MAT 110 Differential Calculus & Coordinate Geometry

Week 2

Topics: ① Limit Definition → Differentiability Test
② Basic Rules of Differentiation
③ Chain Rule
④ Higher Derivatives
⑤ Techniques of Differentiation
↳ Implicit Differentiation

Definition of Limit

Any function $y=f(x)$ is differentiable at $x=a$ if the following limit exists.

$$\frac{dy}{dx}=f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad \text{OR} \quad \frac{dy}{dx}=f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This is also known as $\boxed{1^{st} \text{ Principal of Derivative}}$.

Example: Use the Limit definition to find the derivative of $f(x)=x^2-1$ at $\boxed{x=2.}$

In general $f'(x)=2x \to f'(2)$
$=2(2)$
$=4$

$$\boxed{\begin{aligned}\text{given } f(x)&=x^2-1\\ f(2)&=2^2-1\\ &=3\end{aligned}}$$

Using limit definition

$$\frac{dy}{dx}=\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$
$$=\lim_{x\to 2}\frac{x^2-1-3}{x-2}$$
$$=\lim_{x\to 2}\frac{x^2-4}{x-2}$$
$$=\lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)}$$
$$=\lim_{x\to 2} x+2=2+2=4$$

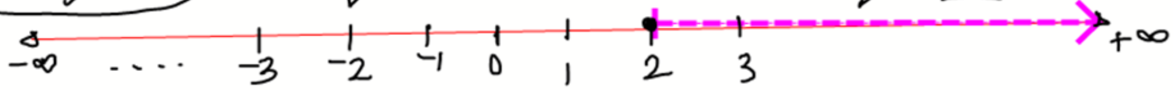
$\therefore f(x)$ is differentiable at $x=2$.

$>$ $<$ ( ) → open interval

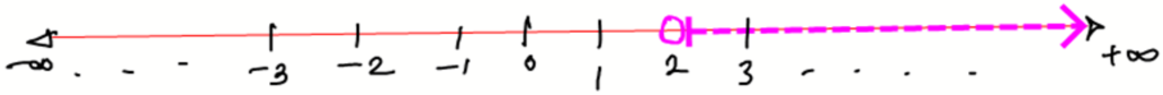
$\geq$ $\leq$ [ ] → closed interval

$x = 2, \dots, 3, \dots$

(i) $x \geq 2 \Rightarrow x = 2$ or $x > 2$



(ii) $x > 2 \Rightarrow x$ is greater than 2



b) $x = 2.00000\,00\,0\,1, 2.1, 3, \dots$

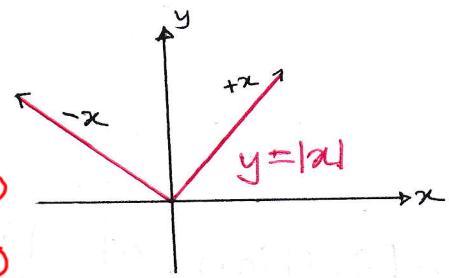
Differentiability & Continuity:

If f is differentiable at $x=a$, then f must be continuous at a.

Example:

To show that $f(x)=|x|$ is continuous at 0, show that

$\lim\limits_{x\to 0} |x| = |0| = 0.$



$x=0,\ x>0$

$$f(x)=|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases} \quad \begin{cases} x>0 \\ x\leq 0 \end{cases}$$

$\boxed{x=0}$

$> < (\ )$ open interval

$\geq \leq [\ ]$ closed interval

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = (-0) = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x) = (0) = 0$$

$\because \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = 0$, hence $f(x)=|x|$ is continuous at $x=0$

for f(0) we should consider f(x)=x since it is a closed interval

Test the Differentiability by 1^{st} Principal of Derivative: (Performing two sided Limit test)

$$f'(x) = f'_-(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{-x-0}{x-0} = \lim_{x\to 0^-} -\frac{x}{x} = \lim_{x\to 0^-} -1 = -1$$

L.H.L

$$f'_+(x) = f'_+(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$$

$$\text{R.H.L} = \lim_{x\to 0^+}\frac{x-0}{x-0} = \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$\because$ L.H.L $\neq$ R.H.L

So two sided limit does not exist

$\therefore$ The derivative does not exist at $x=0$

Continuity $\not\Rightarrow$ Differentiability

Differentiability $\Rightarrow$ Continuity

Test the Differentiability of the following functions:

$\boxed{1}$ $f(x) = \begin{cases} \cos x, & x > 0 \\ -\cos x, & x < 0 \end{cases}$ at $x = 0$

R.H. function

L.H. function

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-\cos x - (-\cos 0^\circ)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-\cos x + 1}{x}$$

