INTERPOLATION

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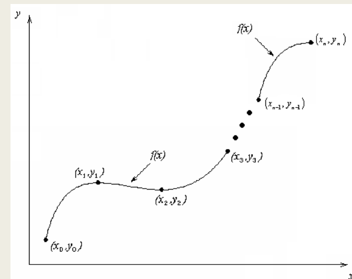
WHAT IS INTERPOLATION?

Given (x_0,y_0), (x_1,y_1), …, (x_n,y_n), finding the value of 'y' at a value of 'x' in (x_0, x_n) is called **interpolation.**



Interpolation depends upon the concepts of operators.

Forward Difference

✓Forward difference is denoted by (delta) Δ

✓Formula of forward difference is

$$\Delta f(x) = f(x+h) - f(x)$$

$x-h$ $\quad$ x $\quad$ $x+h$

Forward

Backward Difference

✓ Backward difference is denoted by (nabla) ∇

✓ Formula of Backward difference is

$$\nabla f(x) = f(x) - f(x-h)$$

$x-h$ $\quad$ x $\quad$ $x+h$

Backward

Shift Operator

- ✓ Shift Operator is denoted by E
- ✓ Formula of Shift Operator is

$$E^n f(x) = f(x + nh)$$

- ✓ If $n = 1$,

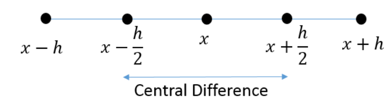
  then $E^1 f(x) = f(x + h)$

Central Difference [δ]

- ✓ Central Operator is denoted by (small delta) [δ]
- ✓ Formula of Central Operator is

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \quad \left(\because E^n f(x) = f(x + nh)\right)$$

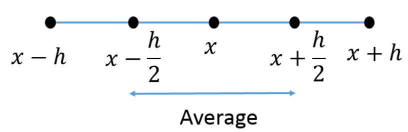


Average Operator

- ✓ Average Operator is denoted by (Mu) μ
- ✓ Formula of Average Operator is

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right]$$

$$\mu = \frac{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}{2} \quad \left(\because E^n f(x) = f(x + nh)\right)$$



Differential Operator [D]

- ✓ Differential Operator is denoted by D
- ✓ Formula of Differential Operator is

$$Df(x) = \frac{d}{dx} f(x) = f'(x) \quad \left(\because D = \frac{d}{dx}\right)$$

Relation Between Operators

$[\Delta, \nabla, E, \delta, \mu, D]$

Prove that $E = 1 + \Delta$

$$
\begin{aligned}
&(1+\Delta)f(x)\\
&= f(x) + \Delta f(x)\\
&= f(x) + f(x+h) - f(x) \qquad (\because \Delta f(x) = f(x+h) - f(x))\\
&= f(x+h)\\
&= E^1\, f(x) \qquad (\because E^1 f(x) = f(x+h))\\
&= Ef(x)
\end{aligned}
$$

$$\Longrightarrow (1+\Delta) = E$$

Prove that $E\nabla = \Delta$

$$
\begin{aligned}
E\nabla\big(f(x)\big) &= E[\nabla f(x)]\\
&= E[f(x) - f(x-h)] \qquad (\because \nabla f(x) = f(x) - f(x-h))\\
&= Ef(x) - Ef(x-h)\\
&= f(x+h) - f(x) \qquad (\because E^n f(x) = f(x+nh))\\
&= \Delta f(x) \qquad (\because \Delta f(x) = f(x+h) - f(x))
\end{aligned}
$$

$$\Longrightarrow E\nabla\big(f(x)\big) = \Delta f(x);\ \forall\, f(x)$$

$$\Longrightarrow E\nabla = \Delta$$

Prove that $\Delta\nabla = \Delta - \nabla$

$$
\begin{aligned}
\Delta\nabla\big(f(x)\big) &= \Delta[\nabla f(x)]\\
&= \Delta[f(x) - f(x-h)] \qquad (\because \nabla f(x) = f(x) - f(x-h))\\
&= \Delta f(x) - \Delta f(x-h)\\
&= \Delta f(x) - [f(x) - f(x-h)]\\
&\qquad (\because \Delta f(x) = f(x+h) - f(x))\\
&= \Delta f(x) - \nabla f(x) \qquad (\because \nabla f(x) = f(x) - f(x-h))\\
&= (\Delta - \nabla) f(x)
\end{aligned}
$$

