Module 4 Fuzzy Logic

Motivation for Fuzzy Theory

- In classical logic, statements are either true or false, and in classical set theory, elements either belong to a set or they do not.
- However, real-world situations often involve uncertainty and vagueness.

Consider the concept of "tall people":

- Suppose we define "tall" as anyone above 6 feet (183 cm).
- In classical set theory, a person who is **5 feet 11 inches (180 cm)** is **not tall**, while someone who is **6 feet 1 inch (185 cm)** is **tall**.
- This rigid classification does not reflect reality, as there is no sharp boundary where a person suddenly becomes "tall." Instead, tallness is **gradual** and **subjective**.

Fuzzy sets solve this problem by allowing degrees of membership. Instead of being simply "tall" or "not tall," we can assign a **membership value** between 0 and 1. For example:

Height (cm)	Classical Set (Tall?)	Fuzzy Set (Membership in "Tall")
160 cm	No	0.1
170 cm	No	0.3
180 cm	No	0.7
183 cm	Yes	0.8
190 cm	Yes	1.0

Here, fuzzy logic models the real-world vagueness better than classical (crisp) sets.

Introduction to Fuzzy Logic

- Fuzzy logic is an extension of classical logic that allows **partial truth values** between 0 and 1.
- It was introduced by **Lotfi Zadeh** in 1965 to handle uncertainty and imprecision in real-world problems.

Key Characteristics of Fuzzy Logic

1. **Degrees of Truth**: Instead of strict true/false values, statements can be **partially true**.
2. **Linguistic Variables**: Uses human-like reasoning, such as "high temperature" or "medium speed," which are inherently fuzzy.
3. **Fuzzy Inference System (FIS)**: Uses fuzzy rules (IF-THEN statements) to make decisions.

Applications of Fuzzy Logic

- **Control Systems**: Washing machines, air conditioners, and cameras use fuzzy logic for adjusting settings.
- **Medical Diagnosis**: Handles uncertain symptoms for better decision-making.
- **Artificial Intelligence**: Used in **natural language processing (NLP)** and **robotics**.

Classical Sets vs. Fuzzy Sets

Classical (Crisp) Sets

A classical set is a well-defined collection of elements where each element either **belongs** to the set (membership = 1) or **does not belong** to the set (membership = 0).

Example

- Define a crisp set **TALL** as the set of people with height **≥ 183 cm**.
- A person who is 185 cm is **in the set** (1), while a person who is 180 cm is **not in the set** (0).

Mathematically:

$$\chi_A(x) = \begin{cases} 1, & x \geq 183 \\ 0, & x < 183 \end{cases}$$

where $\chi_A(x)$ is the membership function of the set **A**.

Fuzzy Sets

A fuzzy set allows for **gradual membership** between 0 and 1. Instead of a sharp boundary, elements can have **partial** membership.

Example

Instead of defining **TALL** as a crisp set, use a fuzzy set with a **membership function**:

$$\mu_T(x) = \begin{cases} 0, & x < 169 \\ \frac{x-170}{20}, & 170 \leq x < 190 \\ 1, & x \geq 190 \end{cases}$$

This means:

- A height of **169 cm** has a membership of **0** (not tall).
- A height of **180 cm** has a membership of **0.5** (somewhat tall).
- A height of **190 cm** has a membership of **1** (fully tall).

Comparison: Crisp Set vs. Fuzzy Set

Feature	Crisp Set	Fuzzy Set
Membership	Strict (0 or 1)	Gradual (0 to 1)
Boundaries	Sharp (well-defined)	Soft (overlapping)
Flexibility	Rigid classification	Handles uncertainty well
Mathematical Representation	Set notation (∈ or ∉)	Membership function μ(x)
Example	A person is "tall" or "not tall"	A person can be "somewhat tall"

Summary

- Fuzzy logic provides a more **realistic** and **flexible** approach to reasoning under uncertainty.
- Unlike classical sets, fuzzy sets allow elements to have **partial membership**, making them useful in real-world applications such as **AI, control systems, and decision-making**.

Crisp Set

- Classical set theory is a branch of mathematical logic that studies sets, which are collections of objects.
- The operations on classical sets include union, intersection, and complement.

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1. Union of Sets (∪)

The **union** of two sets A and B is the set of elements that belong to either A or B or both.

Mathematically,

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Example:

Let $A=\{1,2,3,4\}$ and $B=\{3,4,5,6\}$
Then,
$A \cup B=\{1,2,3,4,5,6\}$
This means the union includes all elements from both sets, without repetition.

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2. Intersection of Sets (∩)

The **intersection** of two sets A and B is the set of elements that are present in both A and B.

Mathematically,

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Example:

Using the same sets:
$A=\{1,2,3,4\}$ and $B=\{3,4,5,6\}$

$A \cap B=\{3,4\}$

Only elements common to both sets are included.

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3. Complement of a Set (A' or A^c)

The **complement** of a set A refers to all elements in the universal set U that are **not in** A.

Mathematically,

$A^c = U - A = \{x \mid x \in U \text{ and } x \notin A\}$

Example:

Let the universal set be U = {1, 2, 3, 4, 5, 6},
and let $A=\{1,2,3\}$.

Then,

A^c = {4, 5, 6}

These are the elements in U but not in A.

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4. Difference of Sets (-)

The **difference** of two sets A−B is the set of elements in A that are not in B.

Mathematically,

$A-B=\{x \mid x\in A \text{ and } x\notin B\}$

Example:

Using A={1,2,3,4}, B = {3, 4, 5, 6},

A−B={1,2}

These elements are in A but not in B.

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5. Symmetric Difference (⊕)

The **symmetric difference** of two sets A and B includes elements that are in **either** A or B but **not in both**.

Mathematically,

$A\oplus B=(A-B)\cup(B-A)$

Example:

For A={1,2,3,4} and B = {3, 4, 5, 6},

$A\oplus B$={1,2,5,6}

This consists of elements that are **not common** in both sets.

Properties of Classical Sets

The following properties hold for any sets A,B, and C:

1. Commutative Property

- **Union:** $A \cup B = B \cup A$
- **Intersection:** $A \cap B = B \cap A$

Example:
For $A=\{1,2\}$ $B = \{2, 3\}$,
$A \cup B=\{1,2,3\}=B \cup A$
$A \cap B=\{2\}=B \cap A$

2. Associative Property

- **Union:** $(A \cup B) \cup C=A \cup (B \cup C)$
- **Intersection:** $(A \cap B) \cap C=A \cap (B \cap C)$

Example:
Let $A=\{1,2\},B=\{2,3\},C=\{3,4\}$

$(A \cup B) \cup C=\{1,2,3\} \cup \{3,4\}=\{1,2,3,4\}$
$A \cup (B \cup C)=\{1,2\} \cup \{2,3,4\}=\{1,2,3,4\}$

Both are equal.

3. Distributive Property

- $A \cap (B \cup C)=(A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C)=(A \cup B) \cap (A \cup C)$

Example:
Using $A=\{1,2,3\}$, $B = \{2, 3, 4\}$, $C=\{3,4,5\}$:

$A \cap (B \cup C)=\{1,2,3\} \cap \{2,3,4,5\}=\{2,3\}$
$(A \cap B) \cup (A \cap C)=\{2,3\} \cup \{3\}=\{2,3\}$

Both are equal.

4. Identity Property

- $A \cup \emptyset=A$
- $A \cap U=A$

Example:
For $A=\{1,2,3\}$
$A \cup \emptyset=\{1,2,3\}$
$A \cap U=A$

5. Idempotent Property

- A∪A=A
- A∩A=A

6. Complement Laws

- $A \cup A^c = U$
- A $\cap A^c = \emptyset$

7. De Morgan's Laws

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$

Example:
If U={1,2,3,4,5}, A = {1, 2}, B={2,3}

$(A \cup B)^c = \{4, 5\} = A^c \cap B^c$

Summary

- Set theory provides a powerful mathematical framework to work with collections of elements.
- The operations **union, intersection, complement, difference,** and **symmetric difference** help define relationships between sets.
- Properties like **commutativity, associativity, distributivity, identity, idempotency, and De Morgan's Laws** provide foundational rules for working with sets.