$\frac{0}{0}$ form; Apply L'Hôspital's law

$$= \lim_{x \to 0^-} \frac{\frac{d}{dx}(-\cos x + 1)}{\frac{d}{dx}(x)}$$

$$= \lim_{x \to 0^-} \frac{-(-\sin x) + 0}{1}$$

$$= \lim_{x \to 0^-} \sin x = \sin 0^\circ = 0$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{\cos x - \cos 0}{x}$$

$$= \lim_{x \to 0^+} \frac{\cos x - 1}{x}$$

$\frac{0}{0}$ form

$$= \lim_{x \to 0^+} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(x)}$$

$$= \lim_{x \to 0^+} \frac{-\sin x - 0}{1}$$

$$= \lim_{x \to 0^+} -\sin 0^\circ = 0$$

$\because$ L.H.L = R.H.L

$\therefore$ $f(x)$ is differentiable at $x = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$

$\boxed{12}$ $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at $x=0$

$x<0$ or $x>0$

$x<0$		$x>0$
$f(x) = x^2 \sin\frac{1}{x}$	$x=0$	$f(x) = x^2 \sin\frac{1}{x}$
	$f(0)=0$	

(ii)

$\because -1 \le \sin x \le 1$

$\Rightarrow -1 \le \sin\left(\frac{1}{x}\right) \le 1$

$\Rightarrow -x \le x \sin\frac{1}{x} \le x \quad [g \le f \le h]$

$\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} (-x) = 0$

$\lim\limits_{x\to 0} h(x) = \lim\limits_{x\to 0} (x) = 0$

by squeeze Theorem

$\lim\limits_{x\to 0} f(x) = 0$

$\Rightarrow \lim\limits_{x\to 0} x \sin\frac{1}{x} = 0$

(i)

$f'_-(x) = \lim\limits_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim\limits_{x\to 0^-} \frac{x^{\cancel{2}} \sin\frac{1}{x} - 0}{\cancel{x}}$

$= \lim\limits_{x\to 0^-} x \sin\frac{1}{x}$

$= 0$ by squeeze Thm

$f'_+(x) = \lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim\limits_{x\to 0^+} \frac{\cancel{x^2} \sin\frac{1}{x} - 0}{\cancel{x}}$

$= \lim\limits_{x\to 0^+} x \sin\frac{1}{x}$

$= 0 \longrightarrow$ by squeeze Thm

$f(0)=0$ given in the question

$\lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^+} f'(x)$

$f(x)$ is differentiable at $x=0$

Hence $f(x)$ is continuous at $x=0$

$\boxed{3}$ $f(x) = \begin{cases} x\cos\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ at $x=0$

$x \neq 0 \rightarrow x<0$ or $x>0$

$0 \longrightarrow f(0)$

$x<0$ $\qquad$ $x>0$

$f(x) = x\cos\frac{1}{x}$ $\qquad$ $x=0$, $f(0)=0$ $\qquad$ $f(x) = x\cos\frac{1}{x}$

(ii) $\because -1 \leq \cos x \leq 1$ $\qquad [g \leq f \leq h]$

$\Longrightarrow -1 \leq \cos\left(\frac{1}{x}\right) \leq 1$

$\lim_{x\to 0} g(x)$

$= \lim_{x\to 0} -1$

$= -1$

$\lim_{x\to 0} h(x)$

$= \lim_{x\to 0} 1$

$= 1$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exists.

(i) consider:

$f_-'(x) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0^-} \frac{x\cos\frac{1}{x} - 0}{x}$

$= \lim_{x\to 0^-} \frac{\cancel{x}\cos\frac{1}{x}}{\cancel{x}}$

$= \lim_{x\to 0^-} \cos\frac{1}{x}$ $\qquad$ use squeeze Thm

$=$ any number between -1 & $+1$

$f_+'(x) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0^+} \frac{x\cos\frac{1}{x} - 0}{x}$

similarly

$= \lim_{x\to 0^+} \cos\frac{1}{x}$

$=$ any number between -1 & $+1$

L.H.L $\neq$ R.H.L

$\therefore f(x)$ is not differentiable at $x=0$

$\boxed{4}$ Let $f(x) = \begin{cases} x^2 - 16x, & x < 9 \quad \to \text{L.H.} \\ 12\sqrt{x}, & x \geq 9 \quad \text{R.H.} \end{cases}$

$x = 9$ or $x > 9$

(i) Is $f(x)$ continuous at $x = 9$?

(ii) Determine whether $f(x)$ is differentiable at $x = 9$.

$\therefore f(9) = 12\sqrt{9} = 12(3) = 36$

(i) $\lim_{x \to 9^-} x^2 - 16x$

$= 9^2 - 16(9) = -63$

$\lim_{x \to 9^+} 12\sqrt{x}$

$= 12\sqrt{9} = 36$

L.H.L $\neq$ R.H.L

$f(x)$ is not continuous at $x = 9$

(ii) $\lim_{x \to 9^-} \frac{f(x) - f(9)}{x - 9}$

$= \lim_{x \to 9^-} \frac{x^2 - 16x - 12\sqrt{9}}{x - 9}$

$\because x < 9$ open interval hence for the input value we need to consider $f(x) = 12\sqrt{x}$ as $x \geq 9$ which mean $x = 9$ or > 9.

