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Newton's Interpolation Formulae

As stated earlier, interpolation is the process of approximating a given function, whose values are known at $N+1$ tabular points, by a suitable polynomial, $P_N(x)$, of degree N which takes the values y_i at $x = x_i$ for $i = 0, 1, \ldots, N$. Note that if the given data has errors, it will also be reflected in the polynomial so obtained.

In the following, we shall use forward and backward differences to obtain polynomial function approximating $y = f(x)$, when the tabular points x_i 's are equally spaced. Let

$$f(x) \approx P_N(x),$$

where the polynomial $P_N(x)$ is given in the following form:

$$\begin{aligned} P_N(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_k(x - x_0)(x - x_1)\cdots(x - x_{k-1}) \\ &\quad + a_N(x - x_0)(x - x_1)\cdots(x - x_{N-1}). \end{aligned} \tag{11.4.1}$$

for some constants $a_0, a_1, \ldots a_N$, to be determined using the fact that $P_N(x_i) = y_i$ for $i = 0, 1, \ldots, N$.

So, for $i = 0$, substitute $x = x_0$ in (11.4.1) to get $P_N(x_0) = y_0$. This gives us $a_0 = y_0$. Next,

$$P_N(x_1) = y_1 \Rightarrow y_1 = a_0 + (x_1 - x_0)a_1.$$

So, $a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$. For $i = 2$, $y_2 = a_0 + (x_2 - x_0)a_1 + (x_2 - x_1)(x_2 - x_0)a_2$, or equivalently

$$2h^2 a_2 = y_2 - y_0 - 2h\left(\frac{\Delta y_0}{h}\right) = y_2 - 2y_1 + y_0 = \Delta^2 y_0.$$

Thus, $a_2 = \frac{\Delta^2 y_0}{2h^2}$. Now, using mathematical induction, we get

$$a_k = \frac{\Delta^k y_0}{k!\, h^k} \quad \text{for} \quad k = 0, 1, 2, \ldots, N.$$

Thus,

$$P_N(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1) + \cdots + \frac{\Delta^k y_0}{k!\, h^k}(x - x_0)\cdots(x - x_{k-1})$$

$$+ \frac{\Delta^N y_0}{N!\, h^N}(x - x_0)\ldots(x - x_{N-1}).$$

As this uses the forward differences, it is called NEWTON'S FORWARD DIFFERENCE FORMULA for interpolation, or simply, forward interpolation formula.

EXERCISE 11.4.1 ***Show that***

$$a_3 = \frac{\Delta^3 y_0}{3!\, h^3} \quad \text{and} \quad a_4 = \frac{\Delta^4 y_0}{4!\, h^2}$$

and in general,

$$a_k = \frac{\Delta^k y_0}{k! h^k}, \quad \text{for} \quad k = 0, 1, 2, \ldots, N.$$

For the sake of numerical calculations, we give below a convenient form of the forward interpolation formula.

Let $u = \frac{x - x_0}{h}$, then

$$x - x_1 = hu + x_0 - (x_0 + h) = h(u - 1), x - x_2 = h(u - 2), \ldots, x - x_k = h(u - k), \quad \text{etc..}$$

With this transformation the above forward interpolation formula is simplified to the following form:

$$P_N(u) = y_0 + \frac{\Delta y_0}{h}(hu) + \frac{\Delta^2 y_0}{2!\, h^2}\{(hu)(h(u-1))\} + \cdots + \frac{\Delta^k y_0 h^k}{k!\, h^k}\left[u(u-1)\cdots(u-k+1)\right]$$

$$+ \cdots + \frac{\Delta^N y_0}{N!\, h^N}\Big[(hu)\big(h(u-1)\big)\cdots\big(h(u-N+1)\big)\Big].$$

$$= y_0 + \Delta y_0(u) + \frac{\Delta^2 y_0}{2!}(u(u-1)) + \cdots + \frac{\Delta^k y_0}{k!}\left[u(u-1)\cdots(u-k+1)\right]$$

$$+ \cdots + \frac{\Delta^N y_0}{N!}\left[u(u-1)\ldots(u-N+1)\right]. \qquad (11.4.2)$$

If N =1, we have a linear interpolation given by

$$f(u) \approx y_0 + \Delta y_0(u). \qquad (11.4.3)$$

For $N = 2$, we get a quadratic interpolating polynomial:

$$f(u) \approx y_0 + \Delta y_0(u) + \frac{\Delta^2 y_0}{2!}[u(u-1)] \qquad (11.4.4)$$

and so on.

It may be pointed out here that if $f(x)$ is a polynomial function of degree N then $P_N(x)$ coincides with $f(x)$ on the given interval. Otherwise, this gives only an approximation to the true values of $f(x)$.

