NUMERICAL METHODS

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Introduction

- By numerical methods we mean certain special type of algebraic methods which give approximate solution to the scientific, engineering and mathematical problems by arithmetic operation on numbers.
- They involve a large number of arithmetic calculations and need fast and efficient computing devices like, calculators, computers and micro electronic machines in their applications.

- They are, generally of iterative nature in which certain mathematical formula are repeatedly applied under similar mathematical conditions to generate a sequence of numbers which are successive approximates to the solution of a problem.
- These methods are affected by three types of errors:-
  - ❑ Round-off error
  - ❑ Truncation error
  - ❑ Inherent error due to finite representation of numbers such as 1/3, 3/7, 5/9 etc.

❑ **Round-off error:** This error is due to rounding of a number to a finite number of decimal places. Every computer has a limitation on the no. of digits it can handle and store. For rounding a number to m-th decimal places, 5 is added to the digit at $(m+1)$th decimal place and first m digits of the resulting number are stored.

❑ **Truncation error:** This error occurs when an infinite series, which is used to approximate a function (for example, $\sin x$), is truncated after a finite number of terms.

❑ **Inherent error:** This error occurs due to the approximate mathematical formulation of the problem and unavailability of precise data.

Three sources of errors give rise to following two errors, (a) Local error and (b) Global error frequently used in numerical method.

(a) Local Error- Amount of error introduced at any computational step of the numerical method.

(b) Global Error- Often called Propagation error, Generated due to the accumulation of previous errors in the numerical method.

- ***Absolute error(e)*:** the difference between the computed (x_c) and true (x_t) values of a number x.

$$e = x_c - x_t$$

- ***Relative true error (e_r)*:**

$$e_r = \frac{x_c - x_t}{x_t} \times 100\% = \frac{e}{x_t} \times 100\%$$

- In practice, the true value is not known, so we cannot get the relative true error.

$$e_i = x_i - x_t$$

where e_i is the error in x at iteration i, and x_i is the computed value of x.

$$e_{i+1} = x_{i+1} - x_t$$

- *Relative error*:

$$\Delta e_i = e_{i+1} - e_i = (x_{i+1} - x_t) - (x_i - x_t) = x_{i+1} - x_i$$

Δe_i is used to measure the error.

Numerical Method Vs Numerical Analysis

- **Numerical method** is a special type of method that is used to obtain an approximate solution to a mathematical problem.
- **Numerical analysis** is the application of an appropriate numerical method to a physical problem in a systematic manner to arrive at a solution and to make interpretation of the said solution with the help of the various fast and efficient computing devices like, algorithms, programs, and computers etc.

Solution of Non-linear Equations

- Some simple equations can be solved analytically:

$$x^2 + 4x + 3 = 0$$

$$\text{Analytic solution} \quad roots = \frac{-4 \pm \sqrt{4^2 - 4(1)(3)}}{2(1)}$$

$$x = -1 \quad and \quad x = -3$$

- Many other equations have no analytical solution:

$$\left.\begin{array}{r} x^9 - 2x^2 + 5 = 0 \\ x = e^{-x} \end{array}\right\} \text{No analytic solution}$$

Methods of finding solutions of non-linear equations

- Direct Methods

Direct methods give the roots of non-linear equations in a finite number of steps. In addition, these methods are capable of giving all the roots at the same time.

For e.g., the roots of the quadratic equation

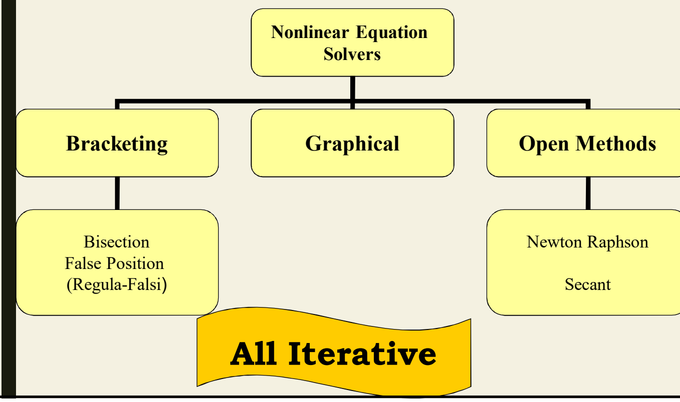
$ax^2+bx+c=0$ where $a \neq 0$

$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- Iterative Methods

Iterative methods also known as trial and error methods, are based on the idea of successive approximations. They start with one or more initial approximations to the root and obtain a sequence of approximations by repeating a fixed sequence of steps till the solution with reasonable accuracy is obtained. Iterative methods, generally, give one root at a time.

