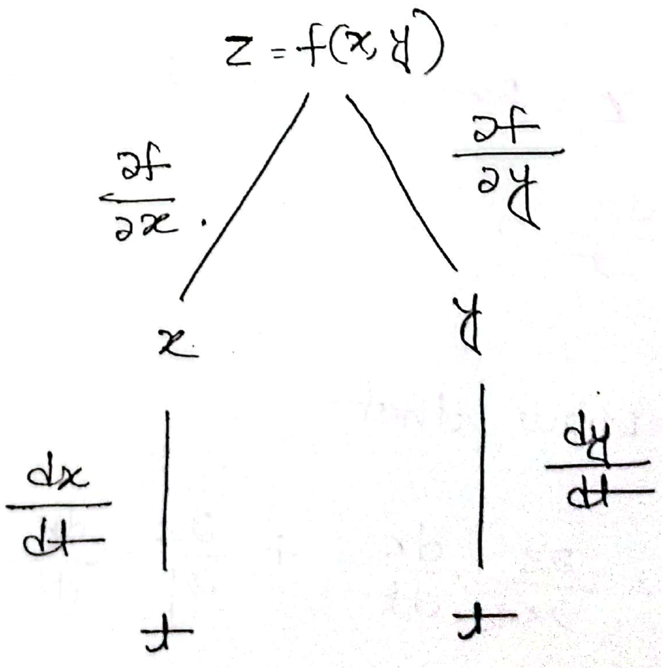
Question :- Show that $u(x,t) = \sin(x-at)$ is a solution of wave equation $u_{tt} = a^2 u_{xx}$.

Chain Rule for Partial Derivatives :-

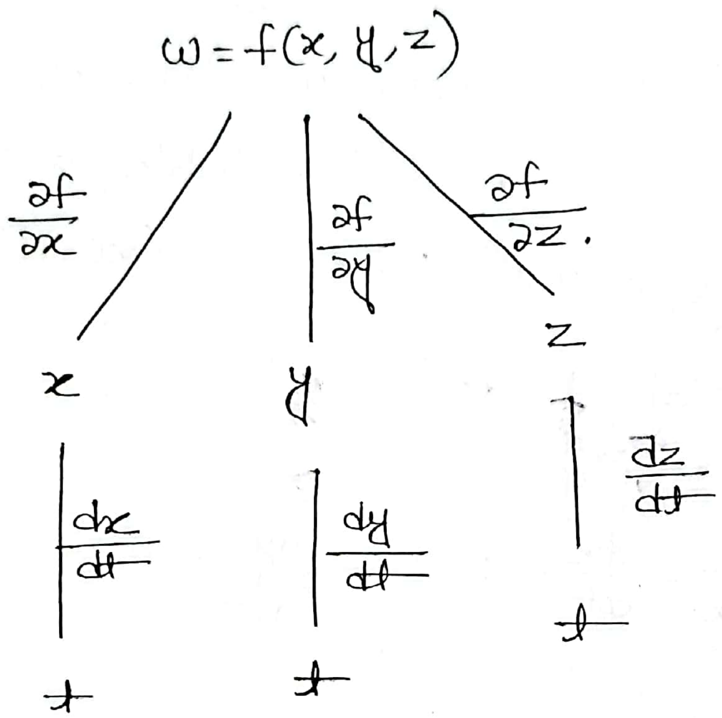
If ~~f(~~ $x = x(t)$ and $y = y(t)$ are differentiable at t and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(t), y(t))$ then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$



In case of three independent variables

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$



Problem :- If $z = 3x^2y^3$, $x = t^4$ and $y = t^2$ then evaluate $\frac{dz}{dt}$.

Solution :- We know that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= 6xy^3 \cdot 4t^3 + 9x^2y^2 \cdot 2t.$$

$$\cancel{= 6 \cdot t^4 \cdot t^6 + 9\, t^8\, t^4}$$

$= 6 \cdot t^4 \cdot (t^2)^3 \cdot 4t^3 + 9 \cdot (t^4)^2 \cdot (t^2)^2 \cdot 2t$

$= 24t^{13} + 18t^{13} = 42t^{13}$

Problem :- If $z = x^2 y$, $x = t^2$, $y = t^3$ then evaluate $\frac{dz}{dt}$.

Solution :- $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

$= 2xy \cdot 2t + x^2 \cdot 3t^2$

$= 2 \cdot t^2 \cdot t^3 \cdot 2t + t^4 \cdot 3t^2$

$= 4t^6 + 3t^6$

$= 7t^6$.

Problem :- ① If $z = 3\cos x - \sin xy$ and $x = \frac{1}{t}$, $y = 3t$ then evaluate $\frac{dz}{dt}$.

② If $z = xye^y$, and $x = t^2$, $y = 5t$ then evaluate $\frac{dz}{dt}$

(3) If $z = x^2y + xy^2$ and $x = 3t$, $y = t^2$ then evaluate

$\frac{dz}{dt}$.

Practice Problem

13.5 $\rightarrow$ 1-10.

