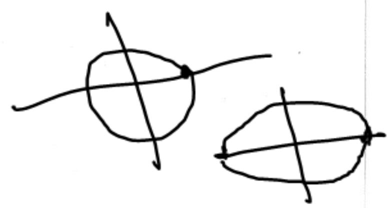
$\boxed{\text{MAT 110}}$ Differential Calculus & Coordinate Geometry

Week 3
→ Parametric differentiation
→ Differentiation with the help of introducing 'ln'
→ Differentiation of inverse function
→ Leibnitz's theorem
→ Gradient & Tangent line

Parametric differentiation:

Ex $x = x(t)$, $y = y(t)$ OR $x = x(\theta)$, $y = y(\theta)$

① consider
$x = 2t$, $y = t^2$ } → pair of these together called parametric eqn
→ t = parameter



② $x = \sin\theta + 2$, $y = \cos\theta - 3$ } → parametric eqn
θ = parameter

If a function is expressed in terms of some parameter, in other words if the equation of the function is given parametrically: considering eqn of an ellipse

$x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$

The chain Rule can be used to find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \rightarrow \frac{dy/dt}{dx/dt}$$ → while $\frac{dx}{dt} \neq 0$

Examples can be found in
Week 2 PDF pg 14, 15
Ex 5, 6, 7

Differentiation with the help of Introducing 'ln'

- ~~The~~ The function can be differentiated by taking 'ln' if:
  The given function is:
  - → product of some different functions
  - → quotient of some different functions
- power of a function is another function
  ↳ see examples in **Week 2** PDF

Ex 4 pg 13
Ex 5 pg 18

product of some different functions:

$y = pqr \qquad p = p(x), \; q = q(x), \; r(x)$

$\ln y = \ln(pqr)$

$= \ln p + \ln q + \ln r$

differentiate w.r.t. x on both sides

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{p}\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} + \frac{1}{r}\frac{dr}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\frac{dq}{dx} + \frac{1}{r}\cdot\frac{dr}{dx}\right]$$

$$\frac{dy}{dx} = pqr\left[\frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} + \frac{1}{r}\frac{dr}{dx}\right]$$

quotient of some different functions:

$$y = \frac{pq}{r}, \quad p = p(x),\ q = q(x),\ r = r(x)$$

$$\ln y = \ln\left(\frac{pq}{r}\right)$$

$$\ln y = \ln p + \ln q - \ln r$$

differentiate w.r.t. x on both sides:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx}\right]$$

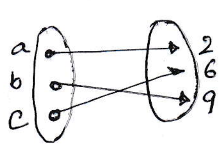
$$\frac{dy}{dx} = \frac{pq}{r}\left[\frac{1}{p}\cdot\frac{dp}{dx} + \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx}\right]$$

Differentiation of Inverse function:

Ex 2, 3 pg 12 week2

- Every function does not have its inverse
- In order to have an inverse, the function must be one-to-one.

A one-to-one function is a function of which the answers never repeat. For example the function $f(x) = x+1$ is a one-to-one function because it produces a different answer for every input.



In general: $\left[f^{-1}(x)\right]' = \dfrac{1}{f'\left[f^{-1}(x)\right]}$

Given an invertible function $f(x)$, the derivative of its inverse function $f^{-1}(x)$ evaluated at $x=a$ is:

$$\boxed{\left[f^{-1}\right]'(a) = \frac{1}{f'\left[f^{-1}(a)\right]}}$$

$$\begin{cases} x = \sin^{-1}\theta \\ \theta = \sin x \end{cases}$$

Let $\underbrace{y = f^{-1}(x)} \Longrightarrow x = f(y)$

$1 = f'(y)\dfrac{dy}{dx}$ differentiate w.r.t. x

$$\boxed{\frac{dy}{dx} = \frac{1}{f'(y)}}$$

$$\therefore \boxed{\left[f^{-1}(x)\right]' = \frac{1}{f'(y)}}$$

$\therefore \dfrac{d}{dx} f^{-1}(x) = \dfrac{dy}{dx} = \dfrac{1}{f'\left[f^{-1}(x)\right]}$ $\quad \because y = f^{-1}(x)$

At the pt $x=a$: $\left[f^{-1}\right]'(a) = \dfrac{1}{f'\left[f^{-1}(a)\right]}$.