$x < 9$ meas x is less than 9, NOT EQUAL to 9. $\therefore$ we cannot substitute $x = 9$ into the function $f(x) = x^2 - 16$.

$= \lim_{x \to 9^-} \frac{x^2 - 16x - 36}{x - 9}$

$= \lim_{x \to 9^-} \frac{x^2 - 18x + 2x - 36}{x - 9}$

$= \lim_{x \to 9^-} \frac{(x - 18)(x + 2)}{(x - 9)}$

$= \frac{(-9)(11)}{0}$

$= \infty$

$\lim_{x \to 9^+} \frac{f(x) - f(9)}{x - 9}$

$= \lim_{x \to 9^+} \frac{12\sqrt{x} - 12\sqrt{9}}{x - 9}$

$= \lim_{x \to 9^+} \frac{12(\sqrt{x} - 3)}{x - 9}$

$= \lim_{x \to 9^+} \frac{12(\sqrt{x} - 3)}{(\sqrt{x} + 3)(\sqrt{x} - 3)}$

$= \lim_{x \to 9^+} \frac{12}{\sqrt{x} + 3}$

$= \frac{12}{6}$

$= 2$

$f(x)$ is not differentiable at $x = 9$

5 Let $f(x)=\begin{cases} x^2 ; & x \le 1 \\ \sqrt{x} ; & x > 1 \end{cases}$

(i) Is $f(x)$ continuous at $x=1$?

(ii) Determine whether $f(x)$ is differentiable at $x=1$.

(i) $\lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} x^2$

$= 1^2 = 1$

$\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} \sqrt{x}$

$= \lim_{x\to 1^+} \sqrt{1} = 1$

L.H.L = R.H.L

$f(x)$ is continuous at $x=1$

(ii) $\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$

for $f(1)$ we will consider $f(x)=x^2 ; x\le 1$

$\left\{ \because f(x)=\sqrt{x} \to x>1 \text{ contains open interval} \right\}$

$= \lim_{x\to 1^-} \frac{x^2-1^2}{x-1}$

$= \lim_{x\to 1^-} \frac{(x+1)(x-1)}{(x-1)}$

$= \lim_{x\to 1^-} x+1$

$= 1+1$

$= 2$

$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1^+} \frac{\sqrt{x}-1^2}{x-1}$

$= \lim_{x\to 1^+} \frac{(\sqrt{x}-1)}{(\sqrt{x}+1)(\sqrt{x}-1)}$

$= \lim_{x\to 1^+} \frac{1}{\sqrt{x}+1}$

$= \frac{1}{\sqrt{1}+1}$

$= \frac{1}{2}$

L.H.L $\neq$ R.H.L

$f(x)$ is not differentiable at $x=1$

It proves that continuity does not imply differentiability.

$\boxed{6}$ Show that

$$f(x) = \begin{cases} x^2+1; & x \le 1 \\ x; & x > 1 \end{cases}$$

(i) is not continuous at $x=1$

(ii) is not differentiable at $x=1$

TRY

Basic Rules of Derivatives

$\frac{dy}{dx}$

Definition: Differentiation means the rate of change of one quantity with respect to another. The speed is calculated as the rate of change of distance with respect to time. This speed at each instant is not the same as the average calculated. Speed is the same as the slope, which is nothing but the instant rate of change of the distance over a period of time.

The ratio of a small change in one quantity with a small change in another quantity which is dependent on the first quantity is called differentiation.

One of the important concepts in calculus is mainly focused on the differentiation of a function.

