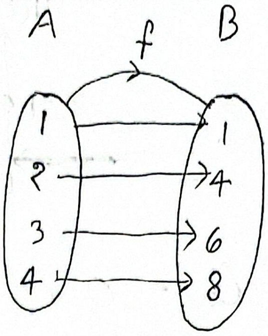
Lec-1 10.2.25

Function : $y = f(x)$



Output

Domain
Input

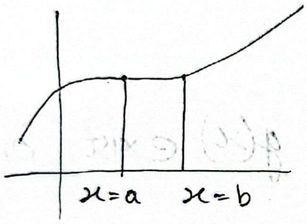
Co-domain
Range

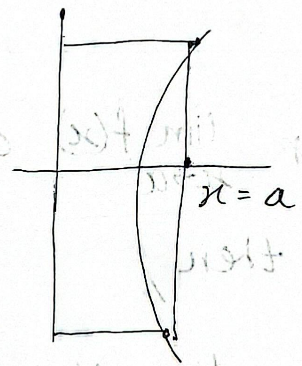
$f(x) = y = x^2$ (Function)

$y^2 = x^2 - 1$

$\Rightarrow y = \pm\sqrt{x^2-1}$ (not function)

↘ 2 output





not function

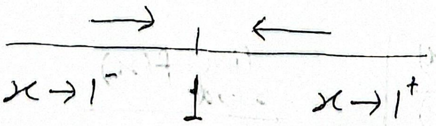
$f(x) = \dfrac{x+2}{x-1}$, $x \neq 1$

$\Rightarrow f(1) = \dfrac{1+2}{1-1}$ = undefined

$x \to 1$, $f(x) = \dfrac{2.9999}{0.9999 - 1} = \dfrac{2.9999}{-0.0001} = 2.99999 --$

$0.9999...$

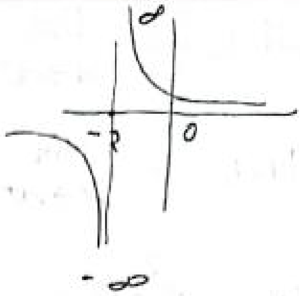
$$\boxed{\lim_{x \to a} f(x) = L}$$



$x \to 1^-$ $x < 1$

$x \to 1^+$ $x > 1$

$\Rightarrow$ L.H.L $= \lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2} \frac{1}{x+2}$



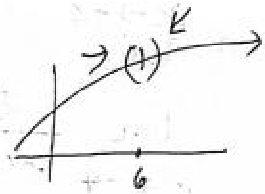
$= -\infty$

R.H.L $= \lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2} x^2-5$

$= -1$

$\therefore$ Limit does not exist.

$\lim\limits_{x \to 6} f(x) = \lim\limits_{x \to 6} \sqrt{x+13}$



$\boxed{\text{Lec-2}}$ 12.2.25

$|x| = \begin{cases} x & ; x > 0 \\ -x & ; x \le 0 \end{cases}$

$|x-2| = \begin{cases} x-2 & ; x-2 > 0 \Rightarrow x > 2 \\ -(x-2) & ; x-2 \le 0 \Rightarrow x \le 2 \end{cases}$

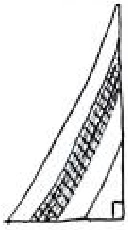
$\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right)$

The squeeze theorem

$g(x) \le f(x) \le h(x)$

if $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$

then $\lim\limits_{x \to a} f(x) = L$



$$2 < 3$$
$$(-1)\,2 > 3\,(-1)$$
$$-2 > -3$$

$\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right)$

$$-1 \le \sin\frac{1}{x} \le 1$$
$$-|x| \le x\sin\frac{1}{x} \le |x|$$
$$-\lim_{x\to 0}|x| \le \lim_{x\to 0} x\sin\frac{1}{x} \le \lim_{x\to 0}|x|$$

Since, $\lim\limits_{x\to 0}|x| = 0$

Applying squeeze theorem we get, $\lim\limits_{x\to 0}|x|\sin\left(\frac{1}{x}\right) = 0$

So, $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$ ✓

Definition of Continuity :-

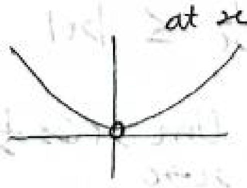
A function $f(x)$ is continuos at $x=a$ if following condition satisfies :-

i. $f(a)$ is defined , $f(x) = \frac{x}{x-1}$, $x=1$ $f(1) = \frac{1}{0}$ undefined

ii. $\lim\limits_{x\to a} f(x)$ exist

iii. $\lim\limits_{x\to a} f(x) = f(a)$

$$f(x) = \begin{cases} x^2 & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$$



$f(0) = 1$

$\lim\limits_{x\to 0} x^2 = 0$

not continuous

Given that, $f(x) = \begin{cases} x^2+1 & ; x>0 \\ 1 & ; x=0 \\ x+1 & ; x<0 \end{cases}$ is it Continous ?

$\Rightarrow f(0) = 1$. [So the function is defined]

$$\left.\begin{aligned} LHL &= \lim_{x\to 0^-} x^2+1 = 1 \\ RHL &= \lim_{x\to 0^+} x^2+1 = 1 \end{aligned}\right\} \text{So limit exists}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1 = f(0)$$

$\therefore$ The function is continuos at $x=0$.

Given that, $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x=0 \end{cases}$ is it Continous ?

$$\Rightarrow \quad -1 \leq \cos\frac{1}{x} \leq 1$$

$$-|x| \leq x\cos\frac{1}{x} \leq |x|$$

$$-\lim_{x\to 0}|x| \leq \lim_{x\to 0} x\cos\frac{1}{x} \leq \lim_{x\to 0}|x|$$

$\because -\lim_{x\to 0}|x| = \lim_{x\to 0}|x| = 0 \quad \therefore \lim_{x\to 0} x\cos\frac{1}{x} = 0$ [Squeeze theorem]

$f(0) = 0 \quad \because \lim_{x\to 0} x\cos\frac{1}{x} = f(0) \quad \therefore$ the function is continues at $x=0$

Q-1 wed

syllabus

i. Limit

2. Continuoty

2 Questions.

Propertise of continuty :-

If the functions of f & g are continuos at $x=a$ then,

1. $f \pm g$ is continuos at $x=a$

2. $f \cdot g$ " " " "

3. f/g " " " " ; $g(a) \neq 0$

* 4. If $\lim_{x \to a} g(x) = L$ and if f is continuous at $x=a$ then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Consider $g(x) = x^2+1$ and $f(x) = \sqrt{x}$ are continuos for $x \geq 0$

If $\lim_{x \to 2} g(x) = 5$ then find $\lim_{x \to 2} f(g(x))$

$\Rightarrow \lim_{x \to 2} f(g(x)) = f\left(\lim_{x \to 2} g(x)\right) = f(5)$

$f(x) = \sqrt{x} \quad \therefore f(5) = \sqrt{5}$

Lec-3

$\lim\limits_{x\to\infty} \dfrac{2-\cos x}{x+3}$ using squeeze teorem :-

$\Rightarrow$

$$-1 \le \cos x \le 1$$

$$(-1)(-1) \ge -\cos x \ge (-1)(1)$$

$$1 \ge -\cos x \ge -1$$

$$2+1 \ge 2-\cos x \ge 2-1$$

$$3 \ge 2-\cos x \ge 1$$

$$1 \le 2-\cos x \le 3$$

$$\cancel{\lim_{x\to\infty} 1 \le \lim_{x\to\infty} 2-\cos x \le}$$

$$\frac{1}{x+3} \le \frac{2-\cos x}{x+3} \le \frac{3}{x+3}$$

$$\lim_{x\to\infty} \frac{1}{|x+3|} \le \lim_{x\to\infty} \frac{2-\cos x}{|x+3|} \le \lim_{x\to\infty} \frac{3}{|x+3|}$$

$$0 \le \underset{0}{\downarrow} \le 0$$

$$\therefore \lim_{x\to\infty} \frac{2-\cos x}{x+3} = 0$$

$$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & ; x \ne 0 \\ 1 & ; x = 0 \end{cases}$$

is the function continuous at $x=0$?

