UCS1524 – Logic Programming

Fundamentals of Propositional Logic

Session Meta Data

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Session Objectives

- Understanding syntax and semantics of propositional logic (PL)
- Learning formal representation of PL, validity, satisfiability and semantic equivalence in PL.

Session Outcomes

- At the end of this session, participants will be able to
  - explain the formal representation of statements in PL and how to check validity of the statements in PL.

Agenda

- Syntax and Semantics of PL
- Validity and satisfiability
- Semantic equivalence

What is logic?

- The philosophical definition is that logic is a description of how one should think.
- In the context of AI, logic is "formal," which means it resembles math in its clarity and lack of ambiguity.
- According to Merriam-Webster dictionary, logic is a science that deals with the principles and criteria of validity of inference and demonstration
- E.g.
  - The arrangement of circuit elements (as in a computer) needed for computation
  - Parsing both natural language and programming language statements

Formal Logic

Formal Logic consists of

- syntax – what is a legal sentence in the logic
- semantics – what is the meaning of a sentence in the logic
- proof theory – formal (syntactic) procedure to construct valid/true sentences

Formal logic provides

- a language to precisely express knowledge, requirements, facts
- a formal way to reason about consequences of given facts rigorously

Where is Formal Logic used in SE?

Programming Languages

- conditional statements
- meaning (semantics) of programs

Requirements and Specification

- rigorous definition of what is to be constructed
- e.g., if we used formal logic for assignments, there would be no questions on what is required, what is optional, and no questions

Computer Hardware

- computers are build out of simple logical gates
- most computer hardware can be specified and understood in propositional logic

Testing and Verification

- rigorously validate that software satisfies its specifications

Algorithms and Optimization

- many complex problems can be reduced to logic and solved effectively using automated decision procedures

Propositional Logic (or Boolean Logic)

Explores simple grammatical connections such as *and*, *or*, and *not* between simplest “atomic sentences”

A = “Paris is the capital of France”

B = “mice chase elephants”

The subject of propositional logic is to declare formally the truth of complex structures from the truth of individual atomic components

A and B

A or B

if A then B

Syntax and Semantics

Syntax

- Rules used to express facts and concepts.
- The way in which linguistic elements (such as words) are put together to form constituents (such as phrases or clauses)
- Determines and restricts how things are written

Semantics

- The study of meanings
- Determine the truth value of the logic formula.
- Determines how syntax is interpreted to give meaning

Syntax of Propositional Logic

An *atomic formula* has a form A_i , where i = 1, 2, 3 …

Formulas are defined inductively as follows:

- All atomic formulas are formulas
- For every formula F, ¬F (called not F) is a formula
- For all formulas F and G, F ∧ G (called and) and F ∨ G (called or) are formulas

Abbreviations

- use A, B, C, … instead of A_1, A_2, …
- use $F_1 \rightarrow F_2$ instead of $\neg F_1 \vee F_2$ (implication)
- use $F_1 \leftrightarrow F_2$ instead of $(F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1)$ (iff)

Syntax of Propositional Logic (PL)

$$
\begin{aligned}
\text{truth\_symbol} &::= \top(\text{true}) \mid \bot(\text{false}) \\
\text{variable} &::= p, q, r, \ldots \\
\text{atom} &::= \text{truth\_symbol} \mid \text{variable} \\
\text{literal} &::= \text{atom} \mid \neg\text{atom} \\
\text{formula} &::= \text{literal} \mid \\
&\quad\ \neg\text{formula} \mid && \text{negation} \\
&\quad\ \text{formula} \wedge \text{formula} \mid && \text{conjunction} \\
&\quad\ \text{formula} \vee \text{formula} \mid && \text{disjunction} \\
&\quad\ \text{formula} \rightarrow \text{formula} \mid && \text{implies} \\
&\quad\ \text{formula} \leftrightarrow \text{formula} && \text{equivalence}
\end{aligned}
$$

