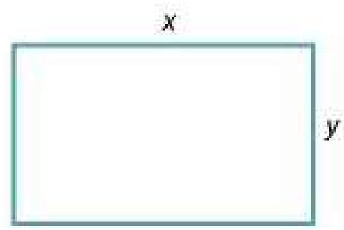
Example

Find the dimensions of a rectangle with perimeter 1000 metres so that the area of the rectangle is a maximum.

Solution

Let the length of the rectangle be x m, the width be y m, and the area be A m^2.



The perimeter of the rectangle is 1000 metres. So

$$1000 = 2x + 2y,$$

and hence

$$y = 500 - x.$$

The area is given by $A = xy$. Thus

$$A(x) = x(500 - x) = 500x - x^2. \qquad (1)$$

Because x and y are lengths, we must have $0 \leq x \leq 500$.

The problem now reduces to finding the value of x in $[0, 500]$ for which A is a maximum. Since A is differentiable, the maximum must occur at an endpoint or a stationary point.

From (1), we have

$$\frac{dA}{dx} = 500 - 2x.$$

Setting $\frac{dA}{dx} = 0$ gives $x = 250$.

Hence, the possible values for A to be a maximum are $x = 0$, $x = 250$ and $x = 500$. Since $A(0) = A(500) = 0$, the maximum value of A occurs when $x = 250$.

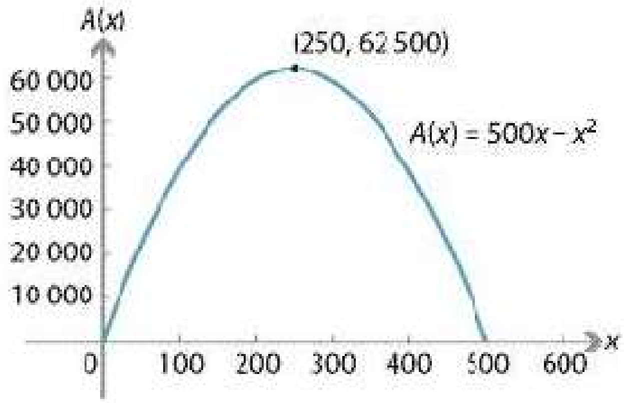
The rectangle is a square with side lengths 250 metres. The maximum area is 62 500 square metres.

1 $\dfrac{dA}{dx} > 0$, for $0 \le x < 250$, and $\dfrac{dA}{dx} < 0$, for $250 < x \le 500$. Hence, there is a local maximum at $x = 250$.

2 $\dfrac{d^2A}{dx^2} = -2 < 0$. This is a second way to see that $x = 250$ is a local maximum.

3 The graph of $A(x) = 500x - x^2$ is a parabola with a negative coefficient of x^2 and a turning point at $x = 250$. This is a third way of establishing the local maximum.

4 It is worth looking at the graph of $A(x)$ against x.



Question:

A farmer has 8 km of fencing wire, and wishes to fence a rectangular piece of land. One boundary of the land is the bank of a straight river. What are the dimensions of the rectangle so that the area is maximised?

5. What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material? Volume of a cylinder is $V = \pi r^2 h$. Surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$

$512 = \pi r^2 h$

$\dfrac{512}{\pi r^2} = h$

$A = 2\pi r^2 + 2\pi r\left(\dfrac{512}{\pi r^2}\right)$

$A = 2\pi r^2 + \dfrac{1024}{r}$

$A' = 4\pi r - \dfrac{1024}{r^2} = 0$

$4\pi r = \dfrac{1024}{r^2}$

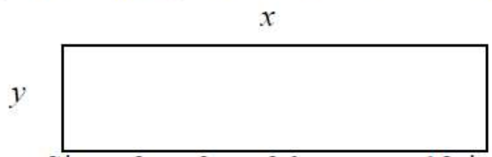
$r^3 = \dfrac{256}{\pi}$

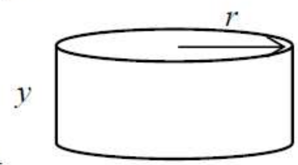
$\boxed{r = \sqrt[3]{\dfrac{256}{\pi}} \approx 4.335 \text{ inches}}$

3) A rectangular sheet of paper with perimeter 36 cm is to be rolled into a cylinder. What are the dimensions of the sheet that give the greatest volume?

Solution:

3) Let x and y be the dimensions of the sheet of paper.





Since $2x+2y=36$, $x+y=18$ is the constraint.