Methods of finding solutions of non-linear equations



Choosing initial approximation

- The best way to choose an initial approximation to the root of equation f(x)=0 is either to plot the function f(x) or to tabulate it. The roots of the equation are the points where the curve representing the function f(x) intersects the x-axis. Therefore, any point in the interval where the function changes its sign can be taken as the initial approximation.
- However, if the equation

f(x)=0

Can be written as

$f_1(x) = f_2(x)$

Then the points of intersection of the graphs of equations

$y= f_1(x)$ and $y = f_2(x)$

Give the root of the equation f(x) = 0. Therefore, any value in the neighborhood of this intersection point can be taken as the initial approximation.

Bisection Method

- If $f(x_3) = 0$, then we have a root at x_3.
- if $f(x_1)$ and $f(x_3)$ are of opposite sign, then the root lies in the interval (x_1, x_3). Thus x_2 is replaced by x_3, and the new interval, which is half of the current interval, is again bisected.
- if $f(x_1)$ and $f(x_3)$ are of same sign, then the root lies in the interval (x_3, x_2). Thus x_1 is replaced by x_3, and the new interval, which is half of the current interval, is again bisected.

Bisection Method

Assumptions:

- $f(x)$ is continuous on $[a, b]$
- $f(a)\,f(b) < 0$

Algorithm:

Loop

1. Compute the mid point $c = (a + b)/2$
2. Evaluate f(c)
3. If $f(a)\,f(c) < 0$ then new interval $[a, c]$
   If $f(a)\,f(c) > 0$ then new interval $[c, b]$

End loop

Example: Given that one root of the non-linear equation

$$x^3 - 4x - 9 = 0$$

lies between 2.625 and 2.75. Find the root correct to four significant places.

Iteration 1:

Sol: Starting with x_1 = **2.625** and x_2 = **2.75**

$f(x_1) = f(2.625) = 2.625^3 - 4x2.625-9 =$ **-1.4121**

$f(x_2) = f(2.75) = 2.75^3 - 4x2.75-9 =$ **0.7969**

Find $x_3 = (2.625+2.75)/2 =$ **2.6875**

$f(x_3) = f(2.6875) = 2.6875^3-4x2.6875-9 =$ **-0.3391**

Replace x_1 by x_3.

The new search interval becomes (**2.6875, 2.75**)

Iteration 2:

Sol: Starting with x_1 = **2.6875** and x_2 = **2.75**

$f(x_1) = f(2.6875) = 2.6875^3-4x2.6875-9=$ **-0.3391**

$f(x_2) = f(2.75) = 2.75^3 - 4x2.75-9 =$ **0.7969**

Find $x_3 = (2.6875+2.75)/2 =$ **2.7186**

$f(x_3) = f(2.7186) = 2.7186^3-4x2.7186-9 =$ **0.2182**

Replace x_2 by x_3.

The new search interval becomes (**2.6875, 2.7186**)

Iteration 3:

Sol: Starting with x_1 = **2.6875** and x_2 = **2.7186**

$f(x_1) = f(2.6875) = 2.6875^3-4x2.6875-9 =$ **-0.3391**

$f(x_2) = f(2.7186) = 2.7186^3-4x2.71865-9 =$ **-0.2182**

Find $x_3 = (2.6875+2.7186)/2 =$ **2.7031**

$f(x_3) = f(2.7031) = 2.7031^3-4x2.7031-9=$ **-0.0615**

Replace x_1 by x_3.

The new search interval becomes (**2.7031, 2.7186**)

Iteration 4:

Sol: Starting with x_1 = **2.7031** and x_2 = **2.7186**

$f(x_1) = f(2.7031) = 2.7031^3 - 4x2.7031 - 9 =$ **-0.0616**

$f(x_2) = f(2.7186) = 2.7186^3 - 4x2.71865 - 9 =$ **-0.2182**

Find $x_3 = (2.7031+2.7186)/2 =$ **2.7108**

$f(x_3) = f(2.7108) = 2.7108^3 - 4x2.7108 - 9 =$ **0.0769**

Replace x_2 by x_3.

