

Fourier Series CT-1

fourier series of a fn

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

(Diagram)

where $a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$

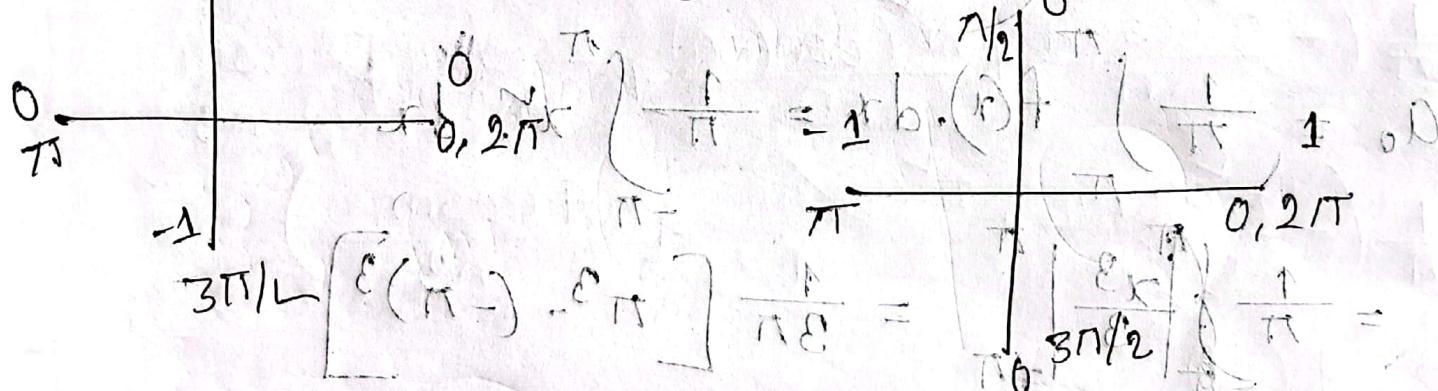
$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

① $\sin n\pi = 0$ $\Rightarrow \cos n\pi = (-1)^n$ $\frac{1}{2} \log$

② $\sin 2n\pi = 0$ $\Rightarrow \cos 2n\pi = 1$ at $x = 0$
 ③ $\cos (2n+1)\pi = -1$

$$\int_{-\pi/2}^{\pi/2} \cos((2n+1)\pi) dx = +1 \cdot \frac{\pi}{2}$$



$$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} (e^{ix} - e^{-ix}) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (e^{ix} - e^{-ix}) \sum_{n=1}^{\infty} \sin nx dx =$$

Ques 0 Above that,

(i) $\int_{-\pi}^{\pi} \cos x dx$

$$x^n = \frac{\pi^n}{n!} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n}, \quad (-\pi < x < \pi)$$

Also, show that,

$$\begin{aligned} \text{i)} \quad \sum \frac{1}{n} n^r &= \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \dots = \frac{\pi^r}{6} \\ \text{ii)} \quad \sum \frac{1}{n^r} &= \frac{1}{1^r} - \frac{1}{2^r} + \frac{1}{3^r} - \dots = \frac{(-1)^{r+1}}{2^r} \pi^r \\ \text{iii)} \quad \sum \frac{1}{(2n-1)^r} &= \frac{1}{1^r} + \frac{1}{3^r} + \frac{1}{5^r} + \dots = \frac{\pi^r}{8} = \pi^r \end{aligned}$$

Solⁿ:

Given function $f(x) = x^r$ in $(-\pi, \pi)$

Fourier series for the given $f(x) = \sum_{n=0}^{\infty} a_n \cos nx$

$$\pi^r = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\begin{aligned} \therefore a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^r dx \\ &= \frac{1}{\pi} \left[\frac{x^{r+1}}{r+1} \right]_{-\pi}^{\pi} = \frac{1}{3\pi} \left[\pi^3 - (-\pi)^3 \right] = \frac{1}{3\pi} (\pi^3 + \pi^3) \\ &= \frac{1}{3\pi} \times 2\pi^3 = \frac{2\pi^4}{3} \end{aligned}$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \text{ b. am n. } \overset{(6)}{=} \text{ and } \cos n(-\pi) = n\pi$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^n \cdot \cos nx dx \quad \text{b. am n. } \overset{7}{=} \left(\frac{1}{\pi} \right) \pi$$

$$= \frac{1}{\pi} \left| x^n \left(\frac{\sin nx}{n} \right) - (2x) \cdot \left(\frac{-\cos nx}{n^2} \right) + 2 \cdot \left(\frac{-\sin nx}{n^3} \right) \right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left| \pi^n \frac{\sin n\pi}{n} - (-\pi)^n \frac{\sin n(-\pi)}{n} + 2 \cdot \frac{\cos n\pi}{n^2} - 2 \cdot \frac{\sin n\pi}{n^3} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\pi^n \frac{\sin n\pi}{n} - (-\pi)^n \frac{\sin n(-\pi)}{n} + \frac{2\pi \cos n\pi}{n^2} \right]$$

