Digital Signals Processing Final Problem Set

December 2023

2 Questions

2.1 Complex Algebra 1

Identify the set of complex numbers that satisfies the condition $z^* = z^{-1}$

- $0i - 1;$
- $-|0 + i|$
- $-\frac{1}{n}|0 + ni|$, *where n is any real number*

2.2 Complex Algebra 2

Simplify the function

$$x[n] = \frac{e^{j\frac{\pi}{2}} + e^{-j\frac{\pi}{2}}}{2} - \frac{e^{j\theta n} - e^{-j\theta n}}{j}$$

$$\longrightarrow \frac{\left[\cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)\right] + \left[\cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)\right]}{2} - \frac{[\cos(\theta n) + j\sin(\theta n)] - [\cos(\theta n) - j\sin(\theta n)]}{j}$$

$$\longrightarrow \frac{[0 + 1j] + [0 - 1j]}{2} - \left[\frac{[\cos(\theta n) + j\sin(\theta n)] - [\cos(\theta n) - j\sin(\theta n)]}{j} \times \frac{-j}{-j}\right]$$

$$\longrightarrow \frac{[0 + 1j] + [0 - 1j]}{2} - \frac{2\sin(\theta n)}{1} = \frac{0}{2} - 2\sin(\theta n) = -2\sin(\theta n)$$

2.3 Periodic Signals

Compute for the minimum period P in samples of the signal $e^{j(M/N)2\pi n}$ for:

$$P = \frac{N}{gcd(M, N)}$$

Answers in red are my final answer/s.

(a) $e^{j2\pi n}$ M=1; N=1 $\longrightarrow P = \frac{1}{gcd(1,1)} = 1$

(b) $e^{j(1/3)2\pi n}$ M=1; N=3 $\longrightarrow P = \frac{3}{gcd(1,3)} = 3$

(c) $e^{j(56/9)2\pi n}$ M=56; N=9 $\longrightarrow P = \frac{9}{gcd(56,9)} = 9$

2.4 Signal Operations 1

Compute the moving average of the signal $y[n]$ from an input signal

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

. Whereas $y[n]$ is defined as

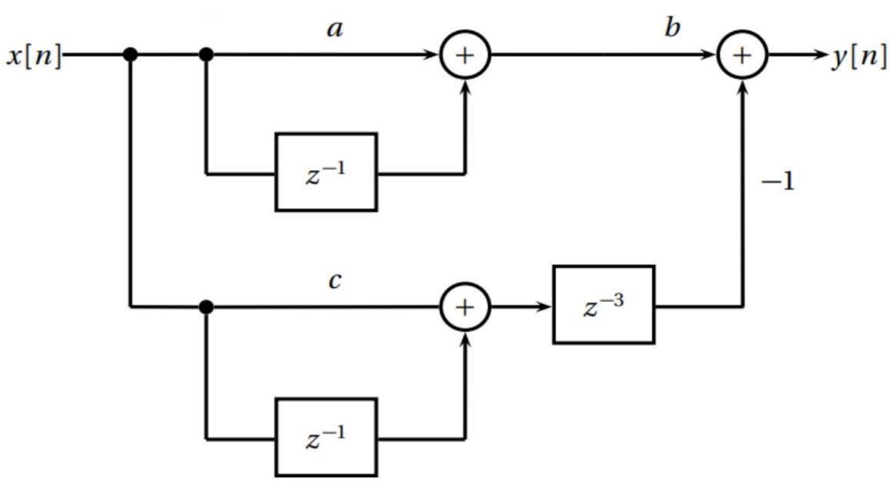
$$\frac{x[n] + x[n-1]}{2}$$

$$\longrightarrow \frac{\left[\delta[n] + 2\delta[n-1] + 3\delta[n-2]\right] + \left[\delta[n-1] + 2\delta[n-2] + 3\delta[n-3]\right]}{2}$$

$$\longrightarrow \frac{\delta n + 3\delta[n-1] + 5\delta[n-2] + 3\delta[n-3]}{2} \longrightarrow \frac{12\delta n - 22\delta}{2} \longrightarrow \frac{2\delta(6n-11)}{2} = \delta(6n-11)$$

2.5 Signal Operations 2

Consider the following block diagram



What is the equation representing the output response $y[n]$?

Answers in red are my final answer/s.

