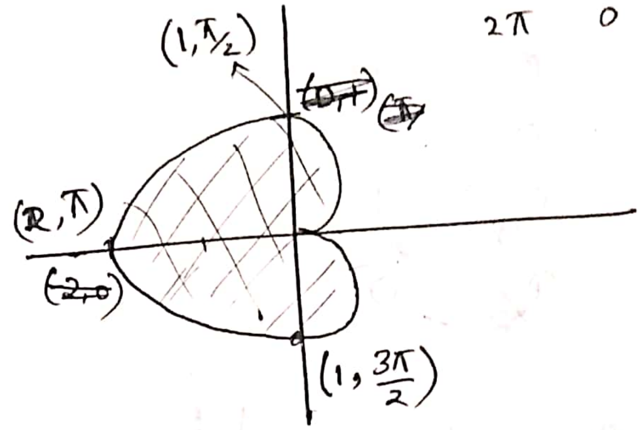
Side note

θ	r
0	0
$\pi/2$	1
π	2
$3\pi/2$	1
2π	0

Q7

The given cardioid

$$r = 1 - \cos\theta$$



$$\text{Area } A = \iint_R 1 \, dA$$

$$= \int_0^{2\pi}\int_0^{1-\cos\theta} r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left. \frac{r^2}{2} \right]_{r=0}^{1-\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin 2\theta \cdot \frac{1}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[(3\pi - 0 + 0) - (0 - 0 - 0)\right]$$

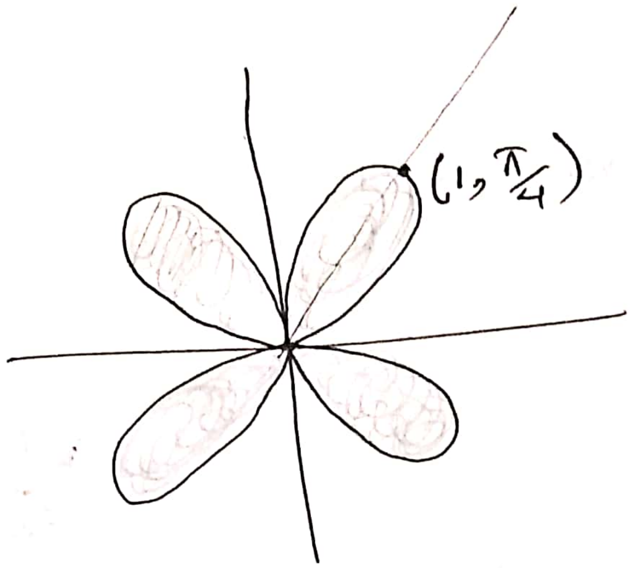
$$= \frac{3\pi}{2} \text{ Ans. } (\text{unit}^2)$$

08

Given equation

$r = \sin 2\theta$

θ	r
0	0
$\pi/4$	1
$\pi/2$	0



$$\text{Area } A = \iint_R dA = 4\int_0^{\pi/2}\int_0^{\sin 2\theta} r \, dr \, d\theta$$

$$= 4\int_0^{\pi/2} \left.\frac{r^2}{2}\right]_{r=0}^{\sin 2\theta} d\theta$$

$$= \int_0^{\pi/2} 2\sin^2 2\theta \, d\theta$$

$$= \int_0^{\pi/2} 1 - \cos 4\theta \, d\theta$$

$$= \left.\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= (\pi/2 - 0) - (0)$$

