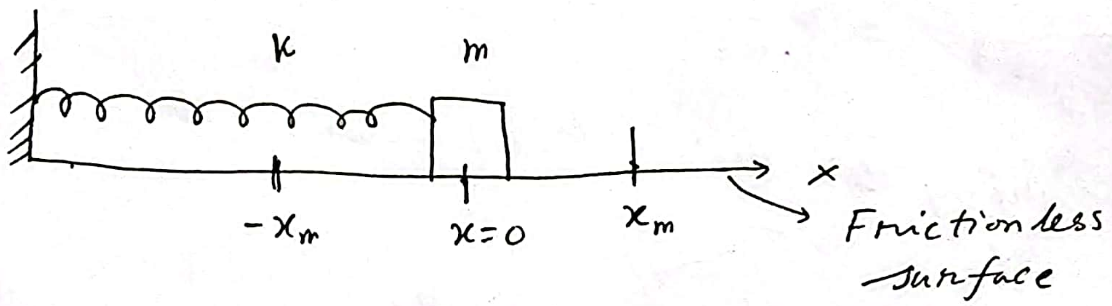
Chap-15 "Oscillations"

Any motion that repeats itself in equal intervals of time is called periodic motion / Harmonic motion.

If a particle in a periodic motion moves back and forth, over the same path, we call the motion is oscillatory or vibratory.

(Spring-Block system)



Time period (T) → Time for completing on cycle

Frequency (f) → To complete number of cycle in per unit time.

$$\boxed{T = \frac{1}{f}}$$

$$[T] = [T]$$

$$[f] = [\frac{1}{T}]$$

$$1\,Hz = 1\,s^{-1}$$

Simple Harmonic Motion (SHM)

→ It's motion is sinusoidal Function of time

For example:

Particle's displacement (or, position), angular frequency

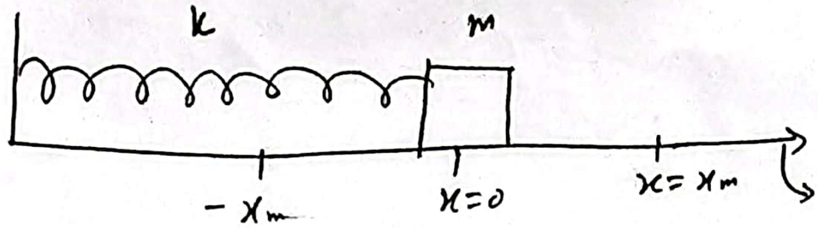
$$x(t) = x_m \sin(\omega t + \delta)$$ phase-constant

displacement at time

or, $x(t) = x_m \cos(\omega t + \delta')$

amplitude

** Spring-Block system:



Frictionless,

Newton's 2nd Law:

$$F = ma$$
$$= m\frac{d^2x}{dt^2} \quad \text{—(1)}$$

According to Hooke's law, $F = -kx$ —(2)

From equation, (1),

$$-kx = m\frac{d^2x}{dt^2}$$
$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$
$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \quad \text{—(3)}$$

Let's consider the solution is,

$$x(t) = e^{\lambda t} \quad \text{—(4)}$$

In equation,

$$\lambda^2 e^{\lambda t} + \frac{k}{m}e^{\lambda t} = 0$$
$$\Rightarrow e^{\lambda t}\left(\lambda^2 + \frac{k}{m}\right) = 0$$
$$e^{\lambda t} \neq 0; \quad \lambda^2 + \frac{k}{m} = 0$$
$$\Rightarrow \lambda = \pm\sqrt{-\frac{k}{m}}$$
$$\therefore \lambda = \pm i\sqrt{\frac{k}{m}}$$