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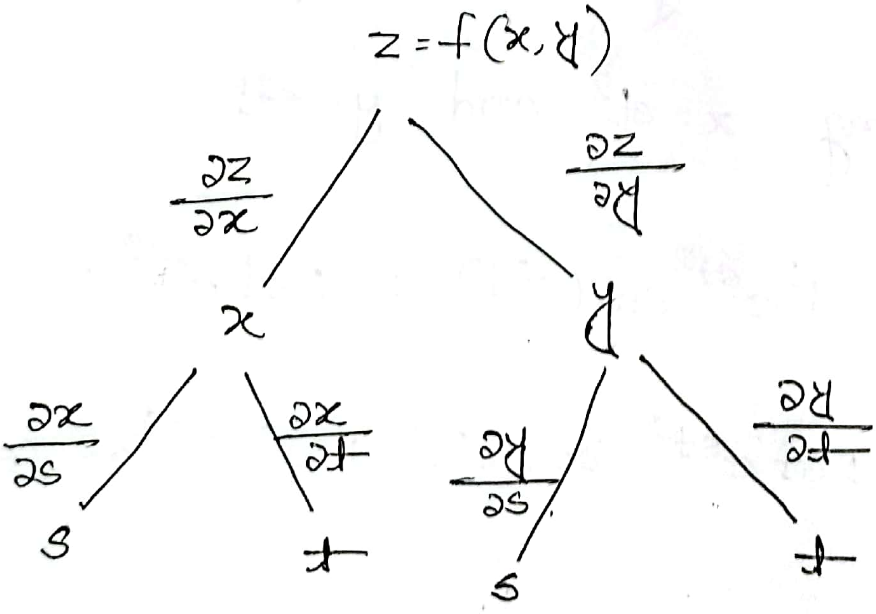
The Chain Rule :- Let us suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = x(s, t)$ and $y = y(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Example :- If $z = e^{x}\sin y$ where $x = st^2$ and $y = s^2 t$ find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution :-

Given ...

$$z = e^x \sin y$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= t^2 e^x \sin y + 2st\, e^x \cos y$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= 2st\, e^x \sin y + s^2 e^x \cos y$$

By substituting $x = st^2$ and $y = s^2t$

$$\frac{\partial z}{\partial s} = t^2 e^{st^2} \sin(s^2t) + 2st\, e^{st^2} \cos(s^2t)$$

$$\frac{\partial z}{\partial t} = 2st\, e^{st^2} \sin(s^2t) + s^2 e^{st^2} \cos(s^2t)$$

To find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, we find the product of partial derivatives along each path from z to s or z to t, and then add these products.

Critical Point :- A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a critical point of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

Saddle Point :- A saddle point of a function is a point in the domain of function where it neither attains a maximum value nor attains a minimum value.

Example :- Find all the critical points of

$$f(x,y) = \frac{x^2}{2} - x + xy^2$$

Solution :- Given that

$$f(x,y) = \frac{x^2}{2} - x + xy^2 .$$

Setting

$f_x(x,y) = 0$ $\qquad\qquad$ $f_y(x,y) = 0 .$

$\Rightarrow x - 1 + y^2 = 0$ $\qquad\qquad$ $\Rightarrow 2xy = 0 .$

Either $x = 0$

or $y = 0 .$

When $x = 0$, $\qquad$ $-1 + y^2 = 0$

$\Rightarrow y = \pm 1 .$

When $y = 0$, $\qquad$ $x - 1 = 0$

$\Rightarrow x = 1 .$

Therefore, the critical points are $(0,1), (0,-1)$ and $(1,0)$.

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Second Partials Test: Let f be a function of two variables with continuous second order partial derivatives ~~in~~ some domain at critical points (x_0, y_0) and let

$$D = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

a) If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0)

b) If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).

c) If $D < 0$, then f has a saddle point at (x_0, y_0)

d) If $D = 0$, then no conclusion can be drawn.

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Example:- Locate all relative extrema and saddle points of $f(x,y) = 4xy - x^4 - y^4$

Solution:- Given that

$$f(x,y) = 4xy - x^4 - y^4.$$

To find out the critical points we set

$f_x(x,y) = 0$ $\qquad\qquad$ $f_y(x,y) = 0$.

$\Rightarrow 4y - 4x^3 = 0$. $\qquad\qquad$ $\Rightarrow 4x - 4y^3 = 0$.

$\Rightarrow y = x^3$ $\qquad\qquad$ $\Rightarrow x = y^3$

$\Rightarrow x = (x^3)^3$

$\Rightarrow x - x^9 = 0$

$\Rightarrow x(1 - x^8) = 0$

$\Rightarrow x = 0, \quad x^8 = 1$

$\Rightarrow x = 0, 1, -1$

When $x = 0$ $\quad y = 0$

When $x = 1$ $\quad y = 1$

When $x = -1$ $\quad y = -1$

The only real critical points are

$(0,0)$, $(1,1)$, $(-1,-1)$.

Here,

$f_{xx} = -12x^2$

$f_{yy} = -12y^2$.

$f_{xy} = 4$.

$$D = f_{xx} f_{yy} - f_{xy}^2 = 144x^2y^2 - 16.$$

Critical Points	D	f_{xx}	Conclusion
$(0,0)$	$-16<0$		Saddle point
$(1,1)$	$128>0$	$-12<0$	Relative maximum
$(-1,-1)$	$128>0$	$-12<0$	Relative maximum

Therefore, $(0,0)$ is saddle point and there exists relative maximum at points $(1,1)$ and $(-1,-1)$. Now the $f(1,1)$ and $f(-1,-1)$ need to be calculated. Ans.