Maxima and Minima (Extrema)

Extrema

- → Relative Maxima/Minima
- → Absolute Maxima/Minima

Now we want to know about critical point and stationary point for better understanding ~~our~~ ~~row~~ to find extrema.

Critical point: We define a critical point for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable.

~~Turning point~~ / **Stationary point**: A stationary point ~~turning~~ of f is also a critical point if $f'(x) = 0$.

Theorem: Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extrema at $x = x_0$, then $x = x_0$ is a critical point of f; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.

e.g. Find the critical point of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Sln. $$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

at $x = 0$, $f'(0)$ is undefined

at $x = 2$, $f'(2) = 0$

So, $x = 0, 2$ are critical points but $x = 2$ is stationary point.

Relative Extrema

For First Derivative Test, we use the sign ~~to~~ change in between two intervals. If the sign of f' changes from "+" to "−" at $x=a$, then there is a relative maximum at that point. If the sign changes from "−" to "+" sign, then there is a relative minimum at that point.

We also use second derivative test for finding relative extrema.

Theorem: (Second derivative test)

Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x_0)=0$ and $f''(x_0)>0$, then f has a relative minimum at x_0.

(b) If $f'(x_0)=0$ and $f''(x_0)<0$, then f has a relative maximum at x_0.

(c) If $f'(x_0)=0$ and $f''(x_0)=0$, then the test is inconclusive; that is, f may have a relative maxima/minimum or neither at x_0.

The second derivative test is often easier to apply than the first derivative test. But second derivative test applies only at $\boxed{\text{stationary points}}$ where the second derivative exists.

Example: [For second derivative Test]

Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

We have $f(x) = 3x^5 - 5x^3$

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$$

$$f''(x) = 60x^3 - 30x = 30x(2x^2-1)$$

For $f'(x)=0$, $\Rightarrow 15x^2(x+1)(x-1)=0$ i.e. $x=0,-1,1$

These three points are called stationary point.

Now

Stationary Point	$30x(2x^2-1)$	$f''(x)$	Conclusion
$x=-1$	$(-30)\cdot 1$	$-$	f has a relative maxi
$x=0$	0	0	Inconclusive
$x=1$	$30\cdot 1$	$+$	f has a relative min.

—x—

Absolute Extrema

Theorem: If a function f is continuous on a finite closed interval $[a,b]$, then f has both an absolute maximum and ~~and~~ an absolute minimum on $[a,b]$.

This theorem is called Extreme-Value Theorem because of existence of extreme value.

* If f has an absolute extrema on an open interval (a,b), then it must occur at a critical point of f.

Procedure for finding the Absolute Extrema

Step 1: Find the critical points of f in (a,b).

Step 2: Evaluate f at all the critical points and at the endpoints a and b.

Step 3: The largest of the values in Step 2 is the absolute maximum value of f on $[a,b]$ and the smallest value is the absolute minimum.

—x—

Example: Find the relative / absolute extrema of the function $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ on an interval $[-5, 5]$.

Sol^n: Here

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$$

$$f'(x) = 12x^3 + 12x^2 - 24x$$

For critical point,

$$f'(x) = 0$$

$$\therefore 12x^3 + 12x^2 - 24x = 0$$

$$x(x+2)(x-1) = 0$$

$$x = 0, -2, 1$$

We know that absolute maxima/minima will be occured on critical point and endpoint of the closed interval.

Therefore,

$$f(0) = 2$$

$$f(-2) = -30$$

$$f(1) = -3$$

$$f(-5) = 1077$$

$$f(5) = 2077$$

Therefore ~~max~~ absolute maximum is 2077 at $x = 5$ and absolute minimum is -30 at $x = -2$.

* If the question mention "Find the interval for maximum and minimum of the given function". Then ~~you~~ we can use the sign change '+' to '−' or '−' to '+' method.