Integration by Parts: (Quiz-1)

Reduction Formulae:

1. $$I_n = \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

H.W.

2. $$I_n = \int \cos^n x \, dx = -\frac{\cos^{n-1} x}{n} \sin x + \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

3. $$I_n = \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}, \quad n \geq 2$$

4. $$I_n = \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}, \quad n \geq 2$$

H.W.

* Obtain the reduction formulae for $\int \sin^n x \, dx$ & evaluate $\int \sin^6 x \, dx$.

$$\boxed{\int u\,dv = uv - \int v\,du} \qquad \frac{dx^n}{dx} = nx^{n-1} \qquad \int dx = x + c$$

Solution: let $I_n = \int \sin^n x \, dx$

$$= \int \underbrace{\sin^{n-1} x}_{u} \, \underbrace{\sin^1 x \, dx}_{dv}$$

let

$u = \sin^{n-1} x$

$\Rightarrow du = (n-1)\sin^{n-2} x \cos x \, dx$

$dv = \sin x \, dx$

$\Rightarrow \int dv = \int \sin x \, dx$

$\Rightarrow v = -\cos x + K$

$$= -\underbrace{\sin^{n-1} x}_{u} \, \underbrace{\cos x}_{v} + \int \cos x (n-1) \sin^{n-2} x \cos x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$I_{n-2} = \int \sin^{n-2} x \, dx$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \left(1 - \sin^2 x\right) dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\left[\int \underbrace{\sin^{n-2} x \, dx}_{I_{n-2}} - \int \underbrace{\sin^n x \, dx}_{I_n}\right]$$

$$I_n = -\sin^{n-1}x\cos x + (n-1)\left[I_{n-2} - I_n\right]$$

$$I_0 = \int dx$$

$$\Rightarrow I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - (n-1)I_n$$

$$\Rightarrow I_n + (n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\boxed{n \in \{0,1,2,3,4,5,\dots\}}$$

$$\Rightarrow \boxed{I_n = \frac{-\sin^{n-1}x\cos x}{n} + \frac{(n-1)}{n}I_{n-2}, \quad n \geq 2}$$

$$I_6 = \int \sin^6 x\,dx$$

$$= \frac{-\sin^5 x\cos x}{6} + \frac{5}{6}I_4 = \frac{-\sin^5 x\cos x}{6} + \frac{5}{6}\left[\frac{-\sin^3 x\cos x}{4} + \frac{3}{4}I_2\right]$$

$$I_0 = \int \sin^0 x\,dx = \int dx = x$$

$$I_6 = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{15}{24} I_2$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{\cancel{15}^{\,5}}{\cancel{24}_{\,8}}\left[-\frac{\sin x\cos x}{2} + \frac{1}{2} I_0\right]$$

$$= -\frac{1}{6}\sin^5\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + C$$

H.W. $\int \sin^8 x\,dx$

$2\sin x\cos x = \sin 2x$

$\sin^2 x = 1-\cos^2 x$, $\cos^2 x = 1-\sin^2 x$, $2\sin^2 x = 1-\cos 2x$

$2\cos^2 x = 1+\cos 2x$

$$\int \sin^m x \cos^n x \, dx$$

(1) if $\boxed{m/n}$ is odd, split the odd powered term

using $\sin^2 x = (1-\cos^2 x)$ / $\cos^2 x = (1-\sin^2 x)$

$\sin x$ $\qquad$ $\cos x$

& $\boxed{\text{opposite term}}$

substitution

$(1-\sin^2 x)^2 \sin x$.

