Lectures on

# Laplace transform

Delivered By


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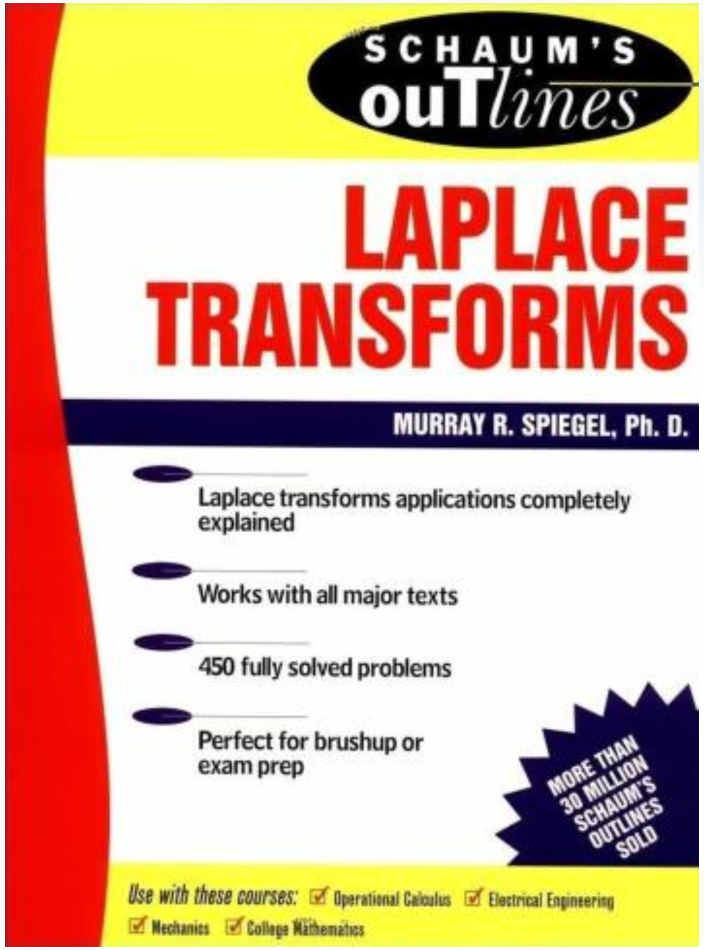
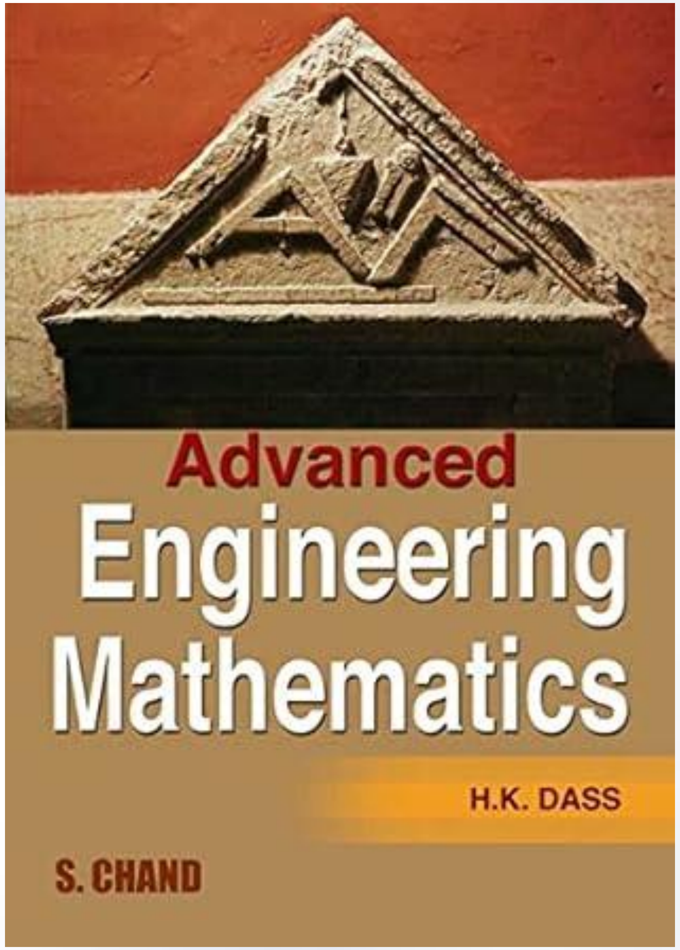
# Outlines

- **Laplace Transformation**
  - Laplace transformation of different functions
  - Inverse Laplace Transformation
  - Application of Laplace Transform

# Reference books



## LAPLACE TRANSFORM

**Definition.** Let $f(t)$ be function defined for all positive values of $t$, then

$$F(s)=\int_0^\infty e^{-st} f(t)\,dt$$

provided the integral exists, is called the **Laplace Transform** of $f(t)$. It is denoted as

$$\boxed{L[f(t)]=F(s)=\int_0^\infty e^{-st} f(t)\,dt}$$

1. $\boxed{L(1)=\frac{1}{s}}$

**Proof.** $L(1) = \int_0^\infty 1\cdot e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[\frac{1}{e^{st}}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$

Hence $L(1) = \frac{1}{s}$ **Proved.**

2. $\boxed{L\left(t^n\right)=\frac{n!}{s^{n+1}}}$ where $n$ and $s$ are positive.