The new search interval becomes (**2.7031, 2.7108**)

Iteration 5:

Sol: Starting with x_1 = **2.7031** and x_2 = **2.7108**

$f(x_1) = f(2.7031) = 2.7031^3 - 4x2.7031 - 9 =$ **-0.0616**

$f(x_2) = f(2.7108) = 2.7108^3 - 4x2.7108 - 9 =$ **0.0769**

Find $x_3 = (2.7031+2.7108)/2 =$ **2.7069**

$f(x_3) = f(2.7069) = 2.7069^3 - 4x2.7069 - 9 =$ **0.0067**

Replace x_2 by x_3.

The new search interval becomes (**2.7031, 2.7069**)

Iteration 6:

Sol: Starting with x_1 = **2.7031** and x_2 = **2.7069**

$f(x_1) = f(2.7031) = 2.7031^3 - 4x2.7031 - 9 =$ **-0.0616**

$f(x_2) = f(2.7069) = 2.7069^3 - 4x2.7069 - 9 =$ **0.0067**

Find $x_3 = (2.7031+2.7069)/2 =$ **2.705**

$f(x_3) = f(2.705) = 2.705^3 - 4x2.705 - 9 =$ **-0.0275**

Replace x_1 by x_3.

The new search interval becomes (**2.705, 2.7069**)

Iteration 7:

Sol: Starting with x_1 = **2.705** and x_2 = **2.7069**

$f(x_1) = f(2.705) = 2.705^3 - 4x2.705 - 9 =$ **-0.0275**

$f(x_2) = f(2.7069) = 2.7069^3 - 4x2.7069 - 9 =$ **0.0067**

Find $x_3 = (2.705+2.7069)/2 =$ **2.7059**

$f(x_3) = f(2.7059) = 2.7059^3 - 4x2.7059 - 9 =$ **-0.0113**

Replace x_1 by x_3.

The new search interval becomes (**2.7059, 2.7069**)

Iteration 8:

Sol: Starting with x_1 = **2.7059** and x_2 = **2.7069**

$f(x_1) = f(2.7059) = 2.7059^3\text{-}4x2.7059\text{-}9 =$ **-0.0113**

$f(x_2) = f(2.7069) = 2.7069^3\text{-}4x2.7069\text{-}9 =$ **0.0067**

Find $x_3 = (2.7059+2.7069)/2 =$ **2.7064**

$f(x_3) = f(2.7064) = 2.7064^3\text{-}4x2.7064\text{-}9=$ **-0.0023**

Replace x_1 by x_3.

The new search interval becomes (**2.7064, 2.7069**)

Iteration 9:

Sol: Starting with x_1 = **2.7064** and x_2 = **2.7069**

$f(x_1) = f(2.7064) = 2.7064^3\text{-}4x2.7064\text{-}9 =$ **-0.0023**

$f(x_2) = f(2.7069) = 2.7069^3\text{-}4x2.7069\text{-}9 =$ **0.0067**

Find $x_3 = (2.7064+2.7069)/2 =$ **2.70665**

$f(x_3) = f(2.70665) = 2.70665^3\text{-}4x2.70665\text{-}9=$ **0.0021**

Replace x_2 by x_3.

The new search interval becomes (**2.7064, 2.7066**)

Iteration 10:

Sol: Starting with x_1 = **2.7064** and x_2 = **2.7066**

$f(x_1) = f(2.7064) = 2.7064^3\text{-}4x2.7064\text{-}9 =$ **-0.0023**

$f(x_2) = f(2.7066) = 2.7066^3\text{-}4x2.7066\text{-}9 =$ **0.0021**

Find $x_3 = (2.7064+2.7066)/2 =$ **2.7065**

$f(x_3) = f(2.7065) = 2.7065^3\text{-}4x2.7065\text{-}9=$ **-0.0005**

Replace x_1 by x_3.

The new search interval becomes (**2.7065, 2.7066**)

Problem: Given that one root of the non-linear equation

$$x^4 - x - 10 = 0$$

lies between 1 and 2. Find the root correct to four significant places.

Problem: Given that one root of the non-linear equation

$$x^4 - x - 10 = 0$$

lies between 1 and 2. Find the root correct to four significant places.

Solution:

Required root is **1.856**.

