(2/3) set

Fourier Analysis

- Fourier Series and Fourier Integral
- Fourier Transform (F.T)
- Application of F.T.

Md. Nuruzzaman sir
01740066434
01836995903
mnmath24@gmail.com

Ref:
1. Fourier Analysis - M.R. Spiegel
2. Advanced Engg. Mathemetics.

Periodic Function:-

$$f(x) = f(x+T)$$

Fourier Series:-

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where,

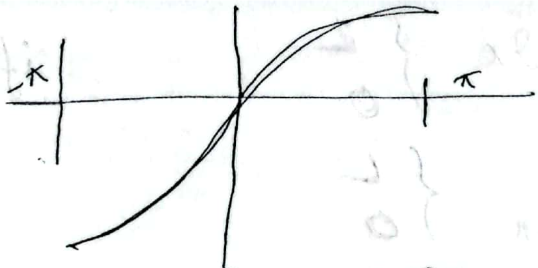
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

- $f(x) \to$ periodic function in $(-L, +L)$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$A = \frac{a_0}{2}$, where a, b fourier coefficient or harmonics.

Ex:



$T = 2\pi$

$L = \pi$

nx

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$= A + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

(#) Basis function:

Property:

(i) Orthogonal → inner product $= 0$ $(u \cdot v = 0)$

2nd lecture 26-10-2024

(ii) → $u \cdot u \neq 0$ and $v \cdot v \neq 0$

$$* f(x) = A + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

$\{1, \cos x, \sin x, \cos 2x, \sin 2x \cdots\}$ fourier basis

Basis function: $\int_{-\pi}^{\pi} f(x) \cdot g(x)\,dx$

$$= \int_{-\pi}^{\pi} 1 \cdot \cos x\,dx = 0$$

$$\int_{-\pi}^{\pi} \sin x \cos x\,dx = 0$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\sin x \cos x\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin 2x\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}[\cos 2x]\right]_{-\pi}^{\pi} = 0$$

And Also

$$\int_{-\pi}^{\pi} \cos x \cos x\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2x)\,dx = \pi$$

a. $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \begin{cases} L & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$

b. $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx \begin{cases} L & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$

c. $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$

If the series $A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$ show th

for $n = 1, 2, 3, \ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$A = a_0/2$$

Proof: Multiplying $f(x) = A + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ —①

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, we get,

$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L} A\cos\frac{m\pi x}{L}\,dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-L}^{L} \cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L}\,dx + b_n \int_{-L}^{L} \sin\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L}\,dx \right\}$$

$a_n = \cos\frac{n\pi x}{L}$

$n = 0$

$$= 0 + a_m \int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{m\pi x}{L}\,dx.$$

$$= a_m L.$$

$$\therefore a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{m\pi x}{L}\,dx$$

$$\therefore a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\,dx$$

Similarly,

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\,dx$$

Now,
Multiplying ① by 1 and integrating form $-L$ to L, we get,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} A\cdot 1 + \sum_{n=1}^{\infty}\left\{\int_{-L}^{L}\cos\frac{n\pi x}{L}\cdot 1\,dx + \int_{-L}^{L}\sin\frac{n\pi x}{L}\cdot 1\,dx\right\}$$

$$= A\int_{-L}^{L} 1\,dx + 0 + 0$$

$$= A\cdot [x]_{-L}^{L}$$

$$= A\cdot 2L$$

$$\Rightarrow A = \frac{1}{2}\cdot\frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$\therefore A = \frac{a_0}{2}$$

a_0

① A

$\cos 0^\circ = 1$, $\sin 0^\circ = 0$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \frac{1}{2}$

$\cos 45^\circ = \sin 45^\circ = \frac{1}{\sqrt{2}}$

$\cos 60^\circ = \frac{1}{2}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 90^\circ = 0$, $\sin 90^\circ = 1$

$\cos n\pi = (-1)^n$, $\sin n\pi$

Problem: Expand the following function in Fourier series:

$$f(x) = \begin{cases} 0 & , -5 < x < 0 \\ 3 & , 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

Soln:

$T = 10$, $2L = 10$ $\therefore L = 5$

a_n, b_n, A

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0} f(x) \cos\frac{n\pi x}{L}\, dx + \int_{0}^{5} f(x)\cos\frac{n\pi x}{L}\, dx\right\}$$

$$= \frac{1}{5}\left\{0 + \int_0^5 3 \cdot \cos\frac{n\pi x}{5}\, dx\right\}$$

$$= \frac{3}{5} \cdot \frac{5}{n\pi}\left[\sin\frac{n\pi x}{5}\right]_0^5$$

$$= \frac{3}{n\pi}\left[\sin n\pi - 0\right]$$

$a_n = 0$

Again: $$b_n = \frac{1}{5}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{5}\, dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0} f(x) \sin\frac{n\pi x}{L}\, dx + \int_0^5 f(x) \sin\frac{n\pi x}{L}\, dx\right\}$$