Examples:

derive the inverse of

(1) $f(x) = \dfrac{e^{-3x}}{x^2+1}$ at $(-1, 0) \to (x, y)$

$$f'(x) = \frac{(x^2+1)(-3e^{-3x}) - e^{-3x}(2x)}{(x^2+1)^2}$$

$$= \frac{-3x^2e^{-3x} - 3e^{-3x} - 2xe^{-3x}}{x^4+2x^2+1}$$

$$= \frac{-(3x^2e^{-3x} + 2xe^{-3x} + 3e^{-3x})}{x^4+2x^2+1}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(y)} = \frac{1}{\dfrac{-(3y^2e^{-3y} + 2ye^{-3y} + 3e^{-3y})}{y^4+2y^2+1}}$$

$$= \frac{-(y^4+2y^2+1)}{3y^2e^{-3y} + 2ye^{-3y} + 3e^{-3y}}$$

$$\left.\frac{d}{dx} f^{-1}(-1)\right|_{\substack{(x,y)\\=(-1,0)}} = \frac{-(0^4+2\,0^2+1)}{3(0)e^{-3(0)} + 2(0)e^{-(0)} + 3e^{-3(0)}} = \frac{-1}{3}$$

$\dfrac{d}{dx} f^{-1}(-1) = \dfrac{1}{f'(y)}$

$(x, y) = (-1, 0)$

(2) Inverse function & chain rule:

Find the derivative of $f(x) = \log(\sin^{-1}(x^2-3x))$.

$$f'(x) = \frac{1}{\sin^{-1}(x^2-3x)} \frac{d}{dx} \sin^{-1}(x^2-3x)$$

$$= \frac{1}{\sin^{-1}(x^2-3x)} \cdot \frac{1}{\sqrt{1-(x^2-3x)^2}} \cdot \frac{d}{dx}(x^2-3x)$$

$$= \frac{1}{\sin^{-1}(x^2-3x)} \cdot \frac{1}{\sqrt{1-(x^2-3x)^2}} \cdot (2x-3)$$

$$= \frac{2x-3}{\sin^{-1}(x^2-3x)\sqrt{1-(x^2-3x)^2}}.$$

(3) Inverse Function & Implicit Differentiation

Find $\frac{dy}{dx}$ for $\cos^{-1}(xy) = x^2$

$$\frac{d}{dx}\left(\cos^{-1}(xy)\right) = \frac{d}{dx}(x^2)$$ [derive both sides w.r.t. x]

$$\frac{-1}{\sqrt{1-(xy)^2}} \frac{d}{dx}(xy) = 2x$$

$$\frac{-\left(1 \cdot y + x \cdot \frac{dy}{dx}\right)}{\sqrt{1-(xy)^2}} = 2x$$

$$\frac{-y - x\frac{dy}{dx}}{\sqrt{1-(xy)^2}} = 2x$$

$$-y - x\frac{dy}{dx} = 2x\sqrt{1-(xy)^2}$$

$$-x\frac{dy}{dx} = 2x\sqrt{1-(xy)^2} + y$$

$$\frac{dy}{dx} = -\frac{1}{x}\left[2x\sqrt{1-(xy)^2} + y\right]$$

$$= -2\sqrt{1-x^2y^2} - \frac{y}{x}$$

$\boxed{4}$ suppose $f(x) = x^5 + 2x^3 + 7x + 1$. Find $[f^{-1}]'(1)$. $\to f^{-1}(1)$

$f'(x) = 5x^4 + 6x^2 + 7 \longrightarrow$ strictly greater than '0' for all x.