$$\therefore A = \pi/2 \text{ unit}^2$$

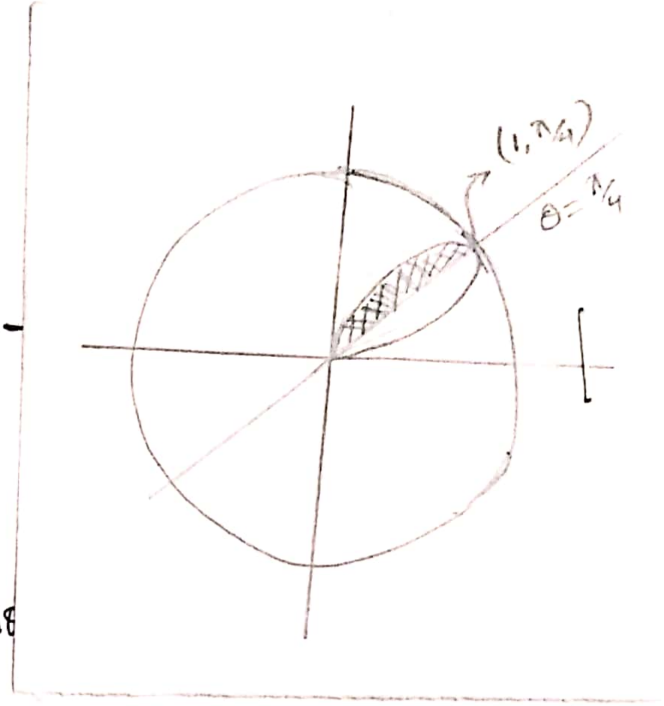
09

Given equations

$r = 1,$

$r = \sin 2\theta$

with $\pi/4 \le \theta \le \pi/2$



$$\text{Area } A = \int_{\pi/2}^{\pi/4} \int_{\sin 2\theta}^{1} r \, dr d\theta$$

$$= \int_{\pi/2}^{\pi/4} \left. \frac{r^2}{2} \right]_{r = \sin 2\theta}^{1} d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi/4} \sin^2 2\theta - 1 \; d\theta$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi/4} \frac{1}{2}(1 - \cos 4\theta) - 1 \; d\theta$$

$$= \frac{1}{4} \int_{\pi/2}^{\pi/4} 1 - \cos 4\theta - 2 \; d\theta$$

$$= \frac{1}{4} \left[-\theta - \sin 4\theta \cdot \frac{1}{4} \right]_{\pi/2}^{\pi/4}$$

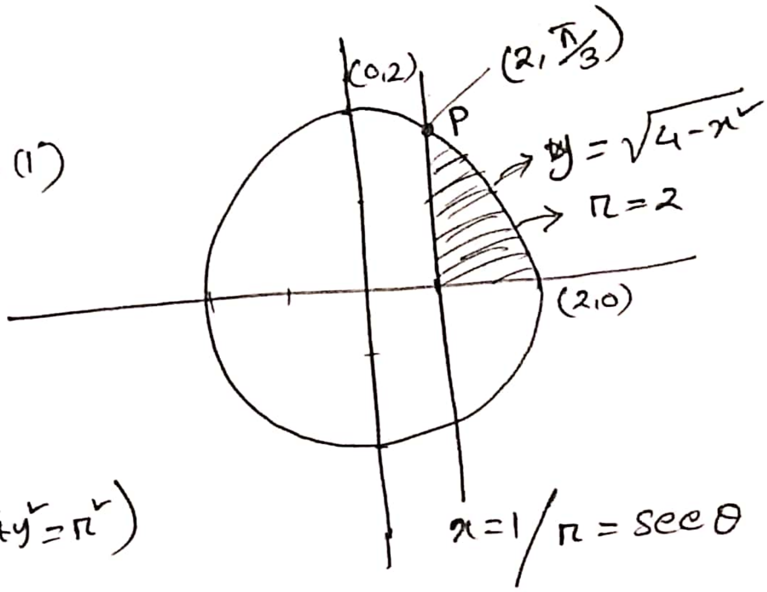
$$= \frac{1}{4} \left[\left(-\pi/4 - 0 \right) - \left(-\pi/2 \;\; 0 \right) \right]$$

$$= \frac{1}{4} \left(-\pi/4 + \pi/2 \right) = \frac{1}{4} \cdot \frac{-\pi + 2\pi}{4} = \frac{\pi}{16} \text{ square unit}$$

Ans.

