Home work Sheet #6

$y = x^n$

Suppose,

$y = x^m$

$y_1 = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)x^{m-3}$

$y_4 = m(m-1)(m-2)(m-3)x^{m-4}$

. . . .

. . . .

$y_n = m(m-1)(m-2)(m-3)$ ...... up to nth function.

$y_n = m(m-1)(m-2)(m-3) \cdots\cdots \{m-(n-1)\} x^{m-n}$

Now,

Suppose,

$\boxed{m = n}$

$y_n = n(n-1)(n-2)(n-3) \cdots\cdots \{n-n+1\} x^{n-n}$

$= n! \times (1 \cdot x^0)$

$= n!$

$\therefore \frac{d}{dx}(x^n) = n!$

$$y = (ax+b)^n$$

suppose, $n = m$

$$\therefore y = (ax+b)^m$$

$$y_1 = am(ax+b)^{m-1}$$

$$y_2 = a^2 \cdot m(m-1)(ax+b)^{m-2}$$

$$y_3 = a^3 \cdot m(m-1)(m-2)(ax+b)^{m-3}$$

$$y_4 = a^4 m(m-1)(m-2)(m-3)(ax+b)^{m-4}$$

$$\vdots$$

$$y_n = a^n \cdot (m)(m-1)(m-2)(m-3) \ldots (m-(n-1)) \cdot (ax+b)^{m-n}$$

Now, if, $m = n$,

then,

$$\cancel{y_n = a^n (m)(m-1)(m-2).}$$

$$y_n = a^n \cdot n(n-1)(n-2)(n-3) \ldots (n-n+1) \cdot (ax+b)^0$$

$$= a^n \cdot n(n-1)(n-2)(n-3) \ldots 1.$$

$$= \boxed{a^n \cdot n!}$$

(4)

$$y = \ln(ax+b)$$

$$y_1 = \frac{1}{ax+b}$$

$$y_2 = \frac{-a}{(ax+b)^2} = (-1) \cdot \frac{a}{(ax+b)^2}$$

$$y_3 = \frac{-a \cdot 2(ax+b) \cdot a}{(ax+b)^4} = (-1)(-2) \cdot \frac{a^2}{(ax+b)^3}$$

$$y_4 = (-1)(-2) \cdot \frac{a^2 \cdot -3(ax+b)^2 \cdot a}{(ax+b)^6} = (-1)(-2)(-3) \cdot \frac{a^3}{(ax+b)^4}$$

$\vdots$

$$\boxed{y_n = (-1)^{n+1} \cdot \frac{(n-1)!\, a^3}{(ax+b)^n}}$$

(5)

$$y = e^{ax}$$

$$y_1 = ae^{ax}$$

$$y_2 = a.ae^{ax}$$

$$y_3 = a.a.a.e^{ax}$$

$$\vdots$$

$$\boxed{y_n = a^n e^{ax}}$$

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$y = \sin(ax+b)$

$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$\cancel{y_2 = a\cos}$

$y_2 = -a^2\sin(ax+b) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$

$y_3 = \cancel{a^2 \sin} -a^2\cos(ax+b) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$

$y_4 = a^4\sin(ax+b) = a^4\sin\left(ax+b+\frac{4\pi}{2}\right)$

$\vdots$

$$\boxed{y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)}$$

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$$y = \cos(ax+b)$$

$$y_1 = -a\sin(ax+b) = a\cos\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos(ax+b) = a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = \cancel{a} - a^3\sin(ax+b) = a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$$

$$y_4 = a^4\cos(ax+b) = a^4\cos\left(ax+b+\frac{4\pi}{2}\right)$$

$$\vdots$$

$$\boxed{y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)}$$

(b) $y = e^{ax}\sin bx$

$$y_1 = ae^{ax}\sin bx + be^{ax}\cos bx$$

$$y_2 = a^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx - b^2e^{ax}\sin bx$$

$$L.H.S = y_2 - 2ay_1 + (a^2+b^2)y$$

$$= a^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx - b^2e^{ax}\sin bx - 2a^2e^{ax}\sin bx - 2abe^{ax}\cos bx + a^2e^{ax}\sin bx + b^2e^{ax}\sin bx$$

$$= 0$$

$$= R.H.S$$

$\therefore$ L.H.S = R.H.S

(c)

