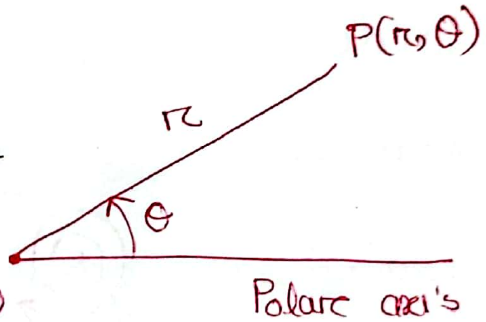
Polar Coordinate System :

A polar coordinate system in a plane consists of a fixed point O, called the pole (or origin) and a ray emanating from the pole, called the polar axis. In such a coordinate system we can associate with each point P in the plane a pair of polar coordinates (r, θ) where r is the distance from P to the pole and θ is an angle from the polar axis to the ray OP. The number r is called the radial coordinate of P and the number θ the angular coordinate (or polar angle) of P.



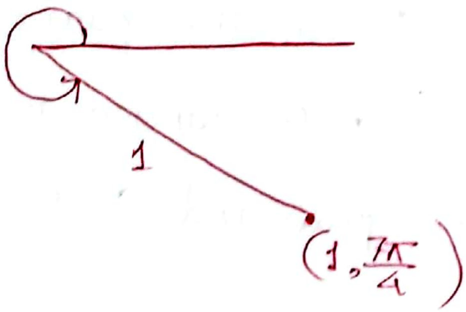
* θ is positive in the counterclockwise direction from the polar axis.

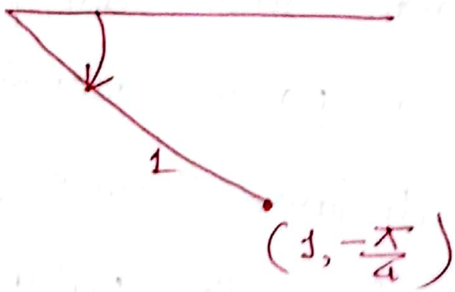
* θ is negative in the clockwise direction from the polar axis.

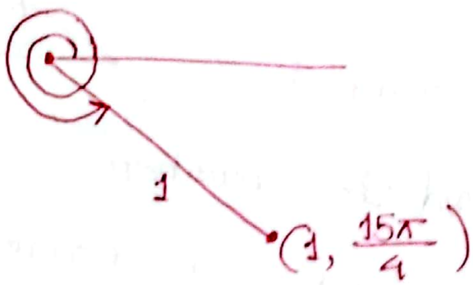
The polar coordinates of a point are not unique. For example, the polar coordinates

$$\left(1, \frac{7\pi}{4}\right), \left(1, -\frac{\pi}{4}\right) \text{ and } \left(1, \frac{15\pi}{4}\right)$$ all

represent the same point.

