Lecture 57:

Increasing, Decreasing and Concavity

Previously we discussed the behaviour of a function at a point. Now we want to discuss behaviour of a function on an interval. The definition of 'increasing', 'decreasing', and 'constant' describe the behaviour of a function on an interval and NOT at a point.

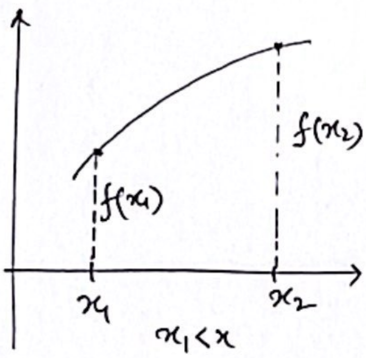
Definition:

Let f be defined on an interval, and let x_1 and x_2 denote two points in that interval.

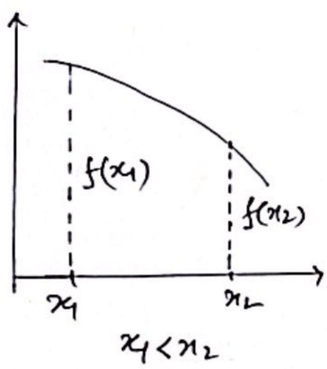
1. f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

2. f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

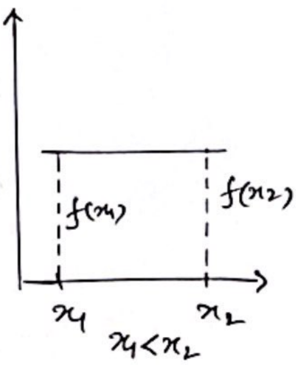
3. f is constant on the interval if $f(x_1) = f(x_2)$ whenever $x_1 < x_2$.



$f(x_1) < f(x_2)$



$f(x_1) > f(x_2)$



$f(x_1) = f(x_2)$

__Theorem:__ Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on (a,b).

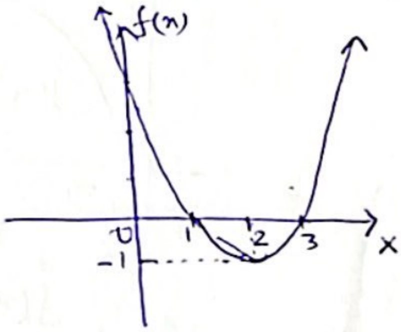
ⓐ If $f'(x) > 0$ for every value of x in (a,b), then f is increasing on $[a,b]$.

ⓑ If $f'(x) < 0$ for every value of x in (a,b), then f is decreasing on $[a,b]$.

ⓒ If $f'(x) = 0$ for every value of x in (a,b), then f is constant on $[a,b]$.

__Example:__ Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.

__Sol^n__



$$f(x) = x^2 - 4x + 3$$

$$f'(x) = 2x - 4 = 2(x-2)$$

$f'(x) < 0$ if x is decreasing

$$2(x-2) < 0 \Rightarrow x - 2 < 0 \Rightarrow x < 2$$

Since f is continuous, by the above theorem we say that f is decreasing on $(-\infty, 2]$.

Similarly, $f'(x) > 0$ if x is increasing

$x > 2$

i.e. f is increasing on $[2, \infty)$.

Concavity

Definition:

If f is differentiable on an open interval, then f is said to be concave up on the open interval if f' is increasing on that interal, and f is said to be concave down on the open interval if f' is decreasing on that interval. $\square$

Theorem:

Let f be twice differentiable on an open interval (a,b).

(a) If $f''(x) > 0 \quad \forall x \in (a,b)$, then f is concave up on (a,b).

(b) If $f''(x) < 0 \quad \forall x \in (a,b)$, then f is concave down on (a,b).

Example: Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing and the intervals on which it is decreasing.

Sol^n: $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

$x^2 + 2x - x - 2$

$x(x+2) - 1(x+2)$

$$f'(x) = 12x^3 + 12x^2 - 24x = 0$$

$$x^3 + x^2 - 2x = 0$$

$$x(x^2 + x - 2) = 0$$

$$x(x+2)(x-1) = 0$$

$$x = 0, \quad x = -2, \quad x = 1$$

$x<2$

$-2 \quad 0 \quad 1$

Now,

Interval	$x(x+2)(x-1)$	$f'(x)$	Conclusion
$x < -2$	$(-)(-)(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	f is decreasing on $[0, 1]$
$x > 1$	$(+)(+)(+)$	$+$	f is increasing on $[1, +\infty)$

