LU-14: FIRST ORDER LOGIC

Outline

- Why FOL?
- Syntax and semantics of FOL

LU Objectives	
To explain syntax and semantics of FOL in detail	
LU Outcomes	CO 3
Apply syntax and semantics of FOL to represent real world statements	

Pros and cons of propositional logic

☺ Propositional logic is declarative

☺ Propositional logic allows partial/disjunctive/negated information

- (unlike most data structures and databases)

☺ Propositional logic is compositional:

☺

- meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

☺ Meaning in propositional logic is context-independent

- (unlike natural language, where meaning depends on context)

☹ Propositional logic has very limited expressive power

- (unlike natural language)
- E.g., cannot say "pits cause breezes in adjacent squares“
  - except by writing one sentence for each square

If it is snowing, it is cold
If it is cold, John is wearing a coat
It is snowing

Therefore, John is wearing a coat

First-order logic

- Whereas propositional logic assumes the world contains facts,
- first-order logic (like natural language) assumes the world contains
-
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  -
  - Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  - Functions: father of, best friend, one more than, plus, …
  -

Syntax of FOL: Basic elements

- Constants KingJohn, 2, Stanford,...
- Predicates Brother, >,...
- Functions Sqrt, LeftLegOf,...
- Variables x, y, a, b,...
- Connectives $\neg, \Rightarrow, \wedge, \vee, \Leftrightarrow$
- Equality =
- Quantifiers $\forall, \exists$

Atomic sentences

Atomic sentence = *predicate* ($term_1$,...,$term_n$)
or $term_1 = term_2$

Term = *function* ($term_1$,...,$term_n$)
or *constant* or *variable*

- E.g., *Brother(KingJohn,RichardTheLionheart)*
- *> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))*

Complex sentences

- Complex sentences are made from atomic sentences using connectives

$$\neg S,\ S_1 \wedge S_2,\ S_1 \vee S_2,\ S_1 \Rightarrow S_2,\ S_1 \Leftrightarrow S_2,$$

E.g. *Sibling(KingJohn,Richard)* $\Rightarrow$ *Sibling(Richard,KingJohn)*

$>(1,2) \vee \leq (1,2)$

$>(1,2) \wedge \neg >(1,2)$

Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains objects (domain elements) and relations among them
-
- Interpretation specifies referents for
  constant symbols → objects

  predicate symbols → relations

  function symbols → functional relations

- An atomic sentence $predicate(term_1,\ldots,term_n)$ is true
  iff the objects referred to by $term_1,\ldots,term_n$
  are in the relation referred to by *predicate*

Truth in first-order logic

Truth example

Consider the interpretation in which
$Richard \rightarrow$ Richard the Lionheart
$John \rightarrow$ the evil King John
$Brother \rightarrow$ the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Models for FOL : Lots!