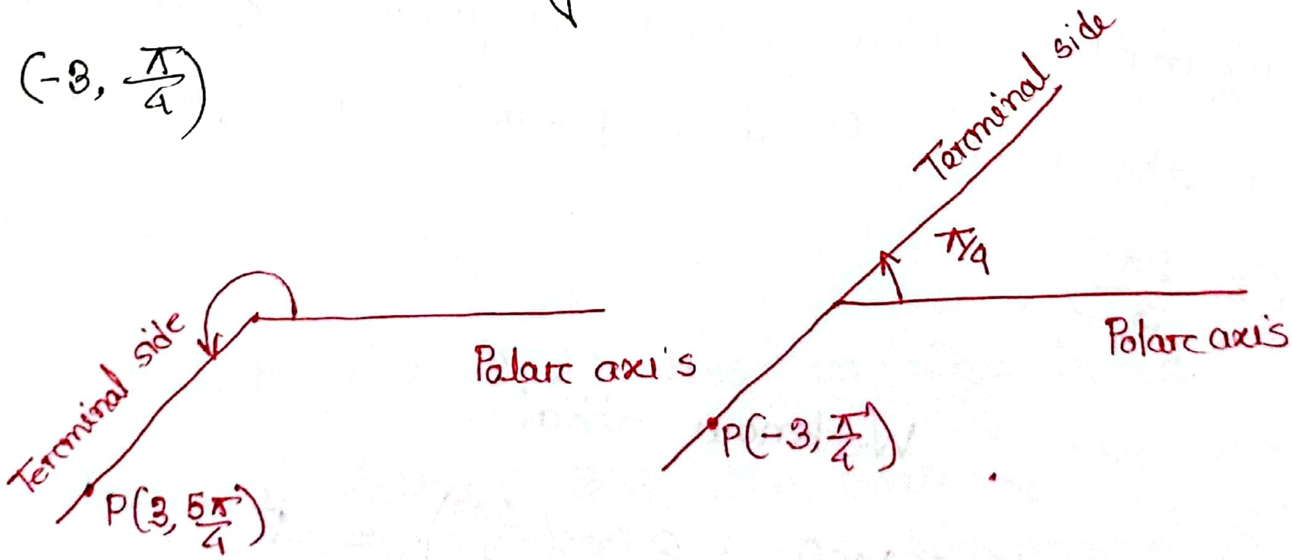






In general, if point P has polar coordinates (r, θ) then $(r, \theta + 2n\pi)$ and $(r, \theta - 2n\pi)$ are also polar coordinates of P for any nonnegative integer n. Thus every point has infinitely many pairs of polar coordinates.

The radial coordinate r of a point P is nonnegative, since it represents the distance from P to the pole. However, it will be convenient to allow non-negative values for r as well. To motivate an appropriate definition we can reach the point $P(3, \frac{5\pi}{4})$ by rotating the polar axis through an angle of $\frac{5\pi}{4}$ and then moving 3 units from the pole along the terminal side of the angle or we can reach the point P by rotating the polar axis through an angle of $\frac{\pi}{4}$ and then moving 3 units from the pole ~~an~~ along the extension of the terminal side. This suggests that the point $(3, \frac{5\pi}{4})$ might also be denoted by $(-3, \frac{\pi}{4})$



In general, the terminal side of the angle $\theta + \pi$ is the extension of the terminal side of θ, therefore $(-r, \theta)$ and $(r, \theta + \pi)$ are polar coordinates of the same point.

Relationship Between Polar and Rectangular Coordinates:-

Polar $\longrightarrow$ Rectangular		Rectangular $\longrightarrow$ Polar	
(r, θ)	(x, y)	(x, y)	(r, θ)

$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x}$

Example:- Find the rectangular coordinates of the point P whose polar coordinates are $\left(6, \frac{2\pi}{3}\right)$.

Solution:- We know that

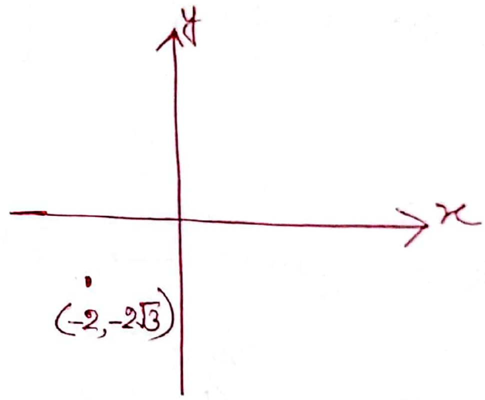
$$x = r\cos\theta = 6\cos\left(\frac{2\pi}{3}\right) = -3$$

$$y = r\sin\theta = 6\sin\left(\frac{2\pi}{3}\right) = 3\sqrt{3}$$

Therefore, the rectangular coordinates of P are $(-3, 3\sqrt{3})$.

Example:- Find polar coordinates of the point P whose rectangular coordinates are $(-2, -2\sqrt{3})$.

