Lecture 2

Some problems

1. $$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$$

$$\lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)(\sqrt{x+1}-1)}$$

$$\lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x+1-1}$$

$$\lim_{x\to 0} \frac{x(\sqrt{x+1}+1)}{x}, \quad x\neq 0$$

$$\lim_{x\to 0} \sqrt{x+1}+1 = 2$$

2. $$\lim_{x\to\infty} \left(\sqrt{x^6+5}-x^3\right)$$

$$\lim_{x\to\infty} \frac{(\sqrt{x^6+5}-x^3)(\sqrt{x^6+5}+x^3)}{(\sqrt{x^6+5}+x^3)}$$

$$\lim_{x\to\infty} \frac{x^6+5-x^6}{\sqrt{x^6+5}+x^3}$$

$$\lim_{x\to\infty} \frac{5}{x^3\left(\sqrt{1+5/x^3}+1\right)}$$

$$\lim_{x\to\infty} \frac{5/x^3}{\sqrt{1+5/x^3}+1} = 0$$

3. $$\lim_{x\to 2} \frac{2x^2-5x+2}{5x^2-7x-6}$$

Do Yourself !!!

A H M Mahbubur Rahman, PhD.
Assistant Professor
MNS Dept.
BRACU

Problem: Let We defined a function

$$f(x) = \begin{cases} \frac{1}{x+2} & , x < -2 \\ x^2 - 5 & , -2 \leq x \leq 3 \\ \sqrt{x+13} & , x > 3 \end{cases}$$

Find the $\lim_{x \to -2} f(x)$ and $\lim_{x \to 3} f(x)$ if it exist.

Solution: at $x = -2$:

$$\text{L.H.L.} = \lim_{x \to -2^-} f(x) = \lim_{x \to -2} \frac{1}{x+2} = -\infty$$

$$\text{R.H.L.} = \lim_{x \to -2^+} f(x) = \lim_{x \to -2} (x^2 - 5) = -1$$

So, L.H.L. $\neq$ R.H.L. thus $\lim_{x \to -2} f(x)$ does not exist.

2nd part:

at $x = 3$:

$$\text{L.H.L.} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3} (x^2 - 5) = 4$$

$$\text{R.H.L.} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} \sqrt{x+13} = \sqrt{16} = 4$$

Thus the limit exists at $x = 3$ and $\lim_{x \to 3} f(x) = 4$

Problem: Estimate the limit: $\lim_{x\to 0} \frac{x}{|x|}$ if it exists and $x \neq 0$.

Solution: Assume that

$$f(x) = \frac{x}{|x|} = \begin{cases} \frac{x}{x} = 1 & , x > 0 \\ \frac{x}{-x} = -1 & , x < 0 \end{cases}$$

Now, at $x = 0$:

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} \frac{x}{-x} = -1$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{x}{x} = 1$$

So, L.H.L. $\neq$ R.H.L. Therefore, $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

Problem: Estimate $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ if exists.

Solution: Do we use the property of limit such as

$$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to b} g(x)$$

The answer is 'NO'. Because the properties of limit exist if the individual limit exists. So, we need another way to solve this problem by using a theorem "Squeeze theorem". Next page we write it.

A H M Mahbubur Rahman, PhD.
Assistant Professor
MNS Dept.
BRACU

The Squeeze Theorem:

Let f, g and h be functions satisfying

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing the number a, ~~with~~ except possibly at a itself. ~~suppose that for every x~~

If g and h have the same limit as x approaches a, say

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then f also has this limit as x approaches a, i.e.

$$\lim_{x \to a} f(x) = L. \qquad \blacksquare$$

Now our previous problem.

Problem: Estimate $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ if exists.

Solution: Assume that $f(x) = x \sin(1/x)$

Since $-1 \le \sin(1/x) \le 1$

$$-|x| \le |x| \sin(1/x) \le |x|$$

$$\lim_{x \to 0} -|x| \le \lim_{x \to 0} |x| \sin(1/x) \le \lim_{x \to 0} |x|$$

again, $|x| \to 0$ as $x \to 0$ i.e. $\lim_{x \to 0} |x| = 0$

Applying the squeeze theorem,

$$\lim_{x \to 0} x \sin(1/x) = 0$$

Problem: Estimate $\lim_{x\to 0} \frac{\tan^{-1}x}{x}$.

Solution: Assume $\tan^{-1}x = h \Rightarrow x = \tan h$

When $x \to 0$, then $h \to 0$

Now $\lim_{x\to 0} \frac{\tan^{-1}x}{x} = \lim_{h\to 0} \frac{h}{\tan h}$

$$= \lim_{h\to 0} \frac{h}{\frac{\sin h}{\cos h}}$$

$$= \lim_{h\to 0} \frac{h}{\sin h} \cdot \cos h$$

$$= \lim_{h\to 0} \frac{h}{\sin h} \cdot \lim_{h\to 0} \cos h$$

$$= 1$$

* Since $\lim_{h\to 0} \frac{h}{\sin h}$ exists. i.e.

