Differential Equations

Differential Equation: An Differential equation containing the derivatives of one or more dependent variables with respecte to one or more independent variables, is said to be a Differential Equation (DE)

For example

1. $\frac{dy}{dx} + 5y = e^x$

2. $\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 3y = 0$

3. $\frac{dx}{dt} + \frac{dy}{dt} = 2x + e^t$

4. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 5y$

5. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

Ordinary Differential Equations: If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single indepen-dent variable, then it is called an ordinary Differential Equations. In above examples 1, 2 and 3 are ordinary differential equations.

Partial Differential Equations: An Equation involving partial derivatives of one or more dependent variables with respect to two or more independent variables is called partial Differential Equation.

For example.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial x^2} = 0$$

Differential Operator D

$$\frac{d^n y}{dx^n} = D^n y$$

$$\frac{d}{dx} = D, \quad \frac{d^2}{\partial x^2} = D^2 \ldots\ldots \quad \frac{d^n}{dx^n} = D^n$$

Prime notation for differential operator

$$y' = \frac{dy}{dx}, \quad y'' = \frac{d^2 y}{dt^2}, \quad |$$

$$Dy = \frac{dy}{dx}, \quad D^2 y = \frac{d^2 y}{dt^2}$$

$$\frac{d^3y}{dx^3} + 2\frac{dy}{dx} = e^y$$

$$D^3y + 2Dy = e^y$$

$$y''' + 2y' = e^y$$

<u>Order of Differential Equation</u> : The order of a differential equation is the order of the highest derivative in the equation.

<u>Degree of Differential Equation</u> : The degree of a differential equation is the power of the Highest Derivative

- $\left(\frac{dy}{dx}\right)^3 + y^4 = \sin x$ → Order 1, Degree 3

- $\left(\frac{d^2u}{dt^2}\right)^3 + \left(\frac{du}{dt}\right)^4 = e^{\sin x}$ → Degree 03, Order 02

- $D^2y + (Dy)^4 + e^y = 7$

  ↳ $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + e^y = 7$ so degree 01, order 02

- $y''' + 3y'' + (7y')^2 = 2$

  ↳ $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \left(7\frac{dy}{dx}\right)^2 = 2$ so degree 1, order 03 n-th order & first degree differential equation given as (General Equation)

$$a_n(x,y)\frac{d^n y}{dx^n} + a_{n-1}(x,y)\frac{d^{n-1} y}{dx^{n-1}} + a_{n-2}(x,y)\frac{d^{n-2} y}{dx^{n-2}} + \cdots$$

$$\cdots + a_1(x,y)\frac{dy}{dx} + a_0(x,y)y = g(x) \quad \text{———} (*)$$

Viz : $y\frac{d^4y}{dx^4} + \sin x\frac{dy}{dx} = e^y$ is a fourth order DE

First Degree & First Order Differential Equation:

General form of first order and first degree Differential Equation.

$$a_1(x,y)\frac{dy}{dx} + a_0(x,y)y = g(x)$$

For example

- $\sin y\frac{dy}{dx} + e^x y = 0$
- $(x+y)\frac{dy}{dx} = \ln y$
- $\frac{dy}{dx} = 0$

· Linear Differential Equation: A DE is said to be linear if it satisfied of following

1) Dependent variable y and its derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ..., $\frac{d^ny}{dx^n}$ all are of first degree

2) No product of y and it derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots$
$\dots \frac{d^ny}{dx^n}$.

3) No non-linear function of y ($\sin y, \cos y, \ln y, e^y \dots$) are present.

For example:

1.) $$x\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = 2$$

It is a DE of order 3 and degree 1 but not Linear as $\left(\frac{dy}{dx}\right)^2$ that is exist.

2) $$x\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{dy}{dx} + y^2 = 0$$

not-linear as y^2 exist.

3. $$2\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 5$$

not-linear as product of y and it derivatives exist.

4.) $$\frac{d^3y}{dx^3} - y\sin x = e^y$$

not-linear as non-linear function of dependent variable exist.

