3D Transformations

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Q. $P = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ $\quad t_x, t_y, t_z = -1$

$\alpha = 30°, \ \beta = 45°$

$M = R_y(45°) \cdot R_x(30°) \cdot T(-1,-1,-1)$

$$= \begin{bmatrix} c45° & 0 & s45° & 0 \\ 0 & 1 & 0 & 0 \\ -s45° & 0 & c45° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c45° & -s45° & 0 \\ 0 & s45° & c45° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c45° & 0 & s45° & 0 \\ 0 & 1 & 0 & 0 \\ -s45° & 0 & c45° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & c45° & -s45° & -c45° + s45° \\ 0 & s45° & c45° & -s45° - c45° \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\to 0$

$-s45° - c45° \to -\sqrt{2}$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 & \frac{-1}{\sqrt{2}} - 1 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/2 & 1/2 & \frac{1}{\sqrt{2}} - 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note:

2-D Rotation = z-axis rotation in 3D

$$\begin{array}{c|c} & \begin{array}{ccc} x & y & \end{array} \\ \begin{array}{c} x \\ y \\ \end{array} & \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \Leftrightarrow \begin{array}{c|c} & \begin{array}{cccc} x & y & z & \end{array} \\ \begin{array}{c} x \\ y \\ z \\ \end{array} & \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

∴ 3-D Rotation about

x-axis:

$$\begin{array}{c|c} & \begin{array}{cccc} x\,(z) & y\,(x) & z\,(y) & \end{array} \\ \begin{array}{c} x\,(z) \\ y\,(x) \\ z\,(y) \\ \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta & -s\theta & 0 \\ 0 & s\theta & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

y-axis:

$$\begin{array}{c|c} & \begin{array}{cccc} x\,(y) & y\,(z) & z\,(x) & \end{array} \\ \begin{array}{c} x\,(y) \\ y\,(z) \\ z\,(x) \\ \end{array} & \begin{bmatrix} c\theta & 0 & s\theta & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta & 0 & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

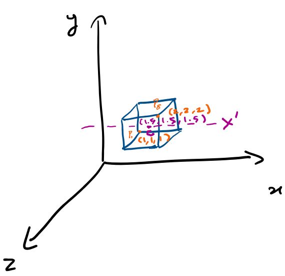
$\Rightarrow P' = M \cdot P$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/2 & 1/2 & \left(-\frac{1}{\sqrt{2}} - 1\right) \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/2 & 1/2 & \left(\frac{1}{\sqrt{2}} - 1\right) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} + \cancel{1} + \frac{1}{2} - \frac{1}{\sqrt{2}} - \cancel{1} \\ \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} + \cancel{1} + \frac{1}{2} + \frac{1}{\sqrt{2}} - \cancel{1} \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} + 1/2 \\ 1/\sqrt{2} \\ -\sqrt{2} + 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.914 \\ 0.707 \\ -0.914 \\ 1 \end{bmatrix}$$

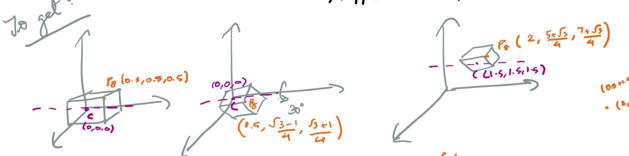
$\therefore P'\left(\frac{1 + 2\sqrt{2}}{2}, \frac{1}{\sqrt{2}}, \frac{1 - 2\sqrt{2}}{2} \right)$

Q.



Rotate by 30° about X';
↳ passes through centroid.
↳ X' ∥ x-axis.

To get:



$\left(0.5 + 1.5, \frac{\sqrt{3}-1}{4} + \frac{3}{2}, \frac{\sqrt{3}+1}{4} + \frac{3}{2}\right)$
$= \left(2, \frac{5+\sqrt{3}}{4}, \frac{7+\sqrt{3}}{4}\right)$

$y' = y\cos 30° - z \sin 30° = \frac{\sqrt{3}}{2} \cdot 0.5 - \frac{1}{2} \cdot 0.5 = \frac{\sqrt{3}-1}{4}$
$z' = y \sin 30° + z \cos 30° = \frac{1}{2} \cdot 0.5 + \frac{\sqrt{3}}{2} \cdot 0.5 = \frac{\sqrt{3}+1}{4}$

C (1.5, 1.5, 1.5)

Apply transformation:

$T(t_x, t_y, t_z) \cdot R_x(\theta) \cdot T(-t_x, t_y, t_z)$

[Translating centroid; rotation by 30°]

$\Rightarrow M = T(1.5, 1.5, 1.5) \cdot R_x(30°) \cdot T(1.5, 1.5, 1.5)$

For each vertex P of the cube, $P' = M \cdot P$

Check:

$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\sqrt{3}}{4} - \frac{1}{4} \\ \frac{1}{4} + \frac{\sqrt{3}}{4} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\sqrt{3}-1}{4} \\ \frac{\sqrt{3}+1}{4} \\ 1 \end{bmatrix}$ ✓

$\begin{bmatrix} 0.5 \\ (\sqrt{3}-1)/4 \\ (\sqrt{3}+1)/4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 + 1.5 \\ (\sqrt{3}-1)/4 + 1.5 \\ (\sqrt{3}+1)/4 + 1.5 \\ 1 \end{bmatrix}$ ✓

$M = \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1.5 \\ & & & -1.5 \end{bmatrix}$

$$\begin{bmatrix}0.5\\0.5\\0.5\\1\end{bmatrix} \quad \begin{bmatrix}0.5\\(\sqrt{3}-1)/4\\(\sqrt{3}+1)/4\\1\end{bmatrix} = \begin{bmatrix}0.5+1.5\\(\sqrt{3}-1)/4+1.5\\(\sqrt{3}+1)/4+1.5\\1\end{bmatrix} \checkmark$$

$$M = \begin{bmatrix}1&0&0&1.5\\0&1&0&1.5\\0&0&1&1.5\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&c30^\circ&-s30^\circ&0\\0&s30^\circ&c30^\circ&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-1.5\\0&1&0&-1.5\\0&0&1&-1.5\\0&0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&1.5\\0&1&0&1.5\\0&0&1&1.5\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&-1.5\\0&c30^\circ&-s30^\circ&-1.5(c30^\circ-s30^\circ)\\0&s30^\circ&c30^\circ&-1.5(s30^\circ+c30^\circ)\\0&0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&\cancel{-1.5+1.5}\ 0\\0&c30^\circ&-s30^\circ&-1.5(c30^\circ-s30^\circ)+1.5\\0&s30^\circ&c30^\circ&-1.5(s30^\circ+c30^\circ)+1.5\\0&0&0&1\end{bmatrix}$$

$c30^\circ = \frac{\sqrt{3}}{2}$, $s30^\circ = \frac{1}{2}$

$$\frac{3}{2}\left(1-\frac{\sqrt{3}-1}{2}\right) = \frac{3}{2}\left(\frac{2-\sqrt{3}+1}{2}\right) = \frac{3(3-\sqrt{3})}{4} = \frac{3\sqrt{3}(\sqrt{3}-1)}{4} = \frac{9-3\sqrt{3}}{4}$$

$$\frac{3}{2}\left(1-\frac{\sqrt{3}+1}{2}\right) = \frac{3}{2}\left(\frac{2-\sqrt{3}-1}{2}\right) = \frac{3(1-\sqrt{3})}{4} = \frac{3-3\sqrt{3}}{4}$$

$$= \begin{bmatrix}1&0&0&0\\0&\sqrt{3}/2&-1/2&3\sqrt{3}(\sqrt{3}-1)/4\\0&1/2&\sqrt{3}/2&3(1-\sqrt{3})/4\\0&0&0&1\end{bmatrix}$$

$3\sqrt{3}(\sqrt{3}-1)/4 \rightarrow A$, $3(1-\sqrt{3})/4 \rightarrow B$

$A = \frac{9-3\sqrt{3}}{4}$

$B = \frac{3-3\sqrt{3}}{4}$

Note the format:

$y_f\left[1-(c30^\circ - s30^\circ)\right]$

$z_f\left[1-(s30^\circ + c30^\circ)\right]$

$$\therefore M = \begin{bmatrix}1&0&0&0\\0&c\theta&-s\theta&y_f[1-(c\theta-s\theta)]\\0&s\theta&c\theta&z_f[1-(c\theta+s\theta)]\\0&0&0&1\end{bmatrix}$$

for rotation about an axis $\parallel^{l}$ to x-axis where (y_f, z_f) lies on the axis.

Check for centroid:

$$\begin{bmatrix}1.5\\1.5\\1.5\\1\end{bmatrix} \rightarrow \begin{bmatrix}1.5\\ \frac{3\sqrt{3}}{4}-\frac{3}{4}+\frac{9-3\sqrt{3}}{4} = \frac{6}{4} = 1.5\\ \frac{3}{4}+\frac{3\sqrt{3}}{4}+\frac{3-3\sqrt{3}}{4} = \frac{6}{4} = 1.5\\1\end{bmatrix} \checkmark$$