(10). Given equations

$$\left.\begin{aligned} x^2+y^2 &= 4 \\ x &= 1 \end{aligned}\right\} \quad (1)$$



~~Parametric eq~~

Polar form of (1)

$$r^2 = 4 \quad (\text{as } x^2+y^2=r^2)$$

$$\therefore r = 2$$

and $\quad x = r\cos\theta = 1$

$$\Rightarrow r = \frac{1}{\cos\theta} = \sec\theta$$

Finding point P : As P is the intersection point of $r = \sec\theta$ and $r = 2$ so solving these equation we get,

$$2 = \sec\theta$$

$$\therefore \theta = \sec^{-1} 2$$

$$\therefore \theta = \pi/3$$

side note

$\sec\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\frac{1}{2}$

$= 60°$

$$\text{Area } A = \int_0^{\pi/3}\int_{\sec\theta}^{2} r \, dr \, d\theta$$

Try by yourself

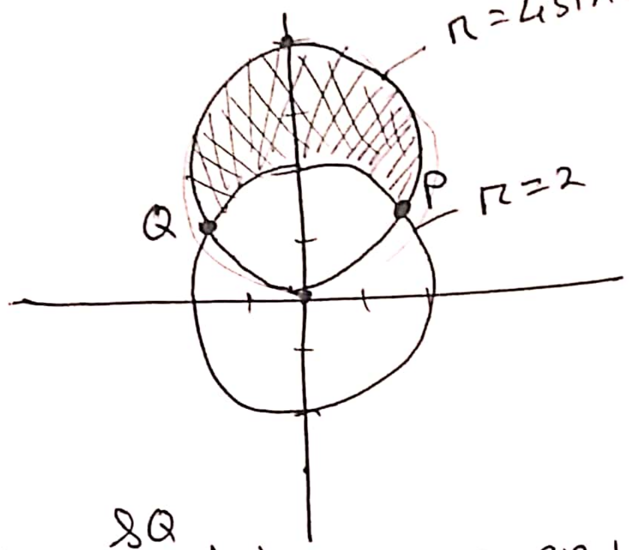
Ans : $\dfrac{4\pi}{3} - \sqrt{3}$

(11) Region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$ is

$$A = \iint_R r \, dr \, d\theta$$

$r = 4\sin\theta$

θ	r
0	0
$\pi/6$	2
$\pi/2$	4



For point P, & Q solving $r=2$ and $r=4\sin\theta$

$$2 = 4\sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2} \quad \therefore \theta = \pi/6$$

~~And $\theta + \pi/2 = \pi/6 + \pi/2 = \frac{\pi+3\pi}{6} = \frac{2\pi}{3}$~~

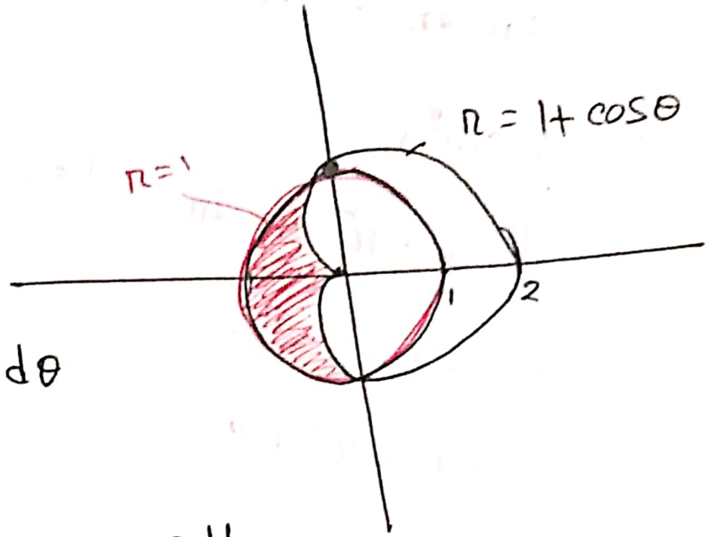
$$\therefore P = P(2, \pi/6)$$

And $Q = Q(2, \pi - \pi/6) = Q(2, \frac{5\pi}{6})$

$$\text{Area } A = \int_{\pi/6}^{5\pi/6} \int_{2}^{4\sin\theta} r \, dr \, d\theta$$

Find the area by your self.