**Proof.** $L(t^n) = \int_0^\infty e^{-st}t^n dt$

Putting $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{dx}{s}$

Thus, we have $L(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} \Rightarrow L(t^n) = \frac{1}{s^{n+1}}\int_0^\infty e^{-x}.x^n\, dx$

$\Rightarrow L(t^n) = \frac{\overline{|n+1}}{s^{n+1}} \Rightarrow L\left(t^n\right) = \frac{n!}{s^{n+1}}$ $\left[\begin{array}{l} \overline{|n+1} = \int_0^\infty e^{-x}.x^n\, dx \\ \text{and} \quad \overline{|n+1} = n! \end{array}\right]$ **Proved.**

**3.** $\boxed{L\left(e^{at}\right)=\dfrac{1}{s-a}}$ , where $s>a$

**Proof.** $L\,(e^{at}) = \int_0^{\infty} e^{-st}.e^{at}dt = \int_0^{\infty} e^{-st+at}.dt$

$$= \int_0^{\infty} e^{(-s+a)t}.dt = \int_0^{\infty} e^{-(s-a)t}.dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^{\infty}$$

$$= \frac{-1}{(s-a)}(0-1) = \frac{1}{s-a}$$

**Proved.**

**6.** $\boxed{L(\sin at)=\dfrac{a}{s^2+a^2}}$

**Proof.** $L\,(\sin at) = L\left[\dfrac{e^{iat}-e^{-iat}}{2i}\right]$ $\left[\because\ \sin at = \dfrac{e^{iat}-e^{-iat}}{2i}\right]$

$$= \frac{1}{2i}\left[L\left(e^{iat}-e^{-iat}\right)\right] = \frac{1}{2i}\left[L\left(e^{iat}\right)-L\left(e^{-iat}\right)\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia}-\frac{1}{s+ia}\right] = \frac{1}{2i}\,\frac{s+ia-s+ia}{s^2+a^2} = \frac{1}{2i}\,\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

**Proved.**

**7.** $\boxed{L\,(\cos at)=\dfrac{s}{s^2+a^2}}$

**Proof.** $L\,(\cos at) = L\left(\dfrac{e^{iat}+e^{-iat}}{2}\right)$ $\left[\because\ \cos at = \dfrac{e^{iat}+e^{-iat}}{2}\right]$

$$= \frac{1}{2}[L(e^{iat}+e^{-iat})] = \frac{1}{2}[L(e^{iat})+L(e^{-iat})] = \frac{1}{2}\left[\frac{1}{s-ia}+\frac{1}{s+ia}\right] = \frac{1}{2}\,\frac{s+ia+s-ia}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

**Proved.**

**4.** $\boxed{\mathbf{L(\cosh at) = \frac{s}{s^2 - a^2}}}$

**Proof.** $L(\cosh at) = L\left[\frac{e^{at} + e^{-at}}{2}\right]$ $\qquad \left(\therefore \cosh at = \frac{e^{at} + e^{-at}}{2}\right)$

$$= \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at}) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad \left[L(e^{at}) = \frac{1}{s-a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

**Proved.**

**5.** $\boxed{\mathbf{L(\sinh at) = \frac{a}{s^2 - a^2}}}$

**Proof.** $L(\sinh\ at) = L\left[\frac{1}{2}\left(e^{at} - e^{-at}\right)\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{a}{s^2-a^2}$$

**Proved.**

## 1. Linearity property.

***Theorem 1-2.*** If $c_1$ and $c_2$ are any constants while $F_1(t)$ and $F_2(t)$ are functions with Laplace transforms $f_1(s)$ and $f_2(s)$ respectively, then

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} \quad = \quad c_1 \mathcal{L}\{F_1(t)\} + c_2 \mathcal{L}\{F_2(t)\} \quad = \quad c_1 f_1(s) + c_2 f_2(s) \tag{2}$$

The result is easily extended to more than two functions.

**Example.**

$$\begin{aligned} \mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} &= 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\} \\ &= 4\left(\frac{2!}{s^3}\right) - 3\left(\frac{s}{s^2+4}\right) + 5\left(\frac{1}{s+1}\right) \\ &= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1} \end{aligned}$$

The symbol $\mathcal{L}$, which transforms $F(t)$ into $f(s)$, is often called the *Laplace transformation operator*. Because of the property of $\mathcal{L}$ expressed in this theorem, we say that $\mathcal{L}$ is a *linear operator* or that it has the *linearity property*.

**2. First translation or shifting property.**

***Theorem 1-3.*** If $\mathcal{L}\{F(t)\} = f(s)$ then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a) \tag{3}$$

**Example.** Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

### 3. Second translation or shifting property.

***Theorem 1-4.*** If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$, then

$$\mathcal{L}\{G(t)\} = e^{-as} f(s) \tag{4}$$

**Example.** Since $\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$, the Laplace transform of the function

$$G(t) = \begin{cases} (t-2)^3 & t > 2 \\ 0 & t < 2 \end{cases}$$

is $6e^{-2s}/s^4$.

4. **Change of scale property.**

***Theorem 1-5.*** If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right) \tag{5}$$

**Example.** Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, we have

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3} \frac{1}{(s/3)^2+1} = \frac{3}{s^2+9}$$

5. **Laplace transform of derivatives.**

***Theorem 1-6.*** If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{F'(t)\} = s\,f(s) - F(0) \tag{6}$$

if $F(t)$ is continuous for $0 \leqq t \leq N$ and of exponential order for $t > N$ while $F'(t)$ is sectionally continuous for $0 \leqq t \leqq N$.

Prove *Theorem 1-9*, Page 4: If $\mathcal{L}\{F(t)\} = f(s)$ then $\mathcal{L}\{F''(t)\} = s^2 f(s) - s\,F(0) - F'(0)$.

## 6. Laplace transform of integrals.

***Theorem 1-11.*** If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s} \tag{11}$$

**Example.** Since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$, we have

$$\mathcal{L}\left\{\int_0^t \sin 2u\,du\right\} = \frac{2}{s(s^2+4)}$$

as can be verified directly.

