CS 4150 : PREREQ. MATERIAL REVIEW

8/30 — 8/31

OUTLINE

1. USEFUL IDENTITIES / BACKGROUND
2. CLOSED FORM OF SUMMATIONS
3. PROOF METHODS & TIPS
4. INDUCTION
5. CONTRADICTION
6. EXAMPLES

LOG RULES / IDENTITIES

- $\log_b(1) = 0$, $\log_b(b) = 1$, $\log_b(0) =$ undefined
- $\log_b(uv) = \log_b u + \log_b v$
- $\log_b \frac{1}{u} = -\log_b u$
- $\log_b u^r = r \cdot \log_b u$
- $\log_b\left(\frac{u}{v}\right) = \log_b u - \log_b v$
- $\log_b a = \dfrac{\log_c a}{\log_c b}$ {change of base}

$\boxed{\begin{array}{c} b > 0,\ b \neq 1 \\ M, N > 0 \end{array}}$

$$\log_a N = x \iff N = a^x$$

$$\ln N = x \iff N = e^x$$

summations (Σ)

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n$$

- $$\sum_{i=1}^{n} c \cdot i = c \cdot \sum_{i=1}^{n} i$$

- $$\sum_{i=1}^{n} i+i = \sum_{i=1}^{n} i + \sum_{i=1}^{n} i$$

- $$\sum_{i=3 \cdot n}^{5n} i = \sum_{i=1}^{5n} i - \sum_{i=1}^{3n-1} i$$

DATA STRUCTURES

- Arrays
- linked lists
- binary search trees
  → balanced bst
- Hash tables
- binary heaps

* know time complexity of various operations (e.g. insertions, deletions, etc)

[See ERICKSON preface for more pre-req.s]

$\cap$:

$\cup$:

$\subseteq$:

$\subsetneq$:

$\subset$:

$\vee$:

$\wedge$:

$\forall$:

$\exists$:

$\equiv$:

$\Rightarrow$:

$\Leftrightarrow$:

$\sum_{i=1}^{n}$:

$\prod_{i=1}^{n}$:

$\lceil x \rceil$: Ceiling

$\lfloor x \rfloor$: floor

$\neg$: negation

$\mathbb{R}$:

$\mathbb{Z}$:

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FINDING CLOSED FORM OF SUMMATIONS

Ex: $\sum_{i=1}^{n} i =$

Σ		n
1	$=1$	1
$1+2$	$=3$	2
$1+2+3$	$=6$	3
$1+2+3+4$	$=10$	4
$1+2+3+4+5$	$=15$	5
		$\vdots$

$$\sum_{i=1}^{n} i = n + \sum_{i=1}^{n-1} i$$

$$= n + (n-1) + \sum_{i=1}^{n-2} i$$

$$= n + (n-1) + (n-2) + \sum_{i=1}^{n-3} i$$

$$\vdots$$

How many terms? $n \longrightarrow n \cdot [\quad ? \quad]$

$$\sum_{i=1}^{n} i \longrightarrow 1+2+3+4+\dots\dots+(n-3)+(n-2)+(n-1)+n$$

$$(n-(n-1))+(n-(n-2))+(n-(n-3))+\dots+(n-3)+(n-2)+(n-1)+n$$

$$(n-(n-3)+(n-3)=n$$

$$(n-(n-2))+(n-2)=n$$

$$(n-(n-1))+(n-1)=n$$

$$(\#\text{ pairs}) \times n + n = \sum_{i=1}^{n} i$$

$$= \frac{(n-1)}{2} \times n + n = \frac{n(n-1)+2n}{2}$$

$$= \frac{n^2 - n + 2n}{2} = \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2} \quad \checkmark$$

TODO: a) prove w/ induction this is correct.
b) whether or not this is $O(n^2)$

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PROOF WRITING REQUIREMENTS

1) Written in English, paragraph format.