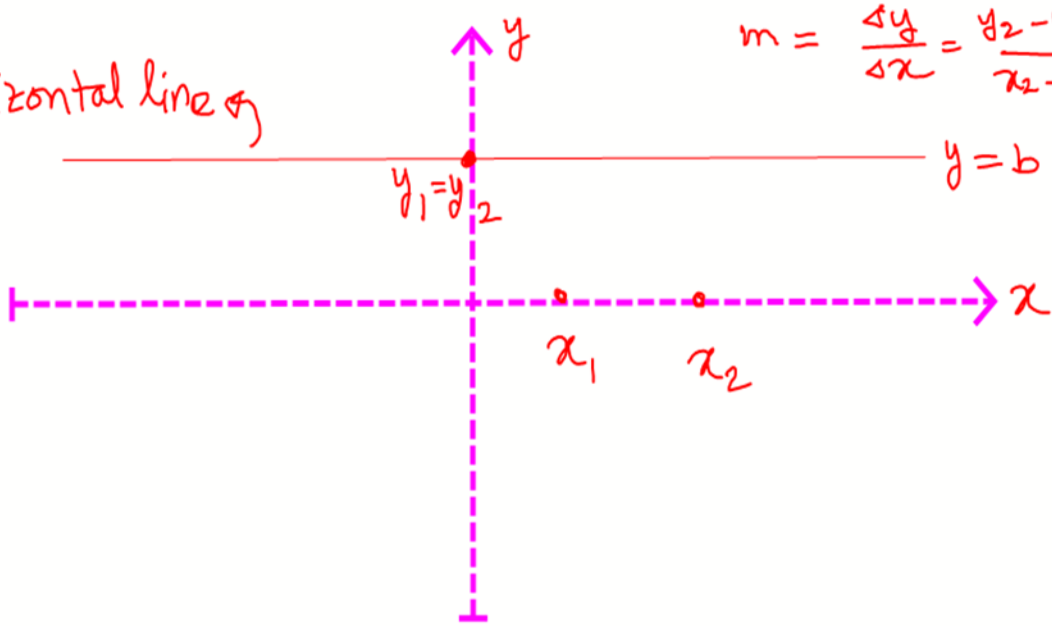
The useful tools that are determined by differentiation are:

- → The maximum or minimum value of a function
- → The velocity and acceleration of moving objects
- → The tangent (slope) of a curve

If $y=f(x)$ is differentiable, then the differentiation is represented as $f'(x)$ or $\frac{dy}{dx}$.

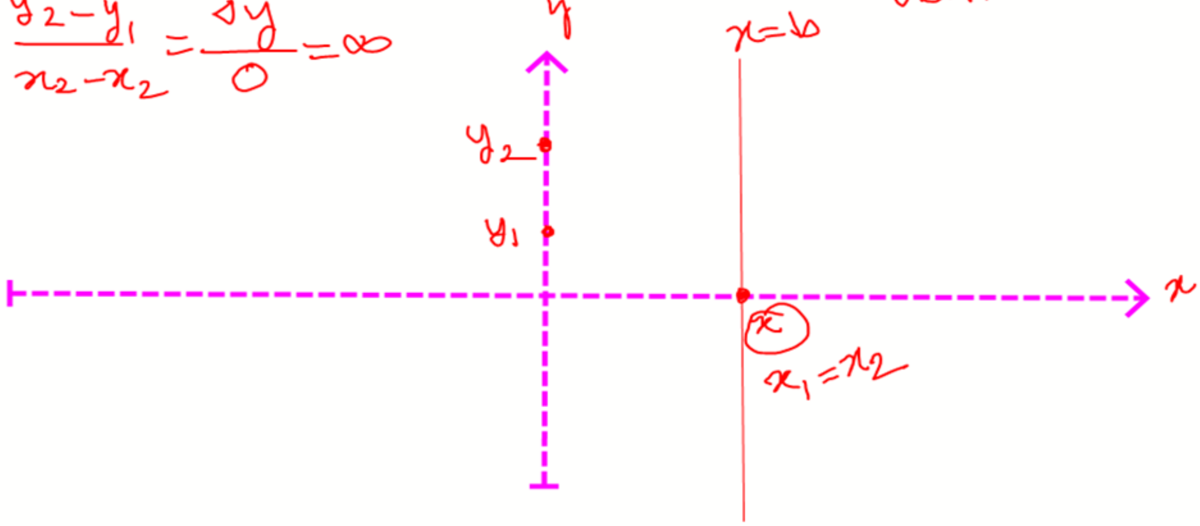
horizontal line eq

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{\Delta x} = 0$$

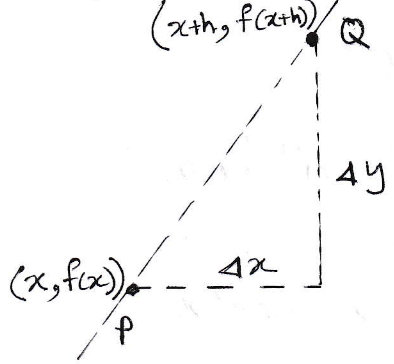


$$m = \frac{y_2 - y_1}{x_2 - x_2} = \frac{\Delta y}{0} = \infty$$

vertical line



slope of the secant line is: $\dfrac{y_2 - y_1}{x_2 - x_1}$

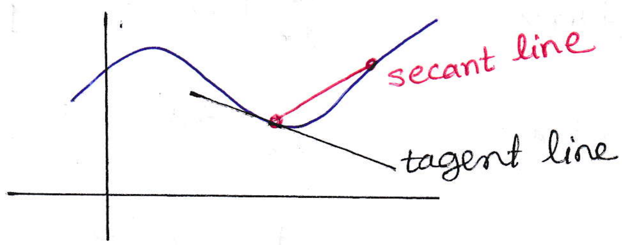


$$\text{slope} = \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x+h) - f(x)}{(x+h) - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

