Fourier Series CT-1

fourier series of a fn

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where, $a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$$

(i) $\sin n\pi = 0$

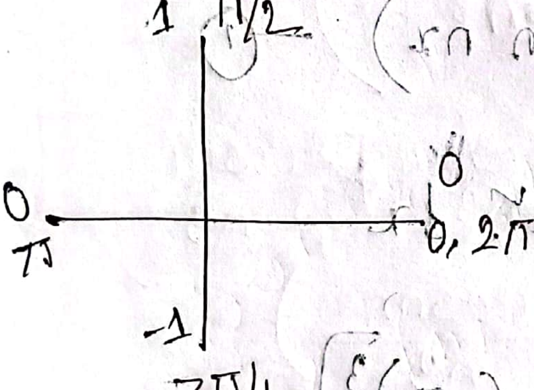
(ii) $\sin 2n\pi = 0$

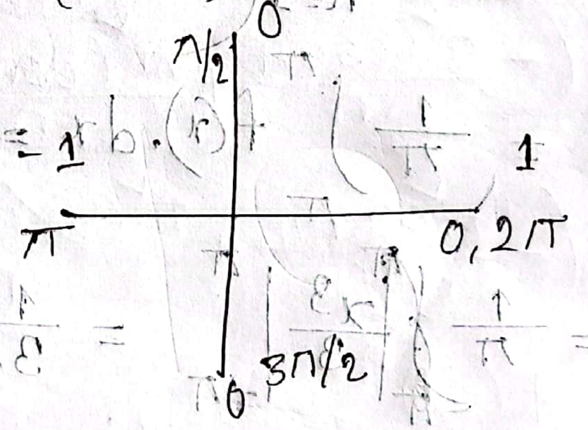
(iii) $\cos n\pi = (-1)^n$

(iv) $\cos 2n\pi = 1$

(v) $\cos (2n+1)\pi = -1$

(vi) $\cos (2n-1)\pi = -1$





Ques① Prove that,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}, \quad (-\pi < x < \pi)$$

Also, show that,

i) $\sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$

iii) $\sum \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solⁿ:

Given function $f(x) = x^2, \; (-\pi < x < \pi)$ ①

Fourier series for the given fⁿ is

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left|\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$$

$$= \frac{1}{3\pi}(\pi^3 + \pi^3)$$

$$= \frac{1}{3\pi} \times 2\pi^3$$

$$= \frac{2\pi^2}{3}$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$\cos n(-\pi) = n\pi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 . \cos nx\, dx$$

$$= \frac{1}{\pi}\left| x^2\left(\frac{\sin nx}{n}\right) - (2x).\left(\frac{-\cos nx}{n^2}\right) + 2.\left(-\frac{\sin nx}{n^3}\right)\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left| x^2\frac{\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2\frac{\sin n\pi}{n} - (-\pi)^2\frac{\sin n(-\pi)}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2(-\pi)\cos n(-\pi)}{n^2} - \frac{2\ .\sin \pi}{\pi^3} - \frac{2\sin n(-\pi)}{(-\pi)^3}\right]$$

$$= \frac{1}{\pi}\left(0 + 0 + \frac{2\pi\cos n\pi}{n^2} + \frac{2\pi\cos n\pi}{n^2}\right)$$

$$= \frac{1}{\cancel{\pi}}\ \frac{4\cancel{\pi}\cos n\pi}{n^2}$$

$$\cos n\pi = (-1)^n$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 . \sin nx . dx$$

$$= \frac{1}{\pi}\left| x^2 \frac{-\cos nx}{n} - 2x.\left(\frac{-\sin nx}{n^2}\right) + 2.\frac{\cos nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \cancel{\frac{1}{\pi}\left(\pi^2 \frac{\cos n\pi}{n} + 2\pi\frac{\sin n\pi}{n^2}\right.} + 2$$

$$= \frac{1}{\pi}\left(\pi^2 \frac{-\cos n\pi}{n} - (-\pi)^2 . \frac{-\cos n(-\pi)}{n} + 2\pi\left(\frac{\cancel{\sin n\pi}}{n^2}\right) - 2(-\pi) . \frac{-\cancel{\sin n(-\pi)}}{n^2} + 2\frac{\cos n\pi}{n^3} - 2\frac{\cos n(-\pi)}{(n)^3}\right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi^2 \cos n\pi}{n} + \frac{\pi^2 \cos n(-\pi)}{n} + \frac{2\cos n\pi}{n^3} + \frac{2\cos n(-\pi)}{n^3}\right)$$