2) Use equations sparingly

3) No pseudocode (but line # references are fine)

PROOFS METHODS : $p \Rightarrow q$ (conditional statement)

WARNING:

Showing that either ①, ②, or ③ are true using one example is NOT sufficient.

↳ For methods ① ② and ③, statements must be true for ALL examples.

① **DIRECT PROOF:** Assume p, conclude q	② **CONTRADICTION:** Assume p and $\neg q$ ↳ somehow get to a contradiction
③ **CONTRAPOSITIVE:** Assume $\neg q$, conclude $\neg p$	**COUNTER EXAMPLE:** ④ (to show $p \Rightarrow q$ is false) Find one example of p and $\neg q$

1.

* General proofs

* can use only one example

[1] Table from Dr. Trefor Bazett's YouTube

PROOF STEPS

1) IDENTIFY LOGICAL STRUCTURE

IF ASSUMPTION $\rightarrow$ $P \Rightarrow q$ $\leftarrow$ CONCLUSION

"then"

e.g. IF x is even THEN x^2 is even

a) Conditional statement

b) BiConditional statement ($P \Leftrightarrow q$)

c) Just a statement: P

d) OR : P or q

e) AND : P and q

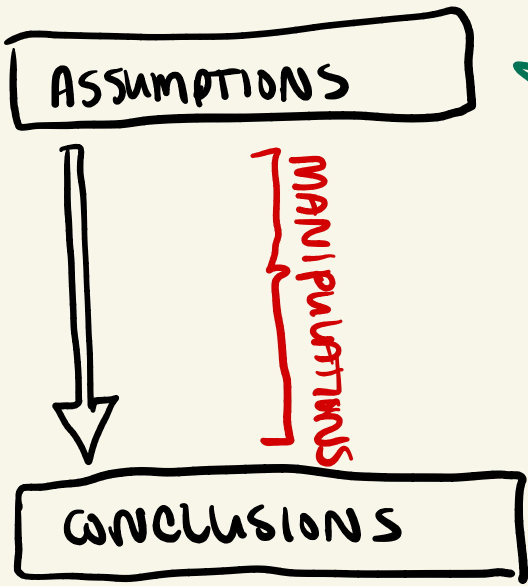
f) Existential : $\exists x, S(x)$

PROOF STEPS

2) FIGURE OUT THE PROPER PROOF METHODS.

(e.g. for $p \Rightarrow q$ stmts: (1) direct, (2) contradiction, (3) contrapositive)

(to show $p \Rightarrow q$ is false: (4) counterexample)

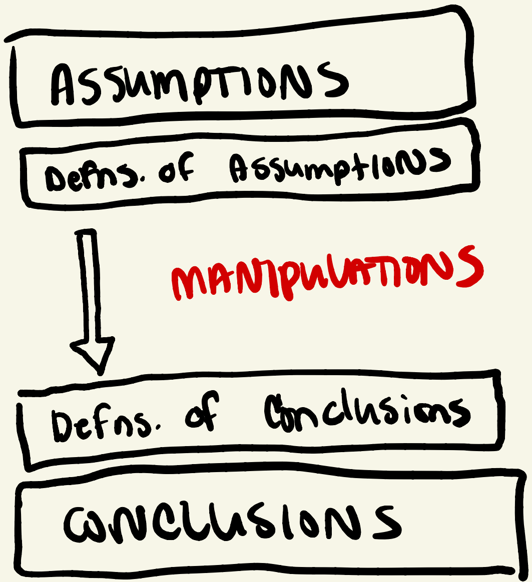


* PROOF METHOD OF CHOICE TELLS YOU WHAT TO PUT IN FOR ASSUMPTIONS & CONCLUSIONS

PROOF STEPS

3) WRITE DOWN ALL MATHEMATICAL DEFNS.