$= \frac{1}{5}\{0 + \int_0^5 3 \sin \frac{n\pi x}{5} dx\}$

$= \frac{-3}{5} \cdot \frac{5}{n\pi} \left[\cos \frac{n\pi x}{5}\right]_0^5$

$= \frac{-3}{n\pi}\left[(-1)^n - 1\right]$

$b_n = \frac{3}{n\pi}\{1-(-1)^n\}$

$\cos n\pi = (-1)^n$

$\boxed{\text{fourier series always gives continuous curbe}}$

And also,

$a_0 = \frac{1}{L}\left\{\int_{-L}^{0} f(x)dx + \int_0^L f(x)dx\right\}$

$= \frac{1}{5}\{0 + \int_0^5 3dx\} = \frac{1}{5}\left[3x\right]_0^5 = \frac{1}{5}\times(15-0)$

$= 3$

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n + b_n \sin \frac{n\pi x}{L}\right)$

$= \frac{3}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}(1-(-1)^n) \sin \frac{n\pi x}{5}$

$a_1 = \frac{1}{L}\int$

$$f(x) = \begin{cases} 2-x & , 0<x<4 \\ x-6 & , 4<x<8 \end{cases}$$ in a Fourier series.

<u>Soln:</u>

$2L = 8 \therefore L = 4$

$$a_n = \frac{1}{4}\left\{\int_0^4 f(x)\cos\frac{n\pi x}{4}dx + \int_4^8 f(x)\cos\frac{n\pi x}{4}dx\right\}$$

$$= \frac{1}{4}\left\{\int_0^4 (2-x)\cos\frac{n\pi x}{4}dx + \int_4^8 (x-6)\cos\frac{n\pi x}{4}dx\right\}$$

$$= \frac{1}{4}\left\{\left[(2-x)\left(\frac{4}{n\pi}\sin\frac{n\pi x}{4}\right) - (-1)\left(-\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right)\right]_0^4 + \right.$$

$$\left.\left[(x-6)\left(\frac{4}{n\pi}\sin\frac{n\pi x}{4}\right) - \left(-\frac{16}{n^2\pi^2}\cos\frac{n\pi x}{4}\right)\right]_4^8\right\}$$

$$= \frac{1}{4}\left\{\left[0 - \frac{16}{n^2\pi^2}(-1)^n - 0 + \frac{16}{n^2\pi^2}\right] + \left[0 + \frac{16}{n^2\pi^2} - 0 - \frac{16}{n^2\pi^2}(-1)^n\right]\right\}$$

$$= \frac{1}{4}\left\{\frac{32}{n^2\pi^2} - \frac{32}{n^2\pi^2}(-1)^n\right\}$$

$$a_n = \frac{8}{n^2\pi^2}\left\{1 - (-1)^n\right\}$$

Again, $$b_n = \frac{1}{4}\left\{\int_0^4 f(x)\sin\frac{n\pi x}{4}dx + \int_4^8 f(x)\sin\frac{n\pi x}{4}dx\right\}$$

$$= \frac{1}{4}\left\{\int_0^4 (2-x)\sin\frac{n\pi x}{4}dx + \int_4^8 (x-6)\sin\frac{n\pi x}{4}dx\right\}$$

$$= \frac{1}{4}\left\{\left[(2-x)\left(-\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right) - (-1)\left(-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right)\right]_0^4 + \left[(x-6)\left(-\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right) - (1)\cdot\left(-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right)\right]_4^8\right\}$$

$$= \frac{1}{4}\left\{\frac{8}{n\pi}(-1)^n - 0 - \frac{8}{n\pi} + 0\right\}$$

$$= \frac{2}{n\pi}(-1)^n - \frac{2}{n\pi}$$

$$b_n = \frac{2}{n\pi}\left\{(-1)^n - 1\right\}$$

$$A = \frac{1}{2}\cdot\frac{1}{4}\left\{\int_0^4 f(x)\,dx + \int_4^8 f(x)\,dx\right\}$$

$$= \frac{1}{8}\left\{\int_0^4 (2-x)\,dx + \int_4^8 (x-6)\,dx\right\}$$

$$\cancel{= \frac{1}{8}\left[-1\right]_0^4 + \left[\;\right]}$$

$$= \frac{1}{8}\left\{\left[-\frac{x^2}{2}\right]_0^4 + \left[\frac{x^2}{2}\right]_4^8\right\}$$

$$= \frac{1}{8}\left[-8 - 0 + 32 - 8\right]$$

$$= 2$$

$$\therefore f(x) = 2 + \sum_{n=1}^{\infty}\left[\frac{8}{n^2\pi^2}\left\{1-(-1)^n\right\}\cos\frac{n\pi x}{L} + \frac{2}{n\pi}\left\{(-1)^n - 1\right\}\sin\frac{n\pi x}{L}\right]$$

