Week 7 — MAT110 Differential Calculus & Coordinate Geometry

→ Maxima Minima of function of Several Variables

→ Taylor Expansion of a two variable function

Maxima Minima of function of Several Variables

Critical Point:

A point (x_0, y_0) in the domain of a function $f(x,y)$ is called a critical point of the function:

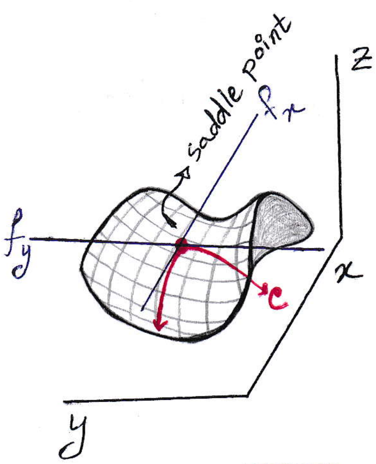
if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

OR

if $f_x(x_0, y_0) = \infty$ and $f_y(x_0, y_0) = \infty$

Saddle Point:

A saddle point is a point in the domain of a function that is a stationary point but not a local extremum on $\underbrace{\text{both axes}}_{(x \,\&\, y \text{ axes})}$



$f' = 0$ → for any horizontal line

If $f(x,y) = z$, x & y are horizontal axes.

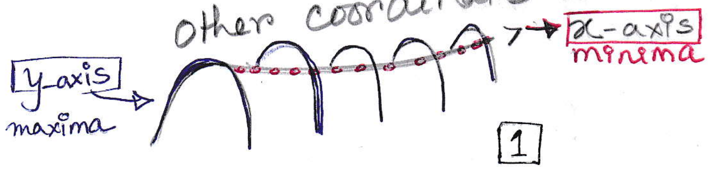
z is vertical axes.

Saddle pt is a maximum in one coordinate & at the same time minimum in the other coordinate or vice versa

$$\left\{\begin{array}{l} f_x(x_0, y_0) = \infty \text{ or} \\ f_y(x_0, y_0) = \infty \\ \text{or both } \infty \end{array}\right.$$

↓

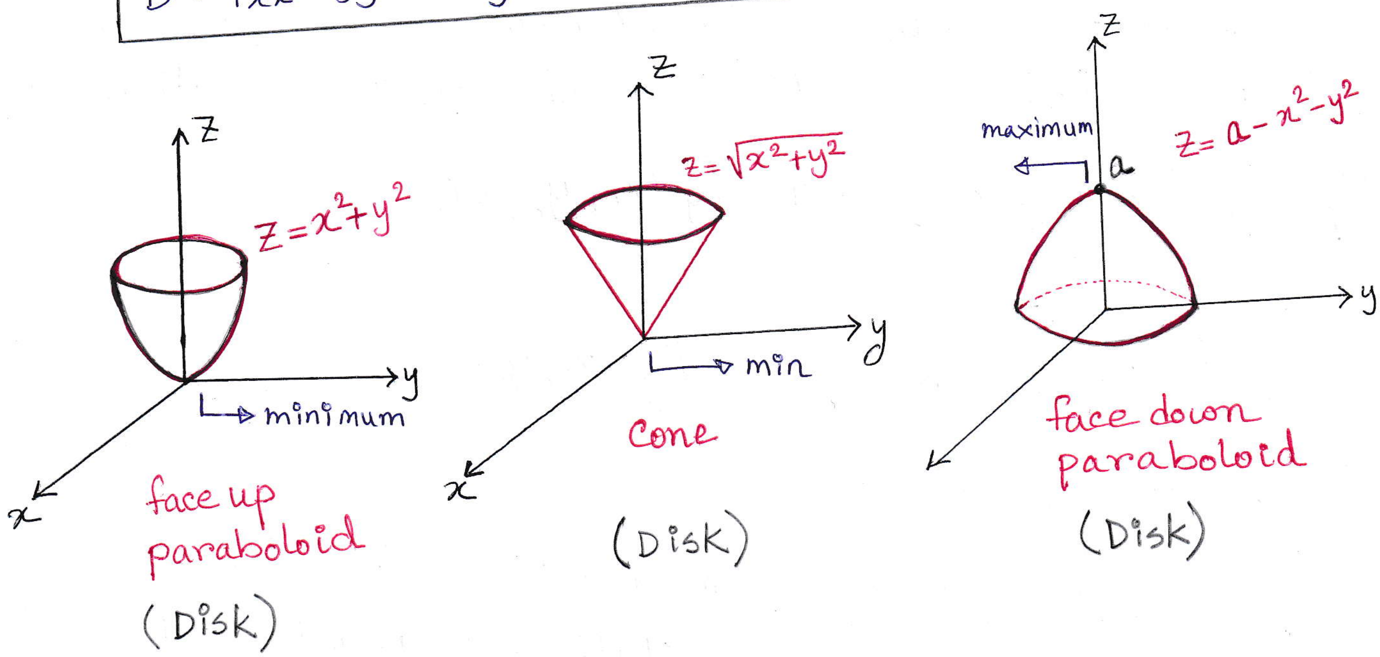
These conditions are not covered in saddle point.