$$(n) \overset{\pi}{=} \frac{2(-\pi) \cos ((-\pi)n20^\circ)}{n^2} - \frac{2 \cdot \sin \pi}{n^3} \overset{\pi n20^\circ}{=} \frac{2 \sin n(-\pi)}{n^3}$$

$$= \frac{1}{\pi} \left(0 + 0 \frac{2n \cos n\pi}{n^2} + \frac{2\pi \cos n\pi}{n^2} \right)$$

$$\overset{(n)n20^\circ}{=} \frac{4\pi \cos n\pi}{n^2} \overset{(n)n20^\circ \pi}{=} \frac{(n)n20^\circ \pi}{n^2} \overset{\cos n\pi = (-1)^n}{=} \frac{4(-1)^{nn20^\circ \pi}}{n^2}$$

$$= \frac{4(-1)^{nn20^\circ \pi}}{n^2} + \frac{n n20^\circ \pi}{n^2} + \frac{n n20^\circ \pi}{n^2} \overset{0}{=} \frac{0}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \text{ where } x = 20^\circ \quad (\text{but } f\left(-\frac{1}{\pi}\right) = 0)$$

$$\pi x = (-\pi, 20^\circ)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x^n \cdot \sin nx dx \quad \text{where } x = 20^\circ$$

$$= \frac{1}{\pi} \left[\frac{1}{n} \left(x^n - \frac{\cos nx}{n} \right) + \left(\frac{x^n \sin nx}{n} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\pi^n - \cos n\pi}{n} + \frac{2\pi \sin n\pi}{n^3} + 2 \right]$$

$$= \frac{1}{\pi} \left(\frac{\pi^n - \cos n\pi}{n} - (-\pi) \cdot \frac{-\cos n(-\pi)}{n^3} + 2 \right)$$

$$= \left(\frac{\pi^n - \cos n\pi}{n^3} + 2 \frac{\cos n\pi}{n^3} - 2 \frac{\cos n(-\pi)}{n^3} \right)$$

$$= \frac{1}{\pi} \left(-\frac{\pi \cos n\pi}{n^3} + \frac{\pi \cos n(-\pi)}{n^3} + \frac{2 \cos n\pi - 2 \cos n(-\pi)}{n^3} \right)$$

$$= \frac{1}{\pi} \left(-\frac{\pi \cos n\pi}{n^3} + \frac{\pi \cos n(-\pi)}{n^3} + \frac{4 \cos n\pi}{n^3} \right)$$

$$= -\frac{4(-1)^n}{n^3}$$

$$= 0$$

Eqn ① become

$$x^r = \frac{\pi b}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

$$x^r = \frac{a_0}{2} + \frac{a_n}{\pi^2} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi b}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

Example: Find the Fourier series of $f(x) = e^{ax}$

in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} f(x) (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} (e^{ax}) dx = \frac{1}{\pi} [e^{ax}]_0^{2\pi} = \frac{e^{2\pi a} - 1}{\pi}$$

$$= \frac{1}{\pi} \left(\frac{e^{2\pi a} - 1}{a} \right) = \frac{1}{\pi} \left(\frac{e^{2\pi a}}{a} - \frac{1}{a} \right)$$

$$= \frac{1}{\pi} \left(\frac{e^{2\pi a} - 1}{a} \right)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} e^{ax} \cos nx dx$$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \cos nx + n \sin nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi a}}{a^2 + n^2} (a \cdot 1) - \frac{1}{a^2 + n^2} (a) \right]$$

$$= \frac{1}{\pi} \cdot \frac{a}{a^2 + n^2} (e^{2\pi a} - 1)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$$

gmodad ①

$$= -\frac{1}{\pi} \int_0^{2\pi} e^{ax} \sin nx dx$$

$$= -\frac{1}{\pi} \left[\frac{e^{ax}}{a^2 + n^2} (a \sin nx - n \cos nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{2a\pi}}{a^2 + n^2} (0 - n) - \frac{1}{a^2 + n^2} (0) \right]$$

$$= -\frac{1}{\pi(a^2 + n^2)} (1 - e^{2a\pi})$$

(π/2, 0)

$$\therefore f(x) = \frac{1}{2\pi} \left(\frac{e^{2a\pi}}{a^2 + 1} \right) + \sum_{n=1}^{\infty} \left[\frac{a}{\pi(a^2 + n^2)} \left(\frac{e^{2a\pi}}{a^2 + n^2} - 1 \right) \right] \frac{\cos nx}{\pi}$$

$$= \left(\frac{1}{2\pi} - \frac{1 - e^{2a\pi}}{\pi(a^2 + 1)} \right) \frac{\cos nx}{\pi}$$