→ OF EVERY WORD IN ASSUMPTIONS & CONCLUSIONS



* WRITING PROOFS w/ THIS SCAFFOLDING makes the manipulations section a bit easier

[1] "SCAFFOLDING" FROM DR. TREFOR BAZETT'S YOUTUBE

EXAMPLE: IF $\underbrace{x \text{ is even}}_{p}$, then $\underbrace{x^2 \text{ is even}}_{q}$

$$p \Rightarrow q$$

ASSUMPTIONS

Defns. of Assumptions

Assume x is even

$x = 2p,\ p \in \mathbb{Z}$

$$\begin{aligned} x^2 &= (2p)^2 \\ &= 4p^2 \\ &= 2 \cdot \underbrace{(2p^2)}_{q} \checkmark \end{aligned}$$

MANIPULATIONS

Defns. of Conclusions

$x^2 = 2q,\ q \in \mathbb{Z}$

CONCLUSIONS

conclude that x^2 is even.

STEP 4) Go from Assumption to the conclusion.

* WANT Assumptions to look like conclusions.

TIPS: (pre-proof)

- Convince yourself of the truth of the statement first
  - work through concrete examples
  - ensure correct understanding of the problem itself

- CAn you draw a picture of what you're trying to prove?

TIPS: (intra-proof)

- "prove ... for all $n \geq 0$" $\equiv$ "... for each $n \geq 0$" $\equiv$ "... for any $n \geq 0$"
  ($\forall$) ($\forall$) ($\forall$)
  - → show problem for any arbitrary n (i.e $\forall n$)
  - → universally quantified statement
  - → THERE ARE ONLY TWO WAYS TO PROVE A UNIVERSALLY QUANTIFIED STATEMENT:
    1) DIRECTLY or
    2) by CONTRADICTION [ERICKSON, APPEN. I]

* STATE your proof method.

- use simple algebra in proof steps & err on the side of showing too many steps than too few.
  - ie. your colleague / boss / TA shouldn't have to do their own calculations to verify a step

Tips: (post-proof)

- Don't get discouraged. If current proof method (e.g. direct, contradiction, etc) isn't working or seems too difficult, try another method.
- Be suspicious & skeptical (are you completely convinced by the proof you just wrote, saw in class / online, etc.)
- Once you think your proof is complete, put yourself in your colleague / boss / TA's shoes and ask:

  a) Am I convinced by this proof?
  b) Can this proof be simplified / condensed?

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4-PARTS TO EVERY INDUCTION PROOF

0) Describe statement to be proved and any ranges on certain variables.

1) The base step: prove one or more base cases.

2) The inductive step: show how the truth of one statement follows from the truth of some previous statement(s)

   2a) state inductive hypothesis (IH)

   2b) verify IH

3) state the precise conclusion that follows by mathematical induction.

(Week) INDUCTION

Let $S(n)$ be a statement involving n. If

(i) $S(1)$ holds, and

(ii) for every $k \geq 1$, $S(k) \rightarrow S(k+1)$,

then for every $n \geq 1$, the statement $S(n)$ holds.

* method for proving universally quantified propositions

→ statements about all elements of a set.

($\forall$)

EXAMPLE:

[For every positive integer n, $(\forall n \in \mathbb{Z}^+)$

$$1+2+3+\ldots+n = \sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$$
]

PROOF:

0) $S(n)$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ {describing our statement $S(n)$ to prove}

1) Base step $(n=1)$: $S(1)$ says that $1 = \frac{1(2)}{2} = 1$ ✓, which is true so $S(1)$ holds.

2a) Inductive Hyp. (IH): For some $k \geq 1$, suppose that

$S(k)$: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ also holds.

2b)

$\longrightarrow$ continued ...

2b) To complete the inductive step, it suffices to verify that the statement:

$$S(k+1): \boxed{\sum_{i=1}^{k+1} i} = \frac{(k+1)(k+2)}{2}$$ also holds.

Goal: show this

$$= \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \quad \{\text{by IH}\}$$

* must use IH & also state when you're using it.

