MAT 110

Taylor & Maclaurine's Polynomials

Taylor's Polynomial:

Named after English mathematician Brook Taylor

It represents a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point such as $x=a$ or in powers of $(x-a)$.

$$P_n(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$$

$$\cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

$$f^{(0)}(a)\frac{(x-a)^0}{0!}$$

$$= \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

This is a higher order differentiation (n^{th} order) of a polynomial series in a certain domain $x=a$.

Examples Expand the following functions into Taylor's polynomial.

① $y = \ln(x)$ in the powers of $(x-2)$. $\boxed{x=2}$

Let $f(x) = \ln(x) \longrightarrow f(2) = \ln 2$

$$f'(x) = \frac{1}{x} = \frac{0!}{x} = \frac{(-1)^0 0!}{x} \longrightarrow f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} = \frac{-1!}{x^2} = \frac{(-1)^1 1!}{x^2} \text{ or } \frac{(-1)^3 1!}{x^2} \longrightarrow f''(2) = -\frac{1}{4}$$

$$f'''(x) = \frac{2}{x^3} = \frac{2!}{x^3} = \frac{(-1)^2 2!}{x^3} \text{ or } \frac{(-1)^4 2!}{x^3} \longrightarrow f'''(2) = \frac{2}{8} = \frac{1}{4}$$

$$f^{IV}(x) = \frac{-6}{x^4} = \frac{-3!}{x^4} = \frac{(-1)^3 3!}{x^4} \text{ or } \frac{(-1)^5 3!}{x^4}.$$

$\vdots$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n} \longrightarrow f^{(n)}(2) = \frac{(-1)^{n-1}(n-1)!}{2^n}$$

$$P_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^n(a)\frac{(x-a)^n}{n!} = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

$$P_n(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!} + f'''(2)\frac{(x-2)^3}{3!} \cdots$$

$$\cdots + f^{(n)}(2)\frac{(x-2)^n}{n!}$$

$$= \ln 2 + \frac{1}{2}(x-2) + \left(-\frac{1}{4}\right)\frac{(x-2)^2}{2} + \left(\frac{1}{4}\right)\frac{(x-2)^3}{6} + \cdots + \frac{(-1)^{n-1}(n-1)!}{2^n}\frac{(x-2)^n}{n!}$$

$$= \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \cdots + \frac{(-1)^{n-1}(x-2)^n}{n 2^n}.$$

$$\boxed{\frac{(n-1)!}{n!} = \frac{(n-1)(n-2)\cdots 1}{n(n-1)(n-2)\cdots 1} = \frac{1}{n}}$$

② $y=e^{ax}$ while $x=1$.

$f(x)=e^{ax} \longrightarrow f(1)=e^{a}$

$f'(x)=ae^{ax} \longrightarrow f'(1)=ae^{a}$

$f''(x)=a^2e^{ax} \longrightarrow f''(1)=a^2e^{a}$

$f'''(x)=a^3e^{ax} \longrightarrow f'''(1)=a^3e^{a}$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(x)=a^ne^{ax} \longrightarrow f^{(n)}(1)=a^ne^{a}$

$$P_n(x)=\sum_{k=0}^{n} f^{(k)}(1)\frac{(x-1)^k}{k!}$$

$$P_n(x)=f(1)+f'(1)(x-1)+f''(1)\frac{(x-1)^2}{2!}+f'''(1)\frac{(x-1)^3}{3!}+\cdots$$

$$\cdots+f^{n}(1)\frac{(x-1)^n}{n!}$$

$$=e^a+ae^a(x-1)+a^2e^a\frac{(x-1)^2}{2}+a^3e^a\frac{(x-1)^3}{6}+\cdots$$

$$\cdots+a^ne^a\frac{(x-1)^n}{n!}$$

Taylor's Polynomial

1. $y = \frac{sinx}{cosx}$ *in the powers of* $(x - \frac{\pi}{2})$: *Evaluate nth term of Taylor's polynomial series.*

$\Rightarrow x = \frac{\pi}{2}$

$$f(x) = \frac{sinx}{cosx}$$

$$f'(x) = \frac{cosxcosx + sinxsinx}{cos^2x} = \frac{1}{cos^2x}$$

$$f''(x) = \frac{-2cosxsinx}{cos^4x} = \frac{-2sinx}{cox^3x}$$

$$f'''(x) = \frac{-2cosxcos^3x - 3cos^2xsinx2sinx}{cos^6x}$$

$$= \frac{-2cos^2x - 6sin^2x}{cos^4x}$$

$f(\frac{\pi}{2}) = undefined$

$f'(\frac{\pi}{2}) = undefined$

$f''(\frac{\pi}{2}) = undefined$

$f'''(\frac{\pi}{2}) = undefined$

Similarly $f^{(n)}(\frac{\pi}{2}) = undefined$

$$P_n(x) = f(\frac{\pi}{2}) + f'(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{f''(\frac{\pi}{2})(x - \frac{\pi}{2})^2}{2!} + \cdots \ldots \ldots \ldots . + \frac{f^{(n)}(\frac{\pi}{2})(x - \frac{\pi}{2})^n}{n!}$$

$$= \infty$$

The series is not defined.