$$y = e^x \sin x$$

$$y_1 = e^x \sin x + e^x \cos x$$

$$y_2 = e^x \sin x + e^x \cos x + e^x \cos x - e^x s$$

$$y_2 = 2e^x \cos x$$

$$y_3 = 2e^x \cos x - 2e^x \sin x$$

$$y_4 = 2e^x \cos x - 2e^x \sin x - 2e^x \sin x$$
$$- 2e^x \cos x$$

$$= -4e^x \sin x$$

$$\therefore \text{L.H.S} = y_4 + 4y$$

$$= -4e^x \sin x + 4e^x \sin x$$

$$= 0$$

$$= \text{R.H.S}$$

Maclaurin polynomial Theorem

If a function f can be differentiated n times at 0, then we defined the n^{th} maclaurin polynomial for f to be

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots\dots + \frac{f^{(n)}(0)}{n!}x^n$$

~~f(x~~

Example : $f(x) = e^x$ [Fined m.p at $P_0, P_1, P_2, P_3 \dots P_n$]

$$f'(x) = f''(x) = f'''(x) = \dots f^{(n)}(x) = e^x$$

$$\therefore f'(0) = f''(0) = f'''(0) = \dots f^{(n)}(0) = e^0 = 1$$

$$P_0(x) = f(0) = 1$$

$$P_1(x) = f(0) + f'(0)x = 1+x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1+x+\frac{1}{2}x^2$$

$$P_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3}x^3 = 1+x+\frac{1}{2}x^2+\frac{1}{3}x^3$$

$$P_n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^{(n)}(0)}{n}x^n = 1+x+\frac{1}{2}x^2 + \dots + \frac{1}{n}x^n$$

Example

n^{th} M.P of (a) $\sin x$ (b) $\cos x$

~~n^{th} derivative of $\sin x \Rightarrow$~~

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$\therefore P_0(x) = f(0) = 0$

$P_1(x) = f(0) + f'(0)x = 0 + x = x$

$P_2(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 = 0 + x + 0 = x$

$P_3(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}\cdot x^3 = x + - \dfrac{x^3}{3!}$

$P_4(x) = f(0) + f'(0)x + f''(0)/2! + \dfrac{f'''(0)}{3!}x^3 + \dfrac{f''''(0)}{4!}x^3 =$

$x - \dfrac{x^3}{3!}$

$P_5 = f(0) + f'(0)x + \dfrac{f''(0)}{2!}\cdot x^2 + \dfrac{f'''(0)x^3}{3!} + \dfrac{f^4(0)x^4}{4!} + \dfrac{f^5(0)x^5}{5!}$

$= x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

$P_6 = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + 0$

$P_7 = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + 0 - \dfrac{x^7}{7!}$

$P_{2n+1} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots + (-1)^n \dfrac{x^{(2n+1)}}{(2n+1)!}$

∞

(b) ~~f(x) cos x~~ (b) cos x

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f(0) = 0$

$f^2(x) = -\cos x \qquad f^2(0) = -1$

$f^3(x) = -\sin x \qquad f^3(0) = 0$

$f^4(x) = -\cos x \qquad f^4(0) = -1$

$f^5(x) = -\sin x \qquad f^5(0) = 0$

$f^6(x) = -\cos x \qquad f^6(x) = -1$

$f^7(x) = -\sin x \qquad f^7(x) = 0$

Now:

$$P_7(x) = f(0) + f'(x) + \frac{f^2(x)\,x^2}{2!} + \frac{f^3(x)\cdot x^3}{3!} + \frac{f^4(x)\cdot x^4}{4!} + \frac{f^5(x)\cdot x^5}{5!} + \frac{f^6(x)\cdot x^6}{6!} + \frac{f^7(x)\cdot x^7}{7!}$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\therefore P_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \cdot \frac{x^{2n}}{2n}$$

Taylor polynomial

If f can be differentiated n times at x_0, then we define the nth taylor polynomial for f about $x = x_0$ to be

$$P_n x = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \dots + \frac{f^n(x_0)(x-x_0)^n}{n!}$$

Example : find first 4 Taylor polynomials for $\ln x$ about $x = 2$

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = \frac{-1}{x^2} \qquad f''(2) = \frac{-1}{4}$