Computing transformed points,

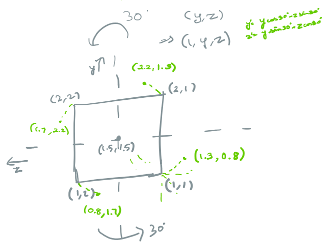
- $P_1' = M.P_1$

$$= \begin{bmatrix}1&0&0&0\\0&\sqrt{3}/2&-1/2&A\\0&1/2&\sqrt{3}/2&B\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{\sqrt{3}-1}{2}+\frac{9-3\sqrt{3}}{4}\\ \frac{\sqrt{3}+1}{2}+\frac{3-3\sqrt{3}}{4}\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{7-\sqrt{3}}{4}\\ \frac{5-\sqrt{3}}{4}\\1\end{bmatrix}$$

$$\sqrt{3}-1+\frac{9-3\sqrt{3}}{4} \qquad \frac{4\sqrt{3}-4+9-3\sqrt{3}}{4} = \frac{5+\sqrt{3}}{4} \checkmark$$

$$\sqrt{3}+1+\frac{3-3\sqrt{3}}{4} \qquad \frac{4\sqrt{3}+4+3-3\sqrt{3}}{4} = \frac{7+\sqrt{3}}{4} \checkmark$$

Understanding the face view of the rotation.



- $P_2' = M.P_2$

$$= \begin{bmatrix}1&0&0&0\\0&\sqrt{3}/2&-1/2&A\\0&1/2&\sqrt{3}/2&B\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\2\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{\sqrt{3}-2}{2}+\frac{9-3\sqrt{3}}{4}\\ \frac{1+2\sqrt{3}}{2}+\frac{3-3\sqrt{3}}{4}\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{5-\sqrt{3}}{4}\\ \frac{5+\sqrt{3}}{4}\\1\end{bmatrix}$$

- $P_3' = M.P_3$

$$= \begin{bmatrix}1&0&0&0\\0&\sqrt{3}/2&-1/2&A\\0&1/2&\sqrt{3}/2&B\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\1\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{2\sqrt{3}-1}{2}+\frac{9-3\sqrt{3}}{4}\\ \frac{2+\sqrt{3}}{2}+\frac{3-3\sqrt{3}}{4}\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{7+\sqrt{3}}{4}\\ \frac{7-\sqrt{3}}{4}\\1\end{bmatrix}$$

- $P_4' = M.P_4$

$$= \begin{bmatrix}1&0&0&0\\0&\sqrt{3}/2&-1/2&A\\0&1/2&\sqrt{3}/2&B\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\2\\1\end{bmatrix} = \begin{bmatrix}1\\ (\sqrt{3}-1)+\frac{9-3\sqrt{3}}{4}\\ (1+\sqrt{3})+\frac{3-3\sqrt{3}}{4}\\1\end{bmatrix} = \begin{bmatrix}1\\ \frac{5+\sqrt{3}}{4}\\ \frac{7+\sqrt{3}}{4}\\1\end{bmatrix}$$

- |||ly,

$$P_5' = \begin{bmatrix}2\\ \frac{7-\sqrt{3}}{4}\\ \frac{5-\sqrt{3}}{4}\\1\end{bmatrix} \quad P_6' = \begin{bmatrix}2\\ \frac{5-\sqrt{3}}{4}\\ \frac{5+\sqrt{3}}{4}\\1\end{bmatrix} \quad P_7' = \begin{bmatrix}2\\ \frac{7+\sqrt{3}}{4}\\ \frac{7-\sqrt{3}}{4}\\1\end{bmatrix} \quad P_8' = \begin{bmatrix}2\\ \frac{5+\sqrt{3}}{4}\\ \frac{7+\sqrt{3}}{4}\\1\end{bmatrix}$$

Notice the pattern!!

$$\begin{bmatrix} \frac{5-\sqrt{3}}{4} \\ 1 \end{bmatrix} \quad \begin{bmatrix} \frac{\;}{4} \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \; \\ 1 \end{bmatrix}$$

Notice the pattern!!

Hence, the vertices of the new cube are:

$P_1\left(1, \frac{7-\sqrt{3}}{4}, \frac{5-\sqrt{3}}{4}\right)$, $P_2\left(1, \frac{5-\sqrt{3}}{4}, \frac{5+\sqrt{3}}{4}\right)$, $P_4\left(1, \frac{5+\sqrt{3}}{4}, \frac{7+\sqrt{3}}{4}\right)$, $P_3\left(1, \frac{7+\sqrt{3}}{4}, \frac{7-\sqrt{3}}{4}\right)$,

$P_5\left(2, \frac{7-\sqrt{3}}{4}, \frac{5-\sqrt{3}}{4}\right)$, $P_6\left(2, \frac{5-\sqrt{3}}{4}, \frac{5+\sqrt{3}}{4}\right)$, $P_8\left(2, \frac{5+\sqrt{3}}{4}, \frac{7+\sqrt{3}}{4}\right)$, $P_7\left(2, \frac{7+\sqrt{3}}{4}, \frac{7-\sqrt{3}}{4}\right)$.