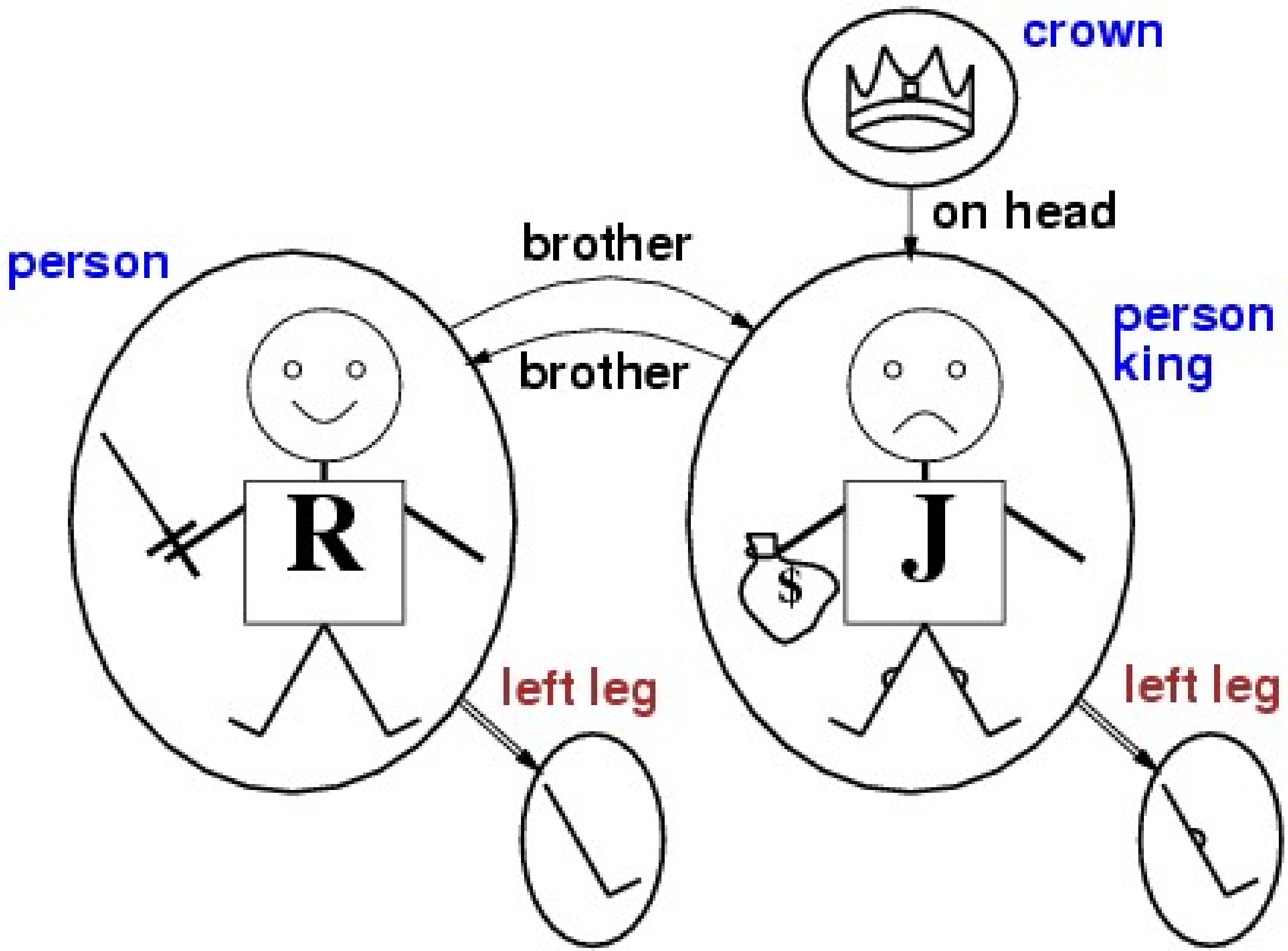
Entailment in propositional logic can be computed by enumerating models

We **can** enumerate the FOL models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
    For each k-ary predicate P_k in the vocabulary
        For each possible k-ary relation on n objects
            For each constant symbol C in the vocabulary
                For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Models for FOL: Example



Sample Model

Objects:

Richard the Lionheart(King of England from 1189 to 1199) ;
his younger brother, the evil King John (who ruled from 1199 to 1215);
the left legs of Richard and John;
a crown.

Relations:

"brother" and "on head" relations are binary relations

Properties:

the "person" property is true of both Richard and John;
the "king" property is true only of John
the "crown" property is true only of the crown.

Functions:

"left leg" is a unary function that includes the following mappings:
Richard the Lionheart → Richard's left leg
King John → John's left leg .

Universal quantification

- $\forall$ *<variables> <sentence>*
- $\forall$ means for all, for everyone

p	q	$p \rightarrow q$
T	T	T
~~T~~	~~F~~	~~F~~
F	T	T
F	F	T

Everyone at Stanford is Intelligent:
$\forall$x At(x, Stanford) $\Rightarrow$ Intelligent(x)

- $\forall$x *P* **is true** in a model *m* iff *P* is true in all possible extended interpretations constructed from the interpretation given in the model where each extended interpretation specifies a domain element to which x refers

Universal quantification

$\forall \langle variables \rangle \ \langle sentence \rangle$

Everyone at Berkeley is smart:
$\forall x \ \ At(x, Berkeley) \ \Rightarrow \ Smart(x)$

$\forall x \ \ P$ is true in a model m iff P is true with x being **each** possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$\begin{aligned} & (At(KingJohn, Berkeley) \ \Rightarrow \ Smart(KingJohn)) \\ \wedge \ & (At(Richard, Berkeley) \ \Rightarrow \ Smart(Richard)) \\ \wedge \ & (At(Berkeley, Berkeley) \ \Rightarrow \ Smart(Berkeley)) \\ \wedge \ & \ldots \end{aligned}$$

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Universal quantification

- Consider King John Model. We can extend the interpretation in five ways:

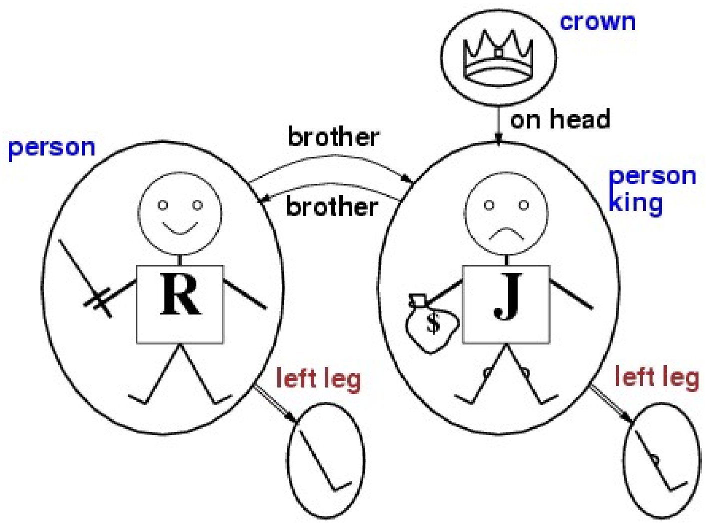
  x → Richard the Lionheart,

  x → King John,

  x → Richard's left leg,

  x → John's left leg,

  x → the crown.



Ex Sentence: All kings are persons

- The universally quantified sentence $\forall$ x King(x) $\Rightarrow$ Person(x) is true in the original model if the sentence is true under each of the five extended interpretations.

Universal quantification

$\forall x\ King(x) \Rightarrow Person(x)$

The universally quantified sentence is equivalent to asserting the following five Sentences :

1. King John is a king ⇒ King John is a person.
   King(King John **) ⇒ Person(**King John **)**
2. Richard the Lionheart is a king ⇒ Richard the Lionheart is a person.
   King(Richard the Lionheart**) ⇒ Person(**Richard the Lionheart **)**
3. Richard's left leg is a king ⇒ Richard's left leg is a person.
   King(Richard's left leg**) ⇒ Person(**Richard's left leg**)**
4. John's left leg is a king ⇒ John's left leg is a person.
   King(John's left leg**) ⇒ Person(**John's left leg**)**
5. The crown is a king ⇒ the crown is a person.
   King(crown**) ⇒ Person(**crown**)**

In our model, King John is the only king, the second sentence asserts that he is a person.
The other four assertions are true in the model, but make no claim whatsoever about the personhood qualifications of legs, crowns, or indeed Richard. This is because none of these objects is a king.
(This makes ⇒ the connective suitable for universal quantifier)

	p	q	p→q
1	T	T	T
	T	F	F
2	F	T	T
3,4,5	$\boxed{F}$	$\boxed{F}$	T

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
-
- Common mistake: using $\wedge$ as the main connective with $\forall$: **(It makes it a overly strong statement)**

  $\forall x\ At(x, Stanford) \wedge Intelligent(x)$

  means "Everyone is at Stanford and everyone is intelligent"

  Use parenthesis to avoid mistake

Existential quantification

- $\exists$<*variables*> <*sentence*>
- $\exists$ means there exists, for some, for at least

Ex:

1. Someone at Stanford is intelligent:

   $\exists x$ At(x, Stanford) $\wedge$ Intelligent(x)

2. King John has a crown on his head.

   $\exists$ x Crown(x) $\wedge$ OnHead(x, John)

$\exists x\ P$ is true in a model *m* iff *P* is true for at least one extended interpretation that assigns x to a domain element.

Existential quantification

At least one of the following assertions is true:

1. Richard the Lionheart is a crown ∧ Richard the Lionheart is on John's head;
2. King John is a crown ∧ King John is on John's head;
3. Richard's left leg is a crown ∧ Richard's left leg is on John's head;
4. John's left leg is a crown ∧ John's left leg is on John's head;
5. The crown is a crown ∧ the crown is on John's head.

The fifth assertion is true in the model

- Roughly speaking, ∃ equivalent to the **disjunction** of **instantiations** of *P*

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$: **(It leads to a very weak statement)**

$$\exists x \text{ At}(x,\text{Stanford}) \Rightarrow \text{Intelligent}(x)$$

is true if there is anyone who is not at Stanford!

($\Rightarrow$ is true even if premise is false)

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Properties of quantifiers

- $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
- $\exists x\ \exists y$ is the same as $\exists y\ \exists x$
- $\exists x\ \forall y$ is not the same as $\forall y\ \exists x$

- $\exists x\ \forall y$ Likes(x,y)
  - "There is a person who likes everyone in the world"
- $\forall y\ \exists x$ Likes(x,y)
  - "Everyone in the world is liked by at least one person"
- Quantifier duality: each can be expressed using the other
- $\forall x$ Likes(x,IceCream) **$\neg\exists x\ \neg$Likes(x,IceCream)**

There is no one who doesn't like icecream

- $\exists x$ Likes(x,Broccoli) **$\neg\forall x\ \neg$Likes(x,Broccoli)**

Not everyone who doesn't like Broccoli

Sample Sentences

- Every gardener likes the sun.
- You can fool some of the people all of the time.
- You can fool all of the people some of the time.
- All purple mushrooms are poisonous.
- No purple mushroom is poisonous.