$\therefore f$ is $\uparrow$ and hence one-to-one

$\therefore f'(y) = 5y^4 + 6y^2 + 7$

$f^{-1}(1) = ?$

$f^{-1}(1) = x$

$1 = f(x)$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(y)} = \frac{1}{5y^4 + 6y^2 + 7}$$

To find $f^{-1}(1)$ we make a table of values $x = -3, -2, -1, \dots, 3, 4 \dots$ into $f(x)$ to see what value of x gives 1.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	...	...	-9	1	11	...	...	...

$f(x) = 1$

then $x = ?$

$\underbrace{? = f^{-1}(1)}_{\text{Question}}$

$\therefore x = 0 \Rightarrow f(x) = 1$

$\therefore f(0) = 1 \Rightarrow f^{-1}(1) = 0$

$$\frac{d}{dx} f^{-1}(x) = [f^{-1}]'(1) = \frac{1}{f'[f^{-1}(1)]} = \frac{1}{f'(0)} \quad \therefore f^{-1}(1) = 0$$

Hence $\dfrac{1}{f'(0)} = \dfrac{1}{5(0^4) + 6(0^2) + 7} = \dfrac{1}{7}$

Leibnitz's Theorem

Find n^{th} differential coefficient of two functions.
(never differentiated before)

If y_0 and x_0 any two initial functions of f such that all their desired differential coefficients exists, then n^{th} differential coefficient of their product is:

$$(y_0 x_0)_n = \underbrace{y_{n+0}}_{\binom{n}{0}} x_0 + \underbrace{n}_{\binom{n}{1}} y_{n-1} x_1 + \underbrace{\frac{n(n-1)}{2!}}_{\binom{n}{2}} x_2 + \underbrace{\frac{n(n-1)(n-2)}{3!}}_{\binom{n}{3}} x_3 + \dots + \underbrace{y_0}_{\binom{n}{n}} x_n$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Definition

n^{th} differential coefficient of 2 functions $y = f(x)$ & $x = f(y)$. If y_0 & x_0 are any two initial functions of f such that

$$(y_0 x_0)_n = y_{n+0} x_0 + n y_{n-1} x_1 + \frac{n(n-1)}{2!} x_2 + \frac{n(n-1)(n-2)}{3!} x_3 + \dots + y_0 x_n$$

$$= y_n x + n y_{n-1} x_1 + \frac{n(n-1)}{2!} y_{n-2} x_2 + \dots + y x_n$$

$$\frac{d^n (yx)}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}}(y) \frac{d^k}{dx^k}(x)$$

Same

$$(yx)_n = \binom{n}{0} y_n x_0 + \binom{n}{1} y_{n-1} x_1 + \binom{n}{2} y_{n-2} x_2 + \dots \dots + \binom{n}{n} y_0 x_n$$

Examples

① Find the 4^{th} derivative of the function $y = e^x \sin x$.

Let $u = \sin x$, $v = e^x$

$$y^{(4)} \xrightarrow{} (uv)^{(4)}$$

$$y^{(4)} = \left(e^x \sin x\right)^{(4)}$$

$$= \binom{4}{0}(\sin x)^{(4)} e^x + \binom{4}{1}(\sin x)^{(3)}(e^x)' + \binom{4}{2}(\sin x)''(e^x)''$$
$$+ \binom{4}{3}(\sin x)'(e^x)''' + \binom{4}{4}(\sin x)(e^x)^{(4)}$$

$$= 1 \cdot \sin x \cdot e^x + 4 \cdot (-\cos x) \cdot e^x + 6(-\sin x)e^x + 4 \cdot \cos x \cdot e^x$$
$$+ 1 \cdot \sin x \cdot e^x$$

$$= -4 e^x \sin x$$

$u = \sin x$
$u' = \cos x$
$u'' = -\sin x$
$u''' = -\cos x$
$u^{(4)} = \sin x$

$V = e^x$
$V' = e^x$
$V'' = e^x$
$V''' = e^x$
$V^{IV} = e^x$

② Find 5^{th} derivative of the function $y = x^2 \sin 2x$ at $x = 0$.

