Lectures on

Fourier analysis

RUET

Delivered By

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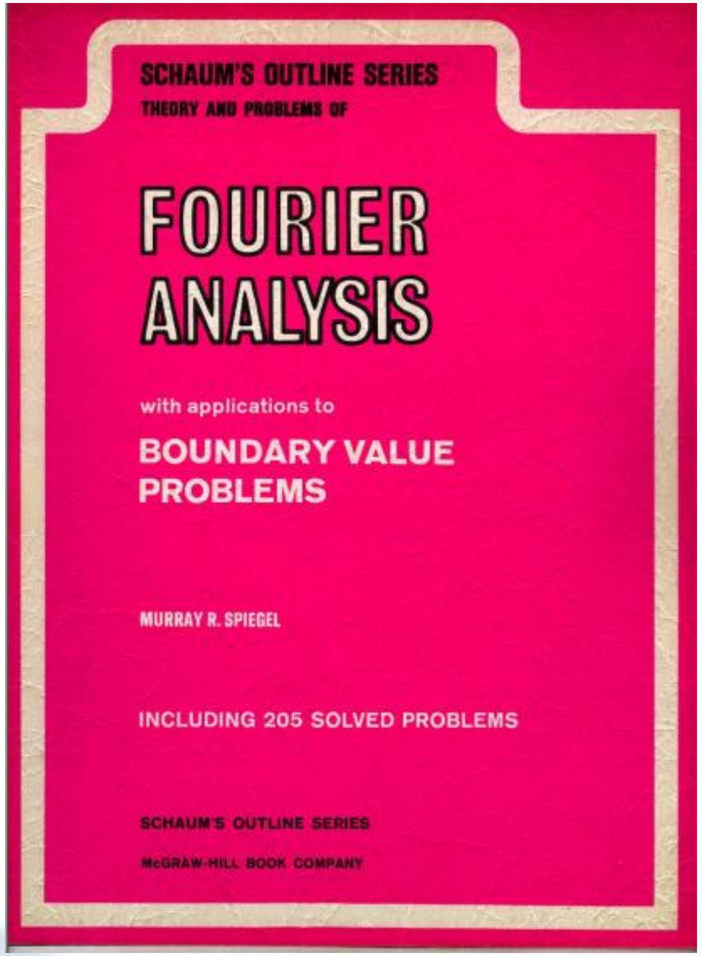
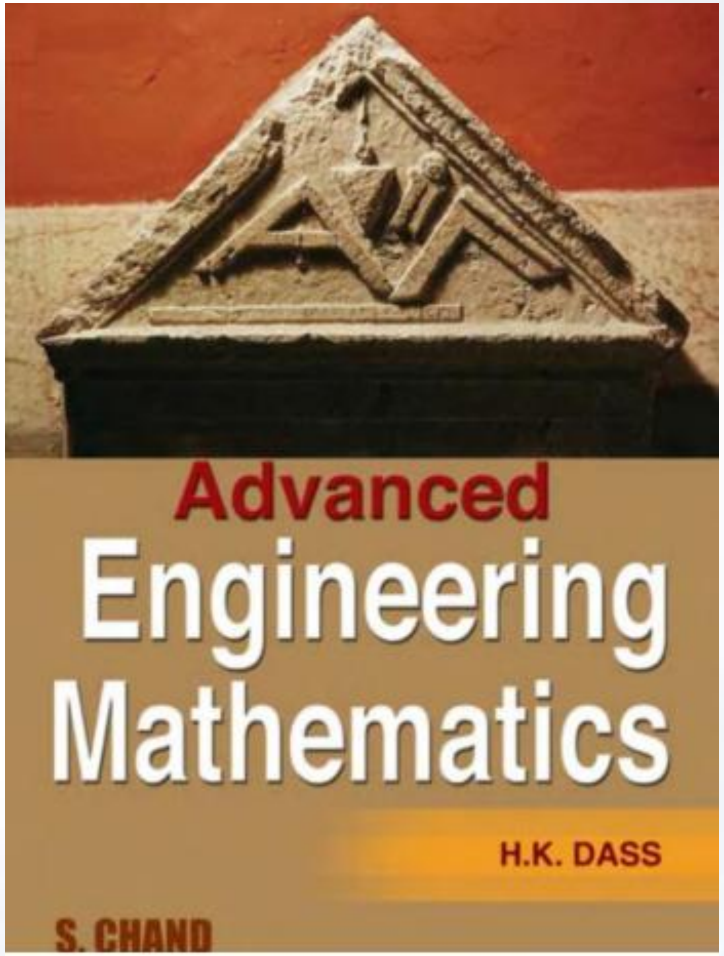
Outlines

- **Fourier series**
  - Periodic function
  - Piecewise continuous function
  - Fourier series
  - Half range Fourier series
  - Application of Fourier series to solve Boundary Value problems

- **Fourier Integral and Fourier Transformation**
  - Fourier Transformation and inverse Fourier Transformation
  - Application of Fourier Transform

Reference books





PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* P or to be *periodic* with period P if for all x, $f(x+P) = f(x)$, where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply *the period* of $f(x)$.

Example 1.

The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

Example 2.

The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

Example 3.

The period of $\tan x$ is π.

Example 4.

A constant has any positive number as a period.

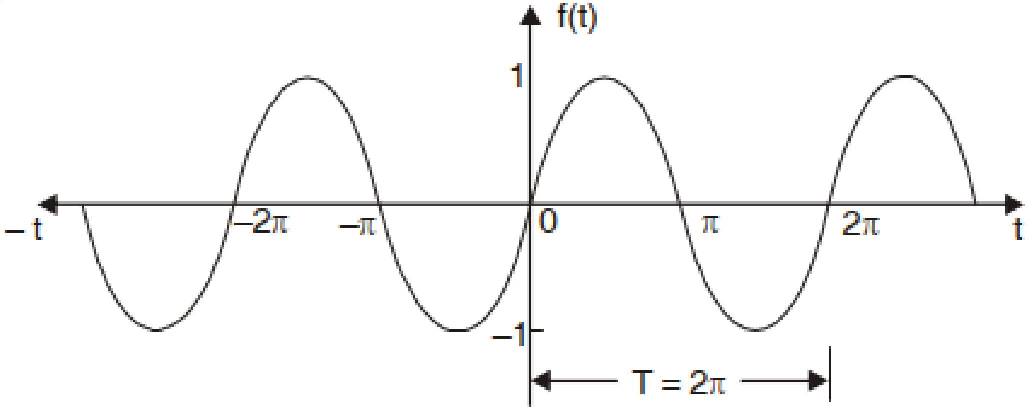
12.1 PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t + T) = f(t + 2T) =$ .... then T is called the period of the function $f(t)$.

For example :

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ so $\sin x$ is a periodic function with the period 2π. This is also called sinusoidal periodic function.



Other examples of periodic functions are shown in the graphs of Fig. 2-1.