(Am)

$$= \left(\frac{1}{2\pi} - \frac{1 - e^{2a\pi}}{\pi(a^2 + 1)} \right) \frac{\cos nx}{\pi}$$

$$= \left(\frac{1}{2\pi} - \frac{(1 - e^{2a\pi})(a^2 + 1)}{\pi(a^2 + 1)^2} \right) \frac{\cos nx}{\pi}$$

$$= \left(\frac{1}{2\pi} - \frac{(1 - e^{2a\pi})(a^2 + 1)}{\pi(a^2 + 1)^2} \right) \frac{\cos nx}{\pi}$$

$$= \left(\frac{1}{2\pi} - \frac{(1 - e^{2a\pi})(a^2 + 1)}{\pi(a^2 + 1)^2} \right) \frac{\cos nx}{\pi}$$

$$\textcircled{11} \quad f(x) = e^{ax} \quad [2\pi \sin(0)] \quad \text{min}(x) \quad \left(\frac{1}{\pi} \right)^{\frac{2\pi}{a}} = 1$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} e^{ax} dx = \frac{1}{\pi} \left[\frac{e^{ax}}{a} \right]_0^{2\pi}$$

$$\left[(n-0) \frac{1}{\pi a^n} - (n-0) \frac{1}{\pi a^n} \right] = \frac{1}{\pi} \left(\frac{e^{2\pi a}}{a} - \frac{1}{a} \right)$$

$$\therefore a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos nx dx$$

$$= \frac{1}{\pi} \left[\int_0^{2\pi} (e^{ax} \cdot \cos nx) dx \right] \Big|_{n=1} + \left(\frac{1}{\pi} \right) \frac{1}{a} = (r)$$

$$= \frac{1}{\pi} \left[e^{ax} \frac{e^{anx} - 1}{a^2 + n^2} (a \cos nx + n \sin nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left(\frac{e^{2\pi a}}{a^2 + n^2} \cdot (a) - \frac{1}{a^2 + n^2} \cdot (a) \right)$$

$$= \frac{1}{\pi} \frac{e^{2\pi a} \cdot a}{a^2 + n^2} \left(e^{2\pi a} - 1 \right)$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{2\pi} e^{inx} \cdot \sin nx \, dx = \sum_{n=1}^{\infty} + \frac{b_n}{\pi} = (r) \\ &\left[\frac{1}{\pi} \int_0^{\pi} \frac{e^{inx}}{a+n} (a \sin nx - n \cos nx) \right]_0^{\pi} = 0 \\ &\left(\frac{1}{\pi} \int_0^{\pi} \frac{e^{inx}}{a+n} (0-n) - \frac{1}{a+n} (0-n) \right) \\ &= \pi(a+n) \left(1 - e^{-2\pi a} \right) \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \left(\frac{e^{2\pi a}}{a} \right) + \sum_{n=1}^{\infty} \left[\frac{a}{\pi(a+n)} \left(1 - e^{-2\pi a} \right) \cos nx \right. \\ &\quad \left. + \left(\frac{n}{a+n} \right) \left(1 - e^{-2\pi a} \right) \sin nx \right] \\ &\quad \left[(p) \cdot \frac{1}{\sqrt{n+0}} \rightarrow (p) \cdot \frac{1}{\sqrt{n+0}} \right] \cdot \frac{1}{\pi} = \\ &\quad \left(\frac{1}{T} \right) \cdot \frac{1}{\sqrt{n+0}} \cdot \frac{1}{\pi} = \end{aligned}$$

Ques: 2 Obtain the fourier series for $f(x) = e^x$ on the interval $0 < x < 2\pi$.

$$e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \cos nx \sin nx \right) \frac{1}{n\pi}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} e^x dx = \frac{1}{\pi} \left[-e^{-x} \right]_0^{2\pi} = \frac{1}{\pi} (1 - e^{-2\pi}) \\ &= \frac{1}{\pi} (1 - e^{-2\pi}) \end{aligned}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} e^x \cos nx dx = \frac{1}{\pi} (1 - e^{-2\pi})$$

$$= \frac{1}{\pi} \int_0^{2\pi} (1/e^x) \cdot \cos nx dx = \frac{1}{\pi} \left[\frac{e^{nx}}{1+n^2} (-\cos nx + n \sin nx) \right]$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (-1) - \frac{1}{1+n^2} (-1) \right] = \frac{1}{\pi} \left[\frac{(e^{2\pi}-1)}{1+n^2} \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{1+n^2} (1 - e^{-2\pi}) \right] = \frac{1}{\pi} \left[\frac{(e^{2\pi}-1)}{1+n^2} \right]$$

$$= \left(\frac{1}{\pi(e^{2\pi}-1)} (1 - e^{-2\pi}) \right) \frac{1}{\pi} = \left[\frac{(e^{2\pi}-1)}{\pi(e^{2\pi}-1)} \right] \frac{1}{\pi} = \frac{1}{\pi}$$

$$\begin{aligned} \therefore b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cdot \sin nx \, dx \quad (\text{for } x \in [0, 2\pi]) \\ &= \frac{1}{\pi} \left[\frac{e^{-x}}{1+n^2} (-\sin nx - n \cos nx) \right]_0^{2\pi} \\ &= \frac{1}{\pi} \left[\frac{e^{-2\pi}}{1+n^2} (-n) - \frac{1}{1+n^2} \cdot (-n) \right] \\ &= \frac{n}{\pi(1+n^2)} \cdot \left(1 - e^{-2\pi} \right) = \frac{n}{\pi} \cdot \left(1 - e^{-2\pi} \right) \end{aligned}$$

$$\therefore f(x) = \frac{1}{2\pi} \left(1 - e^{-2\pi} \right) + \sum_{n=1}^{\infty} \left[\frac{1}{\pi(1+n^2)} (1 - e^{-2\pi}) \cdot \cos nx \right]$$