7. **Multiplication by $t^n$.**

***Theorem 1-12.*** If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s) = (-1)^n f^{(n)}(s) \tag{12}$$

**Example.** Since $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$, we have

$$\mathcal{L}\{te^{2t}\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

$$\mathcal{L}\{t^2e^{2t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$$

**8. Division by $t$.**

***Theorem 1-13.*** If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du \qquad (13)$$

provided $\lim_{t \to 0} F(t)/t$ exists.

**Example.** Since $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ and $\lim_{t\to 0} \frac{\sin t}{t} = 1$, we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{du}{u^2+1} = \tan^{-1}(1/s)$$

## 9. Periodic functions.

***Theorem 1-14.*** Let $F(t)$ have period $T > 0$ so that $F(t+T) = F(t)$ [see Fig. 1-2].

Then
$$\mathcal{L}\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}} \tag{14}$$

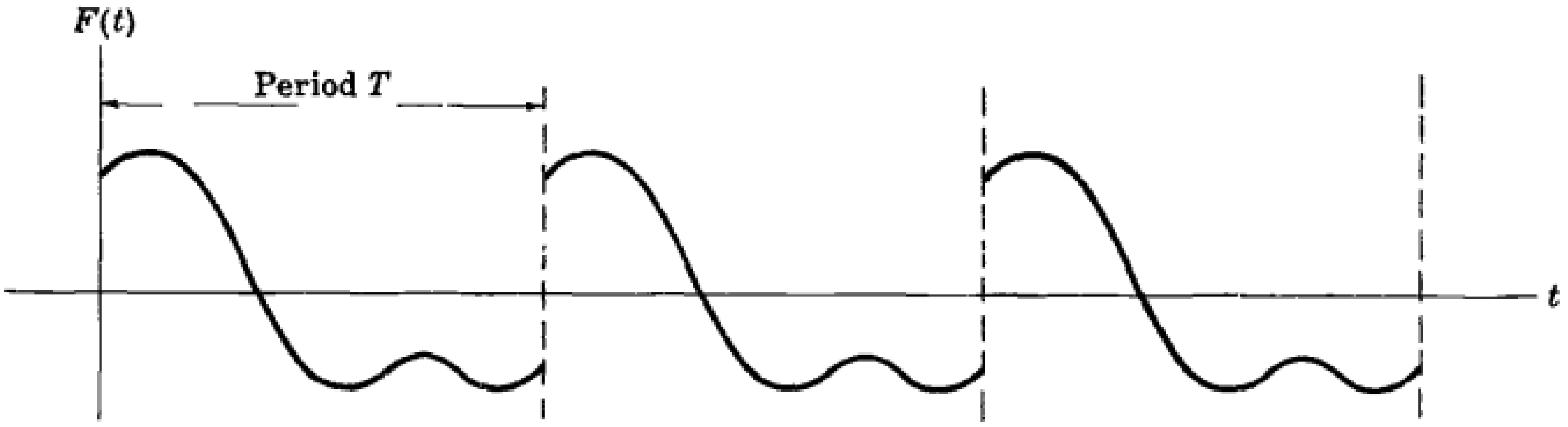



**Fig. 1-2**

**Example 23.** *Find the Laplace transform of the waveform*

$$f(t) = \left(\frac{2t}{3}\right), 0 \le t \le 3.$$

**Solution.**

$$L\left[f(t)\right] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$L\left[\frac{2t}{3}\right] = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st} \left(\frac{2}{3} t\right) dt = \frac{1}{1-e^{-3s}} \frac{2}{3} \left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3} \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] = \frac{2}{3} . \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)} + \frac{2}{3s^2}$$

**Ans.**

## 13.18 FORMULATION OF LAPLACE TRANSFORM

| *S.No.* | $f(t)$ | $F(s)$ |
|---|---|---|
| **1.** | $e^{at}$ | $\frac{1}{s-a}$ |
| **2.** | $t^n$ | $\frac{\lceil n+1}{s^{n+1}}$ or $\frac{n!}{s^{n+1}}$ |
| **3.** | $\sin at$ | $\frac{a}{s^2+a^2}$ |
| **4.** | $\cos at$ | $\frac{s}{s^2+a^2}$ |
| **5.** | $\sinh at$ | $\frac{a}{s^2-a^2}$ |
| **6.** | $\cosh at$ | $\frac{s}{s^2-a^2}$ |
| **9.** | $e^{bt}\sin at$ | $\frac{a}{(s-b)^2+a^2}$ |
| **10.** | $e^{bt}\cos at$ | $\frac{s-b}{(s-b)^2+a^2}$ |
| **11.** | $\frac{t}{2a}\sin at$ | $\frac{s}{(s^2+a^2)^2}$ |
| **12.** | $t\cos at$ | $\frac{s^2-a^2}{(s^2+a^2)^2}$ |

## 13.19 PROPERTIES OF LAPLACE TRANSFORM

| S.No. | Property | $f(t)$ | $F(s)$ |
|---|---|---|---|
| **1.** | Scaling | $f(at)$ | $\frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0$ |
| **2.** | Derivative | $\frac{df(t)}{dt}$<br>$\frac{d^2 f(t)}{dt^2}$<br>$\frac{d^3 f(t)}{dt^3}$ | $s\,F(s) - f(0), \quad s > 0$<br>$s^2 F(s) - sf(0) - f'(0), \quad s > 0$<br>$s^3 F(s) - s^2 f(0) - s\,f'(0) - f''(0), s > 0$ |
| **3.** | Integral | $\int_0^t f(t)\,dt$ | $\frac{1}{s}F(s), \quad s > 0$ |
| **6.** | First shifting | $e^{-at} f(t)$ | $F(s+a)$ |
| **7.** | Second shifting | $f(t)\,u(t-a)$ | $e^{-a} L\,f(t+a)$ |
| **8.** | Multiplication by $t$ | $t\,f(t)$ | $-\frac{d}{ds}F(s)$ |
| **9.** | Multiplication by $t^n$ | $t^n f(t)$ | $(-1)^n \frac{d^n}{ds^n} F(s)$ |
| **10.** | Division by $t$ | $\frac{1}{t} f(t)$ | $\int_s^\infty F(s)\,ds$ |
| **11.** | Periodic function | $f(t)$ | $\frac{\int_0^T e^{-st} f(t)}{1 - e^{-st}} \quad f(t+T) = f(t)$ |
| **12.** | Convolution | $f(t) * g(t)$ | $F(s)\,G(s)$ |