In equation (4)

$$x(t) = Ae^{i\sqrt{\frac{k}{m}}t} + Be^{-i\sqrt{\frac{k}{m}}t} \quad \text{—(5)}$$

at, $t = 0$; $x(0) = x_m$

$$x(0) = A + B = x_m \quad \text{— (6)}$$

at, $t = 0$; $\frac{dx(0)}{dt} = 0$

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left(A e^{i\sqrt{\frac{k}{m}}t} + B e^{-i\sqrt{\frac{k}{m}}t}\right)$$

$$= A i \sqrt{\frac{k}{m}}\, e^{i\sqrt{\frac{k}{m}}t} - B i \sqrt{\frac{k}{m}}\, e^{-i\sqrt{\frac{k}{m}}t}$$

$$\frac{dx(0)}{dt} = A i \sqrt{\frac{k}{m}} - B i \sqrt{\frac{k}{m}}$$

$$\Rightarrow 0 = A i \sqrt{\frac{k}{m}} - B i \sqrt{\frac{k}{m}}$$

$$\Rightarrow A - B = 0$$

$$\Rightarrow A = B \quad \text{— (7)}$$

From equation (6), $A + A = x_m$

$$\Rightarrow A = \frac{x_m}{2} = B$$

From equation (5),

$$x(t) = \frac{x_m}{2} e^{i\sqrt{\frac{k}{m}}t} + \frac{x_m}{2} e^{-i\sqrt{\frac{k}{m}}t}$$

$$= x_m \left(\frac{e^{i\sqrt{\frac{k}{m}}t} + e^{-i\sqrt{\frac{k}{m}}t}}{2}\right)$$

$$\boxed{x(t) = x_m \cos\sqrt{\frac{k}{m}}\, t} \quad \text{— (8)}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

The sol^n of this differential equation is -

$$x(t) = x_m \cos\sqrt{\frac{k}{m}}\,t$$

or, $x(t) = x_m \sin\left(\sqrt{\frac{k}{m}}\,t + \delta\right)$

Periodic function,

$f(t+T) = f(t) \rightarrow T =$ period

$\sin t \rightarrow$ period $T = 2\pi$

$\sin(2\pi t) \rightarrow$ Period $T = \frac{2\pi}{2\pi}$

$\sin\left(\sqrt{\frac{k}{m}}\,t\right) \rightarrow T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}}$$

$$T = \frac{2\pi}{\omega}$$

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

$$\sqrt{\frac{Nm^{-1}}{kg}} = \sqrt{\frac{kg\,ms^{-2}m^{-1}}{kg}} = \sqrt{\frac{1}{s^2}} = \frac{1}{s} = s^{-1}$$

$T = 2\pi\sqrt{\frac{m}{k}}$; $\omega = \sqrt{\frac{k}{m}}$

$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$$\boxed{x(t) = x_m \sin(\omega t + \delta)} \quad ; \quad \omega = \sqrt{\frac{k}{m}}$$

$$\frac{d}{dt} x(t) = x_m (\cos(\omega t + \delta))$$

$$\boxed{v(t) = x_m \omega \cos(\omega t + \delta)}$$

$$\frac{d}{dt} v(t) = -\omega^2 x_m \cancel{\cos} \sin(\omega t + \delta)$$

$$\Rightarrow a = -\omega^2 x_m \sin(\omega t + \delta)$$

$$\Rightarrow \boxed{a = -\omega x(t)}$$

$$\boxed{a \sim -x(t)}$$

$$v(t) = \omega x_m \cos(\omega t + \delta)$$

$$= \omega \sqrt{x_m^2 \cos^2(\omega t + \delta)}$$

$$= \omega \sqrt{x_m^2 - x_m^2 \sin^2(\omega t + \delta)}$$

$$\boxed{v(t) = \omega \sqrt{x_m^2 - x^2(t)}}$$

Energy of in SHM:

$$U(t) = \frac{1}{2} k x^2$$

$$= \frac{1}{2} k x_m^2 \sin^2(\omega t + \delta)$$

$$K(t) = \frac{1}{2} m v^2 = \frac{1}{2} m \omega^2 x_m^2 \cos^2(\omega t + \delta)$$