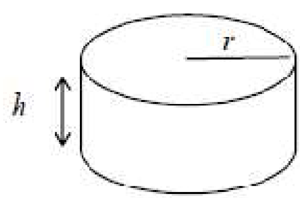
The radius is given by $2\pi r = x$, so $r=\dfrac{x}{2\pi}$, and the volume is $V=\pi r^2 y=\dfrac{x^2 y}{4\pi}$. Using $y=18-x$, $V=\dfrac{x^2(18-x)}{4\pi}$ is the function to be optimized. $\dfrac{dV}{dt}=\dfrac{36x-3x^2}{4\pi}$, so critical numbers are $x=0, 12$. Maximum volume occurs when $x=12$, (Why?) so dimensions are 6 cm by 12 cm and the volume is $\dfrac{216}{\pi}$ cm³.

8) A closed cylindrical container is to have a volume of 300 π in³. The material for the top and bottom of the container will cost \$2 per in², and the material for the sides will cost \$6 per in². Find the dimensions of the container of least cost.

a)Draw a picture, label variables and write down a constrained optimization problem that models this problem. **(5 Pts)**

b) Using calculus, solve the problem in part (a) to find the dimensions.

8)



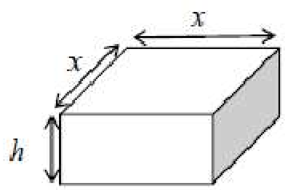
Volume: $V=\pi r^2 h$ Cost: $\$2(2\pi r^2)+\$6(2\pi r h)$

So Problem is minimize cost $C=4\pi r^2+12\pi r h$ subject to the constraint $V=\pi r^2 h=300\pi$ and so $r^2 h=300$.

b) Solving this last equation for h gives: $h=\dfrac{300}{r^2}$, which when substituted into the cost equation yields $C=4\pi r^2+\dfrac{3600\pi}{r}$. The geometry gives $r\in(0,\infty)$. To minimize the cost we determine critical numbers from $C'=8\pi r-\dfrac{3600\pi}{r^2}=0$ hence $r^3=450$ so the critical number is $r=(450)^{1/3}$. This gives $h=\dfrac{300}{(450)^{2/3}}=\dfrac{2(450)^{2/3}}{3}$ in. Since $C''=8\pi+\dfrac{7200}{r^3}\pi=20\pi>0$, the dimensions yield the minimum cost. The cylinder should have a radius $r=(450)^{1/3}$ in, and a height of $h=\dfrac{300}{(450)^{2/3}}$ in order to minimize the cost.

9) A closed rectangular container with a square base is to have a volume of 300 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$6 per in^2. Find the dimensions of the container of least cost.

9) A closed rectangular container with a square base is to have a volume of 300 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$6 per in^2. Find the dimensions of the container of least cost. **(20 Points)**



Volume: $V = x^2 h$ Cost: $\$2(2x^2) + \$6(4xh)$

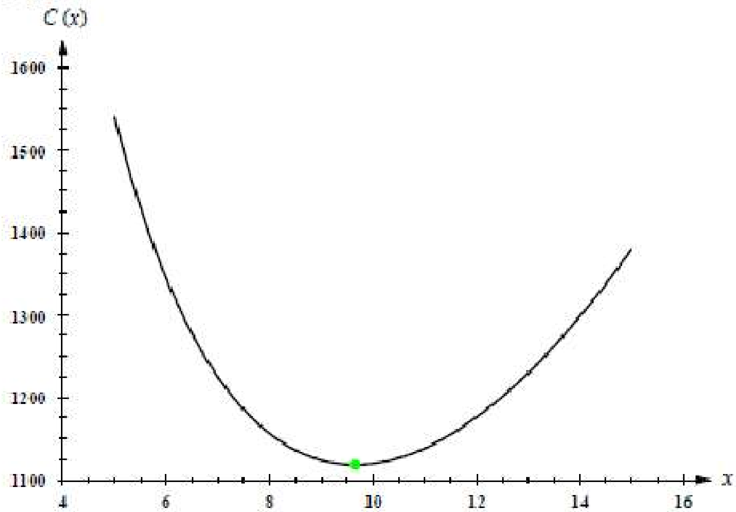
So Problem is minimize $C = 4x^2 + 24xh$ subject to the constraint $x^2 h = 300$ Solving this last equation for h gives: $h = \dfrac{300}{x^2}$, which when substituted into the cost equation yields $C = 4x^2 + \dfrac{7200}{x}$. The geometry gives $x \in (0, \infty)$.

Since $C' = 8x - \dfrac{7200}{x^2} = 0$ gives $x^3 = 900$ whose solution is $x = 900^{1/3} \approx 9.65$. This gives $h = \dfrac{300}{900^{2/3}} = \dfrac{900^{1/3}}{3} \approx 3.22$ in. Note $x^2 h = 900^{2/3} \cdot \dfrac{900^{1/3}}{3} = \dfrac{900}{3} = 300$ in^3 so the volume is

$$C = 4x^2 + \frac{7200}{x}$$

correct. And since $C'' = 8 + \frac{14,400}{x^3} = 8 + 16 > 0$, the dimensions yield the minimum cost. The box should have a square base of side length $900^{1/3}$ in, and a height of $\frac{900^{1/3}}{3}$ in.