$$\boxed{\frac{0}{0},\ \frac{\infty}{\infty},\ 1^{\infty},\ \infty^{\infty}}$$

$$\Rightarrow \lim_{x\to 0}\frac{e^{1/x^2}}{e^{1/x^2}-1} = \lim_{x\to 0}\frac{e^{1/x^2}\cdot\frac{1}{e^{1/x^2}}}{\frac{1}{e^{1/x^2}}\left(e^{1/x^2}-1\right)} = \lim_{x\to 0}\frac{1}{1-\frac{1}{e^{1/x^2}}}$$

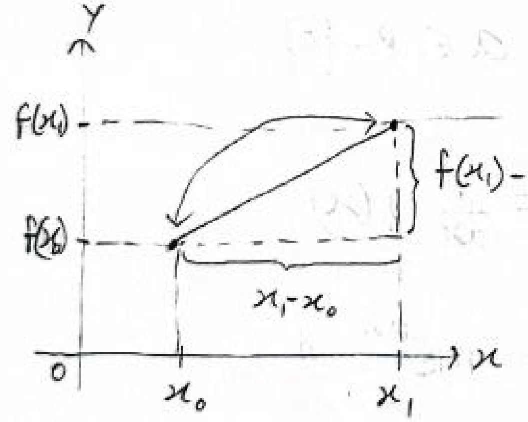
$$= \frac{1}{1-0} = 1$$

Differentiation

Rate of change $\quad$ slope $m = \frac{\Delta y}{\Delta x} = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{\text{rise}}{\text{run}}$

Average rate of change, $r_{avg} = \frac{f(x_1)-f(x_0)}{x_1-x_0}$

Instateneous rate of change at x_0, $r_{in} = \lim_{x_1\to x_0}\frac{f(x_1)-f(x_0)}{x_1-x_0}$



$h = x_1 - x_0$, $\quad x_1 = h + x_0$

$x_1 \to x_0$, $\quad h \to 0$

$$f(x_0) = r_{in} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

is called derivative of $f(x)$ in terds to x at $x = x_0$ for all x

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivative of following function by definition :-

$f(x) = x^2$, $f(x+h) = (x+h)^2$

$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + 0 = 2x$

Notation :- $\frac{df}{dx}$, f', f_x , f_0 , $\dot{f}$

Propentise :-

1. $\frac{d}{dx}[c\, f(x)] = c \frac{d}{dx} f(x)$

2. $\frac{d}{dx} x^p = p x^{p-1}$, $a^{p\in R}$, $a \in R - \{0\}$

3. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

4. $\frac{d}{dx} e^{mx} = e^{mx} \cdot \frac{d}{dx}(mx) = m e^{mx} \cdot \frac{d}{dx}(x)$

Find the value of K for which the given function is continuous at $x = 1$,

$$f(x) = \begin{cases} 7x-2 \; ; & x \le 1 \\ Kx^2 \; ; & x > 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (7x-2) = 5$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} Kx^2 = K$$

If $K=5$, then the function is continuous at $x=1$.

$f(x) = 7x^{-6} \cancel{+} 5\sqrt{x}$

$$\Rightarrow f'(x) = \frac{d}{dx}7x^{-6} - \frac{d}{dx}5\sqrt{x} = 7(-6)x^{-6-1} - \frac{5}{2\sqrt{x}}$$

$$= -42x^{-7} - \frac{5}{2\sqrt{x}}$$

Lec-4

10. 2. 25

Product rule :-

If $f(x)$ & $g(x)$ both are differentiable so, $f \cdot g$ are,

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

or

$$(f\cdot g)' = f\cdot g' + g\cdot f'$$

Find $\frac{dy}{dx}$, where $y=(4x^3+1)(7x^2+x)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[(4x^3+1)(7x^2+x)\right]$$

$$= (4x^3+1)\frac{d}{dx}(7x^2+x) + (7x^2+x)\frac{d}{dx}(4x^3+1)$$

$$= (4x^3+1)(14x+1) + (7x^2+x)(12x^2)$$

$$\left.\frac{dy}{dx}\right|_{x=2} = (32+1)(28+1) + (28+2)\cdot 48$$

$$= 2397$$

Quitient Rule :

If f & g are both differrentiable and $g(x) \neq 0$

so, $\frac{f(x)}{g(x)}$ also differrentiable :-

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

$\boxed{\text{or}}$

$$\left(\frac{f}{g}\right)' = \frac{g.f' - fg'}{g^2}$$

Trigonometric Function :-

1. $\frac{d}{dx}\sin x = \cos x$
2. $\frac{d}{dx}\cos x = -\sin x$
3. $\frac{d}{dx}\tan x = \sec^2 x$
4. $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$
5. $\frac{d\cancel{x}}{dx}\sec x = \sec x \cdot \tan x$
6. $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x$

$y = \frac{\sin x}{1+\cos x} =$

Chain Rule :-

f & g are both differetiable . say $y = f(g(x))$ & $g(x) = u$

$y = f(u)$

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x)$$

$$\boxed{\frac{dy}{dx} = f'(g(x)) \cdot g'(x)}$$

logarithmus naturalis

ln

next quiz important

$f(x) = \sqrt{x+1}$, $g(x) = x^2 - 2$ Find $F'(x)$ where $F(x) = f(g(x))$

$\Rightarrow F'(x) = (f(g(x)))' = f'(g(x)) \cdot g'(x)$

$f(x) = \sqrt{x+1}$

$y = \sin(2x^2+1)$

$\frac{dy}{dx} = \cos(2x^2+1) \cdot 4x = 4x\cos(2x^2+1)$

$\frac{d}{d\theta}\cos 2\theta = -\sin 2\theta \frac{d}{d\theta}(2\theta) = -2\sin(2\theta)$

Lec-5

24.2.25

$f(x) = x^2+1$

f'

Implecit Function:

$x^2y + \sin y = 2xy$

$\frac{d}{dx}[x^2y + \sin y] = \frac{d}{dx}2xy$

$\frac{d}{dx}x^2y + \frac{d}{dx}\sin y = 2\frac{d}{dx}xy$

$x^2\frac{dy}{dx} + y2x + \cos y\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$

$$\frac{dy}{dx}\left[x^2+\cos y+2x\right] = 2y+2xy$$

$$\frac{dy}{dx} = \frac{2y-2xy}{x^2+\cos y-2x}$$

Logarithms & Exponents

i. $\frac{d}{dx}\ln x = \frac{1}{x}$

ii. $\frac{d}{dx}\log_b x = \frac{1}{\ln b}\frac{d}{dx}\ln x = \frac{1}{x\ln b}$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

$$= \frac{\ln x}{\ln b}$$

iii. $\frac{d}{dx}e^{mx} = e^{mx}\frac{d}{dx}mx = me^{mx}$

iv. $\frac{d}{dx}b^x = b^x\ln b$

Say, $y=(x^2+1)^{\sin x}$, $\ln y = \sin \ln(x^2+1)$, $\frac{dy}{dx} = ?$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\sin x\ln(x^2+1)\right] = \frac{\sin x\cdot 2x}{x^2+1} + \cos x\ln(x^2+1)$$

$$\frac{dy}{dx} = y\left\{\frac{2x\sin x}{x^2+1} + \cos x\ln(x^2+1)\right\}$$

@

$y = (x^2+1)^{\sin x}$

Inverse trigonometric Function

i. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

ii. $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$

iii. $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

iv. $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

v. $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

v. $\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$

$y = \sin^{-1}x^3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(x^3) = \frac{1}{\sqrt{1-x^6}} \cdot 3x^2$