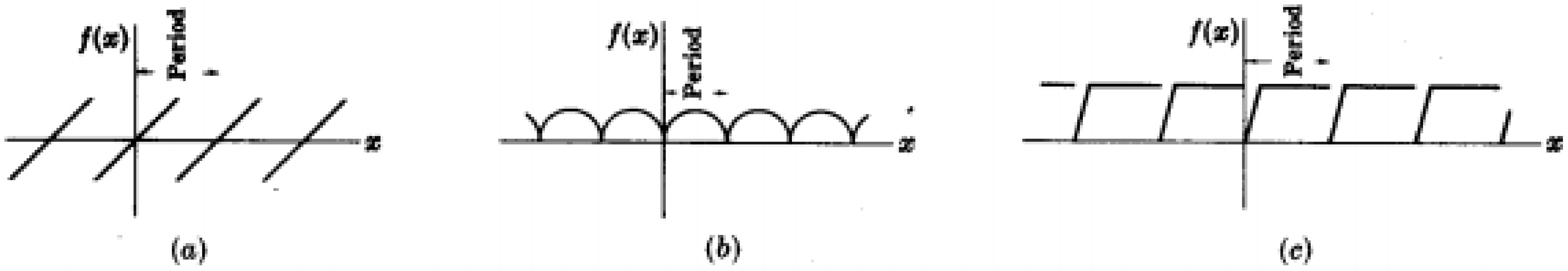


Fig. 2-1

2.1. Graph each of the following functions.

(a) $f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases}$ Period = 10

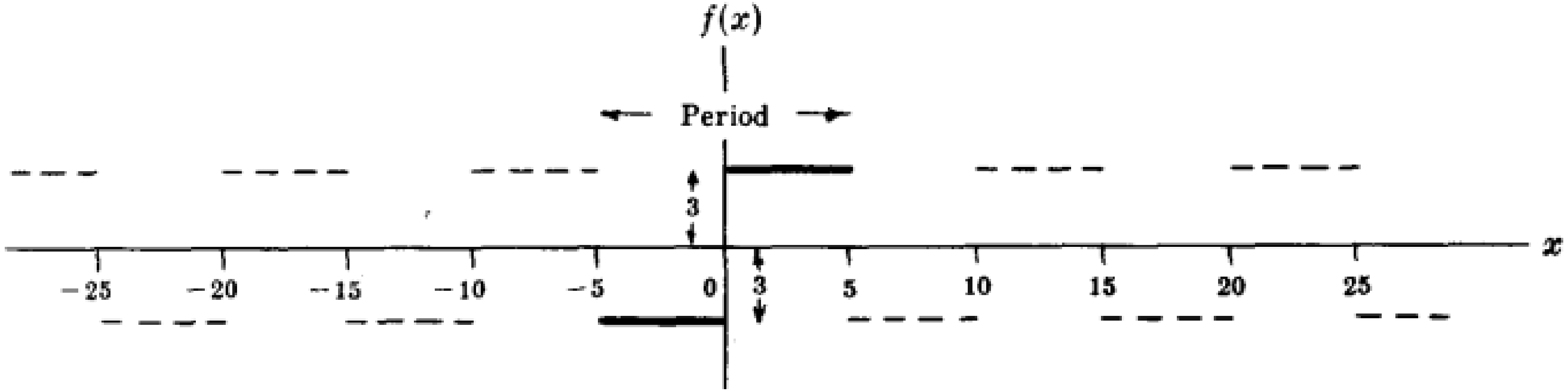


Fig. 2-3

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 2-3 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, etc. These values are the *discontinuities* of $f(x)$.

(b) $$f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases} \quad \text{Period} = 2\pi$$

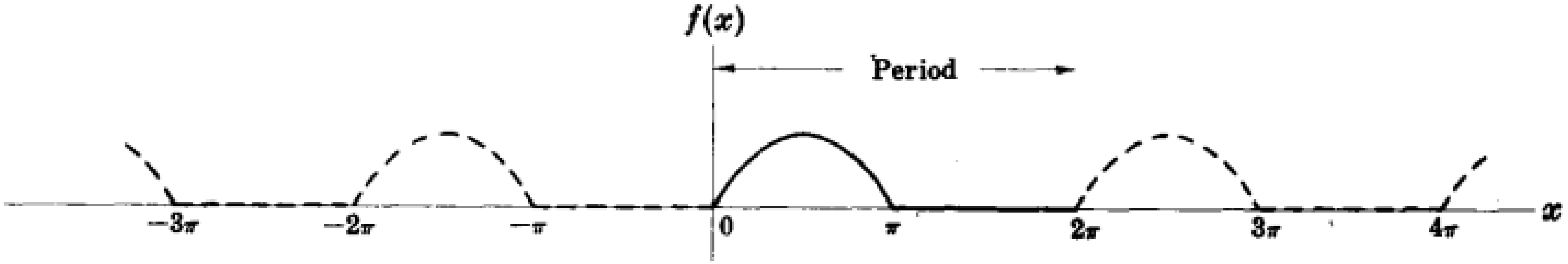


Fig. 2-4

Refer to Fig. 2-4 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

$$(c) \quad f(x) = \begin{cases} 0 & 0 \leqq x < 2 \\ 1 & 2 \leqq x < 4 \\ 0 & 4 \leqq x < 6 \end{cases} \quad \text{Period} = 6$$

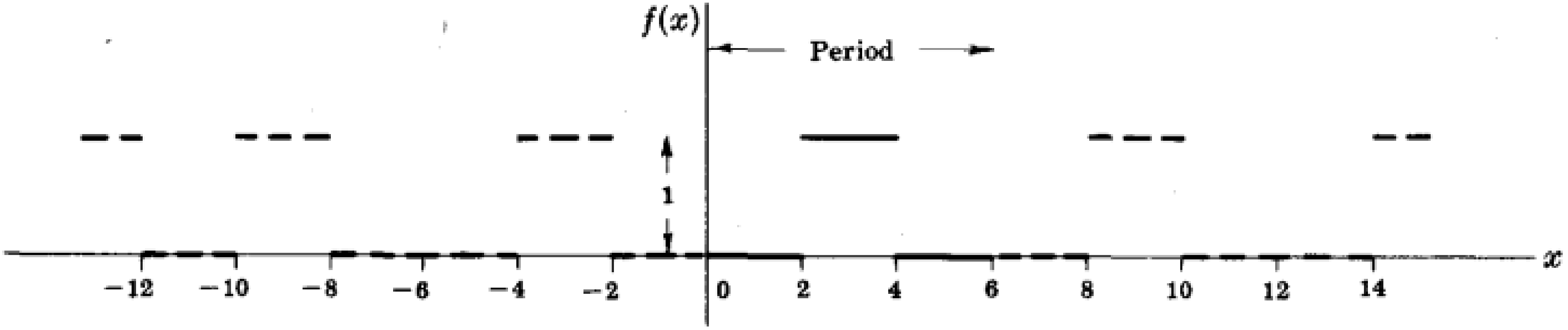


Fig. 2-5

Refer to Fig. 2-5 above. Note that $f(x)$ is defined for all x and is discontinuous at $x = \pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \ldots$.

DEFINITION OF FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

DIRICHLET CONDITIONS

Theorem 2-1: Suppose that

(i) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$

(ii) $f(x)$ is periodic with period $2L$

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$

Then the series (*1*) with coefficients (*2*) or (*3*) converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

For a proof see Problems 2.18–2.23.

According to this result we can write

$$f(x) \quad = \quad \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{4}$$

at any point of continuity x. However, if x is a point of discontinuity, then the left side is replaced by $\frac{1}{2}[f(x+0)+f(x-0)]$, so that the series converges to the mean value of $f(x+0)$ and $f(x-0)$.

PIECEWISE CONTINUOUS FUNCTIONS

A function $f(x)$ is said to be *piecewise continuous* in an interval if (i) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and (ii) the limits of $f(x)$ as x approaches the endpoints of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has at most a finite number of finite discontinuities. An example of a piecewise continuous function is shown in Fig. 2-2. The functions of Fig. 2-1(a) and (c) are piecewise continuous. The function of Fig. 2-1(b) is continuous.

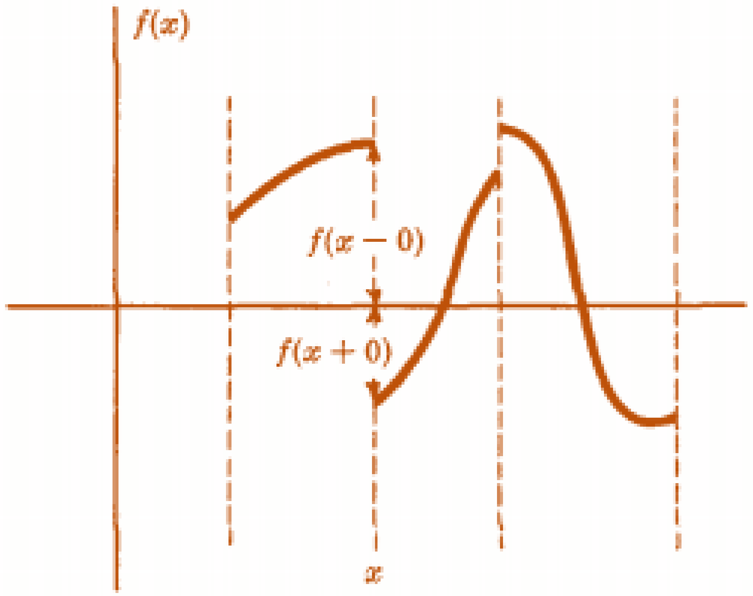


Fig. 2-2

The *limit of* $f(x)$ *from the right* or the *right-hand limit* of $f(x)$ is often denoted by $\lim_{\epsilon \to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$. Similarly, the *limit of* $f(x)$ *from the left* or the *left-hand limit* of $f(x)$ is denoted by $\lim_{\epsilon \to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values $f(x+0)$ and $f(x-0)$ at the point x in Fig. 2-2 are as indicated. The fact that $\epsilon \to 0$ and $\epsilon > 0$ is sometimes indicated briefly by $\epsilon \to 0+$. Thus, for example, $\lim_{\epsilon \to 0+} f(x+\epsilon) = f(x+0)$, $\lim_{\epsilon \to 0+} f(x-\epsilon) = f(x-0)$.

