Week 6, Differential Calculus & Coordinate Geometry

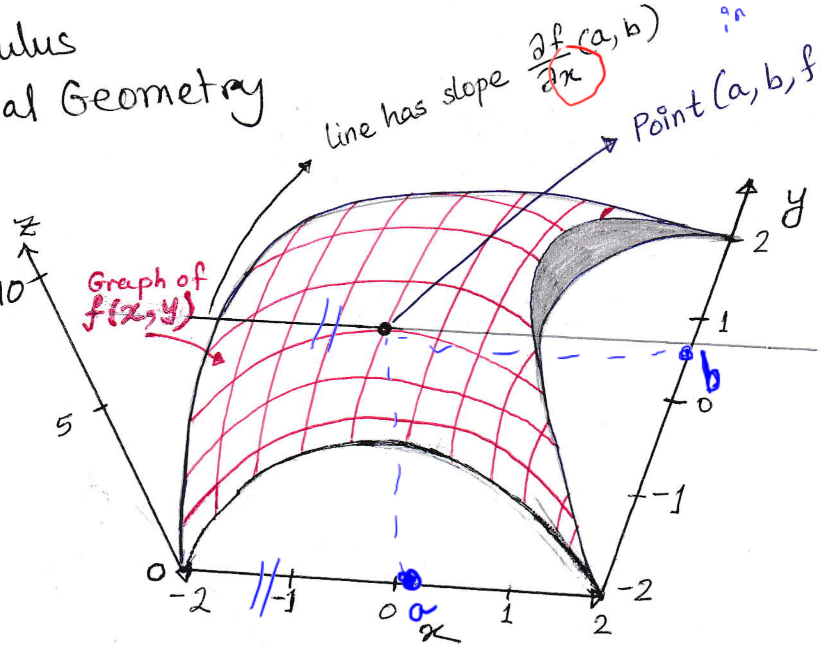
→ Partial Derivatives
→ Chain Rule of Partial Derivative

PAPTIAL DERIVATIVES

In mathematics a partial derivative of a function of several variables is it's derivative w.r.t. (with respect to) one of those variables, with the others held constant (as opposed to the total derivative, in which all variables are allowed to vary)

Partial Derivatives are used in:

→ Vector Calculus
→ Differential Geometry



x, variable
y → constant

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

y - variable
x - constant

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Compare the limit def^n of one variable:

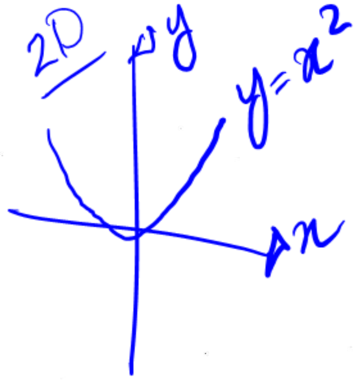
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

or $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

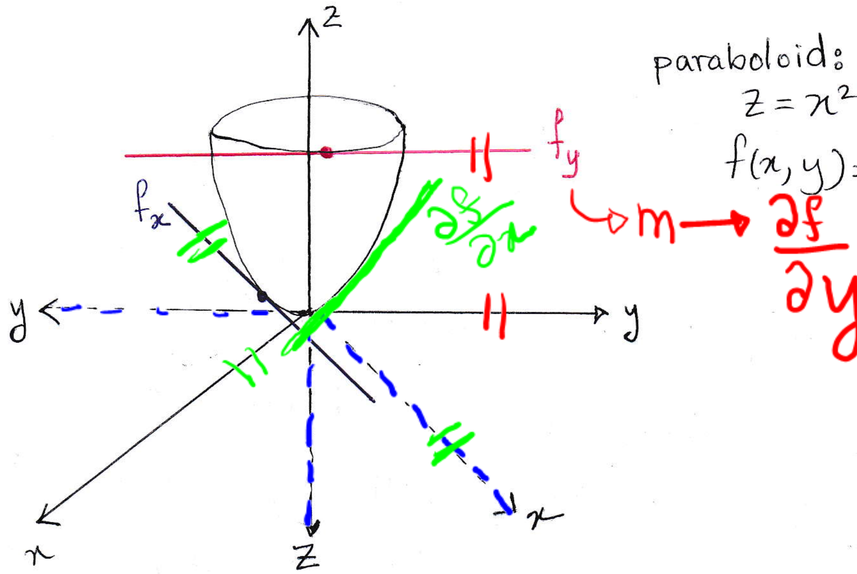
$$\lim_{x \to a} \frac{f(x) - f(a)}{(x-a)}$$