L'Hsosptials Rule:

f & g bot differentiable & $\lim_{x\to a} f(x) = 0$ $\lim_{x\to a} g(x) = 0$

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$\lim_{x \to 0} \frac{e^x - 1}{x^3} = \lim_{x \to 0} \frac{e^x}{3x^2} = +\infty$

$\infty + \infty, \ 0 \cdot \infty, \ 0^0, \ 1^\infty, \ \infty^\infty \to \frac{0}{0}, \frac{\infty}{\infty}$

$y = \lim_{x \to \frac{\pi}{4}} (1 - \tan x) \sec 2x \qquad [0, \infty]$

$= \lim_{x \to \frac{\pi}{4}} \frac{(1 - \tan x)}{\cos 2x}$

$= \lim_{x \to \frac{\pi}{4}} \frac{-\sec^2 x}{-2\sin 2x}$

$= \frac{-(\sqrt{2})^2}{-2} = \frac{-2}{-2} = 1$

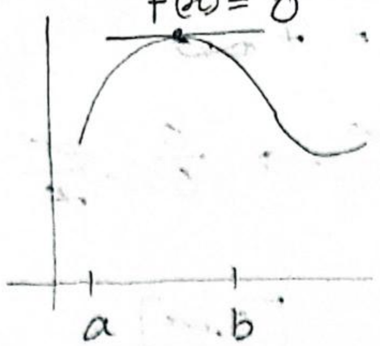
Lec-7 3·3·25

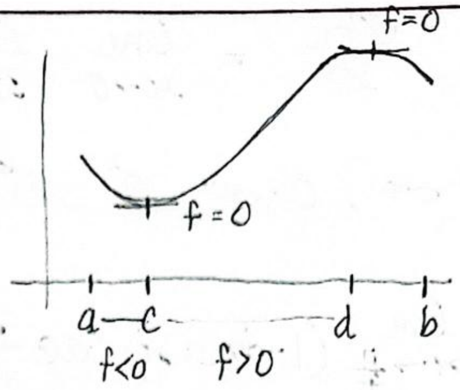
Increasing, Decreasing, Continuity

i. If $f'(x) > 0 \ \forall x \in (a, b)$, then f is increasing on $[a, b]$

ii. If $f'(x) < 0 \ \forall x \in (a, b)$, then f is decreasing on $[a, b]$

iii. If $f'(x) = 0 \ \forall x \in (a, b)$ then f is constant on $[a, b]$





四 Find the intervals on which $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ is increasing, decreasing, and concavity.

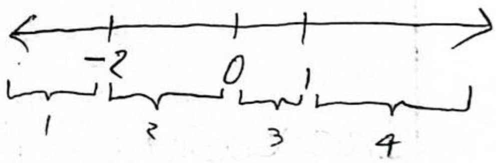
$\Rightarrow F(x) = f'(x) = 12x^3 + 12x^2 - 24x$

For critical point, $f'(x) = 0$.

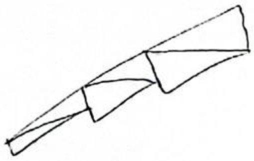
$$12x^3 + 12x^2 - 24x = 0$$

$$12x(x+2)(x-1) = 0$$

$x = 0, -2, 1$



Interval	$12x(x+2)(x-1)$		$f'(x)$	Conclusion
$x < -2$	$(-)(-)(-)$		$-$	f is decreasing on $(-\infty, \ldots$
$-2 < x < 0$	$(-)(+)(-)$	$x=-2$ min	$+$	f is increasing on $[-2, \ldots$
$0 < x < 1$	$(+)(+)(-)$	$x=0$ max	$-$	f " decreasing on $[0, 1 \ldots$
$x > 1$	$(+)(+)(+)$	$x=1$ min	$+$	f " " on $[1, \infty \ldots$



Concavity

Concave up

$f''(x) > 0$

Concave down

$f''(x) < 0$

i. If $f''(x) > 0 \quad \forall x \in (a,b)$ then f is concave up on (a,b)

ii. If $f''(x) < 0$ 〃 〃 〃 f is 〃 down on (a,b)

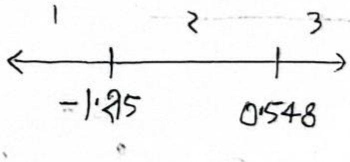
$F(x) = f'(x)$

$F'(x) = f''(x) = 36x^2 + 24x - 24 = 0$

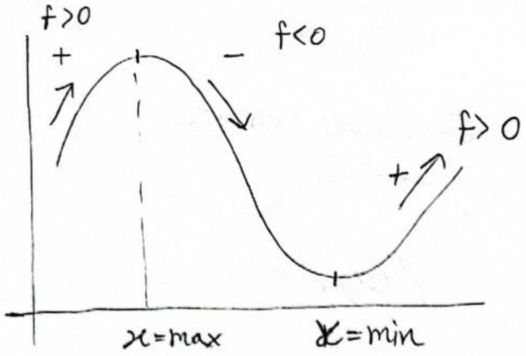
$$36x^2 + 24x - 24 = 0$$

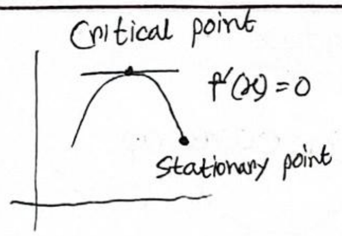
$$x = \frac{-1 \pm \sqrt{7}}{3}$$

$$x = \begin{cases} \frac{-1-\sqrt{7}}{3} = -1.215 \\ \frac{-1+\sqrt{7}}{3} = 0.548 \end{cases}$$



Interval	$f''(x)$	conclusion
$x < -1.215$	+	f is concave up on $(-\infty, -1.215)$
$-1.215 < x < 0.548$	-	f is 〃 down 〃 $(-1.215, 0.548)$
$x > 0.548$	+	up $(0.548, +\infty)$





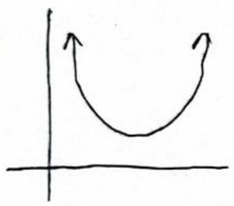
Lec-8

Extrema (max/min value)

- → relative extrema
- → Absolute extrema

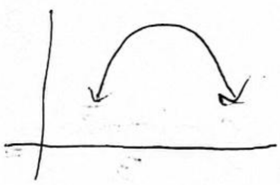
+ to - max

- to + min



Absolute maxina = X

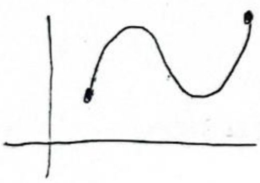
" Minimia = Maybe



A. max = m

A.min = X

Closed



has Absolute maxima/Minima

Need Boundary point & critical point for absolute maxima & minim

Quiz-2
Wed

i. Differentiability
ii. Technic of differrention
iii. Increasing, decreasing concavity
iv. Max / Min.

