#11

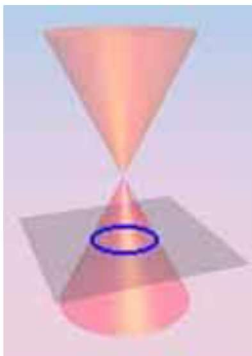
আমার ছেলেরা গনিত বুঝে বুঝে করে।
অংক তো মুদির দোকানদারও পারে।

– রেজা স্যার।

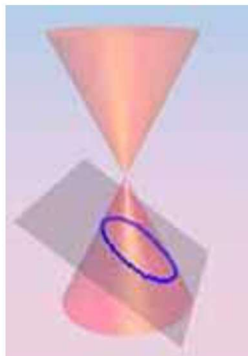
CONICS

Definition:

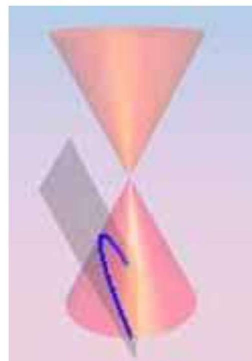
A conic section is the intersection of a plane and a cone. By changing the angle and location of intersection, we can produce a circle, ellipse, parabola or hyperbola; or in the special case when the plane touches the vertex: a point, line or 2 intersecting lines.



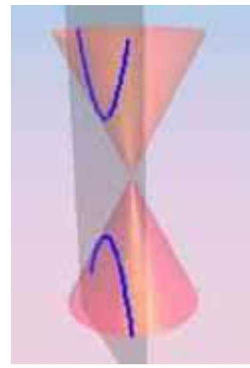
CIRCLE



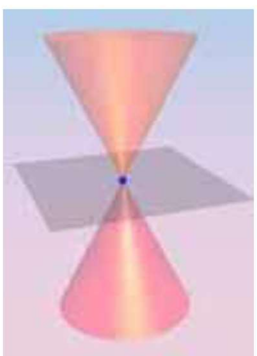
ELLIPSE



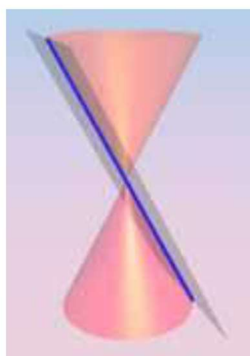
PARABOLA



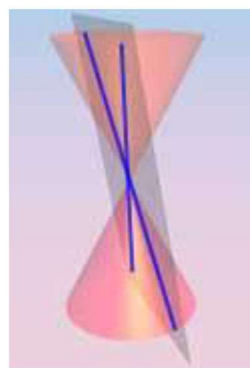
HYPERBOLA



POINT



LINE



PAIR of STRAIGHT LINES

General Equation of Conic: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Detail Classification of Conics

Case	Condition of Invariants	Types of Locus	Required form of equation	Reduces form of equation
Proper Conic: $\Delta \neq 0$	$C > 0, I$ and Δ opposite in sign; a=b, h=0	**Circle**	$x^2+y^2+2gx+2fy+c=0$	
	$C > 0, I$ and Δ opposite in sign	**Ellipse**	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$a'x^2 + b'y^2 + \frac{\Delta}{C} = 0$
	$C < 0$	**Hyperbola**	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$a'x^2 + b'y^2 + \frac{\Delta}{C} = 0$
	$C = 0$	**Parabola**	$y^2 = 4ax$	$Y^2 = 4AX$
	$C > 0, I$ and Δ same sign	**No real locus**		
Degenerate Conic: $\Delta = 0$		**A pair of straight lines**	$ax^2+2hxy+by^2+2gx+2fy+c=0$	

Determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

$C = ab - h^2$

$I = a + b$

Δ, ***C***, and ***I*** are invariants under transformation.

<u>Invariant:</u> a relationship that is not changed by a designed mathematical operation such as transformation of coordinates (unchanged/constant/not varying).