$$\Longrightarrow \Delta\nabla\big(f(x)\big) = (\Delta - \nabla)f(x);\quad \forall\, f(x)$$

$$\Longrightarrow \Delta\nabla = \Delta - \nabla$$

Prove that $E = e^{hD}$

$$
\begin{aligned}
\mathrm{Ef(x)} &= \mathrm{f(x+h)} \\
&= \mathrm{f(x)} + \mathrm{hf'(x)} + \frac{\mathrm{h}^2}{2!}\mathrm{f''(x)} + \cdots \quad \text{(By Taylor's expansion)} \\
&= \mathrm{f(x)} + \mathrm{hDf(x)} + \frac{\mathrm{h}^2}{2!}\mathrm{D}^2\mathrm{f(x)} + \cdots \quad (\because f'(x) = Df(x)) \\
&= \left[1 + \mathrm{hD} + \frac{\mathrm{h}^2}{2!}\mathrm{D}^2 + \cdots\right]\mathrm{f(x)}
\end{aligned}
$$

$\Longrightarrow \mathrm{Ef(x)} = \mathrm{e}^{\mathrm{hD}}\mathrm{f(x)}$ $\quad (\because e^x = 1 + x + \frac{x^2}{2!} + \ \ldots..)$

$\Longrightarrow \mathrm{E} = \mathrm{e}^{\mathrm{hD}}$

Some More Relations

- $\nabla = 1 - E^{-1}$
- $\Delta = \nabla E$
- $\Delta\nabla = \Delta - \nabla$
- $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$
- $(1+\Delta)(1-\nabla) = 1$
- $\mu\delta = \frac{1}{2}(\Delta + \nabla)$

Interpolation

✓ Interpolation is the process of computing intermediate value of a function $y = f(x)$ from a given set of tabular values of the function.

x	x_0	x_1	x_2	…	x_n
y	y_0	y_1	y_2	…	y_n

✓ e.g.
The population of the town is given below. Estimate the population for the year 1895 and 1930 using suitable interpolation.

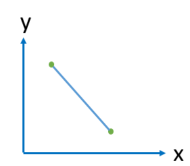
year	1891	1901	1911	1921	1931
Population in thousand	46	66	81	93	101

Interpolation

✓ Linear polynomial

For a two points data :

Linear Polynomial (first order) connecting two points data



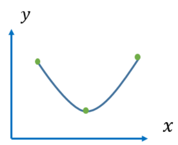
Linear interpolation

Interpolation

- ✓ Quadratic polynomial

  For a three points data :

  Quadratic Polynomial (second order) connecting three points data



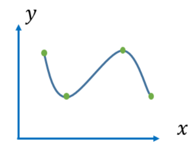
Quadratic interpolation

Interpolation

- ✓ Cubic polynomial

  For a four points data :

  Cubic Polynomial (third order) connecting four points data



Cubic interpolation

Interpolation

$x_0, x_1, x_2, \dots, x_n$

Equally spaced Argument

Unequally spaced Argument

e.g.

X(Arguments)	4	6	8	10
Y	1	3	8	16

Here, Difference between two consecutive value is same. E.g., $h = 6 - 4 = 2$

e.g.

X(Arguments)	1	2	7	8
Y	1	5	5	4

Here, Difference between two consecutive value is not same.

- ✓ We can find Interpolation for Equally spaced data from following methods :

1. Newton's Forward Difference Formula
2. Newton's Backward Difference Formula
3. Stirling's Formula

E.g.
The population of the town is given below. Estimate the population for the year 1895, 1910 & 1930 using suitable interpolation.

year	1891	1901	1911	1921	1931
Population in thousand	46	66	81	93	101

✓ We can find Interpolation for Unequally spaced data from following methods :

1. Newton's Divided Difference Formula
2. Lagrange's interpolation Formula

E.g.
Compute $f(9.2)$ by using interpolation method from the following data.

x	9	9.5	11
$f(x)$	2.1972	2.2513	2.3979

Newton's Forward Difference Formula

✓ $f(x) = y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \ldots\ldots$

Where $p = \frac{x-x_0}{h}$

Newton's Forward Difference Table

x	f(x) = y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		$\Delta y_3 = y_4 - y_3$			
x_4	y_4				

$f(x) = y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \ldots$; where $p = \frac{x-x_0}{h}$