(12) inside the circle $r=1$ and out side the cardioid $r=1+\cos\theta$



$$\text{Area } A = \int_{\pi/2}^{3\pi/2}\int_{1+\cos\theta}^{1} r \, dr \, d\theta$$

Do it by your self.

(13) Given integral

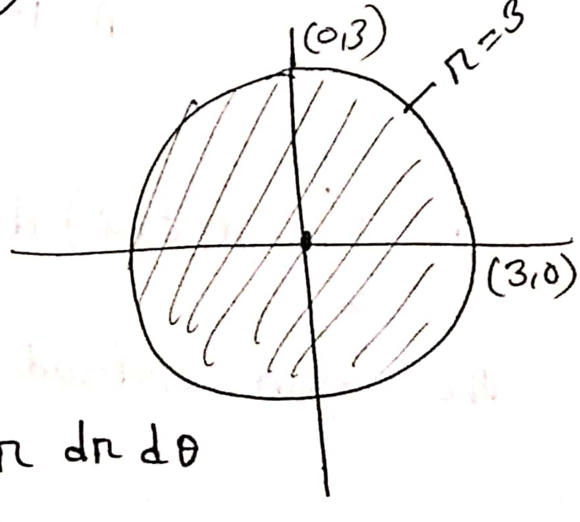
$$\iint_R \sin(x^2+y^2)\, dA \quad \text{———} (1)$$

Using polar coordinate system (1) can express as

$$\iint_R \sin(r^2)\, dA$$

Now the region enclosed by the circle

$$x^2+y^2=9$$



in polar form

$$r^2 = 9$$

$$\therefore r = 3$$

$$\therefore \text{Volume } V = \int_0^{2\pi}\int_0^{3} \sin(r^2) . r \, dr \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^3 \sin(r^2).\, 2r\, dr\, d\theta$$

Let $r^2 = u$

$2r\, dr = du$

$\therefore$ ~~$\int \sin u$~~

$\therefore \int \sin r^2\, 2r\, dr = \int \sin u\, du$

$= -\cos u + c$

$= -\cos r^2 + c$

$\therefore \cancel{\frac{1}{2}\int_0^{2\pi}\int_0^3 -\cos(r^2)\Big]_{r=0}^{3} d\theta}$

$$\therefore \frac{1}{2}\int_0^{2\pi}\int_0^3 \sin(r^2).\, 2r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi} -\cos(r^2)\Big]_{r=0}^{3} d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}(\cos 9 - 1)\, d\theta$$

$$= -\frac{1}{2}(\cos 9 - 1)\,\theta\Big]_{\theta=0}^{2\pi}$$

$$= -\frac{1}{2}(\cos 9 - 1)\, 2\pi + 0$$

$$= \pi(1 - \cos 9)$$ ~~squat~~ Answer

$\therefore \iint_R \sin(x^2+y^2)\, dA = \pi(1-\cos 9)$ when R is the region enclosed by $x^2+y^2=9$.