<u>Locus:</u> the set of all points, lines, or surfaces that satisfy a given requirement.

Conic Section
Practice Sheet – Mat 110

Problems :

1. Reduce the equation to its standard form:

   i) $x^2 - 4xy + y^2 + 8x + 2y - 5 = 0$

   ii) $8x^2 + 4xy + 5y^2 - 16x - 14y + 13 = 0$

   iii) $x^2 - 6xy + 9y^2 - 2x - 3y + 1 = 0$

   iv) $x^2 - 6xy + 9y^2 + 4x + 8y + 15 = 0$

   v) $4x^2 - 24xy - 6y^2 + 4x - 12y + 1 = 0$

   vi) $3x^2 + 2xy + 3y^2 + 2x - 6y + 12\frac{1}{2} = 0$

2. Find the center of the conic $x^2 - 4xy + y^2 + 8x + 2y - 5 = 0$

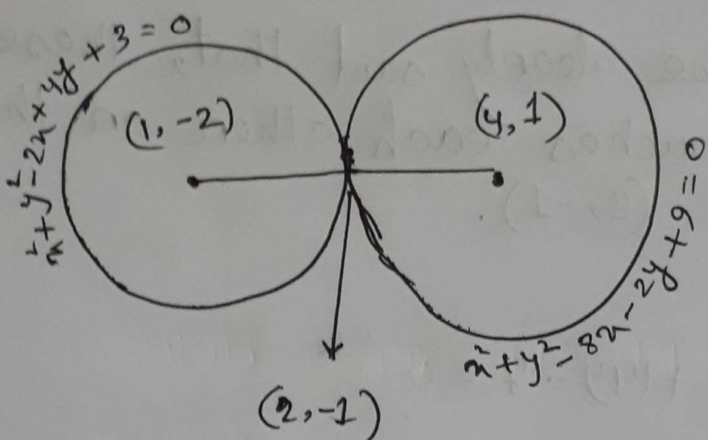
Home work sheet 11

① $c_1 : x^2+y^2-2x+4y+3=0$ —— (I)

$c_2 : x^2+y^2-8x-2y+9=0$ —— (II)

① ⇒ $g_1 = 1$; $f_1 = -2$

$g_2 = 4$; $f_2 = 1$



radius of c1 : $\sqrt{1+4-3} = \sqrt{2}$

Distence from center to the point $(2,-1)$

$\cancel{\sqrt{(1-4)^2+(-2}}$ $\sqrt{(1-2)^2+(-2+1)^2}$

$= \sqrt{1+1} = \sqrt{2}$

∴ In both case, the radius is same.

Radius of c2 : $\sqrt{16+1-9} = 2\sqrt{2}$

Distence from center $(4,1)$ to the point $(2,-1)$:

$\sqrt{(4-2)^2+(1+1)^2} = \sqrt{4+4} = 2\sqrt{2}$

In both case, the radius is same.

Again,
distence between the two center of
the circle:

$$\sqrt{(1-4)^2+(-2-1)^2} = \sqrt{9+9} = 3\sqrt{2}$$

Summition of the radius of two circle:

$$\sqrt{2}+2\sqrt{2} = 3\sqrt{2}$$

$\therefore$ It can be clearly said that, these two circle touches each other on the ~~po~~ point $(2,-1)$.