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2} \checkmark$$

Hence $S(k) \to S(k+1)$ is proved, completing the inductive step.

3) $\xrightarrow[\text{continued...}]{}$

3) Conclusion: By the principal of mathematical induction, for each $n \geq 1$, the statement $S(n)$ is true.

$\blacksquare$

Tip

- If statement to be proved has the variable n, use a different variable (say k), for the inductive step.
  - n is varying, whereas k is fixed
  - using the same variable can lead to confusion

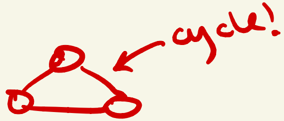
e.g. • If $S(n)$ is stated, inductive step could be $S(k) \to S(k+1)$ GOOD

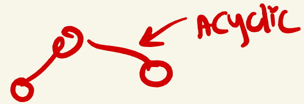
• Some would say: " Assume the statement is true for $n=k$ (IH); to be proved is the statement for $n=k+1$.

Could get confusing!

EXAMPLE: For a $(\forall)$ graph G on $n \geq 1$ vertices, prove that if:

G is connected and acyclic $\Rightarrow$ G is connected and has $n-1$ edges.
(i.e. G is a tree)





proof (weak induction): By induction on n (the # of vertices), for $n \geq 1$

Let $S(n)$ be the statement that any $(\forall)$ tree on n vertices has $n-1$ edges.

$n-1=m$

BASE STEP ($n=1$): The only tree w/ a single vertex has $1-1=0$ edges, so $S(1)$ holds.

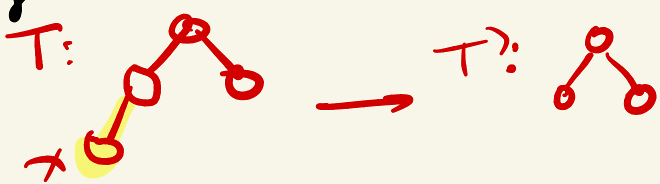
$n=1$
$m=0$

Lemma A: Every tree contains at least one leaf.

INDUCTIVE STEP:

(IH) For some fixed $k \geq 1$, suppose that $S(k)$ is true.

Now let T be a tree on $k+1$ vertices. By Lemma A, let $x \in V(T)$ be a leaf, and form T' by deleting x (and the edge incident w/ x).



*removed exactly 1 vertex and 1 edge.

Then $|V(T')| = k$, and so by $S(k)$ (IH), T' has $k-1$ edges.

Together with the edge removed, this shows that

$$|E(T)| = k = (k+1) - 1 = |V(T)| - 1,$$ = n − 1 edges ✓

showing that $S(k+1)$ follows, completing the inductive step.

$V(T)$ has $k+1$ vertices

Since by IH, $|E(T')| = k-1$ & we only removed 1 edge from T to get T'.

Conclusion: By mathematical induction, for every $n \geq 1$, $S(n)$ holds.

(strong) INDUCTION

> Let $S(n)$ denote a statement involving an integer n. If
>
> (i) $S(k)$ is true and
>
> (ii) for every $m \geq k$, $[S(k) \wedge S(k+1) \wedge \ldots \wedge S(m)] \rightarrow S(m+1)$
>
> then for every $n \geq k$, the statement $S(n)$ is true.

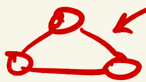
* Certain proofs require more "power"

⟶ done by strengthening the inductive hypothesis

- <u>All</u> weak inductive proofs can be written as strong induction.

EXAMPLE (AGAIN): For a ($\forall$) graph G on $n \geq 1$ vertices, prove that if:

G is connected and acyclic $\Rightarrow$ G is connected and has $n-1$ edges.
(i.e. G is a tree)



STATEMENT $S(n)$ and BASE step remain the same.

