Rule :2 If $\dfrac{\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)}{M} = g(y)$, $IF = e^{\int g(y)\,dy}$

Ex: $\underbrace{(y^4 + 2y)}\,dx + \underbrace{(xy^3 + 2y^4 - 4x)}\,dy = 0 \quad \ldots (1)$

$M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$\dfrac{\partial M}{\partial y} = 4y^3 + 2 \qquad \dfrac{\partial N}{\partial x} = y^3 - 4$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ it isnot exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \frac{-3y^3 - 6}{y(y^3+2)}$$

$$= -3/y = g(y)$$

$$IF = e^{\int -3/y\, dy} = e^{-3\ln y} = y^{-3}$$

Multiplying (i) with IF,

$$y^{-3}(y^4+2y)\,dx + y^{-3}(xy^3 + 2y^4 - 4x)\,dy = 0$$

$$\Rightarrow \underbrace{\left(y + \frac{2}{y^2}\right)}_{M_1} dx + \underbrace{\left(x + 2y - \frac{4x}{y^3}\right)}_{N_1} dy = 0$$

$$\int M_1\,dx = xy + \frac{2x}{y^2} + c_1 \,, \quad \int N_1\,dy = xy + y^2 + \frac{2x}{y^2} + c_2$$

$$\boxed{xy + \frac{2x}{y^2} + y^2 = c}$$

Rule 3: If $\boxed{M \& N}$ are $\boxed{\text{homogeneous}}$ functions in x & y, then $\boxed{IF = \frac{1}{Mx+Ny}}$

{1st order + homogeneous/non-homogeneous + linear/non-linear (D.E.)}

Definition of homogeneous function: let $f(x,y)$ be a two variable function in Domain D. $t \in D$

$$f(tx,ty) = t^k f(x,y)$$, kth degree of homogeneous function.

Ex 1 $f(x,y) = x^2 + y^2$

$$f(tx,ty) = t^2x^2 + t^2y^2 = t^2(x^2+y^2) = t^2 f(x,y).$$

2nd degree homogeneous.

Ex 2 $f(x,y) = x^2 + y^2 + 2x$

$$f(tx,ty) = t^2x^2 + t^2y^2 + 2tx$$

$$f(x,y) = x^2 + y^2 + 2xy$$

$$f(tx,ty) = t^2x^2 + t^2y^2 + 2t^2xy = t^2(x^2+y^2+2xy) = t^2 f(x,y)$$

Rule-3

$$(x^3+y^3)\,dx - xy^2\,dy = 0 \quad \cdots \quad (1)$$

$$\boxed{M = x^3+y^3 \,,\quad N = -xy^2}$$

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2} = f(x,y) \; (\text{let})$$

$$f(tx,ty) = \frac{t^3x^3+t^3y^3}{t^3xy^2} = t^0 f(x,y) \,,\quad 0 \text{ degree}$$

homogeneous

$$\frac{\partial M}{\partial y} = 3y^2 \,,\quad \frac{\partial N}{\partial x} = -y^2 \,,\quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

it is not exact.

$$IF = \frac{1}{Mx+Ny} = \frac{1}{x(x^3+y^3)+y(-xy^2)} = \frac{1}{x^4+\cancel{xy^3}-\cancel{xy^3}}$$

$$= \frac{1}{x^4}$$

Multiply IF with (1'),

$$\underbrace{\left(\frac{1}{x}+\frac{y^3}{x^4}\right)}_{M_1}dx - \underbrace{\frac{y^2}{x^3}}_{N_1}dy = 0$$

$$\int M_1\,dx = \ln|x| - \frac{y^3}{3x^3} + c_1$$

$$\int N_1\,dy = -\frac{y^3}{3x^3} + c_2$$

$$\boxed{\ln|x| - \frac{y^3}{3x^3} = C}$$

10. $(x^3+y^3)\,dx + 3xy^2\,dy = 0$

Rule-4: If $y\,\underline{f(x,y)}\,dx + x\,\underline{g(x,y)}\,dy = 0$

$$IF = \frac{1}{Mx - Ny}$$

Ex: $y\,\underline{(1+xy)}\,dx + x\,\underline{(1-xy)}\,dy = 0 \quad \cdots \quad (1)$

$M = y(1+xy)$, $N = x(1-xy)$, $\frac{\partial M}{\partial y} = 1+2xy$

$\frac{\partial N}{\partial x} = 1-2xy$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, it is not exact

$$IF = \frac{1}{Mx - Ny} = \frac{1}{xy + x^2y^2 - xy + x^2y^2} = \frac{1}{2x^2y^2}$$

$x^2y^2 \neq 0$

Multiplying (1') with IF,

$$\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\left(\frac{1}{2xy^2}-\frac{1}{2y}\right)dy=0 \quad \text{(Verify exact !!)}$$

$$\Rightarrow \int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx-\int\frac{1}{2y}dy=c$$

$$\Rightarrow -\frac{1}{2xy}+\frac{1}{2}\ln x-\frac{1}{2}\ln y=c$$

$$\Rightarrow -\frac{1}{xy}+\ln x-\ln y=c_1, \quad c_1=2c$$

$$\Rightarrow \ln x-\ln y=c_1+\frac{1}{xy}$$

$$\Rightarrow \ln\left(\frac{x}{y}\right) = c_1 + \frac{1}{xy}$$

$$\Rightarrow \left(\frac{x}{y}\right) = e^{c_1 + \frac{1}{xy}}$$

$$\Rightarrow \frac{x}{y} = e^{c_1} \cdot e^{\frac{1}{xy}}$$

$$\Rightarrow \frac{x}{y} = A e^{\frac{1}{xy}} \quad , \quad A = e^{c_1}$$

$$\Rightarrow \boxed{y = Bx e^{-\frac{1}{xy}}} \quad , \quad B = \frac{1}{A}.$$

H.W.

$$y(x^2y^2 + xy)dx + x(x^2y^2 - 1)dy = 0$$

nth order Homogeneous linear O.D.E. with constant coefficients: (100%)

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_1 \frac{dy}{dx} + a_0 = 0$$

let $y = e^{mx}$ be trial solution

$$y(x) = ?? \longrightarrow \text{General Solution.}$$

nth-order non-homogeneous linear ODE with constant coefficients:

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 y + a_0 = f(x)$$

$y_c \rightarrow$ Complementary solution (homogeneous part)

$y_p \rightarrow$ Particular Solution

General solution $\leftarrow$ $y = y_c + y_p$