The Second Partial Test

Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0) and

let $$\boxed{\begin{aligned} D(x_0, y_0) &= f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) \\ D &= f_{xx} f_{yy} - f_{xy}^2 \quad \text{at } (x_0, y_0) \end{aligned}}$$



a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ then f has a relative minimum at (x_0, y_0).

b) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then f has a relative maximum at (x_0, y_0).

c) If $D < 0$ then f has a saddle point at (x_0, y_0)

d) If $D = 0$ then no conclusion

Anton's Calculus 10th Ed.

Ch 13.8

Locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x,y) = y^2 + xy + 3y + 2x + 3$

$f_x(x,y) = y+2$

$f_x = 0 \Rightarrow y+2=0$

$\boxed{y=-2}$

$f_y(x,y) = 2y+x+3$

$f_y = 0 \Rightarrow 2y+x+3=0$

$\Rightarrow 2(-2)+x=-3$

$\Rightarrow x=-3+4$

$\Rightarrow \boxed{x=1}$

$(1,-2)$ is the only critical point

$f_x(x,y) = y+2$

$f_{xx}(x,y) = 0$

$f_{xy}(x,y) = 1$

$f_y(x,y) = 2y+x+3$

$f_{yy}(x,y) = 2$

$D = f_{xx}(1,-2)\, f_{yy}(1,-2) - f_{xy}^2(1,-2)$

$= (0)(2) - 1^2 = -1 < 0$

$\therefore D<0$ then f has a saddle pt at $(1,-2)$

10. $f(x,y) = x^2 + xy - 2y - 2x + 1$

$f_x(x,y) = 2x + y - 2$ $\qquad$ $f_y(x,y) = x - 2$

$f_x = 0 \Rightarrow 2x + y - 2 = 0$ $\qquad$ $f_y = 0 \Rightarrow x - 2 = 0$

$2x + y = 2$ $\qquad$ $\boxed{x = 2}$

$\therefore 2(2) + y = 2$

$\boxed{y = -2}$

$\therefore (2,-2)$ is the only critical point

$f_x(x,y) = 2x + y - 2$ $\qquad$ $f_y(x,y) = x - 2$

$f_{xx}(x,y) = 2$ $\qquad$ $f_{yy}(x,y) = 0$

$f_{xy}(x,y) = 1$

$D = f_{xx}(2,-2)\, f_{yy}(2,-2) - f_{xy}^2(2,-2)$

$= (2)(0) - 1^2 = -1 < 0$

$\therefore D < 0$ then f has a saddle point at $(2,-2)$.