6. Find $\mathcal{L}\{4e^{5t} + 6t^3 - 3\sin 4t + 2\cos 2t\}$.

8. Find (a) $\mathcal{L}\{t^2 e^{3t}\}$, (b) $\mathcal{L}\{e^{-2t}\sin 4t\}$, (c) $\mathcal{L}\{e^{4t}\cosh 5t\}$, (d) $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$.

**Example 13.** Find the Laplace transform of $\int_0^t \frac{\sin t}{t}\,dt$.

**Solution :** We know that $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$ (say).

Now $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty F(u)\,du = \int_s^\infty \frac{1}{u^2+1}\,du$

$$= \left[\tan^{-1} u\right]_s^\infty = \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = \tan^{-1}\frac{1}{s}$$

Since $\tan^{-1}s + \cot^{-1}s = \frac{\pi}{2}$.

Now we also know that $\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s}$.

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s} f(s) = \frac{1}{s}.\tan^{-1}\frac{1}{s}.$$

**12.** Given that $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$, find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$.

By Problem 11,

$$\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \frac{1}{a}\tan^{-1}\{1/(s/a)\} = \frac{1}{a}\tan^{-1}(a/s)$$

Then $\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}(a/s)$.

**Example 8.** *Find the Laplace transform of $t^2$ cos at*

**Solution.** $L(\cos at) = \frac{a}{s^2+a^2}$

$$L\left(t^2 \cos at\right) = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right] = \frac{d}{ds}\frac{\left(s^2+a^2\right).1 - s(2s)}{\left(s^2+a^2\right)^2} = \frac{d}{ds}\frac{a^2-s^2}{\left(s^2+a^2\right)^2}$$

$$= \frac{\left(s^2+a^2\right)^2(-2s) - \left(a^2-s^2\right).2\left(s^2+a^2\right)(2s)}{\left(s^2+a^2\right)^4} = \frac{-2s^3-2a^2s-4a^2s+4s^3}{\left(s^2+a^2\right)^3}$$

$$= \frac{2s\left(s^2-3a^2\right)}{\left(s^2+a^2\right)^3}$$

**Ans.**

**Example 9.** *Obtain the Laplace transform of*

$$t^2 e^t . \sin 4t$$

**Solution.** $L(\sin 4t) = \dfrac{4}{s^2+16}$, $L(e^t \sin 4t) = \dfrac{4}{(s-1)^2+16}$

$$L(t\, e^t \sin 4t) = -\frac{d}{ds}\frac{4}{s^2-2s+17} = \frac{4(2s-2)}{\left(s^2-2s+17\right)^2}$$

$$L(t^2 e^t \sin 4t) = -4\frac{d}{ds}\frac{2s-2}{\left(s^2-2s+17\right)^2}$$

$$= -4\frac{\left(s^2-2s+17\right)^2 2-(2s-2)2\left(s^2-2s+17\right)(2s-2)}{\left(s^2-2s+17\right)^4}$$

$$= \frac{-4\left(2s^2-4s+34-8s^2+16s-8\right)}{\left(s^2-2s+17\right)^3}$$

$$= \frac{-4\left(-6s^2+12s+26\right)}{\left(s^2-2s+17\right)^3} = \frac{8\left(3s^2-6s-13\right)}{\left(s^2-2s+17\right)^3}$$

**Ans.**

**24.** (*a*) Graph the function

$$F(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

extended periodically with period $2\pi$.

(*b*) Find $\mathcal{L}\{F(t)\}$.

(*a*) The graph appears in Fig. 1-5.