$$= \frac{1}{\pi}\left(-\frac{\pi^2 \cancel{\cos n\pi}}{n} + \frac{\pi^2 \cancel{\cos n\pi}}{n} + \frac{4\cos n\pi}{n^3}\right)$$

$$= \cancel{\frac{4(-1)^n}{n^3}}$$

$$= 0$$

eq^n ① become

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$$

$$x^2 = \frac{a_0}{2} + a_n \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$$

<u>Example:</u> find the fourier series of $f(x) = e^{ax}$ in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cancel{\int_0^{2\pi} f(x)} (a_n \cos nx + b_n \sin nx) \quad \text{—①}$$

$$a_0 = \frac{1}{\cancel{2}\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\,dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{e^{2a\pi} - 1}{a}\right) = \frac{1}{\pi}\left(\frac{e^{2\pi a}}{a} - \frac{1}{a}\right)$$

$$= \frac{1}{\pi}\left(\frac{e^{2\pi a} - 1}{a}\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi a}}{a^2+n^2}(a.1) - \frac{1}{a^2+n^2}(a)\right]$$

$$= \frac{1}{\pi}\cdot\frac{a}{(a^2+n^2)}\left(e^{2\pi a} - 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}(0-n) - \frac{1}{a^2+n^2}(-n)\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left(1-e^{2\pi a}\right)$$

$$\therefore f(x) = \frac{1}{2\pi}\left(\frac{e^{2\pi a}-1}{a}\right) + \sum_{n=1}^{\infty}\left[\frac{a}{\pi(a^2+n^2)}\left(e^{2\pi a}-1\right)\cos nx + \frac{n}{\pi(a^2+n^2)}\left(1-e^{2\pi a}\right)\sin nx\right]$$

(Ans)

(11) $f(x) = e^{ax} \quad [2\pi < x < 0]$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x).dx = \frac{1}{\pi}\int_0^{2\pi} e^{ax}.dx = \frac{1}{\pi}\left[\frac{e^{ax}}{a}\right]_0^{2\pi}$

$= \frac{1}{\pi}\left(\frac{e^{2\pi a}}{a} - \frac{1}{a}\right)$

$= \frac{1}{\pi}\left(\frac{e^{2\pi a} - 1}{a}\right)$

$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx.dx$

$= \frac{1}{\pi}\int_0^{2\pi} e^{ax}.\cos nx.dx$

$= \frac{1}{\pi}\left[\cancel{e^{ax}} \frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx)\right]_0^{2\pi}$

$= \frac{1}{\pi}\left[\frac{e^{2a\pi}}{a^2+n^2}.(a) - \frac{1}{a^2+n^2}.(a)\right]$

$= \frac{1}{\pi}\frac{\cancel{e^{2\pi a}}.a}{a^2+n^2}\left(e^{2\pi a} - 1\right)$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, . \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{ax} . \sin nx . \, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\left(a\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi a}}{a^2+n^2}(0-n) - \frac{1}{a^2+n^2}(0-n)\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left(1-e^{2\pi a}\right)$$

$$\therefore f(x) = \frac{1}{2\pi}\left(\frac{e^{2\pi a}-1}{a}\right) + \sum_{n=1}^{\infty}\left[\frac{a}{\pi(a^2+n^2)}\left(e^{2\pi a}-1\right)\cos nx + \frac{n}{\pi(a^2+n^2)}\left(1-e^{2\pi a}\right)\sin nx\right]$$

Ques: 2 Obtain the fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

$\therefore e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$\therefore a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx$

$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \, dx = \frac{1}{\pi} \left[\frac{-e^{-x}}{1} \right]_0^{2\pi}$

$= \frac{1}{\pi} \left(-e^{-2\pi} + 1 \right) = \frac{1}{\pi} \left(1 - e^{-2\pi} \right)$

$\therefore a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$

$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx \, dx = \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \cos nx + n \sin nx) \right]$

$= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\cos nx + n \sin nx) \right]_0^{2\pi}$