11. $f(x,y) = x^2 + xy + y^2 - 3x$

$f_x(x,y) = 2x + y - 3$

$f_x = 0 \Rightarrow 2x + y - 3 = 0$

$\Rightarrow 2x + y = 3$ — (i)

$f_y(x,y) = x + 2y$

$f_y = 0 \Rightarrow x + 2y = 0$ — (ii)

(i) $- 2$(ii) $\Rightarrow -3y = 3$

$\boxed{y = -1}$

Substitute $y = -1$ into (i)

$2x + (-1) = 3$

$2x = 3 + 1 = 4$

$\boxed{x = 2}$

$\therefore (2,-1)$ is the only critical point

$f_x = 2x + y - 3$

$f_{xx} = 2$

$f_{xy} = 1$

$f_y = x + 2y$

$f_{yy} = 2$

$D = f_{xx}(2,-1)\, f_{yy}(2,-1) - f_{xy}^2(2,-1)$

$= (2)(2) - 1^2 = 3 > 0$

$\therefore D > 0$ and $f_{xx} > 0$,

hence f has a relative minimum at $(2,-1)$.

12. $f(x,y) = xy - x^3 - y^2$

$f_x = y - 3x^2$

$f_x = 0 \Rightarrow y - 3x^2 = 0$

$\therefore y - 3(2y)^2 = 0$

$y - 12y^2 = 0$

$y(1 - 12y) = 0$

$y = 0 \qquad y = \frac{1}{12}$

$f_y = x - 2y$

$f_y = 0 \Rightarrow x - 2y = 0$

$x = 2y$

$y = 0 \Rightarrow x = 2(0) = 0$

$y = \frac{1}{12} \Rightarrow x = 2\left(\frac{1}{12}\right) = \frac{1}{6}$

$\therefore (0,0)$ and $\left(\frac{1}{6}, \frac{1}{12}\right)$ are critical points

$f_{xx} = -6x$

$f_{xy} = 1$

$f_{yy} = -2$

At $(0,0)$ we have:

$D = f_{xx}(0,0)\, f_{yy}(0,0) - f_{xy}^2(0,0)$

$= [-6(0)][-2] - [1]^2$

$= -1 < 0$; $\therefore D < 0$ hence f has a saddle pt at $(0,0)$.

At $\left(\frac{1}{6}, \frac{1}{12}\right)$

$D = f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) f_{yy}\left(\frac{1}{6}, \frac{1}{12}\right) - f_{xy}^2\left(\frac{1}{6}, \frac{1}{12}\right)$

$= \left[-6\left(\frac{1}{6}\right)\right][-2] - [1]^2$

$= 2 - 1 = 1 > 0$

$f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) = -6\left(\frac{1}{6}\right)$

$= -1$

$\therefore f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) < 0$

$D > 0$

$\therefore D > 0$ & $f_{xx}\left(\frac{1}{6}, \frac{1}{12}\right) < 0$ hence f has relative maximum at $\left(\frac{1}{6}, \frac{1}{12}\right)$.