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantages

- Slow to converge
- Good intermediate approximations may be discarded

FALSE POSITION METHOD

- Though the bisection method guarantees that iterative process will converge, but convergence is slow. The false position method, also known as ***regula-falsi*** or ***method of linear interpolation***, is similar to the bisection method but faster than it.
- It also starts with two initial approximations to the root say x_1 and x_2, for which f(x) has opposite signs, and then by linear interpolation the next approximation is determined.
- To describe its working assume that x_1 and x_2 are two initial approximations to root for which f(x) has opposite signs. Join the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ by a straight line. The point where this line intersects the x-axis is the next approximations to the root. Let us suppose that the line intersects the x-axis at x_3.

Note:-

1. **interpolation** is a method of constructing new data points within the range of a discrete set of known data points

2. **Convergence** A numerical method to solve equations will be a long process. We would like to know, if the method will lead to a solution (close to the exact solution) or will lead us away from the solution. If the method, leads to the solution, then we say that the method is convergent. Otherwise, the method is said to be divergent.

There are 3 possibilities;

1. If $f(x_3)= 0$, then we have a root at x_3.
2. If $f(x_1)$ and $f(x_3)$ are of opposite sign, then the root lies in the interval (x_1,x_3). Thus x_2 is replaced by x_3 and the iterative procedure is repeated.
3. If $f(x_1)$ and $f(x_3)$ are of same sign, then the root lies in the interval (x_3,x_2). Thus x_1 is replaced by x_3 and the iterative procedure is repeated.

The iterative procedure terminates when the size of the search of the search interval becomes less than the prescribed tolerance.

For derivation of the computational formula to find the intersection point, refer to figure.

Slope of the joining the points$(x_3,0)$ and $(x_2, f(x_2))$ is given by

$$\tan\theta = f(x_2)/x_2- x_3$$

Slope of the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$\tan\theta = f(x_2)-f(x_1)/x_2- x_1$$

Since the line joining the points $(x_3,0)$ and $(x_2, f(x_2))$ is a part of the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$, therefore

$$f(x_2)/ x_2 - x_3 = f(x_2)-f(x_1)/ x_2 - x_1$$

Solving for x_3, we get

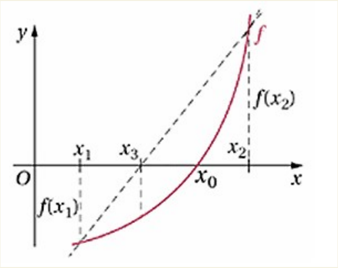
$$x_3 = x_1 f(x_2)-x_2f(x_1)/f(x_2)-f(x_1)$$

Which gives the next approximation to the root.

In general, the $(i+1)^{th}$ approximation to the root is given by the formula.

$$x_{i+1} = \frac{x_{i-1} f(x_i)-x_i f(x_{i-1})}{f(x_i)-f(x_{i-1})}$$

- The expression $f(x_i)-f(x_{i-1})$ represents the slope of the curve in current interval. And if the value of this expression becomes very small, it will introduce a large amount of error in the new approximation.
- Therefore, for computer implementation, we must put a limit on the permissible slope of the curve so that the iteration procedures does not fall in an endless loop.



Example 1:- Given that one of the root of a non-linear equation

$$X^3-2X-5=0$$

lies in the interval (1.75 , 2.5). Find the root correct to four significant digits.

Sol: Since we want the solution correct to four significant digits, the iterative process will be terminated as soon as the successive iterations produces no change at first four significant positions or the function vanishes at new approximation.

Iteration 1: Starting with x_1=1.75 and x_2=2.5

$f(x_1) = f(1.75) = 1.75^3 - 2x1.75 - 5 = -3.1406$

$f(x_2) = f(2.5) = 2.5^3 - 2x2.5 - 5 = 5.625$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1)=$

$1.75 \text{ x } 5.625-2.5 \text{ x } (-3.1406)/ 5.625 – (-3.1406)= 2.0187$

$f(x_3) = f(2.0187) = 2.0187^3-2 \text{ x } 2.0187 – 5 = -0.8109$

Since $f(x_1) \text{ x } f(x_3) = -3.1406 \text{ x } (-0.8109) > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (2.0187, 2.5).

