NUMERICAL INTEGRATION

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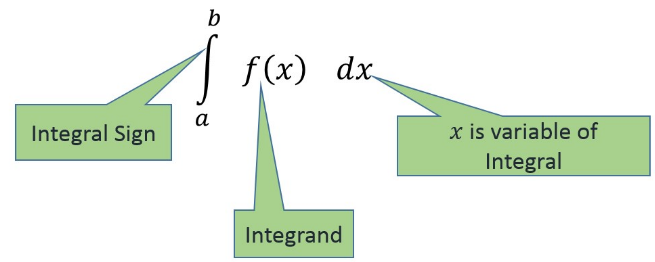
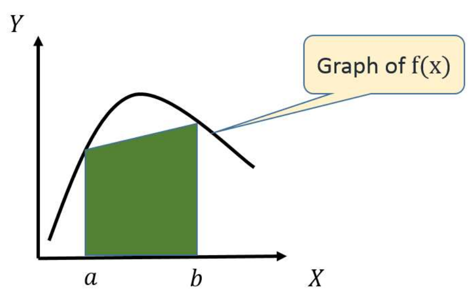
What is Integration?

- Integration is process of evaluating an indefinite integral **or** definite integral.



Graphical Representation of Integral



The “integral” of $f(x)$ from a *to* b is the area under the graph of function $f(x)$

Integration

- Fundamental theorem of calculus :

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Where $F'(x) = f(x)$

This method of evaluating definite integrals is called the **analytical method.**

e.g. $\int_2^4 x\ dx$

$$= \left[\frac{x^2}{2}\right]_2^4 = \frac{16}{2} - \frac{4}{2} = \frac{12}{2} = 6$$

Why Numerical Integration?

There are many functions $sinx^2$, $\frac{e^x}{x}$, etc

$$\int \frac{sinx^2}{x}\,dx$$

Solve it, if you can

- ✓ whose integration analytically is impossible .
- ✓ Also some integrals, whose evaluation would be very complicated.
- ✓ In many cases, analytic methods fails.
- ✓ In such cases, Numerical methods are used.

Numerical Integration

Consider the observation table

x	x_0	x_1	x_2	x_3		x_{n-1}	x_n
$y = f(x)$	y_0	y_1	y_2	y_3		y_{n-1}	y_n

The process of evaluating definite integral $\int_{x_0}^{x_n} f(x)\ dx$ from set of tabulated values of $f(x)$ is called Numerical integration.

Methods of Numerical Integration

1. Trapezoidal Rule
2. Simpson's 1/3 Rule
3. Simpson's 3/8 Rule
4. Weddle's Rule

Trapezoidal Rule

- ➢ Consider the observation table

Formula :

x	x_0	x_1	x_2	x_3		x_{n-1}	x_n
y	y_0	y_1	y_2	y_3		y_{n-1}	y_n

$$\int_a^b f(x)\ dx = \frac{h}{2}[\ (F+L)\ +2(rest\ of\ all)]$$

$$\boxed{= \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})]}$$

$F = first\ obs. = y_0$
$L = last\ obs. = y_n$

$Where, h = \frac{b-a}{n}$

- $x_n = x_0 + nh$
- $n\ is\ no\ of\ sub-intervals.$

Example 1:

Evaluate $\int_0^1 e^x\, dx$ using trapezoidal rule with $n = 10$.

Solution :

Here $a = 0$, b $= 1$ & $n = 10$

Now,

$$h = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10} = 0.1$$

Let, $y = f(x) = e^x$

x	0	0.1	0.2	0.3	0.4
y	1	1.1052	1.2214	1.3499	1.4918

0.5	0.6	0.7	0.8	0.9	1.0
1.6487	1.8221	2.0138	2.2255	2.4596	2.7183

$y(x) = e^x \Rightarrow y(0) = e^0 = 1$

$y(0.1) = e^{0.1} = 1.1052$

$y(0.2) = e^{0.2} = 1.2214$

$y(0.3) = e^{0.3} = 1.3499$

$y(0.4) = e^{0.4} = 1.4918$

$y(0.5) = e^{0.5} = 1.6487$

$y(0.6) = e^{0.6} = 1.8221$

$y(0.7) = e^{0.7} = 2.0138$

$y(0.8) = e^{0.8} = 2.2255$

$y(0.9) = e^{0.9} = 2.4596$

$y(1.0) = e^{1.0} = 2.7183$

x	0	0.1	0.2	0.3	0.4
y	$y_0 = 1$	$y_1 = 1.1052$	$y_2 = 1.2214$	$y_3 = 1.3499$	$y_4 = 1.4918$

0.5	0.6	0.7	0.8	0.9	1.0
$y_5 = 1.6487$	$y_6 = 1.8221$	$y_7 = 2.0138$	$y_8 = 2.2255$	$y_9 = 2.4596$	$y_{10} = 2.7183$

By the Trapezoidal rule,

$$\int_0^1 e^x\, dx = \frac{h}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$

$$= \frac{0.1}{2}[(1 + 2.7183) + 2(1.1052 + 1.2214 + 1.3499 + 1.4918 + 1.6487 + 1.8221 + 2.0138 + 2.2255 + 2.4596)]$$

$$= 1.7197$$

Example 2:

Evaluate $\int_0^\pi sinx\, dx$ using trapezoidal rule with $n = 10$.

