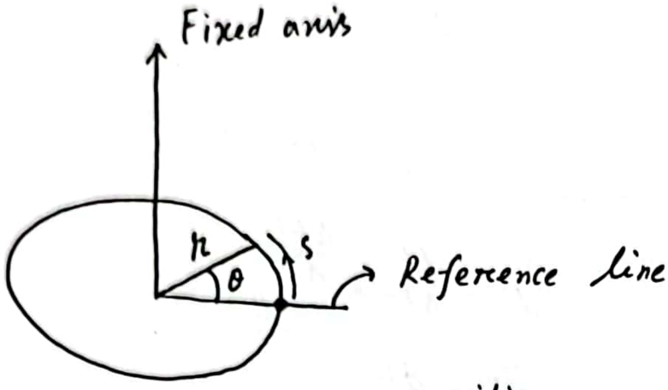
(Chapter-10)

"Rotation"



Angular position ; $\theta = \boxed{\frac{s}{r}}$ → arc length of a circular path
→ radius of circular path
dimensionless

Angular displacement, $\theta_2 - \theta_1 = \Delta\theta$

$\Delta\theta \rightarrow +ve$ (Anti-clockwise)

$\Delta\theta \rightarrow -ve$ (clock wise

unit → radian or degree

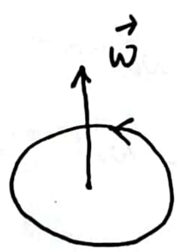
Avg-angular velocity, $\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1}$

$= \frac{\Delta\theta}{\Delta t}$

Instantaneous angular velocity/ angular velocity,

$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$

$= \frac{d\theta}{dt}$



$[\omega] = [\frac{1}{T}]$

unit → radian/sec

or, rev/sec

1 rev = 2π radians

Avg- angular acceleration, $\bar{\alpha} = \dfrac{\omega_2 - \omega_1}{t_2 - t_1}$

$$= \frac{\Delta \omega}{\Delta t}$$

Instantaneous angular acceleration / angular acceleration,

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t}$$

$$= \frac{d\omega}{dt}$$

$$\boxed{\alpha = \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}}$$

$$[\alpha] = [\frac{1}{T^2}]$$

unit → radian / sec^2

Linear		Rotational
x	⟷	θ
v	⟷	ω
a	⟷	α

Equation of motion for constant acceleration and constant angular acceleration

Linear Equation

$v = v_0 + at$

$x - x_0 = v_0 t + \frac{1}{2} a t^2$

$v^2 = v_0^2 + 2a(x - x_0)$

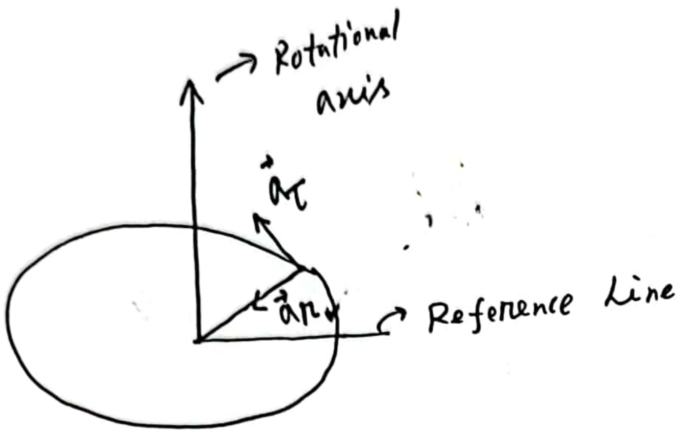
Angular equation

$\omega = \omega_0 + \alpha t$

$\theta - \theta_0 = \omega_0 t + \frac{1}{2} \alpha t^2$

$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Relating Linear and Angular variables



$$\theta = \frac{s}{r}$$

$$\Rightarrow s = r\theta$$

$$\Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt}$$

$$\Rightarrow v = r\omega$$

$$\Rightarrow \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$\Rightarrow \boxed{a_T = r\alpha}$$

