Lecture 3

Problem: Given that $f(x)=\begin{cases} 2x+3 & , x \le 4 \\ 7+\frac{16}{x} & , x>4 \end{cases}$. Does the function continuous at $x=4$?

$$\text{L.H.L.} = \lim_{x\to 4^-} f(x)$$

since $f(4) = 2.4+3 = 8+3 = 11$, defined

$$\lim_{x\to 4} (2x+3) = 11$$

$$\text{R.H.L.} = \lim_{x\to 4^+} f(x)$$

$$\lim_{x\to 4} \left(7+\frac{16}{x}\right) = 11$$

Thus $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4) = 11$

Therefore $f(x)$ is continuous at $x=4$.

Problem: Find a value of the constant K for which the following function is continuous at $x=1$:

$$f(x) = \begin{cases} 7x-2 & , x \le 1 \\ Kx^2 & , x>1 \end{cases}$$

Sol^n:

First we want to ~~define~~ show that the function $f(x)$ is defined at $x=1$.

$$f(1) = 7.1-2 = 5$$

Now, $\text{L.H.L.} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} (7x-2) = 5$

$$R.H.L = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1} Kx^2 = K$$

since the function is continuous at $x=1$ so,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = 5$$

Which means. $\lim_{x \to 1^+} f(x) = 5$

$$\lim_{x \to 1} K = 5 \Rightarrow \boxed{K = 5}$$

<u>Problem:</u> Find a value of the constant K s.t. it will make the following function continuous at $x = 2$:

$$f(x) = \begin{cases} Kx^2 & , x \leq 2 \\ 2x+K & , x > 2 \end{cases}.$$

<u>Solⁿ:</u> $\boxed{\text{DO YOURSELF !!!}}$

Recall the Rates of Change

We need to look at the rate of change problem.

Here we are going to consider a function $f(x)$ that represents some quantity that varies as x varies.

What we want to do here is determine just how fast $f(x)$ is changing at some point from $x = x_1$ to $x = x_0$ over the interval $[x_0, x_1]$. This is called average rate of change of $f(x)$ at $x = x_0$ over the interval $[x_0, x_1]$ and we write

$$r_{ave} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\text{change in } f(x)}{\text{Change in } x}$$

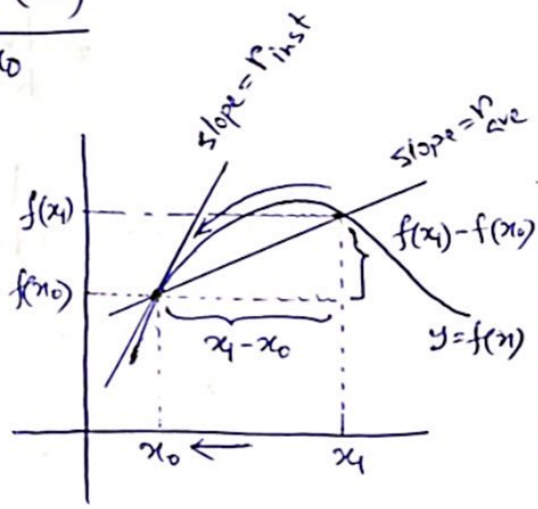
To estimate the instantaneous rate of change at $x = x_0$, we need to do is to choose values of x getting closer and closer to $x = x_0$.

For instantaneous rate of change we can write,

$$r_{inst} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

let $h = x_1 - x_0$, so

$$r_{inst} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



If the previous limit exists i.e. $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists, then it can be interpreted as the slope of the tangent line to the curve $y=f(x)$ at $x=x_0$.

This limit is so important that it has a special notation:

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Definition of derivative

The function f' defined by the formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is called the derivative of f with respect to x. $\square$

Example: Find the derivative with respect to x of $f(x)=x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h}, \quad h\neq 0 \\
&= 2x
\end{aligned}$$

$\square$

Problem: Show that
$$f(x)=\begin{cases} x^2+1 & , x\le 1 \\ 2x & , x>1 \end{cases}$$
is continuous and differentiable at $x=1$.

Sol^n: If the function is differentiable at $x=1$, then it must be continuous at $x=1$.

$\underline{f(x)=x^2+1}$

$f(1)=1^2+1=2$

$f(1+h)=(1+h)^2+1$
$=1+2h+h^2+1$
$=2+2h+h^2$

Now,
$$f'_-(x)=\lim_{h\to 0^-}\frac{f(x+h)-f(x)}{h}$$
$$f'_-(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{\cancel{2}+2h+h^2-\cancel{2}}{h}$$
$$=\lim_{h\to 0}\frac{h(2+h)}{h}=2$$

$\underline{f(x)=2x}$

$f(1)=2$

$f(1+h)=2(1+h)$
$=2+2h$

and,
$$f'_+(x)=\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$$
$$f'_+(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$\lim_{h\to 0}\frac{\cancel{2}+2h-\cancel{2}}{h}$$
$$\lim_{h\to 0}\frac{2h}{h}=2$$

Thus $f'_-(1)=f'_+(1)$

Therefore $f(x)$ is differentiable at $x=1$.