24. Given integral

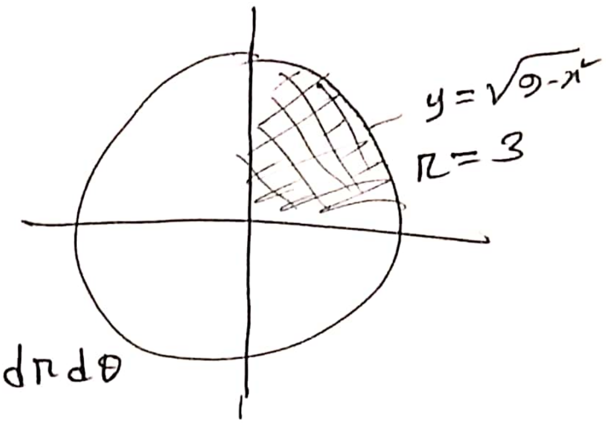
$$\iint_R \sqrt{9-x^2-y^2}\, dA \quad \text{——— (1)}$$

; R: in the 1st quadrant within the circle $x^2+y^2=9$

Using polar form (1) can written as

$$\iint_R \sqrt{9-r^2}\, dA$$

$$= \int_0^{\pi/2}\int_0^{r} \sqrt{9-r^2} \times r\, dr\, d\theta$$



$$= \int_0^{\pi/2}\int_0^{r} r\sqrt{9-r^2}\, dr\, d\theta$$

try by your self

Answer : $\frac{9\pi}{2}$

25. Given integral

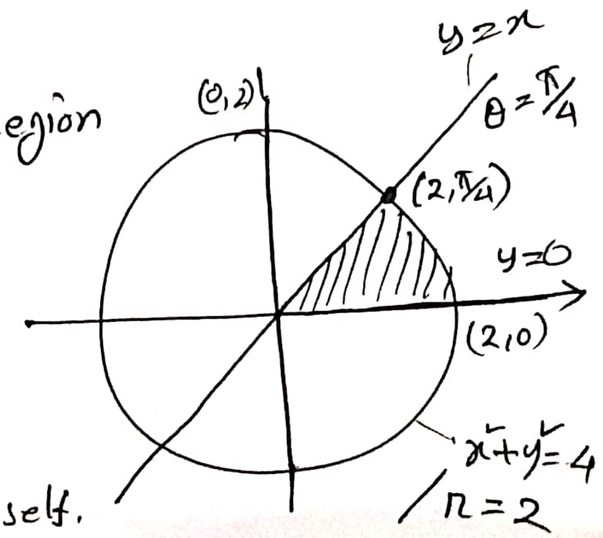
$$\iint_R \frac{1}{1+x^2+y^2}\, dA$$

; Region $y=0$, $y=x$, $x^2+y^2=4$

Putting $r^2=x^2+y^2$ we get

$$\iint_R \frac{1}{1+r^2}\, dA$$

Region



$$\therefore \iint_R \frac{1}{1+r^2}\, dA = \int_0^{\pi/4}\int_0^{2} \frac{1}{1+r^2}\, r\, dr\, d\theta$$

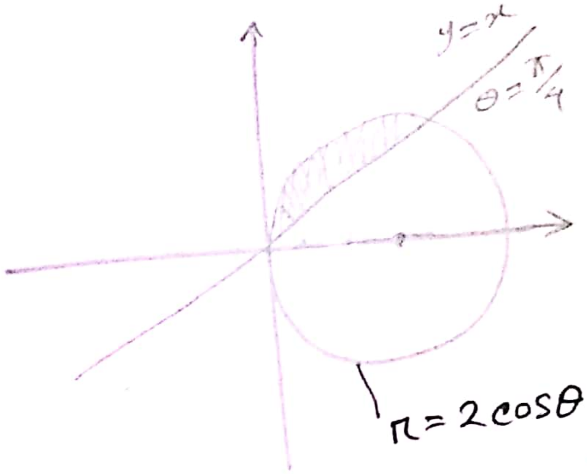
try by your self.

