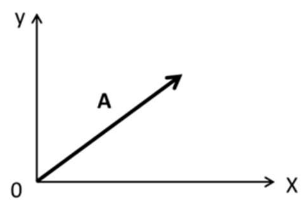
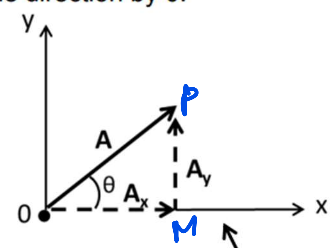
Mathematically it is expressed (in a rectangular coordinates (x,y) as:

With the magnitude expressed by the length of A:



With the magnitude expressed by the length of A: and the direction by θ:



Vector quantities can be DECOMPOSED into components as illustrated

With MAGNITUDE:

$$|\mathbf{A}| = \sqrt{|\mathbf{A}_x|^2 + |\mathbf{A}_y|^2} = \sqrt{A_x^2 + A_y^2}$$

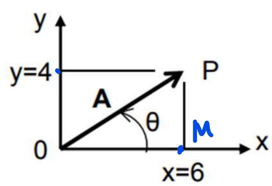
and DIRECTION:

$$\tan\theta = \frac{A_y}{A_x}$$

Example 3.1: A vector **A** in Figure 3.2(b) has its two components along the x- and y-axis with respective magnitudes of 6 units and 4 units. Find the magnitude and direction of the vector **A**.

A_x A_y

Solution: Let us first illustrate the vector **A** in the x-y plane:



The vector A may be expressed in terms of unit vectors **I** and **j** as:

A = 6**i** + 4**J**

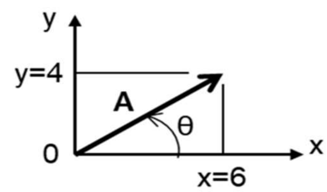
$\vec{A} = A_x\hat{i} + A_y\hat{j}$
$= 6\hat{i} + 4\hat{j}$

And the magnitude of vector **A** is:

A = $|\boldsymbol{A}|$ = A = $\sqrt{A_x^2 + A_y^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 7.21\ units$

and the angle θ is obtained by: $tan\theta = \frac{A_y}{A_x} = \frac{4}{6}$ = 0.67

Example 3.5



Determine the angle θ of a position vector **A** = 6**i** + 4**j** in an x-y plane.

Solution: We may express the vector A in the form of:

$$\mathbf{A} = 6\mathbf{i} + 4\mathbf{j}$$

with **i** and **j** to be the respective unit vectors along the x- and y-coordinates with the magnitudes: x = 6 units and y =4 units.

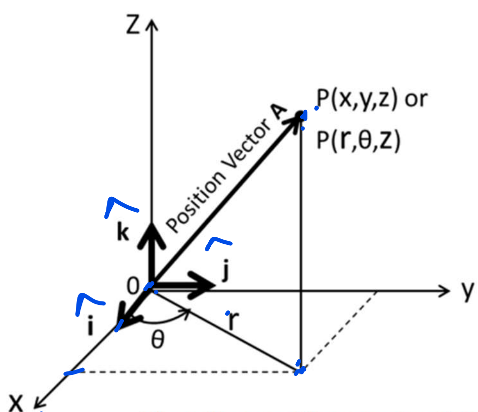
We may thus compute the magnitude of the vector A to be:

$$|A| = \sqrt{x^2 + y^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 7.21\,units$$

The angle θ may be calculated to be:

$$cos\theta = \frac{x}{|A|} = \frac{6}{7.21} = 0.832 \quad \text{or} \quad \theta = 33.68^o$$

3.2 Vectors expressed in terms of Unit Vectors in Rectangular coordinate Systems - A simple and convenient way to express vector quantities



Let: **i** = unit vector along the x-axis
j = unit vector along the y-axis
k = unit vector along the z-axis

in a rectangular coordinate system (x,y,z), or a cylindrical polar coordinate system (r,θ,z).

All unit vectors **i**, **j** and **k** have a **magnitudes of 1.0 (i.e. unit)**

Then the **position vector A** (with it "root" coincides with th origin of the coordinate system) expressed in the following form:

$$\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where x = magnitude of the component of Vector A in the x-coordinate
y = magnitude of the component of Vector A in the y-coordinate
z = magnitude of the component of Vector A in the z-coordinate

We may thus evaluate the magnitude of the vector **A** to be the sum of the magnitudes of all its components as:

$$|\mathbf{A}| = A = \sqrt{\left(\sqrt{x^2 + y^2}\right)^2 + z^2} = \sqrt{x^2 + y^2 + z^2}$$

x z y- z

Example 3.3 If vectors **A** = 2**i** +4**k** and **B** = 5**j** +6**k**, determine: (a) what planes do these two vectors exist, and (b) their respective magnitudes. (c) the summation of these two vectors

Solution:

(a) Vector **A** may be expressed as: **A** = 2**i** +0**j** + 4**k**, so it is positioned in the x-z plane in Figure 3.3. Vector **B** on the other hand may be expressed as: **B** = 0**i** + 5**j** +6**k** with no value along the x-coordinate. So, it is positioned in the y-z plane in a rectangular coordinate system.

(b) The magnitude of vector **A** is: $|\mathbf{A}| = A = \sqrt{2^2 + 4^2} = \sqrt{20} = 4.47$

and the magnitude of vector **B** is: $|\mathbf{B}| = B = \sqrt{5^2 + 6^2} = \sqrt{61} = 7.81$

(c) The addition of these two vectors is:

$\vec{A} + \vec{B} = \cancel{|A| + |B|} = (2+0)\boldsymbol{i} + (0+5)\mathbf{j} + (4+6)\mathbf{k}$ = 2**i** + 5**j** + 10**k** .

EXAMPLE. Find the dot product of $\vec{a} = (1, -1, \sqrt{3})$ and $\vec{b} = (2, -1, 0)$. Also find the angle between the two vectors.