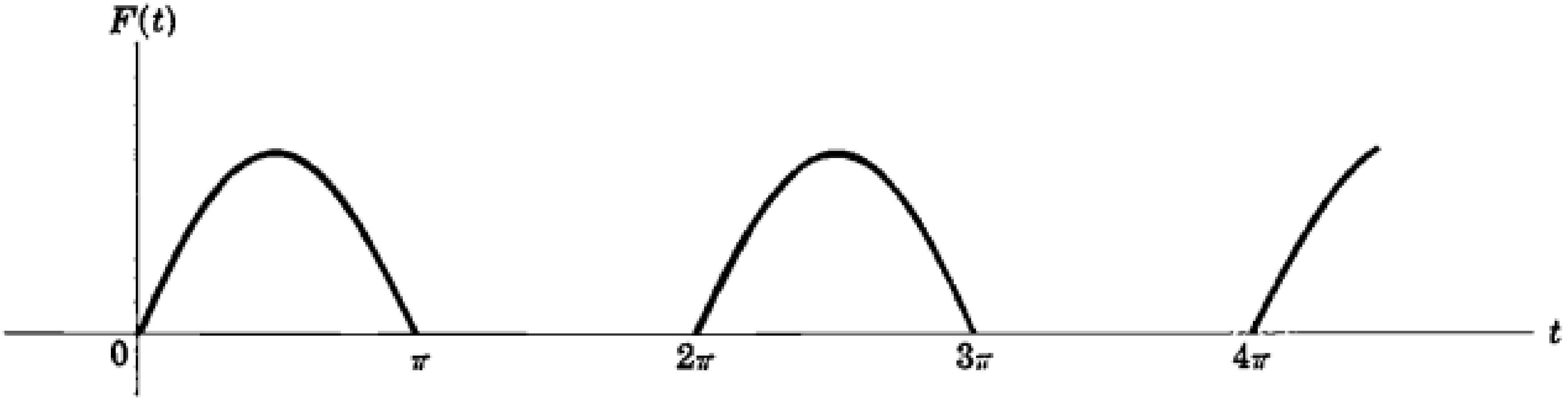



**Fig. 1-5**

(*b*) By Problem 23, since $T = 2\pi$, we have

$$
\begin{aligned}
\mathcal{L}\{F(t)\} &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\, dt \\
&= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt \\
&= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{e^{-st}(-s \sin t - \cos t)}{s^2+1} \right\} \Bigg|_0^{\pi} \\
&= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{1+e^{-\pi s}}{s^2+1} \right\} \quad = \quad \frac{1}{(1-e^{-\pi s})(s^2+1)}
\end{aligned}
$$

using the integral (*1*) of Problem 2, Page 11.

The graph of the function $F(t)$ is often called a *half wave rectified sine curve.*

4. Find $\mathcal{L}\{F(t)\}$ if $F(t) = \begin{cases} 5 & 0 < t < 3 \\ 0 & t > 3 \end{cases}$

By definition,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = \int_0^3 e^{-st}(5)\,dt + \int_3^\infty e^{-st}(0)\,dt$$

$$= 5\int_0^3 e^{-st}\,dt = 5\left.\frac{e^{-st}}{-s}\right|_0^3 = \frac{5(1-e^{-3s})}{s}$$

**45. Evaluate** (a) $\displaystyle\int_0^\infty t\,e^{-2t}\cos t\,dt$, (b) $\displaystyle\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt$.

(a) By Problem 19,

$$\begin{aligned}\mathcal{L}\{t\cos t\} &= \int_0^\infty t\,e^{-st}\cos t\,dt \\ &= -\frac{d}{ds}\mathcal{L}\{\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}\end{aligned}$$

Then letting $s=2$, we find $\displaystyle\int_0^\infty te^{-2t}\cos t\,dt = \frac{3}{25}$.

(b) If $F(t) = e^{-t} - e^{-3t}$, then $f(s) = \mathcal{L}\{F(t)\} = \frac{1}{s+1} - \frac{1}{s+3}$. Thus by Problem 21,

$$\mathcal{L}\left\{\frac{e^{-t} - e^{-3t}}{t}\right\} = \int_s^\infty \left\{\frac{1}{u+1} - \frac{1}{u+3}\right\} du$$

or

$$\int_0^\infty e^{-st}\left(\frac{e^{-t} - e^{-3t}}{t}\right) dt = \ln\left(\frac{s+3}{s+1}\right)$$

Taking the limit as $s \to 0+$, we find $\displaystyle\int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt = \ln 3.$

**The error function**

The error function, denoted by erf (t), is defined by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2}\,du$$

**Heaviside's unit step function.**

The **Heaviside's unit step function** about a point a, denoted by H (t – a), is defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

**The gamma function**

If n > 0, the gamma function is defined by

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt$$

**The unit impulse Function** (or **Dirac delta function)**

The unit impulse function is defined by

$$F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 \leq t \leq \epsilon \\ 0, & t > \epsilon \end{cases}$$

# Inverse Laplace transform

## DEFINITION OF INVERSE LAPLACE TRANSFORM

If the Laplace transform of a function $F(t)$ is $f(s)$, i.e. if $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an *inverse Laplace transform* of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$ where $\mathcal{L}^{-1}$ is called the *inverse Laplace transformation operator*.

**Example.** Since $\mathcal{L}\{e^{-3t}\} = \dfrac{1}{s+3}$ we can write

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

# Table of Inverse Laplace Transforms

| | $f(s)$ | $\mathcal{L}^{-1}\{f(s)\} = F(t)$ |
|---|---|---|
| 1. | $\frac{1}{s}$ | $1$ |
| 2. | $\frac{1}{s^2}$ | $t$ |
| 3. | $\frac{1}{s^{n+1}} \quad n = 0, 1, 2, \ldots$ | $\frac{t^n}{n!}$ |
| 4. | $\frac{1}{s-a}$ | $e^{at}$ |
| 5. | $\frac{1}{s^2+a^2}$ | $\frac{\sin at}{a}$ |
| 6. | $\frac{s}{s^2+a^2}$ | $\cos at$ |
| 7. | $\frac{1}{s^2-a^2}$ | $\frac{\sinh at}{a}$ |
| 8. | $\frac{s}{s^2-a^2}$ | $\cosh at$ |

## 1. Linearity property

**Theorem 1.** If $\mathcal{L}\{F_1(t)\} = f_1(s)$ and $\mathcal{L}\{F_2(t)\} = f_2(s)$ and $c_1$ and $c_2$ are any two constants, then

$$\mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} = c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\}$$
$$= c_1 F_1(t) + c_2 F_2(t).$$

## 2. First translation (or shifting) property

**Theorem 2.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at} F(t).$$

## 3. Second translation (or shifting) property

**Theorem 3.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = G(t) \text{ where } G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