$$f''(x) = 36x^2 + 24x - 24 = 0 \quad \text{(make intervals)}$$

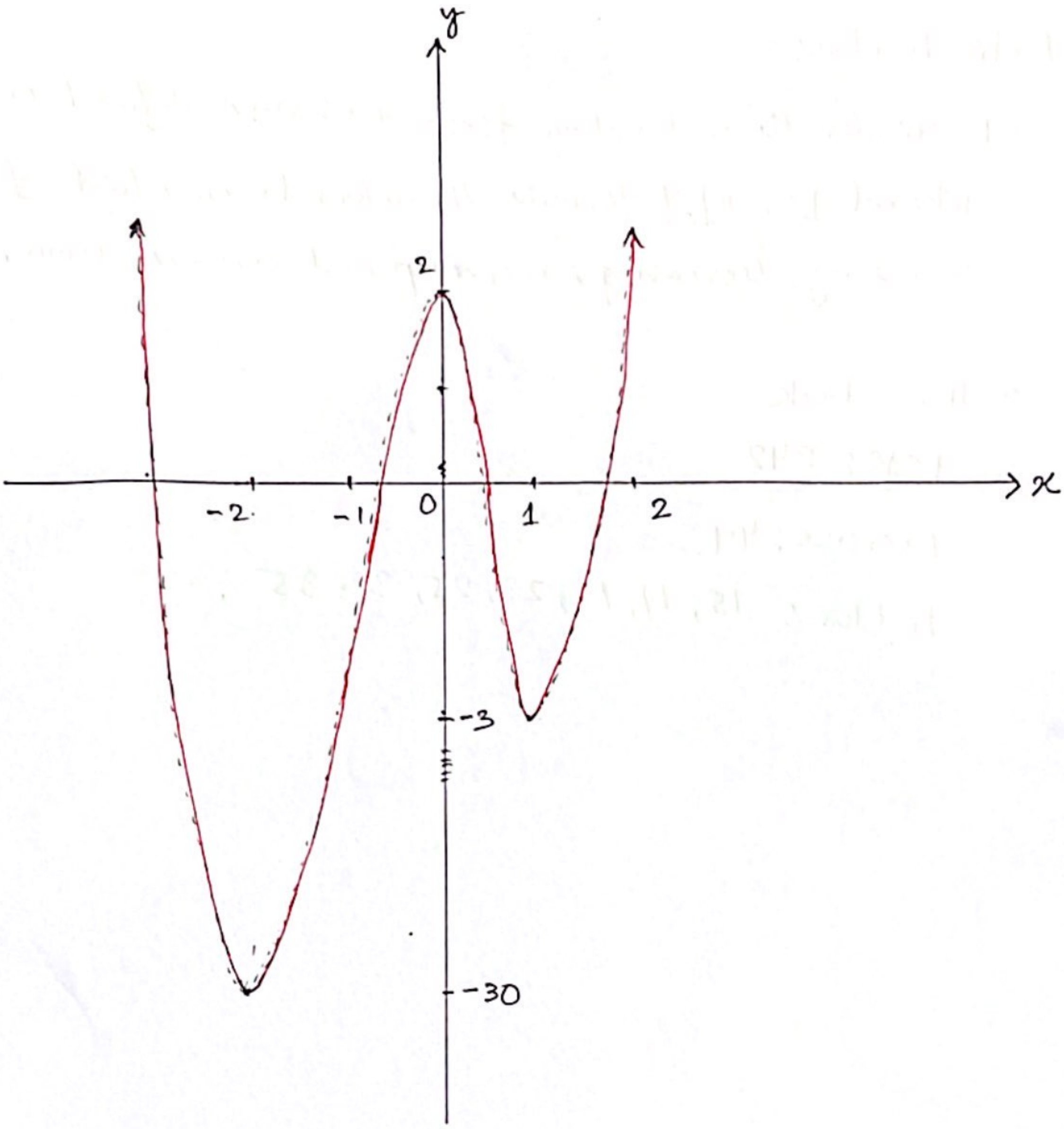
3 $\qquad$ 3

$\overset{\text{''}}{0.5485}$

$$3x^2 + 2x - 2 = 0 \qquad x = \frac{-1+\sqrt{7}}{3}, \ \frac{-1-\sqrt{7}}{3} = -1.215$$

Now,

Interval	$x(3x+2) - 2$	$f''(x)$	Conclusion
$x < \frac{-1-\sqrt{7}}{3}$		$+$	f is concave up on $\left(-\infty, \frac{-1-\sqrt{7}}{3}\right)$
$\frac{-1-\sqrt{7}}{3} < x < \frac{-1+\sqrt{7}}{3}$		$-$	f is concave down on $\left(\frac{-1-\sqrt{7}}{3}, \frac{-1+\sqrt{7}}{3}\right)$
$x > \frac{-1+\sqrt{7}}{3}$		$+$	f is concave up on $\left(\frac{-1+\sqrt{7}}{3}, +\infty\right)$



y
x
2
-2
-1
0
1
2
-3
-30

Extra Problems:

1. Consider ~~the~~ a function $f(x) = x + 2\sin x$ defined on the interval $[0, 2\pi]$. Determine the intervals on which f is increasing, decreasing, concave up and concave down.

2. From Book

   Page: 242

   Exercise: 4.1

   Problem: 15, 17, 19, 23, 25, 33, 35.