Iteration 2: Now we take x_1 = 2.0187 and x_2= 2.5

$f(x_1) = f(2.0187) = 2.0187^3 – 2 \times 2.0187 - 5 = -0.8109$

$f(x_2) = f(2.5) = 2.5^3 - 2x2.5 - 5 = 5.625$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1) =$

2.0187 x 5.625-2.5 x (-0.8109)/ 5.625 – (-0.8109)= 2.0793

$f(x_3) = f(2.0793) = 2.0793^3 -2 \times 2.0793 – 5 = -0.1688$

Since $f(x_1) \times f(x_3) = -0.8109 \times (-0.1688) > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (2.0793, 2.5).

Iteration 3: Now we take x_1 = 2.0793 and x_2= 2.5

$f(x_1) = f(2.0793) = 2.0793^3 – 2 \times 2.0793 - 5 = -0.1688$

$f(x_2) = f(2.5) = 2.5^3 - 2x2.5 - 5 = 5.625$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1) =$
2.0793 x 5.625-2.5 x (-0.1688)/ 5.625 – (-0.1688)= 2.0916

$f(x_3) = f(2.0916) = 2.0916^3 -2 \times 2.0916 – 5 = -0.0329$

Since $f(x_1) \times f(x_3) = -0.1688 \times (-0.0329) > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (2.0916, 2.5).

Iteration 4: Now we take x_1 = 2.0916 and x_2= 2.5

$f(x_1) = f(2.0916) = 2.0916^3 – 2 \times 2.0916 - 5 = -0.0329$

$f(x_2) = f(2.5) = 2.5^3 - 2x2.5 - 5 = 5.625$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1) =$
2.0916 x 5.625-2.5 x (-0.329)/ 5.625 – (-0.329)= 2.0940

$f(x_3) = f(2.0940) = 2.0940^3 -2 \times 2.0940 – 5 = -0.0062$

Since $f(x_1) \times f(x_3) = -0.03299 \times (-0.0062) > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (2.0940, 2.5).

Iteration 5: Now we take x_1 = 2.0940 and x_2= 2.5

$f(x_1) = f(2.0940) = 2.0940^3 – 2 \times 2.0940 - 5 = -0.0329$

$f(x_2) = f(2.5) = 2.5^3 - 2x2.5 - 5 = 5.625$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1) =$
2.0940 x 5.625-2.5 x (-0.0062)/ 5.625 – (-0.0062)= 2.0944

$f(x_3) = f(2.0944) = 2.0944^3 -2 \times 2.0944 – 5 = -0.0017$

Since $f(x_1) \times f(x_3) = -0.0062 \times (-0.0017) > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (2.0944, 2.5).

Iteration 6: Now we take $x_1 = 2.0944$ and $x_2 = 2.5$

$f(x_1) = f(2.0944) = 2.0944^3 - 2 \times 2.0944 - 5 = -0.0017$

$f(x_2) = f(2.5) = 2.5^3 - 2x2.5 - 5 = 5.625$

$x_3 = x_1 f(x_2) - x_2 f(x_1)/f(x_2) - f(x_1) =$

$2.0944 \times 5.625 - 2.5 \times (-0.0017)/ 5.625 - (-0.0017) = 2.0945$

$f(x_3) = f(2.0945) = 2.0945^3 - 2 \times 2.0945 - 5 = -0.0006$

Since $f(x_1) \times f(x_3) = -0.0017 \times (-0.0006) > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (2.0945, 2.5).

Observe that iteration 4,5, and 6 produce no change at four significant positions in the successive approximations to the root. Therefore, we take x=2.094 as the desired solution correct to four significant digits.

Example 2:- Find the root of equation

$$\cos x - 3x + 1 = 0$$

correct to three decimal positions using False position method.

Sol: Since we want the solution correct to three decimal positions, the iterative process be terminated as soon as the successive iterations produce no changes at first three decimal positions or the function vanishes at new approximation.

Now $f(0) = \cos(0) - 3X0 + 1 = 2.0$

$f(1) = \cos(1) - 3X1 + 1 = -1.4597$

Since f(0) is +ve and f(1) is –ve, therefore one root lies between 0 and 1.