Example

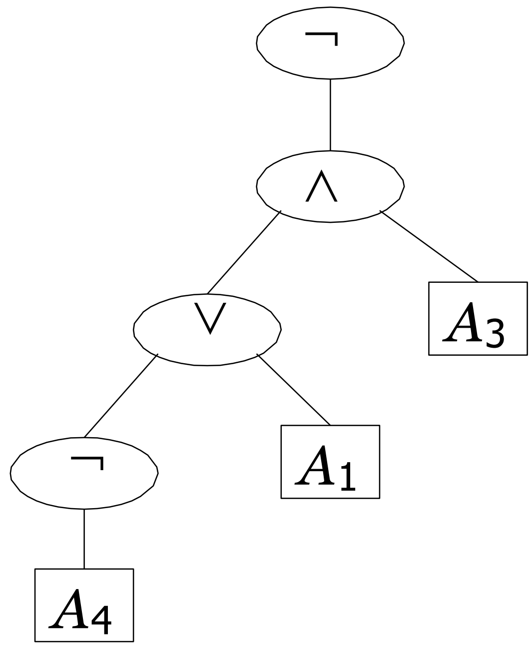
$$F = \neg((A_5 \wedge A_6) \vee \neg A_3)$$

Sub-formulas are

$$\begin{aligned} F, ((A_5 \wedge A_6) \vee \neg A_3), \\ A_5 \wedge A_6, \neg A_3, \\ A_5, A_6, A_3 \end{aligned}$$

Syntax tree of a formula

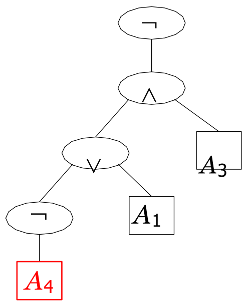
Every formula can be represented by a syntax tree. Example: $F = \neg((\neg A_4 \vee A_1) \wedge A_3)$



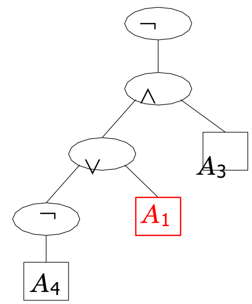
Subformulas

The subformulas of a formula are the formulas corresponding to the subtrees of its syntax tree.

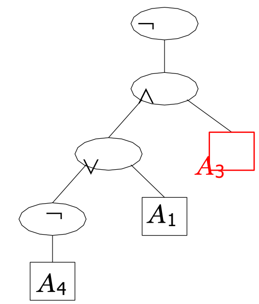
A_4



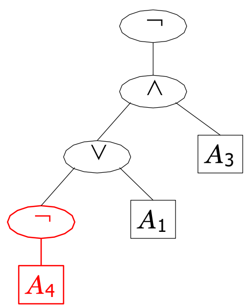
A_1



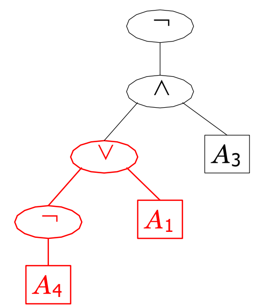
A_3



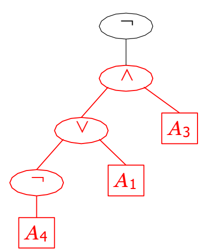
$\neg A_4$



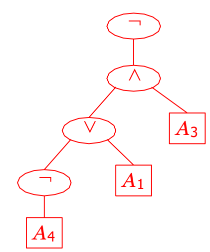
$(\neg A_4 \vee A_1)$



$((\neg A_4 \vee A_1) \wedge A_3)$



$\neg((\neg A_4 \vee A_1) \wedge A_3)$



Semantics of propositional logic

We start with two truth values: {0, 1}

- 0 stands for False, and 1 stands for True

Let **D** be any subset of the *atomic* formulas

An *assignment* **A** is a map **D** → {0, 1}

- **A** assigns True or False to every atomic in **D**

Let **E** ⊇ **D** be set of formulas built from **D** using propositional connectives