Solution:- We know that



$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-2)^2 + (-2\sqrt{3})^2}$$

$$= \sqrt{4+12} = 4$$

$$\tan\theta = \frac{y}{x}$$

$$\Rightarrow \tan\theta = \frac{-2\sqrt{3}}{-2}$$

$$\Rightarrow \tan\theta = \sqrt{3}.$$

The point $(-2, -2\sqrt{3})$ lies in the third quadrant, it follows that the angle satisfying the requirement $0 \leq \theta \leq 2\pi$ is $\theta = \frac{4\pi}{3}$. Thus $(r, \theta) = \left(4, \frac{4\pi}{3}\right)$

Transformation of Equations :-

Polar $\longrightarrow$ Rectangular

$$r = \frac{6}{3\cos\theta + 2\sin\theta}$$

$\Rightarrow 3r\cos\theta + 2r\sin\theta = 6.$

$\Rightarrow 3x + 2y = 6.$ $\left[\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right]$

Rectangular $\longrightarrow$ Polar

$9xy = 4.$

$\Rightarrow 9r\cos\theta \cdot r\sin\theta = 4.$

$\Rightarrow \frac{9}{2} r^2 (2\cos\theta\sin\theta) = 4$

$\Rightarrow r^2 \sin 2\theta = \frac{8}{9}.$

Practice Problem :-

10.2 $\rightarrow$ 3, 4, 5, 9, 10, 11, 12

Cylindrical Coordinate System :-

Three coordinates are required to establish the ~~to~~ location of a point in 3-space. In a rectangular coordinate system, the coordinates can be any real numbers, but in cylindrical and spherical coordinate systems there are restrictions on the allowable values of the coordinates.

Relationship between ~~Rectangular~~ Cardesian and cylindrical Coordinates :-

Cylindrical → ~~Rectangular~~ Cartesian

$(r, \theta, z) \rightarrow (x, y, z)$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

~~Rectangular~~ Cardesian → Cylindrical

$r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$

$z = z.$

Example :- Find the cylindrical coordinates of point P whose ~~rectangular~~ cardesian coordinates are $(4, -4, 4\sqrt{6})$.

Solution:- We know that

$$r = \sqrt{x^2+y^2} = \sqrt{(4)^2+(-4)^2} = \sqrt{16+16}$$
$$= \sqrt{32} = 4\sqrt{2}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{-4}{4} \Rightarrow \tan\theta = -1.$$

$$\Rightarrow \theta = \frac{7\pi}{4}.$$

[since the given point has a negative y-coordinate].

$z = 4\sqrt{6}$.

$\therefore$ The cylindrical coordinates are $\left(4\sqrt{2}, \frac{7\pi}{4}, 4\sqrt{6}\right)$.

Example:- Find the cylindrical coordinates of the point P whose ~~rectangular~~ cartesian coordinates are $\left(4\sqrt{2}, \frac{7\pi}{4}, 4\sqrt{6}\right)$.

Solution:- We know that,

$$x = r\cos\theta$$
$$= 4\sqrt{2}\cos\left(\frac{7\pi}{4}\right) = 4\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 4.$$

$y = r\sin\theta$

$= 4\sqrt{2}\,\sin\left(\frac{7\pi}{4}\right).$

$= 4\sqrt{2}\cdot\left(\frac{-1}{\sqrt{2}}\right)$

$= -4$

$z = z = 4\sqrt{6}.$

Therefore, the cylindrical coordinates are $(4, -4, 4\sqrt{6})$.

Transformation of Equation :-

Cartesian
~~Rectangular~~ → cylindrical

$2x + 3y + 4z = 1.$

$\Rightarrow 2r\cos\theta + 3r\sin\theta + 4z = 1.$

$\Rightarrow 4z = 1 - 2r\cos\theta - 3r\sin\theta$

$\Rightarrow z = \dfrac{1 - 2r\cos\theta - 3r\sin\theta}{4}.$

cylindrical $\rightarrow$ Rectangular

$$r^2\cos 2\theta = z.$$

$$\Rightarrow r^2(1+2\cos^2\theta - 1) = z.$$

$$\Rightarrow r^2 + 2r^2\cos^2\theta = z$$

$$\Rightarrow r^2 + 2(r\cos\theta)^2 = z.$$

$$\Rightarrow x^2+y^2+2x^2 = z.$$

$$r^2\cos 2\theta = z.$$

$$\Rightarrow r^2(2\cos^2\theta - 1) = z.$$

$$\Rightarrow 2r^2\cos^2\theta - r^2 = z$$

$$\Rightarrow 2x^2 - (x^2+y^2) = z.$$

$$\Rightarrow x^2 - y^2 = z.$$

Spherical Coordinates :-

Let's recall the cylindrical coordinates.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z.$$

Now we substitute $r = \rho\sin\varphi$ and $z = \rho\cos\varphi$ in these above equation.

