Chapter 1: Formal Logic

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Logic: The Foundation of Reasoning

- **Definition**: the foundation for the organized, careful method of thinking that characterizes any reasoned activity
- It is the study of reasoning: specifically concerned if it is true or false

- A statement (or proposition) is a sentence that is either true or false but not both.
- Which ones are statements?
  1. 5 is greater than 10.
  2. What is your favorite pet's name?
  3. You are a genius.
  4. All mathematicians wear sandals.
- 1 and 4 are statements. 2 and 3 are not statements.

- Convention: when there is a non-specific item in the statement, then it is not considered a statement
  - For example, "X+2 is greater than 0" is not a statement.

- Usually, we use capital letters A, B, and C to represent statements

Practice

- Which of the following sentences are statements?
  - The moon is made of green cheese.
  - He is certainly a tall man.
  - Next year interest rates will fall.
  - $X - 4 = 0$

Statements and Logic

- Logic focuses on the relationship between statements as opposed to the content of any particular statement.

- An example to illustrate how logic helps us:
  A. All mathematicians love apples.
  B. Anyone who loves apples is smart.
  C. Therefore, all mathematicians are smart.

- Logic cannot help us to determine the individual truth of the above statements, however, if statements A and B are true, what can we say about C?

- Logical methods are used in mathematics to prove theorems and in computer science to prove that programs do what they are supposed to do.

Logical Connectives and Truth Values

- Compound statement
  - It is made of statement variables (such as A, B, and C) and logical connectives (such as $\wedge$, $\vee$, $\leftrightarrow$, $\rightarrow$).
  - Its truth value depends on the truth values of its components and their relationships (logical connectives).

- Logical connectives
  - Negation — A′, not A
  - conjunction — $A \wedge B$, A and B
  - disjunction — $A \vee B$, A or B
  - implication — $A \rightarrow B$, if A, then B
  - Equivalence — $A \leftrightarrow B$, A if and only if B

- A truth table is a table which displays the truth values of a compound statement that correspond to all different combinations of truth values of the statement variables.

Connective #1: Negation

- If A is a statement variable, the negation of A is "not A" and is denoted A′.
- It has the opposite truth value from A: if A is true, then A′ is false; if A is false, then A′ is true.
- True(T) is usually 1; False(F) is usually 0
- Other forms: "It is false that A …", "It is not true that A …", etc.
- Unary connective, instead of binary connective

A	A'
T	F
F	T

Connective #2: Conjunction

- If A and B are statement variables, the conjunction of A and B is $A \wedge B$, which is read "A and B"
- $A \wedge B$ is true when both A and B are true. $A \wedge B$ is false when at least one of A or B is false.
- A and B are called the conjuncts of $A \wedge B$.
- English words: and; but; also; in addition; moreover

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Connective #3: Disjunction

- If A and B are statement variables, the disjunction of A and B is A V B, which is read "A or B"
- A V B is true when at least one of A or B is true. A V B is false when both A and B are false
- A and B are called the disjuncts of A V B
- English word: or

A	B	A V B
T	T	T
T	F	T
F	T	T
F	F	F

Examples

- A = It will rain tomorrow. What’s A’?
  - It is false that it will rain tomorrow.
  - It will not rain tomorrow.
- B = Peter is tall and thin. What’s B’?
  - (1) Peter is not tall or he is not thin.
  - (2) Peter is short and fat.
  - (3) Peter is short or fat.
- C = The river is shallow or polluted. C’?
  - (1) The river is neither shallow nor polluted.
  - (2) The river is not shallow or not polluted.
  - (3) The river is deep and unpolluted.

De Morgan's Laws

- $(A \vee B)' = A' \wedge B'$
- $(A \wedge B)' = A' \vee B'$

Connective #4: Implication

- If A and B are statement variables, the symbolic form of "if A then B" is $A \rightarrow B$.
- Here A is called the hypothesis/antecedent statement and B is called the conclusion/consequent statement.
- "If A then B'' is false when A is true and B is false, and it is true otherwise.
- Other forms:
  - A implies B.
  - B if A.
  - Whenever A, B
  - A, therefore B.
  - B follows from A.
  - A is a sufficient condition for B.
  - B is a necessary condition for A.
  - A only if B.