Ex:

$$\int \sin^3 x \cos^4 x \, dx$$

$$= \int \sin^2 x \, \sin x \cos^4 x \, dx$$

$$= \int (1-\cos^2 x)\sin x \cos^4 x \, dx$$

(ii) If m & n are odd split either of them using above formula. $\int \sin^4 x \cos^5 x\,dx$ — HW

Example Type-I $\int \sin^3 x \cos^4 x\,dx$

let $\cos x = u$

$\Rightarrow -\sin x\,dx = du$

$\Rightarrow \sin x\,dx = -du$

$= \int \sin^2 x \sin x \cos^4 x\,dx$

$= \int (1-\cos^2 x)\cos^4 x \sin x\,dx$

$= \int (1-u^2)\,u^4\,(-du) = \int (u^2-1)\,u^4\,du$

$= \int u^6\,du - \int u^4\,du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$

Ex:

$$\int \sin^3 x \cos^7 x \, dx \qquad \int \sin^3 x \cos^5 x \, dx = \text{H.W.}$$

let $\cos x = u$

$$= \int \sin^2 x \sin x \cos x \, dx$$

$$= \int (1-\cos^2 x) \cos x \sin x \, dx$$

$\int \sin^3 x \cos^5 x \, dx$ H.W.

$$= \int (1-\cos^2 x) \sin x \cos^5 x \, dx$$

let

$\cos x = u$

$\Rightarrow -\sin x \, dx = du$

(iii) If $\boxed{m \,\&\, n}$ are even, $\boxed{2\sin^2 x = (1-\cos 2x)}$, $2\cos^2 x = (1+\cos 2x)$

then $\boxed{\text{reduction formula}}$. $\sin^2 x = \dfrac{1-\cos 2x}{2}$ $\int \sin^8 x \cos^4 x\, dx$

H.W.

$\sin^2 2x = 1 - \cos^2 2x$

$$\int \sin^6 x \cos^6 x \, dx$$

$$= \int (\sin^2 x)^3 (\cos^2 x)^3 \, dx$$

$$= \int \left\{\tfrac{1}{2}(1-\cos 2x)\right\}^3 \left\{\tfrac{1}{2}(1+\cos 2x)\right\}^3 dx$$

$$= \tfrac{1}{64}\int \left\{(1-\cos 2x)(1+\cos 2x)\right\}^3 dx \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \tfrac{1}{64}\int (1-\cos^2 2x)^3 \, dx$$

$$= \frac{1}{64}\int (\sin^2 2x)^3\,dx$$

$$= \frac{1}{64}\int \sin^6 2x\,dx$$

$$= \frac{1}{64}\cdot\frac{1}{2}\int \sin^6 u\,du$$

$$= \frac{1}{128}\int \sin^6 u\,du$$

let

$\Rightarrow 2x = u$

$\Rightarrow dx = \frac{du}{2}$

$\Rightarrow I_6 =$

H.W.

Reduction

$\int \sin^4 x \cos^5 x \, dx$, $\int \sin^4 x \cos^4 x \, dx$. H.W.

Formula

1. $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

2. $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

3. $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

4. $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.

Ex:

$$\int \sin 4x \sin 9x \, dx$$

$$\boxed{\cos(-\theta) = \cos\theta}$$

$$\int \cos ax \, dx = \frac{\sin ax}{a} + C$$

$$= \frac{1}{2}\int 2 \sin 4x \sin 9x \, dx$$

$$= \frac{1}{2}\int \left[\cos(4x-9x) - \cos(4x+9x)\right] dx$$

$$= \frac{1}{2}\int \cos(4x-9x)\, dx - \frac{1}{2}\int \cos(4x+9x)\, dx$$

$$= \frac{1}{2}\,\frac{\sin 5x}{5} - \frac{1}{2}\,\frac{\sin 13x}{13} + C$$

$I_n = \int \sec^n x\,dx$, $\int \sec^5 x\,dx$ $\quad \int u\,dv = uv - \int v\,du$

let $I_n = \int \sec^n x\,dx$

$$= \int \underbrace{\sec^{n-2} x}_{u}\ \underbrace{\sec^2 x\,dx}_{dv}$$

$$= \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \sec^1 x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \Big[\underbrace{\int \sec^n x\,dx}_{I_n} - \underbrace{\int \sec^{n-2} x\,dx}_{I_{n-2}} \Big]$$