SOLUTION:

$$\begin{aligned}
\vec{a}.\vec{b} &= 1.2 + (-1)(-1) + \sqrt{3}.0 = 3 \\
|\vec{a}| &= \sqrt{1+1+3} = \sqrt{5} \\
|\vec{b}| &= \sqrt{4+1} = \sqrt{5} \\
\cos\theta &= \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = 3/5 \\
\theta &= \cos^{-1}(\frac{3}{5})
\end{aligned}$$

$\vec{a}.\vec{b} = |a|.|b| \cos\theta.$

$\Rightarrow 3 = \sqrt{5}.\sqrt{5}\cos\theta$

$\Rightarrow \cos\theta = \frac{3}{5}$

$\Rightarrow \cos\theta = \cos^{-1}(\frac{3}{5})$

PROPOSITION *Let $\vec{a}$ and $\vec{b}$ be non-zero vectors. The vectors, $\vec{a}$ and $\vec{b}$, are* orthogonal / *perpendicular to each other if and only if $\vec{a} \cdot \vec{b} = 0$.*

SOLUTION: Let $\vec{a}$ and $\vec{b}$ be non zero vectors, then $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$. Let θ be the angle between the two vectors.

$$\begin{aligned}
& \vec{a} \cdot \vec{b} = 0 \\
\Leftrightarrow \quad & |\vec{a}|.|\vec{b}| \cos\theta = 0 \\
\Leftrightarrow \quad & \cos\theta = 0; \text{ Since } |\vec{a}| \neq 0, |\vec{b}| \neq 0, \text{ we can divide the equation by them.} \\
\Leftrightarrow \quad & \theta = \pi/2
\end{aligned}$$

Therefore the angle between $\vec{a}$ and $\vec{b}$ is 90 degrees.

EXAMPLE. Determine if the vectors $\vec{a} = (1, -1, 1)$ and $\vec{b} = (1, 2, 1)$ are perpendicular to each other.

$$\vec{a} \cdot \vec{b} = (1)(1) + (-1)(2) + (1)(1) = 0$$

Therefore by the above proporsition, the vectors are perpendicular to each other.

Example 3.7 Determine (a) the result of dot product of the two vectors: **A** = 2**i** + 7**j** + 15**k** and **B** = 21**i** + 31**j** + 41**k**, and (b) the angle between these two vectors

Solution: (a) By using the above expression, we may get the result of the dot product of vectors **A** and **B** to be: **A●B** = 2x21 + 7x31+ 15x41 = 874

(b) We need to compute the magnitudes of both vectors **A** = 16.67 and **B** = 55.52 units, which lead to the angle θ between vectors **A** and **B** to be:

$|A| = \sqrt{2^2+7^2+15^2}$

$|B| = \sqrt{21^2+31^2+41^2}$

$$\cos\theta = \frac{\overset{\vec{A}.\vec{B}}{\cancel{A_xB_x+A_yB_y+A_zB_z}}}{|AB|} = \frac{2\times21+7\times31+15\times41}{16.67\times55.52} = \frac{874}{925.52} = 0.94433 \qquad \therefore \theta = 19.21^o$$

Mathematical expression of Cross product of vectors:

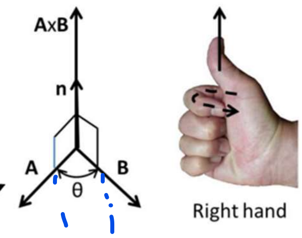
Cross product of two vectors applies to have both vectors lie on a plane but with the result of this Product in the direction perpendicular to the plane of these two vectors, as described in physical Situations illustrated in the preceding slide.

Cross products of two vectors A and B can be expressed as:

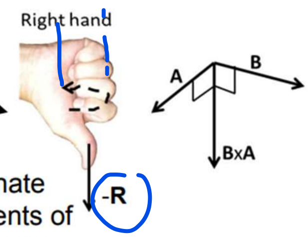
$$\boldsymbol{A} \times \boldsymbol{B} = \mathbf{R}$$

where the result of the cross product of vectors **A** and **B** is a Vector **R.**

*The resultant vector **R** is along the direction that is perpendicular* To the plane on which the vectors **A** and **B** lie.



One will realize that $\boldsymbol{A} \times \boldsymbol{B} \neq \boldsymbol{B} \times \boldsymbol{A}$. ***the case*** $\mathbf{B} \times \boldsymbol{A} = -\boldsymbol{R}$.



Cross product of vectors involving unit vectors:

A = A_x**i** + A_y**j** + A_z**k** and vector **B** = B_x**i** + B_y**j** + B_z**k** in a rectangular coordinate system with A_x, A_y and A_z, B_x, B_y and B_z being the magnitude of components of vector **A** and **B** along the x-, y- and z-coordinates respectively. We will have::

$$\mathbf{R} = \mathbf{A}x\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Cross Product

DEFINITION. *Given 2 vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ in 3-dimensional space the* **cross product** *$\vec{a} \times \vec{b}$ is a vector defined as follows:*

$$\vec{a} \times \vec{b} = (|\vec{a}|.|\vec{b}| \sin\theta)\vec{n}$$

where $\vec{n}$ is a unit vector which is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, determined by the right-hand-rule. i.e. place your right hand on $\vec{a}$ and curl it towards $\vec{b}$, $\vec{n}$ now points in the direction of your thumb. This is the geometric definition of the cross-product.

We also have an algebraic definition of the cross-product, but before we can define it, we need to be able to compute determinants!

EXAMPLE. Consider the two vectors $\vec{a} = (2, 1, 0)$ and $\vec{b} = (-1, 1, 0)$ in 3-dimensional space. Find the cross-product of the two vectors and verify that it is indeed perpendicular to both $\vec{a}$ and $\vec{b}$. Also find the magnitude of $\vec{a} \times \vec{b}$.

SOLUTION:

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} \vec{k} \\
&= 0\vec{i} - 0\vec{j} + 3\vec{k} \\
&= 0(1, 0, 0) - 0(0, 1, 0) + 3(0, 0, 1) \\
&= (0, 0, 3)
\end{aligned}$$

Observe that the two vectors lie in the XY-plane and the cross product lies on the Z-axis and is indeed perpendicular to the plane containing the two vectors. But we can verify this more concretely using the dot product. Let $\vec{n} = \vec{a} \times \vec{b} = (0, 0, 3)$.

$$\begin{aligned}
\vec{n} \cdot \vec{a} &= (2, 1, 0).(0, 0, 3) = 0 + 0 + 0 = 0. \\
\vec{n} \cdot \vec{b} &= (-1, 1, 0).(0, 0, 3) = 0 + 0 + 0 = 0.
\end{aligned}$$

Since both the dot products are zero, we can conclude that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
Finally the magnitude of $\vec{a} \times \vec{b}$ is given by,

$$|\vec{a} \times \vec{b}| = \sqrt{0+0+9} = 3.$$

EXAMPLE. Find a unit vector in the direction of the following vectors:

(i) $\vec{v}_1 = (2, -3)$.