Extended assignment **A**': **E** → {0, 1} extends **A** from atomic formulas to all formulas

Semantics of propositional logic

For an atomic formula A_i in **D**: **A’**$(A_i) = $ **A**(A_i)

A’(F ∧ G)	= 1	if **A’**(F) = 1 and A’(G) = 1
	= 0	otherwise
A’(F ∨ G)	= 1	if **A’**(F) = 1 or A’(G) = 1
	= 0	otherwise
A’(¬F)	= 1	if **A’**(F) = 0
	= 0	otherwise

Exercise: Define Extended Assignment

$$F = \neg(A \wedge B) \vee C$$

$$\mathcal{A}(A) = 1$$
$$\mathcal{A}(B) = 1$$
$$\mathcal{A}(C) = 0$$

Is F true or false under A'?

Truth Tables for Basic Operators

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}'(F \wedge G)$
0	0	0
0	1	0
1	0	0
1	1	1

$\mathcal{A}(F)$	$\mathcal{A}(G)$	$\mathcal{A}'(F \vee G)$
0	0	0
0	1	1
1	0	1
1	1	1

An extended assignment A' extends the truth table from atomic propositions to propositional formulas

$\mathcal{A}(F)$	$\mathcal{A}'(\neg F)$
0	1
1	0

Formula

$$F = \neg(A \wedge B) \vee C$$

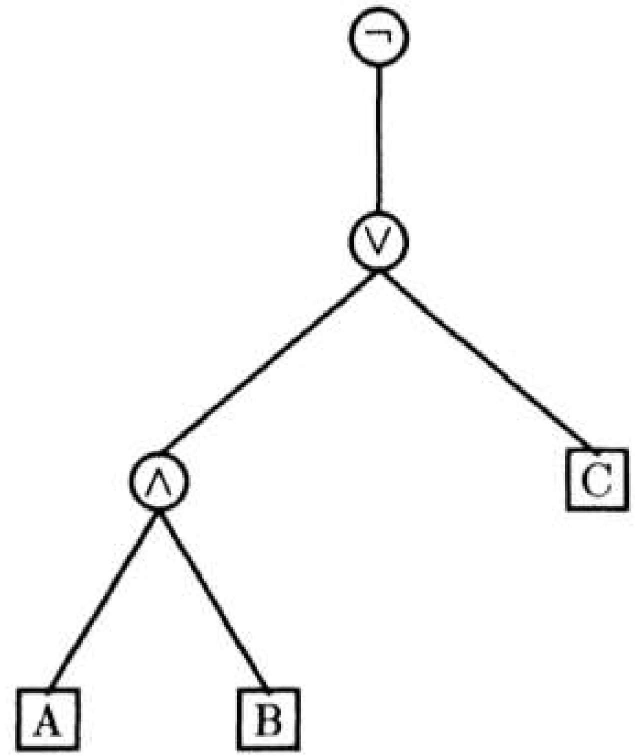
Assignment

$$\mathcal{A}(A) = 1$$

$$\mathcal{A}(B) = 1$$

$$\mathcal{A}(C) = 0$$

Abstract Syntax Tree (AST)



Abbreviations

$$
\begin{array}{rcl}
A, B, C, & & \\
P, Q, R, \text{ or } \ldots & \text{for} & A_1, A_2, A_3 \ldots \\
 & & \\
(F_1 \to F_2) & \text{for} & (\neg F_1 \vee F_2) \\
(F_1 \leftrightarrow F_2) & \text{for} & ((F_1 \wedge F_2) \vee (\neg F_1 \wedge \neg F_2)) \\
(\bigvee_{i=1}^{n} F_i) & \text{for} & (\ldots((F_1 \vee F_2) \vee F_3) \vee \ldots \vee F_n) \\
(\bigwedge_{i=1}^{n} F_i) & \text{for} & (\ldots((F_1 \wedge F_2) \wedge F_3) \wedge \ldots \wedge F_n) \\
\top \text{ or } \mathtt{true} \text{ or } 1 & \text{for} & (A_1 \vee \neg A_1) \\
\bot \text{ or } \mathtt{false} \text{ or } 0 & \text{for} & (A_1 \wedge \neg A_1)
\end{array}
$$