Find the absolute extrema of the function:

$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ on the interval $[-5, 5]$

$\Rightarrow$ For Critical point, $f'(x) = 0$

$$\Rightarrow 12x^3 + 12x^2 - 24x = 0$$

$$\Rightarrow 12x(x+2)(x-1) = 0$$

$$x = 0, -2, 1$$

For absolute extrema:

$f(0) = 2$

$f(-2) = -30$

$f(1) = -3$

$f(-5) = 1077$

$f(5) = 2077$

Absolute maxima value is 2077 at $x = 5$

And absolute minima value is -30 at -2

* Successive Differentiation

$$y_1 = \frac{dy}{dx}$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy_1}{dx}$$

calculus always ln

If $y = \log(x+\sqrt{x^2+1})$, show that $(1+x^2)y_2 + xy_1 = 0$

$\Rightarrow y = \log(x+\sqrt{x^2+1})$

$$y_1 = \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x\right]$$

$$y_1 = \cancel{\frac{x}{(x+\sqrt{x^2+1})\sqrt{x^2+1}}} \qquad \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$$

$$y_1 = \frac{\frac{\cancel{x+\sqrt{x^2+1}}}{\sqrt{x^2+1}}}{\cancel{x+\sqrt{x^2+1}}}$$

$$y_1 = \frac{1}{\sqrt{x^2+1}}$$

$$y_1\sqrt{x^2+1} = 1$$

$$y_1^2(x^2+1) = 1$$

Differnetiate both side in terms of x

uv $\rightarrow$ $(x^2+1)2y_1y_2 + y_1^2(2x+0) = 0$

$$2y_1\{(1+x^2)y_2 + y_1x\} = 0$$

$\therefore (1+x^2)y_2 + y_1x = 0$ (Showed)

88

Find n^{th} differential Coefficient of $\frac{1}{(1+x)^2}$

$\Rightarrow y = (1+x)^{-2}$

$y_1 = (-2)(1+x)^{-3}$

$y_2 = (-3)(-2)(1+x)^{-4} = (-1)^2 2\cdot3\cdot\cancel{4}\,(1+x)^{-(2+2)}$

$y_3 = (-2)(-3)(-4)(1+x)^{-5} = (-1)^3 2\cdot3\cdot4\,(1+x)^{-(3+2)}$

$y_2 = (-1)^2\, 3!\,(1+x)^{-(2+2)} = (-1)^2 (2+1)!\,(1+x)^{-(2+2)}$

$y_3 = (-1)^3\, 4!\,(1+x)^{-(3+2)} = (-1)^3 (3+1)!\,(1+x)^{-(3+2)}$

So, $y_n = (-1)^n (n+1)!\,(1+x)^{-(n+2)}$

Tylors polynominal :

$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$

Lec-9

10.3.25

$$y_1 = \frac{dy}{dx} \qquad y_2 = \frac{dy_1}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

If $y = a\cos(\log x) + b\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$

$$y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$$

$$xy_1 = -a\sin\log x + b\cos(\log x)$$

$$xy_2 + y_1 \cdot 1 = -a\cos(\log)\frac{1}{x} - b\sin(\log)\frac{1}{x}$$

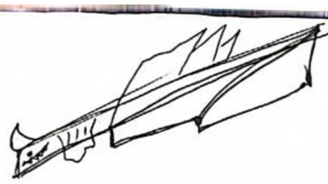
$$x(xy_2 + y_1) = -\underbrace{[a\cos(\log x) + b\sin(\log x)]}_{y}$$

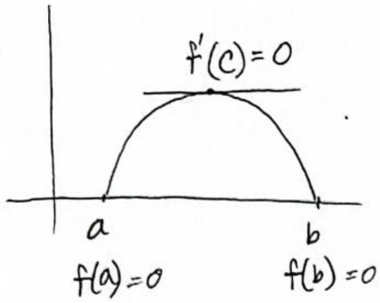
$$x^2y_2 + xy_1 = -y$$

$$\therefore x^2y_2 + xy_1 + y = 0 \quad \text{[showed]}$$

* Rolles Theorem:

Consider f be continuous on closed interval $[a,b]$ and differentiable on (a,b). If $f(a) = 0$ and $f(b) = 0$, then there is at ~~be~~ least one point C is (a,b) such that $f'(c) = 0$.





Verify that the hypothesis of Rolles' theorem are satisfy on the given interval and find all values of C in that interval that satisfy the conclusion of the theorem.

$$f(x) = x^2 - 5x + 4 \quad \text{on } [1,4]$$

$\Rightarrow f(1) = 1^2 - 5 \cdot 1 + 4 = 0$

$f(4) = 4^2 - 5 \cdot 4 + 4 = 0$

$f'(x) = 2x - 5 = 0$

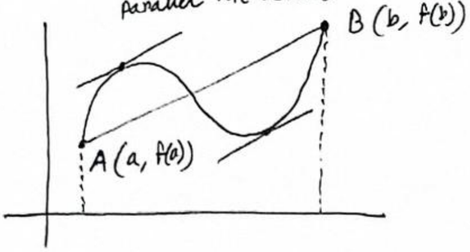
$f'(c) = 2C - 5 = 0 \quad \therefore \quad C = 5/2 \in (1,4)$

$C = 1 \notin (1,4)$
$C = 4 \notin (1,4)$

* The mean value theorem :-

Consider f be continuous on closed interval $[a,b]$ and differentiable on (a,b) Then there is at least one point C in (a,b) such that $f'(C) = \frac{f(b) - f(a)}{b-a}$ → Average velocity

Instatenuous velocity



Determine all the numbers 'C' which satisfy the Conclusion of the Mean value Theorem for the following function. $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$

* every polynominial function is differrentiable and continuous.

$\Rightarrow$ $f'(x) = 3x^2 + 4x - 1$

$f(-1) = 1^3 + 2 \cdot 1^2 + 1 = 2$

$f(2) = 2^3 + 2 \cdot 2^2 - 2 = 14$

According to MVT

$f'(c) = 3c^2 + 4c - 1 = \frac{14-2}{2-(-1)} = \frac{12}{3} = 4$

$3c^2 + 4c - 5 = 0$, $c = \frac{-4 \pm \sqrt{76}}{6} = 0.7863 \in (-1, 2)$

$-2.1196 \notin (-1, 2)$

四 Suppose we know that $f(x)$ is continuous and differentiable on $[6, 15]$ let's also suppose, we know that $f(6) = -2$ & $f'(x) \leq 10$. What is the largest possible value for $f(15)$?

$\Rightarrow \; f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$f'(c) \leq 10$

By MVT, $f'(c) = \cancel{16} \dfrac{f(15) - f(6)}{15-6} = \dfrac{f(15) + 2}{9}$

$9f'(c) = f(15) + 2$

A/Q, $f'(x) \leq 10$

So, in particular we know that $f'(c) \leq 10$

Thus $f(15) = 9f'(c) - 2 \leq 9 \cdot 10 - 2 = 88$

Which means the possible largest value of $f(15)$ is 88.

Lec-10

12.3.25

Maclorin Polynominal :

If f can be differrentiable in time at 0, then we define the n^{th} Maclorin polynominal for f to be

$$P_n(x) = \underbrace{\underbrace{f(0)}_{P_0} + f'(0)x}_{P_1} + \frac{f''(0)}{2!}x^2 + \cdots$$

with P_2 marking $f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$.

