Lecture: 10

Partial Derivatives

Recall that given a function of one variable, $f(x)$, the derivative, $f'(x)$, represents the rate of change of the function as x changes. This is an important interpretation of derivatives and we are not going to want to lose it with functions of more than one variable.

What do we do if we only want one of the variables to change, or if we want more than one of them to change?

In this lecture we are going to concentrate exclusively on only changing one of the variables at a time, while the remaining variable(s) are held fixed. We called it partial derivative of a function with respect to the changing variable.

For example, $f_x(x,y) = 4xy^3$ or $f_y(x,y) = 6x^2y^2$

Here 1st ~~example~~; ~~the~~ called it partial derivative of $f(x,y)$ with respect to x and y is a constant.

Definition:

The formal definition of the two partial derivatives we looked at below,

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}, \quad \text{and}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Notation:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[f(x,y)\right] = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[f(x,y)\right] = D_y f$$

Example: Find all of the first order partial derivatives for the following function.

1. $f(x,y) = x^4 + 6\sqrt{y} - 10$

Sol^n:

$$f_x(x,y) = \frac{\partial}{\partial x}\left[f(x,y)\right] = \frac{\partial}{\partial x}\left(x^4+6\sqrt{y}-10\right) = 4x^3$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(x^4+6\sqrt{y}-10\right) = \frac{6}{2\sqrt{y}} = \frac{3}{\sqrt{y}}$$

2. $w = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

Sol^n: Here $w(x,y,z) = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$

$$\frac{\partial w}{\partial x} = 2xy + 43.$$

$$\frac{\partial w}{\partial y} = x^2\frac{\partial}{\partial y}y - 10z^3\frac{\partial}{\partial y}(y^2) + 0 - 7\frac{\partial}{\partial y}\tan(4y)$$

$$= x^2 - 10z^3 \cdot 2y - 7\cdot\sec^2(4y)\cdot 4$$

$$= x^2 - 20yz^3 - 28\sec^2(4y).$$

$$\frac{\partial w}{\partial z} = 0 - 10y^2\frac{\partial}{\partial z}(z^3) + 0 - 0$$

$$= -30y^2z^2$$

Example: $f(x,y) = \cos\left(\frac{4}{x}\right)e^{x^2y - 5y^3}$. Find $f_x(x,y)$.

$$f_x(x,y) = \partial\cos\left(\frac{4}{x}\right)\frac{\partial}{\partial x}\left(e^{x^2y-5y^3}\right) + e^{x^2y-5y^3}\frac{\partial}{\partial x}\cos\left(\frac{4}{x}\right)$$

$$= \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}\frac{\partial}{\partial x}(x^2y - 5y^3) + e^{x^2y-5y^3}\left(-\sin\frac{4}{x}\right)\frac{\partial}{\partial x}\left(\frac{4}{x}\right)$$

$$= \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}\cdot(2xy) - e^{x^2y-5y^3}\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)$$

$$= 2xy\cos\left(\frac{4}{x}\right)e^{x^2y-5y^3} + \frac{4}{x^2}\sin\left(\frac{4}{x}\right)e^{x^2y-5y^3}.$$

Find $f_y(x,y) \rightarrow \boxed{\text{yourself}}$

Example: let $z = e^{3x} \sin y$. Find $\left.\frac{\partial z}{\partial x}\right|_{(x,0)}$ and $\left.\frac{\partial z}{\partial y}\right|_{(\ln 3, 0)}$

Solⁿ:

$$\frac{\partial z}{\partial x} = \sin y \frac{\partial}{\partial x} e^{3x}$$
$$= \sin y \cdot 3 \cdot e^{3x}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(x,0)} = \sin 0 \cdot 3e^{3x} = 0$$

$$\frac{\partial z}{\partial y} = e^{3x} \frac{\partial}{\partial y}(\sin y) = e^{3x} \cos y$$

$$\left.\frac{\partial z}{\partial y}\right|_{(\ln 3, 0)} = e^{3\ln 3} \cos 0 = e^{3\ln 3} = e^{\ln 3^3} = 27.$$

TRY YOUR SELF

1. let $z = e^{3x} \sin y$. Find $\left.\frac{\partial z}{\partial x}\right|_{(\ln 3, 0)}$

2. let $f(x,y) = xe^{-y} + 5y$.

   (a) Find the slope of the surface $z = f(x,y)$ in the x-direction at the point $(4,0)$. [hints: page - 930 Example: 5]

Higher-order partial Derivatives

1. $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$

2. $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$

3. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cancel{f_{yx}}\ f_{xy}$

4. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$

Example: Find the second-order partial derivatives of

$$f(x,y) = x^2y^3 + x^4y.$$

Sol^n:

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad , \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy^3 + 4x^3y\right) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x^2y^2 + x^4\right) = 6x^2y.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 + x^4\right) = \underline{6xy^2 + 4x^3}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy^3 + 4x^3y\right) = \underline{6xy^2 + 4x^3}$$

So, look both are same i.e. $\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$

* Wave-equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $c>0$

* Laplace's equation: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

* Heat equation: $\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2}$, $c>0$

* Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where $u(x,y)$ and $v(x,y)$ be two functions.

Chain Rules for P.D.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

where $x = x(t)$, $y = y(t)$ and $z = f(x,y) = f(x(t), y(t))$.

If $w = f(x,y,z)$ and $x = x(t)$, $y = y(t)$, and $z = z(t)$ then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}.$$

Example: Suppose that $w = \sqrt{x^2+y^2+z^2}$, $x = \cos\theta$, $y = \sin\theta$ and $z = \tan\theta$. Find $\frac{dw}{d\theta}$ when $\theta = \frac{\pi}{4}$.

Sol^n: We have

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{1}{2\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}(x^2+y^2+z^2) = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}$$

Similarly, $\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(\cos\theta) = -\sin\theta, \quad \frac{\partial y}{\partial \theta} = \cos\theta, \quad \frac{\partial z}{\partial \theta} = \sec^2\theta$$

$$\frac{dw}{d\theta} = \frac{x}{\sqrt{x^2+y^2+z^2}}(-\sin\theta) + \frac{y}{\sqrt{x^2+y^2+z^2}}\cos\theta + \frac{z}{\sqrt{x^2+y^2+z^2}}\sec^2\theta$$

Continue ...

$\theta = \frac{\pi}{4}$

$x = \cos\theta = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$; $y = \sin\theta = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$,

$z = \tan\left(\frac{\pi}{4}\right) = 1$

So, $x = \frac{1}{\sqrt{2}}$, $y = \frac{1}{\sqrt{2}}$, $z = 1$ substituting these in the above eqn.

$$\frac{dw}{d\theta} = \cancel{\frac{1/\sqrt{2}}{\sqrt{\frac{1}{2}+\frac{1}{2}+1}} \cdot \left(-\frac{1}{\sqrt{2}}\right)} + \cancel{\frac{1/\sqrt{2}}{\sqrt{\frac{1}{2}+\frac{1}{2}+1}} \left(\frac{1}{\sqrt{2}}\right)} + \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}+1}} \cdot \underset{(1+1)}{(1+\sec^2\theta)}$$

$$= \frac{2}{\sqrt{\frac{4}{2}}} = \frac{2}{\sqrt{2}} = \sqrt{2} \qquad \blacksquare$$

Example: Given that $z = e^{xy}$, $x = 2u+v$, $y = \frac{u}{v}$.

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ using the chain rule.