13. $f(x, y) = x^2 + y^2 + \frac{2}{xy}$

$f_x = 2x + \frac{2}{y}(-1)x^{-2} = 2x - \frac{2}{x^2 y}$

$f_y = 2y + \frac{2}{x}(-1)y^{-2} = 2y - \frac{2}{xy^2}$

$f_x = 0 \Rightarrow 2x - \frac{2}{x^2 y} = 0$

$\frac{2x^3 y - 2}{x^2 y} = 0$

$2x^3 y - 2 = 0$; $\boxed{x \neq 0 \quad y \neq 0}$

$x^3 y - 1 = 0$

$x^3 y = 1$

$\boxed{y = \frac{1}{x^3}}$

$x = -1 \Rightarrow y = \frac{1}{(-1)^3} = -1$

$x = +1 \Rightarrow y = \frac{1}{(+1)^3} = 1$

$f_y = 0 \Rightarrow 2y - \frac{2}{xy^2} = 0$

$\frac{2xy^3 - 2}{xy^2} = 0$

$2xy^3 - 2 = 0$; $\boxed{x \neq 0 \quad y \neq 0}$

$xy^3 - 1 = 0$

$xy^3 = 1$

$x\left(\frac{1}{x^3}\right)^3 = 1 \quad \because y = \frac{1}{x^3}$

$x \cdot \frac{1}{x^9} = 1$

$\frac{1}{x^8} = 1$

$x^8 = 1$

$x = \pm 1$

$f_{xx} = 2 - \frac{2}{y}(-2x^{-3}) = 2 + \frac{4}{x^3 y}$

$f_{yy} = 2 - \frac{2}{x}(-2y^{-3}) = 2 + \frac{4}{xy^3}$

$f_{xy} = \frac{-2}{x^2}(-y^{-2}) = \frac{2}{x^2 y^2}$

$\therefore$ Critical points are $(1, 1), (-1, -1), (0, 0)$

Critical points (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx} f_{yy} - f_{xy}^2$	Decision
$(0,0)$	∞	∞	∞	$D = \infty$	
$(1, 1)$	$6 > 0$	6	2	$D = 32 > 0$	minimum
$(-1, -1)$	$6 > 0$	6	2	$D = 32 > 0$	minimum

14. $f(x,y) = xe^y$

$f_x = e^y$

$f_x = 0 \Rightarrow e^y = 0$ — (i)

$f_y = xe^y$

$f_y = 0 \Rightarrow xe^y = 0$ — (ii)

compare (i) & (ii) : $e^y = xe^y$

$e^y - xe^y = 0$

$e^y(1-x) = 0$

$e^y = 0$

$\ln e^y = \ln 0$

$y \ln e = \ln 0$

$y = \infty$

$1 - x = 0$

$\boxed{x = 1}$

no critical point.

15. $f(x,y) = x^2 + y - e^y$

$f_x = 2x$

$f_x = 0 \Rightarrow 2x = 0$

$\boxed{x = 0}$

$f_{xx} = 2$

$f_{xy} = 0$

$f_y = 1 - e^y$

$f_y = 0 \Rightarrow 1 - e^y = 0$

$e^y = 1$

$\ln e^y = \ln 1$

$y \ln e = \ln 1$

$\boxed{y = 0}$

$\therefore$ critical point at $(0, 0)$

$\because f_y = 1 - e^y$

$f_{yy} = -e^y$

$D = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}^2(0,0)$

$= (2)(-1) - 0^2 = -2 < 0$

$\because D < 0$, f has a saddle pt at $(0,0)$.

17. $f(x,y) = e^x \sin y$

$f_x = e^x \sin y$

$f_x = 0 \Rightarrow e^x \sin y = 0$ — (i)

$f_y = e^x \cos y$

$f_y = 0 \Rightarrow e^x \cos y = 0$ — (ii)

Compare (i) & (ii)

$e^x \sin y = e^x \cos y$

$e^x \sin y - e^x \cos y = 0$

$e^x(\sin y - \cos y) = 0$

$e^x = 0$

$\ln e^x = \ln 0$

$x = \infty$

$\sin y - \cos y = 0$

$\sin y = \cos y$

$\frac{\sin y}{\cos y} = 1$

$\tan y = \tan \frac{\pi}{4}$

$y = \frac{\pi}{4}$

no critical point.

16. $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

$f_x = y - 2x^{-2}$

$f_x = 0 \Rightarrow y - \frac{2}{x^2} = 0$

$\frac{x^2 y - 2}{x^2} = 0$

$x^2 y - 2 = 0$ $\boxed{x \neq 0}$

$x^2 y = 2$

$\boxed{y = \frac{2}{x^2}}$

$y = \frac{2}{1^2}$ $\because x = 1$

$\boxed{y = 2}$

$f_y = x - 4y^{-2}$

$f_y = 0 \Rightarrow x - 4y^{-2} = 0$

$\Rightarrow x - \frac{4}{y^2} = 0$

$\frac{xy^2 - 4}{y^2} = 0$

$xy^2 - 4 = 0$; $\boxed{y \neq 0}$

$x\left(\frac{2}{x^2}\right)^2 - 4 = 0$ $\because y = \frac{2}{x^2}$

$x\left(\frac{4}{x^4}\right) - 4 = 0$

$\frac{4}{x^3} = 4 \Rightarrow \frac{1}{x^3} = 1$

$\Rightarrow x^3 = 1$

$\Rightarrow \boxed{x = 1}$

$\therefore$ $(0,0)$ & $(1,2)$ are critical points and $(1,2)$ is the stationary point