Sample Sentences

Every gardener likes the sun.

$\forall x\ \text{gardener}(x) \rightarrow \text{likes}(x,\text{Sun})$

You can fool some of the people all of the time.

$\exists x\ \forall t\ \ \text{person}(x) \wedge \text{time}(t) \rightarrow \text{can-fool}(x,t)$

You can fool all of the people some of the time.

$\forall x\ \exists t\ (\text{person}(x) \rightarrow \text{time}(t) \wedge \text{can-fool}(x,t))$

$\forall x\ (\text{person}(x) \rightarrow \exists t\ (\text{time}(t) \wedge \text{can-fool}(x,t)))$

Equivalent

All purple mushrooms are poisonous.

$\forall x\ (\text{mushroom}(x) \wedge \text{purple}(x)) \rightarrow \text{poisonous}(x)$

No purple mushroom is poisonous.

$\neg\exists x\ \text{purple}(x) \wedge \text{mushroom}(x) \wedge \text{poisonous}(x)$

$\forall x\ \ (\text{mushroom}(x) \wedge \text{purple}(x)) \rightarrow \neg\text{poisonous}(x)$

Equivalent

Sample Sentences

- Deb is not tall.

- All kings are persons.

Sample Sentences

- Deb is not tall.

  $\neg$ tall(Deb)
- All kings are persons.

  ($\forall$ x) King(x) => Person(x)

Nested quantifiers

For example, “Brothers are siblings” can be written as

- $\forall x \forall y$ Brother $(x, y) \Rightarrow$ Sibling(x, y)

Consecutive quantifiers of the same type can be written as one quantifier with several variables.

- For example, to say that siblinghood is a symmetric relationship, we can write

$$\forall x, y \text{ Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x)$$

Nested quantifiers

- The order of quantification is therefore very important
- ❖ "Everybody likes somebody" means that for every person, there is someone that person loves:

  $\forall x \exists y \, Likes(x, y)$.

- ❖ On the other hand, to say "There is someone who is liked by everyone," we write

  $\exists y \forall x \, Likes(x, y)$.

Connections between ∀ and ∃

- The two quantifiers are actually intimately connected with each other, through negation.
- Asserting that everyone dislike war is the same as asserting there does not exist someone who likes them, and vice versa:

$\forall\, x\ \neg Likes(x, War)$ is equivalent to $\neg\exists\, x\ Likes(x, War)$

Connections between $\forall$ and $\exists$

- "Everyone likes ice cream" means that there is no one who does not like ice cream:

$\forall$ x Likes(x, IceCream) is equivalent to $\neg\exists$ x $\neg$Likes(x, IceCream)

De Morgan rules

1) $\forall x \neg P \equiv \neg\exists x\ P$
2) $\neg\forall x\ P \equiv \exists x \neg P$
3) $\forall x\ P \equiv \neg\exists x \neg P$
4) $\exists x\ P \equiv \neg\forall x \neg P$
5) $\neg(P \lor Q) \equiv \neg P \land \neg Q$
6) $\neg(P \land Q) \equiv \neg P \lor \neg Q$
7) $P \land Q \equiv \neg(\neg P \lor \neg Q)$
8) $P \lor Q \equiv \neg(\neg P \land \neg Q)$

Equality

The **equality symbol to signify that two terms** refer to the same object.

For example,

Father (John)=Henry

says that the object referred to by Father (John) and the object referred to by Henry are the same.

- It can also be used with negation to insist that two terms are not the same object.

Equality

- To say that Richard has at least two brothers, we would write

$\exists$ x, y Brother (x,Richard) $\wedge$ Brother (y,Richard) $\wedge \neg$(x=y) .

Equality

Equality

$term_1 = term_2$ is true under a given interpretation
if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ and $\forall x \;\; \times(Sqrt(x), Sqrt(x)) = x$ are satisfiable
$2 = 2$ is valid

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow\; [\neg(x = y) \land \exists m, f \;\; \neg(m = f) \land Parent(m, x) \land Parent(f, x) \land Parent(m, y) \land Parent(f, y)]$$