**Proof :** By definition of Laplace transform, we have

$$f(s) = \int_0^\infty e^{-st} F(t)dt.$$

## 4. Change of scale property

**Theorem 4.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right).$$

## 5. Inverse Laplace transform of derivatives

**Theorem 5.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ then

$$\mathcal{L}^{-1}\{f^n(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\} = (-1)^n t^n F(t).$$

where $n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots$

## 6. Inverse Laplace transform of integrals

**Theorem 6.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, Then

$$\mathcal{L}^{-1}\left\{\int_s^{\infty} f(u)\,du\right\} = \frac{F(t)}{t}$$

## 7. Multiplication by $s^n$

**Theorem 7.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(o) = o$, then

$$\mathcal{L}^{-1}\{sf(s)\} = F'(t)$$

## 8. Division by s

**Theorem 8.** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du.$$

## 2.7 Convolution theorem (or convolution property)

**Statement :** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\,du = F * G.$$

**Linearity property.**

***Theorem 2-2.*** If $c_1$ and $c_2$ are any constants while $f_1(s)$ and $f_2(s)$ are the Laplace transforms of $F_1(t)$ and $F_2(t)$ respectively, then

$$\begin{aligned} \mathcal{L}^{-1}\{c_1 f_1(s) + c_2 f_2(s)\} &= c_1 \mathcal{L}^{-1}\{f_1(s)\} + c_2 \mathcal{L}^{-1}\{f_2(s)\} \\ &= c_1 F_1(t) + c_2 F_2(t) \end{aligned} \tag{1}$$

**Example.**

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} &= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} \\ &\quad + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t \end{aligned}$$

**First translation or shifting property.**

*Theorem 2-3.* If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

**Example.** Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\} = \frac{1}{2}e^t \sin 2t$$

**Change of scale property.**

***Theorem 2-5.*** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k} F\left(\frac{t}{k}\right)$$

**Example.** Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$, we have

$$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\frac{4t}{2} = \frac{1}{2}\cos 2t$$

as is verified directly.

**The Convolution property.**

***Theorem 2-10.*** If $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $\mathcal{L}^{-1}\{g(s)\} = G(t)$, then

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du = F * G \tag{11}$$

We call $F * G$ the *convolution* or *faltung* of $F$ and $G$, and the theorem is called the *convolution theorem* or *property.*

**Example.** Since $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\} = \int_0^t e^u\,e^{2(t-u)}\,du = e^{2t} - e^t$$

**Problems**

Find (a) $\mathcal{L}^{-1}\left\{\dfrac{5s+4}{s^3} - \dfrac{2s-18}{s^2+9} + \dfrac{24-30\sqrt{s}}{s^4}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{6}{2s-3} - \dfrac{3+4s}{9s^2-16} + \dfrac{8-6s}{16s^2+9}\right\}$.

(a) $$\mathcal{L}^{-1}\left\{\frac{5s+4}{s^3} - \frac{2s-18}{s^2+9} + \frac{24-30\sqrt{s}}{s^4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{5}{s^2} + \frac{4}{s^3} - \frac{2s}{s^2+9} + \frac{18}{s^2+9} + \frac{24}{s^4} - \frac{30}{s^{7/2}}\right\}$$

$$= 5t + 4(t^2/2!) - 2\cos 3t + 18(\tfrac{1}{3}\sin 3t) + 24(t^3/3!) - 30\{t^{5/2}/\Gamma(7/2)\}$$

$$= 5t + 2t^2 - 2\cos 3t + 6\sin 3t + 4t^3 - 16t^{5/2}/\sqrt{\pi}$$

since $\Gamma(7/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{15}{8}\sqrt{\pi}$.

(b) $$\mathcal{L}^{-1}\left\{\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} + \frac{8-6s}{16s^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{s-3/2} - \frac{1}{3}\left(\frac{1}{s^2-16/9}\right) - \frac{4}{9}\left(\frac{s}{s^2-16/9}\right) + \frac{1}{2}\left(\frac{1}{s^2+9/16}\right) - \frac{3}{8}\left(\frac{s}{s^2+9/16}\right)\right\}$$

$$= 3e^{3t/2} - \tfrac{1}{4}\sinh 4t/3 - \tfrac{4}{9}\cosh 4t/3 + \tfrac{2}{3}\sin 3t/4 - \tfrac{3}{8}\cos 3t/4$$

Find each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s+12}{s^2+8s+16}\right\}$

(c) $\mathcal{L}^{-1}\left\{\dfrac{3s+7}{s^2-2s-3}\right\}$

(d) $\mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{2s+3}}\right\}$

(a)

$$\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$$

$$= 6\,e^{2t}\cos 4t + 2\,e^{2t}\sin 4t = 2\,e^{2t}(3\cos 4t + \sin 4t)$$

(*b*)
$$\mathcal{L}^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\} = \mathcal{L}^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}$$
$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} - 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s+4)^2}\right\}$$
$$= 4\,e^{-4t} - 4t\,e^{-4t} = 4\,e^{-4t}(1-t)$$

(*c*)
$$\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{3s+7}{(s-1)^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{3(s-1)+10}{(s-1)^2-4}\right\}$$
$$= 3\,\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2-4}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2-4}\right\}$$
$$= 3\,e^t\cosh 2t + 5\,e^t\sinh 2t = e^t(3\cosh 2t + 5\sinh 2t)$$
$$= 4\,e^{3t} - e^{-t}$$

Evaluate each of the following by use of the convolution theorem.

(a) $\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.