(3) $f(x) = (1-x^2)$ where $-\pi < x < \pi$

$$\begin{aligned} \therefore a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (1-x^2) \, dx \\ &= \frac{1}{\pi} \left[(1-x^2) \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left[(1-2\pi) - (1+2\pi) \right] = \frac{1}{\pi} (1-2\pi - 1-2\pi) \\ &= \frac{1}{\pi} \times -4\pi = -4 \end{aligned}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - n\pi) dx$$

$$(\pi^2 - \pi + \pi^2 - \pi) \cdot \frac{C(1)}{\sqrt{n}}$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi}$$

$$(N^2 - N + N^2 - N) \cdot \frac{C(1)}{\sqrt{n}}$$

$$= \frac{1}{\pi} \left[\frac{\pi^2}{2} - \frac{\pi^3}{3} \right] - \frac{1}{\pi} \left[\frac{\pi^2}{2} + \frac{\pi^3}{3} \right]$$

$$= \frac{1}{\pi} \times \frac{-2\pi^3}{3} = \frac{1}{\pi} \left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3} \right] =$$

$$= \frac{1}{\pi} \times \frac{2\pi^3}{3} = \frac{-2\pi^3}{3} =$$

$$a_n = \left[\frac{m200}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cos nx dx - \frac{m200 - (n\pi - \pi)}{\pi} \right] \frac{1}{n}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi^2) \cos(n\pi) dx + \frac{\pi m200 - (\pi\pi - \pi)}{\pi} =$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} (x - \pi^2) \cdot \frac{\sin(n\pi)}{n} dx - (-2)^n \right] =$$

$$= \frac{1}{\pi} \left[\frac{(x - \pi^2) \cdot \sin(n\pi)}{n} - \frac{(1-2\pi) \cdot \cos(n\pi)}{n^2} + (-2)^n \cdot \frac{-\sin(n\pi)}{n^3} \right] =$$

$$= \frac{1}{\pi} \left[\frac{(-\pi - \pi^2) \cdot \sin(n\pi)}{n} - \frac{(1-2\pi) \cdot \cos(n\pi)}{n^2} + (-2)^n \cdot \frac{-\sin(n\pi)}{n^3} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-\pi - \pi^2) \cdot \sin(n\pi)}{n} + \frac{2\sin(n\pi)}{n^3} - \frac{2\sin(n\pi)}{n^3} \right]$$

$$= \frac{1}{\pi} \left[(1-2\pi) \cdot \frac{(-1)^n}{n^2} - (1+2\pi) \cdot \frac{(-1)^n}{n^2} \right]$$

$$= \frac{(-1)^n}{\pi n^2} (x - 2\pi - x - 2\pi)$$

$$= \frac{(-1)^n}{\pi n^2} (-4\pi)$$

$$= -4 \frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi) \sin nx dx = -\frac{\pi \cos n\pi}{n} \times \frac{1}{\pi}$$

$$= \frac{1}{\pi} \left[(\pi - \pi^2) \frac{-\cos n\pi}{n} - (1 + 2\pi) \frac{-\sin n\pi}{n^2} + (-2) \cdot \frac{\cos n\pi}{n^3} \right]$$

$$= \frac{1}{\pi} \left[(\pi - \pi^2) \frac{-\cos n\pi}{n} + (-\pi - \pi^2) \frac{\sin n\pi}{n^2} + (1 + 2\pi) \cdot \frac{\sin n\pi}{n^3} - (1 + 2\pi) \frac{-\sin(-n\pi)}{n^3} - \frac{2 \cos n\pi}{n^3} + \frac{2 \cos n(-\pi)}{n^3} \right]$$

$$\pi = \frac{1}{\pi} \left[\frac{n \sin 20^\circ}{\pi} \left(\frac{(-1)^n - 1}{n} + \frac{\cos n\pi}{n} \right) + (-\pi - \pi^2) \frac{\cos n\pi}{n^3} + (1 + 2\pi) \frac{\sin n\pi}{n^3} \right]$$

$$= \frac{1}{\pi} \left[\left(\frac{(-1)^n \sin 20^\circ}{n} + \frac{(1 + 2\pi) \sin n\pi}{n^3} \right) - \left(\frac{(-1)^n - 1}{n} + \frac{(-\pi - \pi^2) \cos n\pi}{n^3} \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{(-\pi + \pi^L)(-1)^n}{n} + \frac{(-\pi - \pi^L)(-1)^n}{n} \right] \quad \text{To solve diff. eqn. of } f(x) \text{ given above}$$

$$= \frac{(-1)^n}{n\pi} (-\pi + \pi^L - \pi - \pi^L)$$

$$= \frac{(-1)^n}{n\pi} (-2\pi) \quad \text{Top sw.}$$

$$\therefore = \frac{-2(-1)^n}{n\pi} \quad \text{rb. 1 term 203 (D)}$$

$$\therefore (x-x^L) = \frac{-4(-1)^n}{2n} + \sum$$

$$\therefore (x-x^L) = -\frac{\pi^L}{3} + \sum_{n=1}^{\infty} \left[\left(\frac{-4(-1)^n}{n\pi} \right) \cos nx + \frac{-2(-1)^n}{n\pi} \sin nx \right]$$

