Propositional Logic

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Overview of the session

- Definition of Logic
- Definition of Propositional Logic
  - Syntax
  - Semantics

What is a Logic?

Logic is defined by:

- **A set of sentences**
  - A sentence is constructed from a set of primitives according to syntax rules.
- **A set of interpretations**
  - An interpretation gives a semantic to primitives. It associates primitives with values.
- **The valuation (meaning) function V**
  - Assigns a value (typically the truth value) to a given sentence under some interpretation

$$V : \text{sentence} \times \text{interpretation} \rightarrow \{True, False\}$$

Propositional Logic

Propositional logic: a formal language for representing knowledge and for making logical inferences

- A **proposition** is a statement that is either true or false.
- A **compound proposition** can be created from other propositions using logical connectives
- The **truth of a compound proposition** is defined by truth values of elementary propositions and the meaning of connectives.
- The **truth table for a compound proposition**: table with entries (rows) for all possible combinations of truth values of elementary propositions.

Propositional Logic

- **Examples (cont.):**
  - How are you?
    - **a question is not a proposition**
  - $x + 5 = 3$
    - **since x is not specified, neither true nor false**
  - 2 is a prime number.
    - **(T)**
  - She is very talented.
    - **since she is not specified, neither true nor false**
  - There are other life forms on other planets in the universe.
    - **either T or F**

Propositional Logic - Syntax

- **Formally propositional logic P:**
  - is defined **by Syntax+interpretation+semantics of P**

Syntax:

- **Symbols (alphabet)** in P:
  - **Constants**: ***True, False***
  - **Propositional symbols**

    Examples:
    - *P*
    - *Pitt is located in the Oakland section of Pittsburgh.,*
    - *It rains outside,* etc.
  - **A set of connectives**:

$$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$$

Propositional Logic - Syntax

Sentences in the propositional logic:

- **Atomic sentences:**
  - **Constructed from constants and propositional symbols**
  - True, False are (atomic) sentences
  - P, Q or *Light in the room is on*, *It rains outside* are (atomic) sentences
- **Composite sentences:**
  - **Constructed from valid sentences via connectives**
  - If A, B are sentences then

    $\neg A \quad (A \wedge B) \quad (A \vee B) \quad (A \Rightarrow B) \quad (A \Leftrightarrow B)$

    or $(A \vee B) \wedge (A \vee \neg B)$

    are sentences

Compound Propositions

- Let p: 2 is a prime ….. **T**

  q: 6 is a prime ….. **F**
- Determine **the truth value** of the following statements:

  $\neg p$:

  $p \wedge q$:

  $p \wedge \neg q$:

  $p \vee q$:

  $p \oplus q$:

  $p \rightarrow q$:

  $q \rightarrow p$:

Compound Propositions

- Let p: 2 is a prime ….. **T**

  q: 6 is a prime ….. **F**

- Determine **the truth value** of the following statements:

  $\neg p$: **F**

  $p \wedge q$: **F**

  $p \wedge \neg q$: **T**

  $p \vee q$: **T**

  $p \oplus q$: **T**

  $p \rightarrow q$: **F**

  $q \rightarrow p$: **T**

Definition of Propositional Logic

SYNTAX (what is a **formula**?)**:**

- Vocabulary consists of a set $\mathcal{P}$ of propositional variables, usually denoted by (subscripted) $p, q, r, \ldots$
- The set of propositional formulas over $\mathcal{P}$ is defined as:
  - Every propositional variable is a formula
  - If F is a formula, $\neg F$ is also a formula
  - If F and G are formulas, $(F \wedge G)$ is also a formula
  - If F and G are formulas, $(F \vee G)$ is also a formula
  - Nothing else is a formula
- Formulas are usually denoted by (subscripted) $F, G, H, \ldots$
- Examples:

  p $\quad\quad$ $\neg p$ $\quad\quad$ $(p \vee q)$ $\quad\quad$ $\neg(p \wedge q)$