Example 1:

Construct forward difference table for following data.(1, 1.1), (2,4.2), (3,9.3), (4,16.4).

x	1	2	3	4
y	1.1	4.2	9.3	16.4

Solution:

x	y	Δ	Δ^2	Δ^3
1	$y_0 = 1.1$			
		$\Delta y_0 = y_1 - y_0 = 3.1$		
2	$y_1 = 4.2$		2	
		$\Delta y_1 = y_2 - y_1 = 5.1$		0
3	$y_2 = 9.3$		2	
		$\Delta y_2 = y_3 - y_2 = 7.1$		
4	$y_3 = 16.4$			

Example 2:

Find $sin\ 52^0$ using the following values.

x	$sin45^\circ$	$sin50^\circ$	$sin55^\circ$	$sin60^\circ$
$f(x)$	0.7071	0.7660	0.8192	0.8660

Solution:

x	$\sin x$	Δ	Δ^2	Δ^3
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

Here, $h = 5$ & $x = 52°$

$$p = \frac{x - x_0}{h} = \frac{52 - 45}{5} = 1.4°$$

x	$\sin x$	Δ	Δ^2	Δ^3
45	0.7071			
		0.0589		
50	0.7660		-0.0057	
		0.0532		-0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

Using Newton's Forward

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ \dots\dots\dots$$

$$y(x) = 0.7071 + 1.4(0.0589) + \frac{(1.4)(0.4)(-0.0057)}{2} + \frac{(1.4)(0.4)(-0.6)(-0.0007)}{6}$$

Example 3:

Construct Newton's forward interpolation polynomial for the following data. Use it to find the value of $x = 5$.

x	4	6	8	10
y	1	3	8	16

Solution:

x	y	Δ	Δ^2	Δ^3
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

Here, $h = 2$, $x_0 = 4$ & $x = 5$

$$p = \frac{x - x_0}{h} = \frac{5 - 4}{2} = 0.5$$

x	y	Δ	Δ^2	Δ^3
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

Using Newton's Forward Interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ \dots\dots\dots$$

$$y(x) = 1 + (0.5)(2) + \frac{(0.5)(-0.5)(3)}{2} + \frac{(0.5)(-0.5)(-1.5)(0)}{6}$$

$y(5) \approx 1.625$

Now, Here, $h = 2,\ x_0 = 4\ \&\ x = x, p = \frac{x-x_0}{h} = \frac{x-4}{2}$

Using Newton's Forward Interpolation formula,

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots \ldots$$

$$y(x) = 1 + \frac{(x-4)(2)}{2} + \frac{(x-4)}{2}\left(\frac{x-4}{2} - 1\right)\left(\frac{3}{2}\right)$$

$$y(x) = 1 + (x-4) + (x-4)(x-6)\left(\frac{3}{8}\right)$$

$$y(x) = \frac{8 + (8x - 32) + (x^2 - 10x + 24)(3)}{8}$$

$$y(x) = \frac{3x^2 - 22x + 48}{8}$$

Exercise:

1. Using Newton's Forward Interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence, find $f(2)$.

x	0	5	10	15
$f(x)$	14	379	1444	3584

Ans: $f(x) = \frac{1}{2}[x^3 + 13x^2 + 56x + 28]$; $f(2) = 100$

2. Construct Newton's Forward interpolation polynomial for the following data. Use it to find the value of y for $x = 5$.

x	4	6	8	10
$f(x)$	1	3	8	16

Ans: $f(x) = \frac{1}{8}[3x^2 - 22x + 48]$; $f(5) = 1.625$

Newton's Backward Difference Formula

✓ $f(x) = y$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

$$\text{Where, } p = \frac{x - x_n}{h}$$

Newton's Backward Difference Table

x	$f(x) = y$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
x_0	y_0				
		$\nabla y_1 = y_1 - y_0$			
x_1	y_1		$\nabla^2 y_2$		
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		$\nabla y_4 = y_4 - y_3$			
x_4	y_4				

$$f(x) = y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Example 1:

Construct backward difference table from the following data

x	4	6	8	10
y	1	3	8	16

Solution:

x	y	∇	∇^2	∇^3
4	1			
		2		
6	3		3	
		5		0
8	8		3	
		8		
10	16			

Example 2:

The area of circle of diameter d is given by

x	80	85	90	95	100
y	5026	5674	6361	7088	7854

Use suitable interpolation to find area of circle of diameter 98. .Also Calculate the error.