Maclaurine's Polynomial:

Named after Scottish mathematician Colin Maclaurine. It is a special case of Taylor polynomial while $a=0$.

If f can be differentiated n times at 0, then we define the n^{th} Maclaurine's polynomial for f to be:

$$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}$$

$$f(0) + f'(0)(x-0) + f''(0)\frac{(x-0)^2}{2!} + f'''(0)\frac{(x-0)^3}{3!} + \cdots + f^{(n)}(0)\frac{(x-0)^n}{n!}$$

Taylor's polynomial while $a=0$

$$= \sum_{k=0}^{n} f^{k}(0)\frac{x^k}{k!}$$

Examples Expand the following functions into Maclaurine's polynomial.

(1) $y = e^x$ in the powers of x.

$f(x) = e^x$ ———— $f(0) = e^0 = 1$

$f'(x) = e^x$ ———— $f'(0) = 1$

$f''(x) = e^x$ ———— $f''(0) = 1$

$f'''(x) = e^x$ ———— $f'''(0) = 1$

$f^{IV}(x) = e^x$

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

$f^{(n)}(x) = e^x$ $\qquad\qquad$ $f^{(n)}(0) = 1$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

② $y = \log_e(x+1)$ in the powers of x

$f(x) = \ln(x+1) \longrightarrow f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{(x+1)} = (x+1)^{-1} = (-1)^0 0!\,(x+1)^{-1} \longrightarrow f'(0) = \frac{1}{0+1} = 1$

$f''(x) = -(x+1)^{-2} = (-1)^1 1!\,(x+1)^{-2} \longrightarrow f''(0) = -(0+1)^{-2} = -1$

$f'''(x) = 2(x+1)^{-3} = (-1)^2 2!\,(x+1)^{-3} \longrightarrow f'''(0) = 2(0+1)^{-3} = 2$

$f^{IV}(x) = -6(x+1)^{-4} = (-1)^3 3!\,(x+1)^{-4}$

$f^{V}(x) = 24(x+1)^{-5} = (-1)^4 4!\,(x+1)^{-5}$

$\vdots$

$n \in \mathbb{N}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(x+1)^{-n} \longrightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!\,(0+1)^{-n}$

$= (-1)^{n-1}(n-1)!\,1^{-n}$

$= \frac{(-1)^{n-1}(n-1)!}{1^n}$

$= (-1)^{n-1}(n-1)!$

$\boxed{1^n = 1 \quad \text{but } 1^\infty = \infty \quad n \in \mathbb{R}}$

$P_n(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^n(0)\frac{x^n}{n!}$

$= 0 + x + (-1)\frac{x^2}{2} + (2)\frac{x^3}{6} + \cdots \cdots + (-1)^{n-1}(n-1)!\,\frac{x^n}{n!}$

$= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n}.$

Note: $\frac{(n-1)!}{n!} = \frac{1}{n}$

$x=0$ Note $\cos 0^\circ = 1$

Maclaurine's Polynomial

1. $y = \frac{sinx}{cosx}$ *in the powers of x; Evaluate nth term of Maclaurine's polynomial series.*

$f(x) = \frac{sinx}{cosx} = tanx$
$f'(x) = sec^2x = 1 + tan^2x$
$f''(x) = 2tanxsec^2x = 2tanx(1 + tan^2x)$
$f'''(x) = 2sec^2x + 6tan^2xsec^2x = 2 + 8tan^2x + 6tan^4x$ Since $sec^2x = 1 + tan^2x$
$f^{(4)}(x) = 16tanxsec^2x + 24tan^3xsec^2x = 16tanx + 40tan^3x + 24tan^5x$
Since $sec^2x = 1 + tan^2x$

$f^{(5)}(x) = 16sec^2x + 120tan^2xsec^2x + 120tan^4xsec^2x$
$= 16 + 136tan^2x + 240tan^4x + 120tan^6x$

$\tan 0^\circ = 0$

$f(0) = 0$
$f'(0) = 1 + 0 = 1$
$f''(0) = 0$
$f'''(0) = 2$
$f^{(4)}(0) = 0$
$f^{(5)}(0) = 16$

Since there is no pattern, we are unable to find $f^{(n)}(x)$ and hence $f^{(n)}(0)$

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \frac{f^{(5)}(0)x^5}{5!} + \cdots$$

$$= 0 + x + 0 + \frac{2x^3}{6} + 0 + \frac{16x^5}{120} + 0 \ \ldots\ldots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \ldots..$$

Infinite series