26

$\iint_R 2y\,dA$: R: first quadrant bounded above by the circle $(x-1)^2+y^2=1$ and below by the line $y=x$

Given integral in polar form

$$\iint_R 2r\sin\theta\, dA$$



The region bounded by

$y=x$ on $\theta=\pi/4$

and

$(x-1)^2+y^2=1$

$\Rightarrow x^2-2x+1+y^2=1$

$\Rightarrow \cancel{r^2=2x}$

$\Rightarrow x^2+y^2=2x$

$r^2=2r\cos\theta$

$\therefore r=2\cos\theta$

$$\therefore \iint_R 2r\sin\theta\, dA = \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} 2r\sin\theta\, r\, dr\, d\theta$$

try by your self Ans: $\cancel{16/3}$ $1/3$

(27) $\int_0^1\int_0^{\sqrt{1-x^2}} x^2+y^2\, dy\, dx$

Let

$x=r\sin\theta$

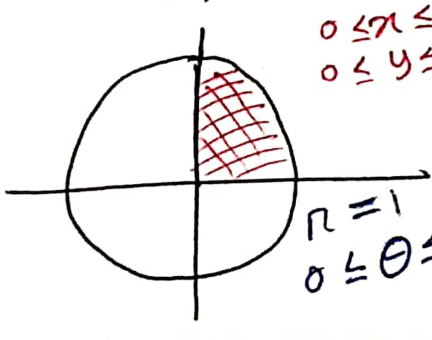
$y=r\cos\theta$

$\therefore x^2+y^2=r^2$

$0\le x\le 1$

$0\le y\le \sqrt{1-x^2}$

$= \iint_R x^2+y^2\, dA$



$r=1$

$0\le\theta\le\pi/2$

$= \int_0^{\pi/2}\int_0^1 r^2\, r\, dr\, d\theta$

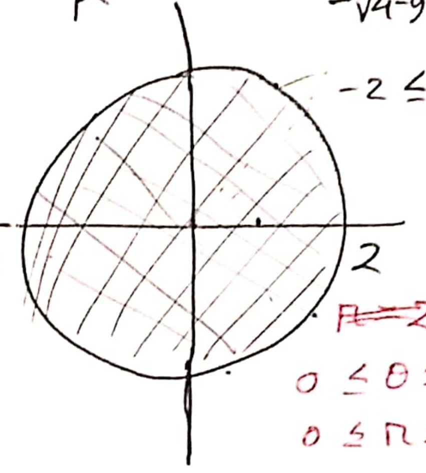
Ans: $\pi/8$

28

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy = \iint_R e^{-(x^2+y^2)}\,dA$$

$-\sqrt{4-y^2} \le y \le \sqrt{4-y^2}$

$-2 \le x \le 2$



$$\therefore \iint_R e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^{2} e^{-r^2}\, r\,dr\,d\theta$$

~~r = 2~~

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

try yourself :

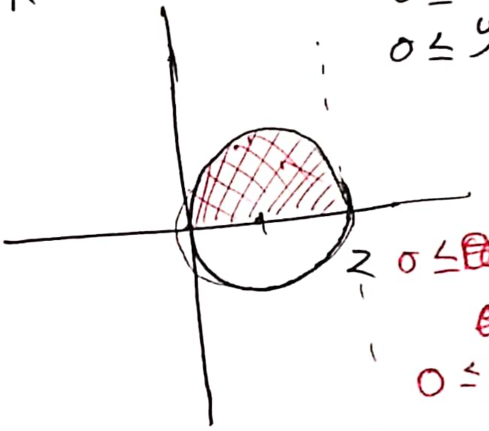
Ans : $(1-e^{-4})\pi$

29

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \iint_R \sqrt{x^2+y^2}\,dA$$

$0 \le x \le 2$

$0 \le y \le \sqrt{2x-x^2}$



$0 \le \theta \le \pi/2$

$0 \le r \le 2\cos\theta$

$y^2 = 2x - x^2$

$\Rightarrow y^2 = -(x^2 - 2x + 1 - 1)$

$\Rightarrow y^2 = -(x-1)^2 + 1$

$\Rightarrow y^2 + (x-1)^2 = 1$

$(x-1)^2 + y^2 = 1$

or $y^2 + x^2 = 2x$

$r^2 = 2r\cos\theta$

$\therefore r = 2\cos\theta$

$$\therefore \iint_R \sqrt{x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^{2\cos\theta} \sqrt{r^2}\; r\,dr\,d\theta$$

try by yourself.

