}(x, y) = \mu_{\underline{A} \times \underline{B}}(x, y) = \min(\, \mu_{\underline{A}}(x), \mu_{\underline{B}}(y)\,)$$

A binary fuzzy relation $\underline{R}(X, Y)$ is called a **bipartite graph** if $X \neq Y$.

A binary fuzzy relation $\underline{R}(X, Y)$ is called **directed graph** or **digraph** if X = Y. , which is denoted as $\underline{R}(X, X) = \underline{R}(X^2)$

Let $\underline{A} = \{a_1, a_2, \ldots, a_n\}$ and $\underline{B} = \{b_1, b_2, .., b_m\}$, then the fuzzy relation between $\underline{A}$ and $\underline{B}$ is described by the **fuzzy relation matrix** as,

$$\begin{bmatrix} \mu_{R(a_1,b_1)} & \mu_{R(a_1,b_2)} & \cdot & \cdot & \mu_{R(a_1,b_m)} \\ \mu_{R(a_2,b_1)} & \mu_{R(a_2,b_2)} & \cdot & \cdot & \mu_{R(a_2,b_m)} \\ \cdot & \cdot & & \cdot\ \cdot & \cdot \\ \cdot & \cdot & & \cdot\ \cdot & \cdot \\ \mu_{R(a_n,b_1)} & \mu_{R(a_n,b_2)} & \cdot & \cdot & \mu_{R(a_n,b_m)} \end{bmatrix}$$

Fuzzy relation matrix

We can also consider fuzzy relation as a mapping from the cartesian space (X, Y) to the interval [0, 1]. The strength of this mapping is represented by the membership function of the relation for every tuple $\mu_{R(x,\, y)}$

Example:

Given $\underline{A}$ = { (a_1, 0.2), (a_2, 0.7), (a_3, 0.4) } and $\underline{B}$ = { (b_1, 0.5), (b_2, 0.6)}, find the relation over $\underline{A}$ x $\underline{B}$

$$\bar{R} = \bar{A} \times \bar{B} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Cartesian product

Fuzzy relation

Fuzzy relations are very important because they can describe interactions between variables.

Example: A simple example of a binary fuzzy relation on X = {1, 2, 3}, called ”approximately equal” can be defined as

$\underline{R}(1, 1) = \underline{R}(2, 2) = \underline{R}(3, 3) = 1$

$\underline{R}(1, 2) = \underline{R}(2, 1) = \underline{R}(2, 3) = \underline{R}(3, 2) = 0.8$

$\underline{R}(1, 3) = \underline{R}(3, 1) = 0.3$

The membership function and relation matrix of $\underline{R}$ are given by

$$R(x, y) = \begin{cases} 1, & \text{if} x = y \\ 0.7, & \text{if} |x - y| = 1 \\ 0.3, & \text{if} |x - y| = 2 \end{cases}$$

$$\bar{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1.0 & 0.7 & 0.3 \\ 0.7 & 1.0 & 0.7 \\ 0.3 & 0.7 & 1.0 \end{bmatrix} \end{array}$$

Operations on fuzzy relation:

For our discussion, we will be using the following two relation matrices:

$$\bar{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0.7 \\ 0.0 & 0.8 & 0.0 & 0.0 \\ 0.9 & 1.0 & 0.7 & 0.8 \end{bmatrix} \end{array}$$

$$\bar{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.0 & 0.9 & 0.6 \\ 0.9 & 0.4 & 0.5 & 0.7 \\ 0.3 & 0.0 & 0.8 & 0.5 \end{bmatrix} \end{array}$$

Union:

$\underline{R} \cup \underline{S} = \{ (a, b), \mu_{\underline{A} \cup \underline{B}}(a, b) \}$

$\mu_{\underline{R} \cup \underline{S}}(a, b) = \max(\mu_{\underline{R}}(a, b), \mu_{\underline{S}}(a, b))$

$\mu_{\underline{R} \cup \underline{S}}(x_1, y_1) = \max(\mu_{\underline{R}}(x_1, y_1), \mu_{\underline{S}}(x_1, y_1))$

$= \max(0.8, 0.4) = 0.8$

$\mu_{\underline{R} \cup \underline{S}}(x_1, y_2) = \max(\mu_{\underline{R}}(x_1, y_2), \mu_{\underline{S}}(x_1, y_2))$

$= \max(0.1, 0.0) = 0.1$

$\mu_{\underline{R} \cup \underline{S}}(x_1, y_3) = \max(\mu_{\underline{R}}(x_1, y_3), \mu_{\underline{S}}(x_1, y_3))$

$= \max(0.1, 0.9) = 0.9$

$\mu_{\underline{R} \cup \underline{S}}(x_1, y_4) = \max(\mu_{\underline{R}}(x_1, y_4), \mu_{\underline{S}}(x_1, y_4))$

$\max(0.7, 0.6) = 0.7$

.

.

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$\mu_{\underline{R} \cup \underline{S}}(x_3, y_4) = \max(\mu_{\underline{R}}(x_3, y_4), \mu_{\underline{S}}(x_3, y_4))$

$= \max(0.8, 0.5) = 0.8$

Thus, the final matrix for union operation would be,

$$\bar{R} \cup \bar{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.9 & 0.7 \\ 0.9 & 0.8 & 0.5 & 0.7 \\ 0.9 & 1.0 & 0.8 & 0.8 \end{bmatrix} \end{array}$$

Union of fuzzy relations

Intersection:

$\underline{R} \cap \underline{S} = \{ (a, b), \mu_{\underline{A} \cap \underline{B}}(a, b) \}$

$\mu_{\underline{R} \cap \underline{S}}(a, b) = \min(\mu_{\underline{R}}(a, b), \mu_{\underline{S}}(a, b))$

$\mu_{\underline{R} \cap \underline{S}}(x_1, y_1) = \min(\mu_{\underline{R}}(x_1, y_1), \mu_{\underline{S}}(x_1, y_1))$

$= \min(0.8, 0.4) = 0.4$

$\mu_{\underline{R} \cap \underline{S}}(x_1, y_2) = \min(\mu_{\underline{R}}(x_1, y_2), \mu_{\underline{S}}(x_1, y_2))$

$= \max(0.1, 0.0) = 0.0$

$\mu_{\underline{R} \cap \underline{S}}(x_1, y_3) = \min(\mu_{\underline{R}}(x_1, y_3), \mu_{\underline{S}}(x_1, y_3))$

$= \max(0.1, 0.9) = 0.1$

$\mu_{\underline{R} \cap \underline{S}}(x_1, y_4) = \min(\mu_{\underline{R}}(x_1, y_4), \mu_{\underline{S}}(x_1, y_4))$

$\max(0.7, 0.6) = 0.6$

.

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$\mu_{\underline{R} \cap \underline{S}}(x_3, y_4) = \min(\mu_{\underline{R}}(x_3, y_4), \mu_{\underline{S}}(x_3, y_4))$

$= \max(0.8, 0.5) = 0.5$

$$\bar{R} \cap \bar{S} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.4 & 0.0 & 0.1 & 0.6 \\ 0.0 & 0.4 & 0.0 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.5 \end{bmatrix} \end{array}$$

Intersection of relation

Complement:

$\underline{R}^c = \{ (a, b), \mu_{\underline{R}}{}^c(a, b) \}$

$\mu_{\underline{R}}{}^{c}(a, b) = 1 - \mu_{\underline{R}}(a, b)$

$\mu_{\underline{R}}{}^{c}(x_1, y_1) = 1 - \mu_{\underline{R}}(x_1, y_1) = 1 - 0.8 = 0.2$

$\mu_{\underline{R}}{}^{c}(x_1, y_2) = 1 - \mu_{\underline{R}}(x_1, y_2) = 1 - 0.1 = 0.9$

$\mu_{\underline{R}}{}^{c}(x_1, y_3) = 1 - \mu_{\underline{R}}(x_1, y_3) = 1 - 0.1 = 0.9$

.