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Another Form of Implication

- Representation of If-Then as Or
- A' = "You do your homework"
- B = "You will flunk"
- A' V B = "Either you do your homework or you will flunk"
- $A \rightarrow B$ = "If you do not do your homework, then you will flunk"
- Thus, $A \rightarrow B \equiv A' \vee B$
- Example: write the negation of the following statement:
  - If the food is good, then the service is excellent.

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

A	B	A′	A′ V B
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Example of Implication

- Your friend says: “If I pass my math test, then I’ll go to the movie Friday”
  - If your friend doesn’t pass the test, then whether he or she goes to the movie or not, you could not claim that the remark was false.
  - In logic, the truth value of $A \rightarrow B$ is true if A is false and B is true.

- “I’ll be there only if it rains.”
  - So, if I’m there, you know it rains. That means “I’m there” is a sufficient condition for “it rains”.

Connective #5: Equivalence

- If A and B are statement variables, the symbolic form of "A if, and only if, B" and is denoted $A \leftrightarrow B$.
- It is true if both A and B have the same truth values and is false if A and B have opposite truth values.
- Other forms: "A is necessary and sufficient for B", "A is equivalent to B", "A if and only if B".

Note: $A \leftrightarrow B$ is a short form for $(A \rightarrow B) \wedge (B \rightarrow A)$

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

A	B	$A \rightarrow B$	$B \rightarrow A$	$(A \rightarrow B) \wedge (B \rightarrow A)$
T	T			
T	F			
F	T			
F	F			

Practice

- What's the truth value of the following:
  - 8 is even or 6 is odd.
  - 8 is even and 6 is odd.
  - If 8 is odd, then 6 is odd.
  - If 8 is even, then 6 is even.

Practice

- Write the negation
  - If the food is good, then the service is excellent.
  - Either the food is good or the service is excellent.
  - Neither the food is good nor the service excellent.

Well-Formed Formula (wff)

- We can string statement letters, connectives, and parentheses (or brackets) together to form new expressions
- A legitimate string is called a well-formed formula, or wff

- For example, $(A \rightarrow B) \vee (B \rightarrow A)$ is a wff,

  but $A)) \vee B(\rightarrow C)$ is not

- Formally:

  (1) All propositional variables and the constants True and False are wffs.

  (2) If A and B are wffs, then A', B', $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \leftrightarrow B$, (A'), (B'), $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are wffs.

Well-Formed Formula

- To reduce the number of parentheses, an order is stipulated in which the connectives can be applied, called the order of precedence which is as follows:
  - Connectives within parentheses, innermost parentheses first
  - Negation (')
  - Conjunction ($\wedge$)
  - Disjunction ($\vee$)
  - Implication ($\rightarrow$)
  - Equivalence ($\leftrightarrow$)

- Hence, $A \vee B \rightarrow C$ is the same as $(A \vee B) \rightarrow C$

Practice

- Common English has many ways to describe logical connectives. Write a wff for each of the following expressions
  - Either A or B
  - Neither A nor B

Practice

- The main connective is the one that is applied last
- Construct truth tables for the following wffs and identify what is the main connective.
  - $A \lor A' \rightarrow B \wedge B'$
  - $(A \rightarrow B) \rightarrow B' \wedge C$

Tautology and Contradiction

- Letters like P, Q, R, S, etc. are used for representing wffs, e.g. $[(A \vee B) \wedge C'] \rightarrow A' \vee C$ can be represented by $P \rightarrow Q$ where P is the wff $(A \vee B) \wedge C'$ and Q represents $A' \vee C$

- Definition of tautology:
  - A wff which is intrinsically true, i.e. no matter what the truth value of the statements that comprise the wff.
  - e.g. It will rain today or it will not rain today ($A \vee A'$)
  - e.g. $P \leftrightarrow Q$ where P is $A \rightarrow B$ and Q is $A' \vee B$

