In a similar manner, we can find the fourth derivative, fifth derivative and, in genera
derivative of y w. r. t. x by differentiating successively the given function y w. r. t. x f
five times and n times.

Following notations are generally used for the successive derivatives of y w. r. t. x :

	First derivative	**Second derivative**	**Third derivative**	**....**	**n^{th} derivative**
or	y_1	y_2	y_3	....	y_n
or	$f'(x)$	$f''(x)$	$f'''(x)$	....	$f^n(x)$
or	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	....	$\frac{d^ny}{dx^n}$
or	Dy	D^2y	D^3y	....	D^ny

Example 1 : If $\sqrt{x+y}+\sqrt{y-x}=c$, prove that $y_2=\frac{2}{c^2}$.

Solution : Given that $\sqrt{x+y}+\sqrt{y-x}=c$

Squaring both sides, we get

$$x+y+y-x+2\sqrt{y^2-x^2}=c^2$$

This $\Rightarrow 2\sqrt{y^2-x^2}=c^2-2y$

Again, squaring both sides, we get

$$4\left(y^2-x^2\right)=c^4-4c^2y+4y^2$$

This $\Rightarrow 4x^2-4c^2y+c^4=0$

Differentiating both sides w. r. t. x, we get

$$8x-4c^2y_1=0$$

This $\Rightarrow 2x-c^2y_1=0$

Again, differentiating both sides w. r. t. x, we get

$$2-c^2y_2=0$$

This $\Rightarrow y_2=\frac{2}{c^2}$.

Example 2 : If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$.

Solution : Given that $y = a\cos(\log x) + b\sin(\log x)$.

Differentiating both sides w. r. t. x, we get

$$y_1 = -a\sin(\log x)\,.\,\frac{1}{x} + b\cos(\log x)\,.\,\frac{1}{x}$$

This $\Rightarrow xy_1 = -a\sin(\log x) + b\cos(\log x)$

Again, differentiating both sides w. r. t. x, we get

$$xy_2 + y_1 = -a\cos(\log x)\,.\,\frac{1}{x} - b\sin(\log x)\,.\,\frac{1}{x}$$

This $\Rightarrow x^2y_2 + xy_1 = -a\cos(\log x) - b\sin(\log x)$

$$= -(a\cos(\log x) + b\sin(\log x))$$

$$= -y \quad (\because y = a\cos(\log x) + b\sin(\log x) \text{ given})$$

This $\Rightarrow x^2y_2 + xy_1 + y = 0.$

Example 3 : If $y = e^{a\sin^{-1}x}$, prove that $(1-x^2)y_2 - xy_1 = a^2y$.

Solution : Given that $y = e^{a\sin^{-1}x}$.

Differentiating both sides w. r. t. x, we get

$$y_1 = e^{a\sin^{-1}x}\left(a\frac{1}{\sqrt{1-x^2}}\right)$$

This $\Rightarrow y_1 = \dfrac{ay}{\sqrt{1-x^2}} \qquad \left(\because y = e^{a\sin^{-1}x} \text{ given}\right)$

$$\Rightarrow y_1\sqrt{1-x^2} = ay$$

Squaring both sides, we get

$$y_1^2(1-x^2) = a^2y^2$$

Again, differentiating both sides w. r. t. x, we get

$$2y_1y_2(1-x^2) + y_1^2(-2x) = a^2(2yy_1)$$

This $\Rightarrow 2y_1\left[(1-x^2)y_2 - xy_1\right] = 2y_1(a^2y)$

$$\Rightarrow (1-x^2)y_2 - xy_1 = a^2y.$$

Example 5 : Find n^{th} differential coefficient of $\frac{1}{(1+x)^2}$.

Solution : Let $y=\frac{1}{(1+x)^2}$

Differentiating both sides w. r. t. x successively, we get

$$y_1=(-2)(1+x)^{-3}$$

$$y_2=(-2)(-3)(1+x)^{-4}$$

$$y_3=(-2)(-3)(-4)(1+x)^{-5}$$

....

....