(ii) formula:-

$$A + \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (1)$$

Show that $\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$

$$(i) a_n = \frac{1}{L} \int_L^L f(x) \cos \frac{n\pi x}{L} dx$$

$$(ii) b_n = \frac{1}{L} \int_L^L f(x) \sin \frac{n\pi x}{L} dx$$

$$(iii) A = \frac{a_0}{2}$$

Proof: Multiplying both sides of $\int f(x) dx$ by $\frac{m\pi x}{L}$ and integrating from $-L$ to L , we get,

$$\int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx = \int_{-L}^L A \cdot \cos \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx + b_n \int_{-L}^L \sin \frac{n\pi x}{L} \cdot \cos \frac{m\pi x}{L} dx \right\}$$

$$\therefore \frac{a_m}{a_m} = \frac{a_m L}{\frac{1}{L} \int_{-L}^L f(x) \cdot \cos \frac{m\pi x}{L} dx} \text{ if } m=1, 2, 3, \dots$$

(b) Multiplying both sides of eqn ① by $\sin \frac{m\pi x}{L}$

$$\begin{aligned} & \int_{-L}^L f(x) \cdot \sin \frac{m\pi x}{L} dx = A \int_{-L}^L \sin \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-L}^L \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx + b_n \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \right\} \\ & = b_m L \end{aligned}$$

$$\therefore b_m = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx$$

proved

③ for A :

$$\int_{-L}^L f(x) \cdot 1 \cdot dx = \int_{-L}^L A \cdot 1 \cdot dx + a_0 + 0$$

$$= A \cdot (x) \Big|_{-L}^L$$

$$\left[0 - 0 \right] \frac{\pi}{\pi n} =$$

$$0 \times \frac{\pi}{\pi n} =$$

$$0 =$$

$$0 = a_0$$

$$\therefore \text{sub. } \frac{N\pi n}{a} = n^2 \frac{AL}{a} \left(1 + b \frac{N\pi n}{a} \cos(\theta) + \left(-\frac{1}{a} \right)^n \sin(\theta) \right)$$

$$A = \frac{1}{2L} \int_{-L}^L f(x) \cdot dx$$

$$\text{sub. } \frac{N\pi n}{a} \text{ in } \frac{AL}{a} \left(1 + b \frac{N\pi n}{a} \cos(\theta) + 0 \right) \frac{1}{a} =$$

$$A = -\frac{a_0}{2}$$

proved

$$\left[\frac{a_0}{\pi n} + \frac{N\pi n \cos 20^\circ}{a} \right] \frac{\pi}{\pi n} =$$

Expand the following function in fourier series:

$$f(x) = \begin{cases} 0 & ; -5 < x < 0 \\ 3 & ; 0 < x < 25 \end{cases}$$

$$\text{period } T = 10 \quad (10 - (-5)) = 15 \quad \frac{\pi}{\pi n} =$$

Solve:

$$T = 10, \quad 2L = 10, \quad \therefore L = 5$$

$$a_n = \frac{1}{5} \int_{-5}^0 f(x) \cdot \cos \left(\frac{n\pi x}{5} \right) dx + \int_0^5 f(x) \cdot \cos \left(\frac{n\pi x}{5} \right) dx$$

$$= \frac{1}{5} \left\{ 0 + \int_0^5 3 \cos \left(\frac{n\pi x}{5} \right) dx + 0 \right\} \frac{1}{5}$$

$$= -\frac{3}{5} \left[\frac{\sin \frac{n\pi x}{5}}{\frac{n\pi}{5}} \right]_0^5 \frac{1}{5}$$

$$= -\frac{3}{5} \times \frac{5}{n\pi} \left[\sin \frac{n\pi x}{5} \right]_0^5$$

$$= \frac{3}{n\pi} [0 - 0]$$

$$= \frac{3}{n\pi} \times 0$$

$$\therefore a_n = 0$$

$$\therefore b_n = \frac{1}{5} \int_0^5 f(x) \sin \frac{n\pi x}{5} dx + \int_{-5}^0 f(x) \sin \frac{n\pi x}{5} du$$

$$= \frac{1}{5} \left\{ 0 + \int_0^5 3 \sin \frac{n\pi u}{5} du \right\} = A$$

$$= \frac{3}{5} \left[\frac{-\cos n\pi x}{5} \Big|_0^5 \right] = A$$

$$= \frac{3}{5} \left[\frac{-\cos n\pi x}{5} \Big|_0^5 \right] = A$$

$$= \frac{3}{n\pi} (-(-1)^n + 1)$$

$$= \frac{3}{n\pi} (1 - (-1)^n)$$

$$x \in \left[-\frac{\pi}{5}, \frac{\pi}{5} \right]. (x) \int_0^0 f(x) dx + \int_0^5 f(x) dx = 0$$