Proof (strong induction)

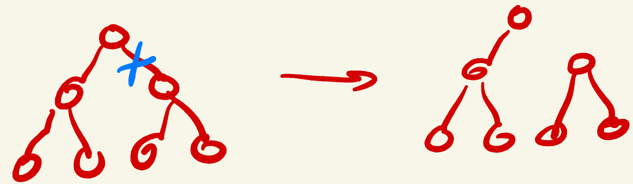
By induction on n (the # of vertices), for $n \geq 1$
let $S(n)$ be the statement that any ($\forall$) tree on n vertices has $n-1$ edges.

BASE STEP ($n=1$): The only tree w/ a single vertex has $1-1=0$ edges, so $S(1)$ holds.

$n-1=m$

$0 \quad n=1, \; m=0$

Lemma B: Every edge in a tree is a bridge, and removal of any edge creates two trees.



INDUCTIVE STEP:

(IH) Fix $k \geq 1$ and suppose that $S(1), \ldots, S(k)$ are all true.

Let T be a tree on $k+1$ vertices.

* Need to prove T has k edges.

By Lemma B, let $\alpha = \{x, y\}$ be a bridge in T;

removal of α produces two trees, say T_1 and T_2.

T_1 and T_2 has at most k vertices.

By the IHs, $S(|V(T_1)|)$ and $S(|V(T_2)|)$,

T_1 has $|V(T_1)|-1$ edges and T_2 has $|V(T_2)|-1$ edges.

Since T_1 and T_2 together have $k+1$ vertices,

α

$$\begin{aligned} |E(T)| &= |E(T_1)| + |E(T_2)| + 1 \\ &= |V(T_1)| - 1 + |V(T_2)| - 1 + 1 \\ &= (k+1) - 1 - 1 + 1 = k \end{aligned}$$

T has k edges as required!

completing the proof of $S(k+1)$.

Conclusion: By mathematical induction, for every $n \geq 1$, $S(n)$ holds. $\square$

WEAK OR STRONG INDUCTION?

For a graph G on $n \geq 1$ vertices, prove that if:

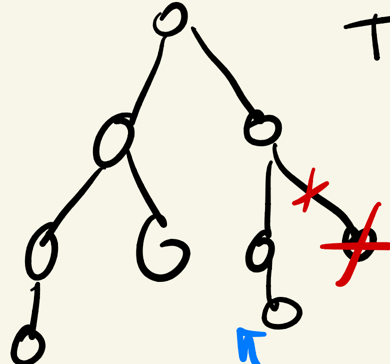
G is connected and acyclic $\Rightarrow$ G is connected and has $n-1$ edges.
(i.e. G is a tree)

- Choice (weak/strong) depends on the idea behind the proof.

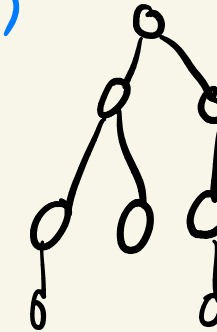
WEAK PROOF: used Lemma A and deleted one leaf producing another tree w/ one fewer vertex and one fewer edge.
↳ IH needs to only apply to the remaining tree on $n-1$ vertices and $n-2$ edges.

STRONG proof: If one didn't think to delete a leaf vertex, instead one could delete any edge and get two smaller trees.
↳ To apply any IH to smaller trees, one must assume IH holds for all smaller trees.

WEAK



T : n vertices (k+1)
n-1 edges

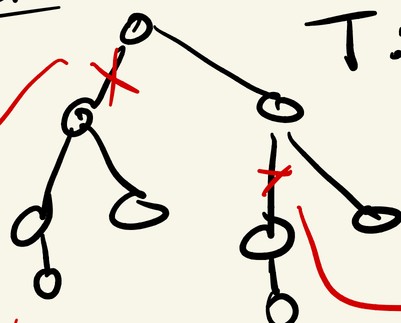


T' : always has n-1 (k) vertices and n-2 edges.

so assuming S(k)

is enough to show S(k+1)

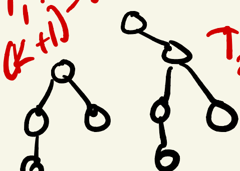
STRONG



T : (k+1) vertices
(k+1)-1 edges.