If we are given additional point x_{N+1} also, then the error, denoted by $R_N(x) = |P_N(x) - f(x)|$, is estimated by

$$R_N(x) \simeq \left|\frac{\Delta^{N+1} y_0}{h^{N+1}(N+1)!}(x-x_0)\cdots(x-x_N)\right|.$$

Similarly, if we assume, $P_N(x)$ is of the form

$$P_N(x) = b_0 + b_1(x-x_N) + b_1(x-x_N)(x-x_{N-1}) + \cdots + b_N(x-x_N)(x-x_{N-1})\cdots(x-x_1),$$

then using the fact that $P_N(x_i) = y_i$, we have

$$b_0 = y_N$$

$$\begin{aligned} b_1 &= \frac{1}{h}(y_N - y_{N-1}) = \frac{1}{h}\nabla y_N \\ b_2 &= \frac{y_N - 2y_{N-1} + y_{N-2}}{2h^2} = \frac{1}{2h^2}(\nabla^2 y_N) \\ &\vdots \\ b_k &= \frac{1}{k!\,h^k}\nabla^k y_N. \end{aligned}$$

Thus, using backward differences and the transformation $x = x_N + hu$, we obtain the Newton's backward interpolation formula as follows:

$$P_N(u) = y_N + u\nabla y_N + \frac{u(u+1)}{2!}\nabla^2 y_N + \cdots + \frac{u(u+1)\cdots(u+N-1)}{N!}\nabla^N y_N. \qquad (11.4.5)$$

EXERCISE 11.4.2 ***Derive the Newton's backward interpolation formula (11.4.5) for*** $N = 3$.

Remark 11.4.3 *If the interpolating point lies closer to the beginning of the interval then one uses the Newton's forward formula and if it lies towards the end of the interval then Newton's backward formula is used.*

Remark 11.4.4 *For a given set of n tabular points, in general, all the n points need not be used for interpolating polynomial. In fact N is so chosen that* N^{th} *forward/backward difference almost remains constant. Thus N is less than or equal to n.*

EXAMPLE 11.4.5

1. Obtain the Newton's forward interpolating polynomial, $P_5(x)$ for the following tabular data and interpolate the value of the function at $x = 0.0045$.

x	0	0.001	0.002	0.003	0.004	0.005
y	1.121	1.123	1.1255	1.127	1.128	1.1285

Solution: For this data, we have the Forward difference difference table

x_i	y_i	Δy_i	$\Delta^2 y_3$	$\Delta^3 y_i$	$\Delta^4 y_i$	$\Delta^5 y_i$	

0	1.121	0.002	0.0005	-0.0015	0.002	-.0025	
.001	1.123	0.0025	-0.0010	0.0005	-0.0005		
.002	1.1255	0.0015	-0.0005	0.0			
.003	1.127	0.001	-0.0005				
.004	1.128	0.0005					
.005	1.1285						

Thus, for $x = x_0 + hu$, where $x_0 = 0$, $h = 0.001$ and $u = \frac{x - x_0}{h}$, we get

$$P_5(x) = 1.121 + u \times .002 + \frac{u(u-1)}{2}(.0005) + \frac{u(u-1)(u-2)}{3!} \times (-.0015)$$
$$+\frac{u(u-1)(u-2)(u-3)}{4!}(.002) + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!} \times (-.0025).$$

Thus,

$$\begin{aligned} P_5(0.0045) &= P_5(0 + 0.001 \times 4.5) \\ &= 1.121 + 0.002 \times 4.5 + \frac{0.0005}{2} \times 4.5 \times 3.5 - \frac{0.0015}{6} \times 4.5 \times 3.5 \times 2.5 \\ &\quad + \frac{0.002}{24} \times 4.5 \times 3.5 \times 2.5 \times 1.5 - \frac{0.0025}{120} \times 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \\ &= 1.12840045. \end{aligned}$$

2. Using the following table for $\tan x$, approximate its value at 0.71. Also, find an error estimate (Note $\tan(0.71) = 0.85953$).

x_i	0.70	72	0.74	0.76	0.78	
$\tan x_i$	0.84229	0.87707	0.91309	0.95045	0.98926	

Solution: As the point $x = 0.71$ lies towards the initial tabular values, we shall use Newton's Forward formula. The forward difference table is:

x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$		
0.70	0.84229	0.03478	0.00124	0.0001	0.00001		

0.72	0.87707	0.03602	0.00134	0.00011			
0.74	0.91309	0.03736	0.00145				
0.76	0.95045	0.03881					
0.78	0.98926						

In the above table, we note that $\Delta^3 y$ is almost constant, so we shall attempt 3^{rd} degree polynomial interpolation.

