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xaktly | Calculus | Limits

Squeeze theorem

Squeezing a function between functions

The squeeze theorem is another way to solve for tricky limits. It works by finding two functions, f(x) and g(x), that are, for every x in their domains, greater than and less than the target function, h(x), respectively. If f(x) and g(x) have the same limit at some value of interest, say x_0, then so must h(x). More precisely, the theorem says

We seek

$$\lim_{x \to x_o} h(x).$$
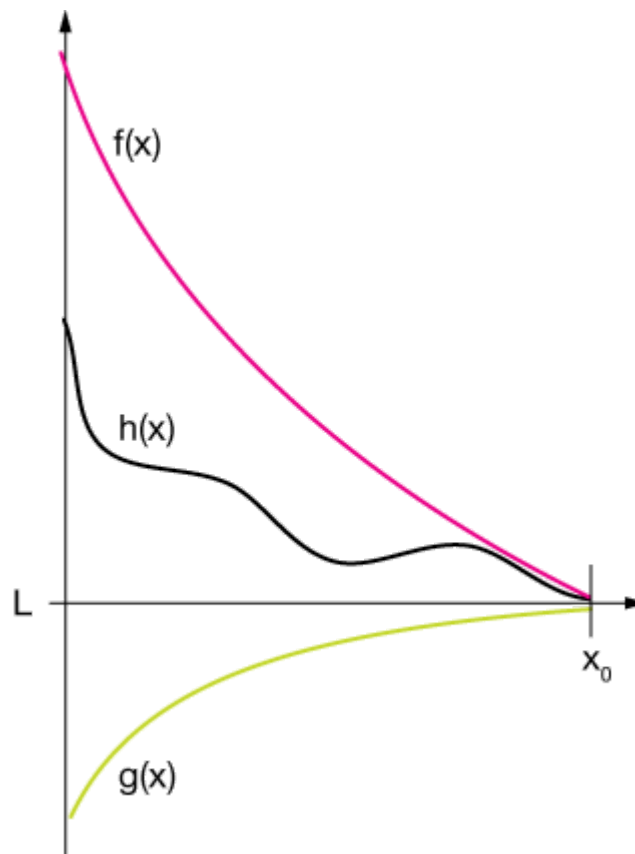
Let $f(x) > h(x)$ for all x in the domains of f(x) and h(x), and let $g(x) < h(x)$ for all x in the domains of g(x) and h(x). Then if

$$\lim_{x \to x_o} f(x) = L \ \text{ and } \ \lim_{x \to x_o} g(x) = L,$$

then

$$\lim_{x \to x_o} h(x) = L$$

because it is "squeezed" between the two functions with known limits.

# Example 1: $\lim_{x\to 0} x^2 sin\left(\frac{1}{x}\right)$

this is a difficult limit to find, but the squeeze theorem makes it easy. First, consider that

$$-1 < sin\left(\frac{1}{x}\right) < 1$$

for all x. Now because $x^2 \geq 0,$ we can multiply everything by x^2 to obtain

$$-x^2 < x^2 sin\left(\frac{1}{x}\right) < x^2$$

This is our recipe for the squeeze theorem, ilustrated in the figure. For all x, the function is less than $y = x^2$ and greater than $y = -x^2,$ which both have a value of zero at x = 0, therefore the value of our function at x = 0 is zero.
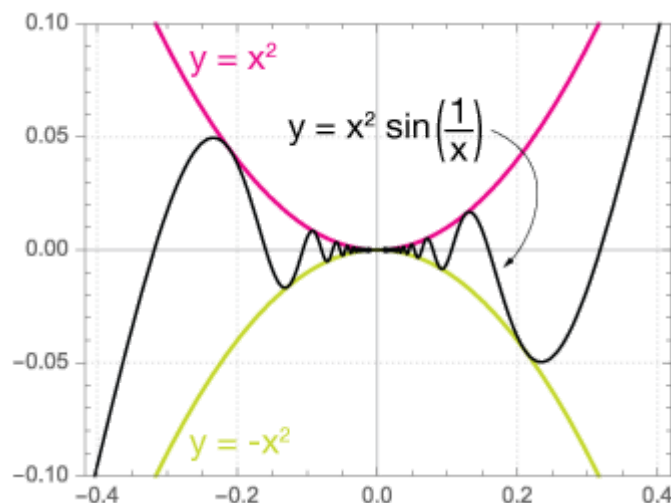
Example 2

We can also use the squeeze theorem to prove the trigonometric limit $\lim_{x\to 0} \frac{sin(x)}{x} = 1.$