$$f_x = y - 2x^{-2} \qquad f_y = x - 4y^{-2}$$

$$f_{xx} = 4x^{-3} \qquad f_{yy} = 8y^{-3} = \frac{8}{y^3}$$

$$f_{xy} = 1$$

Considering the stationary pt $(1,2)$

$$D = f_{xx}(1,2)\, f_{yy}(1,2) - f^2_{xy}(1,2)$$

$$= \left[4(1)^{-3}\right]\left[\frac{8}{(2^3)}\right] - [1]^2$$

$$= (4)(1) - 1 = 3 > 0$$

$$f_{xx}(1,2) = \frac{4}{x^3} = \frac{4}{1^3} = 4 > 0$$

$\because D > 0$ & $f_{xx}(1,2) > 0$ $\therefore$ f has relative minimum at $(1,2)$.

18. $f(x,y) = y \sin x$

$f_x = y \cos x$

$f_x = 0 \Rightarrow y \cos x = 0$

$\Rightarrow y \cos 0^\circ = 0 \quad \because x = 0$

$\Rightarrow y = 0$

$f_y = \sin x$

$f_y = 0 \Rightarrow \sin x = 0$

$\Rightarrow x = \sin^{-1} 0$

$\Rightarrow \boxed{x = 0}$

$\therefore$ The critical pt is $(0,0)$

$f_{xx} = -y \sin x \qquad f_y = \sin x$

$f_{xy} = \cos x \qquad f_{yy} = 0$

$$D = f_{xx}(0,0)\, f_{yy}(0,0) - f^2_{xy}(0,0)$$

$$= [-(0)\sin 0][0] - [\cos(0)]^2 = (0)(1) - 1^2 = -1 < 0$$

$\because D < 0$, then f has a saddle point at $(0,0)$

19. $f(x,y) = e^{-(x^2+y^2+2x)}$

$f_x = e^{-x^2-y^2-2x}(-2x-2)$

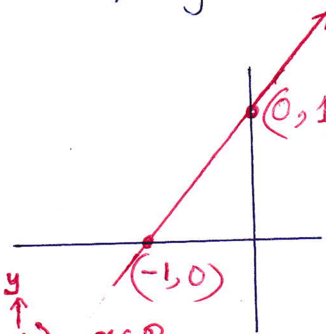
$f_x = 0 \Rightarrow e^{-(x^2+y^2+2x)}(-2x-2) = 0$

$\underbrace{(2x+2)e^{-(x^2+y^2+2x)} = 0}_{\text{(i)}}$

$f_y = e^{-x^2-y^2-2x}(-2y)$

$f_y = 0 \Rightarrow -2y\,e^{-x^2-y^2-2x} = 0$

$\Rightarrow \underbrace{2y\,e^{-(x^2+y^2+2x)} = 0}_{\text{(ii)}}$



Compare (i) & (ii) $\quad 2x+2 = 2y$

$x + 1 = y$

The solution can be written in the form $(x, x+1)$, $x \in \mathbb{R}$ $\quad \overset{y}{\uparrow}\ x+1=y$

Consider two possible solutions $(-1,0), (0,1)$.

at $(-1,0)$ (i) $e^{-(1-2)}(-2+2) = 0$ ✓

(ii) $e^{-(1-2)}(0) = 0$ ✓

$(-1,0)$ satisfies eqn (i) & (ii)

at $(0,1)$ (i) $e^{-(1)}\,2 \neq 0$

(ii) $e^{-(1)}\,2 \neq 0$

$(0,1)$ does not satisfy either (i) or (ii)

$\therefore (-1,0)$ is the only critical point.