$$a_0 = \frac{1}{5} \left\{ 0 + \frac{3}{5} \left[\frac{-\cos n\pi x}{5} \Big|_0^5 \right] \right\} = 0$$

$$= \frac{1}{5} \left[\frac{3}{5} \left[\frac{-\cos n\pi x}{5} \Big|_0^5 \right] \right] = 0$$

$$= \frac{1}{5} \times 5 \times 3 = 3$$

$$\begin{aligned}
 f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right\} \\
 &= \frac{3}{2} + \sum_{n=1}^{\infty} \left\{ 0 + \frac{3}{n\pi} (1 - (-1)^n) \sin \frac{n\pi x}{L} \right\} \\
 &= \frac{3}{2} + \sum_{n=1}^{\infty} \left[\frac{3}{n\pi} (1 - (-1)^n) \sin \frac{n\pi x}{L} \right] \quad (\text{Ans})
 \end{aligned}$$

25(a) Find the Fourier series of $f(x) = x$, $0 < x < 2\pi$

and sketch its graph from $x = 0$ to $x = 4\pi$.

Sol:

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} x dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{2\pi} = \frac{1}{\pi} \times \frac{(2\pi)^2}{2} = 2\pi$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} x \cos nx dx \\
 &= \frac{1}{\pi} \left[x \left[\frac{\sin nx}{n} \right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin nx}{n} dx \right] = \frac{1}{\pi} \left[x \frac{\sin nx}{n} \Big|_0^{2\pi} + \frac{\cos nx}{n^2} \Big|_0^{2\pi} \right] \\
 &= \frac{1}{\pi} \left[(0+1) - \frac{1}{n^2} (1-1) \right] = \frac{1}{\pi n^2} (1-1) = 0
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx \\
 &= \frac{1}{\pi} \left[x \left[-\frac{\cos nx}{n} \right]_0^{2\pi} + \frac{\sin nx}{n^2} \Big|_0^{2\pi} \right] = \frac{1}{\pi} \left[-\frac{1}{n} (1-1) + \frac{1}{n^2} (0-0) \right] = -\frac{1}{n\pi}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[-\frac{2\pi \cos 2n\pi}{n\pi + 1} + 0 + \frac{1}{n\pi} \right]_{n=1}^{\infty} + \frac{\epsilon}{2} = (r) \\
 &= \frac{1}{\pi} \left[\left(-\frac{2\pi}{n\pi + 1} + \frac{1}{n\pi} \right) \frac{\epsilon}{\pi n} + 0 \right]_{n=1}^{\infty} + \frac{\epsilon}{2} = \\
 &= \frac{1}{\pi} \times \frac{-2\pi + 1}{n} \\
 &= -\frac{2\pi}{\pi n} \left\{ (1) - 1 \right\} \frac{\epsilon}{\pi n} \Big|_{n=1}^{\infty} + \frac{\epsilon}{2} = \\
 &\left(\frac{1}{\pi} - \frac{1}{\pi} \right) \frac{1}{\pi} \times -\frac{2\pi}{n}
 \end{aligned}$$

\Rightarrow $\frac{2}{\pi n}$ \rightarrow (i) \Rightarrow to estimate remain of sum by

1. $\sum_{n=1}^{\infty} \left[0 \cdot \cos nx + \frac{2}{n\pi} \sin nx \right]$ denote bnd

$$\left| \frac{2}{n\pi} \sin nx \right| = \frac{2}{n\pi} \leq \frac{2}{\pi} \left(\frac{1}{n} \right) = \frac{1}{\pi} = o.b. (r)$$

2.6 Fourier Series of $f(x) \left[0 \leq x \leq 2\pi \right]$

if the period is 2π .

$$2b = 2\pi \Rightarrow b = \frac{1}{\pi} \left(\frac{1}{\pi} \times b \times 2\pi \right) = (r)$$

$$\begin{aligned}
 a_0 &= \frac{1}{\pi} \int_{0}^{2\pi} f(x) dx = \frac{1}{\pi} \int_{0}^{2\pi} x^3 dx = \frac{1}{\pi} \left[\frac{x^4}{4} \right]_{0}^{2\pi} = \\
 &= \frac{1}{\pi} \times \frac{8\pi^3}{3} = \frac{8}{3}\pi^2
 \end{aligned}$$