Introduction to Fuzzy Sets

- In a classical set theory, an element either **belongs (1)** or **does not belong (0)** to a set.
- In contrast, in fuzzy set theory, an element **can partially belong** to a set with a membership value in the range **[0,1]**.

Membership Function (μ)

A **membership function (μA(x))** defines the degree of membership of an element x in a fuzzy set A. The membership function assigns a value between **0 and 1**, indicating how much the element belongs to the set.

Operations on Fuzzy Sets

Just like classical set operations (union, intersection, complement), fuzzy sets have their own operations, but they use **membership functions** instead of strict inclusion.

1. Union (OR Operation)

The **union** of two fuzzy sets A and B is defined by the **maximum** of their membership values.

$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$

Example

Let's consider two fuzzy sets:

- **A**:
  $\mu_A(x) = \{(15, 1.0), (20, 0.8), (25, 0.5), (30, 0.2)\}$
- **B:
  $\mu_B(x) = \{(15, 0.9), (20, 0.7), (25, 0.6), (30, 0.3)\}$

Now, the **union** of A and B:

$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$

Age	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A\cup B}(x)$
15	1.0	0.9	1.0
20	0.8	0.7	0.8
25	0.5	0.6	0.6
30	0.2	0.3	0.3

2. Intersection (AND Operation)

The **intersection** of two fuzzy sets A and B is defined by the **minimum** of their membership values.

$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$

Example

Using the same sets:

$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$

Age	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A\cap B}(x)$
15	1.0	0.9	0.9
20	0.8	0.7	0.7
25	0.5	0.6	0.5
30	0.2	0.3	0.2

3. Complement (NOT Operation)

The **complement** of a fuzzy set A is defined as:

$\mu_{\neg A}(x) = 1 - \mu_A(x)$

Example

Age	$\mu_A(x)$	$\mu_{\neg A}(x)$
15	1.0	0.0
20	0.8	0.2
25	0.5	0.5
30	0.2	0.8

4. Algebraic Sum

The **algebraic sum** of two fuzzy sets (A) and (B) is defined as (A+B):

$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$

This operation combines the membership values of (A) and (B) in a way that the result is always between 0 and 1.

Example

Let (A) and (B) be two fuzzy sets with membership values:

(x)	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A+B}(x)$
1	0.3	0.5	0.3 + 0.5 - (0.3)(0.5) = 0.65
2	0.7	0.2	0.7 + 0.2 - (0.7)(0.2) = 0.76
3	0.8	0.6	0.8 + 0.6 - (0.8)(0.6) = 0.92

5. Algebraic Product

The **algebraic product** of two fuzzy sets (A) and (B) is defined as, A.B:

$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$

This operation represents the **joint membership** of (x) in both (A) and (B).

Example

Using the same fuzzy sets (A) and (B):

(x)	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A \cdot B}(x)$
1	0.3	0.5	0.3 * 0.5 = 0.15
2	0.7	0.2	0.7 * 0.2 = 0.14
3	0.8	0.6	0.8 * 0.6 = 0.48

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6. Bounded Sum

The **bounded sum** of two fuzzy sets (A) and (B) is defined as:

$\mu_{A \oplus B}(x) = \min(1, \mu_A(x) + \mu_B(x))$

This operation ensures that the resulting membership value does not exceed 1.

Example

Using the same fuzzy sets (A) and (B):

(x)	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A \oplus B}(x)$
1	0.3	0.5	min(1, 0.3 + 0.5) = 0.8
2	0.7	0.2	min(1, 0.7 + 0.2) = 0.9
3	0.8	0.6	min(1, 0.8 + 0.6) = 1.0

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7. Bounded Difference

The **bounded difference** of two fuzzy sets (A) and (B) is defined as:

$\mu_{A \ominus B}(x) = \max(0, \mu_A(x) - \mu_B(x))$

This operation ensures that the resulting membership value is not less than 0.

Example

Using the same fuzzy sets (A) and (B):

(x)	$\mu_A(x)$	$\mu_B(x)$	$\mu_{A \ominus B}(x)$
1	0.3	0.5	max(0, 0.3 - 0.5) = 0.0
2	0.7	0.2	max(0, 0.7 - 0.2) = 0.5
3	0.8	0.6	max(0, 0.8 - 0.6) = 0.2

Some Important Properties of Fuzzy Set

- **Union (OR):** $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
- **Intersection (AND):** $\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
- **Complement (NOT):** $\mu_{\neg \tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$

1. Commutative Property

The commutative property holds true for fuzzy sets.

- **Union:** $\tilde{A} \cup \tilde{B} = \tilde{B} \cup \tilde{A}$
  - $\mu_{\tilde{A} \cup \tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \max(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)) = \mu_{\tilde{B} \cup \tilde{A}}(x)$
- **Intersection:** $\tilde{A} \cap \tilde{B} = \tilde{B} \cap \tilde{A}$
  - $\mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \min(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x)) = \mu_{\tilde{B} \cap \tilde{A}}(x)$

2. Associative Property

The associative property also holds for standard fuzzy set operations.

- **Union:** $(\tilde{A} \cup \tilde{B}) \cup \tilde{C} = \tilde{A} \cup (\tilde{B} \cup \tilde{C})$
  - $\mu_{(\tilde{A} \cup \tilde{B}) \cup \tilde{C}}(x) = \max(\max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x)) = \max(\mu_{\tilde{A}}(x), \max(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = \mu_{\tilde{A} \cup (\tilde{B} \cup \tilde{C})}(x)$
- **Intersection:** $(\tilde{A} \cap \tilde{B}) \cap \tilde{C} = \tilde{A} \cap (\tilde{B} \cap \tilde{C})$
  - $\mu_{(\tilde{A} \cap \tilde{B}) \cap \tilde{C}}(x) = \min(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x)) = \min(\mu_{\tilde{A}}(x), \min(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = \mu_{\tilde{A} \cap (\tilde{B} \cap \tilde{C})}(x)$

3. Distributive Property

The distributive property holds for standard fuzzy set operations:

- **Intersection over Union:** $\tilde{A} \cap (\tilde{B} \cup \tilde{C}) = (\tilde{A} \cap \tilde{B}) \cup (\tilde{A} \cap \tilde{C})$
  - $\mu_{\tilde{A}\cap(\tilde{B}\cup\tilde{C})}(x) = \min(\mu_{\tilde{A}}(x), \max(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)))$
  - $\mu_{(\tilde{A}\cap\tilde{B})\cup(\tilde{A}\cap\tilde{C})}(x) = \max(\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \min(\mu_{\tilde{A}}(x), \mu_{\tilde{C}}(x)))$
  - These two expressions are equivalent.
- **Union over Intersection:** $\tilde{A} \cup (\tilde{B} \cap \tilde{C}) = (\tilde{A} \cup \tilde{B}) \cap (\tilde{A} \cup \tilde{C})$
  - $\mu_{\tilde{A}\cup(\tilde{B}\cap\tilde{C})}(x) = \max(\mu_{\tilde{A}}(x), \min(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)))$
  - $\mu_{(\tilde{A}\cup\tilde{B})\cap(\tilde{A}\cup\tilde{C})}(x) = \min(\max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \max(\mu_{\tilde{A}}(x), \mu_{\tilde{C}}(x)))$
  - These two expressions are equivalent.

4. Identity Property

Identity properties depend on the definition of the null set and universal set in the fuzzy context.

