Chain Rule

(i) $y = f[g(x)]$

$y' = f'(g(x))\, g'(x)$

Ex $y = 4\cos(x^3)$

$y' = -4\sin(x^3)(3x^2)$

$= -12x^2 \sin(x^3)$

(ii) $y = f(g(x))$ $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Let $g(x) = u$ $\quad y = f(u)$

$g'(x) = \dfrac{du}{dx}$ $\quad \dfrac{dy}{du} = f'(u)$

Ex $y = 4\cos(x^3)$ $\quad$ let

$u = x^3$

$y = 4\cos(u)$ $\quad \dfrac{du}{dx} = 3x^2$

$\dfrac{dy}{du} = -4\sin u$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= (-4\sin u)(3x^2)$

$= -4(3x^2)\sin u$

$= -12x^2 \sin(x^3).$

(vi) Let, $y = \cot^{-1}\left(\frac{1+x}{1-x}\right)$

Differentiating with respect to x,

$$\frac{dy}{dx} = \frac{-1}{1+\left(\frac{1+x}{1-x}\right)^2} \times \frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{-\cancel{(1-x)^2}^{\,1}}{1-\cancel{2x}+x^2+1+\cancel{2x}+x^2} \times \frac{(1-x)\cdot 1-(1+x)\cdot(-1)}{\cancel{(1-x)^2}}$$

$$= \frac{-1}{2+2x^2} \times \frac{1-\cancel{x}+1+\cancel{x}}{1}$$

$$= \frac{-1}{1+x^2} \quad \text{Ans}$$

OR

[Let, $x = \tan\theta \Rightarrow \theta = \tan^{-1}$

(viii) Let, $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\sin 2A = \frac{2A}{1+A^2}$

$y = \sin^{-1}(\sin 2x)$
$= 2x$

$\frac{dy}{dx} = 2$

Differentiating w.r.to. x,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \frac{4x^2}{(1+x^2)^2}}} \cdot \frac{(1+x^2)\cdot 2 - 2x \cdot 2x}{(1+x^2)^2}$$

$$= \frac{(1+x^2)}{\sqrt{1+2x^2+x^4-4x^2}} \cdot \frac{2+2x^2-4x^2}{(1+x^2)^2}$$

$$= \frac{1}{\sqrt{1-2x^2+x^4}} \cdot \frac{2-2x^2}{1+x^2}$$

$$= \frac{1}{\sqrt{(1-x^2)^2}} \cdot \frac{2(1-x^2)}{1+x^2}$$

$$= \frac{2}{1+x^2} \quad \text{Ans}$$

(ix) Let, $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{2x}{1-x^2}\right)^2} \cdot \frac{(1-x^2)\cdot 2 - 2x(-2x)}{(1-x^2)^2}$$

$$= \frac{(1-\cancel{x^2})^2}{1-2x^2+x^4+4x^2} \cdot \frac{2-2x^2+4x^2}{(1-\cancel{x^2})^2}$$

$$= \frac{2+2x^2}{1+2x^2+x^4}$$

$$= \frac{2(1+x^2)}{(1+x^2)^2}$$

$$= \frac{2}{(1+x^2)}$$ Ans

let, $x = \tan\theta$

$\therefore \frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \cancel{2\tan 2\theta}$

(x) Let, $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Differentiating w. r. to x,

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2} \cdot \frac{\sqrt{1-x^2}\cdot 1 - x\cdot\frac{1}{2\sqrt{1-}}}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{(1-\cancel{x^2})}{1-\cancel{x^2}+\cancel{x^2}} \cdot \frac{\sqrt{1-x^2}+\frac{x^2}{\sqrt{1-x^2}}}{(1-\cancel{x^2})}$$

(i) Let, $y = (\sin x)^{\ln x}$

Taking ln on both sides,

$$\ln y = \ln(\sin x)^{\ln x}$$

$$= \ln x \cdot \ln(\sin x)$$

Differentiating w. r. to x,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x \cdot \frac{1}{\sin x} \cdot \cos x + \frac{1}{x} \ln(\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y \left[\ln x \cdot \cot x + \frac{1}{x} \ln(\sin x) \right]$$

$$= (\sin x)^{\ln x} \left[\ln x \cdot \cot x + \frac{1}{x} \ln(\sin x) \right]$$

