Example 1.47 (Positive and negative divergence). The field $\vec{\mathbf{F}} = x\hat{\imath} + y\hat{\jmath}$ has positive divergence $\vec{\nabla} \cdot \vec{\mathbf{F}} = 1 + 1 + 0 = 2 > 0$, thus we see in the left plot of Figure 11 that it is somewhat spreading. The field $\vec{\mathbf{G}} = (-x - 2y)\hat{\imath} + (2x - y)\hat{\jmath}$ has negative divergence $\vec{\nabla} \cdot \vec{\mathbf{G}} = -1 - 1 + 0 = -2 < 0$ and we see from the right plot that the field is converging.

$\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)$

$= 1 + 1$

$= 2$

$\frac{\partial}{\partial x}(-x-2y) + \frac{\partial}{\partial y}(2x-y)$

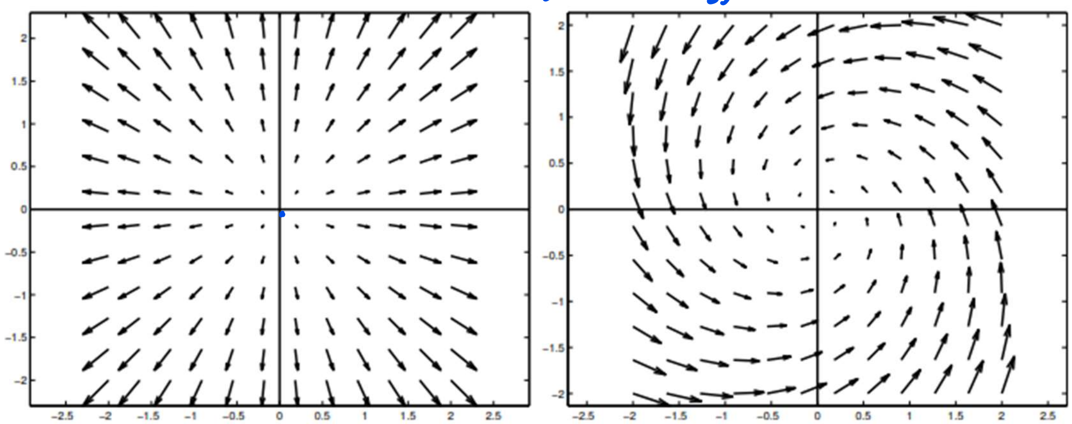


Figure 11: *A representation of the fields* $\vec{\mathbf{F}} = x\hat{\imath} + y\hat{\jmath}$ *(left) and* $\vec{\mathbf{G}} = (-x - 2y)\hat{\imath} + (2x - y)\hat{\jmath}$ *(right) from Example 1.47.* $\vec{\mathbf{F}}$ *has positive divergence and* $\vec{\mathbf{G}}$ *negative.*

Intuitively, the value of the divergence of a vector field at a particular point gives a measure of the "net mass flow" or "flux density" of the vector field in or out of that point. To understand what such a statement means, imagine that the vector field $\mathbf{F}$ represents velocity of a fluid. If $\nabla \cdot \mathbf{F}$ is zero at a point, then the rate at which fluid is flowing into that point is equal to the rate at which fluid is flowing out. Positive divergence at a point signifies more fluid flowing out than in, while negative divergence signifies just the opposite. We will make these assertions more precise, even prove them, when we have some integral vector calculus at our disposal. For now, however, we remark that a vector field $\mathbf{F}$ such that $\nabla \cdot \mathbf{F} = 0$ everywhere is called **incompressible** or **solenoidal**.

EXAMPLE 2 The vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ has

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2.$$

This vector field is shown in Figure 3.38. At any point in $\mathbf{R}^2$, the arrow whose tail is at that point is longer than the arrow whose head is there. Hence, there is greater flow *away* from each point than into it; that is, $\mathbf{F}$ is "diverging" at every point. (Thus, we see the origin of the term "divergence.")