11. Expand $f(x) = x^2, \quad 0 < x < 2\pi$ in a Fourier Series.

Sol^n: $2L = 2\pi$

$\therefore L = \pi$

Now, $a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$

$$= \frac{1}{\pi}\int_{0}^{2\pi} x^2 \cdot \cos nx \, dx$$

$$= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + 4\pi \cdot \frac{1}{n^2} - 0 - 0\right]$$

$$\therefore a_n = \frac{4}{n^2}$$

And, $A_0 = \frac{1}{2} \cdot \frac{1}{\pi}\int_{0}^{2\pi} x^2 \, dx$

$$= \frac{1}{2\pi} \cdot \left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{8 \cdot \pi^3}{3} - 0\right]$$

$$= \frac{4\pi^2}{3}$$

Again, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cdot \sin nx \, dx$$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + 2x\cdot\frac{\sin nx}{n^2} + 2\cdot\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-4\pi^2\cdot\frac{1}{n} + 0 + \frac{2}{n^3} - \frac{2}{n^3}\right]$$

$$= -\frac{4\pi}{n}$$

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

$$(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

Ans

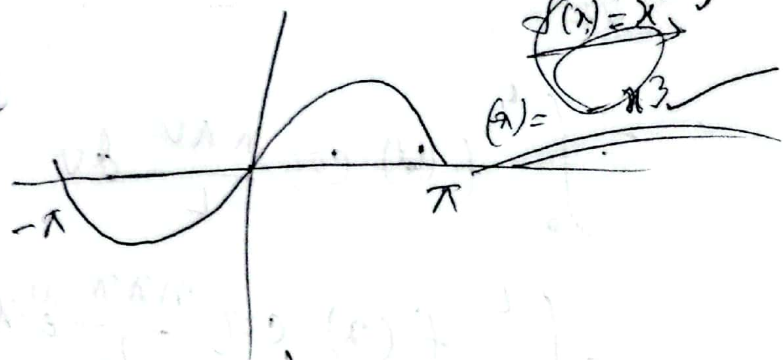
B $f(x) \to$ even $\to$ $\boxed{f(-x)=f(x)}$

$f(x) \to$ odd $\to$ $\boxed{f(-x)=-f(x)}$

$f(-x) = -\sin$

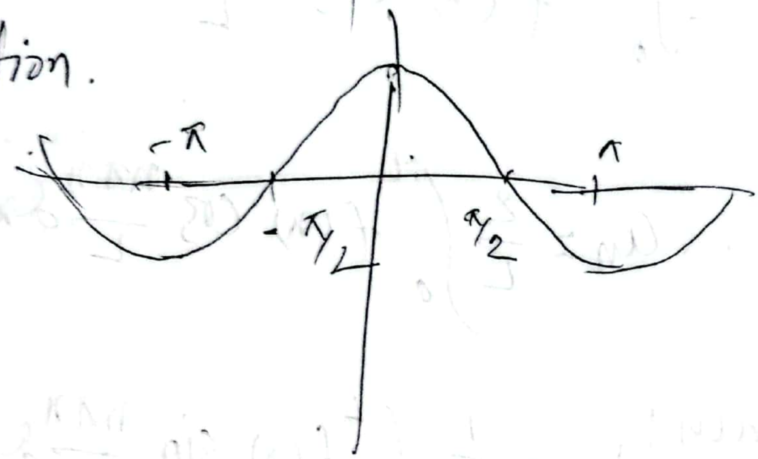
$\sin(x) \longrightarrow$ odd function

$\therefore \sin(-x) = -\sin x$



$\cos(x) \longrightarrow$ even function.

$\therefore \cos(-x) = \cos x$



* If $f(x)$ is even show that $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

and $b_n = 0$.