- **Union:** $\tilde{A} \cup \emptyset = \tilde{A}$
  - Where $\emptyset$ is the fuzzy null set, defined as $\mu_{\emptyset}(x) = 0$ for all x.
  - $\mu_{\tilde{A}\cup\emptyset}(x) = \max(\mu_{\tilde{A}}(x), 0) = \mu_{\tilde{A}}(x)$
- **Intersection:** $\tilde{A} \cap \tilde{U} = \tilde{A}$
  - Where $\tilde{U}$ is the fuzzy universal set, defined as $\mu_{\tilde{U}}(x) = 1$ for all x.
  - $\mu_{\tilde{A}\cap\tilde{U}}(x) = \min(\mu_{\tilde{A}}(x), 1) = \mu_{\tilde{A}}(x)$

De Morgan's Laws

De Morgan's Laws also hold in Fuzzy Logic

- $\neg(\tilde{A} \cup \tilde{B}) = \neg\tilde{A} \cap \neg\tilde{B}$
  - $\mu_{\neg(\tilde{A}\cup\tilde{B})}(x) = 1 - \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
  - $\mu_{\neg\tilde{A}\cap\neg\tilde{B}}(x) = \min(1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x))$
  - These two expressions are equivalent.
- $\neg(\tilde{A} \cap \tilde{B}) = \neg\tilde{A} \cup \neg\tilde{B}$
  - $\mu_{\neg(\tilde{A}\cap\tilde{B})}(x) = 1 - \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
  - $\mu_{\neg\tilde{A}\cup\neg\tilde{B}}(x) = \max(1 - \mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x))$
  - These two expressions are equivalent.

What is a Relation?

A **relation** is a way to **connect** elements from one set to another. It describes **how elements are related**.

Example:

Let’s say we have two sets:

- **Set X (People):** {Alice, Bob, Charlie}
- **Set Y (Cities):** {New York, London, Tokyo}

We can define a **relation** "lives in" between people and cities:

R={(Alice,NewYork),(Bob,London),(Charlie,Tokyo)}

This means:

- Alice **lives in** New York
- Bob **lives in** London
- Charlie **lives in** Tokyo

A relation connects elements from **one set to another**.

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a. What is a Fuzzy Relation?

In fuzzy relations, instead of saying **"Yes (1)" or "No (0)"**, we assign a **degree of relationship** (a value between **0 and 1**).

Example 1: (Fuzzy Relation): People liking of cities

Let’s say people **like** certain cities to different degrees:

Person → City	New York	London	Tokyo
Alice	1.0	0.5	0.1
Bob	0.7	1.0	0.2
Charlie	0.2	0.8	1.0

- Alice **loves New York (1.0), somewhat likes London (0.5), and dislikes Tokyo (0.3)**.
- Bob **likes London the most (1.0) and Tokyo the least (0.4)**.
- Charlie **likes Tokyo the most (1.0) and New York the least (0.2)**.

So, a **fuzzy relation allows for partial relationships** rather than strict Yes/No (Crisp relation).

Example 2: Fuzzy Relation :

A fuzzy relation of **"How much people like different movies"**:

$$R = \begin{bmatrix} 0.9 & 0.5 & 0.2 \\ 0.7 & 0.8 & 0.4 \\ 0.3 & 0.6 & 1.0 \end{bmatrix}$$

Now, we **compare people to different things (movies, cities, etc.)**.

Person → Movie	Movie 1	Movie 2	Movie 3
Alice	0.9	0.5	0.2
Bob	0.7	0.8	0.4
Charlie	0.3	0.6	1.0

So, a **fuzzy set** describes a **single property** (tallness, age, etc.), while a **fuzzy relation** connects two sets.

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b. Summary

1. **Fuzzy sets** describe **how much** something belongs to **one** set.
   - Example: How much someone is **tall** or **young**.
2. **Fuzzy relations** describe **how much** elements in **one set relate** to elements in **another set**.
   - Example: How much people **like** different cities or movies.

Cardinality of a Fuzzy Set

In classical (crisp) set theory, the **cardinality** of a set A (denoted as |A|) is simply the number of elements in the set.

For example, if:

A = {1, 2, 3, 4}

Then the **cardinality** of A is:

|A| = 4

However, in **fuzzy set theory**, each element has a **degree of membership** in the set, so the **cardinality must consider these degrees**.

a.Definition of Fuzzy Cardinality

The **cardinality of a fuzzy set** A (denoted as |A|) is given by:

$|A| = \sum_{x \in A} \mu_A(x)$

where:

- $\mu_A(x)$ is the **membership function** of element xx.
- The sum is taken over all elements in A.
- This is also called the **sigma-count** of the fuzzy set.

Example 1: Cardinality of a Fuzzy Set

Consider a fuzzy set A that represents "Tall people":

A = {(Alice, 0.7), (Bob, 0.5), (Charlie, 0.9)}

To find the **cardinality** of A:

|A| = 0.7 + 0.5 + 0.9 = 2.1

So, the fuzzy set contains **2.1 "units" of tallness**, instead of just counting elements as in a classical set.

Role of Cartesian Products in Fuzzy Relations

In **fuzzy relations**, the **Cartesian product** of two fuzzy sets is used to define **how elements from one set relate to elements of another set**. It forms the basis for creating **fuzzy relation matrices**, which represent the degree of association between elements.

What is a Cartesian Product?

The **Cartesian product** of two sets X and Y, denoted as $X \times Y$, is the set of **all possible ordered pairs** (x, y), where x∈X and y∈Y.

Example (Crisp Cartesian Product)

Let:

X={a,b},Y={1,2}

Then, their Cartesian product is:

X×Y={(a,1),(a,2),(b,1),(b,2)}

This means every element in X is **paired** with every element in Y.

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a. Cartesian Product in Fuzzy Relations

In **fuzzy set theory**, the Cartesian product is used to **create fuzzy relations**, where **each pair is assigned a membership value** in **[0,1]**.

Definition of a Fuzzy Relation

A fuzzy relation R between fuzzy sets A and B is defined on their Cartesian product:

R:X×Y→[0,1]

Here, each pair (x, y) gets a **membership value** $\mu_R(x,y)$, which shows **how strongly x is related to y**.

Example (Fuzzy Cartesian Product)

Let:

A = {(a, 0.8), (b, 0.5)} B={(1,0.7),(2,0.6)}

Their **Cartesian product** creates a **fuzzy relation matrix**:

$$R = \begin{bmatrix} \mu_R(a,1) & \mu_R(a,2) \\ \mu_R(b,1) & \mu_R(b,2) \end{bmatrix}$$

A common way to define the membership values in the relation is using the **minimum operator**:

$\mu_R(x,y) = \min(\mu_A(x), \mu_B(y))$

Using this rule:

X×Y	$\mu_A(x)$	$\mu_B(y)$	$\mu_R(x,y) = \min(\mu_A, \mu_B)$
(a,1)	0.8	0.7	**0.7**
(a,2)	0.8	0.6	**0.6**
(b,1)	0.5	0.7	**0.5**
(b,2)	0.5	0.6	**0.5**

So, the **fuzzy relation matrix** is:

$$R = \begin{bmatrix} 0.7 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}$$

This matrix tells us:

- **(a,1) has a relation strength of 0.7**
- **(b,2) has a relation strength of 0.5**, etc.

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b. Importance of Cartesian Products in Fuzzy Relations

(i) Helps Define Relations Between Sets

- Cartesian product provides a **foundation** for representing **pairwise relationships** in a structured way.
- In **crisp relations**, the pairs either exist (1) or do not (0).
- In **fuzzy relations**, each pair gets a **degree of association**.

(ii) Forms the Basis for Fuzzy Matrices

- Fuzzy relations are often represented as **matrices**, where each entry shows the **strength of the relation**.
- This is useful in **decision-making, pattern recognition, and AI systems**.

(iii) Supports Fuzzy Composition

- Cartesian products are used in **max-min composition**, a key operation in **fuzzy logic and fuzzy inference systems**.