Let $u = \sin 2x$, $v = x^2$

$$(uv)^{(5)} = y^{(5)} = u_5 v_0 + n u_4 v' + \frac{n(n-1)}{2!} u_3 v'' + \frac{n(n-1)(n-2)}{3!} u_2 v'''$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!} u_1 v^{IV} + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} u_0 v^{V}$$

Same

OR $= \binom{5}{0} u^{V} v_0 + \binom{5}{1} u^{IV} v' + \binom{5}{2} u''' v'' + \binom{5}{3} u'' v''' + \binom{5}{4} u' v^{IV} + \binom{5}{5} u v^{V}$

$$= 32 \sin 2x (x^2) + 5(16 \sin 2x)(2x)$$
$$+ 10(-8 \cos 2x)(2) + 0 + 0 + 0$$

$$= 32 x^2 \sin 2x + 160 x \sin 2x - 160 \cos 2x$$

$$y^{(5)}(0) = 32(0)\sin(0) + 160(0)\sin(0) - 160\cos(0)$$
$$= -160$$

$u = \sin 2x$ ✓
$u' = 2\cos 2x$ ✓
$u'' = -4 \sin 2x$ ✓
$u''' = -8 \cos 2x$ ✓
$u^{IV} = 16 \sin 2x$ ✓
$u^{V} = 32 \sin 2x$ ✓

$v = x^2$ ✓
$v' = 2x$ ✓
$v'' = 2$ ✓
$v''' = 0$ ✓
$v^{IV} = 0$ ✓
$v^{V} = 0$ ✓

③ Find $\frac{d^4y}{dx^4}\left(x^3(1+x^2)\right)$

$y_0 = x^3$ | $x_0 = 1+x^2$
$y_1 = 3x^2$ | $x_1 = 2x$
$y_2 = 6x$ | $x_2 = 2$
$y_3 = 6$ | $x_3 = 0$
$y_4 = 0$ | $x_4 = 0$

$$(yx)_n = y_n x + n y_{n-1} x_1 + \frac{n(n-1)}{2!} y_{n-2} x_2 + \frac{n(n-1)(n-2)}{3!} y_{n-3} x_3 + \frac{n(n-1)(n-2)(n-3)}{4!} y_{n-4} x_4$$

$$= \binom{4}{0} y_4 x_0 + \binom{4}{1} y_3 x_1 + \binom{4}{2} y_2 x_2 + \binom{4}{3} y_1 x_3 + \binom{4}{4} y_0 x_4$$

$$(yx)_4 = y_4 x + 4 y_{4-1} x_1 + \frac{4(4-1)}{2!} y_2 x_2 + \frac{4(3)(2)}{3!} y_1 x_3 + \frac{4(3)(2)(1)}{4!} y_0 x_4$$

skip

$$= 0(x) + 4(6)(2x) + 6(6x)(2) + 4(3x^2)(0) + x^3(0)$$

$$= 48x + 72x + 0 + 0$$

$$= 120x$$

Solve the following:

$\boxed{1}$ If $y = \tan^{-1}x$ then show that $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

$y = \tan^{-1}x$

$y_1 = \frac{1}{1+x^2}$

$(1+x^2)y_1 = 1$

$(1+x^2)y_1 - 1 = 0$

$(1+x^2)y_2 + 2x\,y_1 = 0$ derive both sides ⟶ (*)

$$(y_0 x_0)_n = y_{n+0}\,x_0 + n\,y_{n-1}\,x_1 + \frac{n(n-1)}{2!} y_{n-2} x_2 + \cdots$$