(ii) $\vec{v}_2 = (1, -2, 5)$.

(iii) $\vec{v}_3 = (1, 1, 1, 1)$.

SOLUTION: To get a unit vector in the direction of a given vector, all we need to do is scale the size of the vector by the inverse of its length.

(i) $\vec{v}_1 = (2, -3)$.
The length of $\vec{v}_1$ is $|\vec{v}_1| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$.
A unit vector u_1 in the direction of v_1 may then be obtained as follows:

$u_1 = \frac{v_1}{|v_1|}$

$$u_1 = \frac{1}{\sqrt{13}}(2, -3) = (\frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}).$$

(ii) $\vec{v}_2 = (1, -2, 5)$.
The length of $\vec{v}_2$ is $|\vec{v}_2| = \sqrt{1^2 + (-2)^2 + 5^2} = \sqrt{30}$.
A unit vector u_2 in the direction of v_2 may then be obtained as follows:

$$u_2 = \frac{1}{\sqrt{30}}(1, -2, 5) = (\frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}}, \frac{5}{\sqrt{30}}).$$

(iii) $\vec{v}_3 = (1, 1, 1, 1)$.

The length of $\vec{v}_3$ is $|\vec{v}_3| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$.

A unit vector u_3 in the direction of v_3 may then be obtained as follows:

$$u_3 = \frac{1}{2}(1, 1, 1, 1) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

EXAMPLE. Consider the two vectors $\vec{a} = (2, 1, -4)$ and $\vec{b} = (3, -2, 5)$ in 3-dimensional space. Find the cross-product of the two vectors.

SOLUTION:

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -4 \\ 3 & -2 & 5 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -4 \\ -2 & 5 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \vec{k} \\
&= -3\vec{i} - 22\vec{j} - 7\vec{k} \\
&= (-3, -22, -7)
\end{aligned}$$

6. Let $\vec{u} = (2, 3, 5)$, $\vec{v} = (1, 1, -1)$. Find

(a) The unit vector in the direction of $\vec{v} = (1, 1, -1)$ is,

$$\frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}(1, 1, -1) = \frac{1}{\sqrt{3}}(1, 1, -1) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}).$$

(b)

$$3\vec{u} - 5\vec{v} = 3(2, 3, 5) - 5(1, 1, -1) = (6, 9, 15) - (5, 5, -5) = (1, 4, 20).$$

(c)

$$\vec{u} \cdot \vec{v} = (2, 3, 5).(1, 1, -1) = 2 + 3 - 5 = 0.$$

(d)

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \vec{k} \\
&= (3(-1) - (5)(1))\vec{i} - (2(-1) - (5)(1))\vec{j} + ((2)(1) - (3)(1))\vec{k} \\
&= -8\vec{i} + 7\vec{j} - \vec{k} \\
&= (-8, 7, -1)
\end{aligned}$$

7. Determine if $\vec{u} = (-2, 2, 1, -1)$ is perpendicular to $\vec{v} = (-2, -3, 1, 1)$.

SOLUTION:

$$\vec{u}.\vec{v} = (-2)(-2) + (2)(-3) + (1)(1) + (-1)(1) = -2 \neq 0.$$

Hence the two vectors are not perpendicular.

Example 3.10

If vectors **A** = **i** – **j** +2**k** and **B** = 2**i** + 3**j** - 4**k**, determine **A** x **B = C**.

Solution

We may use Equation (3.18) for the solution to be;

$$\mathbf{A}x\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 3 & 4 \end{vmatrix} = \mathbf{C}$$

in which the vector **C = [(-1x4-(2x3]i + (1x4-2x2)j + [1x3-(-1x2)]k = -10i +5k**

Example 3.12

If a position vector **r** in a rectangular coordinate system has both its magnitude and direction varying with time t, and its two components $\mathbf{r_x}$ and $\mathbf{r_y}$ vary with time according to functions:

$r_x = 1 – t^2$ and $r_y = 1+2t$ respectively.

Determine the rate of variation of the position vector with respect to time variable t.

Solution:

We may express the position vector **r** in the following form:

$$\mathbf{r}(t) = \mathbf{r_x}(t) + \mathbf{r_y}(t) = r_x(t)\,\mathbf{i} + r_y(t)\,\mathbf{j}$$

in which **i** and **j** are the unit vectors along the x- and y-coordinate respectively.

The rate of change of the position vector **r**(t) with respect to variable t may be obtained as::

$$\frac{d\mathbf{r}(t)}{dt} = \frac{dr_x(t)}{dt}\mathbf{i} + \frac{dr_y(t)}{dt}\mathbf{j} = \left[\frac{d}{dt}\left(1-t^2\right)\right]\mathbf{i} + \left[\frac{d}{dt}\left(1+2t\right)\right]\mathbf{j} = (-2t)\mathbf{i} + 2\mathbf{j}$$

Derivatives of the products of vectors:

$$\frac{\partial}{\partial x}(\mathbf{A}\bullet\mathbf{B}) = \mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\bullet\mathbf{B} \qquad \frac{\partial}{\partial x}(\mathbf{A}\mathrm{x}\mathbf{B}) = \mathbf{A}\mathrm{x}\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\mathrm{x}\mathbf{B}$$

$$\frac{\partial}{\partial y}(\mathbf{A}\bullet\mathbf{B}) = \mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{A}}{\partial y}\bullet\mathbf{B} \qquad \frac{\partial}{\partial y}(\mathbf{A}\mathrm{x}\mathbf{B}) = \mathbf{A}\mathrm{x}\frac{\partial \mathbf{B}}{\partial y} + \frac{\partial \mathbf{A}}{\partial y}\mathrm{x}\mathbf{B}$$

$$\frac{\partial}{\partial z}(\mathbf{A}\bullet\mathbf{B}) = \mathbf{A}\bullet\frac{\partial \mathbf{B}}{\partial z} + \frac{\partial \mathbf{A}}{\partial z}\bullet\mathbf{B} \qquad \frac{\partial}{\partial z}(\mathbf{A}\mathrm{x}\mathbf{B}) = \mathbf{A}\mathrm{x}\frac{\partial \mathbf{B}}{\partial z} + \frac{\partial \mathbf{A}}{\partial z}\mathrm{x}\mathbf{B}$$

Example 3.13

Determine d**A** if vector function **A**(x,y,z) **=** (x^2 siny) **i +** (z^2 cosy) **j –** (xy^2) **k**.