$\lim_{h\to 0} \frac{h}{\sin h} = 1$

A HM Mahbubur Rahman, PhD.
Assistant Professor
MNS Dept.
BRACU

Some Extra Problems

A [Using Squeeze Th^m]

1. Compute: $\lim\limits_{x\to\infty} \dfrac{2-\cos x}{x+3}$ Ans: 0

2. Compute $\lim\limits_{x\to\infty} \dfrac{x^2(2+\sin^2 x)}{x+100}$ Ans: ∞ (does not exist)

3. Compute $\lim\limits_{x\to 0^-} x^3\cos\left(\dfrac{2}{x}\right)$ Ans: 0

4. Compute $\lim\limits_{x\to -\infty} \dfrac{5x^2-\sin(3x)}{x^2+10}$. Ans: 5

B. Find the limit if it exist, otherwise explain why it does not exist:

(a) $\lim\limits_{x\to -3} \dfrac{|x^2-9|\,(x+2)}{x^2+7x+12}$

(b) $\lim\limits_{x\to 4} \dfrac{16-x^2}{x^3-5x^2+4x}$

(3) $\lim\limits_{x\to 0} \dfrac{x}{3-\sqrt{x+9}}$

(4) $\lim\limits_{x\to 0} f(x)$ where $f(x)=\begin{cases} e^{-|x|/2}, & -1<x<0 \\ x^2, & 0<x<2 \end{cases}$.

(5) $\lim\limits_{t\to -1} \dfrac{t+1}{|t+1|}$

(6) $\lim\limits_{x\to -3} \dfrac{\sqrt{2x+22}-4}{x+3}$.

Continuity

Now we defined a function

$$f(x) = \begin{cases} x^2, & x < 0 \\ x-0, & x \geq 0 \end{cases}$$

Since when $x=0$, $f(0)=0$

Now $\quad \text{L.H.L.} = \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} x^2 = 0$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} x = 0$$

So, L.H.L. = R.H.L which implies limit exists and we have a limiting value at $x=0$.

Here we also define $f(0)=0$.

We use here extra information which is function is defined at $x=0$ when extra condition also satisfy then we say the function is continuous at that point. Now we write the definition.

Definition of Continuity:

A function f is said to be continuous at $x=a$ provided the following conditions are satisfied:

① $f(a)$ is defined

② $\lim\limits_{x \to a} f(x)$ exists

and ③ $\lim\limits_{x \to a} f(x) = f(a)$.

A H M Mahbubur Rahman, PhD.
Assistant Professor
MNS Dept.
BRACU

Example: Given a function,

$$f(x) = \begin{cases} x^2+1 & , x>0 \\ 1 & , x=0 \\ x+1 & , x<0 \end{cases}$$

Show that the function $f(x)$ is continuous at $x=0$.

Solⁿ at $x=0$:

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} (x+1) = 1$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} x^2+1 = 1$$

L.H.L. = R.H.L.. i.e. limit exist.

Since $f(0) = 1$ i.e. $f(0)$ is defined.

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1 = f(0)$.

Therefore, $f(x)$ is continuous at $x=0$.

Problem: Given that $f(x) = \begin{cases} e^{-|x|/2} & , -1<x\leq 0 \\ x^2 & , 0<x<2 \end{cases}$

Does the above function continuous at $x=0$?

Sol: First we want to show that function is defined at $x=0$

$$f(0) = e^{-|0|/2} = e^0 = 1$$

Now for limit exists.

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} e^{-|x|/2} = \cancel{\lim_{x\to 0}} e = 1$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} x^2 = 0$$

So, L.H.L. $\neq$ R.H.L i.e. limit doesnot exist.

Therefore, the above function is not continuous at $x=0$.

Properties of Continuity

If the functions f and g are continuous at a then

1. $f \pm g$ is continuous at a.
2. $f \cdot g$ is continuous at a.
3. f/g is continuous at a ~~if~~ and ~~(g)~~ $g(a) \neq 0$.
4. If $\lim_{x\to a} g(x) = L$, and if f is continuous then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

A H M Mahbubur Rahman, Ph.D
Assistant Professor
MNS Dept.
BRACU

Some Extra Problems :

1. Given that $f(x) = \begin{cases} x^2 - 6x + 10 & , x \geq 2 \\ 4 - x & , x < 2 \end{cases}$,

Test the continuity at $x = 2$.

2. Given that $g(x) = \begin{cases} 2x+3 & , x \leq 4 \\ 7 + \frac{16}{x} & , x > 4 \end{cases}$.

Does the function , $g(x)$, continuous at $x = 4$?

3. Test the continuity of the following function at $x=0$:

$$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2} - 1} & ; \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}$$

Problem: Find a value of the constant K for which the following function is continuous at $x=1$:

$$f(x) = \begin{cases} 7x-2 & , x \le 1 \\ Kx^2 & , x > 1 \end{cases}$$

Sol^n: [First we show the function is defined at $x=1$.

$$f(1) = 7.1 - 2 = 5$$] (No need)

Now, $\text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} (7x-2) = 5$

$\text{R.H.L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} Kx^2 = K$

Since the function is continuous at $x=1$ so,

$\text{L.H.L.} = \text{R.H.L}$

i.e. $K = 5$ ✻.

Problem: Find a value of the constant K s.t. it will make the following function continuous at $x=2$:

$$f(x) = \begin{cases} Kx^2 & , x \le 2 \\ 2x+K & , x > 2 \end{cases}$$

Sol: Do Yourself !!!

A H M Mahbubur Rahman, PhD
Assistant Professor
MNS Dept.
BRACU