A secant line is a line that intersects a curve at a minimum of two distinct points.



tangent line: slope of a curve

Rules:

① Addition: $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

② product: $\frac{d}{dx}(u \times v) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

③ Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \dfrac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$; $v \neq 0$

④ Constant: $\frac{d}{dx}[c] = 0$; $\frac{d}{dx}[cu] = c\frac{d}{dx}(u)$

⑤ Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Important Formulae

$\frac{d}{dx} c = 0$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx} x^n = nx^{n-1}$

$\frac{d}{dx} a^x = a^x \ln a$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x = -\csc^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

$\frac{d}{dx} \csc x = -\csc x \cot x$

$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$

$\frac{d}{dx} \ln|x| = \frac{1}{x}$

$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$

$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

Chain Rule

Let $y = f(g(x))$ composite function

↳ a function that is written inside another function.

$y' = f'(g(x))\, g'(x)$

Consider $y = 4\cos(x^3)$

$$\frac{dy}{dx} = 4(-\sin x^3)(3x^2)$$

$$= -12x^2 \sin x^3.$$

Another way to approach chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$ while $y = f(\underbrace{g(x)}_{u})$ introducing

if $y = f(u)$ and $u = g(x)$

$f(g(x)) = f(u)$

Consider $y = 4\cos x^3$

Let $u = x^3$ $\qquad$ $y = 4\cos x^3$

$\qquad\qquad\qquad\qquad\quad = 4\cos u$

$$\boxed{\frac{du}{dx}} = 3x^2 \qquad \boxed{\frac{dy}{du}} = -4\sin u$$

$$\frac{dy}{dx} = \boxed{\frac{dy}{du}} \cdot \boxed{\frac{du}{dx}}$$

$$= (-4\sin u)(3x^2)$$

$$= (-4\sin x^3)(3x^2)$$

$$= -12x^2 \sin x^3.$$

Note:

$\cos^3 x = (\cos x)^3$

$\cos x^3 \neq (\cos x)^3$

Examples on Chain Rule:

$x^n = n x^{n-1}$

1) $y = \operatorname{cosec}^3 x$

$= (\operatorname{cosec} x)^3$

$\frac{d}{dx}(\operatorname{cosec} x)$

$$\frac{dy}{dx} = 3\operatorname{cosec}^2 x(-\operatorname{cosec} x \cot x)$$

$$= -3\operatorname{cosec}^3 x \cot x$$

$\frac{d}{dx}[\operatorname{cosec} x]$

$\boxed{2}$ $\quad y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\quad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

f, g, $\to x$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\underbrace{\frac{1-x^2}{1+x^2}}_{g}\right)^2}} \times \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$\frac{d}{dx}\left(\frac{u}{v}\right)$

$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$

$$= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}} \times \frac{\cancel{-2x}-2x^3-2x+2x^3}{(1+x^2)^2}$$

$\sqrt[3]{a} = a^{1/3}$

$\sqrt{a} = a^{1/2}$

$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\times\frac{d}{c}$

$$= \frac{-\sqrt{(1+x^2)^2}}{\sqrt{1+2x^2+x^4-1+2x^2-x^4}} \times \frac{-4x}{(1+x^2)^2}$$

$\sqrt{a^2} = a$

$$= \frac{-(\cancel{1+x^2})}{\sqrt{4x^2}} \times \frac{-4x}{(1+x^2)^{\cancel{2}\,1}} = \frac{-4x}{2x(1+x^2)}$$

$$= \frac{2}{1+x^2}$$

$=$ [redacted] $=$ [redacted]

$\boxed{3}$ $\quad y = \sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$ $\quad \to f$, $\to g$, $\to h$

$[f(g(h))]' $

$f'(g(h))\,g'(h)\,h'$

$\frac{d}{dx}x^n = nx^{n-1}$

$$= \left[\sin\left\{2\left[\tan^{-1}\left(\left(\frac{1-x}{1+x}\right)^{1/2}\right)\right]\right\}\right]$$

$$\frac{dy}{dx} = \underbrace{\cos\left\{2\tan^{-1}\left(\frac{1-x}{1+x}\right)^{1/2}\right\}}\ \cancel{2}\ \underbrace{\frac{1}{1+\left[\left(\frac{1-x}{1+x}\right)^{1/2}\right]^2}}\cdot\underbrace{\frac{1}{\cancel{2}}\left(\frac{1-x}{1+x}\right)^{1/2-1}}$$

$$\cdot\underbrace{\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}}$$

$\frac{d}{dx}\left(\frac{u}{v}\right)$

$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$$= \cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}\frac{1}{1+\frac{1-x}{1+x}}\left(\frac{1-x}{1+x}\right)^{-1/2}\frac{-1-x-1+x}{(1+x)^2}$$

$$= \cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}\cdot\frac{1}{\frac{1+x+1-x}{1+x}}\cdot\sqrt{\frac{1+x}{1-x}}\cdot\frac{-2}{(1+x)^2}$$

$$= \cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}\cdot\frac{\cancel{1+x}}{\cancel{2}}\cdot\frac{\cancel{\sqrt{1+x}}}{\sqrt{1-x}}\cdot\frac{(-\cancel{2})}{(1+x)^{\cancel{2}\,1}}$$

$$\sqrt{1+x}$$

$$= -\cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}\frac{1}{\sqrt{1-x^2}}.$$