5.) $\frac{d^4y}{dx^4} + \sin x\frac{dy}{dx} = x^2$ is a linear fourth order DE.

Homogeneous Linear DE

A linear DE is said to be homogeneous if $g(x)=0$ [Each term of the equation contains either y or its derivatives]. If $g(x) \neq 0$ then it is non-homogeneous

For example.

1.) $\frac{d^2y}{dx^2} + 2y - 2x = 0$

$\hookrightarrow \frac{d^2y}{dx^2} + 2y = 2x$

$\hookrightarrow$ non-homogeneous Linear DE of order 2

2.) $\frac{d^3y}{dx^3} + y\sin x = 0$

$\hookrightarrow$ homogeneous Linear DE of order 3.

3) $\frac{dy}{dx} + 2xy = 0$

$\hookrightarrow$ Homogeneous Linear DE of order 1.

4) $\frac{dy}{dx} = x$

$\hookrightarrow$ Non-homogeneous linear DE of order 1

That is non-homogeneous if there exist a term containing only independent variable x

Solution of DE

A function $y=f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation. That is solution of a DE is a function.

For example, $y=x^2+2$ is a solution of $\frac{dy}{dx}-2x=0$

Verifying Solution:

Given $y=x^2+2$

$$\frac{dy}{dx}=2x$$

Then Left side of DE

$$\frac{dy}{dx}-2x = 2x-2x = 0$$

$= $ Right side of DE

$\because$ $y=x^2+2$ satisfy the DE $y'-2x=0$ so $y=x^2+2$ is a solution of that DE.

Also, $y=x^2+9$ is a solution of the DE (verify it)

Example: Show that $y=e^x$ is a solution of

$$\frac{d^2y}{dx^2}-\frac{dy}{dx}=0$$

Solution: Given $y=e^x \Rightarrow \frac{dy}{dx}=e^x, \ \frac{d^2y}{dx^2}=e^x$

LHS: $\frac{d^2y}{dx^2}-\frac{dy}{dx} = e^x-e^x = 0 = $ RHS (showed)

Example: Verify that, the function $y = e^{3x}\cos 2x$ is a solution of the DE $y'' - 6y' + 13y = 0$

Solution:

We have

$$y = e^{3x}\cos 2x$$

$$y' = -2e^{3x}\sin 2x + 3e^{3x}\cos 2x$$

$$y'' = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$$

So. $L.H.S = y'' - 6y' + 13y$

$$= (5e^{3x}\cos 2x - 12e^{3x}\sin 2x) - 6(-2e^{3x}\sin 2x + 3e^{3x}\cos 2x) + 13(e^{3x}\cos 2x)$$

$$= 0 = RHS$$

$\therefore$ $y = e^{3x}\cos 2x$ is a solution of $y'' - 6y' + 13y = 0$

Example: Show that $y = c\ln x$ is a ~~constant~~ solution of of $\ln x \frac{dy}{dx} - y/x = 0$.

Try by yourself.

Solving Differential Equation's (First Order)

- Separable variables
- Linear DE, Integrating factor
- Exact DE,
- A non-exact DE made exact

2.2 Separable Varibles: A first order DE of the form

$$\frac{dy}{dx} = g(x) \cdot f(y)$$

is said to be seperable.

For example:

\# $\frac{dy}{dx} = x + \sin y$

↳ not seprable

\# $\frac{dy}{dx} = e^x \sin x \, e^{x+y} \cdot y$

↳ separable

as $\frac{dy}{dx} = e^x \sin x \cdot e^x \cdot e^y y$

$= (e^{2x} \sin x)(y e^y)$

$\therefore \frac{dy}{dx} = g(x) f(y)$

So. For solving a separable DE

$$\frac{1}{f(y)} dy = g(x)\, dx$$

$$\Rightarrow \int \frac{1}{f(y)} dy = \int g(x)\, dx \quad \text{[integrating}$$