Ans

find $\frac{dy}{dx}$

2\. (ii) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Now, let, $u = (\sin x)^{\cos x}$ — (i) and $v = (\cos x)^{\sin x}$ — (ii)

Then, $y = u + v$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{—— (iii)}$$

Taking ln on both sides of (i)

$$\ln u = \ln (\sin x)^{\cos x}$$

$$= \cos x \ln(\sin x)$$

$\boxed{e^{\ln x} = x}$

$$\therefore \frac{1}{u} \cdot \frac{du}{dx} = \cos x \cdot \frac{1}{\sin x} \cdot \cos x - \sin x \cdot \ln(\sin x)$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^{\cos x} \left[\cos x \cdot \cot x - \sin x (\ln \sin x) \right]$$

Now, Taking ln on both sides of (ii)

$$\ln v = \ln (\cos x)^{\sin x} = \sin x \ln(\cos x)$$

$$\therefore \frac{1}{v} \cdot \frac{dv}{dx} = \sin x \cdot \frac{1}{\cos x} \cdot (-\sin x) + \cos x \cdot \ln(\cos x)$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x} \left[-\sin x \cdot \tan x + \cos x \cdot \ln \cos x \right]$$

$$\therefore \text{(iii)} \Rightarrow, \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\sin x)^{\cos x} \left(\cos x \cot x - \sin x \ln(\sin x) \right)$$

3 (i) Let, ~~y =~~

$$3x^4 - x^2y + 2y^3 = 0$$

Differentiating with respect to x,

$$12x^3 - 2xy - x^2 \cdot \frac{dy}{dx} + 6y^2 \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(-x^2 + 6y^2) = 2xy - 12x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(y - 6x^2)}{6y^2 - x^2}$$ Ans

3. (ii) $x^3+y^3+4x^2y-25=0$

Differentiating w.r.to x,

$3x^2+3y^2\cdot\frac{dy}{dx}+4x^2\frac{dy}{dx}+8xy=0$

$\Rightarrow \frac{dy}{dx}(4x^2+3y^2)=-(8xy+3x^2)$

$\therefore \frac{dy}{dx}=\frac{-x(8y+3x)}{4x^2+3y^2}$ Ans

Find $\frac{dy}{dx}$

8. (iii) $x^y=y^x$ Implicit

Taking ln on both sindes

$\ln x^y=\ln y^x$

$\Rightarrow y\ln x=x\ln y$

Differentiating w.r. to x,

$\frac{1}{x}y+\ln x\cdot\frac{dy}{dx}=x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\ln y$

$\Rightarrow \frac{dy}{dx}\left(\ln x-\frac{x}{y}\right)=\ln y-\frac{y}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{\ln y-y/x}{\ln x-x/y}=\frac{y(x\ln y-y)}{x(y\ln x-x)}$ Ans

(i) $x = a\cos^3\theta$, $y = a\sin^3\theta$ → find $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}}$$ using chain rule

Now, $x = a\cos^3\theta$

$$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta \cdot (-\sin\theta)$$

$$= -3a\cos^2\theta \cdot \sin\theta$$

Again, $y = a\sin^3\theta$

$$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta \cdot \cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \cancel{3}\cancel{a}\sin^2\theta \cdot \cos\theta \cdot \frac{1}{-\cancel{3}\cancel{a}\cos^2\theta \cdot \sin\theta}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$$= -\tan\theta$$ Ans

$x = \sin^2\theta$, $y = \tan\theta$

$\frac{1}{\cos\theta} = \sec\theta$

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$$\therefore \frac{dx}{d\theta} = 2\sin\theta \cdot \cos\theta, \quad \frac{dy}{d\theta} = \sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \sec^2\theta \cdot \frac{1}{2\sin\theta \cdot \cos\theta}$$

(i) $x = a\cos^3\theta$, $y = a\sin^3\theta$ → find $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}}$$ using chain rule

Now, $x = a\cos^3\theta$

$$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta\,.(-\sin\theta)$$

$$= -3a\cos^2\theta\,.\sin\theta$$

Again, $y = a\sin^3\theta$

$$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta\,.\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= \cancel{3a}\sin^2\theta\,.\cos\theta\,.\frac{1}{-\cancel{3a}\cos^2\theta\,.\sin\theta}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$$= -\tan\theta$$ Ans