Proving this limit using the squeeze theorem involves drawing a sector of the unit circle and two triangles related to it. The areas of the triangle can be placed in order to form an inequality, which can then be manipulated into the desired squeeze theorem statment. It begins with this sector.
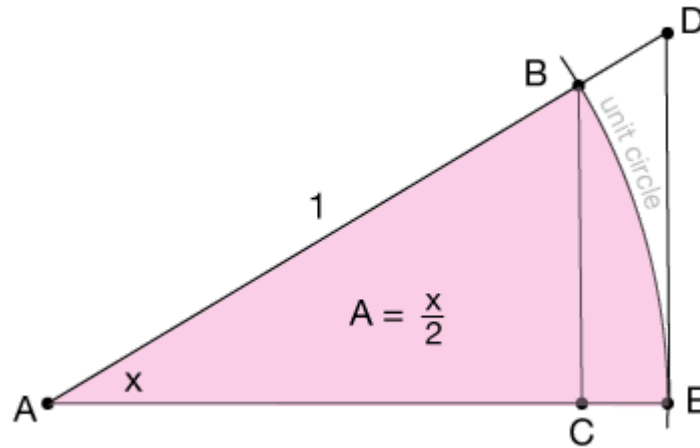
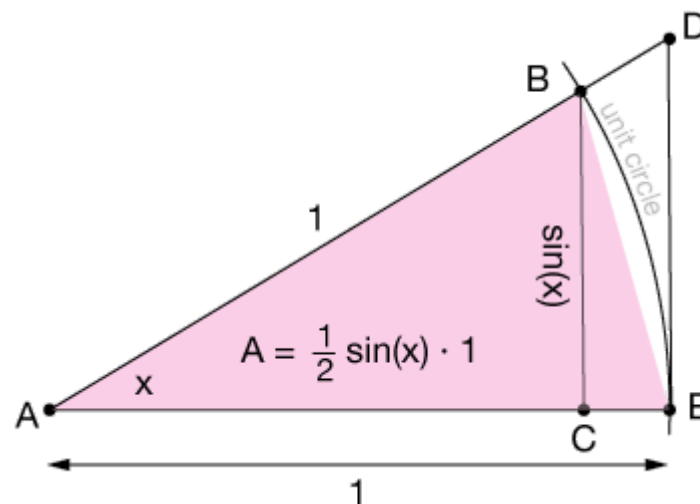Now the area of a sector is found by taking its proportion to the area and angle (radians) of a full circle:

$$\frac{A}{\pi r^2} = \frac{x}{2\pi}$$

which can be rearranged to give

$$A = \frac{xr^2}{2} \;\to\; = \frac{x}{2},$$

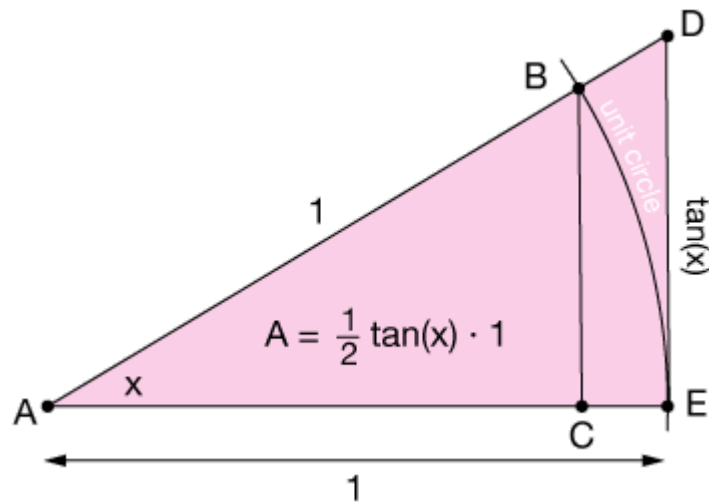where in the last step, we realize that $r = 1$ for the unit circle.

Now our second triangle, the smallest of the three, is $\triangle$ ABE, shown here:

The height and width of this triangle are sin(x) and 1, respectively, giving an area of

$$a = \frac{1}{2} sin(x) \cdot 1$$

The largest triangle, ΔADE, is shown here

The tangent of angle x is the length of the opposite side divided by the adjacent, which has length 1, so the opposite side has length $tan(x)$. Thus the area of the largest triangle is

$$a = \frac{1}{2}tan(x) \cdot 1.$$

Now we can place these areas in order using inequalities:

$$\frac{1}{2}sin(x) \leq \frac{x}{2} \leq \frac{1}{2}tan(x)$$

Multiplication by 2 and conversion of tan(x) to sin(x)/cos(x) gives

$$sin(x) \leq x \leq \frac{sin(x)}{cos(x)}$$

Taking the reciprocal (and remembering to switch the inequalities) gives

$$\frac{1}{sin(x)} \geq \frac{1}{x} \geq \frac{cos(x)}{sin(x)}$$

Multiplication through by $sin(x)$ gives

$$1 \geq \frac{sin(x)}{x} \geq cos(x)$$

Now taking the limit as x → 0 squeezes the center term between 1 and 1, which requires it to also converge to a value of 1.

Example 3

Use the squeeze theorem to find the limit $\lim_{x \to \infty} \frac{cos(x)}{x-1}$.