Solution:

d	A	∇	∇^2	∇^3	∇^4
80	5026				
		648			
85	5674		39		
		687		1	
90	6361		40		-2
		727		-1	
95	7088		39		
		766			
100	7854				

here $h = 5,\ x = 98\ \&\ x_n = 100$

$$p = \frac{x - x_n}{h} = \frac{98 - 100}{5} = -0.4$$

d	A	∇	∇^2	∇^3	∇^4
80	5026				
		648			
85	5674		39		
		687		1	
90	6361		40		-2
		727		-1	
95	7088		39		
		766			
100	7854				

Using backward interpolation ,

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \ \dots\dots$$

$$= 7854 + (-0.4)(766) + \frac{(-0.4)(0.6)(39)}{2!} + \frac{(-0.4)(0.6)(1.6)(-1)}{3!}$$

$$+ \frac{(-0.4)(0.6)(1.6)(2.6)(-2)}{4!}$$

$$\boxed{y \approx 7543.0672}$$

Now exact Area= $\pi r^2 = \pi(49)^2 = 7542.9640$

error= $|exact - App| = |7542.9640 - 7543.0672|$

error= 0.1032

Example 3:

The population of the town is given below. Estimate the population for the year 1895 and 1930 using suitable interpolation.

year	1891	1901	1911	1921	1931
Population in thousand	46	66	81	93	101

Solution:

year	Population	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

here, $h = 10$, $x = 1895$ & $x_0 = 1891$

$$p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

Using Newton's Forward interpolation,

$$y(1895) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots \ldots$$

$$= 46 + (0.4)(20) + \frac{(0.4)(-0.6)(-5)}{2!} + \frac{(0.4)(-0.6)(-1.6)(2)}{3!}$$

$$+ \frac{(0.4)(-0.6)(-1.6)(-2.6)(-3)}{4!}$$

$$= 54.8528$$

$x = 1930, x_n = 1931$ & $h = 10$

$$p = \frac{x - x_n}{h} = \frac{1930 - 1931}{10} = -0.1$$

Using Newton's Backward interpolation,

$$y(1931) = y_n + p\nabla y_n + \frac{p(p+1)\nabla^2 y_n}{2!} + \cdots$$

$$= 101 + (-0.1)(8) + \frac{(-0.1)(0.9)(-4)}{2!} + \frac{(-0.1)(0.9)(1.9)(-1)}{3!}$$

$$+ \frac{(-0.1)(0.9)(1.9)(2.9)(-3)}{4!}$$

$$= 100.4705$$

Exercise:

1. **Using Newton's Backward Interpolation formula, find the polynomial $f(x)$ satisfying the following data. Hence, find $f(1.28)$.**

x	0	1	2	3
$f(x)$	1	0	1	10

Ans: $f(x) = x^3 - 2x^2 + 1$; $f(4) = 33$

2. **From the following table of half-yearly premium for policies maturing at different ages, estimate the premium at the age of 63.**

Age	45	50	55	60	65
Premium (in $)	114.84	96.16	83.32	74.48	68.48

Ans: $f(63) = 70.5851$

Newton's Divided Difference

✓ $[x_0, x_1] = \Delta y_0 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ (Δ Divided delta)

✓ $[x_0, x_1, x_2] = \Delta^2 y_0 = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

✓ $[x_0, x_1, \ldots, x_n] = \Delta^n y_0 = \frac{[x_1, x_2, \ldots, x_n] - [x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$

Newton's Divided Difference Formula

✓$f(x) = y$
$$= y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$

OR

✓$f(x) = y$
$$= y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)(x - x_1)\,\Delta y_0^2 + \cdots$$

Newton's Divided Difference Table

x	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	y_0				
		$\Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0}$			
x_1	y_1		$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0}$		
		$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\Delta^3 y_0 = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0}$	
x_2	y_2		$\Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}$		$\Delta^4 y_0 = \frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4 - x_0}$
		$\Delta y_2 = \frac{y_3 - y_2}{x_3 - x_2}$		$\Delta^3 y_1 = \frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4 - x_1}$	
x_3	y_3		$\Delta^2 y_2 = \frac{\Delta y_3 - \Delta y_2}{x_4 - x_2}$		
		$\Delta y_3 = \frac{y_4 - y_3}{x_4 - x_3}$			
x_4	y_4				

$$f(x) = y = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)(x - x_1)\,\Delta^2 y_0 + \cdots$$

Example 1:

If $f(x) = \frac{1}{x}$, find the divided differences $[a, b]$ and $[a, b, c]$.