Soln: We have, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$= \frac{1}{L}\left\{\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\right\}$

Now $\int_{-L}^{0} f(x) \cancel{dx} \cos \frac{n\pi x}{L}\, dx$

Let $x = -\cancel{\ } u$

$\Rightarrow dx = -d\cancel{\ }u$

$x = 0, \quad u = 0$

$x = L, \quad u = L$

$= \int_{L}^{0} f(-u) \cos \frac{(n\pi u)}{L} (-du)$

$= \int_{0}^{L} f(\cancel{x}) \cos \frac{n\pi u}{L}\, du$

$= \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$

$\therefore a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$

Again; $b_n = \frac{1}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$

$= \frac{1}{L} \left\{ \int_{-L}^{0} f(x) \sin \frac{n\pi x}{L}\, dx + \int_{0}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \right\}$

Now, $\int_{-L}^{0} f(x) \sin \frac{n\pi x}{L}\, dx$

$= \int_{-L}^{0} f(-u) \sin \frac{-n\pi u}{L} (-du)$

$= -\int_{0}^{L} f(u) \sin \frac{n\pi u}{L}\, du$

$\therefore b_n = 0$

$$= \frac{1}{\pi}\Big\{\Big[0 + \pi^3 \cdot \frac{(-1)^n}{n} + 0 - 6\pi \frac{(-1)^n}{n^3}\Big] + \Big[-\pi^3 \cdot \frac{(-1)^n}{n}$$

$$+ 0 + 6\pi \frac{(-1)^n}{n^3} - 0 - 0\Big]$$

$$= \frac{1}{\pi}\Big[\pi^3 \cdot \frac{(-1)^n}{n} - 6\pi \frac{(-1)^n}{n^3} - \pi^3 \cdot \frac{(-1)^n}{n} + 6\pi \frac{(-1)^n}{n^3}\Big]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cdot \sin \frac{n\pi x}{L}\, dx$$

$$= \frac{1}{\pi}\Big[-x^3 \frac{\cos nx}{n} + 3x^2 \frac{\sin nx}{n^2} + 6x \frac{\cos nx}{n^3} - 6 \frac{\sin nx}{n^4}\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\Big[-\pi^3 \cdot \cancel{\frac{(-1)^n}{n}} + 0 + 6\pi \cdot \cancel{\frac{(-1)^n}{n^3}} - 0 + \pi^3 \cdot \cancel{\frac{(-1)^n}{n}} + 0 - 6\pi \cdot \cancel{\frac{(-1)^n}{n^3}}\Big]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos nx\, dx$$

Wa

$$= \frac{1}{\pi}\Big[x^3 \cdot \frac{\sin nx}{n} + 3x^2 \frac{\cos nx}{n^2} - 6x \frac{\sin nx}{n^3} + 6 \frac{\cos nx}{n^4}\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\Big[0 + 3\pi^2 \frac{(-1)^n}{n^2} - 0 + 6 \frac{(-1)^n}{n^4}$$

Now,
$$A = \frac{1}{2}\cdot\frac{1}{L}\left\{\int_{-L}^{0} x^2 dx + \int_{0}^{L} x^2 dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x^3}{3}\right]_{-\pi}^{0} + \left[\frac{x^3}{3}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right)$$

$$= \frac{\pi^2}{3}$$

$$\therefore f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4}{n^2}(-1)^n \cos nx\right\}$$

Ans:

(11) $f(x) = x^3,\ -\pi < x < \pi$ obtain fourier series.

Soln: $f(-x) = -x^3$ $\therefore$ $f(x)$ is odd function

So, a_n and $a_0 = 0$

Now,
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} x^3 \sin nx\, dx + \int_{0}^{\pi} x^3 \sin nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-x^3\frac{\cos nx}{n} + 3x^2\frac{\sin nx}{n^2} + 6x\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}\right]_{-\pi}^{0} + \left[-x^3\frac{\cos nx}{n} + 3x^2\frac{\sin nx}{n^2} + 6x\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}\right]_{0}^{\pi}\right\}$$

Obtain Fourier series for

(i) $f(x) = x^2, \; -\pi < x < \pi$

(ii) $f(x) = x^3, \; -\pi < x < \pi$

Soln: since $f(x)$ = even, $\quad 2L = 2\pi$

then $b_n = 0$ $\quad \therefore L = \pi$

$$a_n = \frac{2}{L}\int_0^L x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^\pi x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + 2x\frac{\cos nx}{n^2} + 2\frac{\sin nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi.(-1)^n}{n^2} + 0 \mp 0 \mp 0 - 0\right]$$

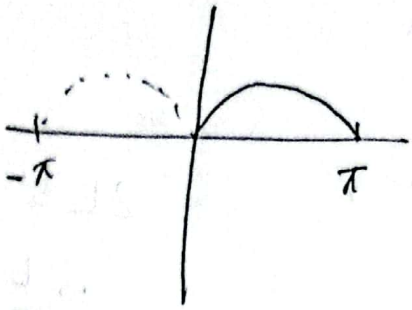
$$= \frac{2}{\pi}\cdot\frac{2\pi(-1)^n}{n^2}$$

$$\therefore a_n = \frac{4}{n^2}(-1)^n$$

Now.

