SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

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✓The matrix notation for following linear system of equation is as follow:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

The above linear system is expressed in the matrix form as $A \cdot X = B$

Elementary Transformation or Operations on a Matrix

Operation	Meaning
R_{ij} or $R_i \leftrightarrow R_j$	Interchange of i^{th} and j^{th} rows
$k \cdot R_i$	Multiplication of all the elements of i^{th} row by non zero scalar k.
$R_{ij}(k)$ or $R_j + k \cdot R_i$	Multiplication of all the elements of i^{th} row by nonzero scalar k and added into j^{th} row.

Row Echelon Form of Matrix

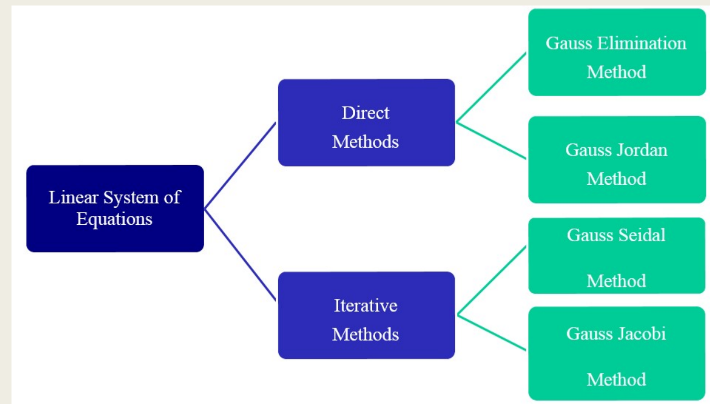
✓To convert the matrix into row echelon form follow the following steps:

1. Every zero row of the matrix occurs below the non zero rows.
2. Arrange all the rows in strictly decreasing order.
3. Make all the entries zero below the leading (first non zero entry of the row) element of 1st row.
4. Repeat step-3 for each row.

Reduced Row Echelon of Matrix

✓To convert the matrix into reduced row echelon form follow the following steps:

1. Convert given matrix into row echelon form.
2. Make all leading elements 1(one).
3. Make all the entries zero above the leading element 1(one) of each row.



Numerical Methods for Solution of a Linear Equation

1. **Direct Methods**
2. **Iterative Methods**

✓ **Direct Methods**

- This method produce the exact solution after a finite number of steps but are subject to errors due to round-off and other factors.
- We will discuss two direct methods :

1.Gauss Elimination method
2.Gauss-Jordan method

✓ **Indirect Method(Iterative Method)**

- In this method, an approximation to the true solution is assumed initially to start method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off error are also smaller.

1.Gauss seidel method

2.Gauss jacobi method

Gauss-Jordan Method

- ✓ This method is modification of the gauss elimination method. This method solves a given system of equation by transforming the coefficient matrix into unit matrix.
- ✓ Steps to solve Gauss-Jordan method:
  1. Write the matrix form of the system of equations.
  2. Write the augmented matrix.
  3. Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.
  4. Write the corresponding linear system of equations to obtain the solution.

Example :

Q : Solve the following set of equations using Gauss Jordan Method

$x + y + z = 5$
$2x + 3y + 5z = 8$
$4x + 5z = 2$

Solution:

The augmented matrix of the system is the following.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

We will now perform row operations until we obtain a matrix in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

From this final matrix, we can read the solution of the system. It is

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Classwork

Q1: Solve the following set of equations using Gauss Jordan Method

$2x + 4y + 6z = 22$
$3x + 8y + 5z = 27$
$-x + y + 2z = 2$

Classwork

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$2x + 4y + 6z = 22$
$3x + 8y + 5z = 27$
$-x + y + 2z = 2$

Answer: $x = 3, y = 1, z = 2$

Classwork

Q2 : Solve the following set of equations using Gauss Jordan Method

$10x - 7y + 3z + 5u = 6$
$-6x + 8y - z - 4u = 5$
$3x + y + 4z + 11u = 2$
$5x - 9y - 2z + 4u = 7$

Classwork

Q2 : Solve the following set of equations using Gauss Jordan Method

$10x - 7y + 3z + 5u = 6$
$-6x + 8y - z - 4u = 5$
$3x + y + 4z + 11u = 2$
$5x - 9y - 2z + 4u = 7$

Answer: $x = 5, y = 4, z = -7, u = 1$

Gauss Jacobi Method

- This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.
- Consider the system of equations.

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

- Where co-efficient matrix $\boldsymbol{A}$ must be diagonally dominant,

$|a_1| \geq |b_1| + |c_1|$

$|b_2| \geq |a_2| + |c_2|$

$|c_3| \geq |a_3| + |b_3| \ldots\ldots(1)$

And the inequality is strictly greater than for at least one row.