Δx

$\Delta x = x_2 - x_1$
$= x - a$



3D



paraboloid:

$$z = x^2 + y^2$$

$$f(x, y) = x^2 + y^2$$

$\frac{d}{dx} f(x) \rightarrow$ 2D

3D Mathematical Notation:

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = f_x \rightarrow \text{slope}$$

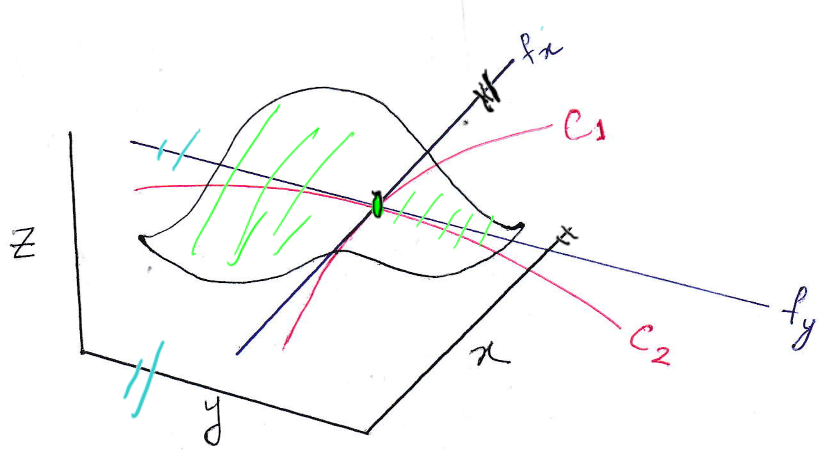
$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y) = f_y \rightarrow \text{slope}$$

$f'(x)$

$\underbrace{f''(x)}$

f_{xx}, f_{yy}, f_{xy}

Geometric Significance



Exercise

[1] Let $f(x,y) = 3x^3y^2$. Find the followings

a) $f_x(x,y) = \frac{\partial}{\partial x}(3x^3y^2) = 3(3x^2)y^2 = 9x^2y^2$

b) $f_y(x,y) = \frac{\partial}{\partial y}(3x^3y^2) = 3x^3(2y) = 6x^3y$

c) $f_x(x,1) = 9x^2(1^2) = 9x^2$ — refer (a)

d) $f_y(1,2) = 6(1)^3(2) = 12$ — refer (b)

[2] Let $f(x,y) = 4x^2 - 2y + 7x^4y^5$. Find the following:

a) f_{xx}

$f(x,y) = 4x^2 - 2y + 7x^4y^5$

$f_x(x,y) = 8x - 0 + 28x^3y^5$

$f_{xx}(x,y) = 8 + 84x^2y^5$

b) f_{yy}

$f(x,y) = 4x^2 - 2y + 7x^4y^5$

$f_y(x,y) = 35x^4y^4 - 2$

$f_{yy}(x,y) = 35x^4\,4y^3 - 0$

$= 140x^4y^3$

c) f_{xy}

$f(x,y) = 4x^2 - 2y + 7x^4y^5$

$f_x(x,y) = 8x + 28x^3y^5$

$f_{xy}(x,y) = 0 + 28x^3\,5y^4$

$= 140x^3y^4$

d) f_{yx}

$f(x,y) = 4x^2 - 2y + 7x^4y^5$

$f_y(x,y) = 35x^4y^4 - 2$

$f_{yx}(x,y) = 35y^4\,4x^3$

$= 140x^3y^4$

$f_{xy} = f_{yx}$

$\underbrace{(x,y)}_{2!}=2$ 0

x,y
y,x

4D

$\boxed{3}$ ~~Let~~ $f(x, y, z) = x^3y^5z^7 + xy^2 + y^3z$, Find the followings:

a) f_{xy}

$f = x^3y^5z^7 + xy^2 + y^3z$

$f_x = 3x^2y^5z^7 + y^2$

$f_{xy} = 15x^2y^4z^7 + 2y$

b) f_{yz}

$f = x^3y^5z^7 + xy^2 + y^3z$

$f_y = 5x^3y^4z^7 + 2xy + 3y^2z$

$f_{yz} = 35x^3y^4z^6 + 3y^2$

c) f_{xz}

$f = x^3y^5z^7 + xy^2 + y^3z$

$f_x = 3x^2y^5z^7 + y^2$

$f_{xz} = 21x^2y^5z^6$

d) f_{zz}

$f = x^3y^5z^7 + xy^2 + y^3z$

$f_z = 7x^3y^5z^6 + y^3$

$\underbrace{(x,y)}_{2!}$ $f_{zz} = 42x^3y^5z^5$

f_{zyy}

e) $f = x^3y^5z^7 + xy^2 + y^3z$

$f_z = 7x^3y^5z^6 + y^3$

$f_{zy} = 35x^3y^4z^6 + 3y^2$

$f_{zyy} = 140x^3y^3z^6 + 6y$

f_{zxy}

f) $f = x^3y^5z^7 + xy^2 + y^3z$

$f_z = 7x^3y^5z^6 + y^3$

$f_{zx} = 21x^2y^5z^6$

$f_{zxy} = 105x^2y^4z^6$

$f_{zxy} = f_{zyx}$

$f_{xy} = f_{yx}$

f_{zyx}

g) $f = x^3y^5z^7 + xy^2 + y^3z$

$f_z = 7x^3y^5z^6 + y^3$

$f_{zy} = 35x^3y^4z^6 + 3y^2$

$f_{zyx} = 105x^2y^4z^6$

h) f_{xxyz}

$f = x^3y^5z^7 + xy^2 + y^3z$

$f_x = 3x^2y^5z^7 + y^2$

$f_{xx} = 6xy^5z^7$

$f_{xxy} = 30xy^4z^7$

$f_{xxyz} = 210xy^4z^6$

$= f_{xyz}$

$\underbrace{(x,y,z)}_{3! = 6}$ $= f_{xzy}$

$= f_{yxz}$

$= f_{yzx}$

$\boxed{4}$ Let $f(x,y,z) = \sqrt{xy} + \ln(x^2 z^3) - x\tan(z)$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Compute f_x, f_{xy}, f_{xyz}, f_z