Solution:

$$dA = \frac{\partial \mathbf{A}}{\partial \mathbf{x}}dx + \frac{\partial \mathbf{A}}{\partial y}dy + \frac{\partial \mathbf{A}}{\partial z}dz = \left[(\sin y)\mathbf{i}\frac{d}{dx}(x^2) - (y^2)\mathbf{k}\frac{dx}{dx}\right]dx + \left[x^2\mathbf{i}\frac{d}{dy}(\sin y) + z^2\mathbf{j}\frac{d}{dy}(\cos y) - x\hat{k}\cdot\frac{d}{dy}(y^2)\right]dy$$

$$+\left[(\cos y)\mathbf{j}\frac{d}{dz}(z^2)\right]dz$$

$$= \left[(2x\sin y)\mathbf{i} - y^2\mathbf{k}\right]dx + \left[(x^2\cos y)\mathbf{i} - (z^2\sin y)\mathbf{j} - 2xy\mathbf{k}\right]dy + \left[(2z\cos y)\mathbf{j}\right]dz$$

$$= (2x\sin y\,dx + x^2\cos y\,dy)\mathbf{i} + (2z\cos dz - z^2\sin y\,dy)\mathbf{j} - (y^2\,dx + 2x\,y\,dy)\mathbf{k}$$

2.1 Planes

Just as it is easy to write the equation of a line in 2D space, it is easy to write the equation of a plane in 3D space.

The point-normal equation of a plane

A vector perpendicular to a plane is said to be *normal to the plane* and is called a *normal vector*, or simply a *normal.*

To write the equation of a plane we need a point $P(x_0, y_0, z_0)$ on the plane and a normal vector $\vec{n} = (a, b, c)$ to the plane.

Let $P = (x_0, y_0, z_0)$ be a point on the plane and $\vec{n}$ be a vector perpendicular to the plane. Then a point $Q(x, y, z)$ lies on the plane,

$\Leftrightarrow$ the vector $\vec{PQ}$ lies on the plane,

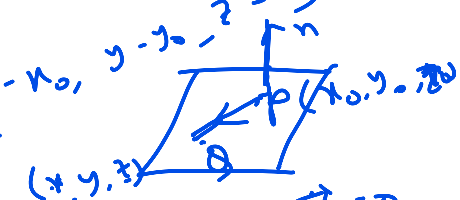
$\Leftrightarrow$ $\vec{PQ}$ and $\vec{n}$ are perpendicular,

$\Leftrightarrow$ $\vec{n} \cdot \vec{PQ} = 0$,

$\Leftrightarrow$ $(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$,

$\Leftrightarrow$ $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

$\vec{PQ} = (x - x_0, y - y_0, z - z_0)$



$\vec{PQ} \cdot \vec{n} = 0$

DEFINITION. *The point-normal equation of a plane that contains the point $P(x_0, y_0, z_0)$ and has normal vector $\vec{n} = (a, b, c)$ is*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

EXAMPLE. Let P be a plane determined by the points $A = (1, 2, 3)$, $B = (2, 3, 4)$, and $C = (-2, 0.3)$. Find a vector which is normal to the plane. Find an equation of the plane.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -3 & -2 & 0 \end{vmatrix}$$

$\vec{AB} = \vec{B} - \vec{A} = (1, 1, 1)$

$\vec{AC} = \vec{C} - \vec{A}$

$= (-3, -2, 0)$

$= (0 + 2)\hat{i} - (0 + 3)\hat{j} + (-2 + 3)\hat{k}$

$= 2\hat{i} - 3\hat{j} + \hat{k} = (2, -3, 1)$

(a, b, c)

$a = 2, b = -3$
$c = 1$

SOLUTION: We need a point on the plane and a normal to the plane. The vector $\vec{AB}\times\vec{AC} = (2,-3,1)$ is a normal to the plane and we take $A = (1,2,3)$ $= (x_0, y_0, z_0)$ as a point on the plane (you can choose B or C instead of A if you want). The equation on the plane in point-normal form is:

$$2(x-1) - 3(y-2) + (z-3) = 0$$

or equivalently,

$$2x - 3y + z = -1$$

Observe that the coefficients of x, y and z are $(2,-3,1)$ which is the normal to the plane.

2.2 Lines

Vector equation of a line

To write the vector equation of a line, we need a point $P(x_0, y_0, z_0)$ on the line and a vector $\vec{v} = (a,b,c)$ that is parallel to the line.

DEFINITION. *The vector equation of a line that contains the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\vec{v} = (a,b,c)$ is:*

$$P + t\vec{v} = \vec{r}, \text{ where } t \text{ is scalar.}$$

or,

$$\begin{aligned} (x_0, y_0, z_0) + t(a,b,c) &= (x,y,z) \\ (x_0 + ta, y_0 + tb, z_0 + tc) &= (x,y,z) \end{aligned}$$

Parametric equation of a line

The parametric equation of a line is derived from the vector equation of a line.

DEFINITION. *The parametric equation of a line that contains the point $P(x_0, y_0, z_0)$ and is parallel to the vector $\vec{v} = (a,b,c)$ is:*

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

EXAMPLE. Let L which passes through the points $\boxed{P(1,1,1)}$ and $Q(3,2,1)$. Find a vector which is parallel to the line. Find the vector-equation and parametric equation of the line.

$= \vec{Q} - \vec{P}$

SOLUTION: The vector $\vec{PQ} = (2,1,0)$ is parallel to the line and we take the point $P(1,1,1)$ on the line.
The vector equation of the line:

$$(1,1,1) + t(2,1,0) = (x,y,z)$$

The parametric equations of the line:

$$\begin{aligned} x &= 1+2t \\ y &= 1+t \\ z &= 1 \end{aligned}$$

~~EXAMPLE. Find the equation of the plane which contains the point $(0,1,2)$ and is perpendicular to the line $(1,1,1)+t(2,1,0)=(x,y,z)$.~~

3.5.3 Gradient, Divergence and Curl

Gradient, divergence and curl are frequently used when dealing with variations of vectors using a vector operator designated by ∇(Pronounced del)defined as follows:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \quad \text{in a rectangular coordinate system}$$

3.5.3.1 Gradient

Gradient relates to the variation of the **magnitudes of vector quantities** with a scalar quantity ϕ, defined by:

$$grad\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Example 3.14-A**:**
Determine the gradient of a scalar quantity $\phi = xy^2z^3$ which is the magnitude of a vector **A** $= A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$:

$$grad\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$= \frac{\partial}{\partial x}(xy^2z^3)\hat{i} + \frac{\partial}{\partial y}(xy^2z^3)\hat{j} + \frac{\partial}{\partial z}(xy^2z^3)\hat{k} = y^2z^3\hat{i} + 2xyz^3\hat{j} + 3xy^2z^2\hat{k}$$

3.5.3.2 **Divergence:**

Divergence of vector function **A**(x,y,z) implies the RATE of "growth" or "contraction" of this vector function in its components along the coordinates. The divergence of the vector function **A(**x,y,z**)** is defined as:

$$div\,\mathbf{A} = \nabla \bullet \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Example 3.14-B**:**

Determine the div (φ**A**) if the gradient of a scalar quantity $\phi = xy^2z^3$ which is the magnitude of a vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$:

$$div\,\mathbf{A} = \nabla \bullet \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

where A_x, A_y and A_z are the magnitude of the components of vector **A** along the x-, y- and z-coordinate respectively.