$$= \frac{1}{2} k x_m^2 \cos^2(\omega t + \delta)$$

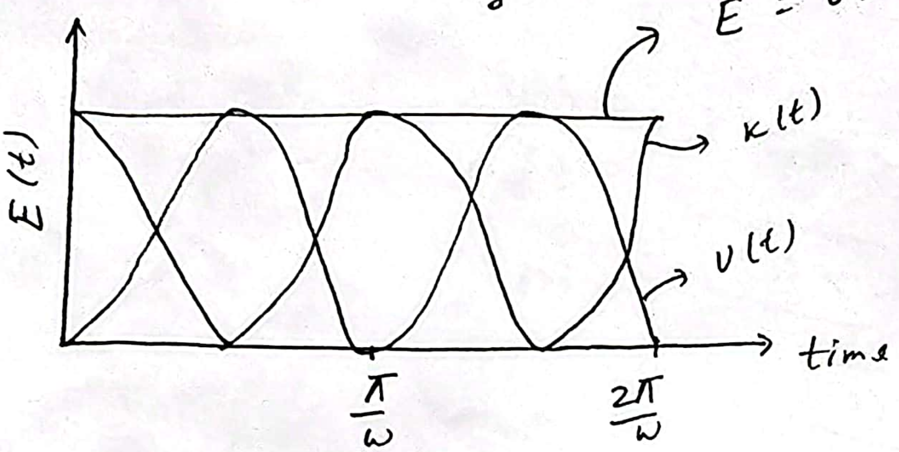
$E(t) = U(t) + K(t)$

$= \frac{1}{2} k x_m^2 \sin^2(\omega t + \delta) + \frac{1}{2} k x_m^2 \cos^2(\omega t + \delta)$

$= \frac{1}{2} k x_m^2 \left(\sin^2(\omega t + \delta) + \cos^2(\omega t + \delta)\right)$

$$\boxed{E(t) = \frac{1}{2} k x_m^2}$$

$\delta = 0$



$U(x) = \frac{1}{2} k x^2$

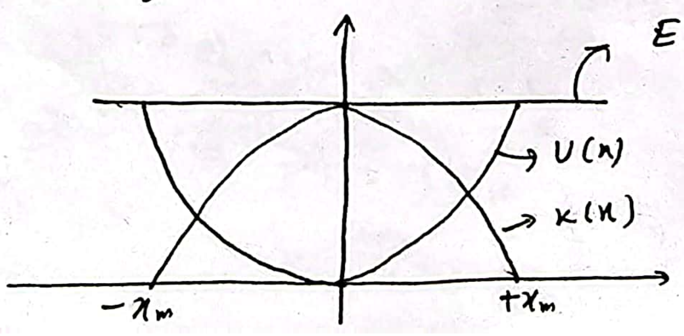
$K(x) = \frac{1}{2} m v^2$

$\cancel{= \frac{1}{2} m \omega^2 x_m \cos^2}$

$= \frac{1}{2} m \omega^2 (x_m^2 - x^2) = \frac{1}{2} k (x_m^2 - x^2)$

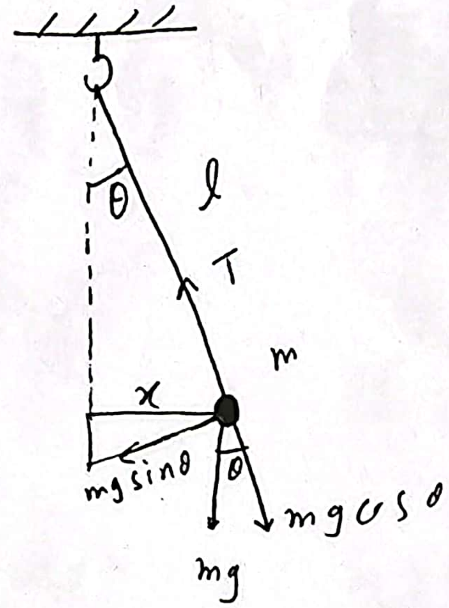
$U(x) + K(x) = \frac{1}{2} k x^2 + \frac{1}{2} k (x_m^2 - x^2)$