Then we get,

$$x = \rho \sin\varphi \cos\theta .$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi .$$

Cardesian ~~Rectangulare~~ $\rightarrow$ Spherical
(x, y, z) $\quad$ (ρ, φ, θ)

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} .$$

$$\tan\theta = \frac{y}{x}$$

Spherical $\rightarrow$ Cardesian ~~Rectangulare~~

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi .$$

Example :- Find the cartesian ~~Rectangulare~~ coordinates of the point with spherical coordinates $\left(4, \frac{\pi}{4}, \frac{\pi}{3}\right)$.

Solution :- We know that

$$x = \rho \sin\varphi \cos\theta$$

$$= 4 \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{3}\right) = \sqrt{2} .$$

$$y = \rho \sin\varphi \sin\theta = 4\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right) = \sqrt{6}$$

$$z = \rho\cos\varphi = 4\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}.$$

The ~~rectangular~~ cartesian coordinates of the point are $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.

Example :- Find the spherical coordinates of the point whose ~~rectangular~~ cardesian coordinates are $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$.

Solution :- We know ~~that~~,

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{2+6+8} = 4$$

$$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{2\sqrt{2}}{4}.$$

$$\Rightarrow \cos\varphi = \frac{1}{\sqrt{2}} \Rightarrow \varphi = \frac{\pi}{4}$$

$$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{\sqrt{6}}{\sqrt{2}}$$

$$\Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$$

$\therefore$ The spherical coordinates are $(4, \pi/4, \pi/3)$.

Transformation of Equations :-

~~Rectangular~~ cartesian → Spherical

$$x^2+y^2-z^2 = 1$$

$$\Rightarrow x^2+y^2+z^2-z^2 = z^2+1$$

$$\Rightarrow x^2+y^2+z^2 = 2z^2+1$$

$$\Rightarrow \rho^2 = 2z^2+1$$

$$\Rightarrow \rho^2 = 2\rho^2\cos^2\varphi+1$$

$$\Rightarrow 1 = 2\cos^2\varphi$$

$$x^2+y^2-z^2=1$$

$$\Rightarrow \rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta - \rho^2\cos^2\varphi = 1$$

$$\Rightarrow \rho^2\sin^2\varphi - \rho^2\cos^2\varphi = 1$$

$$\Rightarrow \rho^2\cos 2\varphi = -1$$

~~Spherical :-~~

Spherical → ~~Rectangular~~ cartesian

$$\rho - 2\sin\varphi\cos\theta = 0$$

$$\Rightarrow \rho^2 - 2\rho\sin\varphi\cos\theta = 0$$

$$\Rightarrow \rho^2 - 2x = 0$$

$$\Rightarrow x^2+y^2+z^2-2x = 0$$

Practice Problem →

11.8 → 1, 2, 3, 4, 5, 6, 7, 8, 19-26,

27-34, 35-46

Spherical to cylindrical

$r = \rho \sin\varphi, \quad \theta = \theta, \quad z = \rho \cos\varphi.$

Cylindrical to spherical

$\rho = \sqrt{r^2+z^2}, \quad \theta = \theta, \quad \tan\varphi = \frac{r}{z}.$

Example:- Find the spherical coordinates of the point whose cylindrical coordinates are $(\sqrt{3}, \frac{\pi}{6}, 3)$.

Solution:- We know that,

$$\rho = \sqrt{r^2+z^2}$$

$$= \sqrt{(\sqrt{3})^2+(3)^2} = \sqrt{12}.$$

$$\theta = \frac{\pi}{4}$$

$$\tan\varphi = \frac{r}{z} \Rightarrow \varphi = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

$$= \frac{\pi}{6}.$$

The spherical coordinates of the point is, $(\sqrt{12}, \frac{\pi}{6}, \frac{\pi}{4})$

Example:- Find the cylindrical coordinates of the point whose spherical coordinates are $(5, \frac{\pi}{4}, \frac{2\pi}{3})$.

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