Iteration 1: Now we take $x_1 = 0$ and $x_2 = 1$

$f(0) = \cos(0) - 3 \times 0 + 1 = 2.0$

$f(1) = \cos(1) - 3x1 + 1 = -1.4579$

$x_3 = x_1 f(x_2) - x_2 f(x_1)/f(x_2) - f(x_1) =$

$0x(-1.4579) - 1 \times 2.0/-1.4597 - 2.0 = 0.5781$

$f(x_3) = f(0.5781) = \cos(0.5781) - 3 \times 0.5781 + 1 = 0.1033$

Thus, the first approximation to the root is 0.5781

Since $f(x_1) \times f(x_3) = 2.0 \times 0.1033 > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (0.5781, 1.0).

Iteration 2: Now we take $x_1 = 0.5781$ and $x_2 = 1$

$f(x_1) = f(0.5781) = \cos(0.5781) - 3 \times 0.5781 + 1 = 0.1033$

$f(x_2) = f(1) = \cos(1) - 3x1 + 1 = -1.4579$

$x_3 = x_1 f(x_2) - x_2 f(x_1)/f(x_2) - f(x_1) =$

$0.5781 \times (-1.4579) - 1 \times 0.1033/-1.4597 - 0.1033 = 0.6060$

$f(x_3) = f(0.6060) = \cos(0.6060) - 3 \times 0.6060 + 1 = 0.0041$

Thus, the second approximation to the root is 0.6060

Since $f(x_1) \times f(x_3) = 0.1033 \times 0.0041 > 0$

Therefore replace x_1 by x_3. The new search interval becomes (0.6060, 1.0).

Iteration 3: Now we take $x_1 = 0.6060$ and $x_2 = 1$

$f(x_1)=f(0.6060) = \cos(0.6060) – 3 \times 0.6060 + 1 = 0.0041$

$f(x_2)=f(1) = \cos(1) - 3x1 + 1 = -1.4579$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1) =$
$0.6060 \times (-1.4579)-1 \times 0.0041/-1.4597 – 0.0041= 0.6071$

$f(x_3) = f(0.6071) = \cos(0.6071)-3 \times 0.6071 + 1 = 0.0002$

Thus, the third approximation to the root is 0.6071

Since $f(x_1) \times f(x_3) = 0.0041 \times 0.0002 > 0$

Therefore replace x_1 by x_3 . The new search interval becomes (0.6071, 1.0).

Iteration 4: Now we take $x_1 = 0.6071$ and $x_2 = 1$

$f(x_1)=f(0.6071) = \cos(0.6071) – 3 \times 0.6071 + 1 = 0.0002$

$f(x_2)=f(1) = \cos(1) - 3x1 + 1 = -1.4579$

$x_3 = x_1f(x_2) – x_2f(x_1)/f(x_2) – f(x_1) =$
$0.6071 \times (-1.4579)-1 \times 0.0002/-1.4597 – 0.0002= 0.6072$

$f(x_3) = f(0.6072) = \cos(0.6072)-3 \times 0.6072 + 1 = 0.0001$

Thus, the third approximation to the root is 0.6072

Since last two iterations produce no change at the first three decimal positions, therefore, we take x=0.607 as the desired solution correct to three decimal positions.

Problem: Given that one root of the non-linear equation

$$x^3 - x - 4 = 0$$

lies between 1 and 2. Find the root correct to four significant places.

Problem: Given that one root of the non-linear equation

$$x^3 - x - 4 = 0$$

lies between 1 and 2. Find the root correct to four significant places.

Solution:

Required root is **1.796**.

Advantages

- *Faster than Bisection method.*
- *Always* ***converges*** *for a single root.*
- No need to calculate a complicated derivative (as in Newton's method).

Disadvantages

- May **converge** slowly for functions with big curvatures.
- Newton-Raphson may be still faster if we can apply it

Newton –Raphson Method

- The **Newton Raphson** method, also known as **Newton's method of tangents**, is one of the fastest iterative methods.
- This method begins with on initial approximation. Here one have to take care while selecting the initial approximation, as it is very sensitive to the initial approximation.
- Once proper choice is made for the initial approximation, it converges faster that **False Position** and the **Secant method.**

Given an initial guess of the root x_0, Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.

Assumptions:

- *$f(x)$ is continuous and the first derivative is known*
- *An initial guess x_0 such that $f'(x_0)\neq 0$ is given*

$Given$: x_i an initial guess of the root of $f(x)=0$

$Question$: How do we obtain a better estimate x_{i+1}?

