1 The Clohessy Wiltshire Model

The relative motion of a chase spacecraft with respect to a target spacecraft that is in a circular orbit about a central body, considered a point mass, can be described by the CW equations of motion

$$\ddot{x} = 3n^2x + 2n\dot{y}
\ddot{y} = -2n\dot{x}
\ddot{z} = -n^2z$$

where the x-axis is along the radius vector of the target spacecraft, the z-axis is along the angular momentum vector of the target spacecraft, and the y-axis completes the right handed system. With this definition, the central body is towards the negative x direction and the y-axis points along the velocity vector of the target spacecraft. Motion along y-axis is considered 'along-track', and motion along the positive and negative z-axis is considered 'out-of-plane' motion.

These equations have a closed form solution given by

$$x(t) = (4 - 3\cos nt)x_0 + \frac{\sin nt}{n}\dot{x}_0 + \frac{2}{n}(1 - \cos nt)\dot{y}_0$$

$$y(t) = 6(\sin nt - nt)x_0 + y_0 - \frac{2}{n}(1 - \cos nt)\dot{x}_0 + \frac{4\sin nt - 3nt}{n}\dot{y}_0$$

$$z(t) = z_0\cos nt + \frac{\dot{z}_0}{n}\sin nt$$

where

$$n = \sqrt{\frac{\mu}{a_t^3}}$$

and a_t is the semi-major axis of the target vehicle's orbit.

The velocities are then can be computed by taking the first time derivative of the position components.

If a state vector is defined as

$$\delta \mathbf{s} = \left(egin{array}{c} \delta \mathbf{r} \ \delta \mathbf{v} \end{array}
ight) = \left(egin{array}{c} x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z} \end{array}
ight)$$

The solution of the system is given by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 4 - 3\cos nt & 0 & 0 & \frac{1}{n}\sin nt & \frac{2}{n}(1 - \cos nt) & 0 \\ 6(\sin nt - nt) & 1 & 0 & -\frac{2}{n}(1 - \cos nt) & \frac{1}{n}(4\sin nt - 3nt) & 0 \\ 0 & 0 & \cos nt & 0 & 0 & \frac{1}{n}\sin nt \\ 3n\sin nt & 0 & 0 & \cos nt & 2\sin nt & 0 \\ 3n\sin nt & 0 & 0 & -2\sin nt & 4\cos nt - 3 & 0 \\ 0 & 0 & -n\sin nt & 0 & 0 & \cos nt \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{pmatrix}$$

The propagation matrix premultiplying the initial condition vector is also referred to as the state transition matrix for the CW equations. This matrix is labeled as $\Phi(t)$, and is a time varying. The system can be partitined as

$$\left(\begin{array}{c} \delta \mathbf{r}(t) \\ \delta \mathbf{v}(t) \end{array}\right) = \left(\begin{array}{cc} \mathbf{M}(t) & \mathbf{N}(t) \\ \mathbf{S}(t) & \mathbf{T}(t) \end{array}\right) \left(\begin{array}{c} \delta \mathbf{r_0} \\ \delta \mathbf{v_0} \end{array}\right)$$

So that the position and velocities are given as

$$\delta \mathbf{r}(t) = \mathbf{M}(t)\delta \mathbf{r}_0 + \mathbf{N}(t)\delta \mathbf{v}_0$$

$$\delta \mathbf{v}(t) = \mathbf{S}(t)\delta \mathbf{r}_0 + \mathbf{T}(t)\delta \mathbf{v}_0$$

In the text (Wiesel, Spaceflight Dynamics), the variables are related to the ones shown as

$$\begin{array}{rcl} \psi & = & nt \\ \delta r & = & x \\ r_0 \delta \theta & = & y \\ \delta z & = & z \\ \Phi_{rr} & = & \mathbf{M} \\ \Phi_{rv} & = & \mathbf{N} \\ \Phi_{vr} & = & \mathbf{S} \\ \Phi_{vv} & = & \mathbf{T} \end{array}$$