The vector field $\mathbf{G} = -x\mathbf{i} - y\mathbf{j}$ points in the direction opposite to the vector field $\mathbf{F}$ of Figure 3.38 (see Figure 3.39), and it should be clear how $\mathbf{G}$'s divergence of -2 is reflected in the diagram. ◆

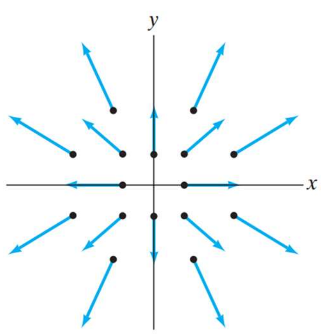


Figure 3.38 The vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of Example 2.

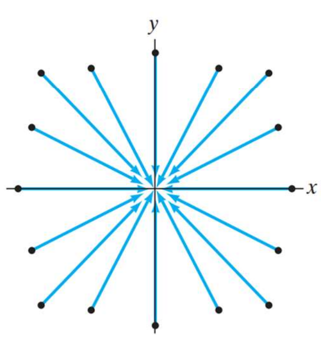


Figure 3.39 The vector field $\mathbf{G} = -x\mathbf{i} - y\mathbf{j}$ of Example 2.

EXAMPLE 3 The constant vector field $\mathbf{F}(x, y, z) = \mathbf{a}$ shown in Figure 3.40 is incompressible. Intuitively, we can see that each point of $\mathbf{R}^3$ has an arrow representing $\mathbf{a}$ with its tail at that point and another arrow, also representing $\mathbf{a}$, with its head there.

The vector field $\mathbf{G} = y\mathbf{i} - x\mathbf{j}$ has

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) \equiv 0.$$

A sketch of $\mathbf{G}$ reveals that it looks like the velocity field of a rotating fluid, without either a source or a sink. (See Figure 3.41.) ◆

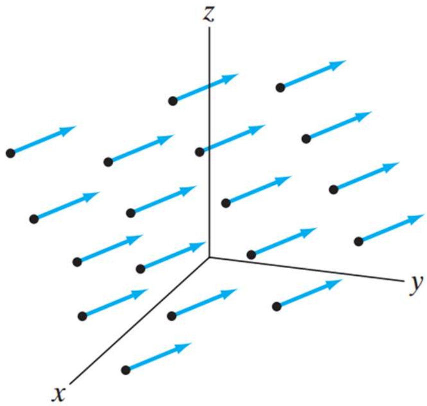


Figure 3.40 The constant vector field $\mathbf{F} = \mathbf{a}$.

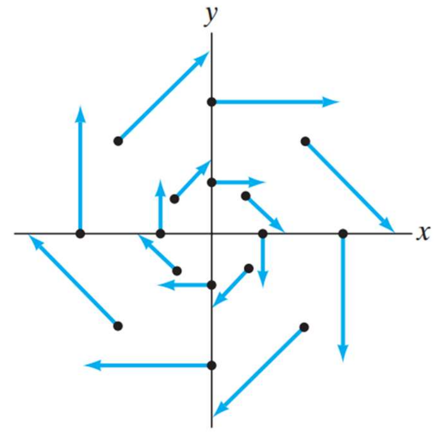


Figure 3.41 The vector field $\mathbf{G} = y\mathbf{i} - x\mathbf{j}$ resembles the velocity field of a rotating fluid.

1.5 Special vector fields and potentials

Quiz

Definition 1.65. Consider a vector field $\vec{\mathbf{F}}$ defined on a domain $D \subset \mathbb{R}^3$.
If $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is called **irrotational** (or curl-free).
If $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is called **solenoidal** (or divergence-free, or incompressible).
If $\vec{\mathbf{F}} = \vec{\nabla}\varphi$ for some scalar field φ, then $\vec{\mathbf{F}}$ is called **conservative** and φ is called **scalar potential** of $\vec{\mathbf{F}}$.
If $\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{A}}$ for some vector field $\vec{\mathbf{A}}$, then $\vec{\mathbf{A}}$ is called **vector potential** of $\vec{\mathbf{F}}$.

In mathematics and physics, a scalar field associates a scalar value to every point in a space.

13. Can you tell in what portions of $\mathbf{R}^2$, the vector fields shown in Figures 3.43–3.46 have positive divergence? Negative divergence?