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$\mu_{\underline{R}}{}^{c}(x_3, y_4) = 1 - \mu_{\underline{R}}(x_3, y_4) = 1 - 0.8 = 0.2$

The complement of relation $\underline{R}$ would be,

$$\bar{R}^c = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.2 & 0.9 & 0.9 & 0.3 \\ 1.0 & 0.2 & 1.0 & 1.0 \\ 0.1 & 0.0 & 0.3 & 0.2 \end{bmatrix} \end{array}$$

Complement of fuzzy relation

Projection:

The projection of $\underline{R}$ on X :

$\prod_X(x) = \sup(\ \underline{R}(x, y) \mid y \in Y)$

The projection of $\underline{R}$ on Y :

$\prod_Y(y) = \sup(\underline{R}(x, y) \mid x \in X)$

sup: Supremum of the set

$$\bar{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.1 & 0.7 \\ 0.0 & 0.8 & 0.0 & 0.0 \\ 0.9 & 1.0 & 0.7 & 0.8 \end{bmatrix} \end{array}$$

The projection of $\underline{R}$ on X :

$\prod_X(x_1) = 0.8$

$\prod_X(x_2) = 0.8$

$\prod_X(x_3) = 1.0$

The projection of $\underline{R}$ on Y :

$\prod_Y(y_1) = 0.9$

$\prod_Y(y_2) = 1.0$

$\prod_Y(y_3) = 0.7$

$\prod_Y(y_4) = 0.8$

Fuzzy composition:

The **fuzzy composition** can be defined just as it is for crisp (binary) relations. Suppose $\underline{R}$ is a fuzzy relation on X × Y, $\underline{S}$ is a fuzzy relation on Y × Z, and $\underline{T}$ is a fuzzy relation on X × Z; then,

Fuzzy Max–Min composition is defined as:

$$\underline{T} = \underline{R} \circ \underline{S} = \mu_{\bar{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\bar{R}}(x,y) \wedge \mu_{\bar{S}}(y,z))$$

$$= max_{y \in Y}\{min(\mu_{\bar{R}}(x,y), \mu_{\bar{S}}(y,z))\}$$

Fuzzy Max–Product composition is defined as:

$$\underline{T} = \underline{R} \circ \underline{S} = \mu_{\bar{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\bar{R}}(x,y) . \mu_{\bar{S}}(y,z))$$

$$= max_{y \in Y}\{(\mu_{\bar{R}}(x,y) \times \mu_{\bar{S}}(y,z))\}$$

Composition of fuzzy relation is defined over two fuzzy_relations.

Fuzzy composition

Fuzzy composition can be defined just as it is for crisp (binary) relations. Suppose $\underline{R}$ is a fuzzy relation on X × Y, $\underline{S}$ is a fuzzy relation on Y × Z, and $\underline{T}$ is a fuzzy relation on X × Z; then,

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$$= max_{y \in Y}\{min(\mu_{\bar{R}}(x,y), \mu_{\bar{S}}(y,z))\}$$

Fuzzy Max–Product composition is defined as:

$$\underline{T} = \underline{R} \circ \underline{S} = \mu_{\bar{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\bar{R}}(x,y) . \mu_{\bar{S}}(y,z))$$

$$= max_{y \in Y}\{(\mu_{\bar{R}}(x,y) \times \mu_{\bar{S}}(y,z))\}$$

Let us try to understand it with the help of an example. The fuzzy max-min approach is identical to that of the crisp_max-min composition.

Example:

$X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$. Consider the following fuzzy relations:

$$\bar{R} = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.6 \\ 0.8 & 0.3 \end{bmatrix} \end{array}$$

Relation $\underline{R}$

$$\bar{S} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

Relation $\underline{S}$

Find the resulting relation, $\underline{T}$ which relates elements of universe X to elements of universe Z, i.e., defined on Cartesian space X × Z

- Using **Max-Min** composition and
- Using **Max-Product** composition

Solution:

So ultimately, we have to find the elements of the matrix,

$$\bar{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \end{array}$$

Composition of relation $\underline{R}$ *and* $\underline{S}$

Max-Min Composition:

Max-min composition is defined as,

$$\underline{T} = \underline{R} \circ \underline{S} = \mu_{\bar{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\bar{R}}(x,y) \wedge \mu_{\bar{S}}(y,z))$$

$$= max_{y \in Y}\{min(\mu_{\bar{R}}(x,y), \mu_{\bar{S}}(y,z))\}$$

From the given relations $\underline{R}$ and $\underline{S}$,

$\mu_{T}(x_1, z_1) = \max(\min(\mu_R(x_1, y_1), \mu_S(y_1, z_1)), \min(\mu_R(x_1, y_2), \mu_S(y_2, z_1)))$

$= \max(\min(0.7, 0.8), \min(0.6, 0.1)) = \max(0.7, 0.1) = 0.7$

$\mu_{T}(x_1, z_2) = \max(\min(\mu_R(x_1, y_1), \mu_S(y_1, z_2)), \min(\mu_R(x_1, y_2), \mu_S(y_2, z_2)))$

$= \max(\min(0.7, 0.5), \min(0.6, 0.6)) = \max(0.5, 0.6) = 0.6$

$\mu_{T}(x_1, z_3) = \max(\min(\mu_R(x_1, y_1), \mu_S(y_1, z_3)), \min(\mu_R(x_1, y_2), \mu_S(y_2, z_3)))$

$= \max(\min(0.7, 0.4), \min(0.6, 0.7)) = \max(0.4, 0.6) = 0.6$

$\mu_{T}(x_2, z_1) = \max(\min(\mu_R(x_2, y_1), \mu_S(y_1, z_1)), \min(\mu_R(x_2, y_2), \mu_S(y_2, z_1)))$

$= \max(\min(0.8, 0.8), \min(0.3, 0.1)) = \max(0.8, 0.1) = 0.8$

$\mu_{T}(x_2, z_2) = \max(\min(\mu_R(x_2, y_1), \mu_S(y_1, z_2)), \min(\mu_R(x_2, y_2), \mu_S(y_2, z_2)))$

$= \max(\min(0.8, 0.5), \min(0.3, 0.6)) = \max(0.5, 0.3) = 0.5$

$\mu_{T}(x_2, z_3) = \max(\min(\mu_R(x_2, y_1), \mu_S(y_1, z_3)), \min(\mu_R(x_2, y_2), \mu_S(y_2, z_3)))$

$= \max(\min(0.8, 0.4), \min(0.3, 0.7)) = \max(0.4, 0.3) = 0.4$

$$\bar{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.7 & 0.6 & 0.6 \\ 0.8 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Max-Min composition of

Max-Product Composition:

$$\underline{T} = \underline{R} \circ \underline{S} = \mu_{\bar{T}}(x,z) = \bigvee_{y \in Y} (\mu_{\bar{R}}(x,y) . \mu_{\bar{S}}(y,z))$$

$$= max_{y \in Y}\{(\mu_{\bar{R}}(x,y) \times \mu_{\bar{S}}(y,z))\}$$

$\mu_T(x_1, z_1) = \max ((\mu_R(x_1, y_1) \times \mu_S(y_1, z_1)), (\mu_R(x_1, y_2) \times \mu_S(y_2, z_1)))$

$= \max((0.7 \times 0.8), (0.6 \times 0.1)) = \max(0.56, 0.06) = 0.56$

$\mu_T(x_1, z_2) = \max ((\mu_R(x_1, y_1) \times \mu_S(y_1, z_2)), (\mu_R(x_1, y_2) \times \mu_S(y_2, z_2)))$

$= \max((0.7 \times 0.5), (0.6 \times 0.6)) = \max(0.35, 0.36) = 0.36$

$\mu_T(x_1, z_3) = \max ((\mu_R(x_1, y_1) \times \mu_S(y_1, z_3)), (\mu_R(x_1, y_2) \times \mu_S(y_2, z_3)))$

$= \max((0.7 \times 0.4), (0.6 \times 0.7)) = \max(0.28, 0.42) = 0.42$

$\mu_T(x_2, z_1) = \max ((\mu_R(x_2, y_1) \times \mu_S(y_1, z_1)), (\mu_R(x_2, y_2) \times \mu_S(y_2, z_1)))$

$= \max((0.8 \times 0.8), \min(0.3 \times 0.1)) = \max(0.64, 0.03) = 0.64$

$\mu_T(x_2, z_2) = \max ((\mu_R(x_2, y_1) \times \mu_S(y_1, z_2)), (\mu_R(x_2, y_2) \times \mu_S(y_2, z_2)))$

$= \max((0.8 \times 0.5), (0.3 \times 0.6)) = \max(0.4, 0.18) = 0.40$

$\mu_T(x_2, z_3) = \max ((\mu_R(x_2, y_1) \times \mu_S(y_1, z_3)), (\mu_R(x_2, y_2) \times \mu_S(y_2, z_3)))$

$= \max((0.8 \times 0.4), (0.3 \times 0.7)) = \max(0.32, 0.21) = 0.32$

$$\bar{T} = \begin{array}{c c} & \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0.56 & 0.36 & 0.42 \\ 0.64 & 0.40 & 0.32 \end{bmatrix} \end{array}$$

Max-Product composition

Fuzzy membership function is used to convert the crisp input provided to the fuzzy inference system. Fuzzy logic itself is not fuzzy, rather it deals with the fuzziness in the data. And this fuzziness in the data is best described by the fuzzy membership function.

A fuzzy inference system is the core part of any fuzzy logic system. Fuzzification is the first step in Fuzzy Inference System.

Formally, a membership function for a fuzzy set A on the universe of discourse X is defined as $\mu_A: X \rightarrow [0, 1]$, where each element of X is mapped to a value between 0 and 1. This value, called **membership value** or **degree of membership**, quantifies the grade of membership of the element in X to the fuzzy set A. Here, X is the universal set and A is the fuzzy set derived from X.