(2) $x^2+y^2-9x+14y-7=0$

$x^2+y^2+15x+14=0$

point: $(2,5)$

$$x^2+y^2-9x+14y-7+K(x^2+y^2+15x+14)=0 \text{ --- (1)}$$

$$4+25-18+70-7+K(4+25+30+14)=0$$

$$K = \frac{74}{73}$$

$$\therefore x^2+y^2-9x+14y-7 \mp \frac{74}{73}(x^2+y^2+15x+14)=0$$

$$73x^2+73y^2-657x+1022y-511 \mp 74x^2 \mp 74y^2$$
$$- \mp 1110x \mp 1036 = 0$$

~~$147x^2+147y^2+453x+1022y+525=0$~~

$$x^2+y^2+1767x-1022y+1547=0$$

3 $x^2+y^2=1$

$$x^2+y^2+2x+4y+1=0$$

$$x^2+y^2-1+kx^2+ky^2+2kx+4ky+k=0$$

$$(1+k)x^2+(1+k)y^2+2kx+4ky+k-1=0$$

$$(g,f) \equiv \left(\frac{-k}{k+1}, \frac{-2k}{k+1}\right)$$

$$r = \sqrt{\frac{k^2}{(k+1)^2}+\frac{4k^2}{(k+1)^2}-\frac{k-1}{k+1}}$$

$$= \sqrt{\frac{4k^2+1}{(k+1)^2}}$$

Now, $x+2y+5=0$

$$\frac{|ax_1^2+by_1^2+c|}{\sqrt{a^2+b^2}} = r$$

$$\frac{\frac{-k}{k+1}-\frac{4k}{k+1}+5}{\sqrt{1+4}} = \sqrt{\frac{4k^2+1}{(k+1)^2}}$$

$$\frac{\frac{-5k+5k+5}{k+1}}{\sqrt{5}} = \sqrt{\frac{4k^2+1}{(k+1)^2}}$$

$$\frac{25}{(k+1)^2 \cdot 5} = \frac{4k^2+1}{(k+1)^2}$$

$$\Rightarrow \frac{5}{(k+1)^2} = \frac{4k^2+1}{(k+1)^2}$$

$4k^2 = 4$ | But $k \neq -1$ [$\because k+1 \neq 0$]

$k = \pm 1$

$\therefore k = 1$

$\therefore \cancel{x^2+y^2}$

$$2x^2 + 2y^2 + 2x + 4y + 1 - 1 = 0$$

$$\boxed{2(x^2+y^2) + 2x + 4y = 0}$$

(4) $x^2+y^2+x+2y+3=0$ — (I)

$x^2+y^2+2x+4y+5=0$ — (II)

$x^2+y^2-7x-8y-9=0$ — (III)

(I) − (II)

$$-x-2y-2=0$$

$$\boxed{x+2y+2=0}$$

(I) − (III)

$$\boxed{8x+10y+12=0}$$

$\rightarrow (x, y) \equiv$

point : $\left(\frac{-2}{3}, \frac{-2}{3}\right)$

Ans.

①(1)

$$x^2 - 6xy + 9y^2 - 2x - 3y + 1 = 0$$

$$\Delta \equiv (1\cdot 9\cdot 1) + \left(-6\cdot -1\cdot \frac{-3}{2}\right) - (1\cdot 9/4) - (9\cdot 1)$$
$$- (1\cdot 9)$$

$$\equiv \frac{-81}{4} < 0$$

Now, $c = ab - h^2 \Rightarrow 9 - (-3)^2 \Rightarrow 0$

$$x^2 - 6x(y+\alpha) + 9(y+\alpha)^2 - 2x - 3(y+\alpha) + 1 = 0$$

Now,

$$-18\alpha y - 3y = 0$$

$$y(18\alpha - 3) = 0$$

$$\therefore 18\alpha = 3$$

$$\alpha = \frac{3}{18} = \frac{1}{6}$$

$$\therefore x^2 - 6xy - \cancel{6}x + 9\left(y + \frac{1}{6}\right)^2 - 2x - 3\left(y + \frac{1}{6}\right) + 1 = 0$$

$$x^2 - 6xy - x + 9\left(y^2 + \frac{y}{3} + \frac{1}{36}\right) - 2x - 3y - \frac{1}{2} + 1 = 0$$

$$\underline{x^2 - 6xy} \underline{- x} + \underline{9y^2} + \cancel{3y} + \frac{1}{4} \underline{- 2x} - \cancel{3y} + \frac{1}{2} = 0$$