3.5.3.3 **Curl:**

The curl of a vector function **A** is related to the "rotation" of this vector. It is defined as:

$$curl\,\mathbf{A} = \nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_y & A_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ A_x & A_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ A_x & A_y \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

Example 3.14-B:

Determine the curl (φ**A**) if the gradient of a scalar quantity $\phi = xy^2z^3$ which is the magnitude of a vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$:

$$curl\,(\phi\mathbf{A}) = \nabla \times (\phi\mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left(\phi A_x\,\mathbf{i} + \phi A_y\,\mathbf{j} + \phi A_z\,\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_y & \phi A_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_y & \phi A_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \phi A_x & \phi A_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \phi A_x & \phi A_y \end{vmatrix}$$

$$= \left(\frac{\partial \phi A_z}{\partial y} - \frac{\partial \phi A_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial \phi A_z}{\partial x} - \frac{\partial \phi A_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial \phi A_y}{\partial x} - \frac{\partial \phi A_x}{\partial y}\right)\mathbf{k}$$

Example 3.15

A rigid body is traveling along the x-axis in a rectangular coordinate system. Assume that the instantaneous position of the body may be represented by a function of $x(t) = 11t^2 - 2t^3$ in meter, in which t is time in second. We further Assume the mass of the body is negligible.

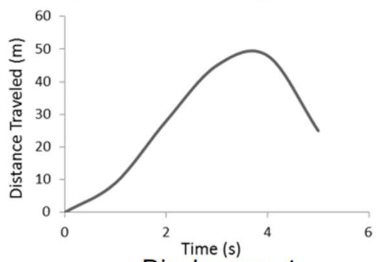
Determine the vector functions of the velocity and acceleration of the moving rigid body.

Solution:

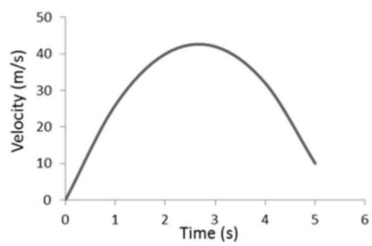
We may determine the magnitude of the velocity vector function **v**(t) and the magnitude of the acceleration vector function **a**(t) as shown below:

$$\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt} = \frac{d}{dt}\left(11t^2 - 2t^3\right) = 22t - 6t^2 \qquad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d}{dt}\left(22t - 6t^2\right) = 22 - 12t$$

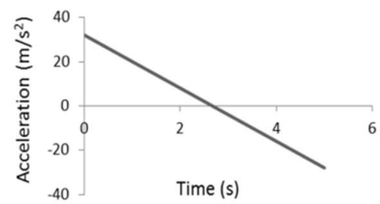
The vector functions of the velocity and acceleration of the moving rigid body thus can be expressed as:: **v**(t) = (22t – 6t²) **i** and **a**(t) = (22-12t) **i** . Graphic solutions are:



Displacement

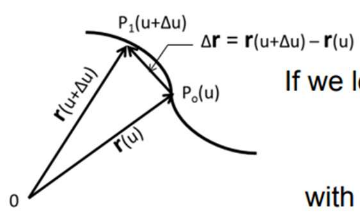


Velocity



Acceleration

3.7.2 **Plane Curvilinear Motion in Rectangular Coordinates-**Cont'd



If we let the RATE of change of the Position vector **r**(t) to be defined as:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

with **i**, **j** = unit vector along the x- and y- coordinates respectively

We will have both the velocity acceleration vectors expressed to be:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \left[\frac{dx(t)}{dt}\right]\mathbf{i} + \left[\frac{dy(t)}{dt}\right]\mathbf{j} \quad \text{and} \quad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \left[\frac{d^2x(t)}{dt^2}\right]\mathbf{i} + \left[\frac{d^2y(t)}{dt^2}\right]\mathbf{j}$$

Example 3.16 If the position of a rigid body moving on a curved path at time t is: **r**(t) = t**i** +2t³**j.** Determine the velocity v(t) and acceleration vector functions at that instant.

$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j}$

$\vec{v} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j}$

Solution: We will first express the components of the position vector from the given expression of r(t) to be: x(t) = t and y(t) = 2t³, from which, we will get the velocity and acceleration vector functions and their respective magnitudes as:

$$\mathbf{v}(t) = \left[\frac{d(t)}{dt}\right]\mathbf{i} + \left[\frac{d(2t^3)}{dt}\right]\mathbf{j} = 1\mathbf{i} + 6t^2\mathbf{j} \quad \text{and} \quad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \left[\frac{d(1)}{dt}\right]\mathbf{i} + \left[\frac{d(6t^2)}{dt}\right]\mathbf{j} = 0\mathbf{i} + 12t\,\mathbf{j} = 12t\,\mathbf{j}$$

and the magnitudes:

$$v(t) = |\mathbf{v}(t)| = \sqrt{1^2 + (6t^2)^2} = \sqrt{1 + 36t^4} \text{ m/s} \quad \text{and} \quad a(t) = |\mathbf{a}(t)| = \sqrt{(12t)^2} = 12t \text{ m/s}^2$$

4.2 Directional Derivative

For a function of 2 variables $f(x,y)$, we have seen that the function can be used to represent the surface

$$z = f(x,y)$$

and recall the geometric interpretation of the partials:

(i) $f_x(a,b)$-represents the rate of change of the function $f(x,y)$ as we vary x and hold $y=b$ fixed.

(ii) $f_y(a,b)$-represents the rate of change of the function $f(x,y)$ as we vary y and hold $x=a$ fixed.

We now ask, at a point P can we calculate the slope of f in an arbitrary direction?