$f_{xx} = -2e^{-(x^2+y^2+2x)} + (-2x-2)e^{-(x^2+y^2+2x)}(-2x-2)$

$= -2e^{-(x^2+y^2+2x)} + (2x+2)^2 e^{-(x^2+y^2+2x)}$

$f_{xx}(-1,0) = -2e^{-(-1)} + 0 = -2e$

$f_{yy} = -2e^{-(x^2+y^2+2x)} + (-2y)e^{-(x^2+y^2+2x)}(-2y)$

$= -2e^{-(x^2+y^2+2x)} + 4y^2 e^{-(x^2+y^2+2x)}$

$f_{yy}(-1,0) = -2e^{-(1-2)} = -2e$

$$f_{xy} = (-2x-2)e^{-x^2-y^2-2x}[-2y]$$

$$f_{xy}(-1,0) = 0$$

$$D = f_{xx}(-1,0)\, f_{yy}(-1,0) - f_{xy}^2(-1,0)$$

$$= (-2e)(-2e) - 0^2$$

$$= 4e^2 > 0$$

$\because$ $D > 0$ and $f_{xx}(-1,0) = -2e < 0$

$\therefore$ f has relative maximum at $(-1,0)$.

(20) $f(x,y) = xy + \frac{a^3}{x} + \frac{b^3}{y}$ $\quad (a \neq 0, b \neq 0)$

$f_x = y - \frac{a^3}{x^2}$

$f_x = 0 \Rightarrow y - \frac{a^3}{x^2} = 0$

$\frac{x^2y - a^3}{x^2} = 0$

$x^2y - a^3 = 0 ; \boxed{x \neq 0}$

$\boxed{y = \frac{a^3}{x^2}}$

$f_y = x - \frac{b^3}{y^2}$

$f_y = 0 \Rightarrow x - \frac{b^3}{y^2} = 0$

$\frac{xy^2 - b^3}{y^2} = 0$

$xy^2 - b^3 = 0 ; \boxed{y \neq 0}$

$x\left(\frac{a^3}{x^2}\right)^2 - b^3 = 0 \quad \because y = \frac{a^3}{x^2}$

$x\left(\frac{a^6}{x^4}\right) = b^3$

$\frac{a^6}{x^3} = b^3 \Rightarrow x^3 = \frac{a^6}{b^3}$

$\boxed{x = \frac{a^2}{b}}$

$y = \frac{a^3}{\left(\frac{a^2}{b}\right)^2} \quad \because x = \frac{a^2}{b}$

$= a^3 \cdot \frac{b^2}{a^4} \Rightarrow \boxed{y = \frac{b^2}{a}}$

$\therefore$ Critical pts are $(0,0)$, $\left(\frac{a^2}{b}, \frac{b^2}{a}\right)$

$f_x = y - \dfrac{a^3}{x^2}$

$f_{xx} = -a^3(-2)x^{-3} = \dfrac{2a^3}{x^3}$

$f_{xy} = 1$

$f_y = x - \dfrac{b^3}{y^2}$

$f_{yy} = -b^3(-2)y^{-3}$

$= \dfrac{2b^3}{y^3}$

Considering Stationary No.	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$
$\left(\frac{a^2}{b}, \frac{b^2}{a}\right)$	$\dfrac{2a^3}{\frac{a^6}{b^3}} = 2a^3 \cdot \dfrac{b^3}{a^6}$ $= \dfrac{2b^3}{a^3}$	$\dfrac{2b^3}{\left(\frac{b^2}{a}\right)^3} = 2b^3 \cdot \dfrac{a^3}{b^6}$ $= \dfrac{2a^3}{b^3}$	1	$\dfrac{2b^3}{a^3} \cdot \dfrac{2a^3}{b^3} - 1^2$ $= 4 - 1$ $= 3 > 0$

$D > 0$

f has relative maximum at $\left(\frac{a^2}{b}, \frac{b^2}{a}\right)$ if either b is $-ve$ or a is $-ve$.

f has relative minimum at $\left(\frac{a^2}{b}, \frac{b^2}{a}\right)$ if a, b both $+ve$ or if a, b both $-ve$

Week 7 MAT 110 Differential Calculus & Coordinate Geometry

(Continued)

Taylor Expansion of a two variable function

Taylor expansion of single variable function:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