$$x^2 + 9y^2 - 6xy - 3x + \frac{3}{4} = 0$$

$$(x-3y)^2 = 3x - \frac{3}{4}$$

$$(x-3y)^2 = \frac{12x-3}{4}$$

$$\boxed{(x-3y)^2 = 4 \cdot \frac{1}{16}(12x-3)}$$

(11)

$$x^2 - 4xy + y^2 + 8x + 2y + 5 = 0$$

$$\Delta \equiv (1 \cdot 1 \cdot 5) + (-4 \cdot 4 \cdot 1) - (1 \cdot 1) - (1 \cdot 16) - (-5 \cdot 4)$$

$$\equiv -5 - 16 - 1 - 16 + 20 \Rightarrow \cancel{48 > 0} \; -18 < 0$$

Now, $c \equiv ab - h^2$

$$\equiv (1 \cdot 1) - (-2)^2 \Rightarrow -3 < 0$$

$\therefore$ This is a hyperbola equation.

Now, $\frac{\partial z}{\partial x} = 2x - 4y + 8$
$\frac{\partial z}{\partial x} = -4x + 2y + 2$ $\Big\} \rightarrow (2, 3)$

The equation would be :

$x^2 + y^2 - 4xy + \lambda = 0$ where λ is $\frac{\Delta}{c}$

$\therefore \frac{\Delta}{c} = \frac{-18}{-3} = 6$

$\therefore$ The equation ~~would~~ will :

$$x^2 + y^2 - 4xy + 6 = 0$$

Again, $a + b = 2$

$ab = 1 - 4 = -3$

$\therefore (a, b) \equiv (-1, 3), (3, -1)$

∴ The equation will :

$$ax^2+by^2+6=0$$
$$-x^2+3y^2+6=0$$
$$x^2-3y^2 = 6$$
$$\boxed{\frac{x^2}{6}-\frac{y^2}{2}=1}$$

$$ax^2+by^2+6=0$$
$$3x^2-y^2=-6$$
$$\frac{x^2}{-2}-\frac{y^2}{-6}=1$$
$$\boxed{\frac{y^2}{6}-\frac{x^2}{2}=1}$$

Ans.

(III)

$$4x^2 - 24xy - 6y^2 + 4x - 12y + 1 = 0$$

$$D \equiv (4 \cdot -6 \cdot 1) + (-24 \cdot 2 \cdot -6) - (4 \cdot 36) - (-6 \cdot 4) - (1 \cdot 144)$$

$$\equiv -24 + 288 - 144 + 24 - 144$$

$$= 0$$

This fool is nothing but a pair of straight line ☹

(IV) $9x^2 - 4xy + 6y^2 - 10x - 7 = 0$

$$\Delta \equiv (9 \cdot 6 \cdot -7) + (-4 \cdot 5 \cdot 0) - (9 \cdot 0) - (6 \cdot 25) - (-7 \cdot 4)$$

$$\equiv -378 - 150 + 28 \Rightarrow -500$$

$$C \equiv ab - h^2 \equiv (9 \cdot 6) - 4 \Rightarrow 50$$

$$I \equiv a + b \Rightarrow 9 + 6 \Rightarrow 15$$

$\therefore$ This is a elips and

Now, removind x term, the equation would be : $9x^2 - 4xy + 6y^2 + \lambda = 0$

~~$9x^2 + 6y^2 - 10x + \lambda = 0$~~

where, $\lambda = \frac{\Delta}{C} = \frac{-500}{50} = -10$

$\therefore$ The equation is : ~~$9x^2 + 6y^2 - 10x - 10 = 0$~~

$9x^2 - 4xy + 6y^2 - 10 = 0$

~~$9x^2 + 6y^2 - 10x - 10 = 0$~~ $9x^2 - 4xy + 6y^2 - 10 = 0$

The final equation will be :