Recall the definition of the vector function ∇f,

$$\vec{\nabla} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}). = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

We observe that,

$$\begin{aligned} \nabla f \cdot \hat{i} &= f_x \\ \nabla f \cdot \hat{j} &= f_y \end{aligned}$$

$$\vec{\nabla} f \cdot \hat{i} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)(\hat{i}) = \frac{\partial f}{\partial x} = f_x$$

This enables us to calculate the directional derivative in an arbitrary direction, by taking the dot product of ∇f with a unit vector, $\vec{u}$, in the desired direction.

Definition. *The directional derivative of the function f in the direction $\vec{u}$ denoted by $D_{\vec{u}}f$, is defined to be,*

$$D_{\vec{u}}f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|}$$

Example. What is the directional derivative of $f(x,y) = x^2 + xy$, in the direction $\vec{i} + 2\vec{j}$ at the point $(1,1)$?

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$
$$= (2x+y)\hat{i} + x\hat{j}$$

SOLUTION: We first find ∇f.

$$\begin{aligned}
\nabla f &= (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \\
&= (2x + y, x) \\
\nabla f(1,1) &= (3,1)
\end{aligned}$$

Let $u = \vec{i} + 2\vec{j}$.

$$|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}.$$

$$\begin{aligned}
D_{\vec{u}} f(1,1) &= \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} \\
&= \frac{(3,1).(1,2)}{\sqrt{5}} \\
&= \frac{(3)(1) + (1)(2)}{\sqrt{5}}
\end{aligned}$$

2. At any point P, $\nabla f(P)$ **is perpendicular to the level set** through that point.

EXAMPLE. 1. Let $f(x, y) = x^2 + y^2$ and let $P = (1, 2, 5)$. Then P lies on the graph of f since $f(1, 2) = 5$. Find the slope and the direction of the steepest ascent at P on the graph of f

SOLUTION: • We use the first property of the Gradient vector. The direction of the steepest ascent at P on the graph of f is the direction of the gradient vector at the point $(1, 2)$.

$$\begin{aligned} \nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= (2x, 2y) \\ \nabla f(1,2) &= (2, 4). \end{aligned}$$

• The slope of the steepest ascent at P on the graph of f is the magnitude of the gradient vector at the point $(1, 2)$.

$$|\nabla f(1, 2)| = \sqrt{2^2 + 4^2} = \sqrt{20}.$$

2. Find a normal vector to the graph of the equation $f(x, y) = x^2 + y^2$ at the point $(1, 2, 5)$. Hence write an equation for the tangent plane at the point $(1, 2, 5)$.

SOLUTION: We use the second property of the gradient vector. For a function g, $\nabla g(P)$ **is perpendicular to the level set**. So we want our surface $z = x^2 + y^2$ to be the level set of a function.
Therefore we define a new function,

$$g(x, y, z) = x^2 + y^2 - z.$$

Then our surface is the level set

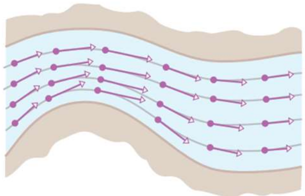
$$\begin{aligned} g(x, y, z) &= 0 \\ x^2 + y^2 - z &= 0 \\ z &= x^2 + y^2 \end{aligned}$$

$$
\begin{aligned}
\nabla g &= (\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) \\
&= (2x, 2y, -1) \\
\nabla g(1,2,5) &= (2,4,-1)
\end{aligned}
$$

By the above property, $\nabla g(P)$ is perpendicular to the level set $g(x,y,z) = 0$. Therefore $\nabla g(P)$ is the required normal vector.
Finally an equation for the tangent plane at the point $(1,2,5)$ on the surface is given by

$$2(x-1) + 4(y-2) - 1(z-5) = 0.$$

VECTOR FIELDS



▲ **Figure 15.1.2**

> **15.1.1 DEFINITION** A ***vector field*** in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane. Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\mathbf{F}(P)$ in 3-space.

Notice that a vector field is really just a vector-valued function. The term "vector field" is commonly used in physics and engineering.

Observe that in this definition there is no reference to a coordinate system. However, for computational purposes it is usually desirable to introduce a coordinate system so that vectors can be assigned components. Specifically, if $\mathbf{F}(P)$ is a vector field in an xy-coordinate system, then the point P will have some coordinates (x, y) and the associated vector will have components that are functions of x and y. Thus, the vector field $\mathbf{F}(P)$ can be expressed as

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

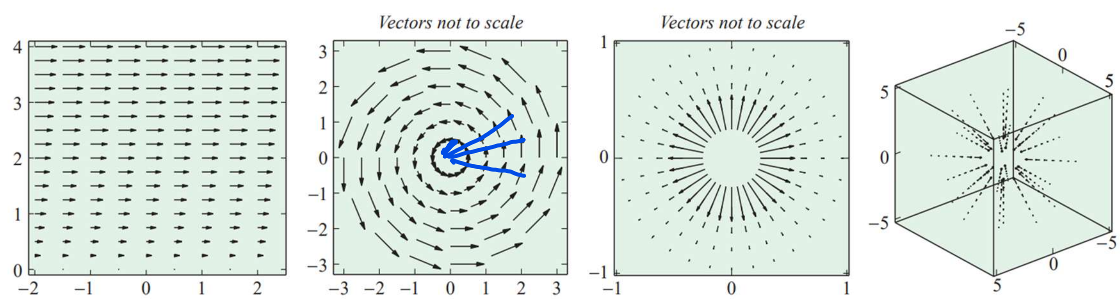
Similarly, in 3-space with an xyz-coordinate system, a vector field $\mathbf{F}(P)$ can be expressed as

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

GRAPHICAL REPRESENTATIONS OF VECTOR FIELDS

A vector field in 2-space can be pictured geometrically by drawing representative field vectors $\mathbf{F}(x, y)$ at some well-chosen points in the xy-plane. But, just as it is usually not possible to describe a plane curve completely by plotting finitely many points, so it is usually not possible to describe a vector field completely by drawing finitely many vectors. Nevertheless, such graphical representations can provide useful information about the general behavior of the field if the vectors are chosen appropriately. However, graphical representations of vector fields require a substantial amount of computation, so they are usually created using computers. Figure 15.1.3 shows four computer-generated vector fields. The vector field in part (a) might describe the velocity of the current in a stream at various depths. At the bottom of the stream the velocity is zero, but the speed of the current increases as the depth decreases. Points at the same depth have the same speed. The vector field in part (b) might describe the velocity at points on a rotating wheel. At the center of the wheel the velocity is zero, but the speed increases with the distance from the center. Points at the same distance from the center have the same speed. The vector field in part (c) might describe the repulsive force of an electrical charge—the closer to the charge, the greater the force of repulsion. Part (d) shows a vector field in 3-space. Such pictures tend to be cluttered and hence are of lesser value than graphical representations of vector fields in 2-space. Note also that the

vectors in parts (b) and (c) are not to scale—their lengths have been compressed for clarity.