$x = \sin^2\theta$, $y = \tan\theta$

$\frac{1}{\cos\theta} = \sec\theta$

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$$\therefore \frac{dx}{d\theta} = 2\sin\theta\,.\cos\theta\,,\quad \frac{dy}{d\theta} = \sec^2\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \sec^2\theta\cdot\frac{1}{2\sin\theta\cdot\cos\theta}$$

(ii) $x^{\sin^{-1}x}$ w.r.to $\sin^{-1}x$

$y \qquad\qquad\qquad z$

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz}$$

let, $y = x^{\sin^{-1}x}$ — (I) and $z = \sin^{-1}x$ — (II)

Taking ln on both sides of (I)

$$\ln y = \sin^{-1}x \ln x$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \sin^{-1}x \cdot \frac{1}{x} + \frac{1}{\sqrt{1-x^2}} \ln x$$

$$\Rightarrow \frac{dy}{dx} = y \left[\frac{1}{x} \sin^{-1}x + \frac{1}{\sqrt{1-x^2}} \ln x \right]$$

$$= x^{\sin^{-1}x} \left[\frac{1}{x} \sin^{-1}x + \frac{1}{\sqrt{1-x^2}} \ln x \right]$$

Differentiating (II) w.r.to x,

$$\frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz}$$

$$= x^{\sin^{-1}x} \left(\frac{1}{x} \sin^{-1}x + \frac{1}{\sqrt{1-x^2}} \ln x \right) \cdot \sqrt{1-x^2}$$

Ans

(i) Let, $y = \ln\sqrt{\frac{1+\sin x}{1-\sin x}} = \ln\left(\frac{1+\sin x}{1-\sin x}\right)^{1/2}$

$= \frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)$

Differentiating with respect to x,

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\frac{1+\sin x}{1-\sin x}}\cdot\frac{(1-\sin x)\cdot\cos x-(1+\sin x)(-\cos x)}{(1-\sin x)^2}$$

$$= \frac{1}{2}\cdot\frac{1-\sin x}{1+\sin x}\cdot\frac{\cos x-\cancel{\sin x\cos x}+\cos x+\cancel{\cos x\sin x}}{(1-\sin x)^2}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{1-\sin x}{1+\sin x}\cdot\frac{\cancel{2}\cos x}{(1-\sin x)^2}$$

$$= \frac{\cos x}{(1+\sin x)(1-\sin x)}$$

$$= \frac{\cos x}{1-\sin^2 x}$$

$$= \frac{\cos x}{\cos^2 x}$$

$$= \frac{1}{\cos x}$$

$= \sec x$ (Ans)

6(i)

Let $y = \ln\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)} = \ln\left(\frac{1-\cos x}{1+\cos x}\right)^{1/2}$

(ii)

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right)$$

Differentiating w.r. to x,

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\frac{1-\cos x}{1+\cos x}}\cdot\frac{(1+\cos x)\cdot(\sin x)-(1-\cos x)\cdot\overset{(-\sin x)}{\cancel{1}}}{(1+\cos x)^2}$$

$$= \frac{1}{2}\cdot\frac{1+\cos x}{1-\cos x}\cdot\frac{\sin x+\cancel{\sin x\cos x}+\sin x-\cancel{\sin x\cos x}}{(1+\cos x)^2}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{1}{1-\cos x}\cdot\frac{\cancel{2}\sin x}{1+\cos x}$$

$$= \frac{\sin x}{1-\cos^2 x}$$

$$= \frac{\sin x}{\sin^2 x}$$

$$= \frac{1}{\sin x}$$

$$= \operatorname{cosec} x.$$

6(i)

ii)

Let $y = \ln\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)} = \ln\left(\frac{1-\cos x}{1+\cos x}\right)^{1/2}$

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right)$$

Differentiating w.r.to.x ,

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\frac{1-\cos x}{1+\cos x}}\cdot\frac{(1+\cos x)\cdot(\sin x)-(1-\cos x)\cdot\overset{(-\sin x)}{1}}{(1+\cos x)^2}$$

$$= \frac{1}{2}\cdot\frac{1+\cos x}{1-\cos x}\cdot\frac{\sin x+\cancel{\sin x\cos x}+\sin x-\cancel{\sin x\cos x}}{(1+\cos x)^2}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{1}{1-\cos x}\cdot\frac{\cancel{2}\sin x}{1+\cos x}$$

$$= \frac{\sin x}{1-\cos^2 x}$$

$$= \frac{\sin x}{\sin^2 x}$$

$$= \frac{1}{\sin x}$$

$$= \operatorname{cosec} x .$$

9 Mat 110

Successive differentiation, Taylor's Theorem & Maclaurin's Theorem in finite & infinite form