(iv) Used in Various Applications

- **Social networks:** Strength of relationships between people.
- **Recommendation systems:** How much a user likes a product.
- **Medical diagnosis:** Relationship between symptoms and diseases.

Fuzzy Composition and Its Types

Fuzzy composition is a key operation in **fuzzy relations**, used to combine two fuzzy relations to obtain a new relation. It is essential in **fuzzy inference systems, decision-**

making, and pattern recognition.

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a. What is Fuzzy Composition?

Fuzzy composition is the process of combining two fuzzy relations **R1 and R2** to form a new relation **R**.

If:

- R1 is a relation between sets **X and Y**
- R2 is a relation between sets **Y and Z**

Then, their **composition** gives a new relation **R** between X and Z.

Mathematically, if:

R1:X×Y
R2:Y×Z

Then, their **composition** gives:

R:X×Z

This means **intermediate elements from Y are eliminated**, and we get a direct relation between X and Z.

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b. Types of Fuzzy Composition

There are **two main types** of fuzzy composition:

(i) Max-Min Composition (Zadeh's Composition)

This is the **most commonly used** fuzzy composition. It is based on:

- **Minimum (min)** operation for matching elements.
- **Maximum (max)** operation for aggregation.

Formula:

For a fuzzy relation R1 (between X and Y) and R2Rbetween Y and Z), the max-min composition is defined as:

$\mu_R(x,z) = \max_{y \in Y} \left(\min(\mu_{R_1}(x,y), \mu_{R_2}(y,z)) \right)$

Example (Max-Min Composition)

Let’s take two fuzzy relations:

Relation R1 (between X and Y):

$$R_1 = \begin{bmatrix} 0.8 & 0.4 \\ 0.6 & 0.9 \end{bmatrix}$$

Relation R2 (between Y and Z):

$$R_2 = \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0.9 \end{bmatrix}$$

Step-by-Step Calculation (Max-Min)

To compute $R = R_1 \circ R_2$, we calculate:

$\mu_R(x, z) = \max_y(\min(\mu_{R_1}(x, y), \mu_{R_2}(y, z)))$

For (x_1, z_1):

$\max(\min(0.8, 0.7), \min(0.4, 0.6)) = \max(0.7, 0.4) = 0.7$

For (x_1, z_2):

$\max(\min(0.8, 0.5), \min(0.4, 0.9)) = \max(0.5, 0.4) = 0.5$

For (x_2, z_1):

$\max(\min(0.6, 0.7), \min(0.9, 0.6)) = \max(0.6, 0.6) = 0.6$

For (x_2, z_2):

$\max(\min(0.6, 0.5), \min(0.9, 0.9)) = \max(0.5, 0.9) = 0.9$

Final Composed Relation R:

$$R = \begin{bmatrix} 0.7 & 0.5 \\ 0.6 & 0.9 \end{bmatrix}$$

This new matrix represents the **fuzzy relation from X to Z**.

(ii) Max-Product Composition

Instead of the **min** operation, this method uses **multiplication**.

Formula:

$\mu_R(x,z) = \max_{y \in Y} \left(\mu_{R_1}(x,y) \times \mu_{R_2}(y,z) \right)$

This is useful when **multiplicative** strength is needed instead of minimum overlap.

Example (Max-Product Composition)

Using the same R1 and R2:

For (x_1, z_1):

$\max(0.8 \times 0.7, 0.4 \times 0.6) = \max(0.56, 0.24) = 0.56$

For (x_1, z_2):

$\max(0.8 \times 0.5, 0.4 \times 0.9) = \max(0.4, 0.36) = 0.4$

For (x_2, z_1):**

$\max(0.6 \times 0.7, 0.9 \times 0.6) = \max(0.42, 0.54) = 0.54$

For (x_2, z_2):

$\max(0.6 \times 0.5, 0.9 \times 0.9) = \max(0.3, 0.81) = 0.81$

Final Max-Product Composition RR:

$$R = \begin{bmatrix} 0.56 & 0.4 \\ 0.54 & 0.81 \end{bmatrix}$$

---

3. Comparison of Max-Min and Max-Product Composition

Feature	Max-Min Composition	Max-Product Composition
Formula	$\max(\min(a,b))$	$\max(a \times b)$
Effect	Chooses the **minimum** matching value	Uses **multiplication** to scale strength
Usage	Common in **fuzzy logic and control systems**	Used in **neural networks and probabilistic models**

Solve:
Two Fuzzy relations R and S are given below. obtain fuzzy relation T ($X \times Z$) as a composition between R and S using Zadeh's Composition and Max-Product composition.

---

4. Applications of Fuzzy Composition

(i) Decision-Making Systems

- Used in **expert systems** where relationships between symptoms and diseases are analyzed.

(ii) Fuzzy Control Systems

- Used in **fuzzy logic controllers** (e.g., **washing machines, air conditioners**).

(iii) Pattern Recognition

- Helps in **matching** data patterns by analyzing **relations between features**.

(iv) AI & Machine Learning

- Used in **fuzzy neural networks** and **image processing**.

---

5. Summary

1. **Fuzzy composition** combines two fuzzy relations to create a **new fuzzy relation**.
2. **Max-Min Composition**: Uses **minimum** for element-wise comparison and **maximum** to aggregate.
3. **Max-Product Composition**: Uses **multiplication** instead of the minimum operation.
4. **Applications** include **fuzzy control, decision-making, and AI**.

Membership Functions

- Fuzzy membership functions (also known as membership functions) are fundamental to fuzzy logic systems, which are used to handle the concept of partial truth, where the truth value may be any number between 0 and 1.
- This is in contrast to classical logic, where propositions are either true (1) or false (0).
- Fuzzy membership functions help in defining the degree of membership of an element in a fuzzy set.

Components of Fuzzy Membership Functions

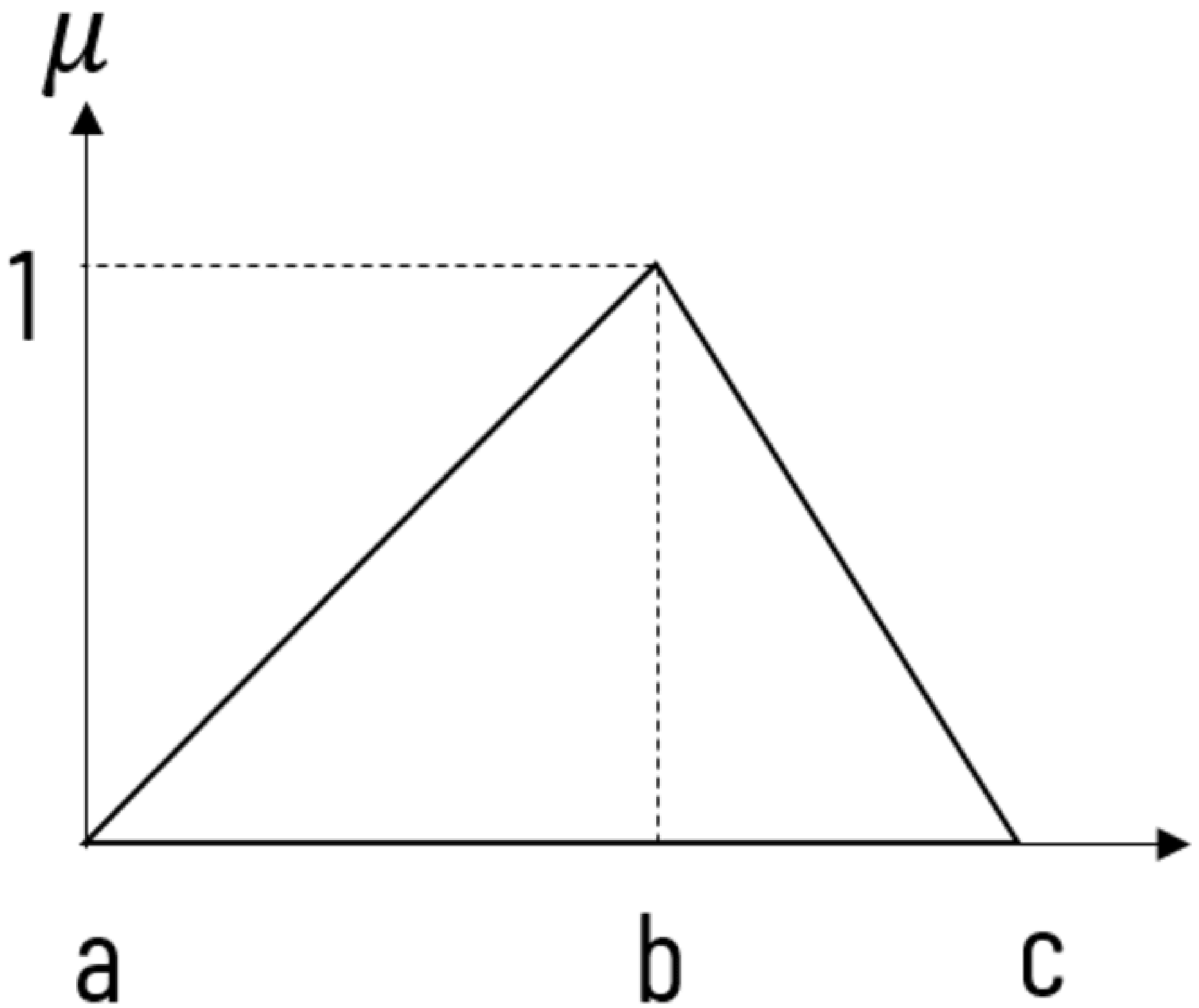
1. **Universe of Discourse (U):**
    - This is the entire range of possible values applicable to a particular fuzzy variable.