$= \frac{1}{2} k x_m^2 = E$



Simple

Simple pendulum



$$\sin\theta = \frac{x}{l}$$

$$\Rightarrow x \simeq l\theta$$

Restoring Force, $F = -mg\sin\theta$ —(1)

$$\sin\theta \simeq \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$0 \leq \theta \leq 4^\circ \rightarrow \sin\theta \simeq \theta$

$$F = -mg\theta$$

$$= -mg\frac{x}{l}$$

$$F = -\left(\frac{mg}{l}\right)x \quad \text{—(2)}$$

Newton's 2nd Law, $F = m\dfrac{d^2x}{dt^2}$ —(3)

$$\therefore -\frac{mg}{l}x = m\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} + \frac{mg}{l}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{g}{l}x = 0$$

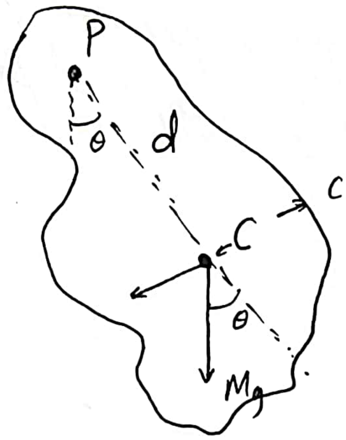
Physic

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$\boxed{T = 2\pi \sqrt{\frac{l}{g}}}$$

Physical Pendulum



Physical Pendulum
↓
Any rigid body Mounted so that it can be swing in a vertical plane about some axis

Restoring torque, $\tau = -Mgd\sin\theta$ —(1)

θ is too small, $\sin\theta \simeq \theta$

$\tau = -Mgd\theta$ —(2)

Hooke's Law for angular displacement,

$\tau = -K\theta$ —(3)

↓
Torsional constant

Comparing (2), (3), $K = Mgd$

Newton's 2nd Law for rotation,

$$\tau = I\alpha$$

$$\Rightarrow -Mgd\theta = I\frac{d^2\theta}{dt^2}$$

$$\Rightarrow I\frac{d^2\theta}{dt^2} + Mgd\theta = 0$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{Mgd}{I}\right)\theta = 0$$

$$\omega = \sqrt{\frac{Mgd}{I}}$$

$$\boxed{T = 2\pi \sqrt{\frac{I}{Mgd}}}$$ → Time period for compound pendulum

$$\theta = \theta_m \sin(\omega t + \delta)$$

$$\frac{d\theta}{dt} = ?$$

$$\frac{d^2\theta}{dt^2} = ?$$

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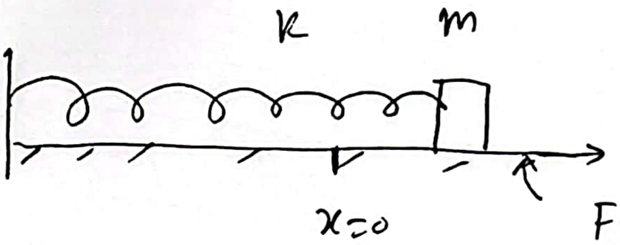
Damped ~~SHM~~ (DHM)



$\to$ damping constant $\to$ unit $kg\,s^{-1}$

Damping Force, $F_d = -bv$

$$F_{net} = ma$$
$$= m\frac{d^2x}{dt^2}$$

$$F_{net} = -kx - bv$$
$$= -kx - b\frac{dx}{dt}$$

$$\therefore \quad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Let, $x(t) = e^{\lambda t}$

$$\frac{d^2}{dt^2}(e^{\lambda t}) + \frac{b}{m}\frac{d}{dt}(e^{\lambda t}) + \frac{k}{m}e^{\lambda t} = 0$$

$$\Rightarrow \lambda^2 e^{\lambda t} + \lambda\frac{b}{m}e^{\lambda t} + \frac{k}{m}e^{\lambda t} = 0$$

$$\Rightarrow e^{\lambda t}\left(\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m}\right) = 0$$

$\Rightarrow$ $e^{\lambda t} \neq 0 \quad \lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$

$$\lambda = \frac{-\frac{b}{m} \pm \sqrt{\left(\frac{b}{m}\right)^2 - 4 \cdot \frac{k}{m}}}{2 \cdot 1}$$

$$= -\frac{b}{2m} \pm \sqrt{\frac{\left(\frac{b}{m}\right)^2 - 4\left(\frac{k}{m}\right)}{4}}$$

$$= -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

$$x(t) = x_m e^{\left(-\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\right) t}$$

$$= x_m e^{-\frac{b}{2m} t} e^{\pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\, t}$$