Proceeding in this way, we get

$$y_n=(-2)(-3)(-4)\ldots\ldots\ldots\ldots(-(n+1))(1+x)^{-(n+2)}$$

i.e., $$y_n=\frac{(-1)^n\,(n+1)!}{(1+x)^{(n+2)}}.$$

Lecture 9

Maclaurin polynomial

Definition: If f can be differentiable n times at 0, then we define the nth Maclaurin polynomial for f to be

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example: Find the Maclaurin polynomials P_0, P_1, P_2, P_3 and P_n for e^x.

Sol^n:

let $f(x) = e^x$

$$f'(x) = e^x = \cancel{f''(x)}$$

$$f''(x) = e^x = f'''(x) = \cdots = f^{(n)}(x)$$

$$f(0) = e^0 = 1, \quad f'(0) = e^0 = 1 = f''(0) = f'''(0) = \cdots = f^{(n)}(0)$$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)x = 1 + 1.x = 1 + x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2!}x^2$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Therefore, $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

Taylor polynomial

Definition: If f can be differentiated n times at x_0, then we define the nth Taylor polynomial for f about $x=x_0$ to be

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Example: Find the nth Taylor polynomial for $\frac{1}{x}$ about $x=1$ and express it in sigma notation.

Sol^n: say, $f(x) = \frac{1}{x} = x^{-1}$ $\qquad f(1) = 1$

$f'(x) = -1 \cdot x^{-2}$ $\qquad f'(1) = (-1)\cdot 1 = -1$

$f''(x) = (-1)(-2)x^{-3} = 2x^{-3}$ $\qquad f''(1) = (-1)(-2)\cdot 1 = 2!$

$f'''(x) = 2(-3)x^{-4}$ $\qquad f'''(1) = (-1)(-2)(-3)\cdot 1 = -3!$

$f^{iv}(x) = 2(-3)(-4)x^{-5}$ $\qquad f^{iv}(1) = (-1)(-2)(-3)(-4) = 4!$

$\vdots$

$f^{(n)}(x) = 2(-3)(-4)(-5)\cdots(-n)\, x^{-(n+1)}$; $\quad f^{(n)}(1) = (-1)(-2)\cdots(-n)\cdot 1 = (-1)^n n!$

So, we can write $f^{(K)}(x) = (-1)^K K!$

~~at~~ $x=1$, the Taylor polynomial for $1/x$:

$$\cancel{\sum_{K=0}^{n} (-1)^K (x-1)^K =}$$

$$\sum_{K=0}^{n} \frac{f^{(n)}(1)}{K!}(x-1)^K = \sum_{K=0}^{n} \frac{(-1)^K K!}{K!}(x-1)^K = \sum_{K=0}^{n} (-1)^K (x-1)^K$$

Maclaurin and Taylor Series

Definition [Taylor Series]

If f has derivatives of all orders at x_0, then we call the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cdots$$

the Taylor series for f about $x = x_0$

For special case $x_0 = 0$, ~~we~~ the series we ~~called~~ Maclaurin series for f.

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

Example: Find the Maclaurin series for $\sin x$.

Sol^n let

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{IV}(x) = \sin x \qquad f^{IV}(0) = 0$

$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$f^{(6)}(x) = -\sin x \qquad f^{(6)}(0) = 0 \qquad f^{(7)}(0) = -1$

$f^{(7)}(x) = -\cos x$

$\cancel{\text{So } P_n(x) = 0 + 1\cdot x + 0 + \frac{-1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \cdots +}$

$\cancel{= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)}$

$P_0(x) = 0$

$P_1(x) = 0 + x$

$P_2(x) = 0 + x + 0$

$P_3(x) = 0 + x + 0 - \frac{x^3}{3!}$

$P_4(x) = 0 + x + 0 - \frac{x^3}{3!} + 0$

$P_5(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!}$

$P_6(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0$

$P_7(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!}$

Odd number we can write $\boxed{2K+1}$

even number we can write $\boxed{2K}$

$$P_{2K+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^K \frac{x^{2K+1}}{(2K+1)!} \quad , K = 0, 1, 2, \cdots$$

Thus, the Maclaurin series for $\sin(x)$ is

$$\sum_{K=0}^{\infty} (-1)^K \frac{x^{2K+1}}{(2K+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^K \frac{x^{2K+1}}{(2K+1)!} + \cdots$$