Differential Equations (D.E.)

A differential equation is any equation which contains derivatives either ordinary derivatives / partial derivatives.

Ex: (i) $\frac{dy}{dx} + y = 3$ (Ordinary Differential Equation (ODE)).

(ii) $u_{tt} = c^2 u_{xx}$ (Wave Equation), let $u(x,t)$ be a function of two variables (P.D.E.)

(iii) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = e^x$ (O.D.E)

Order & Degree of ODEs:

Order: $\boxed{\text{Highest derivative}}$ present in the ODE

Degree: $\boxed{\text{Exponent}}$ of Highest derivative in the ODE.

(i) $\left(\frac{d^2x}{dt^2}\right)^1 + k^2x = 0$ 2nd order, 1st degree ODE.

(ii) $(x^2+y^2)\,dx + 2xy\,dy = 0$, 1st order, 1st degree ODE

$\Rightarrow 2xy\frac{dy}{dx} + (x^2+y^2) = 0$

(iii) $\sqrt{\left(\frac{dy}{dx}\right)^2 + 3y} = \left(\frac{d^2y}{dx^2}\right)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 + 3y = \left(\frac{d^2y}{dx^2}\right)^2$$

2nd order, 2nd degree.

(iv) $3y^2\left(\frac{dy}{dx}\right)^4 + \left(\frac{d^2y}{dx^2}\right)^5 + \underline{\left(\frac{d^3y}{dx^3}\right)^2} = \sin x^2$,

3nd order,

2nd degree.

Classification of ODE:

An ODE can be classified into two ways.

(i) Linearity of the ODE

(ii) Homogeneity of the ODE.

<u>Linearity of the ODE</u>: An ODE is said to be <u>linear</u> if (i) y & its derivatives $\underline{y'}, \underline{y''}, \underline{y'''}, \ldots, \underline{y^{(n)}}$ all have power <u>1</u>.

(ii) the coefficients of the <u>derivatives</u> $a_0(x), a_1(x), a_2(x), \ldots$ all depend on <u>x</u> / <u>constant</u>. $a_0(x)\frac{dy}{dx} + a_1(x)\frac{d^2y}{dx^2} + \cdots + a_n(x) = 0$

(iii) There is no trigonometric term/exponential term of <u>dependent</u> variable.

If an ODE doesnot satisfy these conditions we call it non-linear ODE.

(i) $\cancel{x^3}\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} - 5y = e^x$,

3rd order
1st degree
linear.

(ii) $(y-x)\,dx + 4x\,dy = 0$.

$\Rightarrow 4x\dfrac{dy}{dx} + y - x = 0 \longrightarrow$ 1st order 1st degree & linear.

$\Rightarrow \dfrac{dy}{dx} = \dfrac{x-y}{4x} = f(x,y)$

(iii) $\underline{(1-y)}\dfrac{dy}{dx} + 2y = e^x$, $\longrightarrow$ nonlinear.

(iv) $\dfrac{dy}{dx} + \sin y = 0 \longrightarrow$ nonlinear.

(i) Homogeneity of ODE :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x) = f(x)$$

nth order, 1st degree, linear.

The above ODE is homogeneous if $f(x)=0$ &

non-homogeneous if $f(x)\neq 0$.

(i) $2y'' + 3y' - 5y = 0$ (homogeneous).

(ii) $x^3 y''' + 6y' = e^x$ (non homogeneous).

Separation of Variable :

1st order ODE + linear/nonlinear + homogeneous.

$$\frac{dy}{dx} = f(x,y) \quad , \quad \frac{dy}{dx} = x+y$$

$$\frac{dy}{dx} = 3$$

$$\Rightarrow \boxed{\frac{dy}{dx} = g(x)\,h(y)} \Rightarrow \text{Separation of Variable}$$

$$\boxed{y(x) = ??}$$

$$\Rightarrow \frac{dy}{h(y)} = g(x)\,dx$$

$$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx$$

Ex:

$$(1+x)\,dy - y\,dx = 0 \Rightarrow \boxed{(1+x)\frac{dy}{dx} - y = 0}$$

dependent (dy); independent (dx)

$$\Rightarrow (1+x)\,dy = y\,dx$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{1+x}$$

$\ln(ab) = \ln a + \ln b$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\Rightarrow \ln y = \ln|1+x| + \ln c$$

$$\Rightarrow \ln y = \ln|c(1+x)| \Rightarrow \boxed{y(x) = c(1+x)}.$$

$$y = c(1+x)$$

$y(x) = c(1+x) \xrightarrow{\text{sol}}$ claim: $(1+x)\dfrac{dy}{dx} - y = 0$

$$L.H.S. = (1+x)\frac{d}{dx}\left(c(1+x)\right) - c(1+x)$$

$$= c(1+x) - c(1+x) = 0 = R.H.S.$$

$y(x) = 0 \rightarrow$ verify ??

Ex) $(1+y^2)\,dx + (1+x^2)\,dy = 0$

$\Rightarrow (1+x^2)\,dy = -(1+y^2)\,dx$

$\Rightarrow \int \frac{dy}{1+y^2} = -\int \frac{dx}{1+x^2}$

$\Rightarrow \tan^{-1} y = -\tan^{-1} x + \tan^{-1} c$

$\Rightarrow \tan^{-1} y + \tan^{-1} x = \tan^{-1} c$

$\Rightarrow \tan^{-1} \frac{y+x}{1-xy} = \tan^{-1} c \Rightarrow \frac{y+x}{1-xy} = c$

$\tan^{-1} a + \tan^{-1} b = \tan^{-1} \frac{a+b}{1-ab}$

$\underline{2.2}$ 1-14, 19-26 Dennis G Zill

12. $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$

$\Rightarrow 2y\cos^3 3x\,dy = -\sin 3x\,dx$

$\Rightarrow 2y\,dy = \dfrac{-\sin 3x}{\cos^3 3x}\,dx$

$\Rightarrow \int 2y\,dy = \int \dfrac{du}{3u^3}$

$\Rightarrow 3y^2 = -\dfrac{1}{2u^2} + C$

$\Rightarrow 3y^2 + \dfrac{1}{2\cos^2 3x} = C$. $\Rightarrow$ General Solution

let $\cos 3x = u$

$\Rightarrow -3\sin 3x\,\cancel{dx} = du$

$\Rightarrow -\sin 3x\,dx = \dfrac{du}{3}$

Initial Value Problem:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

23- $\dfrac{dx}{dt} = 4(x^2+1)$, $\boxed{x\left(\frac{\pi}{4}\right) = 1}$ $\Rightarrow t$

$\Rightarrow \dfrac{dx}{4(x^2+1)} = dt$

$\Rightarrow \dfrac{1}{4}\displaystyle\int \frac{dx}{x^2+1} = \int dt$

$\Rightarrow \frac{1}{4}\tan^{-1} x = t + C \quad \cdots (1)$

putting $t = \pi/4$ in (1)

$\frac{1}{4}\tan^{-1} x\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + C$

$\Rightarrow \frac{1}{4}\tan^{-1} 1 = \frac{\pi}{4} + C$

$$c = \frac{1}{4}\cdot\frac{\pi}{4} - \frac{\pi}{4}$$

$$\Rightarrow c = \frac{\pi - 4\pi}{16} = \frac{-3\pi}{16}$$

We put $c = \frac{-3\pi}{4}$ in equation (1)

$$\boxed{\frac{1}{4}\tan^{-1} x = t + \frac{-3\pi}{4}} \Rightarrow \text{Particular Solution}$$