$P_0 = f(0)$, $P_1 = f(0) + f'(0)x$, $P_2 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$

$$P_n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n}{n!}x^n$$

□ Find the Maclorin polynomial P_0, P_1, P_2, P_3 & P_n for e^x

⇒ Say $f(x) = e^x$, $f(0) = e^0 = 1$

$f''(x) = e^x$, $f''(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = 1$

⋮

$f^n(x) = e^x$, $f^n(0) = 1$

$$P_n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^n}{n!}x^n$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

Tylor Polinomial :

If f can be differrentiable in time at $x = x_0$ then we define the n^{th} Taylor polynomial for f at $x=x_0$ to be

$$P_n = f(x_0) + f'(x_0)(x-x_0) + \frac{f''}{2!}(x_0)(x-x_0)^2 + \cdots \frac{f^n}{n!}(x_0)(x-x_0)^n$$

回 Find n^{th} taylor polynomial for $1/x$ about $\boxed{x = 1}$ and express it in sigma notation :-

$\Rightarrow$ Say $f(x) = \frac{1}{x} = x^{-1}$ $\qquad f(1) = 1$

$f'(x) = (-1)x^{-2}$ $\qquad f'(1) = (-1)\cdot 1$

$f''(x) = (-1)(-2)x^{-3}$ $\qquad f''(1) = (-1)(-2) = (-1)^2 \cdot 1 \cdot 2 = (-1)^2\, 2!$

$f'''(x) = (-1)(-2)(-3)x^{-4}$ $\qquad f'''(1) = (-1)(-3) = (-1)^3 \cdot 1 \cdot 2 \cdot 3 = (-1)^3\, 3!$

$\vdots$

$f^n(x) = (-1)(-2)(-3) \cdots (-n)x^{-(n+1)}$

$$f^{(n)}(1) = (-1)(-2)(-3)\cdots(-n)\cdot 1 = (-1)^n n!$$

$P_0 = f(1) = 1$

$P_1 = f(1) + f'(1)(x-1) = 1 + (-1)\cdot(n-1)$

$P_2 = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 1 + (-1)(n-1) + \frac{(-1)^2}{2!}(x-1)^2$

$\vdots$

$P_n = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots + \frac{f^n(1)}{n!}(x-1)^n$

$= 1 + (-1)$

Lec-11

9 · 4 · 25

Partial Derivatives :-

80% P.D
20% Conics

$y = f(x)$ $\qquad \frac{dy}{dx} = f'(x)$

↓

Single

$f(x, y)$

↓

Double

$f(x,y) = x^2 y$

$f_x(x,y) = y\frac{\partial}{\partial x} x^2 = 2x \cdot y$

$f_y(x,y) = x^2 \frac{\partial}{\partial y} y = x^2$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Example: Suppose $f(x,y) = \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}$. Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$.

$$\Rightarrow f_x(x,y) = \cos\left(\frac{4}{x}\right)\frac{\partial}{\partial x}\left(e^{x^2y-5y^3}\right) + e^{x^2y-5y^3}\frac{\partial}{\partial x}\cos\left(\frac{4}{x}\right)$$

$$= \cos\left(\frac{4}{x}\right)e^{x^2y-5y^3}\underbrace{\frac{\partial}{\partial x}(x^2y-5y^3)}_{2xy-0} + e^{x^2y-5y^3}\left(-\sin\frac{4}{x}\right)\frac{\partial}{\partial x}\left(\frac{4}{x}\right)$$

$$-\frac{4}{x^2}$$

$$f_y(x,y) = \cos\frac{4}{x}\frac{\partial}{\partial y}\left(e^{x^2y-5y^3}\right) + e^{x^2y-5y^3}\frac{\partial}{\partial y}\cos\left(\frac{4}{x}\right)$$

$$= \cos\frac{4}{x}\, e^{x^2y-5y^3}\frac{\partial}{\partial y}(x^2y-5y^3) + 0$$

$$= \cos\frac{4}{x}\, e^{x^2y-5y^3}(x^2-15y^2)$$

Find the slop of the surface,

$f(x,y) = xe^{-y} + \ln(x)\,y^2$ in the x- direction at point $(4,0)$

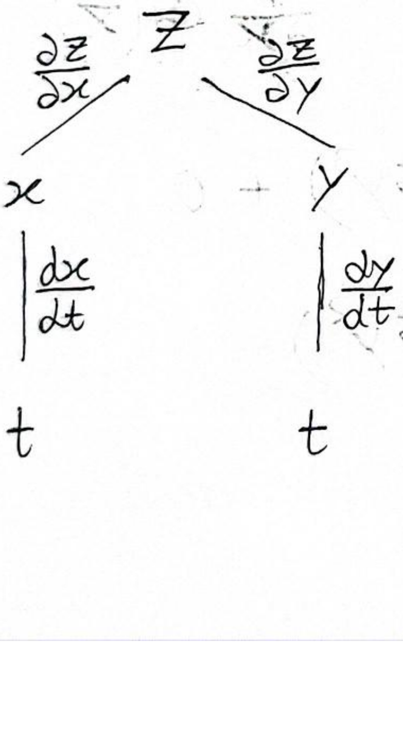
$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(4,0)} = f_x\Big|_{(4,0)}$$

$$f_x(x,y) = e^{-y} + \frac{1}{x}y^2$$

$$f_x(4,0) = e^{0} + \frac{1}{4}\cdot 0 = 1$$

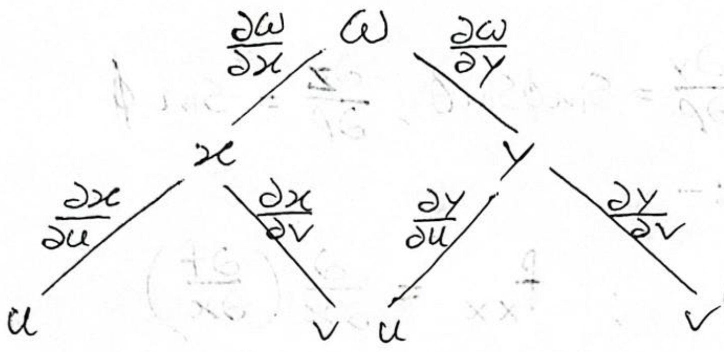
<u>Chain Rule for P.D</u> :-

$$z = f(x,y), \quad x = x(t), \quad y = y(t)$$



$$\boxed{\frac{dz}{dt} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt}}$$

$\omega = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$



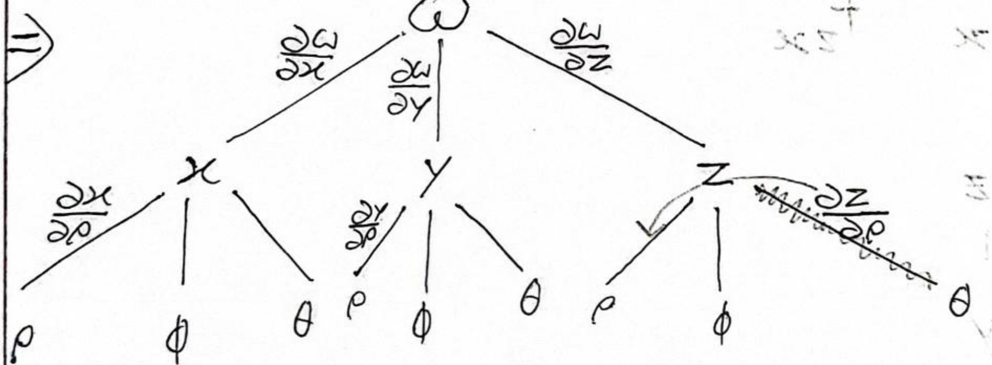
$$\frac{\partial \omega}{\partial u} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial \omega}{\partial v} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Suppose $\omega = x^2 + y^2 - z^2$ and $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$ & $z = \rho \sin\phi$. Find $\frac{\partial \omega}{\partial \rho}$, $\frac{\partial \omega}{\partial \phi}$, $\frac{\partial \omega}{\partial \theta}$

$\Rightarrow$



$$\frac{\partial \omega}{\partial \rho} = \frac{\partial \omega}{\partial x} \cdot \frac{\partial x}{\partial \rho} + \frac{\partial \omega}{\partial y} \cdot \frac{\partial y}{\partial \rho} + \frac{\partial \omega}{\partial z} \cdot \frac{\partial z}{\partial \rho}$$

$$\frac{\partial \omega}{\partial x} = 2x \,, \quad \frac{\partial \omega}{\partial y} = 2y \,, \quad \frac{\partial \omega}{\partial z} = -2z$$

$$\frac{\partial x}{\partial \rho} = \sin\phi\cos\theta \,, \quad \frac{\partial y}{\partial \rho} = \sin\phi\sin\theta \,, \quad \frac{\partial z}{\partial \rho} = \sin\phi$$