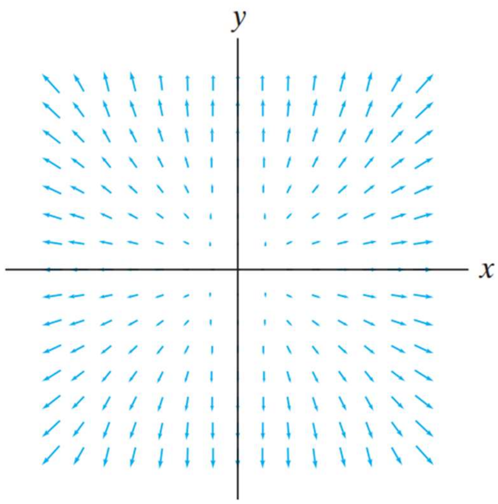
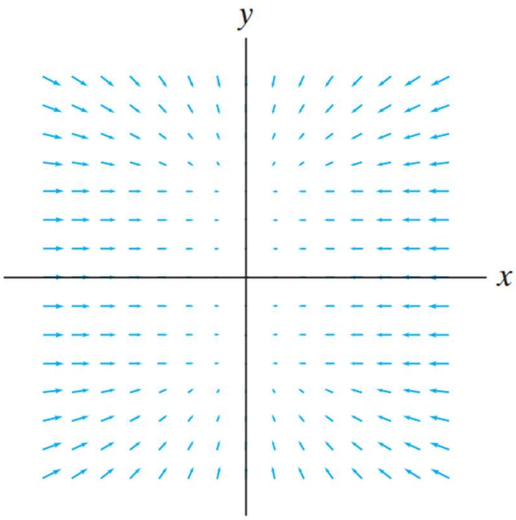


Figure 3.43 Vector field for Exercise 13(a).

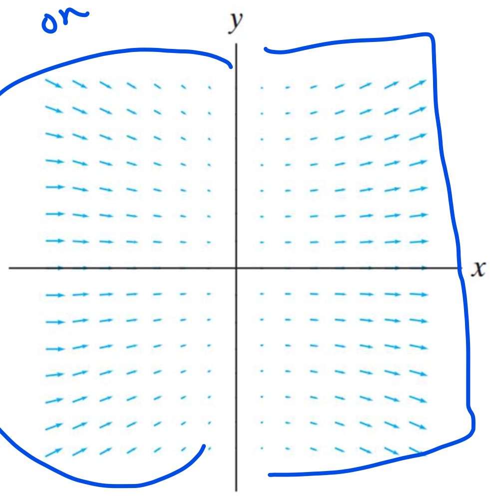
$\operatorname{div} F > 0$ on all of $\mathbb{R}^2$



div F < 0 on all of $\mathbb{R}^2$

Figure 3.44 Vector field for Exercise 13(b).

div F < 0 on $\{(x,y) \mid x < 0\}$



div F > 0 on $\{(x,y) \mid x > 0\}$

Figure 3.45 Vector field for Exercise 13(c).

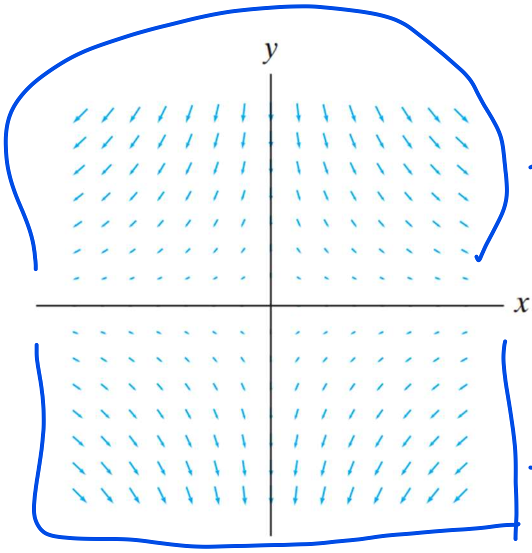


Figure 3.46 Vector field for Exercise 13(d).

div F < 0 on $\{(x,y) \mid y > 0\}$

div F > 0 on $\{(x,y) \mid y < 0\}$.