- Definition of a contradiction:
  - A wff which is intrinsically false, i.e. no matter what the truth value of the statements that comprise the wff.
  - e.g. It will rain today and it will not rain today ($A \wedge A'$)
  - $(A \wedge B) \wedge A'$

- Usually, tautology is represented by 1 and contradiction by 0

Tautological Equivalences

- Two statement forms are called *logically equivalent* if, and only if, they have identical truth values for each row of the truth table.
- The logical equivalence of statement forms P and Q is denoted by writing $P \Leftrightarrow Q$ or $P \equiv Q$.
- Prove by constructing truth table
- $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$

A	B	C	A V B	B V C	(A V B) V C	A V (B V C)
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Some Common Equivalences

Commutative	$A \lor B \Leftrightarrow B \lor A$	$A \land B \Leftrightarrow B \land A$
Associative	$(A \lor B) \lor C \Leftrightarrow A \lor (B \lor C)$	$(A \land B) \land C \Leftrightarrow A \land (B \land C)$
Distributive	$A \lor (B \land C) \Leftrightarrow (A \lor B) \land (A \lor C)$	$A \land (B \lor C) \Leftrightarrow (A \land B) \lor (A \land C)$
Identity	$A \lor 0 \Leftrightarrow A$	$A \land 1 \Leftrightarrow A$
Complement	$A \lor A' \Leftrightarrow 1$	$A \land A' \Leftrightarrow 0$

- The equivalences are listed in pairs, hence they are called dual of each other
- One can be obtained from the other by *replacing* $\lor$ *with* $\land$ *and 0 with 1 or vice versa*
- *How to verify the equivalences?*

Additional Equivalences (1)

- De Morgan's Laws
  - $(A \vee B)' \Leftrightarrow A' \wedge B'$
  - $(A \wedge B)' \Leftrightarrow A' \vee B'$
  - e.g. "Julie likes butter but hates cream"
- Double negation: $(A')' \Leftrightarrow A$
- Rewriting implication: $(A \rightarrow B) \Leftrightarrow A' \vee B$
- Contraposition: $(A \rightarrow B) \Leftrightarrow (B' \rightarrow A')$
- Conditional proof: $A \rightarrow (B \rightarrow C) \Leftrightarrow (A \wedge B) \rightarrow C$
  - "If I miss the train today, then I can arrive only 5 minutes late, assuming that the next train is on time."

Additional Equivalences (2)

- Suppose P and Q are equivalent wffs. Then P can be replaced by Q in any wff R containing P, resulting in a wff R_Q that is equivalent to R

- Example:
  - R: $(A \rightarrow B) \rightarrow B$
  - P: $A \rightarrow B$, Q: $B' \rightarrow A'$
  - R_Q : $(B' \rightarrow A') \rightarrow B$

Logic Connectives in Computer Science

- Web search engine
- Conditional statement in programming use logical connectives with statements
- Example 7 in Section 1.1 (page 11)

```
if ((outflow > inflow) and
   not ((outflow > inflow) and (pressure < 1000)))
        do something;
else
        do something else;
```

- A: outflow > inflow
- B: pressure < 1000
- $A \wedge (A \wedge B)'$
- Can we simply this somehow?

Review of Section 1.1

- Techniques:
  - construct truth tables for compound wffs
  - recognize tautologies and contradictions

- Main ideas:
  - wffs are symbolic representation of statements
  - truth values for compound wffs depend on the truth value of their components and the types of connectives used
  - tautologies are intrinsically true wffs – true for all truth values

Practices

- The negation of "the answer is either 2 or 3":
  - Neither 2 nor 3 is the answer.
  - The answer is not 2 or not 3.
  - The answer is not 2 and it is not 3.

- The negation of "Cucumbers are green and seedy":
  - Cucumbers are not green and not seedy.
  - Cucumbers are not green or not seedy.
  - Cucumbers are green and not seedy.

Practices, Continue…

- Write the negation of the following statement:
  - Either the food is good or the service is excellent.
  - If the price is high, then the food is good and the service is excellent.
  - The processor is fast but the printer is slow.

- Prove the following tautology by converting the left side into the right side
  - $(A \wedge B')' \vee B \leftrightarrow A' \vee B$