Expand $f(x) = \sin x,\ 0 < x < \pi$ in a Fourier Cosine series

Soln:



$T = 2L = 2\pi$

$\therefore L = \pi$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$$

$$\therefore a_n = \frac{-2(1+\cos n\pi)}{\pi(n^2-1)}$$

Fourier Integral

Fourier Integral of a function $f(x)$ is defined in $(-\infty, \infty)$ is

$$f(x) = \int_0^{\infty} \{A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\} d\alpha$$

where,

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\alpha t \, dt$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\alpha t \, dt$$

$$= \frac{1}{\pi}\int_0^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\{\cos\alpha x \cos\alpha t + \sin\alpha x \sin\alpha t\} dt\right\} d\alpha$$

$$= \frac{1}{\pi}\int_{\alpha=0}^{\infty}\left\{\int_{t=\infty}^{\infty} f(t)\cos\alpha(x-t)\, dt\right\} d\alpha$$

Find the Fourier Integral of $f(x) = e^{-kx}$; $x > 0$ and $f(-x) = f(x)$

Soln:

$$~~f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{t=-\infty}^{\infty} f(t)\cos\alpha(x-t)\, dt\right\} d\alpha~~$$

~~when $f(x)$ is even~~

$$~~f(x) = \frac{1}{\pi}\int_{\alpha=0}^{\infty}\left\{\int_{t=0}^{\infty} f(t)\right.~~$$

Solution:

Since $f(x)$ is even function the F.I of the given function is,

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\left\{\int_{t=0}^{\infty} f(t)\cos ut\,dt\right\}\cos ux\,du \quad \text{--- (1)}$$

Now,

$$\int_{t=0}^{\infty} f(t)\cos ut\,dt = \int_0^{\infty} e^{-kt}\cos ut\,dt$$

$$= \left.\frac{e^{-kt}}{k^2+u^2}\left(-k\cos ut + u\sin ut\right)\right|_0^{\infty}$$

$$= 0 - \frac{1}{k^2+u^2}(-k+0)$$

$$= \frac{k}{k^2+u^2}$$

From (1) $\Rightarrow$

$$f(x) = \frac{2}{\pi}\int_{u=0}^{\infty}\frac{k}{k^2+u^2}\cos ux\,du$$

$$\therefore e^{-kx} = \frac{2}{\pi}\int_{u=0}^{\infty}\frac{k}{k^2+u^2}\cos ux\,dx$$

$$\boxed{\therefore \frac{\pi}{2k}e^{-kx} = \int_0^{\infty}\frac{\cos ux}{k^2+u^2}\,dx}$$

H.W ① Find the Fourier Integral of $f(x) = e^{-kx}$, when $f(-x) = -f(x)$ and hence prove that.

$$\int_0^\infty \frac{u \sin ux}{k^2+u^2}\, du = \frac{\pi}{2} e^{-kx}, \quad k>0.$$

⑪ Find the fourier Integral of the function,

$$f(x) = \begin{cases} 0 & , x<0 \\ \frac{1}{2} & x=0 \\ e^{-x} & x>0 \end{cases}$$

$$\boxed{\frac{\pi}{2k} e^{-kx} = \int_0^\infty \frac{\cos ux}{k^2+u^2}\, du}$$

$f(x) = e^{-kx}$, $k>0$, $x>0$ and $f(-x) = -f(x)$.

Sol^n: Since $f(x)$ is an odd function thus the F.I of $f(x)$ is,

$$e^{-kx} = \frac{2}{\pi}\int_{u=0}^{\infty}\left\{\int_{t=0}^{\infty} e^{-kx}\sin ut\,dt\right\}\sin ux\,du \quad \cdots (1)$$

Now, $\int_0^{\infty} e^{-kt}\sin ut\,dt$

$$= \frac{e^{-kt}}{k^2+u^2}\left(-k\sin ut - u\cos ut\right)\Big|_0^{\infty}$$

$$= 0 - \left\{\frac{1}{k^2+u^2}(0-u)\right\}$$

$$= \frac{u}{k^2+u^2}$$

From (1) $\Rightarrow$

$$e^{-kx} = \frac{2}{\pi}\int_{u=0}^{\infty}\frac{u}{k^2+u^2}\sin ux\,du.$$

$$\Rightarrow \frac{\pi}{2}e^{-kx} = \int_0^{\infty}\frac{u\sin ux}{k^2+u^2}\,du.$$

H.W → Find the F.I of $f(x) = \begin{cases} 1 & , |x| < 1 \\ 0 & , |x| > 1 \end{cases}$

and evaluate $\int_0^\infty \frac{\sin ux \cos ux}{u} \, du$.