Taylor Therorem : $f(x+h) \approx f(x) + f'(x)h$

Find h such that $f(x+h)=0$.

$$\Rightarrow h \approx -\frac{f(x)}{f'(x)}$$

Newton – Raphson Formula

A new guess of the root : $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Example: Given the one root of the non-linear equation

$x^3-4x-9=0$

lies between 2.625 and 3.0. Find the root correct to four significant digits.

Sol: Given $f(x) = x^3 - 4x - 9$

$f'(x) = 3x^2 - 4$

Iteration 1: we start with $x_0 = 2.95$

$f(x_0) = f(2.95) = 2.95^3 - 4x\ 2.95 - 9 = 4.8724$

$f'(x_0) = f'(2.95) = 3 \times 2.95^2 - 4 = 22.1075$

$x_1 = x_0 - f(x_0)/f'(x_0)$

$= 2.95 - 4.8724/22.1075 = 2.7296$

Iteration 2: Now taking $x_0 = 2.7296$

$f(x_0) = f(2.7296) = 2.7296^3 - 4x\ 2.7296 - 9 = 0.4191$

$f'(x_0) = f'(2.7296) = 3 \times 2.7296^2 - 4 = 18.3521$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.7296 - \frac{0.4191}{18.3521} = 2.7068$$

Iteration 3: Now taking $x_0 = 2.7068$

$f(x_0) = f(2.\ 7068) = 2.\ 7068^3 - 4x\ 2.7068 - 9 = 0.0049$

$f'(x_0) = f'(2.\ 7068) = 3 \times 2.\ 7068^2 - 4 = 17.9803$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$2.7068 - \frac{0.0049}{17.9803} = 2.7065$$

Iteration 4: Now taking $x_0 = 2.7065$

$f(x_0) = f(2.7065) = 2.7065^3 - 4x\ 2.7065 - 9 = -0.0005$

$f'(x_0) = f'(2.7065) = 3 \times 2.7065^2 - 4 = 17.9754$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$2.7065 - \frac{(-0.0005)}{17.9754} = 2.7065$$

Therefore, we take **x = 2.7065** as the desired solution.

Example: Given the one root of the equation

$$X^3 - X - 4 = 0$$

Find the root correct to four significant digits.

Sol. Given $f(x) = X^3 - X - 4$

Therefore $f'(x) = 3X^2 - 1$

Iteration 1: we start with $x_0 = 2$

$f(2) = 2^3 - 2 - 4 = 2$

$f'(2) = 3 \times 2^2 - 1 = 11$

$$X_1 = 2 - \frac{2}{11} = 1.8182$$

Thus, the first approximation to the root is 1.8182.

Iteration 2: Now taking $x_0 = 1.8182$

$f(x_0) = f(1.8182) = 1.8182^3 - 1.8182 - 4 = 0.1925$

$f'(x_0) = f'(1.8182) = 3 \times 1.8182^2 - 1 = 8.9175$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.8182 - \frac{0.1925}{8.9175} = 1.7966$$

Iteration 3: Now taking $x_0 = 1.7966$

$f(x_0) = f(1.7966) = 1.7966^3 - 1.7966 - 4 = 0.0025$

$f'(x_0) = f'(1.7966) = 3 \times 1.7966^2 - 1 = 8.6833$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.7966 - \frac{0.0025}{8.6833} = 1.7963$$

Iteration 4: Now taking $x_0 = 1.7963$

$f(x_0) = f(1.7963) = 1.7963^3 - 1.7963 - 4 = -0.0001$

$f'(x_0) = f'(1.7963) = 3 \times 1.7963^2 - 1 = 8.6800$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.7963 + \frac{0.0001}{8.6800} = 1.7963$$

Therefore, we take **x = 1.7963** as the desired solution.

Problem: Given that one root of the non-linear equation

$$x^3 + x^2 - 1 = 0$$

lies between 0 and 1. Find the root correct to four significant places.

Problem: Given that one root of the non-linear equation

$$x^3 + x^2 - 1 = 0$$

lies between 0 and 1. Find the root correct to four significant places.

Solution:

Required root is **0.7549**.

Advantages

- Fast convergence!

Disadvantages

- Calculating the required derivative itself $f'(x_{n-1})$ for every iteration may be a costly task for some functions f.
- May not produce a root unless the starting value x_0 is close to the actual root of the function.
- May not produce a root if for instance the iterations get to a point x_{n-1} such that $f'(x_{n-1}) = 0$. Then the method fails!