If ; $\left(\frac{b}{2m}\right)^2 < \frac{k}{m}$ [condition For underdamping damped)

$$x(t) = x_m e^{-\frac{b}{2m} t} e^{\pm i \left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right) t}$$

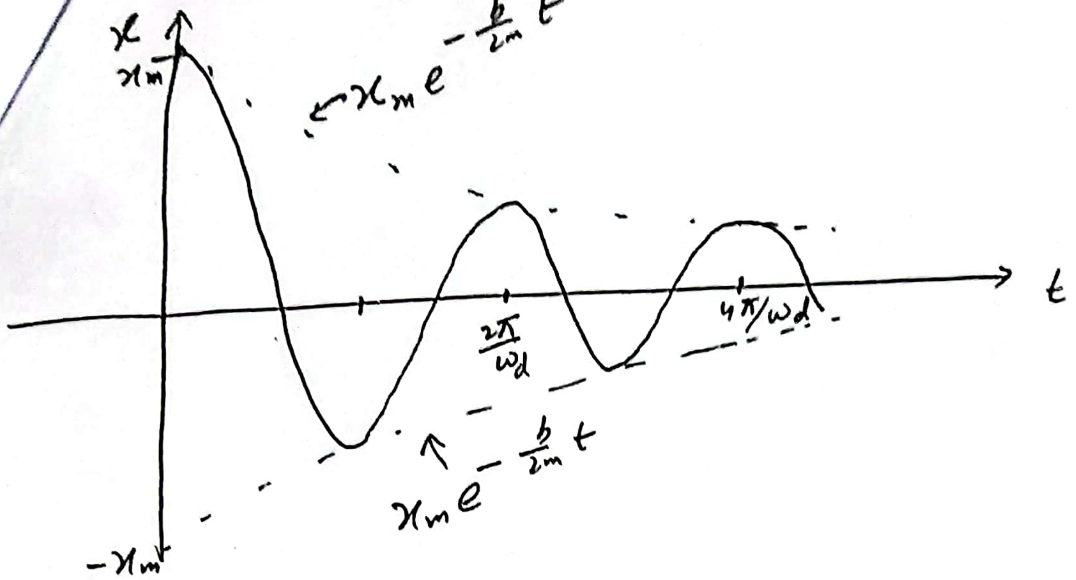
~~Taking Real part~~ $= x_m e^{-\frac{b}{2m} t} \left(\cos\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right) t\right.$

$$\left. + i \sin\left(\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\, t\right)\right)$$

Taking real part,

$$x(t) = x_m e^{-\frac{b}{2m} t} \cos(\omega_d t + \delta)$$

$\omega_d = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$ $\rightarrow$ Damping Frequency



x
x_m
$x_m e^{-\frac{b}{2m}t}$
t
$\frac{2\pi}{\omega_d}$
$4\pi/\omega_d$
$x_m e^{-\frac{b}{2m}t}$
$-x_m$

$$U(t) = \frac{1}{2} k x^2(t)$$

$$= \frac{1}{2} k x_m^2 e^{-\frac{b}{2m}t} \cos^2(\omega_d t + \delta)$$

$$K(t) = \frac{1}{2} m v^2(t)$$

$$= \frac{1}{2} m \left[x_m \left(-\frac{b}{2m} e^{-\frac{b}{2m}t}\right) \cos(\omega_d t + \delta) - x_m \omega e^{-\frac{b}{2m}t} \sin(\omega_d t + \delta) \right]^2$$

For $\left(\frac{b}{2m}\right)^2 << \frac{k}{m}$

$$= \frac{1}{2} m x_m \omega^2 e^{-\frac{b}{m}t} \sin^2(\omega_d t + \delta)$$

$$= \frac{1}{2} k x_m^2 e^{-\frac{b}{m}t} \sin^2(\omega_d t + \delta)$$

$$\boxed{E(t) \simeq \frac{1}{2} k x_m^2 e^{-\frac{b}{m}t}}$$