Note that $x_0 = 0.70,\ h = 0.02$ gives $u = \frac{0.71 - 0.70}{0.02} = 0.5.$ Thus, using forward interpolating polynomial of degree $3,$ we get

$$P_3(u) = 0.84229 + 0.03478u + \frac{0.00124}{2!}u(u-1) + \frac{0.0001}{3!}u(u-1)(u-2).$$

$$\text{Thus,} \qquad \tan(0.71) \approx 0.84229 + 0.03478(0.5) + \frac{0.00124}{2!} \times 0.5 \times (-0.5)$$
$$+\frac{0.0001}{3!} \times 0.5 \times (-0.5) \times (-1.5)$$
$$= 0.859535.$$

An error estimate for the approximate value is

$$\left.\frac{\Delta^4 y_0}{4!}u(u-1)(u-2)(u-3)\right|_{u=0.5} = 0.00000039.$$

Note that exact value of $\tan(0.71)$ (upto 5 decimal place) is $0.85953.$ and the approximate value, obtained using the Newton's interpolating polynomial is very close to this value. This is also reflected by the error estimate given above.

3. Apply 3^{rd} degree interpolation polynomial for the set of values given in Example 11.2.15, to estimate the value of $f(10.3)$ by taking

$$(i)\ \ x_0 = 9.0, \qquad\qquad (ii)\ \ x_0 = 10.0.$$

Also, find approximate value of $f(13.5)$.

Solution: Note that $x = 10.3$ is closer to the values lying in the beginning of tabular values, while $x = 13.5$ is towards the end of tabular values. Therefore, we shall use forward difference formula for $x = 10.3$ and the backward difference formula for $x = 13.5$. Recall that the interpolating polynomial of degree 3 is given by

$$f(x_0 + hu) = y_0 + \Delta y_0 u + \frac{\Delta^2 y_0}{2!} u(u-1) + \frac{\Delta^3 y_0}{3!} u(u-1)(u-2).$$

Therefore,

1. for $x_0 = 9.0$, $h = 1.0$ and $x = 10.3$, we have $u = \frac{10.3 - 9.0}{1} = 1.3$. This gives,

$$\begin{aligned} f(10.3) &\approx 5 + .4 \times 1.3 + \frac{.2}{2!}(1.3) \times .3 + \frac{.0}{3!}(1.3) \times .3 \times (-0.7) \\ &= 5.559. \end{aligned}$$

2. for $x_0 = 10.0$, $h = 1.0$ and $x = 10.3$, we have $u = \frac{10.3 - 10.0}{1} = .3$. This gives,

$$\begin{aligned} f(10.3) &\approx 5.4 + .6 \times .3 + \frac{.2}{2!}(.3) \times (-0.7) + \frac{-0.3}{3!}(.3) \times (-0.7) \times (-1.7) \\ &= 5.54115. \end{aligned}$$

   Note: as $x = 10.3$ is closer to $x = 10.0$, we may expect estimate calculated using $x_0 = 10.0$ to be a better approximation.

3. for $x_0 = 13.5$, we use the backward interpolating polynomial, which gives,

$$f(x_N + hu) \approx y_0 + \nabla y_N u + \frac{\nabla^2 y_N}{2!} u(u+1) + \frac{\Delta^3 y_N}{3!} u(u+1)(u+2).$$

   Therefore, taking $x_N = 14$, $h = 1.0$ and $x = 13.5$, we have $u = \frac{13.5 - 14}{1} = -0.5$. This gives,

$$\begin{aligned} f(13.5) &\approx 8.1 + .6 \times (-0.5) + \frac{-0.1}{2!}(-0.5) \times 0.5 + \frac{0.0}{3!}(-0.5) \times 0.5 \times (1.5) \\ &= 7.8125. \end{aligned}$$

EXERCISE 11.4.6

1. Following data is available for a function $y = f(x)$

x	0	0.2	0.4	0.6	0.8	1.0
y	1.0	0.808	0.664	0.616	0.712	1.0

   Compute the value of the function at $x = 0.3$ and $x = 1.1$

2. The speed of a train, running between two station is measured at different distances from the starting station. If x is the distance in $km.$ from the starting station, then $v(x)$, the speed (in km/hr) of the train at the distance x is given by the following table:

x	0	50	100	150	200	250
v(x)	0	60	80	110	90	0

   Find the approximate speed of the train at the mid point between the two stations.

3. Following table gives the values of the function $S(x) = \int\limits_0^x sin(\frac{\pi}{2}t^2)dt$ at the different values of the tabular points x,

x	0	0.04	0.08	0.12	0.16	0.20
S(x)	0	0.00003	0.00026	0.00090	0.00214	0.00419

   Obtain a fifth degree interpolating polynomial for $S(x)$. Compute $S(0.02)$ and also find an error estimate for it.

4. Following data gives the temperatures (in $^{\circ}C$) between 8.00 am to 8.00 pm. on May 10, 2005 in Kanpur:

Time	8 am	12 noon	4 pm	8pm
Temperature	30	37	43	38

   Obtain Newton's backward interpolating polynomial of degree 3 to compute the temperature in Kanpur on that day at 5.00 pm.

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