$y = f(x) \to 2D$

$z = f(x,y) \to 3D$

$w = f(x,y,z) \to 4D$

$u = f(x_1, x_2, \dots, x_n, x_{n+1}) \to n\text{-}D$

a) $f = \sqrt{xy} + \ln(x^2 z^3) - x\tan(z)$

$$f_x = \frac{1}{2\sqrt{xy}}(y) + \frac{1}{x^2 z^3}(2x z^3) - \tan z$$

$$f_x = \frac{1}{2} x^{-1/2} y^{1/2} + \frac{2}{x} - \tan z$$

b) $$f_{xy} = \frac{1}{2} x^{-1/2} \frac{1}{2} y^{\frac{1}{2}-1} + 0 - 0$$

$$= \frac{1}{4} x^{-1/2} y^{-1/2}$$

$$f_{xy} = \frac{1}{4\sqrt{xy}}$$

c) $f_{xyz} = 0$

d) $f = \sqrt{xy} + \ln(x^2 z^3) - x\tan z$

$$f_z = 0 + \frac{1}{x^2 z^3} x^2 3z^2 - x\sec^2 z$$

$$= \frac{3}{z} - x\sec^2 z$$

Ch 14.5

STEWART

CHAIN RULE of PARTIAL DERIVATIVES

Recall : if $y = f(x)$, $x = g(t)$

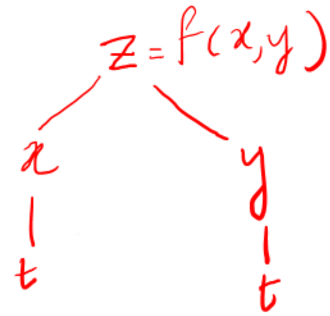
then $\dfrac{dy}{dt} = \left[f(g(t))\right]'$

$= \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

$x = g(t)$ $\quad \dfrac{d\,g(t)}{dt}$

$y = f(x)$ $\quad \dfrac{d\,f(x)}{dx}$

The Chain Rule (case 1) Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and



$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

OR $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \qquad z = f(x,y)$$

Examples: The pressure p (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K (T) and increasing at a rate of 0.1 K/s ($\frac{dT}{dt}$) and the volume is 100 L (V) and increasing at a rate of 0.2 L/s. ($\frac{dV}{dt}$) $\quad \dfrac{dP}{dt}$

$$\frac{dP}{dV}, \frac{dP}{dT}, \frac{dP}{dt}$$

If t = time elapsed in seconds

$T = 300 \qquad V = 100$

$\frac{dT}{dt} = 0.1 \qquad \frac{dV}{dt} = 0.2$

$\frac{1}{V} = V^{-1}$

$P = 8.31 \cdot \frac{1}{V} \cdot T$

$\therefore PV = 8.31\,T$

$\frac{\partial P}{\partial T} = 8.31 \cdot \frac{1}{V}(1)$

$P = 8.31 \frac{T}{V}$

$\frac{\partial P}{\partial T} = \frac{8.31}{V}$

$\frac{\partial P}{\partial V} = 8.31\,T\left(-1V^{-2}\right)$

$\frac{\partial P}{\partial V} = 8.31\,T(-1)V^{-2} = -\frac{8.31\,T}{V^2}$

$$\frac{dP}{dt} = \frac{\partial P}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial P}{\partial V} \cdot \frac{dV}{dt}$$

$$= \frac{8.31}{V} \cdot \frac{dT}{dt} - \frac{8.31\,T}{V^2} \cdot \frac{dV}{dt}$$

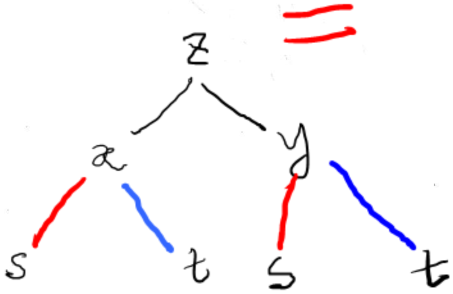
$$= \frac{8.31}{100} \cdot (0.1) - \frac{8.31(300)}{100^2}(0.2)$$

$$= -0.04155$$

The pressure is decreasing at a rate of about 0.042 kPa/s

The chain Rule (Case 2) suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t.