$= \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} (-1) - \frac{1}{1+n^2} (-1) \right]$

$= \frac{1}{\pi(1+n^2)} \left(1 - e^{-2\pi} \right)$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cdot \sin nx \cdot dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{1+n^2}(-n) - \frac{1}{1+n^2}\cdot(-n)\right]$$

$$= \frac{n}{\pi(1+n^2)}\cdot\left(1 - e^{-2\pi}\right)$$

$$\therefore e^{-x} = \frac{1}{2\pi}\left(1 - e^{-2\pi}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{\pi(1+n^2)}\left(1-e^{-2\pi}\right)\cos nx + \frac{n}{\pi(1+n^2)}\left(1-e^{-2\pi}\right)\right]$$

(3) $f(x) = (x - x^2)$ where $-\pi < x < \pi$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2) \cdot dx$$

$$= \frac{1}{\pi}\left[(1-2x)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1-2\pi) - (1+2\pi)\right] = \frac{1}{\pi}(1 - 2\pi - 1 - 2\pi)$$

$$= \frac{1}{\pi} \times -4\pi \quad -4$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cdot dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}-\frac{\pi^3}{3}\right]-\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\times\frac{}{6} \quad \frac{1}{\pi}\left[\cancel{\frac{\pi^2}{2}}-\frac{\pi^3}{3}-\cancel{\frac{\pi^2}{2}}-\frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\times\frac{2\pi^3}{3} = -\frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\cdot dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cdot\cos(nx)\cdot dx$$

$$= \cancel{\frac{1}{\pi}\left|(1-2x)\cdot\frac{\sin nx}{n}-(-2)\cdot\right.}$$

$$= \frac{1}{\pi}\left|(x-x^2)\cdot\frac{\sin nx}{n}-(1-2x)\cdot\frac{-\cos nx}{n^2}+(-2)\cdot\frac{-\sin nx}{n^3}\right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi-\pi^2)\frac{\sin n\pi}{n}-\cancel{(-\pi-\pi^2)\frac{\sin n(-\pi)}{n}}+(1-2\pi)\cdot\frac{\cos n\pi}{n^2}\right.$$

$$\left.-(1+2\pi)\cdot\frac{\cos n(-\pi)}{n^2}+\frac{2\sin n\pi}{n^3}-\frac{2\sin n(-\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[(1-2\pi)\cdot\frac{(-1)^n}{n^2}-(1+2\pi)\cdot\frac{(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{\pi n^2}\left(x - 2\pi - x - 2\pi\right)$$

$$= \frac{(-1)^n}{\pi n^2}(-4\pi)$$

$$= \frac{-4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx \, dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{-\cos nx}{n} - (1 - 2x)\cdot\frac{-\sin nx}{n^2} + (-2)\cdot\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\Big[(\pi - \pi^2)\frac{-\cos n\pi}{n} + (-\pi - \pi^2)\cdot\frac{\cos n\pi}{n} + \cancel{(1-2\pi)\cdot\frac{\sin n\pi}{n^2}}$$

$$\cancel{-(1+2\pi)\frac{-\sin(-n\pi)}{n^2}} - \frac{2\cos n\pi}{n^3} + \frac{2\cos n(-\pi)}{n^3}\Big]$$

$$= \frac{1}{\pi}\Big(\cancel{\frac{\pi\cos n\pi}{n}}\,(\pi - \pi^2)\frac{-\cos n\pi}{n} + (-\pi - \pi^2)\cdot\frac{\cos n\pi}{n}$$

$$\cancel{\frac{2\cos n\pi}{n^3}} + \cancel{\frac{2\cos n\pi}{n^3}}$$

$$= \frac{1}{\pi}\left[(\pi - \pi^2)\frac{-(-1)^n}{n} + (-\pi - \pi^2)\frac{(-1)^n}{n}\right]$$

$$=$$

$$= \frac{1}{\pi}\left[(-\pi+\pi^2)\frac{(-1)^n}{n} + (-\pi-\pi^2)\frac{(-1)^n}{n}\right]$$

$$= \frac{(-1)^n}{n\pi}\left(-\pi+\cancel{\pi^2}-\pi-\cancel{\pi^2}\right)$$

$$= \frac{(-1)^n}{n\pi}(-2\pi)$$

$$= \frac{-2(-1)^n}{n}$$

$$: \cancel{(x-x^2) = \frac{-4(-1)^n}{2n^2} + \Sigma}$$

$$\therefore (x-x^2) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\left(\frac{-4(-1)^n}{n^2}\right)\cos nx + \frac{-2(-1)^n}{n}\sin nx\right]$$