Solution :

x	$f(x)$	First Divided Difference $\Delta f(x)$	Second Divided Difference $\Delta^2 f(x)$
$x_0 = a$	$y_0 = \frac{1}{a}$		
		$\Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a} = -\frac{1}{ab}$	
$x_1 = b$	$y_1 = \frac{1}{b}$		$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \frac{-\frac{1}{bc} + \frac{1}{ab}}{c - a} = \frac{1}{abc}$
		$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\frac{1}{c} - \frac{1}{b}}{c - b} = -\frac{1}{bc}$	
$x_2 = c$	$y_2 = \frac{1}{c}$		

$$\boxed{\Delta y_0 = [\boldsymbol{a}, \boldsymbol{b}] = -\frac{1}{ab}}$$

$$\boxed{\Delta^2 y_0 = [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}] = \frac{1}{abc}}$$

Example 2:

Calculate $y(4.5)$ using Newton's divided difference.

Depth(m)	1.9	3.1	4.2	5.1	5.8
Stress(ksf)	0.3	0.6	0.4	0.9	0.7

Solution:

x	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.9	0.3				
		0.25			
3.1	0.6		−0.1877		
		−0.1818		0.1739	
4.2	0.4		0.3687		−0.1295
		0.5556		−0.3313	
5.1	0.9		−0.5258		
		−0.2857			
5.8	0.7				

Here, $x = 4.5$

	x	f(x) = y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
x_0	1.9	0.3				
			0.25			
x_1	3.1	0.6		−0.1877		
			−0.1818		0.1739	
x_2	4.2	0.4		0.3687		−0.1295
			0.5556		−0.3313	
x_3	5.1	0.9		−0.5258		
			−0.2857			
x_4	5.8	0.7				

$$\mathbf{f(x)} = \mathbf{y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 y_0 + \cdots}$$

$$f(4.5) = \boxed{(0.3)} + (4.5 - 1.9)\boxed{(0.25)} + (4.5 - 1.9)(4.5 - 3.1)\boxed{(-0.1877)} + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)\boxed{(0.1739)} + (4.5 - 1.9)(4.5 - 3.1)(4.5 - 4.2)(4.5 - 5.1)\boxed{(-0.1295)}$$

$$f(4.5) = 0.5415$$

Example 3:

Find $f(8)$ from following data using Newton's Divided difference formula.

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution :

x	f(x) = y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

x	f(x) = y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

$$f(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 y_0 + \cdots$$

$$f(8) = 48 + (8 - 4)(52) + (8 - 4)(8 - 5)(15) + (8 - 4)(8 - 5)(8 - 7)(1) + (8 - 4)(8 - 5)(8 - 7)(8 - 10)(0) + (8 - 4)(8 - 5)(8 - 7)(8 - 10)(8 - 11)(0)$$

$$f(8) = 448$$

Exercise:

1. Evaluate $f(9)$ using Newton's Divided difference Interpolation formula from the following data.

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Ans: $f(9) = 810$

2. Using Newton's Divided difference Interpolation formula, find $f(3)$ from the following table.

x	-1	2	4	5
$f(x)$	-5	13	255	625

Ans: $f(3) = 71$

Lagrange's Interpolation Formula

✓ If data Points are $(x_0, y_0), (x_1, y_1), (x_2, y_2), .., (x_n, y_n)$

Where, $x_0, x_1, x_2, \dots, x_n$ are unequally spaced then,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\dots(x_0-x_n)}\, y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)}\, y_1$$

$$+ \dots + \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\dots(x_n-x_{n-1})}\, y_n$$

Lagrange's Quadratic Interpolation Formula

Explain quadratic Lagrange's interpolation formula.

Explanation :

Let y_0, y_1, y_2 be the value of $f(x)$ corresponding to the non-equally spaced arguments x_0, x_1, x_2.

Constructing a polynomial $p(x)$ of degree 2.