* Higher order P.D :-

$$\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \quad ; \quad f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

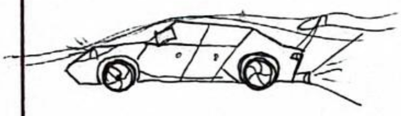
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = f_{yx}$$

$$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$f(x, y, z)$

$$\begin{array}{ccc} f_{yx} & f_{xy} & f_{xz} \\ & \parallel & \parallel \\ f_{yy} & f_{yx} & f_{zx} \\ & & \\ f_{zz} & f_{yz} & \\ & \parallel & \\ & f_{zy} & \end{array}$$



Find the 2^{nd} Order P.D. of the following function :-

$f(x,y) = x^2y^3 + x^4y$

$f_x(x,y) = 2xy^3 + 4x^3y$

$f_{xx} = 2y^3 + 12x^2y$

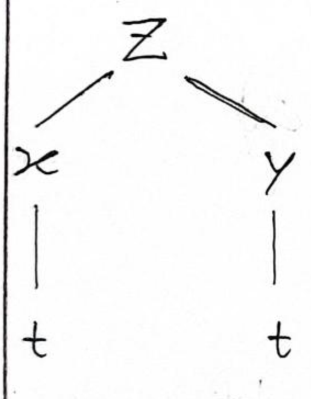
$f_y(x,y) = 3x^2y^2 + x^4$

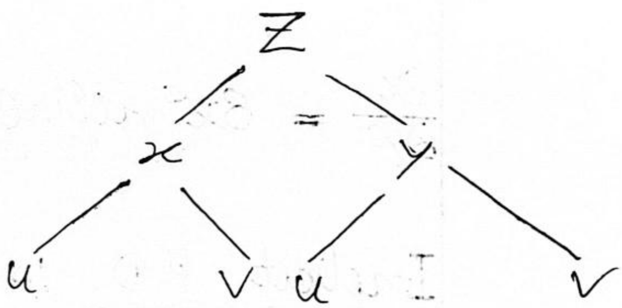
$f_{yy} = 3x^2y$

$f_{xy} = 6xy^2 + 4x^3 = f_{yx}$

Lec-13

16.4.25





$$\frac{dz}{dt} = \frac{\partial z}{\partial x} * \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

↓
Single

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

↓ more

Suppose that $\omega = 4x^2 + 4y^2 - z^2$ and

$x = \rho\sin\phi\sin\theta$, $y = \rho\sin\phi\cos\theta$, $z = \rho\cos\phi$ and

Find $\frac{\partial\omega}{\partial\rho}$, $\frac{\partial\omega}{\partial\phi}$, $\frac{\partial\omega}{\partial\theta}$.

$$\Rightarrow \frac{\partial\omega}{\partial\rho} = \frac{\partial\omega}{\partial x}\cdot\frac{\partial x}{\partial\rho} + \frac{\partial\omega}{\partial y}\cdot\frac{\partial y}{\partial\rho} + \frac{\partial\omega}{\partial z}\cdot\frac{\partial z}{\partial\rho}$$

$\omega = 4x^2 + 4y^2 - z^2$

$\frac{\partial\omega}{\partial x} = 8x$, $\frac{\partial\omega}{\partial y} = 8y$, $\frac{\partial\omega}{\partial z} = -2z$

$\frac{\partial x}{\partial\rho} = \sin\phi\sin\theta$, $\frac{\partial y}{\partial\rho} = \sin\phi\cos\theta$, $\frac{\partial z}{\partial\rho} = \cos\phi$

$$\frac{\partial\omega}{\partial\rho} = 8x\sin\phi\sin\theta + 8y\sin\phi\cos\theta - 2z\cos\phi$$

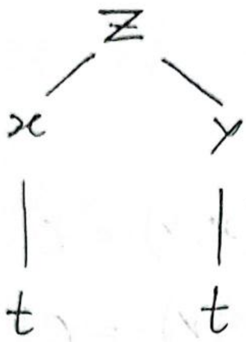
<u>Implicit P.D. :-</u>

Theorem: If the equation $f(x,y)$ defines y implicitly as a differentiable function of x & if $\frac{df}{dy} \neq 0$

then, $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x(x,y)}{f_y(x,y)}$$

$Z = f(x, y) = 0$

$\frac{dz}{dx} = \frac{d}{dx} C = 0$



$\frac{\partial t}{\partial y} \cdot \frac{dy}{dx} = -\frac{\partial t}{\partial x}$

$$\underset{0}{\frac{dz}{dx}} = \frac{\partial z}{\partial x} \cdot \underset{1}{\frac{\partial x}{\partial t}} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$\frac{dy}{dx} = -\frac{\partial t/\partial x}{\partial t/\partial y}$

$\boxed{}$ Consider the sphere $x^2+y^2+z^2 = 1$

Find $\frac{dz}{dx}$ & $\frac{dz}{dy}$ at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$; $\frac{dz}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial z}$

$\Rightarrow$ Let $f(x, y, z) = x^2+y^2+z^2-1$

$\frac{\partial f}{\partial x} = 2x$ $\qquad$ $\frac{\partial z}{\partial x} = -\frac{2x}{2z} = -\frac{x}{z}$

$\frac{\partial f}{\partial z} = 2z$ $\qquad$ $\left.\frac{dz}{dx}\right|_{(2/3,\, 1/3,\, 1/2)} = -\frac{2/3}{2/3} = -1$

Relative max/min for 2 variables :-

$f'(x) = 0$

$f(x, y)$

For critical point $f_x(x, y) = 0$

$f_y(x, y) = 0$

Extreme value theorem or Formats theorem :-

Let f be a function of 2 variables with continuous second-order partial derivatives in some disk central at a critical point (x_0, y_0) and let,

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

$f_{xy} = f_{yx}$

i. If $D > 0$ & $f_{xx}(x_0, y_0) > 0$ then f has a min at (x_0, y_0)

ii. If $D > 0$ & $f_{xx}(x_0, y_0) < 0$ " " " max at (x_0, y_0)

iii. If $D<0$, then f has a saddle point at (x_0, y_0)

Lec-13

21.4.25

Relative extrema :-

say (x_0, y_0) be a critical point.

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

i. if $D>0$ & $f_{xx}(x_0, y_0) > 0$, then f has a minimum value of (x_0, y_0)

ii. if $D>0$ & $f_{xx}(x_0, y_0) < 0$ " f " maximum " "

iii. if $D<0$, then f has a saddle point.

iv. if, $D=0$, No Conclusion.

⊞ Locate all relative extrema & saddle point if any :- of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$.

$\Rightarrow f_x(x,y) = 6x - 2y$

$f_y(x,y) = -2x + 2y - 8$

For critical point :

$f_x(x,y) = 0$ & $f_y(x,y) = 0$

$6x - 2y = 0$ — (i), $-2x + 2y - 8 = 0$ — (ii)

Solving (i) & (ii) $x = 2,\ y = 6$

So $(x,y) = (2,6)$ is the only critical point.