12.4. ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist. (This is not true for Taylor's series).
2. The Fourier series is useful in expanding the periodic functions since outside the closed interval, there exists a periodic extension of the function.
3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.
4. Fourier series of a discontinuous function is not uniformly convergent at all points.
5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

2.2. Prove $\int_{-L}^{L} \sin \frac{k\pi x}{L} dx = \int_{-L}^{L} \cos \frac{k\pi x}{L} dx = 0$ if $k = 1, 2, 3, \ldots$.

2.3. Prove (a) $\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$

(b) $\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0$

where m and n can assume any of the values $1, 2, 3, \ldots$.

2.4. If the series $A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots,$

(a) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (c) $A = \frac{a_0}{2}$.

(a) Multiplying
$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$
by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have
$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \quad \text{if } m \neq 0\end{aligned} \tag{2}$$

Thus
$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(b) Multiplying (1) by $\sin\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have
$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(c) Integration of (1) from $-L$ to L, using Problem 2.2, gives
$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (a), we find $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ and so $A = \frac{a_0}{2}$.

2.5. (a) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \quad \text{Period} = 10$$

(b) Write the corresponding Fourier series.

(c) How should $f(x)$ be defined at $x = -5$, $x = 0$ and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 2-6 below.

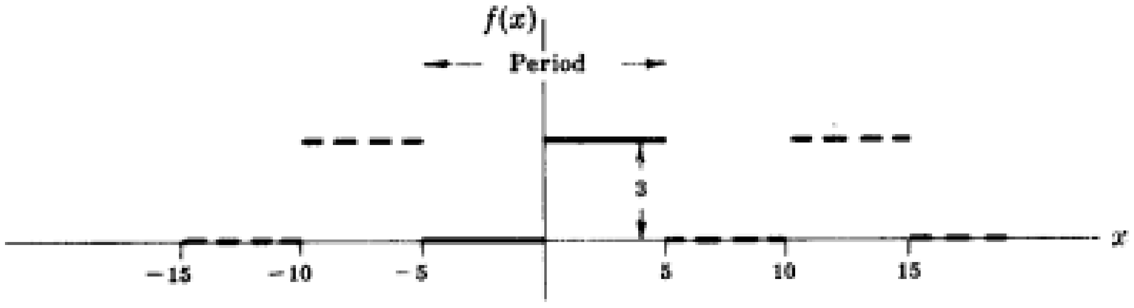


Fig. 2-6

(a) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_0^5 = 0 \qquad \text{if } n \neq 0$$

If $n = 0$, $a_n = a_0 = \frac{3}{5}\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Bigg|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(b) The corresponding Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right)$$

(c) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0 and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$, as seen from the graph. The series will converge to $f(x)$ for $-5 \leqq x \leqq 5$ if we redefine $f(x)$ as follows:

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

Example 1. *Find the Fourier series representing*

$$f(x) = x, \qquad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ to $x = 4\pi$.

Solution. Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots$ ... (1)

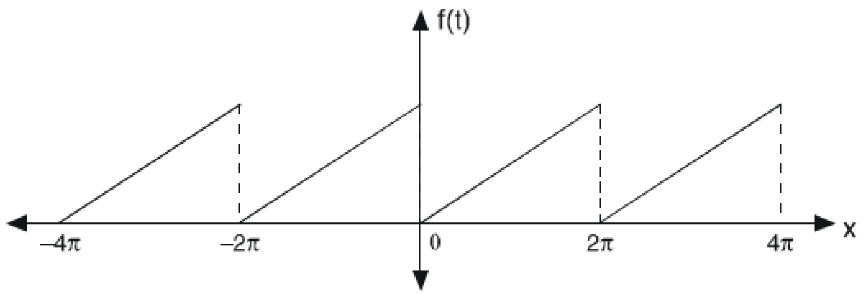
Hence $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$



Substituting the values of a_0, a_n, b_n in (1), we get

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right] \qquad \textbf{Ans.}$$

2.6. Expand $f(x) = x^2,\ 0 < x < 2\pi,$ in a Fourier series if the period is 2π.

The graph of $f(x)$ with period 2π is shown in Fig. 2-7.

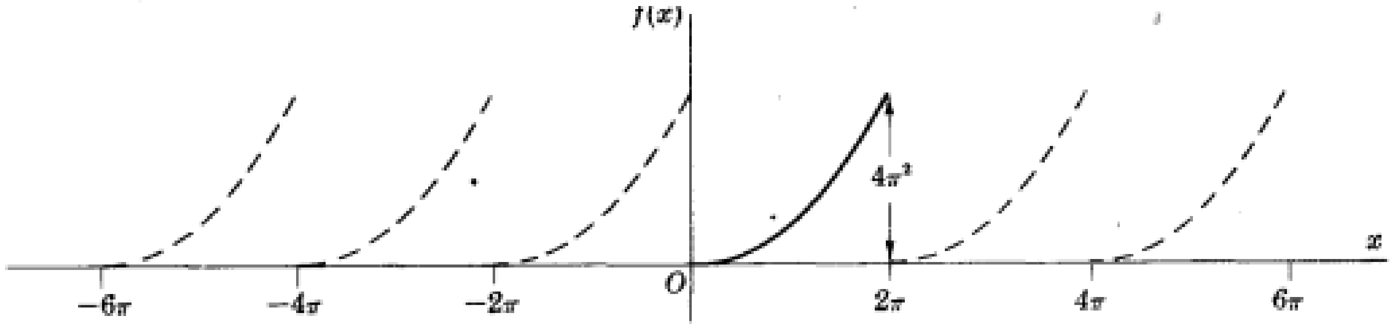


Fig. 2-7

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{4}{n^2}, \qquad n \neq 0$$

If $n = 0$, $\quad a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} x^2\,dx = \dfrac{8\pi^2}{3}.$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $\quad f(x) = x^2 = \dfrac{4\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$ for $\quad 0 < x < 2\pi.$

Example 2. *Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$.*

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

Find a Fourier series to represent, $f(x) = \pi - x$ for $0 < x < 2\pi$.

Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to π and show that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Find a Fourier series to represent: $f(x) = x \sin x$, for $0 < x < 2\pi$.

. *Find the Fourier series for f(x), if* $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

2.34. Graph each of the following functions and find its corresponding Fourier series, using properties of even and odd functions wherever applicable.