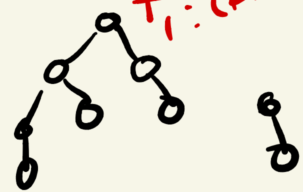
T_1 : (k+1)-5

T_2 : (k+1)-4



T_1 : (k+1)-2

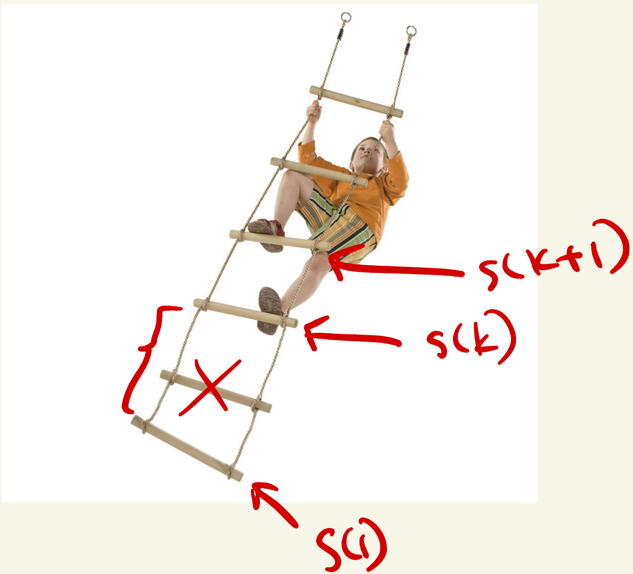
T_2 : (k+1)-7

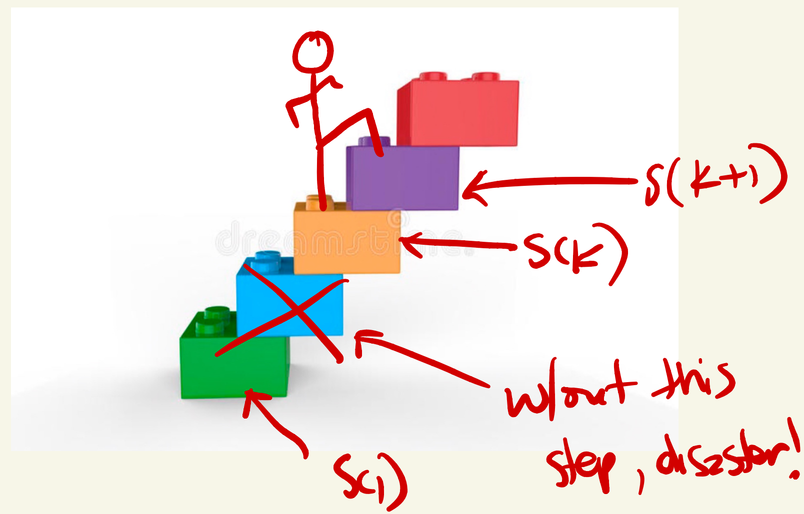


can't simply assume S(k)

to show this

WEAK VS. STRONG INDUCTION





DECISIONS

- Type of induction
  → can't go wrong w/ strong induction
- Which variable to induct on
- How many base cases?

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PROOFS BY CONTRADICTION

② CONTRADICTION:

Assume p and $\neg q$

↳ somehow get to a contradiction

* REMINDER

1 of only 2 ways to prove universally quantified statements ($\forall$)

EXAMPLE

THEOREM: No integer is both even and odd.