Truth tables

Tables for the operators $\rightarrow$, $\leftrightarrow$:

A	B	A	$\rightarrow$	B
0	0	0	1	0
0	1	0	1	1
1	0	1	0	0
1	1	1	1	1

Name: *implication*

Interpretation: If A holds, then B holds.

A	B	A	$\leftrightarrow$	B
0	0	0	1	0
0	1	0	0	1
1	0	1	0	0
1	1	1	1	1

Name: *equivalence*

Interpretation: A holds if and only if B holds.

Formalizing natural language

A device consists of two parts A and B, and a red light. We know that:

- A or B (or both) are broken.
- If A is broken, then B is broken.
- If B is broken and the red light is on, then A is not broken.
- The red light is on.

We use the atomic formulas: Ro (red light on), Ab (A is broken), Bb (B is broken), and formalize this situation by means of the formula

$$((((Ab \vee Bb) \wedge (Ab \rightarrow Bb)) \wedge ((Bb \wedge Ro) \rightarrow \neg Ab))) \wedge Ro)$$

Formalizing natural language

Full truth table:

Ro	Ab	Bb	$((((Ab \vee Bb) \wedge (Ab \rightarrow Bb))) \wedge ((Bb \wedge Ro) \rightarrow \neg Ab)) \wedge Ro)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

Propositional Logic: Semantics

An assignment A is *suitable* for a formula F if A assigns a truth value to every atomic proposition of F

An assignment A is a *model* for F, written A⊨ F, iff

- A is suitable for F
- A’(F) = 1, i.e., F *evaluates to true (or holds)* under A

A formula F is *satisfiable* iff F has a model, otherwise F is *unsatisfiable*
(or contradictory)

A formula F is *valid* (or a tautology), written ⊨ F, iff every suitable assignment for F is a model for F

Determining Satisfiability via a Truth Table

A formula F with n atomic sub-formulas has 2^n suitable assignments

Build a truth table enumerating all assignments

F is satisfiable iff there is at least one entry with 1 in the output

$$
\begin{array}{r|ccccc|c}
 & A_1 & A_2 & \cdots & A_{n-1} & A_n & F \\
\hline
\mathcal{A}_1: & 0 & 0 & & 0 & 0 & \mathcal{A}_1(F) \\
\mathcal{A}_2: & 0 & 0 & & 0 & 1 & \mathcal{A}_2(F) \\
\vdots & & & \ddots & & & \vdots \\
\mathcal{A}_{2^n}: & 1 & 1 & & 1 & 1 & \mathcal{A}_{2^n}(F)
\end{array}
$$

Problem: Is formula F SAT?

$$F = (\neg A \rightarrow (A \rightarrow B))$$

A	B	$\neg A$	$(A \rightarrow B)$	F
0	0	1	1	1
0	1	1	1	1
1	0	0	0	1
1	1	0	1	1

Validity and Unsatisfiability

Theorem:

A formula F is valid if and only if ¬F is unsatifsiable

Proof:

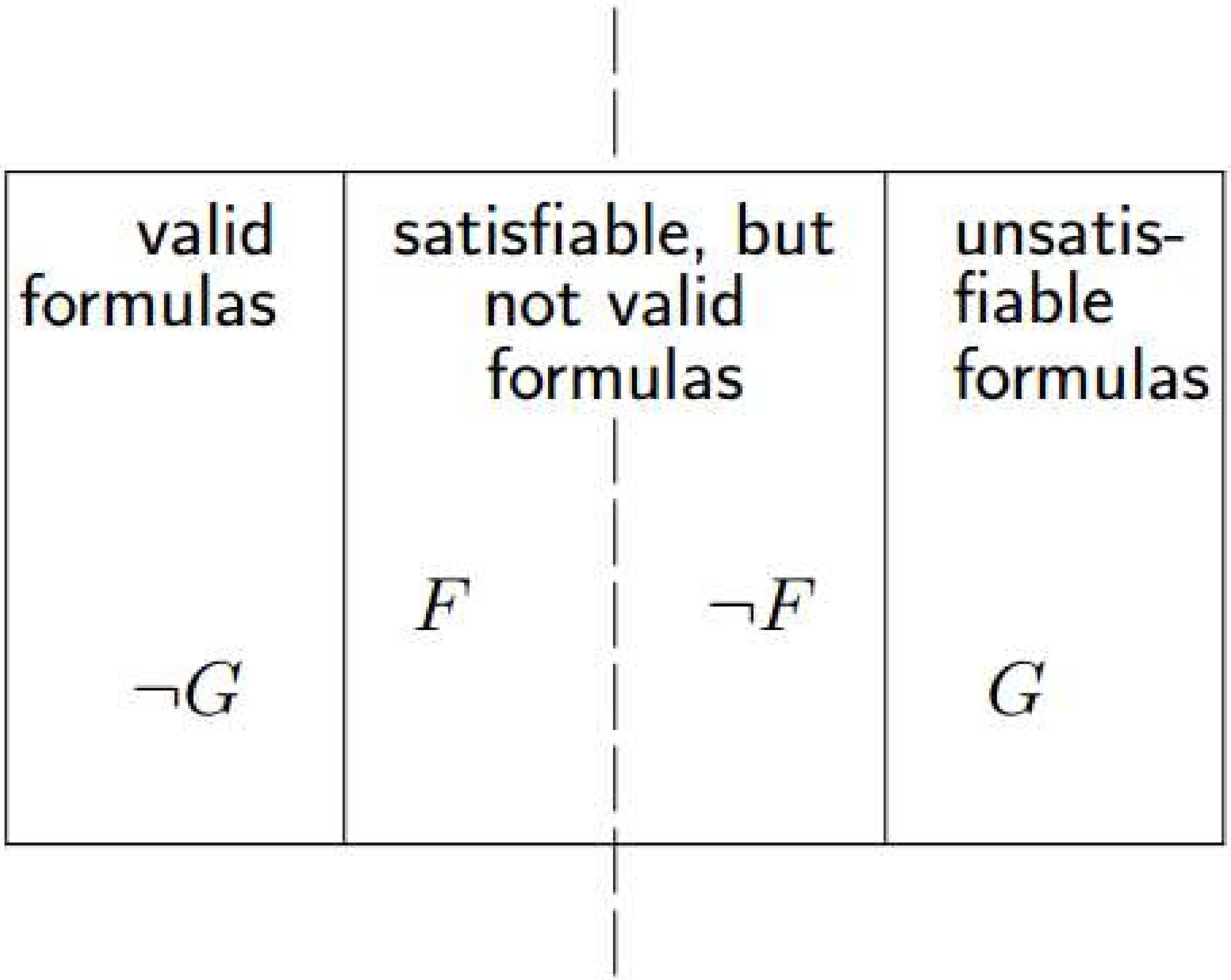
F is valid ⇔ every suitable assignment for F is a model for F

⇔ every suitable assignment for ¬ F is not a model for ¬ F

⇔ ¬ F does not have a model

⇔ ¬ F is unsatisfiable

Mirroring principle



Exercise

Prove or give a counter example

Valid

(a) If $(F \Rightarrow G)$ is *valid* and F is *valid*, then G is *valid*

(b) If $(F \Rightarrow G)$ is *sat* and F is *sat*, then G is *sat*

Not Valid

(c) If $(F \Rightarrow G)$ is *valid* and F is *sat*, then G is *sat*

Valid

Semantic Equivalence

Two formulas F and G are *(semantically) equivalent*, written $F \equiv G$, iff for every assignment A that is suitable for both F and G, $A'(F) = A'(G)$

For example, $(F \wedge G)$ is equivalent to $(G \wedge F)$

Formulas with different atomic propositions can be equivalent

- e.g., all tautologies are equivalent to True
- e.g., all unsatisfiable formulas are equivalent to False

Substitution Theorem

Theorem: Let F and G be equivalent formulas. Let H be a formula in which F occurs as a sub-formula. Let H' be a formula obtained from H by replacing every occurrence of F by G. Then, H and H' are equivalent.