Solution :

Here $a = 0$, b $= \pi$ & $n = 10$

Now,

$$h = \frac{b-a}{n} = \frac{\pi - 0}{10} = \frac{\pi}{10}$$

Let, $y = f(x) = sinx$

x	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$
y	0	0.3090	0.5878	0.8090	0.9511
$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	$10\pi/10$
1.0000	0.9511	0.8090	0.5878	0.3090	0

$y(x) = \sin x \Rightarrow y(0) = \sin 0 = 0$

$y(\pi/10) = \sin \pi/10 = 0.3090$

$y(2\pi/10) = \sin 2\pi/10 = 0.5878$

$y(3\pi/10) = \sin 3\pi/10 = 0.8090$

$y(4\pi/10) = \sin 4\pi/10 = 0.9511$

$y(5\pi/10) = 1$

$y(6\pi/10) = 0.9511$

$y(7\pi/10) = 0.8090$

$y(8\pi/10) = 0.5878$

$y(9\pi/10) = 0.3090$

$y(10\pi/10) = 0$

x	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$
y	0	0.3090	0.5878	0.8090	0.9511
$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	$10\pi/10$
1.0000	0.9511	0.8090	0.5878	0.3090	0

By the Trapezoidal rule,

$$\int_0^{\pi} \sin x\, dx = \frac{h}{2}[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)]$$

$$= \frac{\pi}{20}[(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511 + 1 + 0.9511 + 0.8090 + 0.5878 + 0.309)]$$

$$= \frac{\pi}{20}(12.6276) = 1.9835$$

Simpson's 1/3 Rule

- Consider the observation table

x	x_0	x_1	x_2	x_3	x_4		x_{n-1}	x_n
y	y_0	y_1	y_2	y_3	y_4		y_{n-1}	y_n

$$\int_a^b f(x)\, dx = \frac{h}{3}[(F+L) + 4(odd\ obs.) + 2(even\ obs.)]$$

$$\boxed{= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots) + 2(y_2 + y_4 + \cdots)]}$$

Where, $h = \frac{b-a}{n}$

- *n is no of sub − intervals.*

$F = first\ obs. = y_0$

$L = last\ obs. = y_n$

Example 1:

Evaluate $\int_0^6 \frac{dx}{1+x}$ taking $h = 1$ using Simpson's $\frac{1}{3}$ rule.

Hence Obtain an approximate value of $\log_e 7$.

Solution :

Here $a = 0$, $b = 6$ & $h = 1$

Now,

$$n = \frac{b-a}{h} = \frac{6-0}{1} = 6$$

Let, $y = f(x)$

$$y = \frac{1}{1+x}$$

x	0	1	2	3	4	5	6
y	1	0.5	0.3333	0.2555	0.2000	0.1667	0.1429

$$y(x) = \frac{1}{1+x} \Longrightarrow y(0) = \frac{1}{1+0} = 1$$
$$y(x) = \frac{1}{1+x} \Longrightarrow y(1) = 0.5$$
$$y(x) = \frac{1}{1+x} \Longrightarrow y(2) = 0.3333$$
$$y(x) = \frac{1}{1+x} \Longrightarrow y(3) = 0.2555$$
$$y(x) = \frac{1}{1+x} \Longrightarrow y(4) = 0.2000$$
$$y(x) = \frac{1}{1+x} \Longrightarrow y(5) = 0.1667$$
$$y(x) = \frac{1}{1+x} \Longrightarrow y(6) = 0.1429$$

x	0	1	2	3	4	5	6
y	1	0.5	0.3333	0.2500	0.2000	0.1667	0.1429
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By the Simpson's $\frac{1}{3}$ rule,

$$\int_0^6 \frac{dx}{1+x} = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}[(1 + 0.1429) + 4(0.5 + 0.2500 + 0.1667) + 2(0.3333 + 0.2)]$$

$$= \frac{1}{3}[5.8763] = 1.9588$$

Example 1:

$$\int_0^6 \frac{dx}{1+x} = 1.9588$$

Find the exact solution of integral

By Direct Method,

$$\int_0^6 \frac{dx}{1+x} = [\log(1+x)]_0^6 = \log(7) - \log(1)$$
$$= \log 7$$
$$\int_0^6 \frac{dx}{1+x} = \log 7 = 1.9588$$

Example 2:

Evaluate $\int_{-2}^{6}(1+x^2)^{\frac{3}{2}}\,dx$ using Simpson's $\frac{1}{3}$ rule with taking **six** subintervals. Use four digits after decimal points for calculation.