(a) $f(x) = \begin{cases} 8 & 0 < x < 2 \\ -8 & 2 < x < 4 \end{cases}$ Period 4

(b) $f(x) = \begin{cases} -x & -4 \leqq x \leqq 0 \\ x & 0 \leqq x \leqq 4 \end{cases}$ Period 8

(c) $f(x) = 4x,\ 0 < x < 10,$ Period 10

(d) $f(x) = \begin{cases} 2x & 0 \leqq x \leqq 3 \\ 0 & -3 < x < 0 \end{cases}$ Period 6

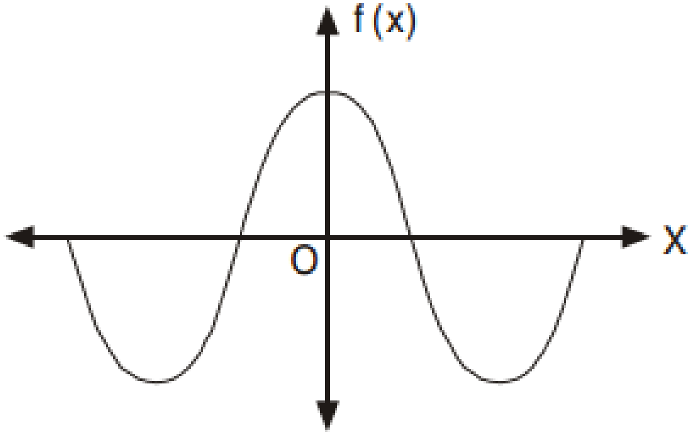
2.35. In each part of Problem 2.34, tell where the discontinuities of $f(x)$ are located and to what value the series converges at these discontinuities.

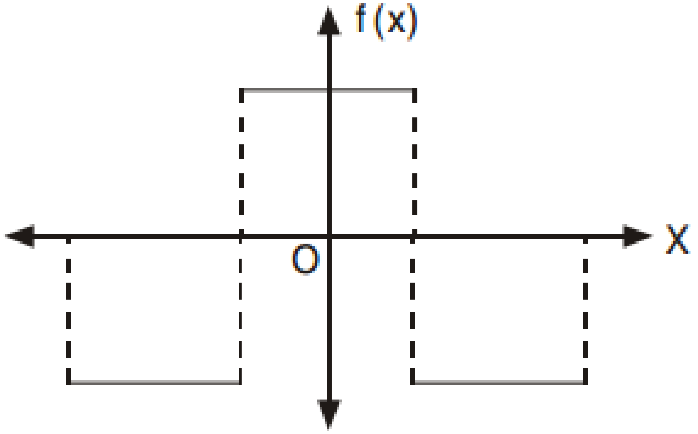
2.36. Expand $f(x) = \begin{cases} 2 - x & 0 < x < 4 \\ x - 6 & 4 < x < 8 \end{cases}$ in a Fourier series of period 8.

12.8 (a) EVEN FUNCTION

A function $f(x)$ is said to be even (or symmetric) function if, $f(-x) = f(x)$

The graph of such a function is symmetrical with respect to y-axis [$f(x)$ axis]. Here y-axis is a mirror for the reflection of the curve.

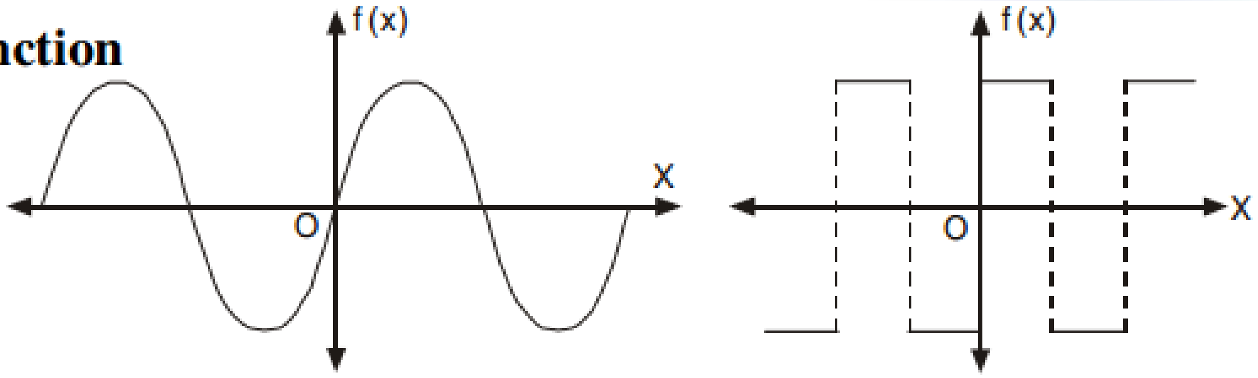




The area under such a curve from $-\pi$ to π is double the area from 0 to π.

$$\therefore \qquad \int_{-\pi}^{\pi} f(x)\, dx = 2 \int_{0}^{\pi} f(x)\, dx$$

(b) Odd Function



A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x) = -f(x)$$

Here the area under the curve from $-\pi$ to π is zero.

$$\int_{-\pi}^{\pi} f(x)\, dx = 0$$

Expansion of an even function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

As $f(x)$ and $\cos nx$ are both even functions .

$\therefore$ The product of $f(x).\cos nx$ is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

As $\sin nx$ is an odd function so $f(x).\sin nx$ is also an odd function. We need not to calculate b_n. It saves our labour a lot.

The series of the even function will contain only cosine terms.

Expansion of an odd function :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \quad [f(x).\cos nx \text{ is odd function.}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

[$f(x).\sin nx$ is even function.]

The series of the odd function will contain only sine terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms

Example 8. *Find the Fourier series expansion of the periodic function of period 2π*

$$f(x) = x^2, \ -\pi \le x \le \pi$$

Hence, find the sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

Solution. $f(x) = x^2, \ -\pi \le x \le \pi$

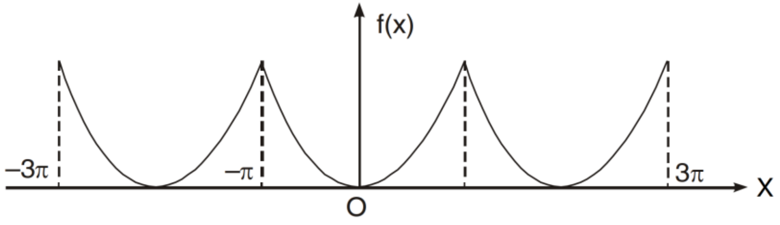
This is an even function. $\therefore \quad b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2\pi\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$



Fourier series is $\quad f(x) = \frac{a_0}{2} + a_1\cos x + a_2 \cos 2x + a_3\cos 3x + \ldots + a_n \cos nx + \ldots$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^3} - \frac{\cos 4x}{4^2} + \ldots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots = \frac{\pi^2}{12}$$

Ans.

Example 9. *Obtain a Fourier expression for*

$$f(x) = x^3 \quad for\ -\pi < x < \pi.$$

HALF-RANGE SERIES,

The given function is defined in the interval $(0, \pi)$ and it is immaterial whatever the function may be outside the interval $(0, \pi)$. To get the series of cosines only we assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$.

$$a_n = \frac{2}{\pi}\int_0^\pi f(x) \cos nx \, dx \quad \text{and} \quad b_n = 0$$

To expand $f(x)$ as a sine series we extend the function in the interval $(-\pi, \pi)$ as an odd function.

$$b_n = \frac{2}{\pi}\int_0^\pi f(x) \sin nx \, dx \quad \text{and} \quad a_n = 0$$

2.12. Expand $f(x) = x$, $0 < x < 2$, in a half-range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 2-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4$, $L = 2$.

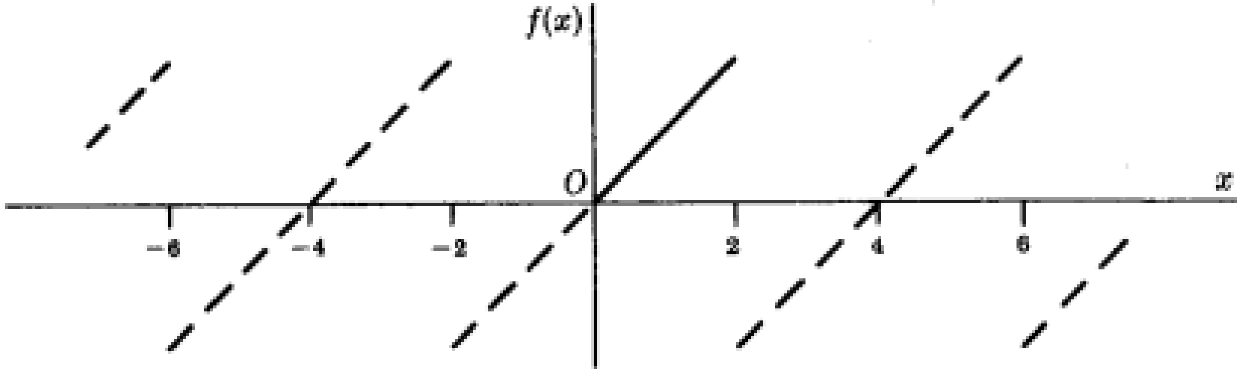


Fig. 2-12

Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then

$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(b) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 2-13 below. This is the *even extension* of $f(x)$. Then $2L = 4$, $L = 2$.