  $(p \wedge (\neg p \vee q))$ $\quad\quad$ $((p \wedge q) \vee (r \vee \neg q))$ $\quad\quad$ $\ldots$

Propositional Logic - Semantics

The semantic gives the meaning to sentences.

the semantics in the propositional logic is defined by:

1. **Interpretation of propositional symbols and constants**
   - Semantics of atomic sentences
2. **Through the meaning of connectives**
   - Meaning (semantics) of composite sentences

Semantic: propositional symbols

A **propositional symbol**

- a statement about the world that is either true or false

  Examples:

  - *Pitt is located in the Oakland section of Pittsburgh*
  - *It rains outside*
  - *Light in the room is on*

- An **interpretation** maps symbols to one of the two values: ***True (T),*** or ***False (F)***, depending on whether the symbol is satisfied in the world

  I: *Light in the room is on ->* ***True,*** *It rains outside ->* ***False***

  I': *Light in the room is on ->* ***False,*** *It rains outside ->* ***False***

Semantic: propositional symbols

The **meaning (value)** of the propositional symbol for a specific interpretation is given by its interpretation

I: *Light in the room is on* **->** ***True,*** *It rains outside* **->** ***False***

V(Light in the room is on, **I***)* = ***True***

V(It rains outside, **I***)* = ***False***

I': *Light in the room is on* **->** ***False,*** *It rains outside* **->** ***False***

V(Light in the room is on, **I'***)* = ***False***

Semantic: constants

- **The meaning (truth) of constants:**
  - True and False constants are always (under any interpretation) assigned the corresponding ***True,False*** value

$$\left.\begin{array}{l} V(True, \mathbf{I}) = \boldsymbol{True} \\ \\ V(False, \mathbf{I}) = \boldsymbol{False} \end{array}\right\} \quad \text{For any interpretation } \mathbf{I}$$

Semantic: composite sentences

- **The meaning (truth value) of complex propositional sentences.**
  - Determined using the standard rules of logic:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
True	*True*	*False*	*True*	*True*	*True*	*True*
True	*False*	*False*	*False*	*True*	*False*	*False*
False	*True*	*True*	*False*	*True*	*True*	*False*
False	*False*	*True*	*False*	*False*	*True*	*True*

Rows define all possible interpretations (worlds)

Semantic: composite sentences – Constructing Truth Table

Constructing the truth table

- **Example:** Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Semantic: composite sentences – Constructing Truth Table

- Example: Construct the truth table for
  $(p \to q) \land (\neg p \leftrightarrow q)$

Rows: all possible combinations of values for elementary propositions: 2^n values

p	q	$\neg p$			$(p \to q) \land (\neg p \leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Semantic: composite sentences – Constructing Truth Table

- Example: Construct the truth table for
  $(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$

Typically the target (unknown) compound proposition and its values

p	q	$\neg p$	$p \rightarrow q$	$\neg p \leftrightarrow q$	$(p \rightarrow q) \wedge (\neg p \leftrightarrow q)$
T	T				
T	F				
F	T				
F	F				

Auxiliary compound propositions and their values

Semantic: composite sentences – Constructing Truth Table

- Examples: Construct a truth table for
  $(p \to q) \land (\neg p \leftrightarrow q)$

p	q	$\neg p$	$p \to q$	$\neg p \leftrightarrow q$	$(p\to q)\land(\neg p\leftrightarrow q)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Definition of Propositional Logic (2)

SEMANTICS (what is an interpretation I, when $I \models F$?)**:**

- An interpretation I over $\mathcal{P}$ is a function $I : \mathcal{P} \rightarrow \{0, 1\}$.
- I satisfies F (written $I \models F$) if and only if $eval_I(F) = 1$.
- $eval_I : Formulas \rightarrow \{0, 1\}$ is a function defined as follows:
  - $eval_I(p) = I(p)$
  - $eval_I(\neg F) = 1 - eval_I(F)$
  - $eval_I(\ (F \wedge G)\ ) = min\{eval_I(F), eval_I(G)\}$
  - $eval_I(\ (F \vee G)\ ) = max\{eval_I(F), eval_I(G)\}$
- If $I \models F$ we say that
  - I is a model of F or, equivalently
  - F is true in I.