    - For example, if we are dealing with temperature, the universe of discourse might be from -50°C to 150°C.
2. **Fuzzy Set (A):**
    - A fuzzy set is defined by a membership function that maps elements of the universe of discourse to a membership degree between 0 and 1.
    - For example, a fuzzy set "warm" could be defined for the temperature variable.
3. **Membership Function (μA(x)):**
    - This function defines the degree of membership of an element x in the fuzzy set A.
    - The function takes an input value x from the universe of discourse and returns a value between 0 and 1, indicating the degree to which x belongs to the set A.
    - Common types of membership functions include:
        - **Triangular Membership Function:** Defined by three parameters {a, b, c}, where a and c are the feet of the triangle and b is the peak.
        - **Trapezoidal Membership Function:** Defined by four parameters {a, b, c, d}, where a and d are the feet and b and c are the shoulders of the trapezoid.
        - **Gaussian Membership Function:** Defined by two parameters {c, σ}, where c is the center and σ is the width.
        - **Sigmoid Membership Function:** Defined by two parameters {a, c}, where a controls the slope and c is the crossover point.

Triangular membership function:

- This is one of the most widely accepted and used membership functions (MF) in fuzzy controller design.
- The triangle which fuzzifies the input can be defined by three parameters a, b and c, where c defines the base and b defines the height of the triangle.

Trivial case:



Here, in the diagram, X-axis represents the input from the process (such as air conditioner, washing machine, etc.) and the Y axis represents the corresponding fuzzy value.

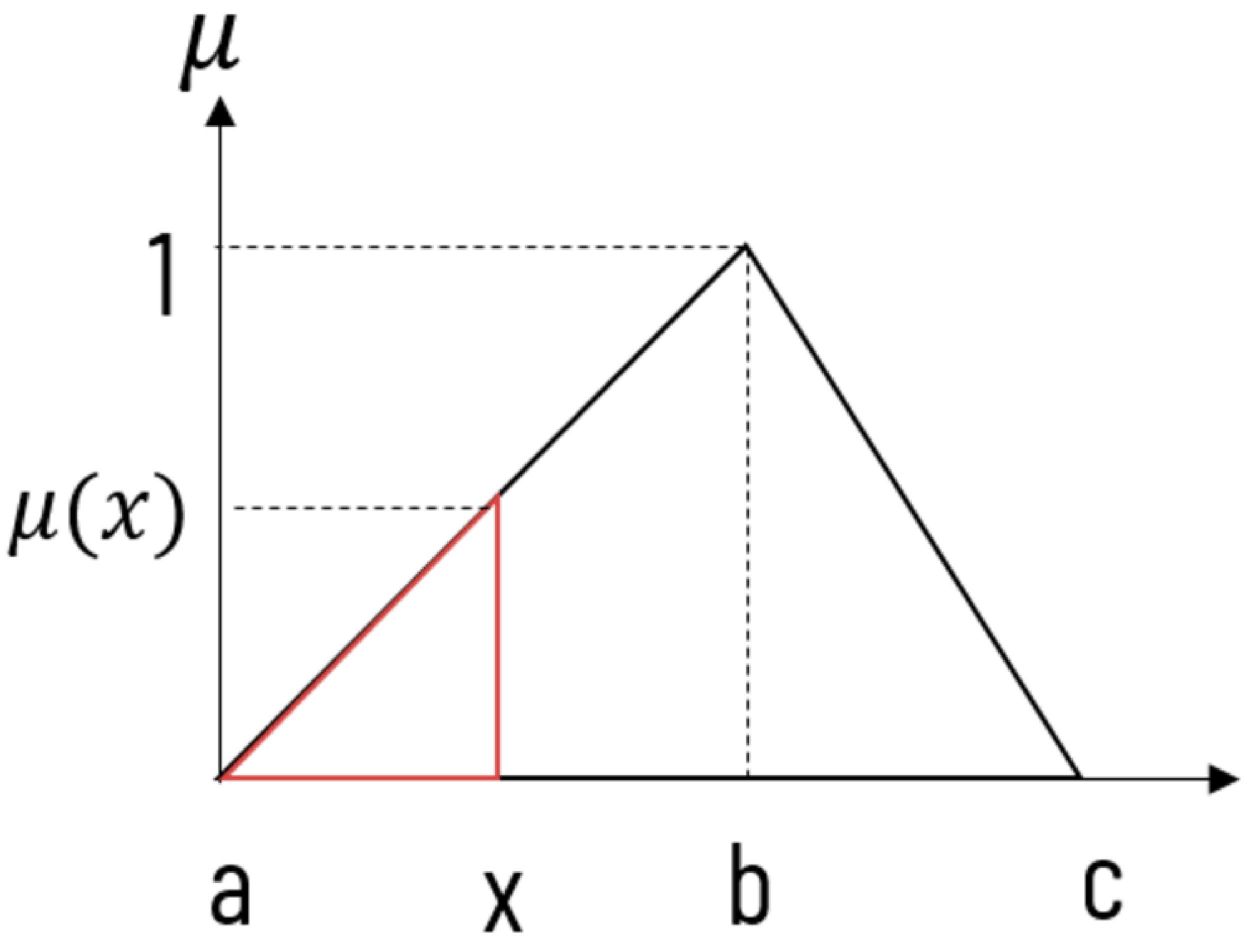
If input x = b, then it is having full membership in the given set. So,

$\mu(x) = 1$, if $x = b$

And if the input is less than a or greater than c, then it does not belong to the fuzzy set at all, and its membership value will be 0

$\mu(x) = 0, \quad x < a$ or $x > c$

x is between a and b:

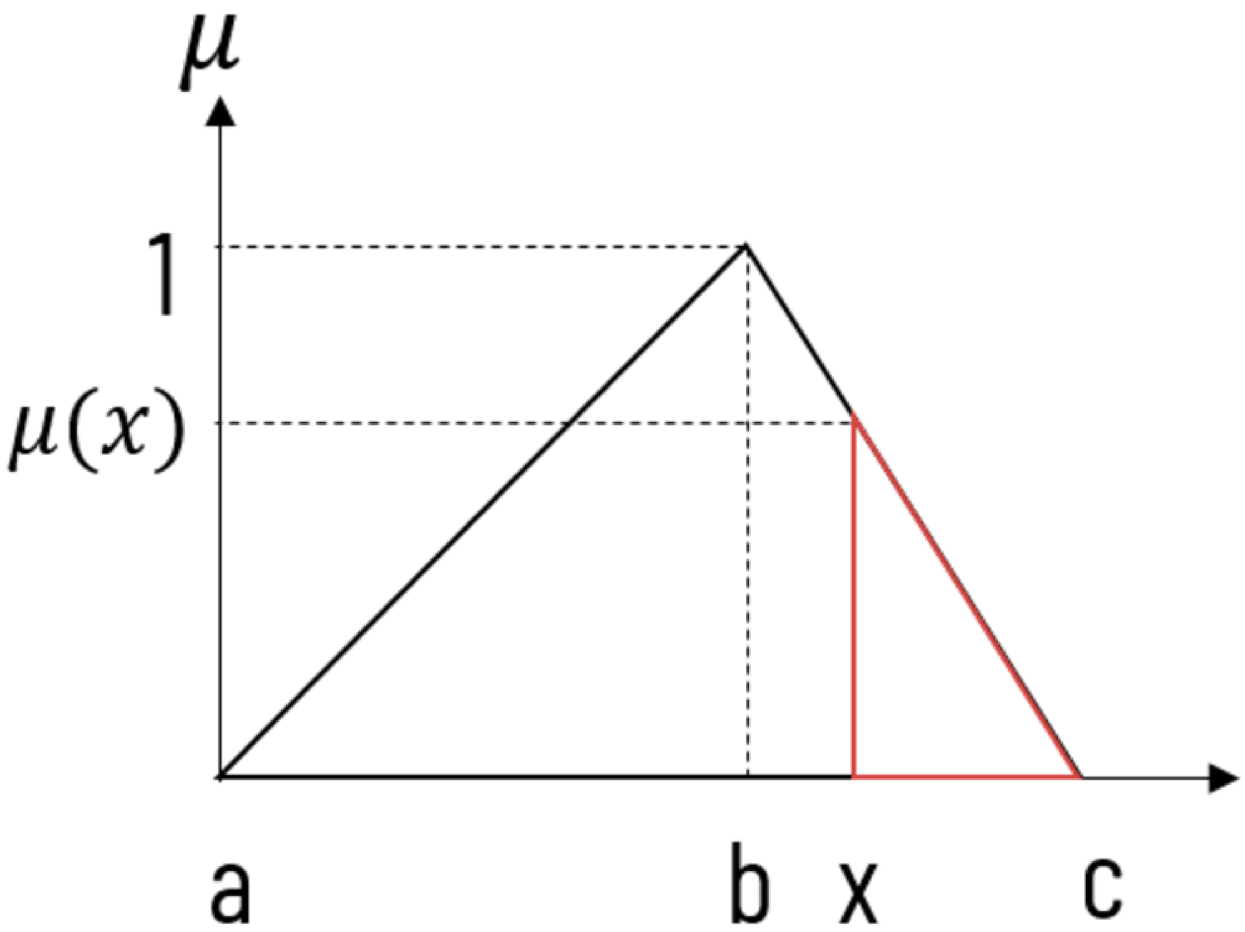


If x is between a and b, as shown in the figure, its membership value varies from 0 to 1. If it is near a, its membership value is close to 0, and if x is near b, its membership value gets close to 1.

We can compute the fuzzy value of x using a similar triangle rule,

$\mu(x) = (x - a) / (b - a), \quad a \le x \le b$

x is between b and c:



If x is between b and c, as shown in the figure, its membership value varies from 0 to 1. If it is near b, its membership value is close to 1, and if x is near c, its membership value gets close to 0.

We can compute the fuzzy value of x using a similar triangle rule,

$\mu(x) = (c - x) / (c - b), \qquad b \leq x \leq c$

Combine all together:

We can combine all the above scenarios in single equation as,

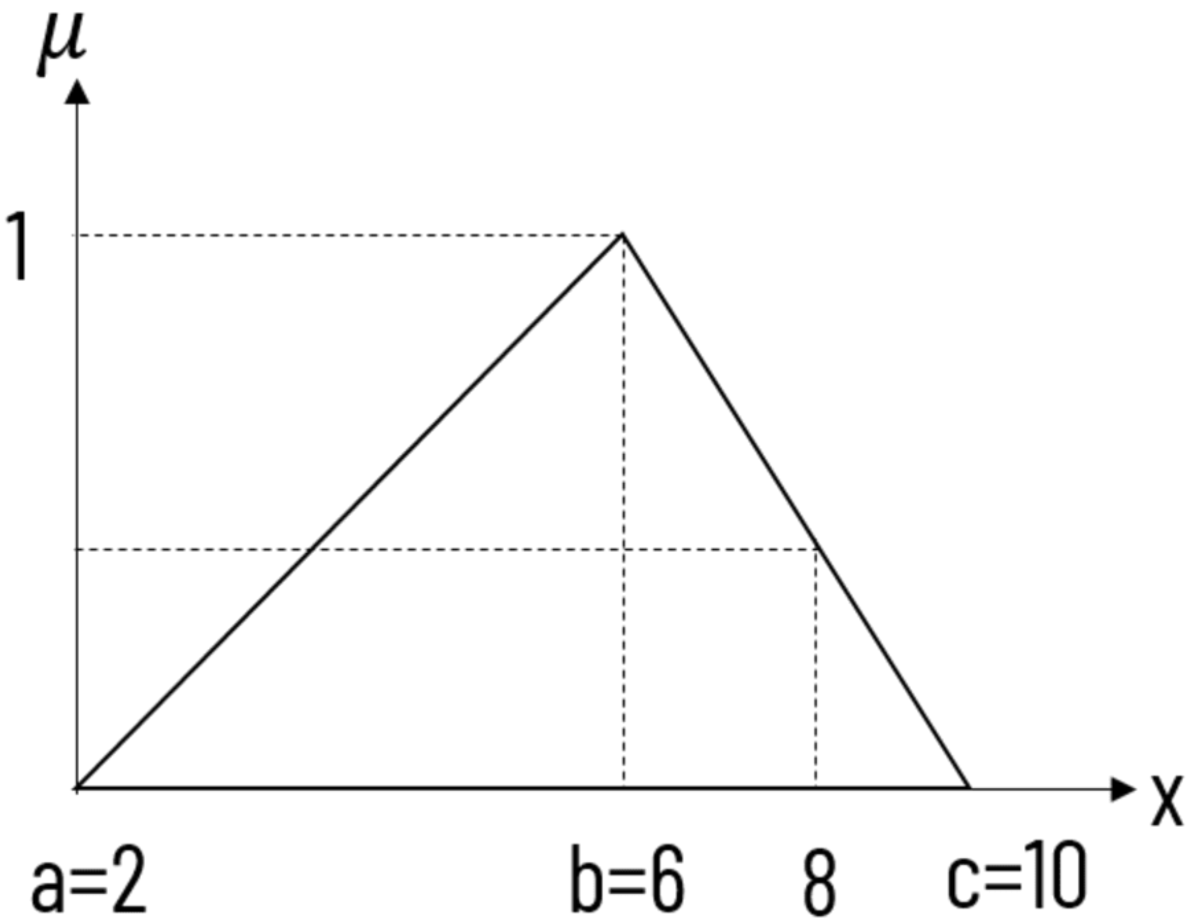
$$\mu_{triangle}(x;a,b,c)=\begin{cases}0, & x\le a\\ \dfrac{x-a}{b-a}, & a\le x\le b\\ \dfrac{c-x}{c-b}, & b\le x\le c\\ 0, & c\le x\end{cases}$$

$$=\max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$

Triangular membership function

Example: Triangular membership function

Determine μ, corresponding to x = 8.0



For the given values of a, b and c, we have to compute the fuzzy value corresponding to x = 8. Using the equation of the triangular membership function,

$$\mu_{triangle}(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$

$$= \max\left(\min\left(\frac{x-2}{6-2},\frac{10-x}{10-6}\right),0\right)$$

$$= \max\left(\min\left(\frac{x-2}{4},\frac{10-x}{4}\right),0\right)$$

$We\ put\ x = 8.0$

$$= \max\left(\min\left(\frac{3}{2},\frac{1}{2}\right),0\right) = \frac{1}{2} = 0.5$$