Solving a separable DE

Example: solve $(1+x)\,dy - y\,dx = 0$

Solution: Given

$$(1+x)\,dy - y\,dx = 0$$

$$\Rightarrow (1+x)\,dy = y\,dx$$

$$\Rightarrow \frac{1}{y}\,dy = \frac{1}{1+x}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx \quad \text{[integrating]}$$

$$\Rightarrow \ln|y| = \ln|1+x| + \ln c \quad [\ln c \text{ arbitrary constant}]$$

$$\Rightarrow y = (1+x) + c$$

Example $\dfrac{dy}{dx} = -\dfrac{x}{y}$

Solution

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = -\int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow x^2 + y^2 = 2c$$

which is required solution.

$$\Rightarrow \quad \frac{dy}{dx} = e^{x} \cdot e^{y}$$

$$\Rightarrow \quad e^{-y}\, dy = e^{x}\, dx$$

$$\Rightarrow \quad \int e^{-y}\, dy = \int e^{x}\, dx$$

$$\Rightarrow \quad -e^{-y} = e^{x} + c$$

$\therefore \quad e^{x} + e^{-y} + c = 0$ is a general solution of ———(1)

given DE

Now from initial condition $y(0) = 1$ that is

when $x=0$ then $y = 1$

From (1)

$$e^{0} - e^{-1} + c = 0$$

$$\therefore \quad c = \frac{1-e}{e}$$

Substituting this values of c in (1)

$e^{x} + e^{-y} + \frac{1-e}{e} = 0$ is the particular solution

of given DE.

Initial value problem (IVP)

An initial value problem (IVP) is an differential equation together with an initial condition which specifies the value of unknown function at a given point in the domain.

For example.

1) $\frac{dy}{dx} = 2xy + 2$ with $y(0) = 2 \longrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$

2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + e^y = 0$ with $y(1) = 5$

$y'(1) = 3$

Example : ~~Solve the initial value problem~~

General solution: & **particular solution:**

A solution having arbitrary constant

Have no any arbitrary constant

Example: Solve the initial value problem

$$\frac{dy}{dx} = e^{x+y} \text{ with } y(0)=1$$

Solution : We have

$$\frac{dy}{dx} = e^{x+y}$$

$$\Rightarrow (-1)(-1) = e^{\frac{-1}{-1}} e^{c}$$

$$\Rightarrow \quad 1 = e^{1} \cdot e^{c}$$

$$\Rightarrow e^{0} = e^{1+c}$$

$$\therefore \; 1+c = 0$$

$$\therefore \; c = -1$$

Substituting $c = -1$ in (1)

$$e^{1/x} xy = e^{-1}$$

$$\Rightarrow \quad y = \frac{e^{-1}}{e^{1/x} . x}$$

$$\Rightarrow \quad y = \frac{1}{x e^{(1+\frac{1}{x})}}$$

which is particular solution.

<u>Example :</u>

Solve $\frac{dy}{dt} + 2y = 1 \; ; \; y(0) = \frac{5}{2}$

Solution: Given

$$\frac{dy}{dt} + 2y = 1$$

$$\Rightarrow \quad dy = (1-2y)\, dt$$

$$\Rightarrow \quad \frac{1}{1-2y} dy = dt$$

$$\Rightarrow \int \frac{1}{1-2y}\,dy = \int dt$$

$$\Rightarrow -\frac{1}{2}\int \frac{-2}{1-2y}\,dy = \int dt$$

$$\Rightarrow -\frac{1}{2}\ln|1-2y| = t + c$$

$$\Rightarrow \ln|1-2y| = -2(t+c)$$

$$\Rightarrow \ln|1-2y| = -2t - 2c$$

$$\Rightarrow \ln|1-2y| = \ln e^{-2t} + \ln c_1 \quad [\ln c_1 = -2c]$$

$$\Rightarrow 1-2y = c_1 e^{-2t} \text{ ——— } (1)$$

From initial condition $t=0 \Rightarrow y = 5/2$

$$1 - 2\cdot\frac{5}{2} = c_1 e^0$$

$$\Rightarrow c_1 = -4$$

Sobstituting this values in (1)

$$1-2y = -4e^{-2t}$$

$$\Rightarrow y = 2e^{-2t} + \frac{1}{2}$$

which is the particular solotion.

