Lecture 10 — Partial Derivatives

Continue ...

Example: Suppose that $\omega = x^2+y^2-z^2$ and $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.

Using appropriate forms of the chain rule to find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$.

Sol^n:

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho} \quad \text{——— (1)}$$

$$w = x^2+y^2-z^2$$

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2y, \quad \frac{\partial w}{\partial z} = -2z$$

$$x=\rho\sin\phi\cos\theta, \quad y=\rho\sin\phi\sin\theta, \quad z=\rho\cos\phi$$

$$\frac{\partial x}{\partial \rho} = \sin\phi\cos\theta \qquad \frac{\partial y}{\partial \rho} = \sin\phi\sin\theta \qquad \frac{\partial z}{\partial \rho} = \cos\phi$$

From (1)

$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x\sin\phi\cos\theta + 2y\sin\phi\sin\theta - 2z\cos\phi \\
&= 2(\rho\sin\phi\cos\theta)\sin\phi\cos\theta + 2(\rho\sin\phi\sin\theta)\sin\phi\sin\theta \\
&\qquad - 2(\rho\cos\phi)\cos\phi \\
&= 2\rho\sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi \\
&= 2\rho\sin^2\phi(\cos^2\theta+\sin^2\theta) - 2\rho\cos^2\phi \\
&= 2\rho(\sin^2\phi - \cos^2\phi) \\
&= -2\rho\cos 2\phi
\end{aligned}$$

$\boxed{\frac{\partial w}{\partial \theta} \text{ do yourself !!!}}$

Implicit Differentiation

Theorem 1:

If the equation $f(x,y) = c$ defines y implicitly as a differentiable function of x, and if $\frac{\partial f}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Example: Given that $x^3 + y^2 x - 3 = 0$, find $\frac{dy}{dx}$.

Solⁿ:

$$f(x,y) = x^3 + y^2 x - 3$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2, \quad \frac{\partial f}{\partial y} = 2xy$$

So, $$\frac{dy}{dx} = -\frac{3x^2+y^2}{2xy}.$$

Theorem 2:

If the equation $f(x,y,z)=c$ defines z implicitly as a differentiable function of x and y, and if $\partial f/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

Example: Consider the sphere $x^2+y^2+z^2=1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

Sol^n:

$$f(x,y,z) = x^2+y^2+z^2-1$$

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{2y}{2z} = -\frac{y}{z}$$

at the point $(\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$.

$$\left.\frac{\partial z}{\partial x}\right|_{(2/3,1/3,2/3)} = -\frac{2/3}{2/3} = -1$$

$$\left.\frac{\partial z}{\partial y}\right|_{(2/3,1/3,2/3)} = -\frac{1/3}{2/3} = -\frac{1}{2}$$

Do yourself:

1. Find $\frac{dz}{dx}$ for $z = x\ln(xy) + y^3$, $y = \cos(x^2+1)$

2. Use the chain rule to find $\frac{dz}{dt}$.

   (a) $z = 3x^2y^3$; $x = t^4$, $y = t^3$

   (b) $z = 5\cos xy - \sin xq$; $x = \frac{1}{t}$, $y = t$, $q = t^2$

   (c) $z = \sqrt{1+x-2yq^4x}$; $x = \ln t$, $y = t$, $q = 3t$

3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ form the ~~bellow~~ equations:

   (a) $x^2 - 3yz^2 + xyz - 2 = 0$

   (b) $ye^x - 5\sin 3z = 4z$

Maxima and Minima for Two Variables

In this lecture we are going to be looking at identifying relative minimums and relative maximums.
The definition of relative extema for functions of two variables is identical to that for functions of one variable we just need to remember now that we are working with functions of two variables.

~~we~~ Recall that. the critical points of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable.
Now ~~for~~ we want to define critical points for two variables.

Critical point

A point (x_0, y_0) in the domain of a function $f(x,y)$ is called a critical point of the function if $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$ or if one or both partial derivatives do not exist at (x_0,y_0).

Theorem:

If f has a relative extreme at a point (x_0,y_0), and if the first-order partial derivatives of f exist at this point, then

$$f_x(x_0,y_0) = 0 \quad \text{and} \quad f_y(x_0,y_0) = 0$$

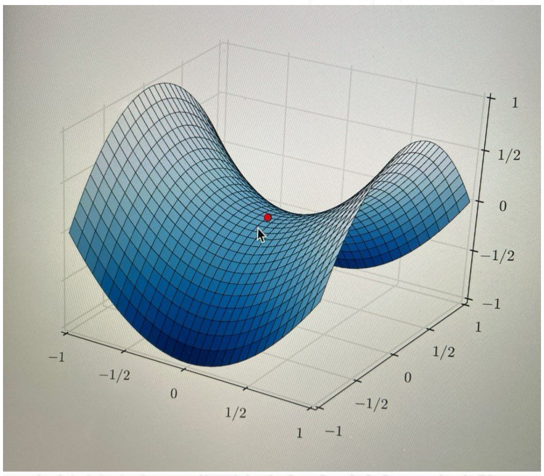
But this theorem ~~used~~ has some drawback. So, we need second partial test.

Theorem (The Second Partial test)

Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let

$$D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)$$

1. If $D>0$ and $f_{xx}(x_0,y_0) > 0$, then f has a relative minimum at (x_0, y_0).
2. If $D>0$ and $f_{xx}(x_0,y_0) < 0$, then f has a relative maximum at (x_0, y_0).
3. If $D<0$, then f has a saddle point at (x_0,y_0).
4. If $D=0$, then no conclusion can be drawn.



Example: Locate all relative extrema and saddle points of

$$f(x,y) = 3x^2 - 2xy + y^2 - 8y$$

Soln.

Here $f_x(x,y) = 6x - 2y$, $f_y(x,y) = -2x + 2y - 8$

For critical points

$$6x - 2y = 0 \quad \text{--- (I)}$$

$$-2x + 2y - 8 = 0 \quad \text{--- (II)}$$

$2y = 6x$

$-2x + 6x = 8$

$4x = 8$

$x = 2$

$y = \frac{6 \cdot 2}{2} = 6$

Solving (I) & (II) we get

$$x = 2, \; y = 6$$

So, $(2,6)$ is the only critical point.

Now, $f_{xx}(x,y) = 6$, $f_{yy}(x,y) = 2$, $f_{xy}(x,y) = -2$

We have

$$D = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}^2(x,y)$$

at point $(2,6)$

$$D = f_{xx}(2,6)\, f_{yy}(2,6) - f_{xy}^2(2,6) = 6 \cdot 2 - (-2)^2$$

$$= 12 - 4 = 8 > 0$$

and $f_{xx}(2,6) = 6 > 0$

So, f has a relative minimum at $(2,6)$.

TRY YOURSELF

1. Find and classify all the critical points of

   ⓐ $f(x,y) = 4 + x^3 + y^3 - 3xy.$

   ⓑ $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2.$

2. Locate all relative extema and saddle points of

$$f(x,y) = 4xy - x^4 - y^4.$$