Thus, x = 8 will be mapped to a fuzzy value of 0.5 using the given triangle fuzzy membership function

Trapezoid Membership Function

Let a, b, c and d represents the x coordinates of the membership function. then

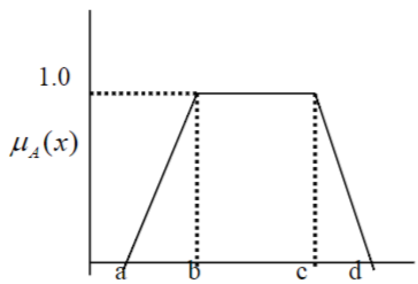
Trapezoid(x; a, b, c, d) = 0 if $x \leq a$;

= (x-a)/ (b-a) if $a \leq x \leq b$

= 1 if $b \leq x \leq c$;

= (d-x)/ (d-c) 0 if $c \leq x \leq d$;

= 0, if $d \leq x$.



$$\mu_{\text{trapezoid}} = \max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right)$$

1. Using your own intuition and definitions of the universe of discourse, plot fuzzy membership functions for "weight of people."

Solution: The universe of discourse is weight of people. Let the weights be in kg, i.e., kilogram. Let the linguistic variables be the following:

$$\text{Very thin (VT)}: W \leq 25$$
$$\text{Thin (T)}: 25 < W \leq 45$$
$$\text{Average (AV)}: 45 < W \leq 60$$
$$\text{Stout (S)}: 60 < W \leq 75$$
$$\text{Very stout (VS)}: W > 75$$

Now plotting the defined linguistic variables using triangular membership functions, we obtain Figure 1.

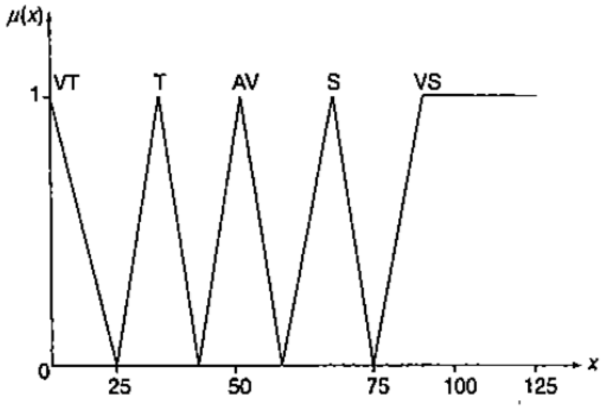


Figure 1 Membership function for weight of people.

Applications of Fuzzy Membership Functions

1. **Control Systems:** Used in various control applications such as temperature control, speed control, and more.
2. **Pattern Recognition:** Helps in classifying patterns based on fuzzy criteria.
3. **Decision Making:** Aids in decision-making processes where information is imprecise or uncertain.
4. **Natural Language Processing:** Used to model the vagueness and ambiguity inherent in human language.

Advantages of Using Fuzzy Membership Functions

- **Flexibility:** Allows for a more nuanced representation of concepts that do not have precise boundaries.
- **Robustness:** Can handle imprecise and incomplete data effectively.
- **Human-like Reasoning:** Mimics human decision-making processes, making it easier to incorporate expert knowledge.

Summary

Fuzzy Membership Functions

- **Definition**:
  - Fuzzy membership functions define the **degree of membership** of an element in a fuzzy set, ranging between 0 and 1.
  - Contrasts with classical logic, where membership is binary (0 or 1).
- **Components**:

1. **Universe of Discourse (U)**:
   - Range of possible values for a fuzzy variable (e.g., temperature: -50°C to 150°C).
2. **Fuzzy Set (A)**:
   - Defined by a membership function that maps elements of U to a membership degree (0 to 1).
3. **Membership Function (μA(x))**:
   - Assigns a value between 0 and 1 to indicate how much an element x belongs to set A.

- **Types of Membership Functions**:
  - **Triangular**: Defined by 3 parameters {a, b, c}.
  - **Trapezoidal**: Defined by 4 parameters {a, b, c, d}.
  - **Gaussian**: Defined by 2 parameters {c, σ}.
  - **Sigmoid**: Defined by 2 parameters {a, c}.
- **Applications**:
  - Control systems (e.g., temperature, speed control).
  - Pattern recognition (classification based on fuzzy criteria).
  - Decision-making (handling imprecise or uncertain information).
  - Natural language processing (modeling vagueness and ambiguity).
- **Advantages**:
  - **Flexibility**: Represents concepts with gradual boundaries.
  - **Robustness**: Handles imprecise and incomplete data effectively.
  - **Human-like Reasoning**: Mimics human decision-making processes.

- Fuzzy membership functions are essential in fuzzy logic systems for representing **partial truth** and handling uncertainty.
- They allow for **gradual membership**, making them useful in real-world applications where binary logic is insufficient.
- Common types include **triangular**, **trapezoidal**, **Gaussian**, and **sigmoid** functions.
- Applications span **control systems**, **pattern recognition**, **decision-making**, and **natural language processing**.
- Advantages include **flexibility**, **robustness**, and **human-like reasoning**.

Defuzzification

- Defuzzification is the process of converting a **fuzzy set** (which has degrees of membership) into a **crisp value** (a single, precise value).

- This is essential in fuzzy logic systems, especially in **control systems** and **decision-making**, where a clear output is required.

---

1. Why is Defuzzification Needed?

- Fuzzy logic allows for **partial truth** (degrees of membership between 0 and 1).
- However, in real-world applications, we often need a **single, crisp value** to act upon (e.g., setting a temperature or speed).
- Defuzzification bridges the gap between **fuzzy reasoning** and **crisp action**.

---

2. Mathematical Framework for Defuzzification

Given a fuzzy set (A) with membership function ($\mu_A(x)$), defuzzification computes a crisp value (x^*) that best represents the fuzzy set.

Mathematically, defuzzification can be represented as:

$$x^* = \text{Defuzzify}(\mu_A(x))$$

where (x^*) is the crisp output.

---

3. Lambda Cuts (α-Cuts)

Lambda cuts (or α-cuts) are a way to **slice** a fuzzy set at a specific membership level (α). They convert a fuzzy set into a **classical (crisp) set** by considering only elements with membership values **greater than or equal to** (α).

Definition:

For a fuzzy set (A) and a threshold (α) (where ($0 \leq \alpha \leq 1$)), the α-cut is defined as:

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$$

Here, (A_α) is a **crisp set** containing all elements with membership values ≥ (α).

Example:

Consider a fuzzy set (A):

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1.0)\}$$

For (α = 0.5):

$$A_{0.5} = \{2, 3, 4\}$$

For (α = 0.8):

$$A_{0.8} = \{3, 4\}$$

---

4. Common Defuzzification Methods

There are several methods to compute the crisp value (x^*). The most commonly used methods are:

(i) Max-Membership Method (Height Method)

- This method selects the value (x^*) where the membership function ($\mu_A(x)$) is **maximum**.
- If there are multiple points with the same maximum membership value, the **average** of those points is taken.