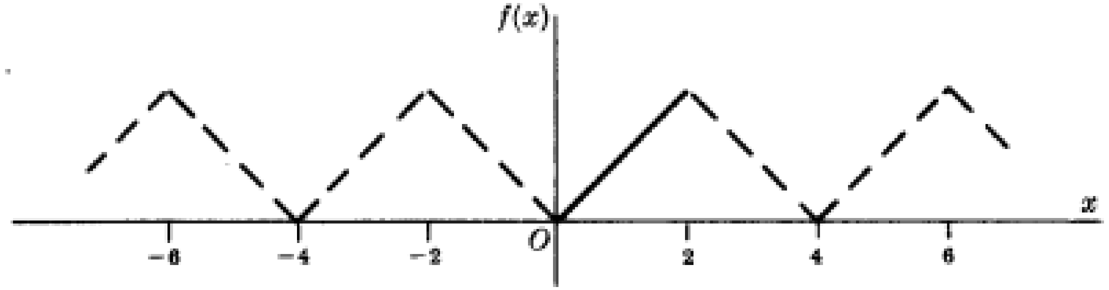


Fig. 2-13

Thus $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2$$

$$= \frac{-4}{n^2\pi^2}(\cos n\pi - 1) \qquad \text{if } n \neq 0$$

If $n = 0$, $\quad a_0 = \int_0^2 x\,dx = 2.$

Then
$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

It should be noted that although both series of (a) and (b) represent $f(x)$ in the interval $0 < x < 2$, the second series converges more rapidly.

2.11. Expand $f(x) = \sin x$, $0 < x < \pi$, in a Fourier cosine series.

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence we extend the definition of $f(x)$ so that it becomes even (dashed part of Fig. 2-11). With this extension, $f(x)$ is defined in an interval of length 2π. Taking the period as 2π, we have $2L = 2\pi$, so that $L = \pi$.

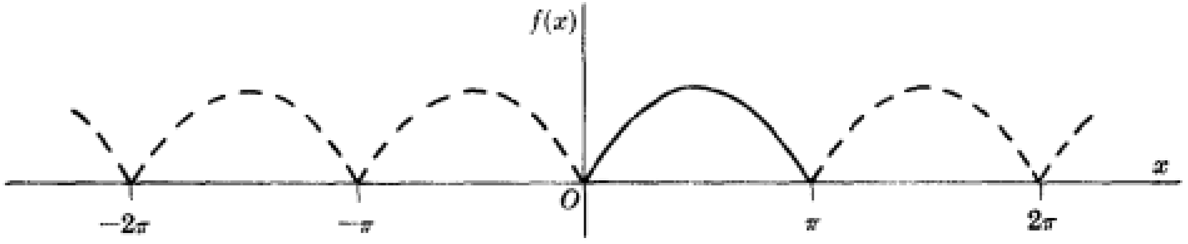


Fig. 2-11

By Problem 2.10, $b_n = 0$ and

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^\pi \sin x \cos nx\,dx \\ &= \frac{1}{\pi}\int_0^\pi \{\sin(x+nx) + \sin(x-nx)\}\,dx = \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^\pi \\ &= \frac{1}{\pi}\left\{\frac{1-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi - 1}{n-1}\right\} = \frac{1}{\pi}\left\{-\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right\} \\ &= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \qquad \text{if } n \neq 1 \end{aligned}$$

For $n = 1$, $\quad a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\,dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Bigg|_0^\pi = 0.$

Then

$$\begin{aligned} f(x) &= \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx \\ &= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right) \end{aligned}$$

Find the half-range sine series for the function

$$f(x) = 2x - 1 \qquad 0 < x < 1.$$

Ans. $-\dfrac{2}{\pi}\left[\sin \pi x + \dfrac{1}{2}\sin 4\pi x + \dfrac{1}{3}\sin 6\pi x + \ldots\right]$

2.37. (*a*) Expand $f(x) = \cos x,\ 0 < x < \pi,$ in a Fourier sine series.

(*b*) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leqq x \leqq \pi$?

2.38. (*a*) Expand in a Fourier series $f(x) = \cos x,\ 0 < x < \pi,$ if the period is π; and (*b*) compare with the result of Problem 2.37, explaining the similarities and differences if any.

2.39. Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8 - x & 4 < x < 8 \end{cases}$ in a series of (*a*) sines, (*b*) cosines.

2.40. Prove that for $0 \leqq x \leqq \pi$,

(*a*) $$x(\pi - x) = \frac{\pi^2}{6} - \left(\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \cdots\right)$$

(*b*) $$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)$$

Fourier Integral

Fourier Integral

THE FOURIER INTEGRAL

Let us assume the following conditions on $f(x)$:

1. $f(x)$ and $f'(x)$ are piecewise continuous in *every* finite interval.
2. $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, i.e. $f(x)$ is absolutely integrable in $(-\infty, \infty)$.

Then *Fourier's integral theorem* states that

$$f(x) = \int_0^{\infty} \{A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x\}\, d\alpha \qquad (1)$$

where

$$\left. \begin{aligned} A(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos \alpha x\, dx \\ B(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin \alpha x\, dx \end{aligned} \right\} \qquad (2)$$

The result (*1*) holds if x is a point of continuity of $f(x)$. If x is a point of discontinuity, we must replace $f(x)$ by $\dfrac{f(x+0)+f(x-0)}{2}$ as in the case of Fourier series. Note that the above conditions are sufficient but not necessary.

The similarity of (*1*) and (*2*) with corresponding results for Fourier series is apparent. The right-hand side of (*1*) is sometimes called a *Fourier integral expansion* of $f(x)$.

EQUIVALENT FORMS OF FOURIER'S INTEGRAL THEOREM

Fourier's integral theorem can also be written in the forms

$$f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\int_{u=-\infty}^{\infty} f(u)\cos\alpha(x-u)\,du\,d\alpha \tag{3}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha(x-u)}\,du\,d\alpha$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,d\alpha \int_{-\infty}^{\infty} f(u)\,e^{-i\alpha u}\,du \tag{4}$$

where it is understood that if $f(x)$ is not continuous at x the left side must be replaced by $\dfrac{f(x+0)+f(x-0)}{2}$.

These results can be simplified somewhat if $f(x)$ is either an odd or an even function, and we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\alpha x\,d\alpha\int_0^{\infty} f(u)\sin\alpha u\,du \qquad \text{if } f(x) \text{ is odd} \tag{5}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\alpha x\,d\alpha\int_0^{\infty} f(u)\cos\alpha u\,du \qquad \text{if } f(x) \text{ is even} \tag{6}$$

The fourier integral theorem can also be written as,

$$f(x) = \frac{1}{\pi}\int_{u=0}^{\infty}\left\{\int_{t=-\infty}^{\infty} f(t)\cos u(x-t)\,dt\right\}du$$

or $$f(x) = \frac{1}{2\pi}\int_{u=-\infty}^{\infty}\left\{\int_{t=-\infty}^{\infty} f(t)\cos u(x-t)\,dt\right\}du.$$

If $f(x)$ is even

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\left\{\int_{t=0}^{\infty} f(t)\cos ut\,dt\right\}\cos ux\,du.$$

If $f(x)$ is odd

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\left\{\int_{t=0}^{\infty} f(t)\sin ut\,dt\right\}\sin ux\,du$$

Example 5. Find the Fourier integral of the function $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = f(x)$ for $k > 0$, and hence prove that $\int_0^\infty \frac{\cos ux}{k^2 + u^2} du = \frac{\pi}{2k} e^{-kx}$. **D. U. M. SC (F) 1991**

Solution : Since $f(-x) = f(x)$, so $f(x)$ is even and for even function we have the Fourier integral

$$f(x) = \frac{2}{\pi}\int_0^\infty f(t)\,dt\int_0^\infty \cos ut \cos ux\,du$$

$$= \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(t)\cos ut\,dt\right]\cos ux\,du \quad (1)$$

Now $\int_0^\infty f(t)\cos ut\,dt = \int_0^\infty e^{-kt}\cos ut\,dt$

$$= \left[\frac{e^{-kt}}{k^2+u^2}(-k\cos ut + u\sin ut)\right]_0^\infty$$

$$= 0 + \frac{k}{k^2+u^2} = \frac{k}{k^2+u^2}.$$

Thus from (1), we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{k}{k^2+u^2}\cos ux\,du$$

$$= \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2}du \ (x > 0, k > 0) \quad (2)$$

which is the required Fourier integral of the function $f(x) = e^{-kx}$.