let $u = \sec^{n-2} x$

$du = (n-2) \sec^{n-3} x \sec x \tan x\,dx$

$dv = \sec^2 x\,dx$

$\Rightarrow \int dv = \int \sec^2 x\,dx$

$\Rightarrow v = \tan x + K$

$n-3+1$

$= n-2$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_1 = \int \sec^1 x \, dx$$

$$= \ln|\sec x + \tan x| + C$$

$$\Rightarrow I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow (1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow I_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2) I_{n-2}}{n-1}, \quad \boxed{n \geq 2}$$

$$\int \sec^5 x \, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3$$

$$I_3 = \frac{1 \sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C$$

$$\int \sec x \, dx = \int \sec x \, \frac{(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

$$\int \frac{f'(x)}{f(x)} \, dx$$

$$= \int \frac{(\sec^2 x + \tan x \sec x)}{\underline{\sec x} + \underline{\tan x}} \, dx$$

$$= \underline{\ln|f(x)| + C}$$

$$= \underline{\ln|\sec x + \tan x|} + C$$

$$\int \tan^{m} x \sec^{n} x\, dx$$

(1) $\boxed{n \text{ is even}}$, $\boxed{\sec^{n-2} x \cdot \sec x}$ → split

$\boxed{\sec^2 x = 1 + \tan^2 x}$ → split, $\tan x = u$

$$\sec^6 x = (\sec^2 x)^3 = (1+\tan^2 x)^3$$

$$\int \tan^3 x \sec^4 x\, dx = \int \tan^3 x \cdot \sec^2 x \cdot \boxed{\sec^2 x\, dx}$$

let $\tan x = u$

$\sec^2 x\, dx = du$

H.W.

$$\int \tan^5 x \sec^8 x\, dx$$

$$= \int \tan^3 x (1+\tan^2 x) \sec^2 x\, dx$$

$$= \int u^3 (1+u^2)\, du = \int u^3\, du + \int u^5\, du$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^5 x}{5} + C$$

$$* \int \tan^3 x \sec^6 x \, dx$$

let $\tan x = u$.

$$= \int \tan^3 x \sec^4 x \underline{\sec^2 x \, dx}$$

$$= \int \tan^3 x \left(\sec^2 x\right)^2 \sec^2 x \, dx$$

$$= \int \tan^3 x \left(1+\tan^2 x\right)^2 \sec^2 x \, dx$$

H.W.

(ii) If $\boxed{m \& n}$ both are odd, $\boxed{\sec x \tan x} \longrightarrow$ split

$$\boxed{\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2}, \quad \text{let } \boxed{\sec x = u}$$

$$\int \tan^5 x \sec^3 x \, dx$$

$$= \int \tan^4 x \sec^2 x \tan x \sec x \, dx$$

$$= \int (\tan^2 x)^2 \sec^2 x \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \tan x \sec x \, dx$$

$$= \int (u^2 - 1)^2 u^2 \, du$$

let $\boxed{\sec x = u}$

$\Rightarrow \sec x \tan x \, dx = du$

$$\int \tan^7 x \sec^3 x \, dx$$

$$= \int \tan^6 x \sec^2 x \sec x \tan x \, dx$$

$$= (\tan^2 x)^3 \sec^2 x \sec x \tan x \, dx$$

$$= (\sec^2 x - 1)^3 \underset{u}{\sec^2 x} \underset{du}{\sec x \tan x \, dx}$$

H.W.

$= \int u^2 (u^4 - 2u^2 + 1)\, du$

$= \int (u^6 - 2u^4 + u^2)\, du = \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C.$

(iii) If m is even & n is odd

We convert $\tan^2 x$ into $(\sec^2 x - 1)$ & then apply

$\boxed{\text{Reduction}}$

$\int \tan^6 x \sec^3 x\, dx$ H.W.

$\int \tan^4 x \sec x\, dx$

$= \int (\sec^2 x - 1)^2 \sec x\, dx = \int (\sec^4 x - 2\sec^2 x + 1) \sec x\, dx$

$= \int (\sec^5 x - 2\sec^3 x + \sec x)\, dx$

Now Reduction Apply