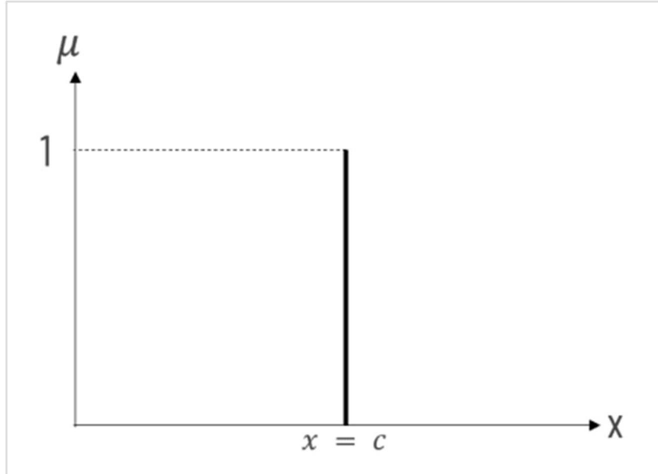
The fuzzy membership function is the graphical way of visualizing the degree of membership of any value in a given fuzzy set. In the graph, X-axis represents the universe of discourse and the Y-axis represents the degree of membership in the range [0, 1]

In the following discussion, we will see various fuzzy membership functions. These functions are mathematically very simple. Fuzzy logic is meant to deal with the fuzziness, so the use of complex membership functions would not add much precision to the final output.

Fuzzy Membership Function:

Singleton membership function:

The Singleton membership function assigns membership value 1 to a particular value of x and assigns value 0 to the rest of all. It is represented by the impulse function as shown.



Singleton membership function

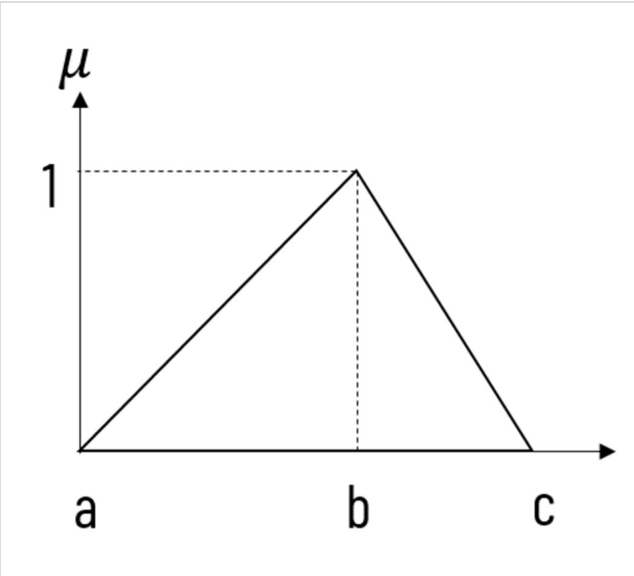
Mathematically it is formulated as,

$$\mu(x) = \begin{cases} 1, & if\ x = c \\ 0, & otherwise \end{cases}$$

Triangular membership function:

This is one of the most widely accepted and used membership functions (MF) in fuzzy controller design. The triangle which fuzzifies the input can be defined by three parameters a, b and c, where c defines the base and b defines the height of the triangle.

Trivial case:



Here, in the diagram, X-axis represents the input from the process (such as air conditioner, washing machine, etc.) and the Y axis represents the corresponding fuzzy value.

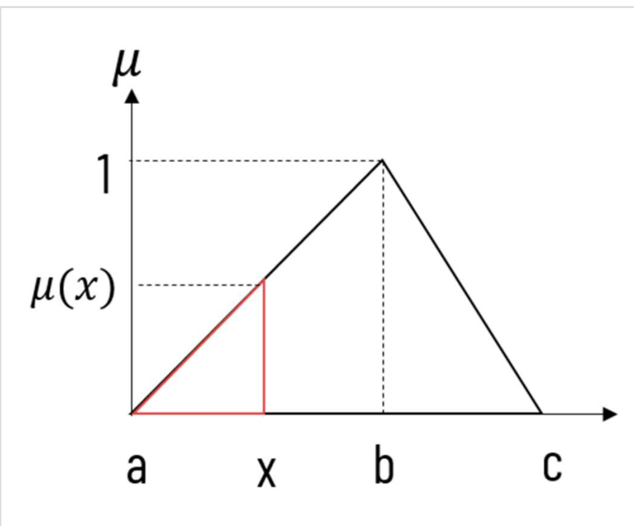
If input x = b, then it is having full membership in the given set. So,

$$\mu(x) = 1, \text{ if } x = b$$

And if the input is less than a or greater than b, then it does belong to the fuzzy set at all, and its membership value will be 0

$$\mu(x) = 0, \quad x < a \text{ or } x > c$$

x is between a and b:

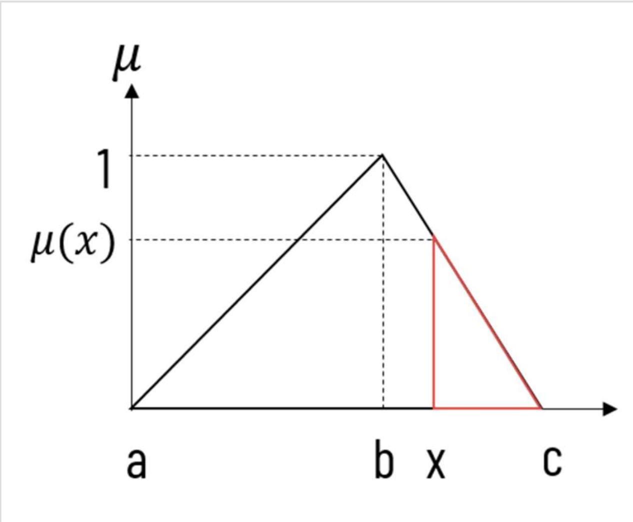


If x is between a and b, as shown in the figure, its membership value varies from 0 to 1. If it is near a, its membership value is close to 0, and if x is near b, its membership value gets close to 1.

We can compute the fuzzy value of x using a similar triangle rule,

$$\mu(x) = (x - a) / (b - a), \quad a \le x \le b$$

x is between b and c:



If x is between b and c, as shown in the figure, its membership value varies from 0 to 1. If it is near b, its membership value is close to 1, and if x is near c, its membership value gets close to 0.

We can compute the fuzzy value of x using a similar triangle rule,

$\mu(x) = (c - x) / (c - b)$, $b \leq x \leq c$

Combine all together:

We can combine all the above scenarios in single equation as,

$$\mu_{triangle}(x; a, b, c) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & c \leq x \end{cases}$$

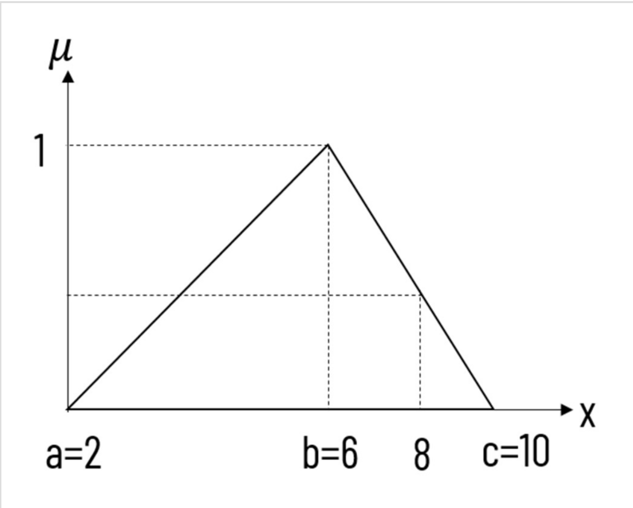
$$= \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Triangular membership function

Example: Triangular membership function

Determine μ, corresponding to x = 8.0

For the given values of a, b and c, we have to compute the fuzzy value corresponding to x = 8. Using the equation of the triangular membership function,



$$\mu_{triangle}(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{6-2}, \frac{10-x}{10-6}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{4}, \frac{10-x}{4}\right), 0\right)$$

We put $x = 8.0$

$$= \max\left(\min\left(\frac{3}{2}, \frac{1}{2}\right), 0\right) = \frac{1}{2} = 0.5$$

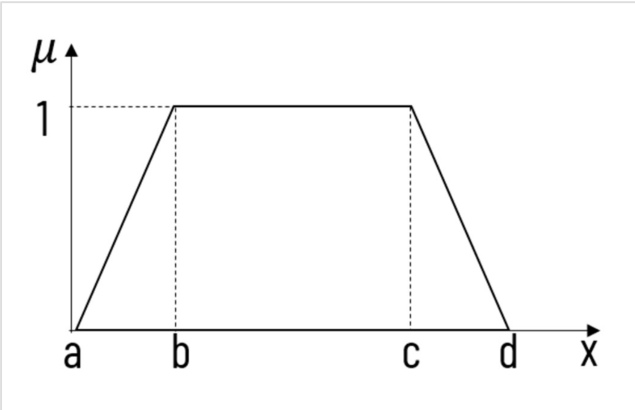
Thus, x = 8 will be mapped to a fuzzy value of 0.5 using the given triangle fuzzy membership function

Trapezoidal membership function:

The trapezoidal membership function is defined by four parameters: a, b, c and d. Span b to c represents the highest membership value that element can take. And if x is between (a, b) or (c, d), then it will have a membership value between 0 and 1.