$\equiv$ 1) Logical structure $\rightarrow$ $\left[\forall n \in \mathbb{Z},\ \neg(n \text{ is even } \& \text{ odd})\right]$ $\leftarrow$ statement P

Proof:

ASSUMPTIONS

Defns. of Assumptions

$\Downarrow$ MANIPULATIONS

Defns. of Conclusions

CONCLUSIONS

2) Assume our statement is false. i.e $\neg P$

$\hookrightarrow \neg(\forall n \in \mathbb{Z},\ \neg(n \text{ is even } \& \text{ odd}))$

$\equiv \exists n \in \mathbb{Z},\ n \text{ is even } \& \ n \text{ is odd}$

*there has to be at least one that's both even & odd

3) Defns: so $\exists k_1, k_2 \in \mathbb{Z},\ n = 2k_1,\ n = 2k_2 + 1$

4) manipulations: $2k_1 = 2k_2 + 1$

$\Rightarrow 2(k_1 - k_2) = 1$

$\Rightarrow \boxed{k_1 - k_2 = \frac{1}{2}}$ *

nonsense! $k_1, k_2 \in \mathbb{Z}$. an integer minus an integer is still an integer. But here we're showing that an int. minus an int. gives a fraction!

$\Rightarrow\Leftarrow$

*Need to derive a conclusion that is nonsense.

Example (cont..)

$$k_1 - k_2 \in \mathbb{Z}$$

and $\Rightarrow\Leftarrow$

$$k_1 - k_2 \notin \mathbb{Z}$$

→ Since the negation of our original statement (p) is nonsense and claiming so leads to a contradiction.

Therefore, our original statement p is true!

↳ Because a stmt. p is either <u>TRUE</u> OR <u>FALSE</u>

i.e. The statement: (no integer is both even & odd) <u>must</u> be true.

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(StoogeSort) Prove by induction (on n) that the following algorithm sorts its input.

Algorithm 2: StoogeSort(A[0, ..., n-1])

---

if $n = 2$ *and* $A[0] > A[1]$ **then**
    swap $A[0]$ and $A[1]$
else if $n > 2$ **then**
    $m = \lceil 2n/3 \rceil$
    StoogeSort$(A[0, ..., m-1])$
    StoogeSort$(A[n-m, ..., n-1])$
    StoogeSort$(A[0, ..., m-1])$

---

2. (StoogeSort)

We prove that StoogeSort correctly sorts the input array A by (strong) induction on n, the length of A.

Base Case (n=2): Consider an array A of size 2. If the first value in A is larger than the second, the if statement on line 1 will trigger the values being swapped on line 2, and A will be sorted. If the second value in A is larger, then A is already sorted (and the if statement in line 1 has no effect). Lines 3-7 do not apply to arrays of size 2, so the base case holds.

Induction Hypothesis: StoogeSort correctly sorts arrays of size k for all $k = 2, ..., n-1$.

Inductive Step: We need to show that StoogeSort correctly sorts arrays of size n. Let $m = \lceil 2n/3 \rceil$ as in line 4. For convenience, we refer to the "thirds" of the array as $A_1 = A[0, \ldots, n-m-1]$, $A_2 = A[n-m, \ldots, m-1]$ and $A_3 = A[m, \ldots, n-1]$. StoogeSort makes three recursive calls on arrays with size $m < n$.

After the first recursive call on line 5, $A_1 \cup A_2$ is correctly sorted (by the induction hypothesis since $|A_1 \cup A_2| < n$). That is, after line 5 we have that

$$A_1[0] \leq ... \leq A_1[n-m-1] \leq A_2[n-m] \leq ... \leq A_2[m-1] \tag{2}$$

Then second recursive call on line 6 correctly sorts $A_2 \cup A_3$ (again by the induction hypothesis since $|A_2 \cup A_3| < n$). That is, we have that