Proof:

(Let's talk about proof by induction first…)

Mathematical Induction (over Natural Numbers)

To proof that a property P(n) holds for all natural numbers n

1. Show that P(0) is true
2. Assume that P(k) is true for some natural number k
   - This assumption is called an **Inductive Hypothesis (IH)**
3. Show that P(k+1) is true using IH from the previous step
4. Conclude that P(n) holds for all natural numbers n
   - P(n) is proven (or established, true) by mathematical induction

Example: Mathematical Induction

Show by induction that the formula for arithmetic series is correct

$$0 + \cdots + n = \frac{n(n+1)}{2}$$

Base Case: P(0) is $0 = \frac{0(0+1)}{2}$

IH: Assume P(k), show P(k+1)

$$
\begin{aligned}
& 0 + \cdots + k + (k+1) \\
= \; & \frac{k(k+1)}{2} + (k+1) \\
= \; & \frac{k(k+1)+2(k+1)}{2} \\
= \; & \frac{(k+1)((k+1)+1)}{2}
\end{aligned}
$$

IH is used in this step

Structural Induction on the formula structure

The definition of a syntax of a formula is an *inductive* definition

- first, define atomic formulas; second, define more complex formulas from simple ones, each next definition uses previous definition recursively

The definition of the semantics of a formula is also inductive

- first, determine value of atomic propositions; second, define values of more complex formulas

The same principle works for proving properties of formulas!

- To show that every formula F satisfies some property S:
- (base case) show that S holds for atomic formulae
- (induction step) assume S holds for an arbitrary fixed formulas F and G. Show that S holds for $(F \wedge G)$, $(F \vee G)$, and $(\neg F)$

Substitution Theorem

Theorem: Let F and G be equivalent formulas. Let H be a formula in which F occurs as a sub-formula. Let H' be a formula obtained from H by replacing every occurrence of F by G. Then, H and H' are equivalent.

Proof: by induction on formula structure

(base case) if H is atomic, then F = H, H' = G, and $F \equiv G$

(inductive step)

(case 1) $H = \neg H_1$

(case 2) $H = H_1 \wedge H_2$

(case 3) $H = H_1 \vee H_2$

Useful Equivalences

$$
\begin{array}{rcll}
(F \wedge F) & \equiv & F & \\
(F \vee F) & \equiv & F & \text{(Idempotency)} \\
(F \wedge G) & \equiv & (G \wedge F) & \\
(F \vee G) & \equiv & (G \vee F) & \text{(Commutativity)} \\
((F \wedge G) \wedge H) & \equiv & (F \wedge (G \wedge H)) & \\
((F \vee G) \vee H) & \equiv & (F \vee (G \vee H)) & \text{(Associativity)} \\
(F \wedge (F \vee G)) & \equiv & F & \\
(F \vee (F \wedge G)) & \equiv & F & \text{(Absorption)} \\
(F \wedge (G \vee H)) & \equiv & ((F \wedge G) \vee (F \wedge H)) & \\
(F \vee (G \wedge H)) & \equiv & ((F \vee G) \wedge (F \vee H)) & \text{(Distributivity)} \\
\neg\neg F & \equiv & F & \text{(Double Negation)}
\end{array}
$$