Taylor expansion around $x=0$

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n,m}\, x^n y^m$$

$$= C_{0,0} + C_{1,0}x + C_{0,1}y + C_{2,0}x^2 + C_{1,1}xy + C_{0,2}y^2 + \cdots$$

The coefficients would be

$$C_{0,0} = f(0,0)$$

$$C_{1,0} = \frac{\partial f}{\partial x}$$

$$C_{0,1} = \frac{\partial f}{\partial y}$$

$$C_{1,1} = \frac{\partial f}{\partial x \partial y}$$

$$C_{2,0} = \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}$$

$$C_{n,0} = \frac{1}{n!}\frac{\partial^{(n)} f}{\partial x^{(n)}}$$

$$C_{0,n} = \frac{1}{n!}\frac{\partial^{(n)} f}{\partial y^{(n)}}$$

Taylor expansion around $(0,0)$:
(Also known as Maclaurine's expansion)

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\overset{(x-0)}{x}\frac{\partial}{\partial x} + \overset{(y-0)}{y}\frac{\partial}{\partial y}\right)^n f(x,y)$$

Taylor expansion around the point (a,b):

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left((x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right)^n f(x,y)$$

<u>Example</u> Compute the 1^{st} and 2^{nd} degree Taylor Polynomials, $L(x,y)$ (1st deg) and $Q(x,y)$ (2nd deg) for $f(x,y) = xe^y + 1$ for (x,y) near the point $(1,0)$

<u>Note</u>

$\frac{\partial}{\partial x} = f_x$; $\frac{\partial}{\partial y} = f_y$; $\frac{\partial}{\partial x \partial y} = f_{xy}$; $\frac{\partial^2}{\partial x^2} = f_{xx}$; $\frac{\partial^2}{\partial y^2} = f_{yy}$

$f(x,y) = xe^y + 1$

$f_x(x,y) = e^y$ → $f_{xx}(x,y) = 0$

$f_y(x,y) = xe^y$ → $f_{yy}(x,y) = xe^y$

$f_{xy}(x,y) = e^y$

Similarly

$f_{yx}(x,y) = e^y$

$f_x(1,0) = e^0 = 1$

$f_y(1,0) = 1 \cdot e^0 = 1$

$f_{xx}(1,0) = 0$

$f_{yy}(1,0) = 1 \cdot e^0 = 1$

$f_{xy}(1,0) = e^0 = 1$

$f(1,0) = 1 \cdot e^0 + 1 = 2$

1^{st} degree :

$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$= 2 + 1(x-1) + 1(y)$

$= 2 + x - 1 + y$

$L(x,y) = 1 + x + y \longrightarrow 1^{st}$ order Taylor Polynomial Approximation

2^{nd} degree :

$Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2!}(x-1)^2 + f_{xy}(1,0)(x-1)(y-0)$ $\quad n=2 \quad n=1$

$+ \frac{f_{yy}(1,0)}{2!}(y-0)^2$

$= (1+x+y) + \frac{0}{2!}(x-1)^2 + 1(x-1)(y-0) + \frac{1}{2!}(y-0)^2$

$= 1 + x + y + 0 + xy - y + \frac{1}{2}y^2$

$Q(x,y) = 1 + x + xy + \frac{1}{2}y^2 \longrightarrow 2^{nd}$ order Taylor Polynomial Approximation

Let $w = f(x,y)$ is a function of two variables. The Taylor series expansion for this function w.r.t a point (a,b) is defined as:

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n,m}(x-a)^n(y-b)^n .$$

where,

$$C_{n,m} = \frac{1}{n!m!}\frac{\partial^{n+m} f}{\partial x^n \partial y^m}|_{(a,b)}.$$

If we expand it, we can write it as:

$$f(x,y) = C_{0,0} + C_{1,0}(x-a) + C_{0,1}(y-b)$$
$$+C_{2,0}(x-a)^2 + C_{1,1}(x-a)(y-b) + C_{0,2}(y-b)^2 .$$
$$+C_{3,0}(x-a)^3 + C_{2,1}(x-a)^2(y-b) + C_{1,2}(y-b)^2(x-a) + C_{0,3}(y-b)^3 + ...$$