Solution :

Here $a = -2$, b $= 6$ & $n = 6$

Now,

$$h = \frac{b-a}{n} = \frac{6+2}{6} = \frac{8}{6} = \frac{4}{3} \neq 1.3333$$

Let, $y = f(x) = (1+x^2)^{\frac{3}{2}}$

x	-2	$-2/3$	$2/3$	2	$10/3$	$14/3$	6
y	11.1803	1.7360	1.7360	11.1803	42.1479	108.7094	225.0622

$y(x) = (1+x^2)^{\frac{3}{2}} \Longrightarrow y(-2) = (1+0)^{\frac{3}{2}} = 11.1803$

$y(x) = (1+x^2)^{\frac{3}{2}} \Longrightarrow y(-2/3) = 1.7360$

$y(2/3) = 1.7360$

$y(2) = 11.1803$

$y(10/3) = 42.1479$

$y(14/3) = 108.7094$

$y(6) = 225.0622$

x	-2	$-2/3$	$2/3$	2	$10/3$	$14/3$	6
y	11.1803	1.7360	1.7360	11.1803	42.1479	108.7094	225.0622
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By the Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 e^x\,dx = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{4}{9}[(11.1803+225.0622)+4(1.736+11.1803+108.7094)+2(1.736+42.1479)]$$

$$= \frac{4}{9}(810.5131) = 360.2280$$

Exercise

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Trapezoidal rule taking $h=1$.

   [Ans: 1.4108]

1. Evaluate by Trapezoidal rule $\int_0^4 \frac{dx}{1+x^4}$ taking h =4. [Ans: 1.081]

2. Evaluate $\int_{-2}^{+2} \frac{3x\ dx}{(4+x)^2}$ by using Simpson's 1/3 rd rule with six intervals. [Ans: -0.710]

3. Evaluate $\int_0^4 e^x\ dx$ by using Simpson's 1/3 rd rule given that $e = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.6$ and compare the result with its actual value.

   [Ans: 53.87]

Simpson's 3/8 Rule

➢ Consider the observation table

x	x_0	x_1	x_2	x_3	x_4		x_{n-1}	x_n
y	y_0	y_1	y_2	y_3	y_4		y_{n-1}	y_n

$$\int_a^b f(x)\ dx = \frac{3h}{8}[(F+L)+2(Multiple\ of\ 3)+3(Not\ Multiple\ of\ 3)\ ]$$

$$\boxed{= \frac{3h}{8}[(y_0+y_n)+2(y_3+y_6+\cdots)+3(y_1+y_2+y_4\ldots)\ ]}$$

$Where, h = \frac{b-a}{n}$

- $n\ is\ no\ of\ sub-intervals.$

$F = first\ obs. = y_0$

$L = last\ obs. = y_n$

Example 1:

Evaluate $\int_0^3 \frac{dx}{1+x}$ taking $n = 6$ using Simpson's 3/8 rule. Hence Obtain an approximate value of $\log_e 2$.Estimate the bound of error involved in the process.

Solution :

Here $a = 0$, b $= 3$ & $n = 6$

Now,

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

Let, $y = f(x) = \frac{1}{1+x}$

x	0	0.5	1	1.5	2	2.5	3
y	1	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500

$$y(x) = \frac{1}{1+x} \Longrightarrow y(0) = \frac{1}{1+0} = 1$$

$$y(x) = \frac{1}{1+x} \Longrightarrow y(0.5) = 0.6667$$

$$y(x) = \frac{1}{1+x} \Longrightarrow y(1) = 0.5000$$

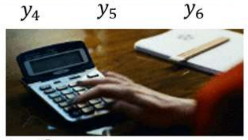
$$y(x) = \frac{1}{1+x} \Longrightarrow y(1.5) = 0.4000$$

$$y(x) = \frac{1}{1+x} \Longrightarrow y(2) = 0.3333$$

$$y(x) = \frac{1}{1+x} \Longrightarrow y(2.5) = 0.2857$$

$$y(x) = \frac{1}{1+x} \Longrightarrow y(3) = 0.2500$$

x	0	0.5	1	1.5	2	2.5	3
y	1	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500
	y_0	y_1	y_2	y_3	y_4	y_5	y_6



By the Simpson's $\frac{3}{8}$ rule,

$$\int_0^3 \frac{dx}{1+x} = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3(0.5)}{8}[(1 + 0.25) + 2(0.4) + 3(0.5 + 0.3333 + 0.2857 + 0.6667)]$$

$$= 1.3888$$

Example 2:

Evaluate $\int_4^{5.2} \log x \; dx$ using Simpson's 3/8 rule.

Solution:

Let $y = f(x) = \log x$

Let $n = 6 \Rightarrow h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

By Simpson's 3/8 rule,

$$\int_4^{5.2} \log x \, dx = \frac{3h}{8}\big((y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\big)$$

$$= \frac{3(0.2)}{8}\big((1.3863 + 1.6487) + 2(1.5261)$$

$$+ 3(1.4351 + 1.4816 + 1.5686 + 1.6094)\big)$$

$$= 1.8278$$