$\mathbf{F}(x, y) = \frac{1}{5}\sqrt{y}\,\mathbf{i}$ $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{10(x^2 + y^2)^{3/2}}$ $\mathbf{F}(x, y, z) = -\dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$

(a)   (b)   (c)   (d)

Example 1.47 (Positive and negative divergence). The field $\vec{\mathbf{F}} = x\hat{\imath} + y\hat{\jmath}$ has positive divergence $\vec{\nabla} \cdot \vec{\mathbf{F}} = 1 + 1 + 0 = 2 > 0$, thus we see in the left plot of Figure 11 that it is somewhat spreading. The field $\vec{\mathbf{G}} = (-x - 2y)\hat{\imath} + (2x - y)\hat{\jmath}$ has negative divergence $\vec{\nabla} \cdot \vec{\mathbf{G}} = -1 - 1 + 0 = -2 < 0$ and we see from the right plot that the field is converging.

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(x) + \frac{\partial f}{\partial y}(y) = 1 + 1 = 2$

$\frac{\partial}{\partial x}(-x-2y) + \frac{\partial}{\partial y}(2x-y)$

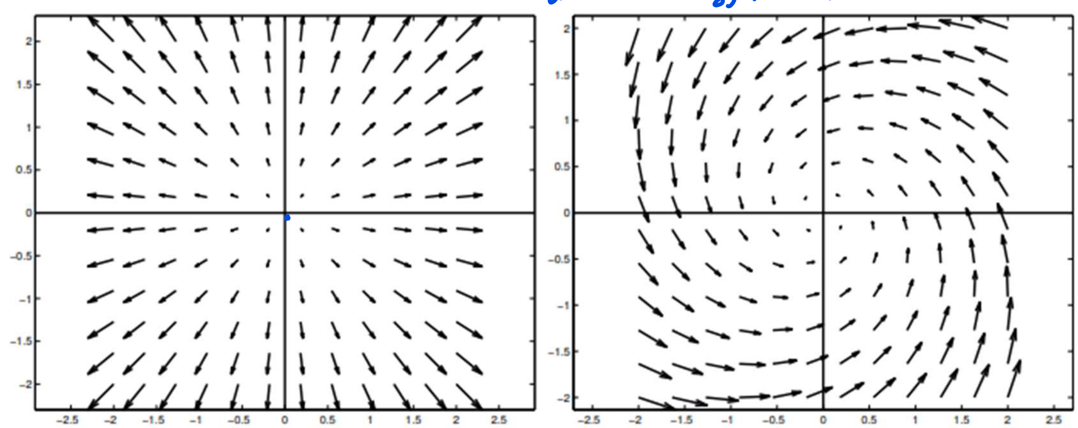


Figure 11: *A representation of the fields $\vec{\mathbf{F}} = x\hat{\imath} + y\hat{\jmath}$ (left) and $\vec{\mathbf{G}} = (-x - 2y)\hat{\imath} + (2x - y)\hat{\jmath}$ (right) from Example 1.47. $\vec{\mathbf{F}}$ has positive divergence and $\vec{\mathbf{G}}$ negative.*

Intuitively, the value of the divergence of a vector field at a particular point gives a measure of the "net mass flow" or "flux density" of the vector field in or out of that point. To understand what such a statement means, imagine that the vector field $\mathbf{F}$ represents velocity of a fluid. If $\nabla \cdot \mathbf{F}$ is zero at a point, then the rate at which fluid is flowing into that point is equal to the rate at which fluid is flowing out. Positive divergence at a point signifies more fluid flowing out than in, while negative divergence signifies just the opposite. We will make these assertions more precise, even prove them, when we have some integral vector calculus at our disposal. For now, however, we remark that a vector field $\mathbf{F}$ such that $\nabla \cdot \mathbf{F} = 0$ everywhere is called **incompressible** or **solenoidal**.

EXAMPLE 2 The vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

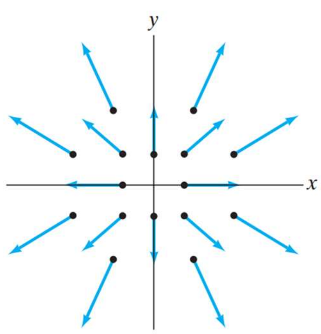


Figure 3.38 The vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of Example 2.

This vector field is shown in Figure 3.38. At any point in $\mathbf{R}^2$, the arrow whose tail is at that point is longer than the arrow whose head is there. Hence, there is greater flow *away* from each point than into it; that is, $\mathbf{F}$ is "diverging" at every point. (Thus, we see the origin of the term "divergence.")

The vector field $\mathbf{G} = -x\mathbf{i} - y\mathbf{j}$ points in the direction opposite to the vector field $\mathbf{F}$ of Figure 3.38 (see Figure 3.39), and it should be clear how $\mathbf{G}$'s divergence of -2 is reflected in the diagram. ◆

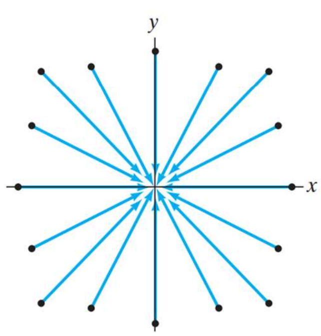


Figure 3.39 The vector field $\mathbf{G} = -x\mathbf{i} - y\mathbf{j}$ of Example 2.

EXAMPLE 3 The constant vector field $\mathbf{F}(x, y, z) = \mathbf{a}$ shown in Figure 3.40 is incompressible. Intuitively, we can see that each point of $\mathbf{R}^3$ has an arrow representing $\mathbf{a}$ with its tail at that point and another arrow, also representing $\mathbf{a}$, with its head there.