Fourier transform

$f(t) = t^2$

F.T

Inverse F.T.

$F(n) =$

Fourier transform of a function $f(x)$ is denoted by $F\{f(x)\}$ or $F(u)$ and defined as.

$$F(u) = \int_0^{\infty} f(x)\, e^{-iux}\, dx$$

and Inverse F.T is

$$F^{-1}\{F(u)\} = f(x) = \frac{1}{2\pi}\int_0^{\infty} F(u)\, e^{iux}\, du.$$

F cosine T. of (x)

$$F_c(n) = \int_0^{\infty} f(x) \cos nx\, dx$$

Inverse

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(n) \cos nx\, dx$$

F. Sine T. of $f(x)$

$$F_s(n) = \int_0^\infty f(x) \sin nx \, dx$$

Inverse,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(n) \sin nx \, dn$$

Finite F. cosine T. of $f(x)$ $0 < x < l$

$$F_c(n) = \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$$

Inverse,

$$f(x) = \frac{1}{L} F_c(\ ) + \frac{2}{L} \sum_{n=1}^{\infty} F_c(n) \cos \frac{n\pi x}{L}$$

finite F sine T. of $f(x)$ $0 < x < l$

$$F_s(n) = \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} \, dx$$

Inverse

$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{L}$$

Find the sine transform of e^{-x}, $x>0$

Sol^n: Given $f(x) = e^{-x}$

we have the sine transform of $f(x)$ is

$$F(n) = \int_0^\infty e^{-x} \sin nx \, dx$$

$$= \frac{e^{-x}}{1+n^2} (-\sin nx - n\cos nx) \Big|_0^\infty$$

$$= 0 - \frac{1}{1+n^2} (0 - n)$$

$$F(n) = \frac{n}{1+n^2}$$

* Find the inverse sine Transform of $\frac{n}{1+n^2}$

Sol^n:

$$F(n) = \frac{n}{1+n^2}$$

Inverse F sine T of $F(n)$ is,

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{n}{1+n^2} \sin nx \, dn$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} e^{-x}$$

$$= e^{-x}$$

$\boxed{H.W}$ 1. Find the F.T of

i. $f(x)=\begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases}$

ii. $f(x)=e^{-|x|}, \quad -\infty<x<\infty$

iii. $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$

2. Solve the integral equation

$$\int_0^\infty f(x)\cos ux\,dx=\begin{cases}1-u, & 0\le u<1\\ 0, & u>1\end{cases}$$

Soln: We have Fourier cosine Transform of any function $f(x)$ is,

$$F(u)=\int_0^\infty f(x)\cos ux\,dx$$

Thus, $F(u)=\begin{cases}1-u, & 0\le u<1\\ 0, & u>1\end{cases}$ ——— ①

Taking Inverse F. cosine T, we get,

$$F^{-1}\{F(u)\}=\frac{2}{\pi}\int_0^\infty F(u)\cos ux\,du$$

$$f(x)=\frac{2}{\pi}\int_0^1 (1-u)\cos ux\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^1 \cos ux\, dx - \int_0^1 u\cos ux\, dx\right\} \quad \text{(ii)}$$

$$= \boxed{\frac{2}{\pi} \quad \frac{\sin x}{x}}$$

Now

$$\int_0^1 \cos ux\, dx = \frac{1}{x}\sin ux\Big|_0^1$$
$$= \frac{\sin x}{x}$$

$$\int_0^1 u\cos ux\, dx = u\frac{\sin ux}{x} - \int 1 \frac{\sin ux}{x}\, dx$$
$$= \left[\frac{u\sin ux}{x} - \frac{1}{x^2}\cos ux\right]_0^1$$
$$= \left(\frac{\sin x}{x} + \frac{\cos x}{x^2} - \frac{1}{x^2}\right)$$

from (ii) $\Rightarrow$

$$= \frac{2}{\pi}\left\{\frac{\sin x}{x} - \frac{\sin x}{x} - \frac{\cos x}{x} + \frac{1}{x^2}\right\}$$

$$f(x) = \frac{2(1-\cos x)}{\pi x^2}$$

* Solve the integral equation

$$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1 - \alpha & , 0 \le \alpha < 1 \\ 0 & \alpha \ge 1 \end{cases}$$