Again putting $f(x)=e^{-kx}$ in (2), we get $e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos ux}{k^2+u^2}.dx$

$$\int_0^\infty \frac{\cos ux}{k^2+u^2}.du = \frac{\pi}{2k}e^{-kx}.$$

Example 6. Find the Fourier integral of the function $f(x) = e^{-kx}$ when $x > 0$ and $f(-x) = -f(x)$ for $k > 0$ and hence prove that

$$\int_0^\infty \frac{u\sin ux}{k^2+u^2}du = \frac{\pi}{2}e^{-kx}, \ k > 0.$$

Solution : Since $f(-x) = -f(x)$, so $f(x)$ is an odd function for which we have Fourier integral $f(x) = \frac{2}{\pi}\int_0^\infty f(t)\,dt\int_0^\infty \sin ut \sin ux\,du$

$$= \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(t)\sin ut\,dt\right]\sin ux\,du \quad (1)$$

Now $\int_0^\infty f(t)\sin ut\,dt = \int_0^\infty e^{-kt}\sin ut\,dt$

$$= \left[\frac{e^{-kt}(-k\sin ut - u\cos ut)}{k^2+u^2}\right]_0^\infty$$

$$= 0 + \frac{u}{k^2+u^2} = \frac{u}{k^2+u^2}$$

Thus from (1), we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u}{k^2+u^2}\sin ux\,du$$

$$= \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{k^2+u^2}du \ (x > 0, k > 0) \quad (2)$$

which is the required Fourier integral of the function $f(x) = e^{-kx}$.

putting $f(x) = e^{-kx}$ in (2), we get

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{k^2+u^2}du.$$

or, $$\int_0^\infty \frac{u\sin ux}{k^2+u^2}du = \frac{\pi}{2}e^{-kx}$$

Example 7. Find the Fourier integral of the function

$$f(x) = \begin{cases} 0 & \text{when } x < 0 \\ \frac{1}{2} & \text{when } x = 0 \\ e^{-x} & \text{when } x > 0. \end{cases}$$

Solution : By the definition of Fourier integral in general, we have

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} \cos u\,(x-t)\, du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{0}^{\infty} \cos u\,(x-t)\, du$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{0}^{\infty} (\cos ux \cos ut + \sin ux \sin ut)\, du$$

$$= \frac{1}{\pi}\left[\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\cos ut\, dt\right\}\cos ux\, du + \int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\sin ut\, dt\right\}\sin ux\, du\right] \quad (1)$$

Now $\int_{-\infty}^{\infty} f(t) \cos ut\, dt = \int_{-\infty}^{0} f(t)\cos ut\, dt + \int_{0}^{\infty} f(t) \cos ut\, dt$

$$= \int_{-\infty}^{0} 0 \cos ut\, dt + \int_{0}^{\infty} e^{-t} \cos ut\, dt$$

$$= 0 + \left[\frac{e^{-t}}{1+u^2}(-\cos ut + u \sin ut)\right]_0^{\infty}$$

$$= \frac{1}{1+u^2} + 0 = \frac{1}{1+u^2}.$$

Similarly, $\int_{-\infty}^{\infty} f(t) \sin ut\, dt = \int_{-\infty}^{0} f(t)\sin ut\, dt + \int_{0}^{\infty} f(t) \sin ut\, dt$

$$= \int_{-\infty}^{0} 0 \sin ut\, dt + \int_{0}^{\infty} e^{-t} \sin ut\, dt$$

$$= 0 + \int_{0}^{\infty} e^{-t} \sin ut\, dt$$

$$= \left[\frac{e^{-t}(-\sin ut - u \cos ut)}{1+u^2}\right]_0^{\infty}$$

$$= 0 + \frac{u}{1+u^2} = \frac{u}{1+u^2}.$$

Putting these values in (1), we get

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty} \frac{\cos ux}{1+u^2}\, du + \int_0^{\infty} \frac{u \sin ux}{1+u^2}\, du\right]$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\frac{\cos ux + u \sin ux}{1+u^2}\right) du \quad (2)$$

putting $x = 0$ in (2), we get

$$f(0) = \frac{1}{\pi}\int_0^{\infty} \frac{du}{1+u^2} = \frac{1}{\pi}\left[\tan^{-1} u\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1} 0\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) = \frac{1}{2}$$

so $f(x) = \frac{1}{2}$ for $x = 0$ is satisfied.

Hence $f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{(\cos ux + u \sin ux)}{1+u^2}\, du$

which is the required Fourier integral of the given function.

Example 1. *Express the function*

$$f(x) = \begin{cases} 1 & when\ |x| \le 1 \\ 0 & when\ |x| > 1 \end{cases}$$

as a Fourier integral. Hence evaluate

$$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda \qquad (Mysore\ 1975S)$$

Solution. The Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos\lambda(t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-1}^1 \cos\lambda(t-x)\, dt\, d\lambda \qquad (\text{since } f(t)=1)$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^1 d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda} d\lambda \qquad \text{By } \sin C + \sin D \text{ formula}$$

Thus $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda$$ **Ans.**

or $$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$$

or $$\int_0^\infty \frac{\sin\lambda \cos\lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Fourier Transform

Fourier transform

Fourier transform of a function $f(x)$ is denoted as $F\{f(x)\}$ or $F(u)$ and defined as

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-iux}\, dx$$

~~Fourier~~ Inverse fourier transform

Inverse fourier transform of $F(u)$ is denoted as $F^{-1}\{F(u)\}$ or $f(x)$ and defined as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u)\, e^{iux}\, du.$$

Fourier cosine transform of $f(x)$

$$F_c(n) = \int_{0}^{\infty} f(x) \cos nx \, dx$$

Inverse Fourier cosine transform of $F_c(n)$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_c(n) \cos nx \, dn$$

Fourier sine transform of $f(x)$

$$F_s(n) = \int_{0}^{\infty} f(x) \sin nx \, dx$$

Inverse Fourier sine transform of $F_s(n)$

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(n) \sin nx \, dn.$$

Finite Fourier Cosine transform of $f(x)$, $0<x<L$

$$F_c(n) = \int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Inverse: $F^{-1}\{F_c(n)\} = f(x) = \frac{1}{L}F_c(0) + \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{L}$.

Finite Fourier Sine transform of $f(x)$, $0<x<L$

$$F_s(n) = \int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Inverse: $F^{-1}\{F_s(n)\} = f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{L}$.

Example 8. Find the Fourier sine transform of e^{-x}, $x \geq 0$.

Solution : By definition of Fourier sine transform of $f(x)$ for $0 < x < \infty$, we have

$$f_s(n) = \int_0^\infty F(x) \sin nx \, dx$$

$$\therefore \quad f_s(n) = \int_0^\infty e^{-x} \sin nx dx \qquad \text{Since } f(x) = e^{-x}$$

$$= \left[-\frac{e^{-x}}{n} \cos nx\right]_0^\infty - \int_0^\infty \frac{e^{-x} \cos nx}{n} dx$$

$$= 0 + \frac{1}{n} - \left[\frac{e^{-x} \sin nx}{n^2}\right]_0^\infty - \int_0^\infty \frac{e^{-x} \sin nx}{n^2} dx$$

$$= \frac{1}{n} - 0 - \frac{1}{n^2} f_s(n)$$

or, $\quad f_s(n) + \frac{1}{n^2} f_s(n) = \frac{1}{n}$

or, $\quad \left(1 + \frac{1}{n^2}\right) f_s(n) = \frac{1}{n}$

or, $\quad \left(\frac{n^2+1}{n^2}\right) f_s(n) = \frac{1}{n}$

or, $\quad f_s(n) = \frac{1}{n} \cdot \frac{n^2}{n^2+1} = \frac{n}{n^2+1}$

$\therefore \quad f_s(n) = \frac{n}{n^2+1}$.