We can apply the triangle MF if elements are in between a to b or c to d.

It is quite obvious to combine all together as,



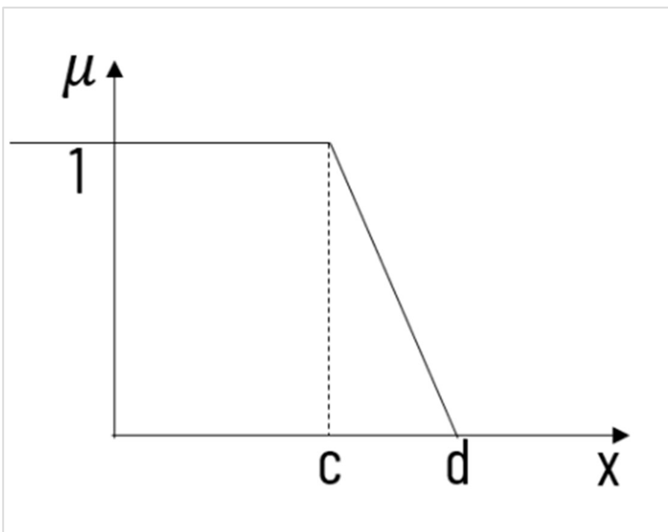
$$\mu_{trapezoidal}(x; a, b, c, d) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c \le x \le d \\ 0, & d \le x \end{cases}$$

$$= \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

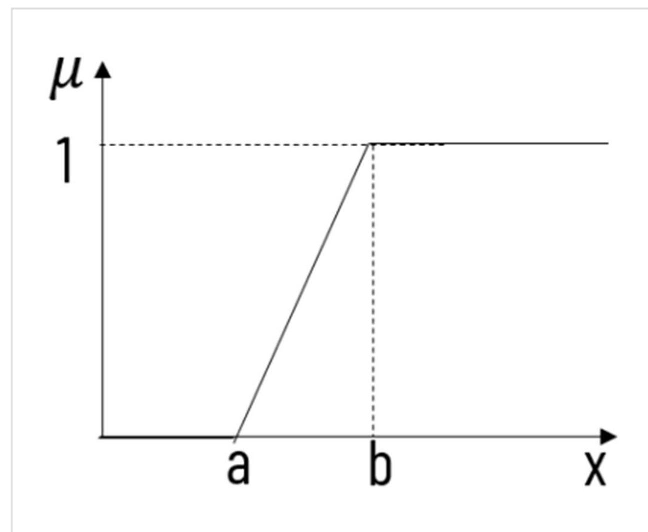
There are two special forms of trapezoidal function based on the openness of function. They are known as R-function (Open right) and L-function (Left open). Shape and parameters of both functions are depicted here:

R-function: it has a = b = -∞

L-function: It has c = d = +∞



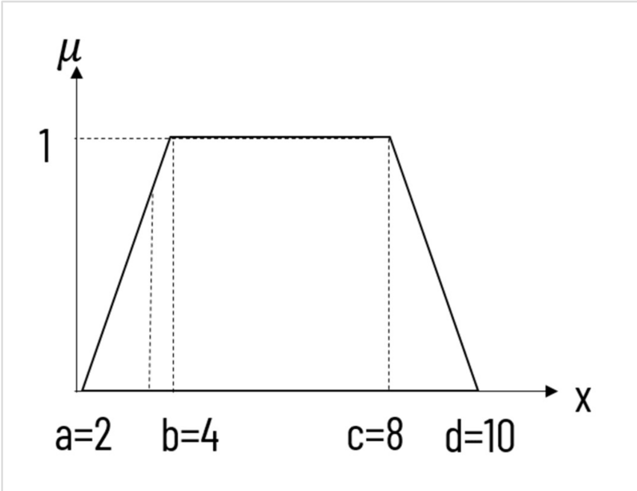
R-function



L-function

Example: Trapezoidal membership function

Determine μ, corresponding to x = 3.5



$$\mu_{trapezoidal}(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right)$$

$$= \max\left(\min\left(\frac{x-2}{4-2},1,\frac{10-x}{10-8}\right),0\right)$$

$$= \max\left(\min\left(\frac{x-2}{2},1,\frac{10-x}{2}\right),0\right)$$

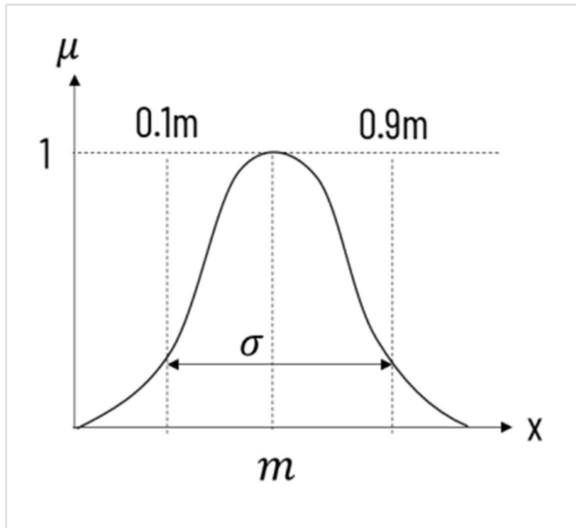
We put $x = 3.5$

$$= \max\left(\min\left(\frac{1.5}{2},1,\frac{6.3}{2}\right),0\right)$$

$$= \max(0.75, 0) = 0.75$$

Gaussian membership function:

A Gaussian MF is specified by two parameters {m, σ} and can be defined as follows.



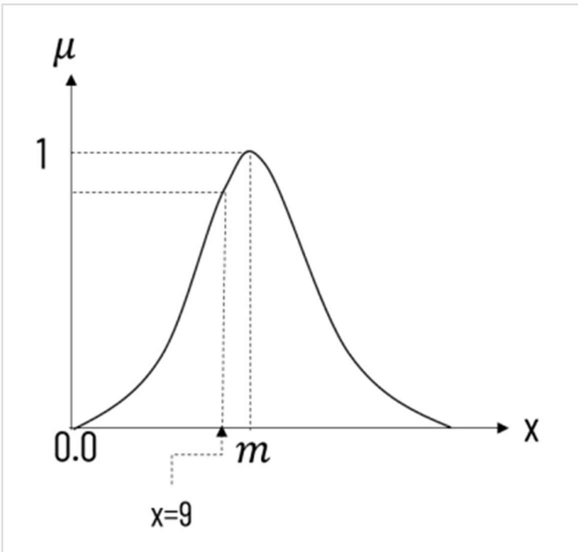
$$\mu_{gaussian}(x;m,\sigma) = e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

Gaussian membership function

In this function, m represents the mean / center of the gaussian curve and σ represents the spread of the curve. This is a more natural way of representing the data distribution, but due to mathematical complexity, it is not much used for fuzzification.

Example: Gaussian membership function

Determine μ corresponding to x = 9, m = 10 and σ = 3.0



$$\mu_{gaussian}(x; m, \sigma) = e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

$Take\ m = 10.0\ and\ \sigma = 3.0$

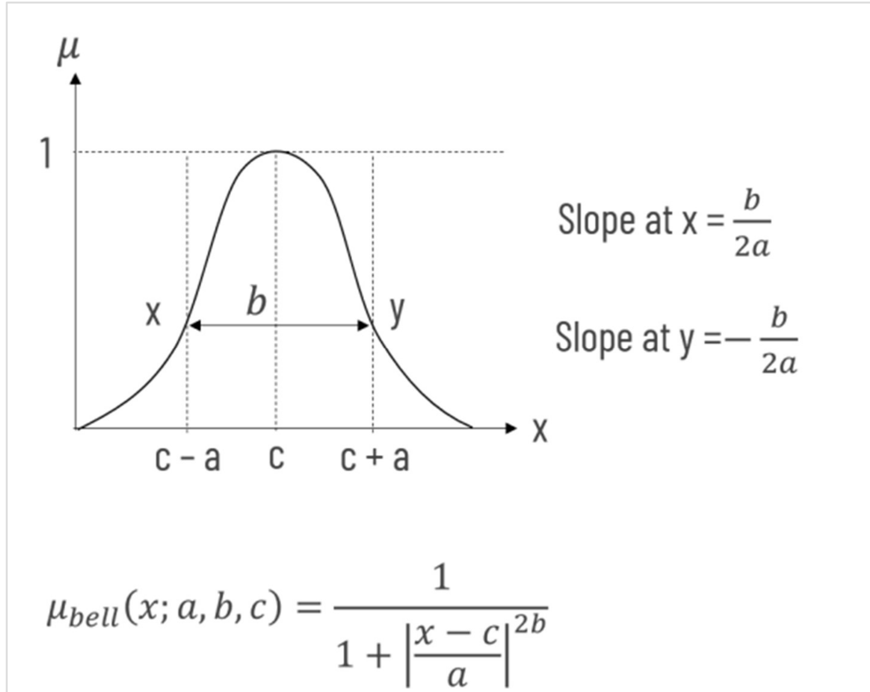
$$\mu_{gaussian} = e^{-\frac{1}{2}\left(\frac{x-10}{3}\right)^2}$$