For Practice

Exercise 2.2 : 1–14 & 19–26

2.3 First Order Linear Equation

Standard form of first order linear equation is

$$\frac{dy}{dx} + P(x)\,y = g(x)$$

Integrating Factor : $I(x) = e^{\int P(x)\,dx}$

Example: $dy - 3y\,dx = 0$

Solution: Given

$$dy - 3y\,dx = 0$$

$$\Rightarrow \frac{dy}{dx} - 3y = 0 \quad \text{———— (1)}$$

$$\therefore P(x) = -3$$

So the integrating factor $I(x) = e^{\int P(x)\,dx} = e^{\int -3\,dx}$

$$\therefore I(x) = e^{-3x}$$

Multiply the equation (1) with $I(x)$ we get

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0$$

$$\Rightarrow \frac{d}{dx}\left(e^{-3x}\cdot y\right) = 0$$

$$\Rightarrow \int \frac{d}{dx}\left(e^{-3x}y\right) dx = 0$$ [integrating on both sides]

$\Rightarrow \; e^{-3x} y = c$

$\therefore \; y = e^{3x} c$ is the solution of given DE.

Solving a linear first order DE :

(I) Put the equation into the standard form

$$\frac{dy}{dx} + P(x) y = g(x)$$

(II) From standard form identify $P(x)$ and then find the integrating factor $I(x) = e^{\int P(x)\, dx}$

(III) Multiply the standard form of the DE by the integrating factor. The left side of the resulting equation is automatically the derivative of the integrating factor and y (the dependent va.)

$$\frac{d}{dx}\left(e^{\int P(x)\, dx} \cdot y\right) = e^{\int P(x)\, dx} f(x)$$

(IV) Integrating both sides of this last equation

Example: solve $x\frac{dy}{dx} - 4y = x^6 e^x$

Solution:

Given

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

$$\Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^5 e^x \text{ ——— } (1)$$

$$\therefore P(x) = -\frac{4}{x}$$

$$\text{IF } I(x) = e^{\int -\frac{4}{x}dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\ln x}$$

$$= e^{\ln x^{-4}} = x^{-4} = \frac{1}{x^4}$$

Multiply (1) by $\frac{1}{x^4}$

$$\cancel{\left(\frac{1}{x^4} - \frac{4}{x}y\right)}$$

$$= \frac{1}{x^4}\left(\frac{dy}{dx} - \frac{4}{x}y\right) = \frac{1}{x^4}\cdot x^5 e^x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^4}\cdot y\right) = xe^x$$

$$\Rightarrow \frac{1}{x^4}y = \int xe^x\,dx$$

$$\Rightarrow \frac{1}{x^4}y = xe^x - e^x + c$$

$$\therefore y = x^5e^x - x^4e^x + x^4c$$

Ans.

Example: Solve $\frac{dy}{dx} + y = x$, $y(0) = 4$

Solution:

Given

$$\frac{dy}{dx} + y = x \quad \text{———} \quad (1)$$

Here $P(x) = 1$

Then IF $I(x) = e^{\int P(x)\,dx} = e^{\int dx} = e^{x}$

Multiply (1) by $I(x)$

$$e^{x}\frac{dy}{dx} + e^{x}y = xe^{x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{x} \cdot y\right) = xe^{x}$$

$$\Rightarrow ye^{x} = \int xe^{x}\,dx$$

$$\Rightarrow ye^{x} = xe^{x} - e^{x} + c$$

From initial condition $x = 0,\ y = 4$

$$4e^{0} = 0 - e^{0} + c$$

$$\Rightarrow c = 5$$

$$\therefore ye^{x} = xe^{x} - e^{x} + 5$$

$\therefore y = x - 1 + 5e^{-x}$ which is the particular sol^n