The vector field $\mathbf{G} = y\mathbf{i} - x\mathbf{j}$ has

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) \equiv 0.$$

A sketch of $\mathbf{G}$ reveals that it looks like the velocity field of a rotating fluid, without either a source or a sink. (See Figure 3.41.) ◆

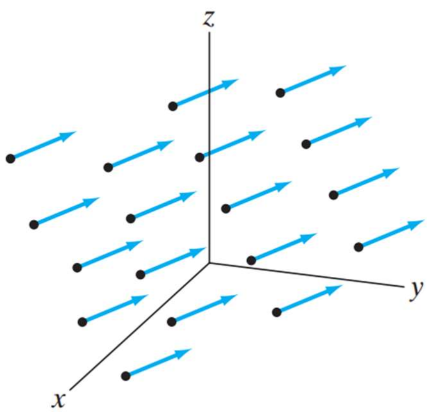


Figure 3.40 The constant vector field $\mathbf{F} = \mathbf{a}$.

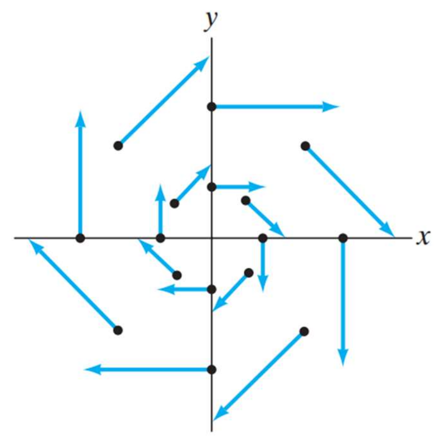


Figure 3.41 The vector field $\mathbf{G} = y\mathbf{i} - x\mathbf{j}$ resembles the velocity field of a rotating fluid.

1.5 Special vector fields and potentials

Quiz

Definition 1.65. Consider a vector field $\vec{\mathbf{F}}$ defined on a domain $D \subset \mathbb{R}^3$.
If $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is called **irrotational** (or curl-free).
If $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is called **solenoidal** (or divergence-free, or incompressible).
If $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ for some scalar field φ, then $\vec{\mathbf{F}}$ is called **conservative** and φ is called **scalar potential** of $\vec{\mathbf{F}}$.
If $\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{A}}$ for some vector field $\vec{\mathbf{A}}$, then $\vec{\mathbf{A}}$ is called **vector potential** of $\vec{\mathbf{F}}$.

13. Can you tell in what portions of $\mathbf{R}^2$, the vector fields shown in Figures 3.43–3.46 have positive divergence? Negative divergence?

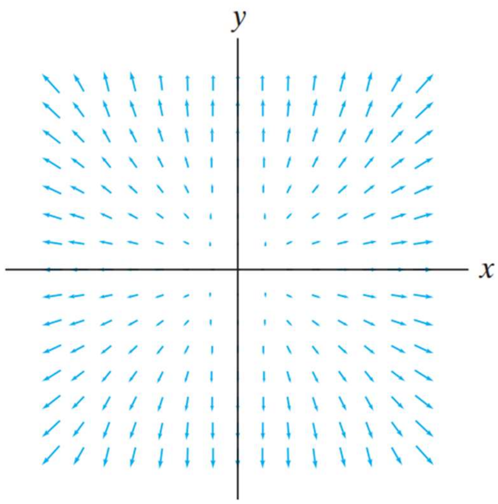


Figure 3.43 Vector field for Exercise 13(a).

div $F > 0$ on all of $\mathbb{R}^2$

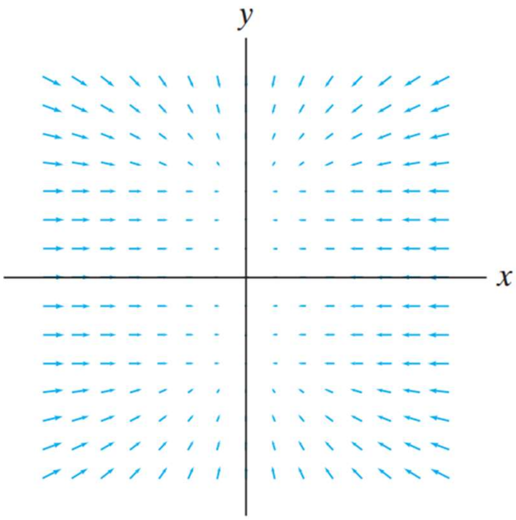
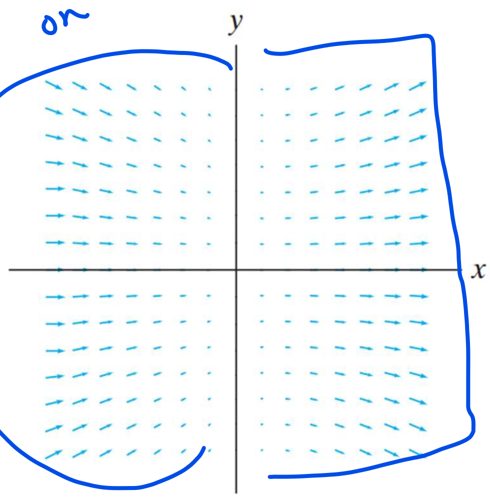


Figure 3.44 Vector field for Exercise 13(b).

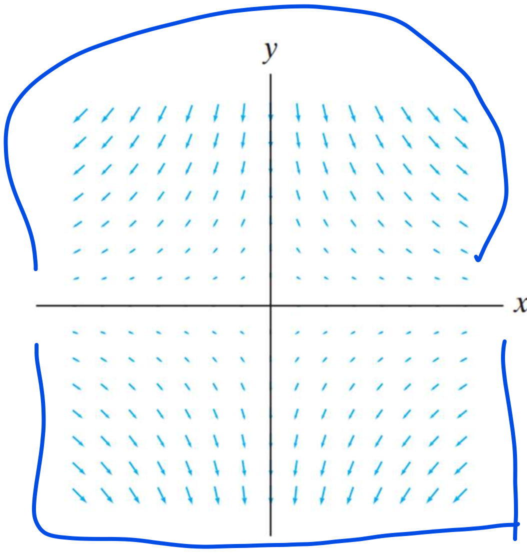
div F < 0 on all of $\mathbb{R}^2$

div F < 0 on $\{(x,y) \mid x < 0\}$



div F > 0 on $\{(x,y) \mid x > 0\}$

Figure 3.45 Vector field for Exercise 13(c).



$\rightarrow$ div F < 0 on $\{(x, y) \mid y > 0\}$

$\rightarrow$ div F > 0 on $\{(x, y) \mid y < 0\}$.

Figure 3.46 Vector field for Exercise 13(d).