$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \frac{1}{4}$

$f^4(x) = \frac{-6}{x^4} \qquad f^4(2) = \frac{-6}{16} \quad \frac{-3}{4}$

$P_0(x) = f(2) = \ln 2$

$P_1(x) = f(2) + f'(2)(x-2) = \ln 2 + \frac{1}{2}(x-2)$

$P_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4} \cdot \frac{(x-2)^2}{2!}$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$

$P_3(x) = \cancel{f(x_0) + f'(x_0)(x-x_0) +}$

$$f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} + \frac{f'''(2)(x-2)^3}{3!}$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4} \cdot \frac{(x-2)^2}{2} + \frac{1}{4} \cdot \frac{1}{3!}(x-3)^3$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-3)^3$$

$$\boxed{\text{Maclaurin and taylor series}}$$

① (1) $f(x) = \sin x$; $f(x_0) = 1$

$f'(x) = \cos x$; $f'(x_0) = 0$

$f''(x) = -\sin x$; $f''(x_0) = -1$

$f^3(x) = -\cos x$; $f'''(x_0) = 0$

$f^4(x) = \sin x$; $f^4(x_0) = 1$

$f^5(x) = \cos x$; $f^5(x_0) = 0$

$f^6(x) = -\sin x$; $f^6(x_0) = -1$

$$\therefore \sum_{K=0}^{\infty} \frac{F^K \cdot (\pi/2)(x-\frac{\pi}{2})^K}{K!} = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)(x-x_0)^2}{2!} + \frac{f^3(x_0)(x-x_0)^3}{3!} + \frac{f^4(x_0)(x-x_0)^4}{4!} + \frac{f^5(x_0)(x-x_0)^5}{5!} + \frac{f^6(x_0)(x-x_0)^6}{6!}$$

$$P x = 1 + 0 - \frac{(x-x_0)^2}{2!} + 0 + \frac{(x-x_0)^4}{4!} + 0 - \frac{(x-x_0)^6}{6!} + \cdots\cdots +$$

$$= 1 - \frac{(x-x_0)^2}{2!} + \frac{(x-x_0)^4}{4!} - \frac{(x-x_0)^6}{6!} + \cdots$$

$$P_{2n} x = 1 - \frac{(x-x_0)^2}{2!} + \frac{(x-x_0)^4}{4!} - \frac{(x-x_0)^6}{6!} + \cdots + (-1)^n \cdot \frac{(x-\ldots}{(2n\ldots}$$

$$P_{2n} x = 1 - \frac{(x-x_0)^2}{2!} + \frac{(x-x_0)^4}{4!} - \frac{(x-x_0)^6}{6!} + \cdots + (-1)^n \frac{(x-x_0)^{2n}}{(2n)!}$$

(II)

$$f(x) = \ln x \qquad f(x_0) = \ln 2$$

$$f'(x) = \frac{1}{x} \qquad f'(x) = \frac{1}{2}$$

$$f''(x) = \frac{-1}{x^2} \qquad f''(x) = \frac{-1}{4}$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(x) = \frac{2}{8} = \frac{1}{4}$$

$$f^{4}(x) = \frac{\cancel{3}-6}{x^4} \qquad f'''(x) = \frac{-6}{\cancel{8}16} = \frac{-3}{4}$$

Now,

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)(x-x_0)^2}{2!} + \frac{f^3(x_0)(x-x_0)^3}{3!} + \cdots\cdots$$

$$= \ln 2 + \frac{1}{2}(x-x_0) \cancel{+} - \frac{1}{8}(x-x_0)^2 + \frac{1}{24}(x-x_0)^3 + \cdots\cdots$$

(2) $y = \ln x$

$\therefore f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$

$f''(x) = \frac{-1}{x^2} \qquad f''(2) = \frac{-1}{4}$

$f'''(x) = \frac{1}{x^3} \qquad f^3(2) = \frac{1}{8}$

By Taylor's theorem,

$$\cancel{P(x) = f(x) + f''(x)\, f'(x-x_0) + \frac{f''(x-x_0)^2}{2!} + \frac{f'''(x-x_0)^3}{3!}}$$

$$+ \frac{f^4(x-x_0)^4}{4!}$$

$$= \cancel{\ln 2 + \frac{1}{2}(x-x_0) - \frac{1}{8}(x-x_0) + \frac{1}{24}}(x-x_0)$$

$$P(x) = f(x) + f'(x)(x-x_0) + \frac{f''(x)(x-x_0)^2}{2!} + \frac{f'''(x)(x-x_0)^3}{3!}$$