Formula:

$$x^* = \arg\max_x(\mu_A(x))$$

Example:

Consider a fuzzy set (A) with membership function:

$$\mu_A(x) = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 0.8)\}$$

Here, the maximum membership value is **0.8**, which occurs at (x = 3) and (x = 4). Using the max-membership method:

$$x^* = \frac{3 + 4}{2} = 3.5$$

(ii) Centroid Method (Center of Gravity)

- This method computes the **center of gravity** of the area under the membership function.
- It is the most commonly used defuzzification method because it provides a **balanced** result.

Formula:

$$x^* = \frac{\int x \cdot \mu_A(x)\, dx}{\int \mu_A(x)\, dx}$$

For discrete data:

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}$$

Example:

Consider a fuzzy set (A) with membership function:

$$\mu_A(x) = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 0.8)\}$$

Using the centroid method:

$$x^* = \frac{(1 \cdot 0.2) + (2 \cdot 0.5) + (3 \cdot 0.8) + (4 \cdot 0.8)}{0.2 + 0.5 + 0.8 + 0.8} = \frac{0.2 + 1.0 + 2.4 + 3.2}{2.3} = \frac{6.8}{2.3} \approx 2.96$$

(iii) Weighted Average Method

- This method computes the weighted average of the **peak values** of the membership function, weighted by their membership degrees.
- It is often used when the membership function has **symmetrical shapes**.

Formula:

$$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}$$

where (x_i) are the **peak values** of the membership function.

Example:

Consider a fuzzy set (A) with membership function:

$$\mu_A(x) = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 0.8)\}$$

Here, the peak values are (x = 3) and (x = 4). Using the weighted average method:

$$x^* = \frac{(3 \cdot 0.8) + (4 \cdot 0.8)}{0.8 + 0.8} = \frac{2.4 + 3.2}{1.6} = \frac{5.6}{1.6} = 3.5$$

---

5. Comparison of Defuzzification Methods

Method	Formula	Advantages	Disadvantages
Max-Membership	$x^* = \arg\max_x(\mu_A(x))$	Simple and fast	Ignores the shape of the membership

Method	Formula	Advantages	Disadvantages
			function
Centroid	$x^* = \frac{\int x \cdot \mu_A(x)\, dx}{\int \mu_A(x)\, dx}$	Balanced and considers the entire shape	Computationally expensive for complex shapes
Weighted Average	$x^* = \frac{\sum_{i=1}^{n} x_i \cdot \mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}$	Simple and works well for symmetrical shapes	Not suitable for asymmetric shapes

---

6. Applications of Defuzzification

- **Control Systems**: Used in fuzzy logic controllers (e.g., washing machines, air conditioners) to determine precise outputs.
- **Decision-Making**: Converts fuzzy decisions into actionable, crisp values.
- **Pattern Recognition**: Helps in classifying data based on fuzzy criteria.
- **AI & Robotics**: Used in fuzzy inference systems for decision-making and control.

---

7. Summary

- Defuzzification converts a **fuzzy set** into a **crisp value**.
- Common methods include **Max-Membership**, **Centroid**, and **Weighted Average**.
- The **Centroid method** is the most widely used due to its balanced approach.
- Defuzzification is essential in **control systems**, **decision-making**, and **AI**.

Truth Values and Truth Tables in Fuzzy Logic

- In classical logic, **truth values** are binary: a statement is either **true (1)** or **false (0)**.
- In **fuzzy logic**, truth values can range between **0 and 1**, representing **degrees of truth**.
- This allows fuzzy logic to handle uncertainty and vagueness in real-world scenarios.

---

1. Fuzzy Truth Values

- In fuzzy logic, a truth value is a **degree of membership** in the interval [0, 1].

- For example:
  - A statement like "The temperature is high" might have a truth value of **0.7**, indicating that it is **70% true**.

---

2. Fuzzy Truth Tables

Fuzzy truth tables extend classical truth tables to handle **partial truth values**. They define the truth values of **logical operations** (AND, OR, NOT) for fuzzy inputs.

Key Fuzzy Logical Operations

1. **AND Operation (Conjunction):**
   - Represented as: $\mu_{A \text{ AND } B}(x) = \min(\mu_A(x), \mu_B(x))$
   - Example: If

$$\mu_A(x) = 0.8 \quad and \quad \mu_B(x) = 0.5, then \quad \mu_{A \text{ AND } B}(x) = \min(0.8, 0.5) = 0.5.$$

2. **OR Operation (Disjunction):**
   - Represented as: ($\mu_{A \text{ OR } B}(x) = \max(\mu_A(x), \mu_B(x))$)
   - Example: If ($\mu_A(x) = 0.8$) and ($\mu_B(x) = 0.5$), then ($\mu_{A \text{ OR } B}(x)$ = $\max(0.8, 0.5)$ = 0.8).
3. **NOT Operation (Negation):**
   - Represented as: ($\mu_{\text{NOT } A}(x) = 1 - \mu_A(x)$)
   - Example: If ($\mu_A(x) = 0.8$), then ($\mu_{\text{NOT } A}(x) = 1 - 0.8 = 0.2$).

Fuzzy Truth Table Example

μ_A	μ_B	$\mu_{A \text{ AND } B}$	($\mu_{A \text{ OR } B}$)	($\mu_{\text{NOT } A}$)
0.8	0.5	0.5	0.8	0.2
0.3	0.7	0.3	0.7	0.7
0.9	0.2	0.2	0.9	0.1

Fuzzy Propositions

A **fuzzy proposition** is a statement that can have a **degree of truth** between 0 and 1. It is used to express **uncertain or vague** information.

Examples of Fuzzy Propositions

1. "The temperature is high."
   - Degree of truth: 0.7 (70% true).
2. "The car is fast."
   - Degree of truth: 0.9 (90% true).
3. "The food is spicy."
   - Degree of truth: 0.4 (40% true).

Formation of Fuzzy Rules

Fuzzy rules are **IF-THEN statements** that describe how input variables are mapped to output variables using fuzzy logic. They are the backbone of **fuzzy inference systems**.

Structure of a Fuzzy Rule

A fuzzy rule has the form:
IF (Condition) THEN (Action)

- **Condition (Antecedent):** A fuzzy proposition describing the input.
- **Action (Consequent):** A fuzzy proposition describing the output.

Examples of Fuzzy Rules

1. **Rule 1:**
   - IF **temperature is high** THEN **fan speed is fast**.
   - Here, "temperature is high" is the antecedent, and "fan speed is fast" is the consequent.
2. **Rule 2:**
   - IF **speed is medium** AND **distance is short** THEN **brake gently**.
   - Here, "speed is medium" and "distance is short" are antecedents, and "brake gently" is the consequent.

Fuzzy Rule Formation Process

1. **Define Input and Output Variables:**
   - Identify the variables involved in the system (e.g., temperature, fan speed).
2. **Fuzzify Inputs:**
   - Use **membership functions** to convert crisp inputs into fuzzy sets (e.g., "high temperature" with a membership value of 0.7).
3. **Construct Rules:**

- Formulate IF-THEN rules based on expert knowledge or data.

4. **Apply Fuzzy Operators:**
    - Use fuzzy AND, OR, and NOT operations to evaluate the rules.
5. **Defuzzify Outputs:**
    - Convert the fuzzy output into a crisp value (e.g., fan speed = 80 RPM).

Summary

- **Fuzzy truth values** represent degrees of truth between 0 and 1.
- **Fuzzy truth tables** define the behavior of logical operations (AND, OR, NOT) for fuzzy inputs.
- **Fuzzy propositions** express statements with partial truth.
- **Fuzzy rules** are IF-THEN statements that map inputs to outputs using fuzzy logic.
- **Fuzzy rule-based systems** are widely used in control systems, decision-making, and AI.

By combining **truth values**, **truth tables**, and **fuzzy rules**, fuzzy logic provides a powerful framework for handling uncertainty and vagueness in real-world applications.