$Put\ x = 9$

$$\mu_{gaussian} = e^{-\frac{1}{2}\left(\frac{9-10}{3}\right)^2} = 0.9459$$

Generalized bell-shaped function:

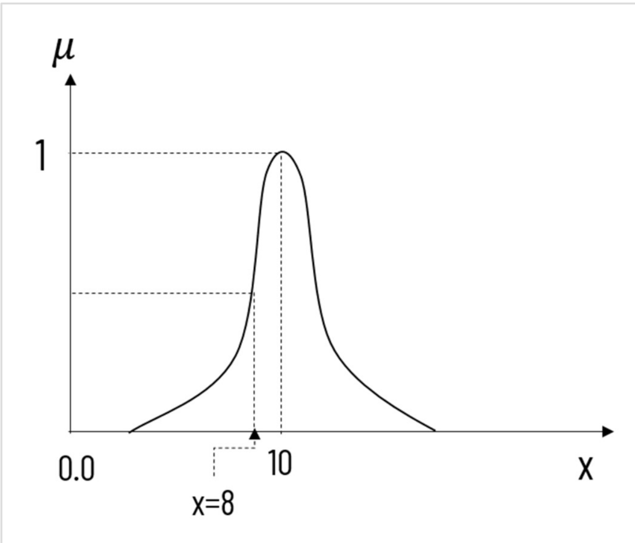
It is also called **Cauchy MF.** A generalized bell MF is specified by three parameters {a, b, c} and can be defined as follows.



Generalized bell shape membership function

Example: Generalized bell shape membership function

Determine μ corresponding to x = 8



Using the above-discussed equation of the generalized bell-shape membership function,

$$\mu_{bell}(x; a, b, c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

$$Take\ c\ = 10, a = 2, b = 3$$

$$\mu_{bell}\ = \frac{1}{1+\left|\frac{x-10}{2}\right|^{6}}$$

$$Put\ x = 8$$

$$\mu_{bell}\ = \frac{1}{1+\left|\frac{8-10}{2}\right|^{6}} = 0.5$$

it is called generalized MF, because by changing the parameters a, b and c, we can produce a family of different membership functions.

The function $\mu(X) = 1 / (1 + x^2)$ can be modelled by setting a = b = 1 and c = 0. Similarly, we can produce other shapes/functions by setting appropriate a, b and c

Sigmoid Membership function:

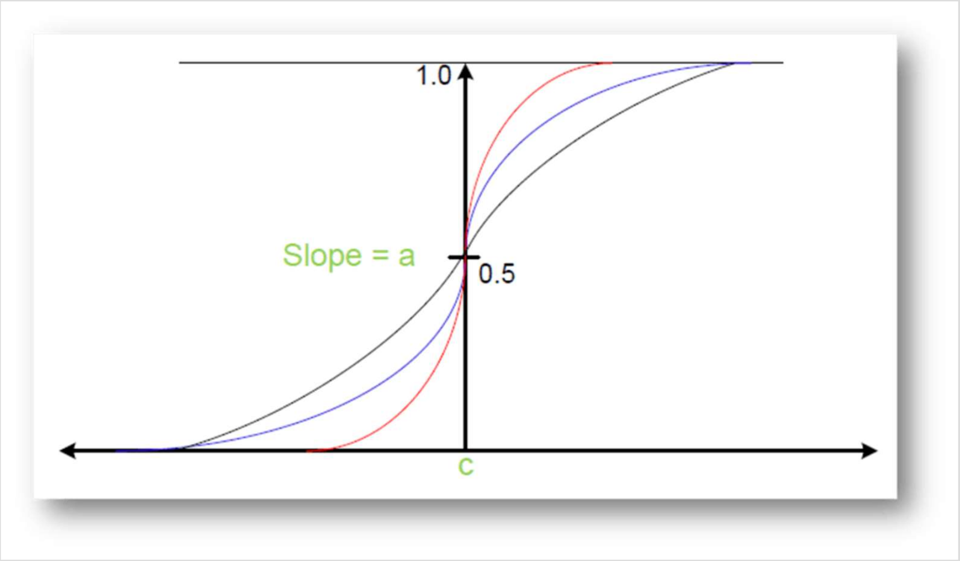
Sigmoid functions are widely used in classification tasks in machine learning. Specifically, it is used in logistic regression and neural networks, where it suppresses the input and maps it between 0 and 1.

It is controlled by parameters a and c. Where a controls the slope at the crossover point $x = c$

Mathematically, it is defined as

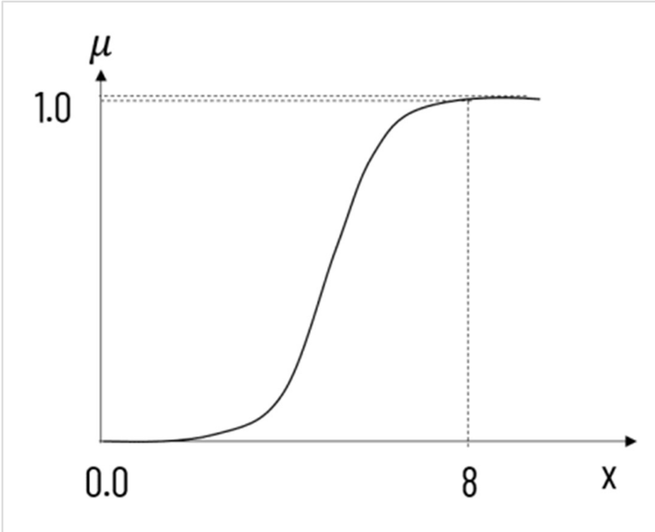
$$\mu_{sigmoid}(x; a, c) = \frac{1}{1 + e^{-a(x - c)}}$$

Graphically, we can represent it as,



Example: Sigmoid function

Determine μ corresponding to x = 8



By using the equation of the sigmoid membership function

$$\mu_{sigmoid}(x; a, c) = \frac{1}{1+e^{-a(x-b)}}$$

$Take\ b = 6, a = 2$

$$\mu_{sigmoid} = \frac{1}{1+e^{-2(x-6)}}$$

$Put\ x = 8$

$$\mu_{sigmoid} = \frac{1}{1+e^{-2(8-6)}} = \frac{1}{1+e^{-4}} = 0.98$$

Fuzzy Inference System (FIS) is a key component of any fuzzy controller. FIS consists of various functional blocks.

The fundamental task of any FIS is to apply the if-then rules on fuzzy input and produce the corresponding fuzzy output. The whole process is based on the computer paradigm including fuzzy set theory, if-then rules and the fuzzy reasoning process.

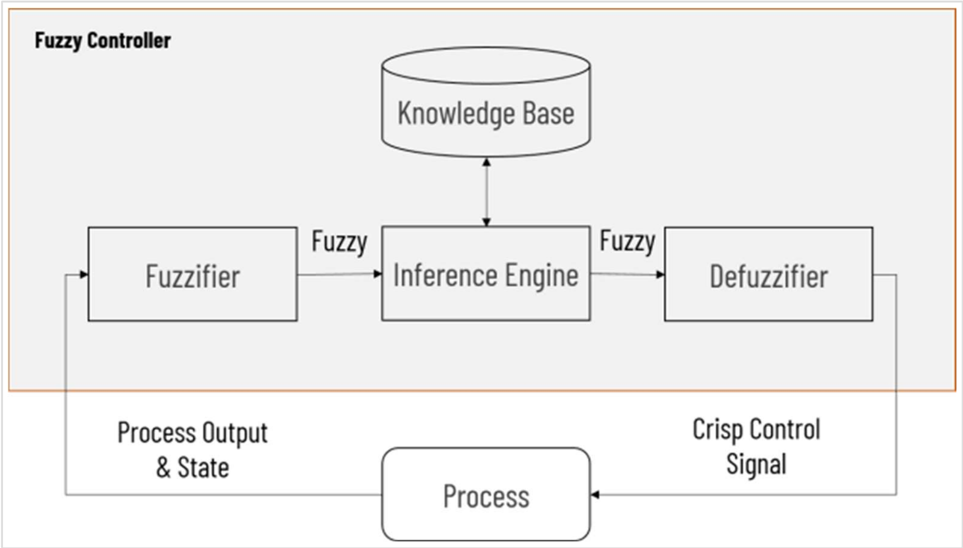
Fuzzy inference (reasoning) is the actual process of ***mapping from a given input to an output using fuzzy logic***.