(A) formula:-

$$A\ \cancel{A+\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \quad \text{—(1)}$$

show that

(i) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\frac{n\pi x}{L}\,dx$

(ii) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

(iii) $A = a_0/2$

proof: Multiplying both side of $f(x)$ by

ⓐ $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, we get,

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L} A\cdot\cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + b_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\cdot\cos\frac{m\pi x}{L}\,dx\right\}$$

$$= a_m L$$

$$\therefore a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\frac{m\pi x}{L}\,dx \quad \text{if } m=1,2,3\ldots$$

ⓑ Multiplying both side of eqⁿ ① by $\sin\frac{m\pi x}{L}$

$$\int_{-L}^{L} f(x)\cdot\sin\frac{m\pi x}{L}\,dx = A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cdot\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= b_m L$$

$$\therefore b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx$$

$\boxed{\text{proved}}$

(c) for A;

$$\int_{-L}^{L} f(x).1.dx = \int_{-L}^{L} A.1.dx + 0 + 0$$

$$= A\,(x)\Big|_{-L}^{L}$$

$$= 2AL$$

∴ $$A = \frac{1}{2L}\int_{-L}^{L} f(x).dx$$

∴ $$A = \frac{a_0}{2}$$ [proved]

(4) Expand the following function in fourier series:—

$$f(x) = \begin{cases} 0 \ ; & -5 < x < 0 \\ 3 \ ; & 0 < x < 5 \end{cases} \quad \text{period} = 10$$

Solve: $T = 10$, $2L = 10$, ∴ $L = 5$

∴ $$a_n = \frac{1}{5}\int_{-5}^{0} f(x)\cos\frac{n\pi x}{5}\,dx + \int_{0}^{5} f(x).\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{5}\left\{0 + \int_{0}^{5} 3\cos\frac{n\pi x}{L}\,dx\right\}$$

$$= \frac{3}{5}\left[\frac{\sin\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right]_0^5$$

$$= \frac{3}{5}\times\frac{5}{n\pi}\left[\sin\frac{n\pi x}{5}\right]_0^5$$

$$= \frac{3}{n\pi}\left[\,0-0\,\right]$$

$$= \frac{3}{n\pi} \times 0$$

$$= 0$$

$$\therefore a_n = 0$$

$$\therefore b_n = \frac{1}{5}\int_{-5}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \int_{0}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{0 + \int_0^5 3\sin\frac{n\pi x}{5}\,dx\right.$$

$$= \frac{3}{5}\left[-\cos\frac{n\pi x}{5} \times \frac{5}{n\pi}\right]_0^5$$

$$= \frac{3}{n\pi}\left[-\cos\frac{n\pi x}{5}\right]_0^5$$

$$= \frac{3}{n\pi}\left(-(-1)^n + 1\right)$$

$$= \frac{3}{n\pi}\left\{1-(-1)^n\right\}$$

$$\therefore a_0 = \frac{1}{5}\int_{-5}^{0} f(x)\,dx + \int_0^5 f(x)\,dx$$

$$= \frac{1}{5}\left\{0 + 3\right|_0^5$$

$$= \frac{1}{5}\left[3x\right]_0^5$$

$$= \frac{1}{5} \times \cancel{15}\,3$$

$$= 3$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\}$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} \left\{ 0 + \frac{3}{n\pi} \{1-(-1)^n\} \sin \frac{n\pi x}{L} \right\}$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty} \left[\frac{3}{n\pi} \{1-(-1)^n\} \sin \frac{n\pi x}{L} \right]$$

(Ans)