Semantics of propositional logic

We start with two truth values: $\{0, 1\}$

- 0 stands for False, and 1 stands for True

Let **D** be any subset of the *atomic* formulas An *assignment* **A** is a map $\mathbf{D} \rightarrow \{0, 1\}$

- **A** assigns True or False to every atomic in **D**

Let $\mathbf{E} \supseteq \mathbf{D}$ be set of formulas built from **D** using propositional connectives

Extended assignment **A**': $\mathbf{E} \rightarrow \{0, 1\}$ extends **A** from atomic formulas to all formulas

- continued on the next slide

Semantics of propositional logic

For an atomic formula A_i in **D**: $\mathbf{A'}(A_i) = \mathbf{A}(A_i)$

$\mathbf{A'}(F \wedge G)$ = 1 if $\mathbf{A'}(F) = 1$ and $A'(G) = 1$

= 0 otherwise

$\mathbf{A'}(F \vee G)$ = 1 if $\mathbf{A'}(F) = 1$ or $A'(G) = 1$

= 0 otherwise

$\mathbf{A'}(\neg F)$ = 1 if $\mathbf{A'}(F) = 0$

= 0 otherwise

We write $\mathcal{A}$ instead of $\hat{\mathcal{A}}$.

Exercise: Define Extended Assignment

$$F = \neg(A \wedge B) \vee C$$

$$\mathcal{A}(A) = 1$$
$$\mathcal{A}(B) = 1$$
$$\mathcal{A}(C) = 0$$

Is F true or false under A'?

Parse Tree

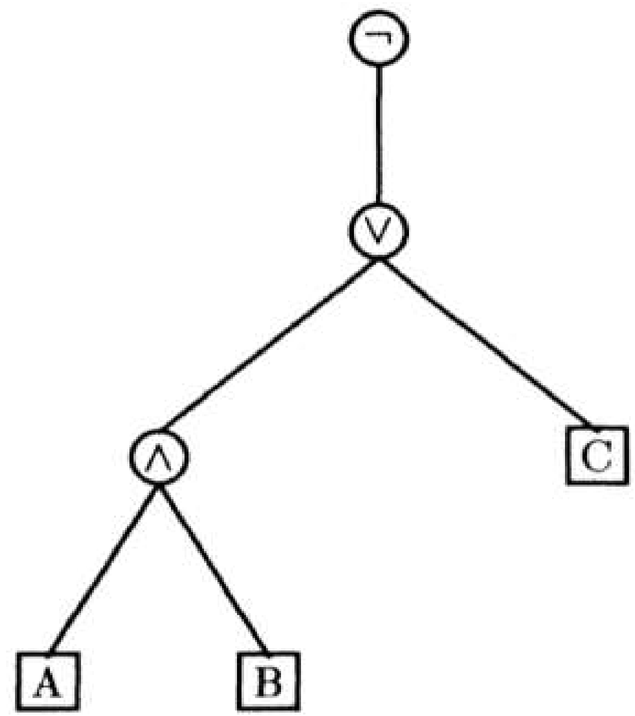
Formula

$$F = \neg(A \wedge B) \vee C$$

Assignment

$$\mathcal{A}(A) = 1$$
$$\mathcal{A}(B) = 1$$
$$\mathcal{A}(C) = 0$$

Abstract Syntax Tree (AST)



Is F true or false under A'?

$$F = (\neg A \rightarrow (A \rightarrow B))$$

A	B	$\neg A$	$(A \rightarrow B)$	F
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	0	1	1

Applications of Proposition Logic

- **Translation of English sentences**
- **Inference and reasoning:**
  - new true propositions are inferred from existing ones
  - Used in Artificial Intelligence:
    - Rule based (expert) systems
    - Automatic theorem provers
- **Design of logic circuit**

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a short trip
- t= We will be home by sunset

Translation

Assume the following sentences:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

$$\neg p \wedge q$$
$$r \rightarrow p$$
$$\neg r \rightarrow s$$
$$s \rightarrow t$$

Denote:

- p = It is sunny this afternoon
- q = it is colder than yesterday
- r = We will go swimming
- s= we will take a canoe trip
- t= We will be home by sunset

Translation

Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a Horror movie.