Convolution Property: of Fourier transform:

Convolution

$f(x)$ and $g(x)$

$$f*g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Statement: If $F\{f(x)\}$ and $F\{g(x)\}$ be the Fourier transform of $f(x)$ and $g(x)$ ~~repu~~ respectively then the F.T of the convolution of $f(x)$ and $g(x)$ is the product of their FT.

$$F\{f(x) * g(x)\} = F\{f(x)\} \cdot F\{g(x)\}$$

Proof: Let $F(u)$ and $G(u)$ be the F.T of $f(x)$ and $g(x)$, respectively,

Then we can write,

$$F(u) = \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt$$

$$G(u) = \int_{-\infty}^{\infty} g(s)\, e^{ius}\, ds$$

Now, $F(u) \cdot G(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, g(s)\, e^{-iu(s+t)}\, ds\, dt$

Let, $s+t = x$ and $t = t$

Then the Jacobian of this transformation is

$$\frac{\partial(t,s)}{\partial(t,x)} = \begin{vmatrix} \frac{\partial t}{\partial t} & \frac{\partial t}{\partial x} \\ \frac{\partial s}{\partial t} & \frac{\partial s}{\partial x} \end{vmatrix}$$

Thus we have,

$$dt\,ds = \frac{\partial(t,s)}{\partial(x,y)}\, dt\, dx$$

$$\frac{\partial(t,s)}{\partial(y,x)} = \begin{vmatrix} \frac{\partial t}{\partial y} & \frac{dt}{dx} \\ \frac{ds}{\partial y} & \frac{ds}{\partial x} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= 1$$

$$dt\,ds = 1\, dt\, dx$$

From ① ⇒

$$F(u)\,G(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, g(x-t)\, e^{-iux}\, dt\, dx$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt\right\} e^{-iux}\, dx$$

$$= \int_{-\infty}^{\infty} \{f * g\}\, e^{-iux}\, dx$$

$$F(u)\cdot G(u) = F\{f*g\}$$

* Solve for $y(x)$ the integral equation

$$\int_{-\infty}^{\infty} \frac{y(u)\,du}{(x-u)^2+a^2} = \frac{1}{x^2+b^2}, \quad 0<a<b$$

Laplace transform

Advanced Engg Mathematics H.K. Dass

$f(t) \Rightarrow F(s)$

$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$

$\boxed{Ex:}$ $f(t) = t$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot t \, \partial t$

$= \left[t \cdot \frac{1}{-s} e^{-st} \right]_0^\infty +$

$f(t) = t^n$

$\mathcal{L}\{(t^n)\} = \int_0^\infty e^{-st} \cdot t^n dt$

$= \int_0^\infty e^{-y} \cdot \left(\frac{y}{s}\right)^n \frac{dy}{s}$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^n \, \partial y$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^{n+1-1} \, dy$

$= \frac{1}{s^{n+1}} \sqrt{n+1}$

$= \frac{\sqrt{n+1}}{s^{n+1}}$

(#) $f(t) = e^{at}$

$$L\{e^{at}\} = \int_0^\infty e^{-st} . e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \frac{-1}{s-a} \left[e^{-(s-a)t} \right]_0^\infty$$

$$= \frac{-1}{s-a} (0-1)$$

$$= \frac{1}{s-a}$$

(#) $f(t) = \cos at$

$$\therefore L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2} (-s\cos at + a \sin at) \right]_0^\infty$$

$$= \frac{1}{s^2+a^2} [0 - 1(-s+0)]$$

$$= \frac{s}{s^2+a^2}$$

$L\{\sin at) = \frac{a}{s^2+a^2}$

Linear property:

$L\{f(t)\}$

~~Def~~

First translation property:

let $L\{f(t)\} = F(s)$

then $L\{e^{at} f(t)\} = F(s-a)$

$L\{1\} = \frac{1}{s}$

$\therefore L\{e^{at} \cdot 1\} = \frac{1}{s-a}$

(ii) $L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{e^{2t} \sin at\} =$

$= \frac{a}{(s-a)^2+a^2}$

Second shifting property:

if $L\{f(t)\} = F(s)$

then $L\{G(t)\} = e^{-as} F(s)$, where, $G(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

Change of scale property

$L\{f(t)\} = F(s)$

then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

$L\{\sin 2t\} = \frac{1}{2} \cdot \frac{4}{s^2+2^2}$

$= \frac{2}{s^2+2^2}$

L.T of derivatives:

if $L\{f(t)\} = F(s)$

then $L\{f'(t)\} = ?$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{st} \cdot f(t)\,dt$$

$$= [0 - f(0)] + s\,F(s)$$

$$\therefore L\{f'(t)\} = s\,F(s) - f(0)$$

Laplace Transformation

Laplace transformation help in solving the differential equations with boundary values without finding the general solution and the values of the arbitrary constants.