25(a) Find the fourier series of $f(x) = x, \; 0 < x < 2\pi$ and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Soln:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x).dx = \frac{1}{\pi} \int_0^{2\pi} x.dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \times \left[2\pi^2 - 0 \right] = \frac{1}{\pi} \times 2\pi^2 = 2\pi$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x).\cos nx.dx = \frac{1}{\pi} \int_0^{2\pi} x.\cos nx.dx$$

$$= \frac{1}{\pi} \left[x \frac{\sin nx}{n} - 1.\frac{-\cos nx}{n^2} \right]_0^{2\pi} = \frac{1}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[(0+1) - \frac{1}{n^2} \right] = \frac{1}{\pi n^2} (1-1) = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \sin nx.dx$$

$$= \frac{1}{\pi} \left[x \frac{-\cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{2\pi} = \frac{1}{\pi} \left[-\frac{1}{n} \right] + \frac{1}{\pi} \left[-\frac{1}{n} + 0 \right]$$

$$= -\frac{1}{n\pi} - \frac{1}{n\pi} =$$

$$= \frac{1}{\pi}\left[-\frac{2\pi \cdot \cos 2n\pi}{n} + 0 + \frac{1}{n}\right]$$

$$= \frac{1}{\pi} \times \left(-\frac{2\pi}{n} + \frac{1}{n}\right)$$

$$= \frac{1}{\pi} \times \frac{-2\pi + 1}{n}$$

$$= -\frac{2\pi}{\pi n} + \frac{1}{n\pi}$$

$$= \frac{1}{\pi} \times -\frac{2\pi}{n}$$

$$= -\frac{2}{n}$$

$$\therefore x = \pi + \sum_{n=1}^{\infty}\left[0 . \cos nx - \frac{2}{n} \sin nx\right]$$

$$= \pi - \frac{2}{n} \sin nx$$

2.6 fourier Series of $f(x) = x^2$, $0 < x < 2\pi$ if the period is 2π

$$2L = 2\pi \therefore L = \pi$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) . dx = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi} \times \frac{8\pi^3}{3} = \frac{8}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) . \cos nx . dx = \frac{1}{\pi}\int_0^{2\pi} x^2 . \cos nx . dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x . \frac{-\cos nx}{n^2} + 2 \frac{-\sin nx}{n^3}\right]_0^{2\pi}$$

(11)

$$= \frac{1}{\pi}\left[4\pi^2 \cancel{\frac{\sin 2n\pi}{n}} - 0 + 4\pi \frac{\cos 2n\pi}{n^2} - 0 - \cancel{\frac{2\sin 2n\pi}{n^3}} + 0\right]$$

$$= \frac{1}{\pi}\left[4\pi \frac{\cos 2n\pi}{n^2}\right]$$

$$= \frac{1}{\pi} \times 4\pi \frac{1}{n^2}$$

$$= \frac{4}{n^2}$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx.dx = \frac{1}{\pi}\int_0^{2\pi} x^2.\sin nx\, dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{-\cos nx}{n} - 2x \frac{-\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-4\pi^2 \frac{\cos 2n\pi}{n} + 0 + 4\pi \cancel{\frac{\sin 2n\pi}{n^2}} + 0 + \frac{2\cos 2n\pi}{n^3} - \frac{2\cos n.0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$= -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^3}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx + \frac{4\pi}{n}\sin nx\right)$$

Ans

(11)

$$= \frac{1}{\pi}\left[4\pi^2 \cancel{\frac{\sin 2n\pi}{n}} - 0 + 4\pi \frac{\cos 2n\pi}{n^2} - 0 - \cancel{\frac{2\sin 2n\pi}{n^3}} + 0\right]$$

$$= \frac{1}{\pi}\left[4\pi \frac{\cos 2n\pi}{n^2}\right]$$

$$= \frac{1}{\pi} \times 4\pi \frac{1}{n^2}$$

$$= \frac{4}{n^2}$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx.dx = \frac{1}{\pi}\int_0^{2\pi} x^2.\sin nx\, dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{-\cos nx}{n} - 2x \frac{-\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-4\pi^2 \frac{\cos 2n\pi}{n} + 0 + 4\pi \cancel{\frac{\sin 2n\pi}{n^2}} + 0 + \frac{2\cos 2n\pi}{n^3} - \frac{2\cos n.0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$= -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx + \frac{4\pi}{n}\sin nx\right)$$

Ans

\# Expand $f(x) = \begin{cases} 2-x \,; & 0<x<4 \\ x-6 \,; & 4<x<8 \end{cases}$ in a fourier series.