$f_{xx}(x,y) = 6,\quad f_{yy}(x,y) = 2,\quad f_{xy}(x,y) = -2$

at $(2,6)$

$D = 6 \cdot 2 - 4 = 8 > 0$

& $f_{xx}(2,6) = 6 > 0$

So, f has a minimum value

$$\boxed{f(2,6) = 3 \cdot 2^2 - 2 \cdot 2 \cdot 6 + 6^2 - 8 \cdot 6 = -24}$$

四 Locate all relative maxima & saddle point (if any) of $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$\Rightarrow f_x(x,y) = 6xy \cancel{+} 6x$ | $f_{xx}(x,y) = 6y - 6$

$f_y(x,y) = 3x^2 + 3y^2 - 6y$ | $f_{yy}(x,y) = 6y - 6$

For Critical point, $6xy - 6x = 0$ — (i)

$3x^2 + 3y^2 - 6y = 0$ — (ii)

(i) $\Rightarrow x(y-1) = 0$

$x = 0,\ y = 1$

(ii) $\Rightarrow y^2 - 2y = 0$ $[\because x = 0]$

$y(y-2) = 0$

$y = 0, 2$

$3x^2 + 3 - 6 = 0$ $[\because y = 1]$

$x^2 = 1$

$x = +1, -1$

$\therefore (x,y) = (0,0); (0,2); \cancel{(0,1)}$

$(1,1); (-1,1)$

at $(0,0)$

$D = (-6)(-6) - 0^2 = 36 > 0$, $f_{xx}(0,0) = -6 < 0$; R max

at $(0,2)$

$D =$

H.W. Find all relative extrema and saddle point (if any) of $f(x, y) = (6x - x^2)(4y - y^2)$

Absolute extrema for 2 variable :-

Find all absolute extrema :-

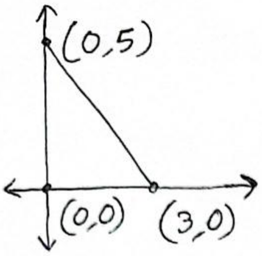
1. Find all the critical points of f that lie in the interior of $\mathbb{R}^2$.

2. Find all the boundary points at which absolute extrema can occur.

3. Evaluate $f(x, y)$ at points obtained in the processing steps. The largest of these values is the absolute extrema and the lowest is the absolute minimum.

例 Find the absolute extrema of $f(x, y) = 3xy - 6x - 3y + 7$ on the closed triangular region $\mathbb{R}^2$ with,

$(0, 0)$; $(3, 0)$ & $(0, 5)$

$\Rightarrow$ 

$f_x(x,y) = 3y - 6 = 0 \ ; y = 2$

$f_y(x,y) = 3x - 3 = 0 \ ; x = 1$

$\therefore$ Critical point $(1, 2)$

For line segment of $(0,0)$, $(3,0)$

here $y = 0$

So, $f(x,0) = 0 - 6x + 7 \qquad f_x(x,0) = -6 \neq 0$

So there is no critical point in this line segment.

For line segment of $(0,0)$, $(0,5)$

here $x = 0$

$f(0,y) = -3y + 7 \qquad f_y(0,y) = -3 \neq 0$

No critical point.

$(0,5)$ & $(3,0)$

$\dfrac{x-0}{0-3} = \dfrac{y-5}{5-0} \Rightarrow y = -\frac{5}{3}x + 5$

$f(x, -\frac{5}{3}x+5) = 3x\left(-\frac{5}{3}x+5\right) - 6x - 3\left(-\frac{5}{3}x+5\right) + 7$

$= -5x^2 + 14x - 8$

$f_x\left(x, -\frac{5}{3}x+5\right) = -10x + 14 = 0$

$$\therefore x = \frac{7}{5}$$

$\therefore y = \frac{2}{3}$

$(x, y) = \left(\frac{7}{5}, \frac{2}{3}\right)$

Lec-14

23·4·25

Lagrange Multipliers :-

$\vec{\nabla} \cdot f(x,y) = \frac{\partial}{\partial x} f \hat{i} + \frac{\partial}{\partial y} f \hat{j}$

$\vec{\nabla} \cdot f = \lambda \vec{\nabla} g$

$\Rightarrow \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \lambda \frac{\partial g}{\partial x}\hat{i} + \lambda \frac{\partial g}{\partial y}\hat{j} + \lambda \frac{\partial g}{\partial z}\hat{k}$

$\frac{\partial}{\partial x} f \hat{i} = \lambda \frac{\partial}{\partial x} g \hat{i}$; $\frac{\partial}{\partial y} f \hat{j} = \lambda \frac{\partial}{\partial y} g \hat{j}$; $\frac{\partial}{\partial z} f \hat{k} = \frac{\partial}{\partial z} g \hat{k}$

$$\boxed{f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z}$$

Use lagrange multipliers to find the max & min max & min values of $f(x,y,z) = 3x+6y+2z$ subject to constaint $2x^2+4y^2+z^2 = 70$.

$\Rightarrow f(x,y,z) = 3x+6y+2z$

$$2x^2+4y^2+z^2 = 70 \quad \text{--- (i)}$$

Say $g(x,y,z) = 2x^2+4y^2+z^2$

$f_x = 3 \quad f_y = 6 \quad f_z = 2 \quad g_x = 4x \quad g_y = 8y \quad g_z = 2z$

Applying lagrange multipliers.

$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_y$

$3 = \lambda 4x \qquad 6 = \lambda 8y \qquad 2 = \lambda 2z$

$x = \frac{3}{4\lambda}$ — (ii) $\qquad y = \frac{3}{\lambda 4}$ — (iii) $\qquad z = \frac{1}{\lambda}$ — (iv)

For eq (i) :-

$2\left(\frac{3}{4\lambda}\right)^2 + 4\left(\frac{3}{4\lambda}\right)^2 + \frac{1}{\lambda} = 70$

$\lambda^2 = \frac{1}{16} \qquad \therefore \lambda = \pm\frac{1}{4}$

For $\lambda = \frac{1}{4}$ $\quad x = 3, \quad y = 3, \quad z = 4$

For $\lambda = -\frac{1}{4}$ $\quad x = -3, \quad y = -3, \quad z = -4$

at $(3, 3, 4)$

$f(3, 3, 4) = 35 \longrightarrow$ Absolute max

at $(-3, -3, -4)$

$f(-3, -3, -4) = -34 \longrightarrow$ Absolate min

$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$ & $x^2 + y^2 + z^2 = 36$,

$\Rightarrow$ Let $g(x, y, z) = x^2 + y^2 + z^2$

$f(x, y, z) = (x^2 - 2x + 1) + (y^2 - 4y + 4) + (z^2 - 4z + 4)$

$f_x = 2x - 2 \qquad f_y = 2y - 4 \qquad f_z = 2z - 4$

$g_x = 2x \qquad g_y = 2y \qquad g_z = 2z$

Applying lagrange Multipliers :-

$2x - 2 = \lambda \cdot 2x \quad \cancel{\Rightarrow 1 - \frac{1}{x} = \lambda \Rightarrow}$

$x =$ $\quad \Rightarrow x - 1 = \lambda x$

$\frac{1}{1-\lambda} = \frac{1}{1-\frac{3}{2}} = \frac{2}{2-3} = -2$

$\Rightarrow 1 - \frac{1}{x} = \lambda \quad \Rightarrow \lambda + 1 = \frac{1}{x} \quad \therefore \boxed{x = \frac{1}{1\cancel{+}\lambda}}$ — (ii)

$2y - 4 = \lambda 2y \quad \Rightarrow y - 2 = \lambda y \quad \Rightarrow 1 - \frac{2}{y} = \lambda$

$\Rightarrow \cancel{\lambda} \quad 1 - \lambda = \cancel{2/y} \qquad \boxed{\therefore y = \frac{2}{1-\lambda}}$ — (iii)

$2z - 4 = \lambda 2z \qquad \boxed{\therefore z = \frac{2}{1-\lambda}}$ — (iv

Applying in (i)

$$\frac{1}{(1\cancel{+}\lambda)^2} + \frac{4}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 36$$

$$\Rightarrow \cancel{\frac{1-2\lambda+\lambda^2+4+8\lambda+4\lambda^2+4+8\lambda+4\lambda^2}{(1+2\lambda+\lambda^2)(1-2\lambda+\lambda^2)}} = 36$$

$$\Rightarrow \frac{\cancel{9}}{(1-\lambda)^2} = \cancel{36}\,4$$

$$\Rightarrow \frac{\cancel{9}\,1}{(1-\lambda)^2} = \cancel{36}\,4 \qquad 1 - \lambda = \pm\frac{1}{2} \qquad \boxed{\therefore \lambda = \frac{1}{2}, \frac{3}{2}}$$