$(y_2)_n = y_{n+2}$ similarly $(y_1)_n = y_{n+1}$

$(y_0)_n = y_n = y_{n+0}$

differentiate (*) n times by Leibnitz's Theorem

$$y_{n+2}(1+x^2) + n\,y_{n+1}(2x) + \frac{n(n-1)}{2!} y_n \cdot 2 + 2x\,y_{n+1} + 2n\,y_n = 0$$

$$(1+x^2)y_{n+2} + 2nx\,y_{n+1} + n^2 y_n - n y_n + 2x\,y_{n+1} + 2n y_n = 0$$

$$(1+x^2)y_{n+2} + 2(n+1)x\,y_{n+1} + n(n+1)y_n = 0$$

$n^2 y_n + n y_n$

$n(n+1)y_n$

$\boxed{2}$ If $y = a\cos(\ln x) + b\sin(\ln x)$ then

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$$

$$y = a\cos(\ln x) + b\sin(\ln x)$$

$$y_1 = -a\sin(\ln x)\frac{1}{x} + b\cos(\ln x)\frac{1}{x}$$

$$x y_1 = -a\sin(\ln x) + b\cos(\ln x) \qquad \text{[multiply by } x\text{]}$$

$$x y_2 + y_1 = -a\cos(\ln x)\frac{1}{x} - b\sin(\ln x)\frac{1}{x} \qquad \text{(derive both sides)}$$

$$x^2 y_2 + x y_1 = -a\cos(\ln x) - b\sin(\ln x) \qquad \text{[multiply by } x\text{]}$$

$$= -(a\cos(\ln x) + b\sin(\ln x))$$

$$x^2 y_2 + x y_1 = -y$$

$$x^2 y_2 + x y_1 + y = 0$$

differentiate both sides n times (apply Leibnitz's Theorem)

$$x^2 y_{n+2} + n\,2x\,y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + x y_{n+1} + (1)n y_n + y_n = 0$$

$$x^2 y_{n+2} + 2nx\,y_{n+1} + n^2 y_n - n y_n + x y_{n+1} + n y_n + y_n = 0$$

$$x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2+1)y_n = 0$$

3 If $y = \cot^{-1}x$ then show that $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

Given $y = \cot^{-1}x$

$$y_1 = -\frac{1}{1+x^2}$$

$$(1+x^2)y_1 = -1$$

$$(1+x^2)y_2 + 2xy_1 = 0 \quad \text{differentiate both sides}$$

$$(1+x^2)y_{n+2} + 2xny_{n+1} + \frac{n(n-1)}{2!}(2)y_n + 2xy_{n+1} + n(2)y_n = 0$$

$$(1+x^2)y_{n+2} + 2nxy_{n+1} + n^2y_n - ny_n + 2xy_{n+1} + 2ny_n = 0$$

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

4 If $y\sqrt{1-x^2} = \sin^{-1}x$ then show that $(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0$

Given $y\sqrt{1-x^2} = \sin^{-1}x$

$$\sqrt{1-x^2}\,y_1 + y\frac{1}{2\sqrt{1-x^2}}(-2x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(differentiate both sides)}$$

$$(1-x^2)y_1 - xy = 1 \quad \text{(multiply by } \sqrt{1-x^2}\text{)}$$

Apply Leibnitz's Theorem

$$(1-x^2)y_{n+1} + n(-2x)y_n + \frac{n(n-1)}{2!}(-2)y_{n-1} - xy_n - n(1)y_{n-1} = 0$$

$$(1-x^2)y_{n+1} - 2nxy_n + n^2y_{n-1} - ny_{n-1} - xy_n - ny_{n-1} = 0$$

$$(1-x^2)y_{n+1} + (-2nx - x)y_n + (-n^2 + n - n)y_{n-1} = 0$$

$$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2y_{n-1} = 0$$

5 If $y = e^{m\sin^{-1}x}$ then show $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Given $y = e^{m\sin^{-1}x}$

$$\ln y = \ln e^{m\sin^{-1}x}$$

$$= m\sin^{-1}x \ln e$$

$$\ln y = m\sin^{-1}x$$

$$\frac{1}{y}\cdot y_1 = m\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = ym$$

$$(1-x^2)y_1^2 = y^2m^2 \quad \text{(Square both sides)}$$

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = m^2\, 2yy_1 \quad \text{(derive both sides)}$$

$$(1-x^2)y_2 - xy_1 = m^2y \quad [\div \text{ by } 2y_1]$$

$$(1-x^2)y_2 - xy_1 - m^2y = 0$$

Apply Leibnitz's Theorem

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} + n(-1)y_n - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2y_n + ny_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

