Rolle's Theorem

Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

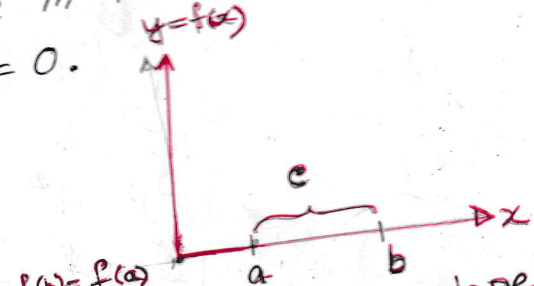
Verify the hypothesis of Rolle's Theorem for the following functions:

1. $f(x) = x^2 - 6x + 8$; $[2, 4]$

check

$f(a) = f(2) = 2^2 - 6(2) + 8 = 0$
$f(b) = f(4) = 4^2 - 6(4) + 8 = 0$

$\therefore$ by Rolle's theorem there is at least one number c in the interval $(2, 4)$ such that $f'(c) = 0$.



$f'(c) = 0 = \text{slope}$

slope of a line on x-axis or parallel to x-axis is '0'.

$f(c) = c^2 - 6c + 8$

$f'(c) = 2c - 6 = 0$

$\Rightarrow 2c = 6$

$\Rightarrow \boxed{c = 3}$

2. $f(x) = \cos x$; $[\pi/2, 3\pi/2]$

check

$f(a) = f(\pi/2) = \cos \pi/2 = 0$
$f(b) = f(3\pi/2) = \cos(\frac{3\pi}{2}) = 0$

$\therefore$ by Rolle's theorem there is at least one number c in the interval $(\pi/2, 3\pi/2)$ such that $f'(c) = 0$

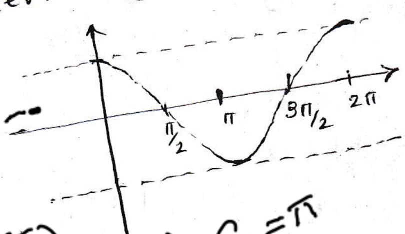
$f(c) = \cos c$

$f'(c) = -\sin c = 0$

$\Rightarrow \sin c = 0$ on the interval $(\frac{\pi}{2}, 3\pi/2)$

$c = \sin^{-1} 0$

$\boxed{c = \pi}$, $2\pi, 3\pi, \cdots$



$\pi \in (\frac{\pi}{2}, \frac{3\pi}{2})$ $\therefore c = \pi$

3. $f(x) = \frac{1}{2}x - \sqrt{x}$; $[0, 4]$

<u>check</u>

$$\left.\begin{aligned} f(a) = f(0) &= \tfrac{1}{2}(0) - \sqrt{0} = 0 \\ f(b) = f(4) &= \tfrac{1}{2}(4) - \sqrt{4} = \tfrac{4}{2} - \sqrt{4} \\ &= 0 \end{aligned}\right\}$$

$\therefore$ by Rolle's theorem there is at least one number c in the interval $(0, 4)$ such that $f'(c) = 0$

$$f(c) = \frac{1}{2}c - \sqrt{c}$$

$$f'(c) = \frac{1}{2} - \frac{1}{2}c^{-1/2} = 0$$

$$\Rightarrow \frac{1}{2} = \frac{1}{2}c^{-1/2}$$

$$\Rightarrow \frac{1}{1} = \frac{1}{\sqrt{c}}$$

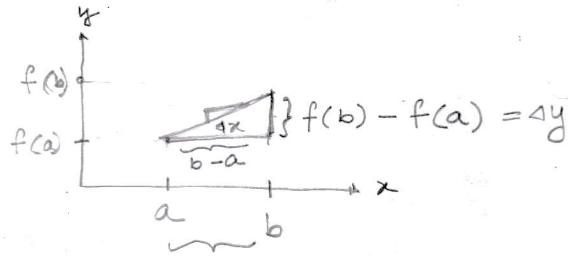
$$\sqrt{c} = 1$$

$$(\sqrt{c})^2 = 1^2$$

$$\boxed{c = 1}$$

$1 \in [0, 4]$

For MVT



$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x} = \text{slope}$$

Mean-Value Theorem

Let f be differentiable on (a,b) and continuous on $[a,b]$.
Then there is at least one number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Verify the hypothesis of Mean value theorem for the following functions:

1. $f(x) = x^3 + x - 4$; $[-1, 2]$

$a = -1$
$b = 2$

$f(a) = f(-1) = (-1)^3 + (-1) - 4$
$= -1 - 1 - 4 = -6$

$f(b) = f(2) = (2)^3 + (2) - 4$
$= 8 + 2 - 4 = 6$

$\therefore$ by mean value theorem there is at least one number c in $(-1, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$f(c) = c^3 + c - 4$

$f'(c) = 3c^2 + 1$

$$3c^2 + 1 = \frac{6-(-6)}{2-(-1)}$$

$$= \frac{12}{3}$$

$3c^2 + 1 = 4$

$3c^2 = 3$

$c^2 = 1$

$c = \pm 1$

$c_1 = -1$, $c_2 = 1$

$(a,b) = (-1,2)$

$\because c \in (a,b)$

$\therefore c = 1$

Ans:

2. $f(x) = \sqrt{x+1}$; $[0, 3]$

$f(a) = f(0) = \sqrt{0+1} = 1$

$f(b) = f(3) = \sqrt{3+1} = 2$

$f(c) = \sqrt{c+1}$

$f'(c) = \frac{1}{2}(c+1)^{-1/2} = \frac{f(b)-f(a)}{b-a} = \frac{2-1}{3-0} = \frac{1}{3}$

$(c+1)^{-1/2} = \frac{2}{3}$

$(c+1)^{1/2} = 3/2$

$\left(\sqrt{(c+1)}\right)^2 = (3/2)^2$

$c+1 = 9/4$

$c = 9/4 - 1 = 5/4$

$\boxed{c = \frac{5}{4}}$

3. $f(x) = \sqrt{25-x^2}$; $[0, 5]$

$f(a) = f(0) = \sqrt{25-0^2} = 5$

$f(b) = f(5) = \sqrt{25-5^2} = 0$

$f(c) = \sqrt{25-c^2}$

$f'(c) = \frac{1}{2}(25-c^2)^{-1/2}(-2c)$

$= -c(25-c^2)^{-1/2} = \frac{f(b)-f(a)}{b-a} = \frac{0-5}{5-0} = -1$

$\frac{c}{\sqrt{25-c^2}} = 1$

$\frac{c^2}{25-c^2} = 1$

$c^2 = 25 - c^2$

$2c^2 = 25$

$c^2 = \frac{25}{2}$

$c = \pm\frac{5}{\sqrt{2}}$

$\pm\frac{5}{\sqrt{2}} \approx 3.53553$
$\approx \pm 3.53553$

$c = -\frac{\sqrt{5}}{2}$ does not exist ~~in~~ over the interval $(0, 5)$

$\therefore$ $\boxed{c = \frac{5}{\sqrt{2}}}$