Useful Equivalences

$$
\begin{aligned}
\neg(F \wedge G) &\equiv (\neg F \vee \neg G) \\
\neg(F \vee G) &\equiv (\neg F \wedge \neg G) && \text{(deMorgan's Laws)}
\end{aligned}
$$

$$
\begin{aligned}
(F \vee G) &\equiv F, \text{ if } F \text{ is a tautology} \\
(F \wedge G) &\equiv G, \text{ if } F \text{ is a tautology} && \text{(Tautology Laws)}
\end{aligned}
$$

$$
\begin{aligned}
(F \vee G) &\equiv G, \text{ if } F \text{ is unsatisfiable} \\
(F \wedge G) &\equiv F, \text{ if } F \text{ is unsatisfiable} && \text{(Unsatisfiability Laws)}
\end{aligned}
$$

We can prove them using structural induction!

Equivalence

- Prove that using substituion theorm
  ((A V (B V C)) ^ (C V ⌐A)) = ((B ^ ⌐ A) V C)

$$
\begin{aligned}
&((A \vee (B \vee C)) \wedge (C \vee \neg A)) \\
&\quad \equiv \quad (((A \vee B) \vee C) \wedge (C \vee \neg A)) && \text{(Associativity and ST)} \\
&\quad \equiv \quad ((C \vee (A \vee B)) \wedge (C \vee \neg A)) && \text{(Commutativity and ST)} \\
&\quad \equiv \quad (C \vee ((A \vee B) \wedge \neg A)) && \text{(Distributivity)} \\
&\quad \equiv \quad (C \vee (\neg A \wedge (A \vee B))) && \text{(Commutativity und ST)} \\
&\quad \equiv \quad (C \vee ((\neg A \wedge A) \vee (\neg A \wedge B))) && \text{(Distributivity and ST)} \\
&\quad \equiv \quad (C \vee (\neg A \wedge B)) && \text{(Unsatisfiability Law and ST)} \\
&\quad \equiv \quad (C \vee (B \wedge \neg A)) && \text{(Commutativity and ST)} \\
&\quad \equiv \quad ((B \wedge \neg A) \vee C) && \text{(Commutativity)}
\end{aligned}
$$

Exercise : Children and Doctors

Formalize and show that the two statements are equivalent

- If the child has temperature or has a bad cough and we reach the doctor, then we call him

$$((T \vee C) \wedge R) \Rightarrow D$$

- If the child has temperature, then we call the doctor, provided we reach him, and, if we reach the doctor then we call him, if the child has a bad cough

$$(R \Rightarrow (T \Rightarrow D)) \wedge (C \Rightarrow (R \Rightarrow D))$$

$((T \vee C) \wedge R) \Rightarrow D$

$((T \wedge R) \vee (C \wedge R) \Rightarrow D)$

$((T \wedge R) \Rightarrow D) \wedge ((C \wedge R) \Rightarrow D)$

$(\neg T \vee \neg R \vee D) \wedge (\neg C \vee \neg R \vee D)$

$(\neg R \vee \neg T \vee D) \wedge (\neg C \vee \neg R \vee D)$

$(\neg R \vee (T \Rightarrow D) \wedge (\neg C \vee (R \Rightarrow D)$

$(R \Rightarrow (T \Rightarrow D)) \wedge (C \Rightarrow (R \Rightarrow D))$

Summary

- What is propositional logic (PL)?
- Syntax of PL
- Semantics of PL
- Validity and satisfibility
- Semantic equivalence in PL

Check your understanding

	Valid	Satisfiable	Unsatisfiable
A			
$A \vee B$			
$A \vee \neg A$			
$A \wedge \neg A$			
$A \rightarrow \neg A$			
$A \rightarrow B$			
$A \rightarrow (B \rightarrow A)$			
$A \rightarrow (A \rightarrow B)$			
$A \leftrightarrow \neg A$			

Check your understanding

Which of the following statements are true?

				Y/N
If	F is valid,	then	F is satisfiable	
If	F is satisfiable,	then	$\neg F$ is satisfiable	
If	F is valid,	then	$\neg F$ is satisfiable	
If	F is unsatisfiable,	dann	$\neg F$ is valid	