FYI



EXAMPLE 1 A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Figure 2 illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in feet). Then we express A in terms of x and y:

$$A = xy$$

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = xy = x(2400 - 2x) = 2400x - 2x^2$$

Note that the largest x can be is 1200 (this uses all the fence for the depth and none for the width) and x can't be negative, so the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leqslant x \leqslant 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$. The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(600) = 720{,}000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720{,}000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all x, so A is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

The corresponding y-value is $y = 2400 - 2(600) = 1200$, so the rectangular field should be 600 ft deep and 1200 ft wide. ■

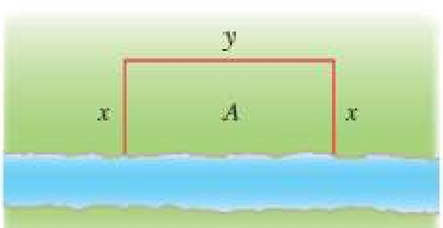
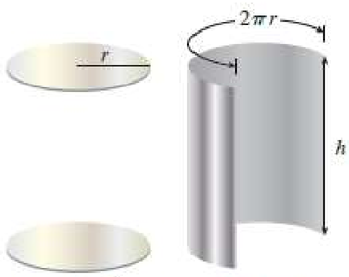


FIGURE 2

FIGURE 3



Area $2(\pi r^2)$ Area $(2\pi r)h$

FIGURE 4

EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

SOLUTION Draw a diagram as in Figure 3, where r is the radius and h the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and h. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We would like to express A in terms of one variable, r. To eliminate h we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm^3. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for A gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know that r must be positive, and there are no limitations on how large r can be. Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

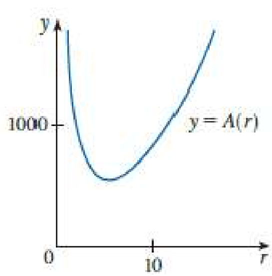


FIGURE 5

In the Applied Project following this section we investigate the most economical shape for a can by taking into account other manufacturing costs.

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

Since the domain of A is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so A is decreasing for *all* r to the left of the critical number and increasing for *all* r to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an *absolute* minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

The value of h corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

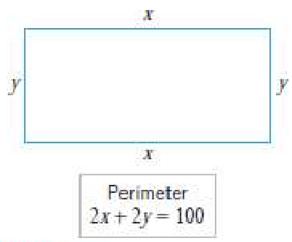
▸ Example 1 A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

Solution. Let

$$x = \text{length of the rectangle (ft)}$$
$$y = \text{width of the rectangle (ft)}$$
$$A = \text{area of the rectangle (ft}^2\text{)}$$

Then

$$A = xy \tag{1}$$



▲ Figure 4.5.1

Since the perimeter of the rectangle is 100 ft, the variables x and y are related by the equation

$$2x + 2y = 100 \quad \text{or} \quad y = 50 - x \tag{2}$$

(See Figure 4.5.1.) Substituting (2) in (1) yields

$$A = x(50 - x) = 50x - x^2 \tag{3}$$

Because x represents a length, it cannot be negative, and because the two sides of length x cannot have a combined length exceeding the total perimeter of 100 ft, the variable x must satisfy

$$0 \le x \le 50 \tag{4}$$

Thus, we have reduced the problem to that of finding the value (or values) of x in $[0, 50]$, for which A is maximum. Since A is a polynomial in x, it is continuous on $[0, 50]$, and so the maximum must occur at an endpoint of this interval or at a critical point.

From (3) we obtain

$$\frac{dA}{dx} = 50 - 2x$$

Setting $dA/dx = 0$ we obtain

$$50 - 2x = 0$$

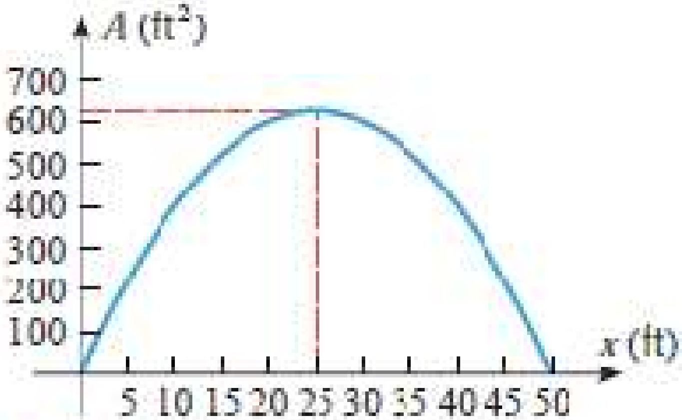
or $x = 25$. Thus, the maximum occurs at one of the values

$$x = 0, \quad x = 25, \quad x = 50$$

Substituting these values in (3) yields Table 4.5.1, which tells us that the maximum area of 625 ft^2 occurs at $x = 25$, which is consistent with the graph of (3) in Figure 4.5.2. From (2) the corresponding value of y is 25, so the rectangle of perimeter 100 ft with greatest area is a square with sides of length 25 ft. ◂