(a) We can write $\dfrac{s}{(s^2+a^2)^2} = \dfrac{s}{s^2+a^2} \cdot \dfrac{1}{s^2+a^2}$. Then since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{\sin at}{a}$, we have by the convolution theorem,

$$
\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} &= \int_0^t \cos au \cdot \frac{\sin a(t-u)}{a}\,du \\
&= \frac{1}{a}\int_0^t (\cos au)(\sin at \cos au - \cos at \sin au)\,du \\
&= \frac{1}{a}\sin at \int_0^t \cos^2 au\,du - \frac{1}{a}\cos at \int_0^t \sin au \cos au\,du \\
&= \frac{1}{a}\sin at \int_0^t \left(\frac{1+\cos 2au}{2}\right) du - \frac{1}{a}\cos at \int_0^t \frac{\sin 2au}{2}\,du \\
&= \frac{1}{a}\sin at \left(\frac{t}{2} + \frac{\sin 2at}{4a}\right) - \frac{1}{a}\cos at \left(\frac{1-\cos 2at}{4a}\right) \\
&= \frac{1}{a}\sin at \left(\frac{t}{2} + \frac{\sin at \cos at}{2a}\right) - \frac{1}{a}\cos at \left(\frac{\sin^2 at}{2a}\right) \\
&= \frac{t \sin at}{2a}
\end{aligned}
$$

(*b*) We have $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$, $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = t\,e^{-t}$. Then by the convolution theorem,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} &= \int_0^t (ue^{-u})(t-u)\,du \\
&= \int_0^t (ut-u^2)\,e^{-u}\,du \\
&= (ut-u^2)(-e^{-u}) - (t-2u)(e^{-u}) + (-2)(-e^{-u})\Big|_0^t \\
&= te^{-t} + 2e^{-t} + t - 2
\end{aligned}$$

Find $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

We have

$$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

Multiply both sides of (1) by $s+1$ and let $s \to -1$; then

$$A = \lim_{s \to -1} \frac{2s^2-4}{(s-2)(s-3)} = -\frac{1}{6}$$

Multiply both sides of (1) by $s-2$ and let $s \to 2$; then

$$B = \lim_{s \to 2} \frac{2s^2-4}{(s+1)(s-3)} = -\frac{4}{3}$$

Multiply both sides of (1) by $s-3$ and let $s \to 3$; then

$$C = \lim_{s \to 3} \frac{2s^2-4}{(s+1)(s-2)} = \frac{7}{2}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/6}{s+1} + \frac{-4/3}{s-2} + \frac{7/2}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

Find $\mathcal{L}^{-1}\left\{\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right\}$.

Find $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$.

. Find $\mathcal{L}^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$.

Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

Use the convolution theorem to find (*a*) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+3)(s-1)}\right\}$, (*b*) $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^2(s-2)}\right\}$.

*Ans.* (*a*) $\frac{1}{4}(e^t - e^{-3t})$, (*b*) $\frac{1}{16}(e^{2t} - e^{-2t} - 4te^{-2t})$

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$. *Ans.* $\frac{1}{2}(\sin t - \cos t + e^{-t})$

Find $\mathcal{L}^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}$. *Ans.* $\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t$

# Solving ODE using Laplace transform

**Example 1.** Solve the following differential equation by using Laplace transform :

$$\frac{dy}{dt} - 3y = o; y(o) = 1.$$

**Solution** : The given differential equation can be written as $y' - 3y = o$ (1)

Taking the Laplace transform of both sides of (1) we get

$\mathcal{L}\{y'\} - 3\,\mathcal{L}\{y\} = \mathcal{L}\{o\}$

or, $sY(s) - y(o) - 3Y(s) = 0$

Since $\mathcal{L}\{F'(t)\} = sf(s) - F(o)$

or, $sY(s) - 1 - 3Y(s) = o$ since $y(o) = 1$

or, $(s - 3)\,Y(s) = 1$

or, $Y(s) = \dfrac{1}{s - 3}$ (2)

Now taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

or, $y(t) = e^{3t}$. since $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$.

**Example 2.** Solve the differential equation

$$y' + 2y = e^t;\ y(o) = 1$$

by using Laplace transform.

**Solution :** The given differential equation is

$$y' + 2y = e^t \qquad (1)$$

Taking the Laplace transform of both sides of (1), we get

$$\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^t\}$$

$$sY(s) - y(o) + 2Y(s) = \frac{1}{s-1}$$

$$\text{or, } sY(s) - 1 + 2Y(s) = \frac{1}{s-1}$$

$$\text{or, } (s+2)Y(s) = 1 + \frac{1}{s-1} = \frac{s}{s-1}$$

$$\text{or, } Y(s) = \frac{s}{(s-1)(s+2)} = \frac{1}{3}\cdot\frac{1}{s-1} + \frac{2}{3}\cdot\frac{1}{s+2} \qquad (2)$$

Now taking inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$\text{or, } y(t) = \frac{1}{3}e^t + \frac{2}{3}e^{-2t}.$$

**Example 4.** Solve the differential equation

$y' + y = \sin t;\ y(o) = 1.$

by using the Laplace transform.

**Solution :** The given differential equation is

$y' + y = \sin t \quad (1)$

Taking the Laplace transform of both sides of (1), we get

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$

or, $sY(s) - y(o) + Y(s) = \dfrac{1}{s^2+1}$

since $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$

or, $sY(s) - 1 + Y(s) = \dfrac{1}{s^2+1}$

or, $(s+1)Y(s) = 1 + \dfrac{1}{s^2+1}$

or, $Y(s) = \dfrac{1}{s+1} + \dfrac{1}{(s+1)(s^2+1)} \quad (2)$

$\dfrac{1}{(s+1)(s^2+1)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+1}$

or, $1 = A(s^2+1) + (Bs+C)(s+1) \quad (3)$

Taking $s = -1$ in (3), we get

$1 = 2A + o, \therefore \boxed{A = \frac{1}{2}}.$

Again taking the coefficients of $s^2$ from both sides of (3),

we get $o = A + B$.