$$\therefore a_0 = \frac{1}{\pi}\left[\int_0^4 f(x).dx + \int_4^8 f(x).dx\right]$$

$$= \frac{1}{\pi}\int_0^4 (2-x).dx + \int_4^8 (x-6)\,dx$$

$$= \frac{1}{\pi}\left[2x - \frac{x^2}{2}\right]_0^4 + \left[\frac{x^2}{2} - 6x\right]_4^8$$

$$= \frac{1}{\pi}\left\{(8-8)-0 + \{(32-48)-(8-24)\}\right\}$$

$$= \frac{1}{\pi}(-16+16) = 0$$

$$\therefore a_n = \frac{1}{\pi}\int_0^4 f(x)\cos nx\,dx + \int_4^8 f(x)\sin nx\,.dx$$

$$= \frac{1}{\pi}\int_0^4 (2-x).\cos nx.dx + \int_4^8 (x-6).\sin nx$$

(iii) $f(x) = \begin{cases} 2-x \; ; 0<x<4 \\ x-6 \; ; 2<x<4 \end{cases}$ in period $= 8$

$L = 4$

$\therefore \cancel{a_0 = \frac{1}{\pi}\int_0^4 (2-x).dx + \int_2 (x-6).dx}$

$\therefore a_n = \frac{1}{4}\int_0^4 f(x).\frac{\cos n\pi x}{L}dx + \int_2^4 f(x).\sin\frac{n\pi x}{L}dx$

$= \frac{1}{4}\int_0^4 (2-x).\cos\frac{n\pi x}{L}dx + \int_2^4 (x-6).\sin\frac{n\pi x}{L}dx$

$= \frac{1}{4}\left[\cancel{(0-1)}(2-x).\frac{\sin n\pi x}{4}.\frac{4}{n\pi} + 1.\frac{-\cancel{\sin}\cos n\pi x}{4}.\frac{16}{n^2\pi^2}\right]_0^4$

$+ \left[(x-6).\frac{-\cos n\pi x}{4}\times\frac{4}{n\pi} - (1).\frac{-\sin n\pi x}{4}\times\frac{16}{n^2\pi^2}\right]_2^4$

$= \frac{1}{4}\left[\frac{4(2-x)\sin n\pi x}{n\pi} - \frac{4}{n^2\pi^2}.\frac{\cos n\pi x}{1}\right]_0^4 + \left[-\frac{(x-6)\cos n\pi x}{n\pi} + \frac{16}{n^2\pi^2}\frac{\sin n\pi x}{4}\right]_2^4$

$= \frac{1}{4}\left[\frac{4(2-4)\sin 4n\pi}{n\pi} - \frac{4}{n^2\pi^2}\frac{\cos 4n\pi}{1}\right] -$

(iii) $f(x) = \begin{cases} 0 ; & -5<x<0 \\ 3 ; & 0<x<5 \end{cases}$ $\quad 2L = 10 \therefore L = 5$

$\therefore a_0 = \frac{1}{\pi}\left[\int_{-5}^{0} f(x).dx + \int_{0}^{5} f(x).dx\right]$

$= \frac{1}{\pi}\left[\int_{-5}^{0} 0 + \int_{0}^{5} 3.dx\right]$

$= \frac{1}{\pi}\left[3x\right]_0^5$

$= \frac{1}{5}(15-0)$

$= 3$

$\therefore a_n = \frac{1}{5}\left[\int_{-5}^{0} f(x).\cos\frac{n\pi x}{L}\,dx + \int_{0}^{5} f(x).\sin\frac{n\pi x}{L}.dx\right]$

$= \frac{1}{5}\left[0 + \int_{0}^{5} 3.\sin\frac{n\pi x}{5}.dx\right]$

$= \frac{1}{5}\left[3.\frac{\cos n\pi x}{5} \times \frac{5}{n\pi}\right]_0^5$

$= \frac{3}{\cancel{5}} \times \cancel{\frac{n\pi}{5}} \frac{5}{n\pi}$

$= \frac{3}{n\pi}\left(\cos\frac{\cancel{5}n\pi}{\cancel{5}} - \cos 0\right)$

$= \frac{3}{n\pi}\left(\cos n\pi - \cos 0\right)$