Solving the system (1) for x, y, z respectively , we obtain

$$\mathrm{x} = \frac{1}{\mathrm{a}_1}(\mathrm{d}_1 - \mathrm{b}_1\mathrm{y} - \mathrm{c}_1\mathrm{z})$$

$$\mathrm{y} = \frac{1}{\mathrm{b}_2}(\mathrm{d}_2 - \mathrm{a}_2\mathrm{x} - \mathrm{c}_2\mathrm{z})$$

$$\mathrm{z} = \frac{1}{\mathrm{c}_3}(\mathrm{d}_3 - \mathrm{a}_3\mathrm{x} - \mathrm{b}_3\mathrm{y}) \ldots\ldots (2)$$

- We start with $x_0 = 0, y_0 = 0$ & $z_0 = 0$ in equ. (2)

$$\therefore x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$\therefore y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$\therefore z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Again substituting these value x_1, y_1, z_1 in Eq. (2), the next approximation is obtained.

This process is continued till the values of x, y, z are obtained to desired degree of accuracy.

Solve by Gauss Jacobi method up to three iteration.
$20x + y - 2z = 17, 2x - 3y + 20z = 25, 3x + 20y - z = -18$

Solution:-

$|20| > |1| + |-2|$
$|-3| \ngeq |2| + |20|$

So, It is not diagonally dominant.

We need to rearrange the equations.

$20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$

$|20| > |1| + |-2|$
$|20| > |3| + |-1|$
$|20| > |2| + |-3|$

So, All Equations are diagonally dominant.

(Make subject x, y, z from diagonally dominant equations.)

Here,

$x = \frac{1}{20}(17 - y + 2z)$

$y = \frac{1}{20}(-18 - 3x + z)$

$z = \frac{1}{20}(25 - 3x + 3y)$

Let the initial values are $x = y = z = 0$.

1^{st} iteration,

$$x^1 = \frac{1}{20}(17 - 0 + 0) = 0.85$$

$$y^1 = \frac{1}{20}(-18 - 0 + 0) = 0.9$$

$$z^1 = \frac{1}{20}(25 - 0 + 0) = 1.25$$

$x = \frac{1}{20}(17 - y + 2z)$

$y = \frac{1}{20}(-18 - 3x + z)$

$z = \frac{1}{20}(25 - 3x + 3y)$

Iteration	x	y	z
1	0.85	1.00	1.25
2	1.02	-0.97	1.03
3	1.00	-1.00	1.00

Ans: $(x, y, z) = (1, -1, 1)$

Classwork

Q : Solve the following set of equations using Gauss Jacobi Method

20x + y - 2z = 17
3x + 20y - z = -18
2x – 3y + 20z = 25

Classwork

Q : Solve the following set of equations using Gauss Jacobi Method

20x + y - 2z = 17
3x + 20y - z = -18
2x – 3y + 20z = 25

Answer: x = 1, y = -1, z =1

Inverse of a Matrix

Let A be a square matrix. A square matrix A^{-1} of equal size such that $A^{-1}A = AA^{-1} = I$ is called the *inverse of A*.

Example. Given matrix A, show matrix A^{-1} is its inverse.

$$A = \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 0 & 1/4 \\ 1/2 & 1/8 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1/2 & 1/8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ and

$$A^{-1}A = \begin{bmatrix} 0 & 1/4 \\ 1/2 & 1/8 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Gauss-Jordan Method for Finding the Inverse of a Matrix

Given the square matrix A.

1. Adjoin the identity matrix I (of the same size) to form the augmented matrix: $[A \mid I]$
2. Use row operations to reduce the matrix to the form: $[I \mid B]$ (if possible)

Matrix B is the inverse of A.

Example. Find the inverse of A. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -3 \\ 2 & 1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 3 & -3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{Step 1}$$

$$\xrightarrow[R_3 + (-2)R_1]{R_2 + R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right] \quad \text{Step 2}$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 3 & -1 & -2 & 0 & 1 \end{array}\right] \quad \text{Step 3}$$

$$\xrightarrow[R_3 + (-3)R_2]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2 & -7/2 & -3/2 & 1 \end{array}\right] \quad \text{Step 4}$$

$$\xrightarrow[\frac{1}{2}R_3]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -7/4 & -3/4 & 1/2 \end{array}\right] \quad \text{Step 5}$$

$$\xrightarrow[R_2 + R_3]{R_1 + (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 11/4 & 3/4 & -1/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & 1/2 \\ 0 & 0 & 1 & -7/4 & -3/4 & 1/2 \end{array}\right] \quad \text{Step 6}$$

$$\xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & 1/2 & 0 \\ 0 & 1 & 0 & -5/4 & -1/4 & 1/2 \\ 0 & 0 & 1 & -7/4 & -3/4 & 1/2 \end{array}\right]$$

Step 7

$$\boxed{A^{-1} = \begin{bmatrix} 3/2 & 1/2 & 0 \\ -5/4 & -1/4 & 1/2 \\ -7/4 & -3/4 & 1/2 \end{bmatrix}}$$

Classwork

Q : Calculate the inverse of the following matrix using Gauss Jordan Method

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Classwork

Q : Calculate the inverse of the following matrix using Gauss Jordan Method

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Answer: $\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$