30

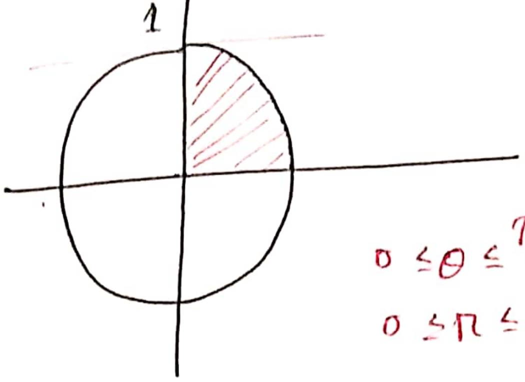
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\, dx\, dy = \iint_R \cos(x^2+y^2)\, dA$$

$0 \le y \le 1$

$0 \le x \le \sqrt{1-y^2}$

$$\therefore \iint_R \cos(x^2+y^2)\, dA = \int_0^{\pi/2} \int_0^1 \cos r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^1 r \cos(r^2)\, dr\, d\theta$$



$0 \le \theta \le \pi/2$

$0 \le r \le 1$

Let $u = r^2$

$\therefore du = 2r\, dr$

Now $\int r \cos(r^2)\, dr = \frac{1}{2} \int \cos u\, du = \frac{1}{2} \sin u + c$

$$\therefore \int_0^{\pi/2} \left[\int_0^1 r \cos r^2\, dr \right] d\theta = \int_0^{\pi/2} \frac{1}{2} \sin r^2 \Big]_{r=0}^{1} d\theta$$

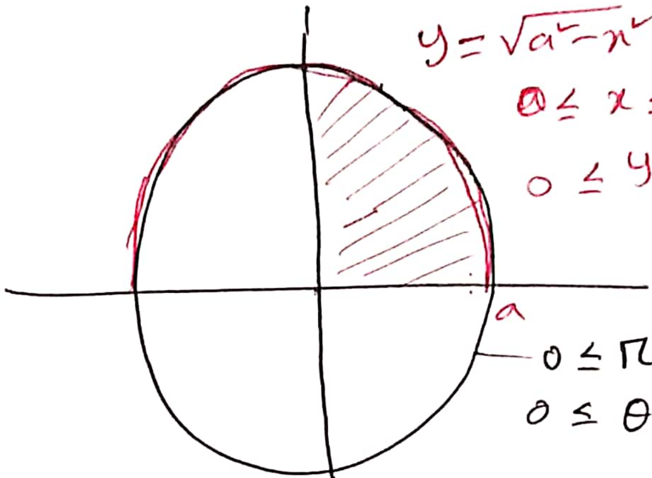
$$= \frac{1}{2} \int_0^{\pi/2} (\sin 1 - 0)\, d\theta$$

$$= \frac{1}{2} \sin 1 \, [\theta]_0^{\pi/2} = \frac{\sin 1}{2} \left(\frac{\pi}{2} - 0 \right)$$

$$= \frac{\pi}{4} \sin 1 \quad \text{Ans.}$$

31

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{1}{(1+x^2+y^2)^{3/2}}\,dy\,dx = \iint_R \frac{1}{(1+x^2+y^2)^{3/2}}\,dA \; ; a>0$$



$y=\sqrt{a^2-x^2}$

$0 \le x \le a$

$0 \le y \le \sqrt{a^2-x^2}$

$0 \le r \le a$

$0 \le \theta \le \pi/2$

$$\therefore \iint_R \frac{1}{(1+x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/2}\int_0^{a} \frac{1}{(1+r^2)^{3/2}}\, r\,dr\,d\theta$$

Let $1+r^2 = u$

$2r\,dr = du$

do as previous one ☺

32

$$\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2+y^2}\,dx\,dy = \iint_R \sqrt{x^2+y^2}\,dA$$