FIS has been successfully applied in fields such as automatic control, data classification, decision analysis, expert systems and many more

And because of its multi-disciplinary nature, the fuzzy inference system is also known as

- Fuzzy-rule-based system
- Fuzzy expert system
- Fuzzy model
- Fuzzy associative memory
- Fuzzy logic controller
- Fuzzy system

The functional block diagram of the fuzzy Inference system is depicted in the following diagram:



Fuzzy inference system

As shown in Figure, a fuzzy controller operates by repeating a cycle of the following four steps :

1. Compare the input variables with the membership functions on the **antecedent** part to obtain the membership values of each linguistic label. (this step is often called *fuzzification.)*
2. Combine (*usually multiplication or* min) the membership values on the **premise** part to get the *firing strength* (*decree of fulfilment) of each rule.*
3. Generate the **qualified consequents** (either fuzzy or crisp) or each rule depending on the firing strength.
4. **Aggregate the qualified consequents** to produce a crisp output. (This step is called *defuzzification*.)

Components of FIS:

Knowledge Base = Data Base + Rule Base

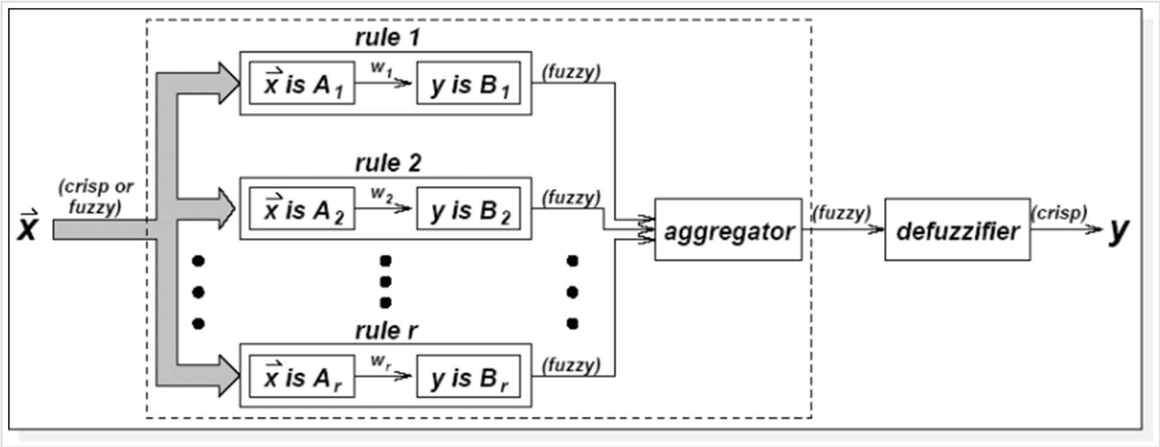
- A database which defines the membership functions of the fuzzy sets used in the fuzzy rules
- A rule base containing a number of fuzzy IF-THEN rules

Fuzzifier:

- Converts the **crisp input** to a **linguistic variable** using the membership functions stored in the fuzzy knowledge base.

Inference Engine:

- Using **If-Then type fuzzy rules** converts the fuzzy input to the **fuzzy output**.



Fuzzy inference process

Defuzzifier:

- Converts the **fuzzy output** of the inference engine to **crisp value** using membership functions analogous to the ones used by the fuzzifier.
- Some commonly used defuzzifying methods:
  - Maxima Methods
  - Weighted Average Method
  - Center of Gravity (CoG) Method
  - Center of Sums (CoS) Method
  - Centroid of area (COA) Method

Fuzzy Inference Method:

The most important two types of fuzzy inference methods:

Linguistic Fuzzy: Mamdani fuzzy_inference (Mamdani and Assilian (1975))

- If I_1 is A AND I_2 is B then O is C
- Mamdani's approach follows linguistic fuzzy modelling and is characterized by its **high interpretability** and **low accuracy**.

Precise Fuzzy Modelling:

- Sugeno or Takagi–Sugeno–Kang or TS fuzzy inference (Sugeno (1985))
- If I_1 is A AND I_2 is B then O is $f(I_1, I_2) = a_1 I_1 + b_1 I_2 + c_1$
- On the other hand, Takagi and Sugeno's approach follows precise fuzzy modelling and obtains **high accuracy** but at the cost of **low interpretability**.

The main difference between the two methods lies in the **consequent of fuzzy rules**.

Fuzzy Rule-Based System:

IF premise (**antecedent**), THEN conclusion (**consequent**)

Canonical forms of Rule-Based System:

- **Rule 1**: If condition C^1 then restriction R^1
- **Rule 2**: If condition C^2 then restriction R^2
- .
- .
- **Rule n**: If condition C^n then restriction R^n

Aggregation of Fuzzy Rules

Conjunctive System of Rules:

- In this rule system, we must satisfy **all the rule**
- The rules are connected by '**and**' connectives
- In this case, the aggregated output, y, is found by the **fuzzy intersection** of all the individual rule consequents, y^i where i=1, 2, 3, …, r
- $y = y^1$ and y^2 and y^3 … y^r
- $y = y^1 \cap y^2 \cap y^3 \cap \ldots \cap y^r$
- $\mu_y(y) = \min[\mu_{y1}(y), \mu_{y2}(y), \ldots, \mu_{yL}(y)]$

Disjunctive System of Rules:

- In this rule system, we should satisfy **at least one rule**
- The rules are connected by the '**or**' connectives
- In this case, the aggregated output, y, is found by the **fuzzy union** of all the individual rule consequents, y^i where i=1, 2, 3, …, r
- $y = y^1$ or y^2 or y^3 … y^r
- $y = y^1 \cup y^2 \cup y^3 \cup \ldots \cup y^r$
- $\mu_y(y) = \max[\mu_{y1}(y), \mu_{y2}(y), \ldots, \mu_{yL}(y)]$

What is Fuzzy Inference System (FIS)? A fuzzy inference system is a key component of any fuzzy logic system. It uses fuzzy set theory, IF-THEN rules and fuzzy reasoning process to find the output corresponding to crisp inputs. Predicates in IF-THEN rules are connected using *and* or *or* logical connectives.

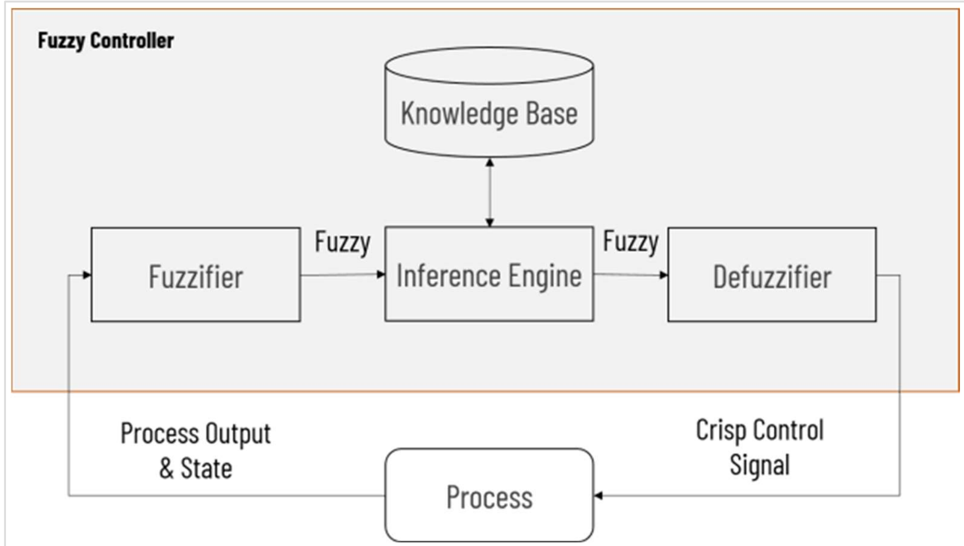
Characteristics of FIS:

- Read crisp value from the process
- Maps the crisp value into fuzzy value using the fuzzy membership function
- Apply IF-THEN rules from the fuzzy rule base and compute fuzzy output
- Convert fuzzy output into crisp by applying some defuzzification methods.

How fuzzy inference system works?

Crisp input of any process (measuring temperature of air conditioner, measuring altitude, attitude, height, angle of direction for aeroplane etc.) is given to the fuzzifier, which applies fuzzy membership function and maps the actual readings into fuzzy value (i.e. the value between 0 to 1).

The inference engine applies fuzzy rules from the knowledge base and produces the fuzzy output, which is again between 0 and 1. This output can not be used directly in any process or system. It needs to be mapped into the original domain. Defuzzifier is the inverse process of fuzzification, it converts the fuzzy output into crisp output, which can be fed to the process. Crisp_sets are internally converted to fuzzy_sets.