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{0}^{2\pi} x^3 \cos nx dx \\
 &= \frac{1}{\pi} \left[\frac{x^3}{n} \left(1 + 0 \right) \right]_{0}^{2\pi} = \frac{1}{\pi} \left[\frac{8}{n} \right] = \frac{8}{n\pi}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[\frac{x^3}{n} + \frac{1}{n} \right]_{0}^{2\pi} = \frac{1}{\pi} \left[\frac{8}{n} + \frac{1}{n} \right] = \frac{9}{n\pi}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\pi} \left[\cancel{\frac{4\pi^2 \sin 2n\pi}{n}} - 0 + \frac{4\pi}{n} \frac{\cos 2n\pi}{n^2} - 0 - \frac{2\sin 2n\pi}{n^3} \right] = \cancel{\left(\dots \right)} \frac{1}{n^3} \\
 &\quad \text{2nd part} \quad 8 > r > 1 \quad (r-1) + 0 \\
 &\approx \frac{1}{\pi} \left[\cancel{\frac{4\pi}{n}} \frac{\cos 2n\pi}{n^2} \right] \left[\text{rb.}(r) + \frac{8}{n} \right] + \text{rb.}(r) + \frac{1}{n} = 0 \\
 &= \frac{1}{\pi} \times \cancel{\frac{4\pi}{n}} \frac{1}{n^2} \\
 &= \frac{4}{n^3} \\
 \therefore b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} x^r \cdot \sin nx dx \\
 &= \frac{1}{\pi} \left[x^r \frac{-\cos nx}{n} - 2x \frac{-\sin nx}{n^2} + 2 \frac{\cos nx}{n^3} \right] \Big|_0^{2\pi} = 0 - (8 \cdot 0) \\
 &= \frac{1}{\pi} \left[-4\pi^2 \frac{\cos 2n\pi}{n} + 0 + 4\pi \frac{\sin 2n\pi}{n^2} + 0 + \frac{2\cos 2n\pi}{n^3} \right] = \\
 &\quad \text{rb. } \text{in } n^2 \text{ (r)} + \text{rb. } n^2 \text{ (r)} + \frac{2\cos 2n\pi}{n^3} = 0 \\
 &= \frac{1}{\pi} \left[-\frac{4\pi^2}{n} + \frac{2}{n^3} \right] = \frac{1}{\pi} \frac{1}{n^3} \left[-\frac{4\pi^2}{n^2} + \frac{2(r-2)}{n^3} \frac{2(+1)^r}{n^3} \right] \\
 &= -\frac{4\pi^2}{n^3}
 \end{aligned}$$

$$f(x) = \frac{4\pi^3}{n^3} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \cos nx + \frac{4\pi}{n} \sin nx \right)$$

Ans:

$$= \frac{1}{\pi} \left[4\pi \frac{\cancel{\sin 2n\pi}}{n} - 0 + 4\pi \frac{\cos 2n\pi}{n^2} - 0 - \frac{2\sin 2n\pi}{n^3} \right] = \left(\cancel{\frac{1}{n^3}} \right) + 0$$

8 > r > 1 < 2 + 8

$$= \frac{1}{\pi} \left[4\pi \frac{\cos 2n\pi}{n^2} \right] \left[\text{rb.}(r) + \frac{8}{\pi} \left(\text{rb.}(3) + \text{rb.}(5) \right) \right] \frac{1}{n^2} = 0$$

$$= \frac{1}{\pi} \times 4\pi \frac{1}{n^2}$$

$$= \frac{4}{n^2}$$

$$\therefore b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} x^r \cdot \sin nx dx$$

$$= \frac{1}{\pi} \left[x^r \frac{-\cos nx}{n} \Big|_0^{2\pi} - 2x \frac{\sin nx}{n} \Big|_0^{2\pi} + 2 \frac{\cos nx}{n^2} \Big|_0^{2\pi} \right] = 0 - (8 \cdot 8)$$

$$= \frac{1}{\pi} \left[-4\pi^r \frac{\cos 2n\pi}{n} + 0 + 4\pi \frac{\sin 2n\pi}{n^2} + 0 + \frac{2 \cdot \cos 2n\pi}{n^3} \right]$$

$$= \text{rb.}_m \text{nrB.}(r) + \text{rb.}_m \text{nrB.}(5) + \frac{2 \cdot \cos 2n\pi}{n^3}$$

$$= \frac{1}{\pi} \left[-\frac{4\pi^r}{n} + \frac{2}{n^3} \right] = \frac{1}{\pi^8} \frac{-\frac{4\pi^r}{n^3} + \frac{2}{n^3}}{\text{rb.}_m \text{nrB.}(2)} \frac{2(+)^r}{n^3}$$

$$= -\frac{4\pi^r}{n}$$

$$\therefore f(x) = \frac{4\pi^3}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx + \frac{4\pi}{n} \sin nx \right)$$

An:

Expand $f(x) = \begin{cases} 2-x; & 0 < x \leq 4 \\ x-6; & 4 < x \leq 8 \end{cases}$ in a Fourier series.