$\longrightarrow a = x[n] + x[n-1];\ c = [x[n-1] + x[n-3]](-1)$

$\longrightarrow [x[n] + x[n-1]] - [x[n-1] + x[n-3]] \quad \longrightarrow y[n] = x[n] - x[n-3]$

2.6 Signal Operations 3

If we represent finite-length signals as vectors in Euclidean space, many operations on signals can be encoded as a matrix-vector multiplication. Consider for example a circular shift in $\mathbb{C}^3$: a delay by one (i.e. a right shift) transforms the signal $\mathbf{x} = [x_0 \quad x_1 \quad x_2]^T$ into $\mathbf{x}' = [x_2 \quad x_0 \quad x_1]^T$ and it can be described by the matrix

$$D = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

so that $\mathbf{x}' = D\mathbf{x}$

Determine the matrix F that implements the one-step-different operator in $\mathbb{C}^3$ i.e. the operator that transforms a signal $\mathbf{x}$ into

$$\frac{1}{\sqrt{2}}[(x_0 + 0.5x_1 - x_2) \quad (x_1 - 2x_0 + x_2) \quad (x_2 - x_1)]^T$$

$$\longrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} (x_0 + 0.5x_1 - x_2) \\ (x_1 - 2x_0 + x_2) \\ (x_2 - x_1) \end{bmatrix} = F\begin{bmatrix} x_0 \\ x_1 \\ x_3 \end{bmatrix} \quad \longrightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} (x_0 + 0.5x_1 - x_2) \\ (x_1 - 2x_0 + x_2) \\ (x_2 - x_1) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\longrightarrow \text{ROW 1: } \frac{1}{\sqrt{2}}(x_0 + 0.5x_1 - x_2) = f_{11}x_0 + f_{12}x_1 + f_{13}x_2$$

$$\longrightarrow \text{ROW 2: } \frac{1}{\sqrt{2}}(x_1 - 2x_0 + x_2) = f_{21}x_0 + f_{22}x_1 + f_{23}x_2$$

$$\longrightarrow \text{ROW 3: } \frac{1}{\sqrt{2}}(x_2 - x_1) = f_{31}x_0 + f_{32}x_1 + f_{33}x_2$$

$$\longrightarrow F = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0.5 & -1 \\ -2 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0.70711 & 0.35355 & -1.41421 \\ -1.41421 & 0.70711 & 0.70711 \\ -0.70711 & -0.70711 & 0 \end{bmatrix}$$

2.7 Frequency Analysis

Given the three signals:

(a) $x_1[n] = e^{j\frac{\pi}{2}n}$

Answers in red are my final answer/s.

(b) $x_2[n] = e^{-j\frac{\pi}{2}n}$

(c) $x_3[n] = \frac{2}{\sqrt{\pi}} e^{j2\pi n}$

Prove that $-\frac{x_1[n]}{x_2[n]} + x_3[n] = \frac{2\sqrt{\pi}+\pi}{\pi}$. Identify the fundamental frequencies of signals 2.7.a, 2.7.b, and 2.7.c. Relate the frequencies you have determined to the equality you have just solved. You may use verbal or computational reasoning.

$$\longrightarrow -\frac{e^{j\frac{\pi}{2}n}}{e^{-j\frac{\pi}{2}n}} + \frac{2}{\sqrt{\pi}} e^{j2\pi n} \quad \longrightarrow -\left(e^{j\frac{\pi}{2}n} \times e^{j\frac{\pi}{2}n}\right) + \frac{2}{\sqrt{\pi}} e^{j2\pi n} \quad \longrightarrow -e^{2j\frac{\pi}{2}n} + \frac{2}{\sqrt{\pi}} e^{j2\pi n}$$

$\longrightarrow -e^{j\pi n} + \frac{2}{\sqrt{\pi}} e^{j2\pi n}$; using $e^{j\pi} + 1 = 0$ $\longrightarrow -(-1) + \frac{2}{\sqrt{\pi}} e^{j2\pi n}$;

obtaining minimum period of $e^{j2\pi n}$ $\longrightarrow 1 + \frac{2}{\sqrt{\pi}}(1) \longrightarrow \left[\frac{2+\sqrt{\pi}}{\sqrt{\pi}}\right] \frac{\sqrt{\pi}}{\sqrt{\pi}} \longrightarrow \frac{2\sqrt{\pi}+\pi}{\pi}$

Answers in red are my final answer/s.