$\cancel{\Rightarrow \frac{1}{2} = 1 - \lambda \quad \therefore \lambda = \frac{1}{2}}$

$x = 2, \quad y = 4, \quad z = 4 \quad (\lambda = \frac{1}{2})$

$x = -2, \quad y = -4, \quad z = -4 \quad (\lambda = \frac{3}{2})$

at $(2, 4, 4)$

$f(2, 4, 4) = 1^2 + 2^2 + 2^2 = 9$ —— (min)

at $(-2, -4, -4)$

$f(-2, -4, -4) = (-3)^2 + (-6)^2 + (-6)^2 = 81$ —— (max)

Lec - 15

Tylors polynomial for 2 variables. (2nd degree)

at $(x, y) = (x_0, y_0)$

f_{xx}
f_{yy}
f_{xy}
f_{yx}

$$f(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$+ \frac{1}{2!}\Big[f_{xx}(x_0, y_0)(x-x_0)^2 + f_{yy}(x_0, y_0)(y-y_0)^2 + \underbrace{f_{xy}(x_0, y_0)(x-x_0)(y-y_0) + f_{yx}(x_0, y_0)(y-y_0)(x-x_0)}_{2f_{xy}(x_0, y_0)(x-x_0)(y-y_0)}\Big]$$

Ex Find the 1st & 2nd degree tylor polynomial for

$f(x, y) = e^x \cos y$ at $(0, \pi/2)$

tylon
laglance }*

$$\Rightarrow f(x,y) = e^{x}\cos y \qquad f(0,\pi) = e^{0}\cos\pi = -1$$

$$f_x(x,y) = e^{x}\cos y \qquad f_x(0,\pi) = -1$$

$$f_{xx}(x,y) = e^{x}\cos y \qquad f_{xx}(0,\pi) = -1$$

$$f_y(x,y) = -e^{x}\sin y \qquad f_y(0,\pi) = 0$$

$$f_{yy}(x,y) = -e^{x}\cos y \qquad f_{yy}(0,\pi) = 1$$

$$f_{xy}(x,y) = -e^{x}\sin y \qquad f_{xy}(0,\pi) = 0$$

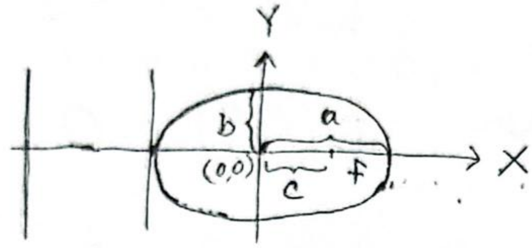
$$\therefore e^{x}\cos y = -1 + (-1)(x-0) + 0\cdot(y-\pi) + \frac{1}{2!}\left\{-1\cdot(x-0)^2 + 1\cdot(y-\pi)^2 + 2\cdot(x-0)(y-\pi)\cdot 0\right\}$$

$$= -1 - x + \frac{1}{2!}\left[-x^2 + (y-\pi)^2\right]$$

Lec-17

Ellipse :-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ , \quad a > b \qquad \boxed{a^2 - b^2 = c^2}$$



Center : $x = 0$, $y = 0$ (x - axis)

1. vertices - $(x = \pm a, \ y = 0)$

   Co- vertices - $(x = 0, \ y = \pm b)$

2. Foci - $(x = \pm c, \ y = 0)$

3. Eccentricity - $e = \frac{c}{a}$

4. Equation of directrix - $x = \pm \frac{a}{e}$

5. Equation of latus rectum - $x = \pm ae$

6. Length of Latus reclum - $LL' = \frac{2b^2}{a}$

P.d
Tylor
Lagrange
max/min

conversion
Conics
Chainrule

Hyperbola :- $\boxed{a^2+b^2=c^2}$ $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Transform the equ :- $x^2+9y^2-6x+18y-18=0$ into standard form of conic and find vertices, foci, equ of directrics, equ of latus rectum, length of latus rectum & eccentricity.

$\Rightarrow$ $x^2+9y^2-6x+18y-18=0$

Comparing this with $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

$A=1$, $B=0$, $C=9$, $D=-6$, $E=18$, $F=-18$

$$\Delta=\begin{vmatrix} A & \frac{1}{2}B & \frac{1}{2}D \\ \frac{1}{2}B & C & \frac{1}{2}E \\ \frac{1}{2}D & \frac{1}{2}E & F \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 0 & 9 & 9 \\ -3 & 9 & -18 \end{vmatrix} =$$

$x^2+9y^2-6x+18y-18=0$

$x-2\cdot3x+9+9(y^2+2y+1)-18-18=0$

$(x-3)^2+9(y+1)^2=36$

$\frac{(x-3)^2}{36}+\frac{(y+1)^2}{4}=1$

$$\frac{(x-3)^2}{6^2} + \frac{(y+1)^2}{2^2} = 1^2$$

$X = x-3$, $Y = y+1$, $a = 6$, $b = 2$

Lec-18

7.5.25

Conversion :- i. Rectangular ii. Cylindrical iii. Sherical

(x, y, z) (r, θ, z) (ρ, θ, ϕ)

Cylindrical → Rectangular: $(r, \theta, z) \rightarrow (x, y, z)$

$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

Rectangular → Cylindrical: $(x, y, z) \rightarrow (r, \theta, z)$

$r = \sqrt{x^2+y^2}$, $\tan\theta = \frac{y}{x}$, $z = z$

Sherical → Cylindrical : $(\rho, \theta, \phi) \rightarrow (r, \theta, z)$

$r = \rho\sin\phi$ $\theta = \theta$ $z = \rho\cos\phi$

Cylindrical → shperical :- $(r, \theta, z) \rightarrow (\rho, \theta, \phi)$

$\rho = \sqrt{r^2 + z^2}$, $\theta = \theta$, $\tan\phi = \frac{r}{z}$

Shperical → Rectangular :-

$x = r\cos\theta = \rho\sin\phi\cos\theta$

$y = r\sin\theta = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

rectangular → shepical :-

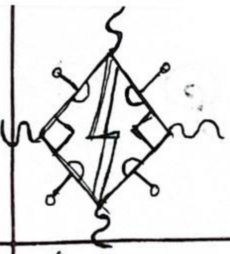
Lec-19

$$\begin{tikzcd}
(x, y, z) \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow, "\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{array}"] & \\
& (r, \theta, z) \arrow[dl, "\begin{array}{l} r = \rho\sin\phi \\ \theta = \theta \\ z = \rho\cos\phi \end{array}"] \\
(\rho, \theta, \phi) &
\end{tikzcd}$$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$



$f_x \quad f_x = \lambda g_x \quad f_y = \lambda g_y \quad f_z = \lambda g_z$

$x = 2\lambda$

$f(x, y, z) = \frac{x}{3} + 3y + z \;, \quad 2x^2 + 4y^2 + z^2 = 35$ —(i)

$g(x, y, z) = 2x^2 + 4y^2 + z^2$

$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z$

$\frac{1}{3} = \lambda 4x \qquad 3 = \lambda 8y \qquad 1 = \lambda \cdot 2z$

$x = \frac{1}{12\lambda}$ —(ii) $\quad y = \frac{3}{8\lambda}$ —(iii) $\quad z = \frac{1}{2\lambda}$ —(iv)

$L(x, y) = f(A, B) + f_x(A, B)(x - A) + f_y(A, B)(y - B)$

$Q(x, y) = \frac{1}{2!}\Big[f_{xx}(A, B)(x - A)^2 + f_{yy}(A, B)(y - B)^2 + 2f_{xy}(A, B)(x - A)(y-$