Let $y = p(x) = A_0(x-x_1)(x-x_2) + A_1(x-x_0)(x-x_2)$

$+A_2(x-x_0)(x-x_1) \dots\dots (1)$

Substituting $x = x_0$ & $y = y_0$ in (1), we have

$y_0 = p(x_0) = A_0(x_0-x_1)(x_0-x_2)$

$$\therefore A_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)}$$

Now,

Substituting $x = x_1$ & $y = y_1$ in (1), we have

$y_1 = p(x_1) = A_1(x_1-x_0)(x_1-x_2)$

$$\therefore A_1 = \frac{y_1}{(x_1-x_0)(x_1-x_2)}$$

In similar way,

Substituting $x = x_2$ & $y = y_2$ in (1), we have

$y_2 = p(x_2) = A_2(x_2-x_0)(x_2-x_1)$

$$\therefore A_2 = \frac{y_2}{(x_2-x_0)(x_2-x_1)}$$

Substituting all the values of A_k; $k = 0,1,2.$

Equation (1) becomes

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,y_0$$

$$+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

which is known as quadratic Lagrange's interpolation formula.

Example 1:

Determine the **interpolating polynomial** of degree three and compute $f(2)$ by using Lagrange's interpolation for the following table.

x	-1	0	1	3
y	2	1	0	-1

Solution :

By Lagrange's interpolation formula,

$y = P(x)$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here,

x	$x_0=-1$	$x_1=0$	$x_2=1$	$x_3=3$
y	$y_0=2$	$y_1=1$	$y_2=0$	$y_3=-1$

$$= \frac{(x-0)(x-1)(x-3)}{(-1-0)(-1-1)(-1-3)}(2) + \frac{(x+1)(x-1)(x-3)}{(0+1)(0-1)(0-3)}(1)$$

$$+\frac{(x+1)(x-0)(x-3)}{(1+1)(1-0)(1-3)}(0) + \frac{(x+1)(x-0)(x-1)}{(3+1)(3-0)(3-1)}(-1)$$

$$= \frac{x(x^2-4x+3)}{(-1)(-2)(-4)}(2) + \frac{(x^2-1)(x-3)}{(1)(-1)(-3)}(1)$$

$$+\frac{x(x^2-2x-3)}{(2)(1)(-2)}(0) + \frac{x(x^2-1)}{(4)(3)(2)}(-1)$$

$$y = \frac{(x^3-4x^2+3x)}{-8}(2) + \frac{(x^3-3x^2-x+3)}{3}(1)$$

$$+0 + \frac{x^3-x}{24}(-1)$$

$$= \frac{(x^3-4x^2+3x)}{24}(-6) + \frac{(x^3-3x^2-x+3)}{24}(8) + \frac{x^3-x}{24}(-1)$$

$$= \frac{-6x^3+24x^2-18x+8x^3-24x^2-8x+24-x^3+x}{24}$$

$$y = f(x) = \frac{1}{24}(x^3-25x+24)$$

$$y = f(2) = -0.75$$

Example 2:

Compute $f(9.2)$ by using Lagrange's interpolation method from the following data.

x	9	9.5	11
$y = f(x)$	2.1972	2.2513	2.3979

Solution :

By Lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0$$

$$+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Here,

x	$x_0 = 9$	$x_1 = 9.5$	$x_2 = 11$
$y = f(x)$	$y_0 = 2.1972$	$y_1 = 2.2513$	$y_2 = 2.3979$

$$= \frac{(9.2-9.5)(9.2-11)}{(9-9.5)(9-11)}(2.1972) + \frac{(9.2-9)(9.2-11)}{(9.5-9)(9.5-11)}(2.2513)$$

$$+\frac{(9.2-9)(9.2-9.5)}{(11-9)(11-9.5)}(2.3979)$$

$$= 1.1865 + 1.0806 - 0.0480$$

$$y(9.2) = 2.2191$$

Exercise

(1) Using Lagrange's formula of fit a polynomial to the data. And hence find $y(2)$.

x	-1	0	2	3
y	8	3	1	12

Ans : $\mathbf{y(x) = \frac{1}{3}[2x^3 + 2x^2 - 15x + 9], y(2) = 1}$

(2) Find the value of $\tan 33^0$ by Lagrange's formula If,
$\tan 30^0 = 0.5774$, $\tan 32^0 = 0.6249$, $\tan 35^0 = 0.7002$,
$\tan 38^0 = 0.7813$.

Ans : $[\mathbf{0.6494}]$