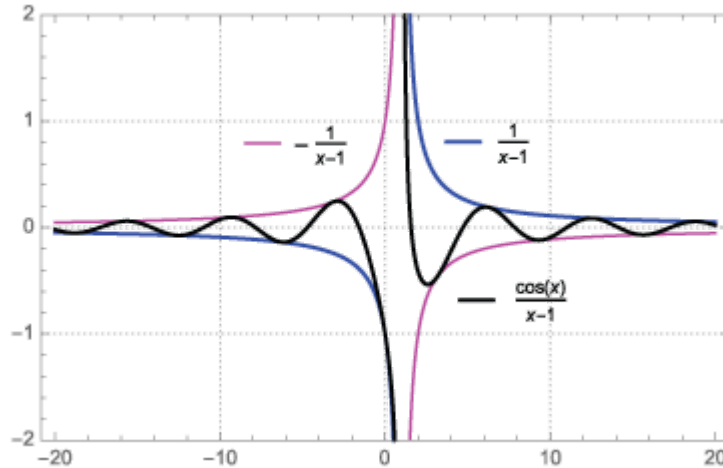**Solution**: To begin, note that

$$-1 \leq cos(x) \leq 1$$

That is, the cosine function oscillates between -1 and 1. Now divide all of these terms by $(x - 1)$:

$$\frac{-1}{x-1} \leq \frac{cos(x)}{x-1} \leq \frac{1}{x-1}$$

The middle term contains the expression for which we seek the limit, and the outer limits are:

$$\lim_{x \to \infty} \frac{-1}{x-1} = \lim_{x \to \infty} \frac{1}{x-1} = 0.$$

So $\lim_{x \to \infty} \frac{cos(x)}{x-1} = 0$ by the squeeze theorem. Here is a graph of the three functions, just so you can get an idea of what's going on.

Example 4

Use the squeeze theorem to find the limit $\lim_{x \to 0} x^2 \left[1 + sin\left(\frac{1}{x}\right)\right].$

Solution: We can begin this limit much as in example 3 above, by noting that the sine function oscillates between -1 and 1:

$$-1 \leq sin\left(\frac{1}{x}\right) \leq 1$$

Now add 1 to all three parts of the inequality to get

$$0 \leq 1 + sin\left(\frac{1}{x}\right) \leq 2$$

One thing we know already is that $1 + sin\left(\frac{1}{x}\right)$ is bounded between 1 and 2, so multiplication by x^2, with our limit as $x \to 0$ should also be bounded. In other words, our squeeze-theorem attempt should work.

Multiplication of each element of the inequality by x^2 gives

$$0 \leq x^2 \left[1 + sin\left(\frac{1}{x}\right)\right] \leq 2x^2$$

The outer limits are both zero, so the limit of interest is squeezed between zero and zero, so it, too, is zero. Here is a graph of the three functions in the neighborhood of $x = 0$, showing that the function (black) is squeezed between $y = 0$ and $y = 2x^2.$
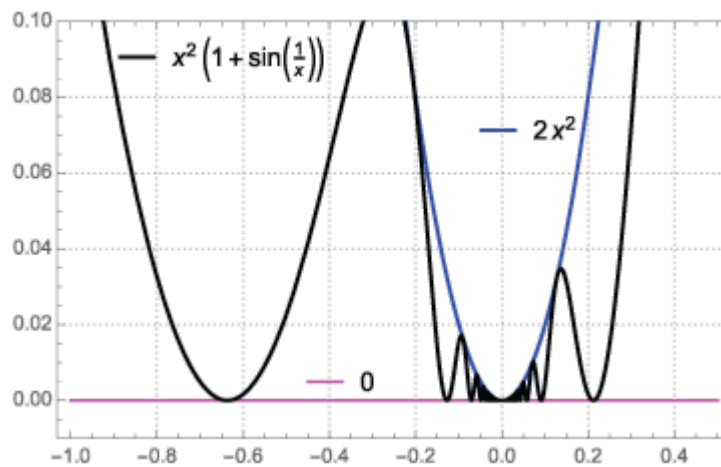
Practice problems

Find the limits using the squeeze theorem.

1.

$$\lim_{a \to \infty} \frac{sin(x)}{a}$$

Hint: begin with $-1 \leq sin(x) \leq 1$.

Solution

2.

$$\lim_{x \to 1} (x^2 + x - 2) \cdot cos\left(\frac{1}{x - 1}\right)$$

Hint: begin with $-1 \leq cos\left(\frac{1}{x-1}\right) \leq 1$

Solution

3.

$$\lim_{x \to 0} x^2 sin\left(\frac{2\pi}{x}\right)$$

Hint: begin with $-1 \leq sin\left(\frac{2\pi}{x}\right) \leq 1$

Solution

4.

$$\lim_{x \to 0} x^2 sin\left(\frac{3}{x}\right) + 2$$

Solution

5.

$$\lim_{x \to 0} x \cdot sin(x)$$

Solution

6.

$$\lim_{x \to 0} \; x^2 \, cos \left(\frac{1}{x^2} \right)$$

Solution