Defination: Let $f(t)$ be function defined for all positive values of t, then,

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$ called Laplace transformation of $f(t)$. It is denoted as,

$$\boxed{L[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt}$$

Important Formulae

1. $L(1) = \frac{1}{s}$

2. $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3 \ldots\ldots$

3. $L(e^{at}) = \frac{1}{s-a} \quad \ldots\ldots (s>a)$

4. $L(\cosh at) = \frac{s}{s^2-a^2} \quad (s^2>a^2)$

5. $L(\sinh at) = \frac{a}{s^2-a^2} \quad (s^2>a^2)$

6. $L(\sin at) = \dfrac{a}{s^2+a^2} \quad (s>0)$

7. $L(\cos at) = \dfrac{s}{s^2+a^2} \quad (s>0)$

Proof

1. $L(1) = \dfrac{1}{s}$

Proof : $L(1) = \displaystyle\int_0^\infty e^{-st}.1\,dt = \left[\dfrac{e^{-st}}{-s}\right]_0^\infty = -\dfrac{1}{s}\left[\dfrac{1}{e^{st}}\right]_0^\infty$

$= -\dfrac{1}{s}[0-1] = \dfrac{1}{s}$

2. $L(t^n) = \dfrac{n!}{s^{n+1}}$

Proof : $L(t^n) = \displaystyle\int_0^\infty e^{-st}.t^n\,dt$

$= \displaystyle\int_0^\infty e^{-x}.\left(\dfrac{x}{s}\right)^n \dfrac{\partial x}{s}$

$= \displaystyle\int_0^\infty e^{-x}.\dfrac{x^n}{s^{n+1}}.\partial x$

$= \dfrac{1}{s^{n+1}}\displaystyle\int_0^\infty e^{-x}.x^n\,\partial x$

$= \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$

Let,

$st = x$

$\therefore t = \dfrac{x}{s}$

$\partial t = \dfrac{1}{s}\partial x$

We know,

$\Gamma(n+1) = n! = \displaystyle\int_0^\infty e^{-x}.x^n\,d$

$\because \sin at = \frac{e^{iat} - e^{-iat}}{2i}$

4. $L(\sin at) = \frac{a}{s^2+a^2}$

Proof: $L(\sin at) = L\left[\frac{e^{iat} - e^{-iat}}{2i}\right]$

$= \frac{1}{2i}\left[L(e^{iat}) - L(e^{-iat})\right]$

$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$

$= \frac{1}{2i}\left[\frac{s+ia-s+ia}{s^2 - i^2a^2}\right]$

$= \frac{1}{2i}\left[\frac{2ia}{s^2+a^2}\right] = \frac{a}{s^2+a^2}$ /proved

5. $L(\cos at) = \frac{s}{s^2+a^2}$

Proof: $L(\cos at) = \int_0^\infty e^{-st} . \cos at \, dt$

$= L\left[\frac{e^{iat} + e^{-iat}}{2}\right]$

$= \frac{1}{2}\left[L(e^{iat}) + L(e^{-ait})\right]$

$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{(s+ai)}\right]$

$$= \frac{1}{2}\left[\frac{(s+ai)+(s-ai)}{s^2-i^2a^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+abi+s^2-abi}{s^2+a^2}\right]$$

$$= \frac{1}{2}\left[\cancel{\frac{2s^2}{s^2+a^2}}\right] = \frac{1}{2}\left(\frac{2s}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$$

$\boxed{\text{Example: 1}}$ Find the Laplace transformation of $f(t)$ defined as

$$f(t) = \begin{cases} \frac{t}{k}, & \text{when } 0<t<k \\ 1, & \text{when } t>k \end{cases}$$

$\boxed{\text{Solution}}$ $$L[f(t)] = \int_0^k e^{-st} \cdot \frac{t}{k}\, dt + \int_k^\infty e^{-st} \cdot 1\, dt$$

$$= \frac{1}{k}$$