Hence the Fourier sine transform of e^{-x} is $\frac{n}{n^2+1}$.

Example 9. Find the inverse Fourier sine transform of $f_s(n) = \dfrac{n}{1+n^2}$.

Example 10. Find the Fourier cosine transform of e^{-x}, $x \geq 0$.

Solution : By definition of Fourier cosine transform of $f(x)$ for $0 < x < \infty$, we have

$$f_c(n) = \int_0^\infty F(x) \cos nx \, dx \qquad (1)$$

$$\therefore \quad f_c(n) = \int_0^\infty e^{-x} \cos nx \, dx \quad \text{Since } F(x) = e^{-x}$$

$$= \left[\frac{e^{-x} \sin nx}{n}\right]_0^\infty + \int_0^\infty \frac{e^{-x} \sin nx}{n} \, dx$$

$$= 0 - \frac{1}{n^2} \left[e^{-x} \cos nx\right]_0^\infty - \frac{1}{n^2} \int_0^\infty e^{-x} \cos nx \, dx$$

$$= 0 + \frac{1}{n^2} - \frac{1}{n^2} f_c(n)$$

or, $$f_c(n) + \frac{1}{n^2} f_c(n) = \frac{1}{n^2}$$

or, $$\frac{(n^2+1)}{n^2} f_c(n) = \frac{1}{n^2}$$

or, $$f_c(n) = \frac{1}{n^2+1}$$

Hence the Fourier cosine transform of e^{-x} is $\frac{1}{n^2+1}$

Example 11. Find the inverse Fourier cosine transform of $f_c(n) = \frac{1}{1+n^2}$

Example 13. Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Example 15. Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Example 14. Find the Fourier transform of $F(x) = e^{-|x|}$ where x belongs to $(-\infty, \infty)$.

Example 18. Solve the integral equation

$$\int_0^\infty F(x)\cos ux\,dx = \begin{cases} 1-u, & 0 \le u < 1 \\ 0, & u > 1 \end{cases}$$

Hence deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}.$

Proof : By definition of the Fourier cosine transform of $F(x)$, the given integral equation gives

$$F_c[F(x)] = f_c(u) = \begin{cases} 1-u, & 0 \le u < 1 \\ 0, & u > 1. \end{cases}$$

Using the corresponding inverse Fourier cosine formula, we have

$$F(x) = \frac{2}{\pi}\int_0^\infty f_c(u)\cos ux\,du$$

$$= \frac{2}{\pi}\left[\int_0^1 f_c(u)\cos ux\,du + \int_1^\infty f_c(u)\cos ux\,du\right]$$

$$= \frac{2}{\pi}\left\{\int_0^1 (1-u)\cos ux\,du + \int_1^\infty o\cos ux\,du\right\}$$

$$= \frac{2}{\pi}\int_0^1 (1-u)\cos ux\,du + o$$

$$= \frac{2}{\pi}\left[(1-u)\frac{\sin ux}{x}\right]_0^1 - \frac{2}{\pi}\int_0^1 (o-1)\frac{\sin ux}{x}\,du$$

$$= o + \frac{2}{\pi}\int_0^1 \frac{\sin ux}{x}\,du$$

$$= \frac{2}{\pi x}\int_0^1 \sin ux\,du$$

$$= \frac{2}{\pi x^2}\Big[(-\cos ux)\Big]_0^1$$

$$= -\frac{2}{\pi x^2}(\cos x - 1) = \frac{2(1-\cos x)}{\pi x^2}.$$

Hence $F(x) = \dfrac{2(1-\cos x)}{\pi x^2}$ which is the required solution.

5.6. Solve the integral equation

$$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & 0 \leqq \alpha \leqq 1 \\ 0 & \alpha > 1 \end{cases}$$

If we write

$$F_S(\alpha) = \int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & 0 \leqq \alpha \leqq 1 \\ 0 & \alpha > 1 \end{cases}$$

then, by *(10)*, page 81,

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_0^\infty F_S(\alpha) \sin \alpha x \, d\alpha \\ &= \frac{2}{\pi} \int_0^1 (1 - \alpha) \sin \alpha x \, d\alpha \\ &= \frac{2(x - \sin x)}{\pi x^2} \end{aligned}$$

The **convolution** of the two functions f (x) and g (x) is defined by $f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$

Convolution property of Fourier transform

Statement : If $F\{f(x)\}$ and $F\{g(x)\}$ are the Fourier transforms of the functions $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i e. $F\{f(x) * g(x)\} = F\{f(x)\}.\, F\{g(x)\}$

By definition of Fourier transform, we have

$$\left.\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(t)\, e^{-iut}\, dt \\ \text{and } G(u) &= \int_{-\infty}^{\infty} g(s)\, e^{-ius}\, ds \end{aligned}\right\} \qquad (1)$$

$$\text{Then } F(u)\,G(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(s)\, e^{-iu(t+s)}\, dt\, ds \qquad (2)$$

Let $t + s = x$ in the double integral (2) which we wish to transform from the variables (t, s) to the variables (t, x). From advanced calculus, we have

$$dtds = \frac{\partial(t, s)}{\partial(t, x)}\, dt\, dx \qquad (3)$$

where the **Jacobian** of the transformation is given by

$$\frac{\partial(t, s)}{\partial(t, x)} = \begin{vmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial t} & \frac{\partial s}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1)$$

Then (2) becomes

$$F(u)\,G(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,e^{-iux}\,dt\,dx$$

$$= \int_{-\infty}^{\infty} e^{-iux}\left[\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt\right]dx$$

$$= F\left\{\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt\right\}$$

$= F\,(f * g)$ Since

$f * g = \int_{-\infty}^{\infty} f(t)\,g(x-t)dt$ is the **convolution** of f and g.

Hence $F\,(f * g) = F(u)\,G(u) = F\{f\}\,.\,G\{g\}$.

5.8. Show that $f * g = g * f$.

Let $x - u = v$. Then

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du = \int_{-\infty}^{\infty} f(x-v)\, g(v)\, dv$$

$$= \int_{-\infty}^{\infty} g(v)\, f(x-v)\, dv = g * f$$

5.10. Solve for $y(x)$ the integral equation

$$\int_{-\infty}^{\infty} \frac{y(u)\,du}{(x-u)^2+a^2} = \frac{1}{x^2+b^2} \qquad 0 < a < b$$

We have

$$\mathcal{F}\left\{\frac{1}{x^2+b^2}\right\} = \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{x^2+b^2}\,dx = 2\int_0^{\infty} \frac{\cos \alpha x}{x^2+b^2}\,dx = \frac{\pi}{b}e^{-b\alpha}$$

on making use of Problem 5.5(*b*). Then, taking the Fourier transform of both sides of the integral equation, we find

$$\mathcal{F}\{y\}\mathcal{F}\left\{\frac{1}{x^2+a^2}\right\} = \mathcal{F}\left\{\frac{1}{x^2+b^2}\right\}$$

i.e. $$Y(\alpha)\frac{\pi}{a}e^{-a\alpha} = \frac{\pi}{b}e^{-b\alpha} \qquad \text{or} \qquad Y(\alpha) = \frac{a}{b}e^{-(b-a)\alpha}$$

Thus $$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}Y(\alpha)\,d\alpha = \frac{a}{b\pi}\int_0^{\infty} e^{-(b-a)\alpha}\cos \alpha x\,d\alpha = \frac{(b-a)a}{b\pi[x^2+(b-a)^2]}$$

Application of Fourier transform
To
solve Boundary value problems

Example 5. Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $0 < x < 6$, $t > 0$, subject to the conditions $U(0, t) = 0$, $U(6, t) = 0$, $U(x, 0) = \begin{cases} 1, & 0 < x < 3 \\ 0, & 3 < x < 6 \end{cases}$

and interpret physically.