Then $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$



Example If $z = e^{x}\sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= (e^{x}\sin y)(t^2) + (e^{x}\cos y)(2st)$$

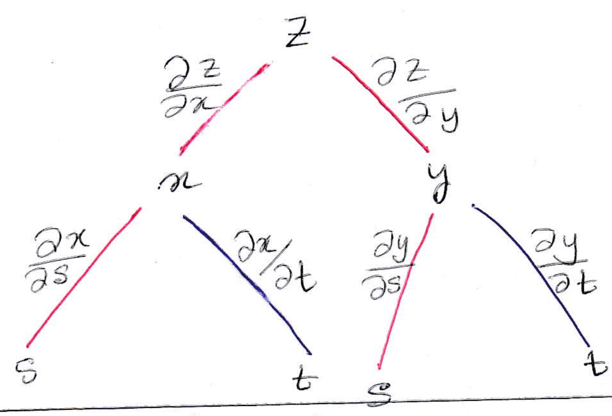
$$= t^2 e^{st^2}\sin(s^2t) + 2st\, e^{st^2}\cos(s^2t)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (e^{x}\sin y)(2st) + (e^{x}\cos y)(s^2)$$

$$= 2st\, e^{st^2}\left(\sin(s^2t) + \cos(s^2t)\right)$$

Tree diagram



Chain Rule (General formula): Suppose that u is differentiable function on the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

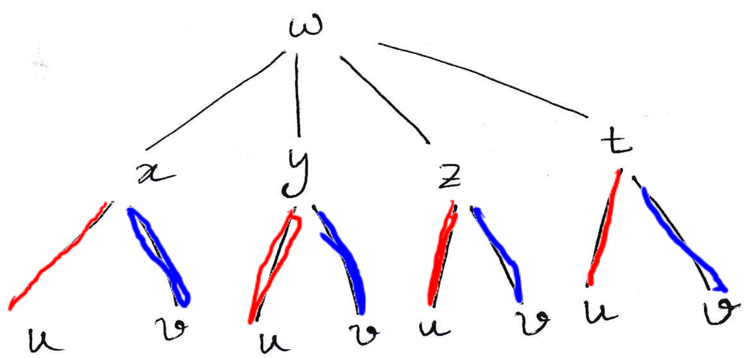
$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Example Write out the chain rule for the case where $w = f(x, y, z, t)$ and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and $t = t(u, v)$.

$n = 4, \quad m = 2$

$$\rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

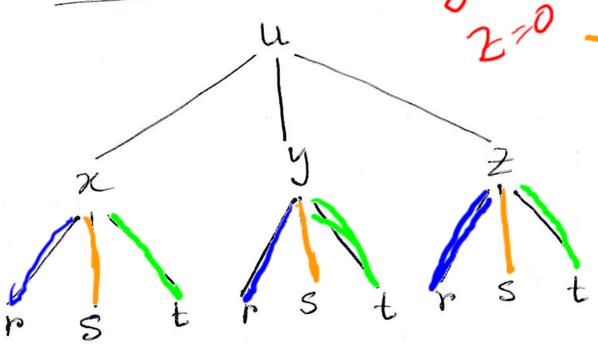
$$\rightarrow \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$



Example If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$ and $z = r^2 s \sin t$, find the value of $\frac{\partial u}{\partial s}$ when $r = 2, s = 1, t = 0$.

Tree diagram

$x = 2$
$y = 2$
$z = 0$



With the help of tree diagram:

$$\rightarrow \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

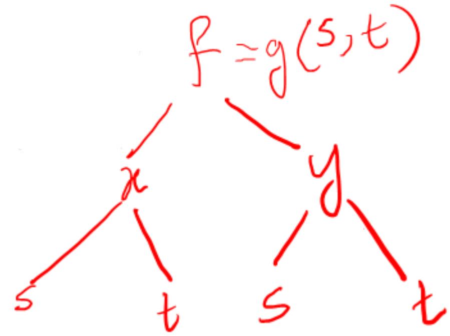
$$= (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t)$$

$$= (64)(2) + (16)(4) + (0)(0)$$

$$= 192$$

Example $g(s,t) = f(s^2-t^2, t^2-s^2)$. f is differentiable. show that g satisfies the equation:

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$



Let $x = s^2 - t^2$, $y = t^2 - s^2$

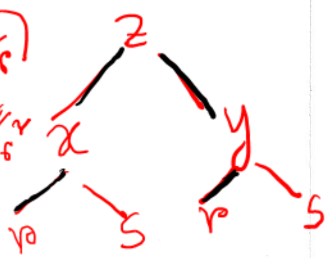
$$\therefore g(s,t) = f(s^2-t^2, t^2-s^2)$$
$$= f(x,y)$$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$= \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$= \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

$$\therefore t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = \left(2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}\right) + \left(-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}\right)$$
$$= 0$$

Example If $z = f(x,y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$, $y = 2rs$ find expressions for (a) $\frac{\partial z}{\partial r}$ and (b) $\frac{\partial^2 z}{\partial r^2}$ $\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial^2 z}{\partial r^2}$



a) $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}(2r) + \frac{\partial z}{\partial y}(2s)$

(b) Applying product Rule to the expression in part (a) we have

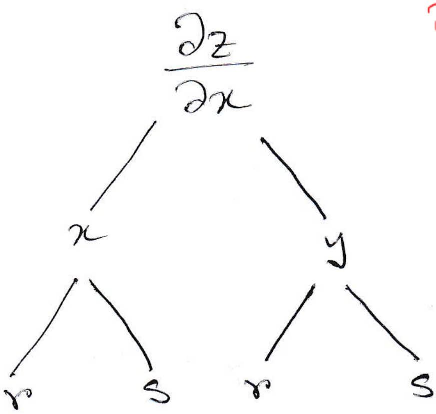
$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\overset{u}{2r}\,\overset{v}{\frac{\partial z}{\partial x}} + 2s\frac{\partial z}{\partial y}\right)$$ [Performing $\frac{\partial}{\partial r}$ on both sides]

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \qquad \text{— (b)}$$

$\frac{\partial}{\partial r}(z)$

$\hookrightarrow \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)$ & $\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$

(b) If we apply chain rule on (a) we have



$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}\right) \quad \left[\text{apply } \frac{\partial}{\partial x} \text{ on both sides of (a)}\right]$$

$$\underline{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}$$

$$= \frac{\partial^2 z}{\partial x^2}(2r) + \frac{\partial^2 z}{\partial y \partial x}(2s) \quad \text{— (i)}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}\right) + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}\right) \quad \left[\text{apply } \frac{\partial}{\partial y} \text{ on both sides of (a)}\right]$$

$$\underline{\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r}$$

$$= \frac{\partial^2 z}{\partial x \partial y}(2r) + \frac{\partial^2 z}{\partial y^2}(2s) \quad \text{— (ii)}$$

Substitute (i) & (ii) into eqn (b)

$$\frac{\partial^2 z}{\partial r^2} = 2\frac{\partial z}{\partial x} + 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}\right) + 2s\left(2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

Implicit Differentiation:

$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \underbrace{\frac{\partial F}{\partial y}}_{\neq 0}\frac{dy}{dx} = 0$$

$F(x,y)=0$ defines y implicitly as a differentiable function of x.
$y=f(x)$, where $F(x,f(x))=0$ for all x in the domain of f.

Apply Chain Rule on $F(x,y)=0$.

So we have $\dfrac{dy}{dx} = -\dfrac{\partial F/\partial x}{\partial F/\partial y} = -\dfrac{F_x}{F_y}$

Example Find y' if $x^3+y^3=6xy$

$$F(x,y)=x^3+y^3-6xy=0$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{3x^2-6y}{3y^2-6x}=-\frac{x^2-2y}{y^2-2x}$$

Example Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3+y^3+z^3+6xyz+4=0$

Let $F(x,y,z)=x^3+y^3+z^3+6xyz+4=0$

$$\therefore \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}$$

$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{3y^2+6xz}{3z^2+6xy}=-\frac{y^2+2xz}{z^2+2xy}$$