$\therefore B = -A = -\frac{1}{2}$. Or, $B = -\frac{1}{2}$

Lastly, taking the constant terms from both sides of (3),

we get

$1 = A + C \quad \therefore \boxed{C = \frac{1}{2}}$

Therefore, $\dfrac{1}{(s+1)(s^2+1)} = \dfrac{1}{2}\cdot\dfrac{1}{s+1} + \dfrac{1}{2}\cdot\dfrac{(-s+1)}{s^2+1}$

$$= \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Thus from (2), we get

$$Y(s) = \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$= \frac{3}{2}\cdot\frac{1}{s+1} - \frac{1}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} \qquad (4)$$

Taking the inverse Laplace transform of both sides of (4),

we get

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

or, $y(t) = \dfrac{3}{2}e^{-t} - \dfrac{1}{2}\cos t + \dfrac{1}{2}\sin t.$

**Example 6.** Solve following differential equation by using Laplace transform : $Y''(t) + Y(t) = t;\ Y(o) = 1,\ Y'(o) = -2$

**[ D. U. H. (T) 1991 ]**

**Solution :** The given differential equation is

$$Y''(t) + Y(t) = t \qquad (1)$$

Taking the Laplace transfrom of both sides of (1) and using the given conditions we get $\mathcal{L}\{Y''(t)\} + \mathcal{L}\{Y(t)\} = \mathcal{L}(t)$

or, $s^2y - sY(o) - Y'(o) + y = \dfrac{1}{s^2}$

or, $s^2y - s + 2 + y = \dfrac{1}{s^2}$

or, $(s^2 + 1)y = s - 2 + \dfrac{1}{s^2} = \dfrac{s^3 - 2s^2 + 1}{s^2}$

or, $y = \dfrac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)}$

Now $\dfrac{s^3 - 2s^2 + 1}{s^2(s^2 + 1)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs + D}{s^2 + 1}$

or, $s^3 - 2s^2 + 1 = As(s^2 + 1) + B(s^2 + 1) + Cs^3 + Ds^2 \qquad (2)$

Equating the coefficients of $s^3$ from both sides of (2), we get $C = 1$. Putting $s = o$, in (2), we get $B = 1$.

Equating the coefficients of $s^2$ from both sides of (2), we get $-2 = B + D$, or, $-2 = 1 + D \therefore D = -3$.

Equating the coefficients of s from both sides of (2), wet ge $+0 = A + 0 \therefore A = 0$

$$\therefore y = 0 + \frac{1}{s^2} + \frac{s-3}{s^2+1} = \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1} \qquad (3)$$

Taking the inverse Laplace transform of both sides of (3), we get

$$\mathcal{L}^{-1}\{y\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

or, $Y(t) = t + \cos t - 3\sin t$ which is the required solution.

**Example 8.** Solve the following differential equation using Laplace transform :

$$Y'' + 9Y = \cos 2t;\ Y(o) = 1,\ Y\left(\frac{\pi}{2}\right) = -1$$

**Solution :** The given differential equation is

$$Y'' + 9Y = \cos 2t \quad (1)$$

Taking Laplace transform of both sides of (1), we get

$$\mathcal{L}\{Y''\} + 9\,\mathcal{L}\{Y\} = \mathcal{L}\{\cos 2t\}$$

$$\text{or, } s^2 y(s) - sY(o) - Y'(o) + 9y(s) = \frac{s}{s^2+4}$$

$$\text{or, } s^2 y(s) - s - c + 9\,y(s) = \frac{s}{s^2+4}$$

where $Y'(o) = c$ (say)

$$\text{or, } (s^2+9)y(s) = s + c + \frac{s}{s^2+4}$$

$$\text{or, } y(s) = \frac{s+c}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

$$\text{or, } y(s) = \frac{s}{s^2+3^2} + \frac{c}{s^2+3^2} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} \quad (2)$$

Taking the inverse Laplace transform of both sides of (2), we get

$$\mathcal{L}^{-1}\{y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{c}{s^2+3^2}\right\}$$

$$+\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

$$\text{or, } Y(t) = \cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t$$

$$= \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t \quad (3)$$

$$\therefore Y\left(\frac{\pi}{2}\right) = \frac{4}{5}\cos\frac{3\pi}{2} + \frac{c}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\cos\pi$$

$$= -\frac{4}{5}\cos\frac{\pi}{2} - \frac{c}{3}\sin\frac{\pi}{2} + \frac{1}{5}\cos\pi$$

$$\text{or, } -1 = o - \frac{c}{3} - \frac{1}{5},\ \text{since } Y\left(\frac{\pi}{2}\right) = -1$$

$$\text{or, } \frac{c}{3} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\therefore c = \frac{12}{5}$$

Thus from (3), we get

$$Y(t) = \frac{4}{5}\cos 3t + \frac{1}{3}\cdot\frac{12}{5}\sin 3t + \frac{1}{5}\cos 2t$$

$$= \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t.$$

which is the required solution.

# Solve the following differential equations:

1. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$, where $y = 2$, $\frac{dy}{dx} = -4$ at $x = 0$. **Ans.** $y = e^{-x}(2\cos 2x - \sin 2x)$

2. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$, given $y = \frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = 6$ at $x = 0$. **Ans.** $y = e^{x} - 3e^{-x} + 2e^{-2x}$

3. $y'' + 2y' + y = te^{-t}$ if $y(0) = 1$, $y'(0) = -2$. **Ans.** $y = \left(1 - t + \frac{t^3}{6}\right)e^{-t}$

4. $\frac{d^2y}{dx^2} + y = x\cos 2x$, where $y = \frac{dy}{dx} = 0$ at $x = 0$. **Ans.** $y = \frac{4}{9}\sin 2x - \frac{5}{9}\sin x - \frac{x}{3}\cos 2x$

5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = x^2e^{2x}$, where $y = 1$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -2$ at $x = 0$.
   **Ans.** $y = e^{2x}(x^2 - 6x + 12) - e^{x}(15x^2 + 7x + 11)$