Solution : The given partial differential equation is $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ (1)

Taking the finite Fourier sine transform (with $l = 6$) of both sides of (1), we get

$$\int_0^6 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{6}\,dx = \int_0^6 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{6}\,dx \quad (2)$$

Let $u = u(n, t) = \int_0^6 U(x, t) \sin\frac{n\pi x}{6}\,dx$

Then $\frac{du}{dt} = \int_0^6 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{6}\,dx$

$$= \int_0^6 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{6}\,dx \text{ using (2)}$$

(On integrating by parts)

$$= \left[\sin\frac{n\pi x}{6}\cdot\frac{\partial U}{\partial x}\right]_0^6 - \frac{n\pi}{6}\int_0^6 \cos\frac{n\pi x}{6}\cdot\frac{\partial U}{\partial x}\,dx$$

$$= 0 - \frac{n\pi}{6}\left[\cos\frac{n\pi x}{6}\cdot U(x, t)\right]_0^6 - \frac{n^2\pi^2}{36}\int_0^6 \sin\frac{n\pi x}{6}\, U(x, t)\,dx$$

$$= 0 - \frac{n\pi}{6}\left[\cos n\pi.\, U(6, t) - U(0, t)\right] - \frac{n^2\pi^2}{36}\int_0^6 U(x, t) \sin\frac{n\pi x}{6}\,dx$$

$$= 0 - \frac{n^2\pi^2}{36}\int_0^6 U(x, t) \sin\frac{n\pi x}{6}\,dx \text{, Since } U(6,t) = U(0,t) = 0$$

$$= -\frac{n^2\pi^2}{36} u, \text{ Since } u = \int_0^6 U(x,t) \sin\frac{n\pi x}{6}\,dx$$

$\therefore \frac{du}{dt} = -\frac{n^2\pi^2}{36} u$. where $u = u(n, t)$

or. $\frac{du}{u} = -\frac{n^2\pi^2}{36}\,dt$

Integrating both sides, we get

$\log u = -\frac{n^2\pi^2}{36} t + \log A$, A being some constant of integration

or, $\log u = \log e^{-\frac{n^2\pi^2 t}{36}} + \log A = \log A e^{-\frac{n^2\pi^2 t}{36}}$.

$$\therefore \quad u = A e^{-\frac{n^2\pi^2 t}{36}} \quad (3)$$

When $t = 0$, $u(n, 0) = Ae^0 = A$

$\therefore \boxed{A = u(n, 0)}$ (4)

Now $u(n, t) = \int_0^6 U(x, t) \sin\frac{n\pi x}{6}\,dx$

$$\therefore u(n, 0) = \int_0^6 U(x, 0) \sin\frac{n\pi x}{6}\,dx$$

$$= \int_0^3 U(x, 0) \sin\frac{n\pi x}{6}\,dx + \int_3^6 U(x, 0) \sin\frac{n\pi x}{6}\,dx$$

$$= \int_0^3 1.\sin\frac{n\pi x}{6}\,dx - \int_3^6 0.\sin\frac{n\pi x}{6}\,dx$$

$$= \int_0^3 \sin\frac{n\pi x}{6}\,dx + 0$$

$$= -\frac{6}{n\pi}\left[\cos\frac{n\pi x}{6}\right]_0^3$$

$$= -\frac{6}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$= \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

Thus from (4), we have

$$A = \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) \quad (5)$$

Putting the value of A in (3), we get

$$u(n, t) = \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2 t}{36}}$$

Taking the inverse Fourier sine transform we get

$$U(x, t) = \frac{2}{6}\sum_{n=1}^{\infty} \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) e^{\frac{-n^2\pi^2 t}{36}} \cdot \sin\frac{n\pi x}{6}$$

$$\text{or,}\quad U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2 t}{36}} \cdot \sin\frac{n\pi x}{6}.$$

Physical interpretation

Physically $U(x, t)$ represents the temperature at any point x at any time t in a bar with the ends $x = 0$ and $x = 6$ kept at zero temperature which is insulated laterally. Initially the temperature in the half bar from $x = 0$ to $x = 3$ is constant equal to 1 unit while the half bar from $x = 3$ to $x = 6$ is at zero temperature.

Example 2. Prove that the solution of the boundary value problem $\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}$

$$U(o,t) = U(2,t) = o,\ t > o$$

$$U(x, o) = x,\ o < x < 2$$

is $U(x, t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2} e^{-\frac{3}{4}n^2\pi^2 t}$.

Proof : The given partial differential equation is

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the finite Fourier sine transform (with $l = 2$) of both sides of (1), we get

$$\int_0^2 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{2} dx = \int_0^2 3\frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{2} dx \quad (2)$$

Let $u = u(n,t) = \int_0^2 U(x,t) \sin\frac{n\pi x}{2} dx$

then $\frac{du}{dt} = \int_0^2 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{2} dx$

$$= \int_0^2 3\frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{2} dx \text{ using (2)}$$

(on integrating by parts)

$$= 3\left[\sin\frac{n\pi x}{2}.\frac{\partial U}{\partial x}\right]_0^2 - \frac{3n\pi}{2}\int_0^2 \cos\frac{n\pi x}{2}.\frac{\partial U}{\partial x} dx$$

$$= 0 - \frac{3n\pi}{2}\left[\cos\frac{n\pi x}{2}.U(x,t)\right]_0^2 - \frac{3n^2\pi^2}{4}\int_0^2 \sin\frac{n\pi x}{2}.U(x,t)\, dx$$

$$= o - \frac{3n^2\pi^2}{4}\int_0^2 U(x,t) \sin\frac{n\pi x}{2} dx$$

Since $U(o, t) = U(2, t) = o$

$$= -\frac{3n^2\pi^2}{4} u, \text{ Since } u = \int_0^2 U(x, t) \sin\frac{n\pi x}{2} dx.$$

$\therefore \quad \frac{du}{dt} = -\frac{3n^2\pi^2}{4} u$ where $u = u(n, t)$.

or, $\quad \frac{du}{u} = -\frac{3n^2\pi^2}{4} dt$

Integrating both sides, we get

$\log u = -\frac{3n^2\pi^2}{4} t + \log A$, where A is an arbitrary constant.

or, $\quad \log u = \log e^{-\frac{3n^2\pi^2 t}{4}} + \log A = \log A e^{-\frac{3n^2\pi^2}{4}t}$

$$\therefore \quad u = u(n, t) = A e^{-\frac{3n^2\pi^2}{4}t} \quad (3)$$

when $t = 0$, $u(n, o) = Ae^{o} = A$

$$\therefore \boxed{A = u(n, o)} \quad (4)$$

Now $u(n, t) = \int_0^2 U(x, t) \sin\frac{n\pi x}{2} dx$

$$\therefore \quad u(n, o) = \int_0^2 U(x, o) \sin\frac{n\pi x}{2} dx$$

$$= \int_0^2 x \sin\frac{n\pi x}{2}\, dx, \text{ Since } U(x, o) = x$$

$$= \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\, dx$$

$$= -\frac{4}{n\pi}\cos n\pi + 0 + \frac{4}{n^2\pi^2}\left[\sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi + 0 = -\frac{4}{n\pi}\cos n\pi.$$

Thus from (4), we have $\boxed{A = -\frac{4}{n\pi}\cos n\pi}$

putting the value of A in (3), we get

$$u(n, t) = -\frac{4}{n\pi}\cos n\pi\, e^{-\frac{3n^2\pi^2}{4}t} \qquad (5)$$

Now taking the inverse finite Fourier sine transform, we get

$$U(x,t) = \frac{2}{2}\sum_{n=1}^{\infty} -\frac{4}{n\pi}\cos n\pi.\, e^{-\frac{3}{4}n^2\pi^2 t}\sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n e^{-\frac{3}{4}n^2\pi^2 t}\sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}.\, e^{-\frac{3}{4}n^2\pi^2 t}$$

which is the required solution.