Table 4.5.1

x	0	25	50
A	0	625	0



▲ **Figure 4.5.2**

▶ **Example 6** Find a point on the curve $y = x^2$ that is closest to the point (18, 0).

Solution. The distance L between (18, 0) and an arbitrary point (x, y) on the curve $y = x^2$ (Figure 4.5.9) is given by

$$L = \sqrt{(x-18)^2 + (y-0)^2}$$

Since (x, y) lies on the curve, x and y satisfy $y = x^2$; thus,

$$L = \sqrt{(x-18)^2 + x^4} \tag{19}$$

Because there are no restrictions on x, the problem reduces to finding a value of x in $(-\infty, +\infty)$ for which (19) is a minimum. The distance L and the square of the distance L^2

are minimized at the same value (see Exercise 68). Thus, the minimum value of L in (19) and the minimum value of

$$S = L^2 = (x - 18)^2 + x^4 \tag{20}$$

occur at the same x-value.

From (20),

$$\frac{dS}{dx} = 2(x - 18) + 4x^3 = 4x^3 + 2x - 36 \tag{21}$$

so the critical points satisfy $4x^3 + 2x - 36 = 0$ or, equivalently,

$$2x^3 + x - 18 = 0 \tag{22}$$

To solve for x we will begin by checking the divisors of -18 to see whether the polynomial on the left side has any integer roots (see Appendix C). These divisors are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$, and ± 18. A check of these values shows that $x = 2$ is a root, so $x - 2$ is a factor of the polynomial. After dividing the polynomial by this factor we can rewrite (22) as

$$(x - 2)(2x^2 + 4x + 9) = 0$$

Thus, the remaining solutions of (22) satisfy the quadratic equation

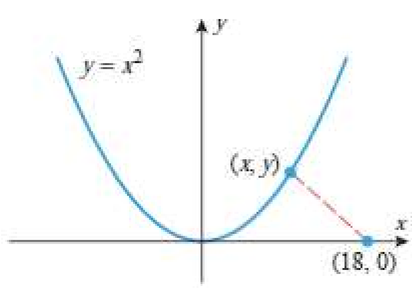
$$2x^2 + 4x + 9 = 0$$

But this equation has no real solutions (using the quadratic formula), so $x = 2$ is the only critical point of S. To determine the nature of this critical point we will use the second derivative test. From (21),

$$\frac{d^2S}{dx^2} = 12x^2 + 2, \quad \text{so} \quad \left.\frac{d^2S}{dx^2}\right|_{x=2} = 50 > 0$$

which shows that a relative minimum occurs at $x = 2$. Since $x = 2$ yields the only relative extremum for L, it follows from Theorem 4.4.4 that an absolute minimum value of L also occurs at $x = 2$. Thus, the point on the curve $y = x^2$ closest to $(18, 0)$ is

$$(x, y) = (x, x^2) = (2, 4) \blacktriangleleft$$



▲ **Figure 4.5.9**

EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

SOLUTION The distance between the point $(1, 4)$ and the point (x, y) is

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

(See Figure 6.) But if (x, y) lies on the parabola, then $x = \frac{1}{2}y^2$, so the expression for d becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get d in terms of x alone.)

Instead of minimizing d, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y-4)^2$$

(You should convince yourself that the minimum of d occurs at the same point as the minimum of d^2, but d^2 is easier to work with.) Note that there is no restriction on y, so the domain is all real numbers. Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y-4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of x is $x = \frac{1}{2}y^2 = 2$. Thus the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. [The distance between the points is $d = \sqrt{f(2)} = \sqrt{5}$.] ■

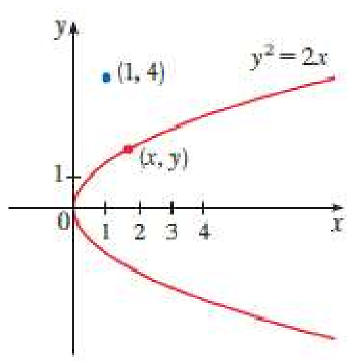


FIGURE 6