Parse:

- **If (you are older than 13 or you are with your parents)then (you can attend a Horror movie)**

Atomic (elementary) propositions:

– A= you are older than 13
– B= you are with your parents
– C=you can attend a Horror movie

- **Translation: ?**

Translation

Assume a sentence:

If you are older than 13 or you are with your parents then you can attend a Horror movie.

Parse:

- **If (you are older than 13 or you are with your parents)then (you can attend a Horror movie)**

Atomic (elementary) propositions:

- **A= you are older than 13**
- **B= you are with your parents**
- **C=you can attend a Horror movie**

- **Translation:** $A \vee B \rightarrow C$

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 1 find logical connectives

Translation

- **General rule for translation.**
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- **Example:**

You can have free coffee if you are senior citizen and it is a Tuesday

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

<u>You can have free coffee</u> if <u>you are senior citizen</u> and <u>it is a Tuesday</u>

a **b** **c**

Step 2 break the sentence into elementary propositions

Translation

- **General rule for translation.**
- Look for patterns corresponding to logical connectives in the sentence and use them to define elementary propositions.

- **Example:**

<u>You can have free coffee</u> if <u>you are senior citizen</u> and <u>it is a Tuesday</u>

a b c

Step 3 rewrite the sentence in propositional logic

$$\mathbf{b} \wedge \mathbf{c} \rightarrow \mathbf{a}$$

Propositional Logic: Semantics

An assignment A is ***suitable*** for a formula F if A assigns a truth value to every atomic proposition of F

An assignment A is a ***model*** for F, written A⊨ F, iff

- A is suitable for F
- A'(F) = 1, i.e., F *evaluates to true (or holds)* under A

A formula F is *satisfiable* iff F has a model, otherwise F is *unsatisfiable* (or contradictory)

A formula F is *valid* (or a tautology), written ⊨ F, iff every suitable assignment for F is a model for F

Formalizing natural language (I)

A device consists of two parts A and B, and a red light. We know that:

- A or B (or both) are broken.
- If A is broken, then B is broken.
- If B is broken and the red light is on, then A is not broken.
- The red light is on.

We use the atomic formulas: Ro (red light on), Ab (A is broken), Bb (B is broken), and formalize this situation by means of the formula

$$((((Ab \vee Bb) \wedge (Ab \rightarrow Bb)) \wedge ((Bb \wedge Ro) \rightarrow \neg Ab))) \wedge Ro)$$

Formalizing natural language (II)

Full truth table:

Ro	Ab	Bb	**F** $((((Ab \vee Bb) \wedge (Ab \rightarrow Bb))) \wedge ((Bb \wedge Ro) \rightarrow \neg Ab)) \wedge Ro)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
$\boxed{1}$	$\boxed{0}$	$\boxed{1}$	$\boxed{1}$
1	1	0	0
1	1	1	0

- I satisfies F (written $I \models F$) if and only if $eval_I(F) = 1$.
- If $I \models F$ we say that
  I is a model of F or, equivalently
  F is true in I.