$ax^2 + by^2 - 10 = 0$

Here,

$a+b = \boxed{a'+b'} \rightarrow 9+6 \Rightarrow 15$

$\therefore a+b = 15$ —— (i)

$ab - h^2 = \boxed{a'b' - h'^2} \rightarrow (9.6) - (-2)^2$

$ab - h^2 = 50$

$ab = 50$ —— (ii)

$b = \frac{50}{a}$ —— (iii)

$a + \frac{50}{a} = 15$

$a^2 + 50 = 15a$

$a^2 - 15a + 50 = 0$

$a^2 - 10a - 5a + 50 = 0$

$a(a-10) - 5(a-10) = 0$

$\boxed{\begin{array}{l} a = 10;\ 5 \\ b = 5;\ 10 \end{array}} \rightarrow$ The equation :

$10x^2 + 5y^2 = 10$

$\boxed{x^2 + \frac{y^2}{2} = 1}$

or, $5x^2 + 10y^2 = 10$

$\boxed{\frac{x^2}{2} + y^2 = 1}$

$\rightarrow$ Ans.

(v) $x^2 - 4xy - 2y^2 + 10x + 4y = 0$

$D \equiv (1 \cdot -2 \cdot 0) + (-4 \cdot 5 \cdot 2) - (1 \cdot 4) - (-2 \cdot 25) - 0$

$\equiv 0 - 40 - 4 + 50$

$\equiv 50 - 44$

$\equiv 6 > 0$

$J = a + b = 1 - 2 = -1$

$\therefore$ ~~This is~~

$C \equiv ab - h^2 \Rightarrow (1 \cdot -2) - (-2)^2$

$\Rightarrow -2 - 4 \Rightarrow -6$

$\therefore$ This equation is hyperbola.

$\lambda = \dfrac{\Delta}{C} \Rightarrow \dfrac{6}{-6} \Rightarrow -1$

$\therefore$ The equation :

$x^2 - 2y^2 - 4xy - 1 = 0$

Final equation : $ax^2 + by^2 - 1 = 0$

$a + b = 1 - 2 = -1$ —— (I)

$ab - h^2 = a'b' - h'^2$

$ab - 0 = -2 - (-2)^2$

$ab = -2 - 4 \Rightarrow -6$ —— (II)

$b = \dfrac{-6}{a}$ —— (III)

$a - \dfrac{6}{a} = -1$

$a^2 - 6 = -a$

$a^2 + a - 6 = 0$

$a^2 + 3a - 2a - 6 = 0$

$a(a+3) - 2(a+3) = 0$

$a = 2, -3$

$b = -3, 2$

∴ The equation is

$\boxed{\text{for } (2, -3)}$

$$2x^2 - 3y^2 = 1$$

$$\frac{x^2}{\frac{1}{2}} - \frac{y^2}{\frac{1}{3}} = 1 \quad \text{——} \quad 1$$

$\boxed{\text{for } (-3, 2)}$

$$-3x^2 + 2y^2 = 1$$

(vi) $x^2+4y^2-2x-16y+1=0$

Here,

$\cancel{D=(1\cdot 0\cdot 1)}$

$D \equiv (1\cdot4\cdot1)+(2f\cdot g\cdot0)-(1\cdot64)-(4\cdot1)-(1\cdot0)$

$\equiv 4-64-4 \Rightarrow -64$

$C \equiv ab-h^2 \Rightarrow 4-0 \Rightarrow 4>0$

~~c>0 and~~

$J = a+b \Rightarrow 5$

$C>0$ and J and D opposite in sign.

$\therefore$ It is ellips.