$$A_2[n-m] \leq ... \leq A_2[m-1] \leq A_3[m] \leq ... \leq A_3[n-1] \tag{3}$$

However, after the execution of line 6, equation 2 may no longer hold. For example, if some value in A_3 was the smallest in the array it is now at $A_2[n-m]$, but is less than $A_1[0]$, which is problematic. Because of this, there is a final recursive call on line 7, to sort $A_1 \cup A_2$ again. This call correctly sorts $A_1 \cup A_2$ by the induction hypothesis since $|A_1 \cup A_2| < n$. Note that A_3 was untouched by the recursive call on line 7 and was correctly sorted prior to the call. Further, after line 5, all $n/3$ values in A_2 are at least every value in A_1. This implies that after line 6, all values in A_3 are at least all values in A_2 *and* at least all values in A_1. We can thus conclude that A is correctly sorted.

FIND CLOSED FORM

$$\sum_{i=1}^{n} i^2 = \theta_n$$

??

$$\theta_n + (n+1)^2 = \theta_{n+1}$$

$$= \sum_{i=1}^{n+1} i^2$$

$$= \sum_{i=0}^{n} (i+1)^2$$

$$= \sum_{i=0}^{n} i^2 + 2i + 1$$

$n=2$: $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$

$\parallel$

$n=2$: $(0+1)^2 + (1+1)^2 + (2+1)^2 = 14$

$1^2 + 2^2 + 3^2$

cont...

$$= \sum_{i=0}^{n} i^2+2i+1 \qquad \{\text{copied over}\}$$

$$= \sum_{i=0}^{n} i^2 + \sum_{i=0}^{n} 2i + \sum_{i=0}^{n} 1$$

$$\Theta_n + 2\sum_{i=0}^{n} i + \underbrace{\left[1+\overbrace{1+\dots+1}^{n}\right]}_{n+1}$$

$$\cancel{\Theta_n} + (n+1)^2 = \cancel{\Theta_n} + 2\sum_{i=0}^{n} i + n+1$$

$$\Rightarrow \sum_{i=0}^{n} i = \frac{1}{2}\cdot(n^2+2n+1) - n - 1 \qquad \{\text{simplify + rearrange}\}$$

$$= \frac{1}{2}(n^2+n) = \frac{n(n+1)}{2} \qquad (!)$$

Since we know that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Observation: trying to find closed form $\sum_{i=1}^{n} i^2$ led us to actually finding $\sum_{i=1}^{n} i$!

now,

if we want to find closed form $\sum_{i=1}^{n} i^2$, lets try to instead find $\sum_{i=1}^{n} i^3$.

cont...

Say $\sum_{i=1}^{n} i^3 = \lambda_n$

$$\lambda_n + (n+1)^3 = \lambda_{n+1}$$

$$= \sum_{i=1}^{n+1} i^3 = \sum_{i=0}^{n} (i+1)^3$$

$$= \sum_{i=0}^{n} k^3 + 3k^2 + 3k + 1$$

$$= \sum_{i=0}^{n} k^3 + 3\sum_{i=0}^{n} k^2 + 3\sum_{i=0}^{n} k + (n+1)$$

$$\cancel{\lambda_n} + (n+1)^3 = \underbrace{\sum_{i=0}^{n} k^3}_{\lambda_n} + 3\underbrace{\sum_{i=0}^{n} k^2}_{\theta_n} + 3\underbrace{\sum_{i=0}^{n} k}_{\frac{n(n+1)}{2}} + (n+1)$$

GOAL

$$n^3+3n^2+3n+1 = 3\theta_n + 3\cdot\frac{n(n+1)}{2} + (n+1)$$

$$\left(n^3+3n^2+2n = 3\theta_n + \frac{3n^2+3n}{2}\right)2$$

*This method is called "perturbing the sum"

$$\Rightarrow 6\theta_n = 2n^3+6n^2+4n-3n^2-3n = 2n^3+3n^2+n \quad *$$

$$\Rightarrow 6\theta_n = \frac{n(2n+3+1)}{6} = \boxed{\frac{n(2n+1)(n+1)}{6} = \sum_{i=1}^{n} i^2}$$