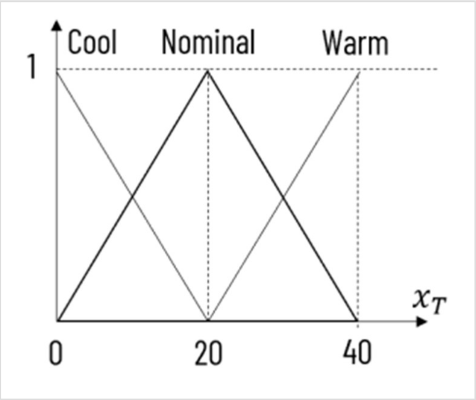
Fuzzy inference system

Let's take the scenario of the air conditioner. For simplicity, we will consider only one parameter which determines the temperature of the air conditioner to be set. Let us consider that the temperature of an air conditioner depends on the parameter called room temperature.

Let us divide the range of room temperature into cool, nominal and warm. Any temperature value may belong to multiple sets with different membership values. The value t = 20 has a membership value of 1 in the nominal set and 0 in the other two. If we move to the right, the membership value in the nominal set decreases and in the warm set, it increases. And as we move left from t = 20, the membership value in the nominal set decreases and in the cool set it increases

Any room temperature is converted to fuzzy value using such fuzzification methods (here we have used triangular membership function, more are discussed later in this article). And that input goes to the inference engine, which will determine what should be the temperature of the air conditioner. Assume that the internal representation of that temperature belongs to the increase and decrease set.

So the generated output might have some membership for both the increase and decrease sets. Assume that it's 0.7 for increased class and 0.15 for decreased class. Although output is computed, it can not be given directly to the controller of AC, because the controller does not understand what is meaning of the membership value of temperature 0.7 in the increase class and 0.15 in the decrease class.



Linguistic representation of temperature

A number of defuzzification methods are there which can be used to convert this fuzzy output into a crisp one. For a particular instance, the crisp output may be -4, which says reducing the temperature of the air conditioner by 4 degrees.

Approaches for Fuzzy Inference System

The following are the popular approaches to fuzzy inference systems. The antecedent part of all rules remains the same, they differ only in the consequent part.

- Mamdani fuzzy inference system
- Takagi-Sugeno fuzzy inference system
- Tsukamoto fuzzy inference system

Inference means to reach a particular conclusion based on some evidence associated with a logic

In 1975, Professor Ebrahim Mamdani of London University built one of the first fuzzy systems to control a steam engine and boiler combination.

He applied a set of fuzzy rules supplied by experienced human operators.

Mamdani fuzzy inference system operates as follow:

1. Determining a set of fuzzy rules
2. Fuzzifying the inputs using the input membership_functions
3. Combining the fuzzified inputs according to the fuzzy rules to establish a rule strength (Fuzzy Operations)
4. Finding the consequence of the rule by combining the rule strength and the output membership function (implication)
5. Combining the consequences to get an output distribution (aggregation)
6. Defuzzifying the output distribution (this step is only if a crisp output (class) is needed).

There are two types of Mamdani fuzzy inference systems:

1. Max-Min inference method
2. Max-Product inference method

Max-Min Inference Method:

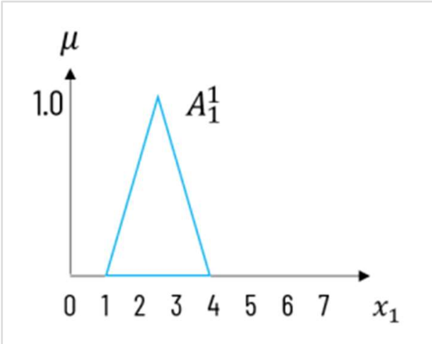
Consider the following rules:

Rule 1: IF x_1 is A_1^1 **and** x_2 is A_2^1 THEN y^1 is B^1

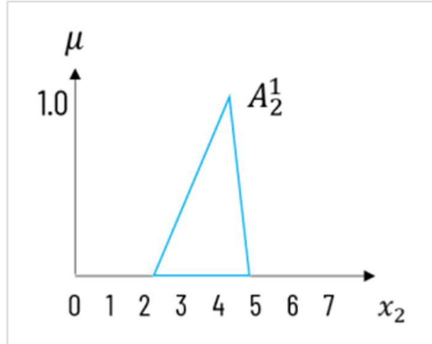
Rule 2: IF x_1 is A_1^2 **or** x_2 is A_2^2 THEN y^2 is B^2

let us compute the output for $x_1 = 2.5$ and $x_2 = 3$

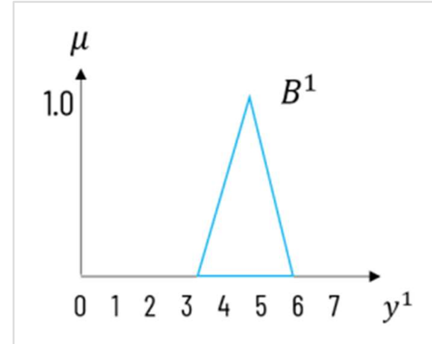
Membership functions for given rules are shown below:



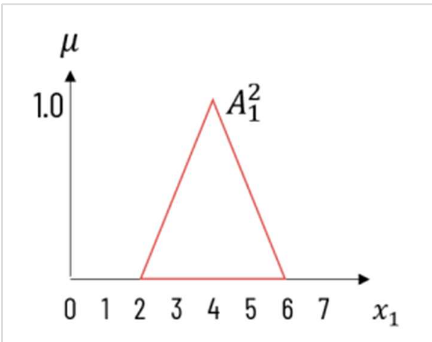
Input Fuzzy set A_1^1



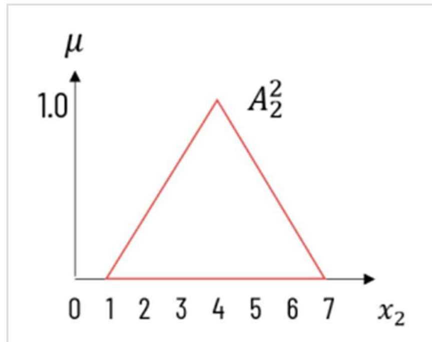
Input Fuzzy set A_2^1



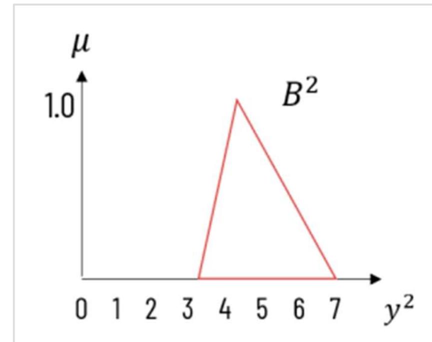
Output Fuzzy set B^1



Input Fuzzy set A_1^2



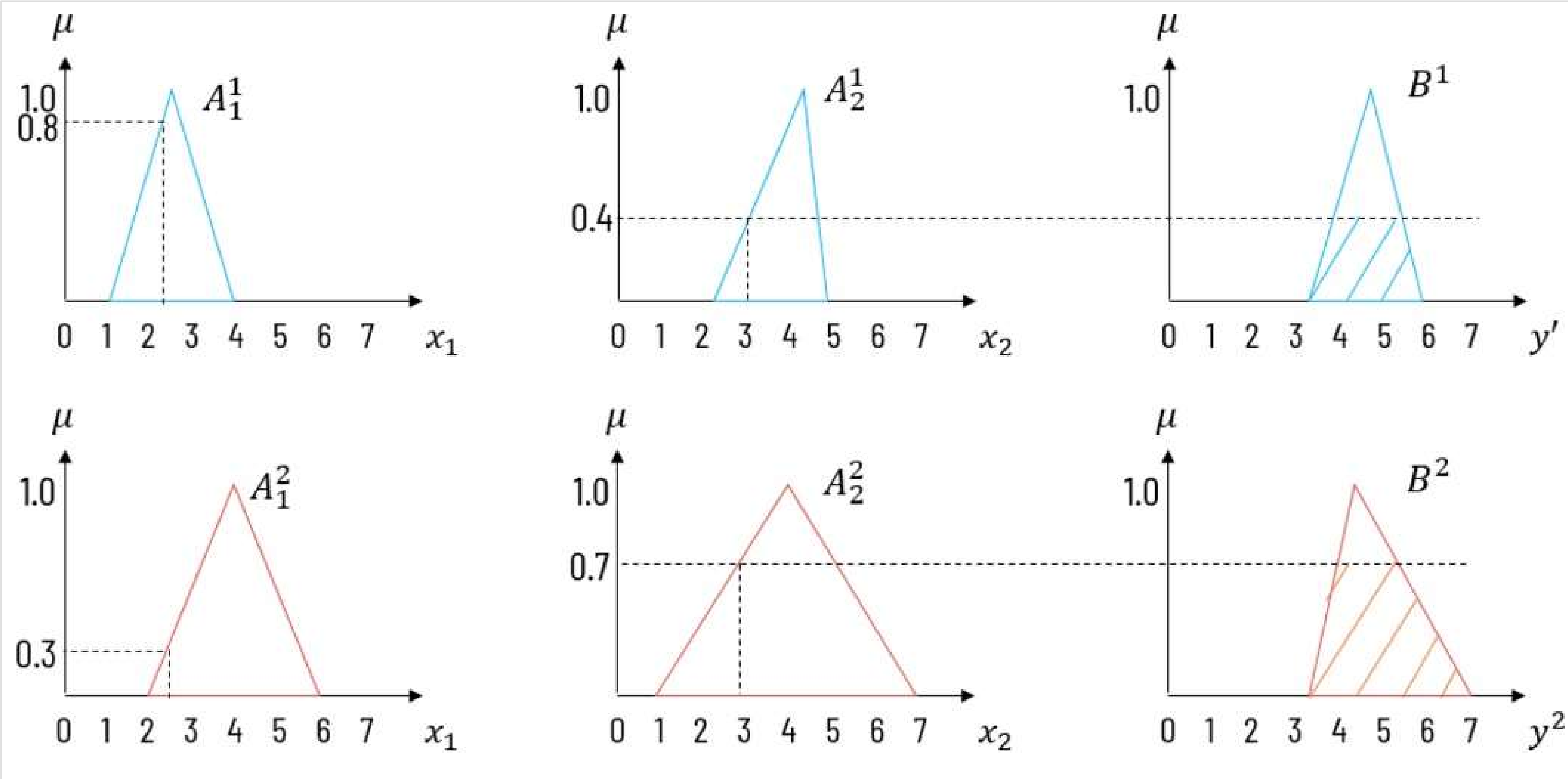
Input Fuzzy set A_2^2



Output Fuzzy set B^2

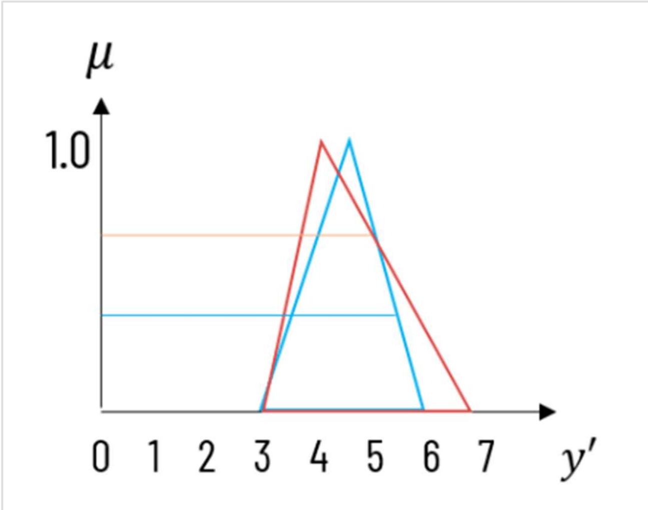
The fuzzy value corresponding to x1 and x2 in both input fuzzy sets is shown in the following figure. Its corresponding membership in the output function is also depicted. As it is a Max-Min inference method, the highest membership value from two input sets is assigned to the corresponding output set.

The calculation of fuzzified value for input crisp value is discussed in later half of this article.



For the first rule, the fuzzy membership value for x1 would be 0.8 and for x2 it would be 0.4. Propositions in the first IF-THEN rule are connected using **and** connective. So we have to take the intersection of fuzzy values, which returns the minimum of them. So the output y1 will have membership 0.4 in the fuzzy output set B^1.

For the second rule, the fuzzy membership value for x1 would be 0.3 and for x2 it would be 0.7. Propositions in the second IF-THEN rule are connected using **or** connective. So we have to take the union of fuzzy values, which returns a maximum of them. So the output y2 will have membership 0.7 in the fuzzy output set B^2.



Aggregated fuzzy output

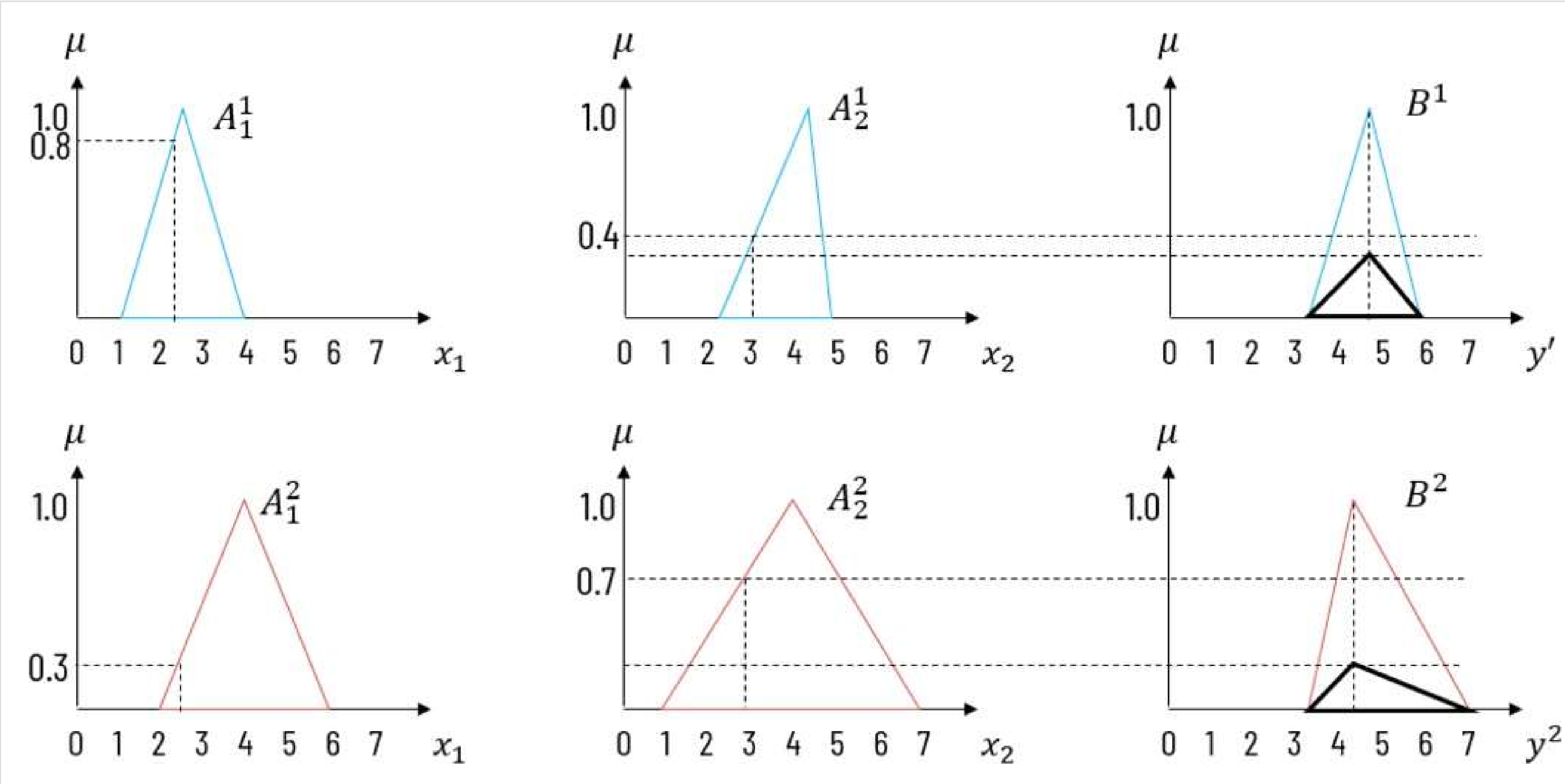
To compute the final crisp output, we shall aggregate the fuzzy output functions as shown in the figure.

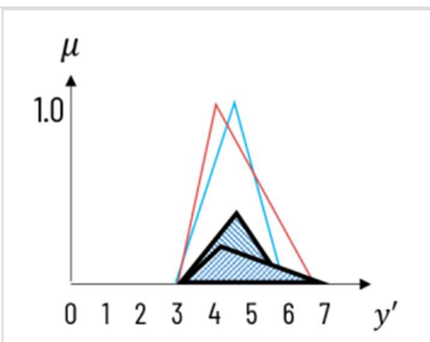
We can apply any defuzzification technique as discussed earlier to find the final crisp output for given case.

Max-Product Inference Method

We will consider the same inputs we used in the max-min inference system. In the max-product inference method, the output function is scaled down to the assigned fuzzy output value, rather than simply clipping the output function as discussed. The rest of the procedures are identical to the max-min inference method

The mapping of the assigned fuzzy value to the output function and the scaling of the output fuzzy function are described in the following figure.





To compute the final crisp output, we shall aggregate the fuzzy output functions as shown in the figure.

We can apply any defuzzification technique as discussed earlier to find the final crisp output for given case.

Aggregated fuzzy output