<u>Problem:</u> Show that $f(x) = \begin{cases} x^2+2 & x \le 1 \\ x+2 & x > 1 \end{cases}$

is continuous but not differentiable at $x = 1$.

$\boxed{\text{Do yourself !!!}}$

Problem: Find the derivative with respect to x of $f(x) = x^3 - x$.

Solⁿ:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - (x+h) - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 1)}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1)$$

$$= 3x^2 - 1$$

$\square$

Techniques of Differentiation

We apply the varios type of techiques for solving derivatives ~~differentiation~~ of functions.

For example:

(i) $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$

(ii) $\frac{d}{dx}[x^r] = rx^{r-1}$, r is any real number.

(iii) $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

Problem: Find $f'(x)$ where

(i) $f(x) = 7x^{-6} - 5\sqrt{x}$ (ii) $f(x) = x^e + \frac{1}{x^{10}}$

Solⁿ: (i) $f(x) = 7x^{-6} - 5\sqrt{x}$

$\sqrt{x} = x^{1/2}$

$$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\left[7x^{-6} - 5\sqrt{x}\right]$$

$$= 7\frac{d}{dx}x^{-6} - 5\frac{d}{dx}\sqrt{x}$$

$$= 7(-6)x^{-6-1} - 5\cdot\frac{1}{2}x^{1/2-1}$$

$$= -42x^{-7} - \frac{5}{2\sqrt{x}}$$

(ii) $$f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\left(x^e + \frac{1}{x^{10}}\right)$$

$$= \frac{d}{dx}x^e + \frac{d}{dx}x^{-10}$$

$$= ex^{e-1} + (-10)x^{-10-1}$$

$$= ex^{e-1} - 10x^{-11}$$

Problem: Find $f''(x)$ or $\frac{d^2}{dx^2} f(x)$.

ⓐ $f(x) = 7x^3 - 5x^2 + x$

$f'(x) = 21x^2 - 10x + 1$

$f''(x) = 42x - 10$

ⓑ $f(x) = (5x^2 - 3)(7x^3 + x)$

TRY YOURSELF

The Product Rule: If f and g are differentiable at x, then so is the product $f \cdot g$ and

$$\frac{d}{dx}[f(x)\, g(x)] = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x).$$

or

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

e.g. Find $\frac{dy}{dx}$ if $y = (4x^2 - 1)(7x^3 + x)$

$$\begin{aligned}\frac{dy}{dx} &= (4x^2-1)\frac{d}{dx}(7x^3+x) + (7x^3+x)\frac{d}{dx}(4x^2-1) \\ &= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) \\ &= 84x^4 + 4x^2 - 21x^2 - 1 + 56x^4 + 8x^2 \\ &= 140x^4 - 9x^2 - 1\end{aligned}$$

The Quotient Rule: If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

or

$$\left(\frac{f}{g}\right)' = \frac{g.f' - f.g'}{g^2}$$

Example: Find $y'(x)$ for $y = \dfrac{x^3+2x^2-1}{x+5}$

Sol^n: $$y = \frac{x^3+2x^2-1}{x+5}$$

$$y' = \frac{d}{dx}\frac{x^3+2x^2-1}{x+5}$$

$$= \frac{(x+5)\frac{d}{dx}(x^3+2x^2-1) - (x^3+2x^2-1)\frac{d}{dx}(x+5)}{(x+5)^2}$$

$$= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)\cdot 1}{(x+5)^2}$$

$$= \frac{3x^3+15x^2+4x^2+20x-x^3-2x^2+1}{(x+5)^2}$$

$$= \frac{2x^3+17x^2+20x+1}{(x+5)^2}$$

Trigonometric Function

~~Problem~~ Some Formula:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \ ; \ \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \ ; \ \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Problem: Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1+\cos x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)$$

$$= \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

□

Problem: Find $f'(x)$ if $f(x) = \dfrac{\sin x \sec x}{1+x\tan x}$.

$$f(x) = \frac{\sin x \frac{1}{\cos x}}{1+x\tan x}$$

$$= \frac{\tan x}{1+x\tan x}$$

$$f'(x) = \frac{(1+x\tan x)\frac{d}{dx}(\tan x) - \tan x \frac{d}{dx}(1+x\tan x)}{(1+x\tan x)^2}$$

$$= \frac{(1+x\tan x)\sec^2 x - \tan x\,(0+\tan x + x\sec^2 x)}{(1+x\tan x)^2}$$

$$= \frac{\sec^2 x + \cancel{x\tan x\sec^2 x} - \tan^2 x - \cancel{x\tan x\sec^2 x}}{(1+x\tan x)^2}$$

$$= \frac{\sec^2 x - \tan^2 x}{(1+x\tan x)^2}$$

$$= \frac{1}{(1+x\tan x)^2} \qquad \square$$