[6] If $\ln y = \tan^{-1} x$ show that $(1+x^2) y_{n+2} + (2nx+2x-1) y_{n+1}$
$+ n(n+1) y_n = 0$

$$\frac{1}{y} y_1 = \frac{1}{1+x^2}$$

$$(1+x^2) y_1 = y$$

$$(1+x^2) y_1 - y = 0$$

$$(1-x^2) y_2 + 2x y_1 - y_1 = 0$$ derive both sides

$$(1-x^2) y_2 + (2x-1) y_1 = 0$$

complete the solution.

[7] Given $y=(\cos^{-1}x)^2$ show that $(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0.$

$$y = (\cos^{-1}x)^2$$

$$y_1 = 2(\cos^{-1}x)\frac{(-1)}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = -2(\cos^{-1}x)$$

$$(1-x^2) y_1^2 = 4(\cos^{-1}x)^2$$ (square both sides)

$$(1-x^2) y_1^2 = 4y \qquad \because y = \cos^{-1}x$$

$$(1-x^2) 2y_1 y_2 + (-2x) y_1^2 - 4y_1 = 0$$

$$(1-x^2) 2y_1 y_2 - 2x y_1^2 - 4y_1 = 0$$

$$(1-x^2) y_2 - x y_1 - 2 = 0 \quad [\div \text{ by } 2y_1]$$

Apply Leibnitz's Theorem

$$(1-x^2) y_{n+2} + n(-2x) y_{n+1} + \frac{n(n-1)}{2!}(-2) y_n - x y_{n+1} + n(-1) y_n = 0$$

$$(1-x^2) y_{n+2} - 2nx y_{n+1} + n^2 y_n - n y_n - x y_{n+1} - n y_n = 0$$

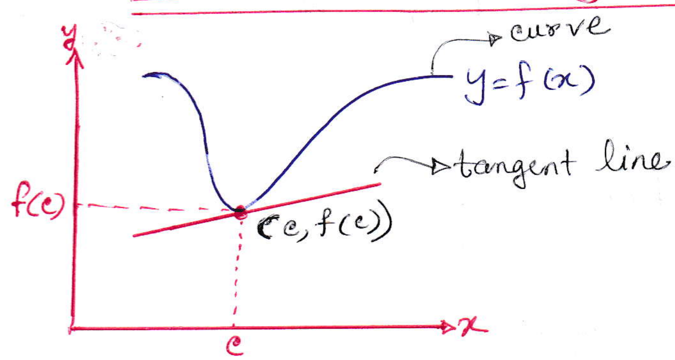
$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - n^2 y_n = 0$$

$\boxed{8}$ If $x = \tan(\ln y)$ show that $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$

Given $x = \tan(\ln y)$

$\ln y = \tan^{-1}(x)$ $\longrightarrow$ same as example $\boxed{6}$

Gradient & Tangent line



→ tangent line intersect the curve $y=f(x)$ only once

→ tangent line is a slope of a curve

The limit definition of derivative is:

$$f'(x) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \quad \text{OR} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

This represents the slope of the tangent to the curve at the point $x=a$.