Contradiction and Tautology

Some composite sentences may always (under any interpretation) evaluate to a single truth value:

- **Contradiction** (always ***False***) / **Unsatisfiable**

$$P \wedge \neg P$$

- **Tautology** (always ***True***)

$$P \vee \neg P$$

$$\left.\begin{aligned} \neg(P \vee Q) &\Leftrightarrow (\neg P \wedge \neg Q) \\ \neg(P \wedge Q) &\Leftrightarrow (\neg P \vee \neg Q) \end{aligned}\right\} \text{DeMorgan's Laws}$$

Model, validity and satisfiability

- A **model (in logic):** An **interpretation is a model** for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

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True	*False*	*True*	*True*	*True*
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Satisfiable sentence

P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

Model, validity and satisfiability

- A **model (in logic):** An **interpretation is a model** for a set of sentences if it assigns true to each sentence in the set.
- A sentence is **satisfiable** if it has a model;
  - There is at least one interpretation under which the sentence can evaluate to True.
- A sentence is **valid** if it is ***True*** in all interpretations
  - i.e., if its negation is **not satisfiable** (leads to contradiction)

			Satisfiable sentence	Valid sentence
P	Q	$P \vee Q$	$(P \vee Q) \wedge \neg Q$	$((P \vee Q) \wedge \neg Q) \Rightarrow P$
True	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
False	*True*	*True*	*False*	*True*
False	*False*	*False*	*False*	*True*

Validity and Unsatisfiability

Theorem:

A formula F is valid if and only if ¬F is unsatifsiable

Proof:

F is valid ⇔ every suitable assignment for F is a model for F

⇔ every suitable assignment for ¬ F is not a model for ¬ F

⇔ ¬ F does not have a model

⇔ ¬ F is unsatisfiable

Small Syntax Extension

- We will write $(F \rightarrow G)$ as an abbreviation for $(\neg F \vee G)$
- Similarly, $(F \leftrightarrow G)$ is an abbreviation of $((F \rightarrow G) \wedge (G \rightarrow F))$

They both capture very intuitive concepts, which ones?

Small Syntax Extension

- We will write $(F \rightarrow G)$ as an abbreviation for $(\neg F \vee G)$
- Similarly, $(F \leftrightarrow G)$ is an abbreviation of $((F \rightarrow G) \wedge (G \rightarrow F))$

They both capture very intuitive concepts, which ones?

- $I \models (F \rightarrow G)$ iff $I \models F$ implies $I \models G$

- $I \models (F \leftrightarrow G)$ iff $I \models F$ and $I \models G$ or
  $I \not\models F$ and $I \not\models G$

  iff $eval_I(F) = eval_I(G)$

Operator (Connective) Precedence

Operator precedence:

LOW	$\leftrightarrow$	binds weaker than
	$\rightarrow$	which binds weaker than
	$\vee$	which binds weaker than
	$\wedge$	which binds weaker than
HIGH	$\neg$	.

So we have

$$A \leftrightarrow B \vee \neg C \rightarrow D \wedge \neg E \equiv (A \leftrightarrow ((B \vee \neg C) \rightarrow (D \wedge \neg E)))$$

But: well chosen parenthesis help to visually parse formulas.

EXCERCISE

1. Draw the parse tree for the following formula:

$$((\neg p) \wedge ((\neg q) \wedge ((\neg r) \wedge ((\neg s) \wedge \top))))$$

List all sub-formulas of the expression.

2. According to the operator precedences, the following formula has a unique reading.

$$\neg p \wedge q \rightarrow \neg r \vee \neg p \rightarrow r$$

   Indicate this reading by writing all parentheses

EXERCISE on Translation

- Assume two elementary statements:

  p: you drive over 65 mph;   q: you get a speeding ticket

- **Translate each of these sentences to logic**

a) you do not drive over 65 mph.

b) you drive over 65 mph, but you don't get a speeding ticket.

c) you will get a speeding ticket if you drive over 65 mph.

d) if you do not drive over 65 mph then you will not get a speeding ticket.

e) driving over 65 mph is sufficient for getting a speeding ticket.

f) you get a speeding ticket, but you do not drive over 65 mph.

References

- Chpater 1 of Logic for Computer Scientists http://www.springerlink.com/content/978-0-8176-4762-9/