$$\therefore a_0 = \frac{1}{\pi} \left[\int_0^4 f(x) dx + \int_4^8 f(x) dx \right] = \frac{1}{\pi} \left[\int_0^4 (2-x) dx + \int_4^8 (x-6) dx \right] = \frac{1}{\pi} \left[\left[2x - \frac{x^2}{2} \right]_0^4 + \left[\frac{x^2}{2} - 6x \right]_4^8 \right] = \frac{1}{\pi} \left[(8-8) - 0 + (32-48) - (8-24) \right] = \frac{1}{\pi} (-16+16) = 0 + 0 + \frac{\pi n(20)}{n} = \frac{64}{2} = 32$$

$$\therefore a_n = \frac{1}{\pi} \left[\int_0^4 f(x) \cos nx dx + \int_4^8 f(x) \sin nx dx \right] = \frac{1}{\pi} \left[\int_0^4 (2-x) \cos nx dx + \int_4^8 (x-6) \sin nx dx \right] = \left[\frac{c}{n} + \frac{\pi n p}{n} \right] = \frac{\pi n p}{n}$$

$$\left(\sin nx \frac{\pi p}{n} + \cos nx \frac{p}{n} \right) \sum_{k=1}^{\infty} + \frac{\pi n p}{n} = (1)$$

$$\text{Q11) } f(x) = \begin{cases} 2-x & 0 < x < 4 \\ x-6 & 2 < x < 4 \end{cases} \quad \text{in period } [0, 8] = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$$L=4$$

$$\therefore a_0 = \frac{1}{\pi} \int_0^4 (2-x) dx + \int_2^4 (x-6) dx = \frac{1}{\pi} \left[\frac{1}{2}x^2 - x \right]_0^4 + \left[\frac{1}{2}x^2 - 6x \right]_2^4 = \dots$$

$$a_n = \frac{1}{\pi} \int_0^4 f(x) \cdot \cos nx dx = \frac{1}{\pi} \left[\int_0^4 f(x) \cdot \sin \frac{n\pi x}{4} dx \right] =$$

$$= \frac{1}{\pi} \int_0^4 (2-x) \cos \frac{n\pi x}{4} dx + \int_2^4 (x-6) \sin \frac{n\pi x}{4} dx =$$

$$= \frac{1}{\pi} \left[(0+) (2-x) \cdot \frac{\sin n\pi x}{n\pi} \cdot \frac{4}{n\pi} + 1 \cdot \frac{-\sin n\pi x}{n\pi} \cdot \frac{16}{n\pi} \right]_0^4$$

$$+ \left[(x-6) \cdot \frac{-\cos n\pi x}{n\pi} \cdot \frac{4}{n\pi} - (-1) \cdot \frac{-\sin n\pi x}{n\pi} \cdot \frac{16}{n\pi} \right]_2^4$$

$$= \frac{1}{\pi} \left[\frac{4(2-4)\sin 4n\pi}{n\pi} - \frac{4}{n\pi} \cdot \frac{\cos 4n\pi}{n\pi} \right]_0^4 + \left[\frac{-1(x-6)\cos n\pi x}{n\pi} \right]_2^4$$

$$= \frac{1}{\pi} \left[\frac{4(2-4)\sin 4n\pi}{n\pi} - \frac{4}{n\pi} \cdot \frac{\cos 4n\pi}{n\pi} \right]_0^4 - \frac{(0-20) - (4-20)}{n\pi}$$

$$\textcircled{1} \quad f(x) = \begin{cases} 0 & ; -5 \leq x \leq 0 \\ 3 & ; 0 \leq x \leq 5 \end{cases} \quad \begin{aligned} 2L &= 10 & x-2 \\ L &= 5 & & \therefore L = 5 \\ 2x &> 2 & x-2 & & \end{aligned}$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-5}^0 f(x) dx + \int_0^5 f(x) dx$$

$$= \frac{1}{\pi} \left[\int_{-5}^0 0 dx + \int_0^5 3 dx \right] \quad \text{[using } f(x) \text{]} \quad \left(\frac{1}{\pi} = 0 \right)$$

$$= \frac{1}{\pi} \left[\int_{-5}^0 3x dx \right] \quad \text{[using } f(x) \text{]} \quad \left(\frac{1}{\pi} = 0 \right)$$

$$P = \frac{1}{5} (15 - 0)$$

$$= \frac{3}{5} \cdot \frac{\pi \sin \frac{n\pi}{5}}{\pi n} \quad \left(\frac{1}{\pi} = 0 \right)$$

$$\therefore a_n = \frac{1}{5} \left[\int_{-5}^0 f(x) \cos \frac{n\pi x}{5} dx + \int_0^5 f(x) \sin \frac{n\pi x}{5} dx \right]$$

$$= \frac{1}{5} \left[0 + 3 \int_0^5 \sin \frac{n\pi x}{5} dx \right] \quad \left(\frac{1}{\pi} = 0 \right)$$

$$= \frac{1}{5} \left[3 \cdot \frac{\cos n\pi x}{n\pi} \Big|_0^5 \right] \quad \left(\frac{1}{\pi} = 0 \right)$$

$$= \frac{3}{5} \times \frac{5}{n\pi} \quad \left(\frac{1}{\pi} = 0 \right)$$

$$= \frac{3}{n\pi} \left(\cos \frac{n\pi}{5} - \cos 0 \right)$$

$$= \frac{3}{n\pi} (\cos n\pi - \cos 0)$$