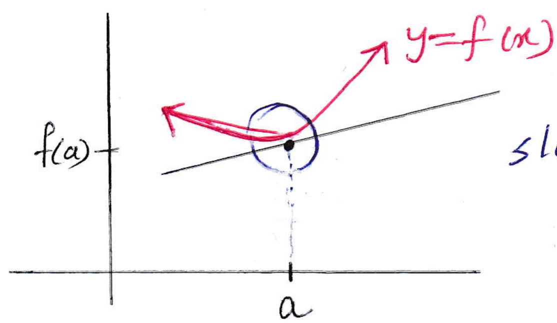
$f'(x)$ = rate of change

m = rate of change

$y = 3x+2$

$x=0$

$f(x)=?$



slope = Gradient : $\frac{\Delta y}{\Delta x}$

$$= \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x)-f(a)}{x-a}.$$

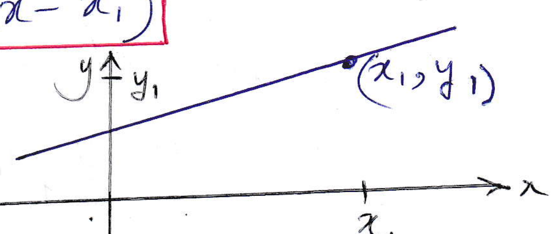
The value of $f'(x)$ at $x=a$ corresponds to the gradient of the tangent to the curve $y=f(x)$ at the pt $x=a$.

Eqn of Straight line passing through (x_1, y_1) is:

$$\boxed{y - y_1 = m(x - x_1)}$$

$\boxed{x_1 = a}$

$(a, f(a))$



$f(x)\Big|_{x=a} = ?$

$f(x) = f(a)$

∴ Eqn of the tangent line to the curve $y=f(x)$ at $x=a$ is:

$$\boxed{y - f(a) = f'(a)(x-a)}$$

Examples:

① Find the eqn of the tangent line of $f(x) = \sqrt{x-2}$

$\downarrow$

at $(3, 1)$ (x_1, y_1)

$x = a = 3$

$\therefore f(x+h) = \sqrt{x+h-2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \cdot \frac{\sqrt{x+h-2} + \sqrt{x-2}}{\sqrt{x+h-2} + \sqrt{x-2}}$$

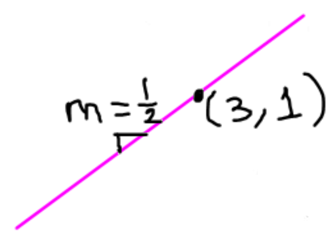
(Rationalizing the ~~denominator~~ numerator with conjugate.)

(algebric manipulation)

$$= \lim_{h \to 0} \frac{(x+h-2) - (x-2)}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$

$$= \lim_{h \to 0} \frac{x+h-2-x+2}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h-2} + \sqrt{x-2})}$$



$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-2} + \sqrt{x-2}}$$

$(x_1, y_1) \rightarrow$ given

$$= \frac{1}{\sqrt{x-2} + \sqrt{x-2}} = \frac{1}{2\sqrt{x-2}} = f'(x)$$

at $(3, 1)$ $\underbrace{f'(x)}_{\rightarrow m} = \frac{1}{2\sqrt{x-2}} = \frac{1}{2\sqrt{3-2}} = \frac{1}{2} = m$

Eqn of tangent line: $y - y_1 = m(x - x_1)$ $\rightarrow y = mx + c$

$y - 1 = \frac{1}{2}(x - 3) \Rightarrow \boxed{y = \frac{1}{2}x - \frac{1}{2}}$

$y_1 = ?$ $\qquad x = a$

② Find the eqn of the line tangent to $f(x) = x^2$ at $x = 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = x^2$

$f(x+h) = (x+h)^2$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h \to 0} \frac{2x+h}{1}$$

$$= 2x + 0$$

$$= 2x$$

$x = 2$

at $x = 2$, $\underbrace{f'(x)}_{f'(a)} = 2x$

$= 2(2)$

$\boxed{\therefore m = 4}$

Eqn of tangent line

[Note: $x = 2 \Rightarrow f(x) = x^2$
$= 2^2 = 4$]
y

$(x_1, y_1) = (2, 4)$

∴ We have found the coordinates $(2, 4)$ for the pt shared by $f(x)$ & the line tangent to $f(x)$ at $x = 2$.

Now $m = f'(x) = 4$ at $x = 2$

$(x_1, y_1) = (2, 4)$

Hence $y - y_1 = m(x - x_1)$

$\Rightarrow y - 4 = 4(x - 2)$

$\Rightarrow y - 4 = 4x - 8$

$\Rightarrow \boxed{y = 4x - 4} \Rightarrow y = mx + c$

This is the eqn for the tangent line.