$$+ \frac{f^4(x-x_0)^4(x)}{4!} + \cdots\cdots$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-3)^3$$

$$+ \cdots\cdots$$

$y = e^{ax}$

$f(x) = e^{ax}$ $\qquad$ $f(1) = e^{a}$

$f'(x) = ae^{ax}$ $\qquad$ $f'(1) = ae^{a}$

$f''(x) = a^2 e^{ax}$ $\qquad$ $f''(1) = a^2 e^{a}$

$f'''(x) = a^3 e^{ax}$ $\qquad$ $f'''(1) = a^3 e^{a}$

Now,

$$P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} +$$

$$\frac{f'(x_0)(x-x_0)^3}{3!} + \cdots\cdots$$

$$= e^{a} + ae^{a}(x-1) + a^2e^{a}\frac{(x-1)^2}{2} +$$

$$\frac{a^3e^{a}}{3!}(x-1)^3 + \cdots\cdots$$

(3) $f(x) = e^{ax}$ $\quad f(0) = e^0 = 1$

$f'(x) = ae^{ax}$ $\quad f'(0) = a \cdot e^0 = a$

$f''(x) = a^2 e^{ax}$ $\quad f''(0) = a^2 \cdot e^0 = a^2$

$f'''(x) = a^3 e^{ax}$ $\quad f'''(0) = a^3$

$$\therefore P_3(x) = f(0) + f'(0)\,x + \cancel{f''(0)} \frac{f''(0)\,x}{2!} + \frac{f'''(0)\,x}{3!} + \frac{f^4(0)\,x}{4!}$$

$$= 1 + a + \frac{a^2}{2!}x + \frac{a^3}{3!}x + \ldots\ldots\ldots$$

$$\cancel{P_n = 1 + ae^{ax} + \frac{a^2}{2!}e^{ax} + \frac{a^3}{3!}e^{ax} + \ldots\ldots \frac{a^n}{n!}e^{ax}}$$

$$P_n = 1 + ax + \frac{a^2}{2!}x + \frac{a^3}{3!}x + \ldots\ldots\ldots + \frac{a^n}{3!}x$$

Cos x $\rightarrow$ Page 6

(4)

$$f(x) = e^x \cos x$$

$$f(0) = 1$$

$$f'(x) = e^x \cos x - e^x \sin x$$

$$= 1$$

$$f''(x) = e^x \cancel{\cos x} - e^x \sin x - e^x \sin x - e^x \cancel{\cos x}$$

$$= -2e^x \sin x$$

$$f''(0) = 0$$

$$f'''(x) = -2e^x \sin x - 2e^x \cos x$$

$$f'''(0) = -2$$

Now,

$$P_0 = f(0) = 1$$

$$P_1 = f(0) + f'(0)x = 1 + x$$

$$P_2 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x$$

$$P_3 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x - \frac{2}{3!}x^3$$

$$f(x) = \ln(1+x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(x) = 2$$

$$f''''(x) = \frac{-6}{(1+x)^4} \qquad f''''(x) = -6$$

$$\therefore P = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^4(0)x^4}{4!}$$

$$= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \cdots$$

$$P_4 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\cdots$$

$$\therefore P_n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\cdots + (-1)^{n+1} \cdot \frac{1}{n}x^n$$

$\cos x$ & $\sin x \longrightarrow$ Page 5 & 6

(1) $f(x) = \sin x$; $f(x_0) = 1$

$f'(x) = \cos x$; $f'(x_0) = 0$

$f''(x) = -\sin x$; $f''(x_0) = -1$

$f^3(x) = -\cos x$; $f'''(x_0) = 0$

$f^4(x) = \sin x$; $f^4(x_0) = 1$

$f^5(x) = \cos x$; $f^5(x_0) = 0$

$f^6(x) = -\sin x$; $f^6(x_0) = -1$

$$\therefore \sum_{K=0}^{\infty} \frac{F^K \cdot (\pi/2)(x-\frac{\pi}{2})^K}{K!} = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)(x-x_0)^2}{2!}$$

$$+ \frac{f^3(x_0)(x-x_0)^3}{3!} + \frac{f^4(x_0)(x-x_0)^4}{4!} + \frac{f^5(x_0)(x-x_0)^5}{5!}$$

$$+ \frac{f^6(x_0)(x-x_0)^6}{6!}$$

$4x$