$x^2+4y^2-2x-16y+1=0$

$x^2+4y^2+K=0$

Here, $K = \frac{\Delta}{C} = \frac{-64}{4} = -14$

$\therefore x^2+y^2=14$

$$\boxed{\frac{x^2}{14}+\frac{4y^2}{14}=\cancel{0}1}$$

$\Downarrow$

$$\frac{x^2}{14}+\frac{y^2}{\frac{14}{4}}=1$$

(VII)

$9x^2 + 24xy + 16y^2 + 22x + 46y + 9 = 0$

Here,

$\Delta \equiv (9 \cdot 16 \cdot 9) + (24 \cdot 11 \cdot 23) - (9 \cdot 23^2) - (16 \cdot 11^2)$

$- (9 \cdot 144)$

$\equiv -625$

$c \equiv ab - h^2 \equiv 0$

$\therefore$ This is a ~~parap~~ parabolla.

$9x^2 + 24x(y+\alpha) + 16(y+\alpha)^2 + 22x + 46(y+\alpha) + 9 = 0$

$32\alpha y + 46y = 0$

$y(32\alpha + 46) = 0$

$\Rightarrow \alpha = \frac{-46}{32} = \frac{-23}{16}$

$9x^2 + 24x\left(y - \frac{23}{16}\right) + 16\left(y - \frac{23}{16}\right)^2 + 22x +$

$46\left(y - \frac{23}{16}\right) + 9 = 0$

$\Rightarrow \left(y - \frac{23}{16}\right)(24x + 16y - 23 + 46) + 9x^2 + 22x + 9 = 0$

$\Rightarrow \left(y - \frac{23}{16}\right)(24x + 16y + 23) + 9x^2 + 22x + 9 = 0$

$\Rightarrow 24xy + 16y^2 + \cancel{23y} - \frac{69x}{2} - 23y - \frac{529}{16} + 9x^2 + 22x + 9 = 0$

$\Rightarrow 9x^2 + 16y^2 + 24xy - \frac{25x}{2} - \frac{385}{16} = 0$

$\Rightarrow 144x^2 + 256y^2 + 384xy - 200x - 385 = 0$

$\Rightarrow (12x)^2 + (16y)^2 + 2\cdot 12x \cdot 16y = 200x - 385$

$\Rightarrow (12x+16y)^2 = 200\left(x - \frac{385}{200}\right)$

$\Rightarrow (12x+16y)^2 = 4\cdot 50\cdot\left(x - \frac{77}{40}\right)$

Ans.

(VIII)

$3x^2 + 2xy + 3y^2 + 2x - 6y + \frac{25}{2} = 0$

$D \equiv \left(3\cdot 3\cdot \frac{25}{2}\right) + (2\cdot 1\cdot -3) - (3\cdot 9) - (3\cdot 1)$
$- \left(\frac{25}{2}\cdot 1\right)$

$\equiv \frac{225}{2} - 6 - 27 - 3 - \frac{25}{2}$

$\equiv 64 > 0$

$C \equiv ab - h^2 \Rightarrow 9 - 1 \Rightarrow 8 > 0$

$I \Rightarrow 3 + 3 \Rightarrow 6$

$c > 0$ and Δ & I are same in position. $\therefore$ It has no real locus.

#
② (i)

$$x^2 - 4xy + y^2 + 8x + 2y - 5 = 0$$

$$z = x^2 + y^2 - 4xy + 8x + 2y - 5$$

$$\frac{\partial z}{\partial x} = 2x - 4y + 8$$

$$\frac{\partial z}{\partial y} = 2y - 4x + 2$$

<u>center</u>: $(2, 3)$

(ii) $x^2 - 2xy + 2y^2 - 3x + 7y - 1 = 0$

$$\frac{\partial z}{\partial x} \Rightarrow 2x - 2y - 3 \quad || \quad \frac{\partial z}{\partial y} \Rightarrow -2x + 4y + 7 = 0$$

<u>center</u>: $\left(\frac{-1}{2}, -2\right)$

(iii) $3x^2 - 7xy - 6y^2 + 3x - 9y + 5 = 0$

$$\frac{\partial z}{\partial x} = 6x - 7y + 3 \quad || \quad \frac{\partial z}{\partial y} = -7x - 12y - 9$$

<u>center</u>: $\left(-9/11, -3/11\right)$