$\boxed{14}$ $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Let $u = (\sin x)^{\cos x}$ — (i) $\quad v = (\cos x)^{\sin x}$ — (ii)

Then $y = u + v$ & $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$u = (\sin x)^{\cos x}$ — (i)

$\ln u = \ln(\sin x)^{\cos x}$ → introduce ln on both sides

$(uv)' = u'v + v'u$

$= \cos x \ln(\sin x)$

$$\frac{1}{u}\cdot\frac{du}{dx} = \cos x\left[\frac{1}{\sin x}(\cos x)\right] + (-\sin x)\ln(\sin x)$$

$$\frac{du}{dx} = u\left[\cos x \cot x - \sin x \ln(\sin x)\right]$$

$$\frac{du}{dx} = (\sin x)^{\cos x}\left[\cos x \cot x - \sin x \ln(\sin x)\right] \text{ — (a)}$$

$$[uv]' = u'v + v'u$$

$$V = (\cos x)^{\sin x} \quad \text{— (ii)}$$

$$\ln V = \ln(\cos x)^{\sin x} = \sin x \ln(\cos x)$$

$$\frac{1}{V} \cdot \frac{dv}{dx} = \cos x(\ln \cos x) + \sin x \frac{1}{\cos x} \cdot (-\sin x)$$

$$\frac{dv}{dx} = V\left[\cos x \ln(\cos x) + \tan x(-\sin x)\right]$$

$$\frac{dv}{dx} = (\cos x)^{\sin x}\left(\cos x \ln(\cos x) - \sin x \tan x\right) \quad \text{— (b)}$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} = (a) + (b)$$

$\boxed{5}$ $x = a\cos^3\theta$; $y = a\sin^3\theta$; Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \qquad x = f(\theta) \;,\; y = f(\theta)$$

$$x = a\cos^3\theta = a(\cos\theta)^3 \qquad y = a\sin^3\theta = a(\sin\theta)^3$$

$$\boxed{\frac{dx}{d\theta}} = 3a\cos^2\theta(-\sin\theta) \qquad \boxed{\frac{dy}{d\theta}} = 3a\sin^2\theta \cdot \cos\theta$$

$$= -3a\sin\theta\cos^2\theta$$

$$\frac{dy}{dx} = \boxed{\frac{dy}{d\theta}} \cdot \boxed{\frac{d\theta}{dx}} = 3a\sin^2\theta \cdot \cos\theta\left(\frac{1}{-3a\sin\theta\cos^2\theta}\right)$$

$$\therefore \frac{d\theta}{dx} = \frac{1}{-3a\sin\theta\cos^2\theta} \qquad = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

6 $x = \sin^2\theta$, $y = \tan\theta$, Find $\frac{dy}{dx}$

$= (\sin\theta)^2$

$\boxed{\frac{dx}{d\theta}} = 2\sin\theta \cdot \cos\theta$

$\boxed{\frac{dy}{d\theta}} = \sec^2\theta$

$x = f(\theta)$
$y = f(\theta)$

$$\frac{dy}{dx} = \boxed{\frac{dy}{d\theta}} \cdot \boxed{\frac{d\theta}{dx}} = \sec^2\theta \cdot \frac{1}{2\sin\theta \cdot \cos\theta}$$

$$\frac{d\theta}{dx} = \frac{1}{2\sin\theta\cos\theta}$$

$$= \frac{1}{2}\sec^2\theta \cdot \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$$

$$= \frac{1}{2}\sec^2\theta \cdot \operatorname{cosec}\theta \cdot \sec\theta$$

$$= \frac{1}{2}\sec^3\theta \operatorname{cosec}\theta$$

7 $x = a\sec^2\theta$, $y = a\tan^2\theta$; find $\frac{dy}{dx}$

$\boxed{\frac{dx}{d\theta}} = 2a\sec\theta \cdot \sec\theta\tan\theta$

$= \boxed{2a\sec^2\theta\tan\theta}$

$\boxed{\frac{dy}{d\theta}} = 2a\tan\theta \cdot \sec^2\theta$

$= \boxed{2a\sec^2\theta\tan\theta}$

$$\frac{dy}{dx} = \boxed{\frac{dy}{d\theta}} \boxed{\frac{d\theta}{dx}} = \boxed{2a\sec^2\theta\tan\theta} \cdot \boxed{\frac{1}{2a\sec^2\theta\tan\theta}}$$

$$= 1$$

Higher Derivatives

Alternative forms of higher derivatives are shown:

First derivative: $\frac{dy}{dx}$, y', y_1, Dy, $f'(x)$

Second derivative: $\frac{d^2y}{dx^2}$, y'', y_2, D^2y, $f''(x)$

n^{th} derivative: $\frac{d^ny}{dx^n}$, $y^{(n)}$, y_n, D^ny, $f^{(n)}(x)$

Let $y = x^3 + 3\sin x \longrightarrow f(x)$

$$\frac{dy}{dx} = 3x^2 + 3\cos x \longrightarrow 1^{st} \text{ derivative}$$

$$\frac{d^2y}{dx^2} = 6x - 3\sin x \longrightarrow 2^{nd} \text{ derivative}$$

$$\frac{d^3y}{dx^3} = 6 - 3\cos x \longrightarrow 3^{rd} \text{ derivative}$$

Let $y = e^{ax}$

$y_1 = ae^{ax}$

$y_2 = a^2 e^{ax}$

$y_3 = a^3 e^{ax}$

$\vdots$

$y_n = a^n e^{ax} \longrightarrow n^{th}$ derivative

Techniques of Differentiation

✓ Implicit Differention

Implicit functions:

Any function $y = f(x)$ can be presented as $f(x, y) = 0$. $f(x, y) = 0$ is known as Implicit function.

The technique we apply to evaluate $\frac{dy}{dx}$ from $f(x, y) = 0$ is known as Implicit Differentiation.

Examples on Implicit Differentiation:

[1] $xy = 1$ find $\frac{dy}{dx}$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$1 \cdot y + x \cdot \frac{dy}{dx} = 0$$

$$x \cdot \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x^2}$$

[2] Find $\frac{dy}{dx}$

$$5y^2 + \sin y = x^2$$

$$\frac{d}{dx}\left[5y^2 + \sin y\right] = \frac{d}{dx}\left[x^2\right]$$

$$5\frac{d}{dx}y^2 + \frac{d}{dx}\sin y = \frac{d}{dx}\left[x^2\right]$$

$$5\left[2y\frac{dy}{dx}\right] + \cos y \cdot \frac{dy}{dx} = 2x\frac{dx}{dx}$$

$$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}(10y + \cos y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}.$$

[3] $3x^4 - \underline{x^2 y} + 2y^3 = 0$

$$12x^3 - \underline{2xy - x^2\frac{dy}{dx}} + 6y^2 \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-x^2 + 6y^2\right) = 2xy - 12x^3$$

$$= 2x\left(y - 6x^2\right)$$

$$\frac{dy}{dx} = \frac{2x\left(y - 6x^2\right)}{6y^2 - x^2}$$

$\boxed{4}$ $x^3+y^3+4x^2y-25=0$

$$3x^2+3y^2\frac{dy}{dx}+4x^2\frac{dy}{dx}+8x\cdot y-0=0$$

$$\frac{dy}{dx}(3y^2+4x^2)=-8xy-3x^2$$

$$\frac{dy}{dx}=\frac{-(8xy+3x^2)}{3y^2+4x^2}$$

$\boxed{5}$ $x^y=y^x$

$$\ln x^y=\ln y^x$$

$$y\ln x=x\ln y$$

$$\frac{dy}{dx}\cdot\ln x+y\cdot\frac{1}{x}=1\cdot\ln y+x\cdot\frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}\ln x-\frac{x}{y}\frac{dy}{dx}=\ln y-\frac{y}{x}$$

$$\frac{dy}{dx}\left(\ln x-x/y\right)=\ln y-y/x$$

$$\frac{dy}{dx}=\frac{\ln y-y/x}{\ln x-x/y}$$

Examples on Higher order Derivatives:

$\boxed{1}$ $y = x^n$

$y' = n x^{n-1}$

$y'' = n(n-1)x^{n-2}$

$y''' = n(n-1)(n-2)x^{n-3}$

$y^{(4)} = n(n-1)(n-2)(n-3)x^{n-4}$

$\vdots$

$y^{(n)} = n(n-1)(n-2)(n-3)\cdots(n-(n-1))\,x^{n-n}$

$= n(n-1)(n-2)(n-3)\cdots(1)\,x^0$

$= n(n-1)(n-2)(n-3)\cdots(1)$

$= n!$

$\boxed{2}$ $y = (ax+b)^n$

$y' = n(ax+b)^{n-1}a = an(ax+b)^{n-1}$

$y'' = an(n-1)(ax+b)^{n-2}(a) = a^2 n(n-1)(ax+b)^{n-2}$

$y''' = a^3 n(n-1)(n-2)(ax+b)^{n-3}$

$y^{(4)} = a^4 n(n-1)(n-2)(n-3)(ax+b)^{n-4}$

$\vdots$

$y^{(n)} = a^n n(n-1)(n-2)(n-3)\cdots(n-(n-1))(ax+b)^{n-n}$

$= a^n n(n-1)(n-2)(n-3)\cdots(1)(ax+b)^0$

$= a^n n(n-1)(n-2)(n-3)\cdots 1$

$= a^n n!$

$\boxed{3}$ $y = \log_e (ax+b)$

$\Rightarrow y = \ln (ax+b)$

$y' = \dfrac{a}{ax+b} = a(ax+b)^{-1} = 0!\,a^1 (ax+b)^{-1}$

$y'' = a(-1)(ax+b)^{-2}(a) = -a^2 (ax+b)^{-2} = -1!\,a^2(ax+b)^{-2}$

$y''' = 2a^2(ax+b)^{-3}(a) = 2a^3 (ax+b)^{-3} = 2!\,a^3 (ax+b)^{-4} = (-1)^2 2!\,a^3 (ax+b)^{-4}$

$y^{(4)} = -6a^4(ax+b)^{-4} = -3!\,a^4(ax+b)^{-4} = (-1)^3 3!\,a^4(ax+b)^{-4}$

$y^{(5)} = 24a^5 (ax+b)^{-5} = 4!\,a^5 (ax+b)^{-5} = (-1)^4\, 4!\,a^5(ax+b)^{-5}$

NOTE

$(-1)^{\text{odd no.}} = -1$

$(-1)^{\text{even no.}} = +1$

$n = 1, 2, 3, 4, 5, \dots$ natural no.

$(-1)^{n+1} = 1, -1, 1, -1, 1, -1, 1, -1, 1, \dots$

$(-1)^{n-1} = 1, -1, 1, -1, 1, -1, 1, -1, 1, \dots$

$y^{(n)} = (-1)^{n-1} (n-1)!\, a^n (ax+b)^{-n}$

$\boxed{4}$ $y = \dfrac{1}{x+a} = (x+a)^{-1}$

$y' = -(x+a)^{-2} = (-1)^1\, 1\cdot(x+a)^{-2} = (-1)^1\, 1!\,(x+a)^{-2}$

$y'' = +2(x+a)^{-3} = (-1)^2\, 1\cdot 2\cdot (x+a)^{-3} = (-1)^2\, 2!\,(x+a)^{-3}$

$y''' = -6(x+a)^{-3} = (-1)^3\, 1\cdot 2\cdot 3\cdot (x+a)^{-4} = (-1)^3\, 3!\,(x+a)^{-4}$

$\vdots$

$y^{(n)} = (-1)^n\, 1\cdot 2\cdot 3 \cdots n\, (x+a)^{-(n+1)} = (-1)^n\, n!\,(x+a)^{-(n+1)}$

$\boxed{5}$ $y = \sin(ax+b)$

$\boxed{\cos x = \sin\left(x+\frac{\pi}{2}\right)}$

$y' = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$

$y'' = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$

$y''' = a^3\cos\left(ax+b+\frac{\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$

$y^{(n)} = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$

$\boxed{6}$ $y = \cos(ax+b)$

$\boxed{-\sin x = \cos\left(x+\frac{\pi}{2}\right)}$

$y' = -a\sin(ax+b) = a\cos\left(ax+b+\frac{\pi}{2}\right)$

$y'' = -a^2\sin\left(ax+b+\frac{\pi}{2}\right) = a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$

$y''' = -a^3\sin\left(ax+b+\frac{2\pi}{2}\right) = a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$

$\vdots$

$y^{(n)} = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$

$\boxed{7}$ $y = \sin^3 x$

$\Rightarrow y = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$\vdots$

$y^{(n)} = \frac{3}{4}\underbrace{\sin\left(x+\frac{n\pi}{2}\right)} - \frac{1}{4}\underbrace{(3^n)\sin\left(3x+\frac{n\pi}{2}\right)}$

Refer to ex 5

$\sin^3 x = \sin^2 x \sin x$

$= \frac{1}{2}(1-\cos 2x)\sin x$

$= \frac{1}{2}(\sin x - \cos \underbrace{2x}_{a} \underbrace{\sin x}_{b})$

$\cos a \sin b = \frac{1}{2}[\sin(a+b) - \sin(a-b)]$

$= \frac{1}{2}\left[\sin x - \left\{\frac{1}{2}(\sin 3x - \sin x)\right\}\right]$

$= \frac{1}{2}\left[\sin x - \frac{1}{2}\sin 3x + \frac{1}{2}\sin x\right]$

$= \frac{1}{2}\left[\frac{3}{2}\sin x - \frac{1}{2}\sin 3x\right]$

$= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$