Successive Differentiation

Define the n^{th} derivative of the given variable

$y = 3x^4$

$\frac{dy}{dx} = 12x^3$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 36x^2$

$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = 72x$ etc. / $n = 5$ max

The symbols for the successive derivatives are usually as follows

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

$\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$

$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}$

If $y = f(x)$, the successive derrivatives are

$f'(x), f''(x), f'''(x), f^{iv}(x), \ldots, f^{(n)}(x).$

$y', y'', y''', y^{iv}, \ldots, y^{(n)}$

$y_1, y_2, y_3, y_4, \ldots, y_n$

or $\frac{d}{dx}f(x), \frac{d^2}{dx^2}f(x), \frac{d^3}{dx^3}f(x), \frac{d^4}{dx^4}f(x), \ldots, \frac{d^n}{dx^n}f(x)$

Successive Differentiation Exercise

Find the n^{th} derivative of the following functions.

1. $y = x^n$

$y' = nx^{n-1}$

$y^2 = n(n-1)x^{n-2}$

$y^3 = n(n-1)(n-2)x^{n-3}$

$y^4 = n(n-1)(n-2)(n-3)x^{n-4}$

$y^n = n(n-1)(n-2)(n-3)\cdots(n-(n-1))x^{n-n} = n(n-1)(n-2)(n-3)\cdots(1)x^0$

$= n(n-1)(n-2)(n-3)\cdots 1.$

2. $y = (ax+b)^n$

$y' = n(ax+b)^{n-1}(a) = an(ax+b)^{n-1}$

$y'' = an(n-1)(ax+b)^{n-2}(a) = a^2n(n-1)(ax+b)^{n-2}$

$y''' = a^3n(n-1)(n-2)(ax+b)^{n-3}$

$y^{iv} = a^4 n(n-1)(n-2)(n-3)(ax+b)^{n-4}$

$y^{(n)} = a^n n(n-1)(n-2)\cdots\cdots(n-(n-1))(ax+b)^{n-n}$

$= a^n n(n-1)(n-2)\cdots 1 = a^n n!$

3. $y = \log_e(ax+b)$

$y = \ln(ax+b)$

$y' = \frac{a}{ax+b} = a(ax+b)^{-1} = a^1\, 0!\,(ax+b)^{-1} = (-1)^0\, 0!\, a^1 (ax+b)^{-1}$

$y'' = +a(-1)(ax+b)^{-2}(a) = -a^2(ax+b)^{-2} = -1!\,a^2(ax+b)^{-2} = (-1)^1\, 1!\, a^2(ax+b)^{-2}$

$y''' = 2a^2(ax+b)^{-3}(a) = 2a^3(ax+b)^{-3} = 2!\,a^3(ax+b)^{-3} = (-1)^2\, 2!\, a^3(ax+b)^{-3}$

$y^{(iv)} = -6a^4(ax+b)^{-4} = -3!\,a^4(ax+b)^{-4} = (-1)^3\, 3!\, a^4(ax+b)^{-4}$

$y^{(5)} = 24a^5(ax+b)^{-5} = 4!\, a^5(ax+b)^{-5} = (-1)^4\, 4!\, a^5(ax+b)^{-5}$

$y^{(n)} = (-1)(-2)(-3)(-4)\cdots(-(n-1))\cdot a^n(ax+b)^{-n} = (-1)^{n-1}(n-1)!\, a^n(ax+b)^{-n}$

4. $y = \frac{1}{x+a} = (x+a)^{-1}$

$y' = -(x+a)^{-2} = (-1)^1\, 1\,(x+a)^{-2}$

$y'' = +2(x+a)^{-3} = (-1)^2\, 1.2\,(x+a)^{-3}$

$y''' = -6(x+a)^{-4} = (-1)^3\, 1.2.3\,(x+a)^{-4}$

$y^{iv} = 24(x+a)^{-5} = (-1)^4\, 1.2.3.4\,(x+a)^{-5}$

$y^{n} = (-1)^n\, 1.2.3.\cdots n\,(x+a)^{-(n+1)}$

$= (-1)^n\, n!\,(x+a)^{-(n+1)}$