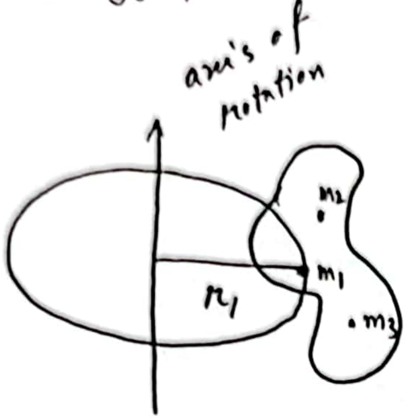
For uniform circular motion,

$$a_r = \frac{v^2}{r}$$

$$= \frac{\omega^2 r^2}{r}$$

$$\boxed{a_r = \omega^2 r}$$

Kinetic Energy of Rotation



$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \cdots + \frac{1}{2} m_n v_n^2$$

$$= \frac{1}{2} \sum_i m_i v_i^2$$

$$v_i = \omega r_i$$

$$= \frac{1}{2} \sum_i m_i (\omega r_i)^2$$

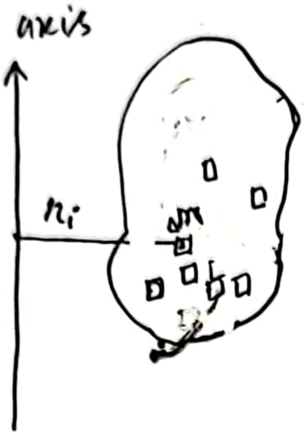
$$= \frac{1}{2} \omega^2 \sum_i m_i r_i^2$$

$$I = \sum_i m_i r_i^2 = \text{moment of inertia}$$

$$\boxed{K = \frac{1}{2} I \omega^2}$$

Linear		Rotational
x	$\longleftrightarrow$	θ
v	$\longleftrightarrow$	ω
a	$\longleftrightarrow$	α
m	$\longleftrightarrow$	I
$K = \frac{1}{2} m v^2$	$\longleftrightarrow$	$K = \frac{1}{2} I \omega^2$

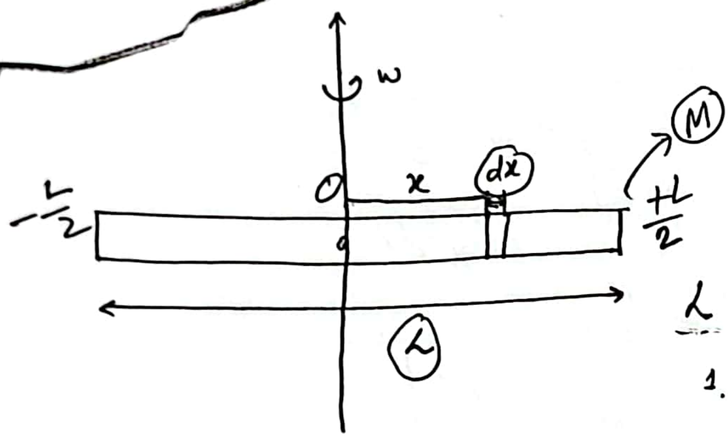
Moment of Inertia / Rotational Inertia $\quad I = \sum_i m_i r_i^2$



$$I = \sum_i \Delta m \, r_i^2$$

If $\Delta m \to 0$

$$\boxed{I = \int r^2 \, dm}$$



L unit length contains, M unit mass

1 " " " $\frac{M}{L}$ " "

$\therefore dx$ " " " $\frac{M}{L} dx$ " "

$$\boxed{dm = \frac{M}{L} dx}$$

Moment of Inertia, $\quad I = \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 \, dm$

$$= \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 \frac{M}{L} dx$$

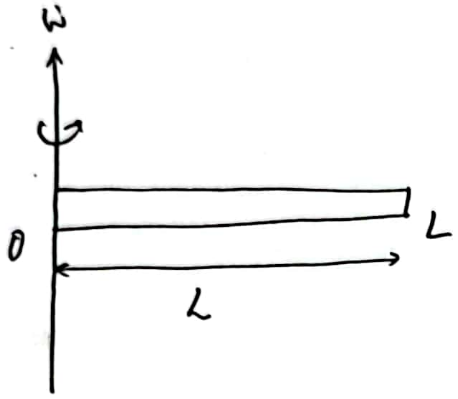
$$= \frac{M}{L} \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 dx$$

$$I = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{+\frac{L}{2}}$$

$$= \frac{M}{L}\,\frac{1}{3}\left[\frac{L^3}{8} + \frac{L^3}{8}\right]$$

$$= \frac{M}{L}\,\frac{1}{3}\,2\,\frac{L^3}{8}$$

$$\boxed{I = \frac{1}{12} M L^2}$$



$$I = \int_0^L x^2\,dm$$

$$= \int_0^L x^2\,\frac{M}{L}\,dx$$

$$= \frac{M}{L}\int_0^L x^2\,dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{1}{3}\,\frac{M}{L}\,L^3$$

$$\boxed{I = \frac{1}{3} M L^2}$$

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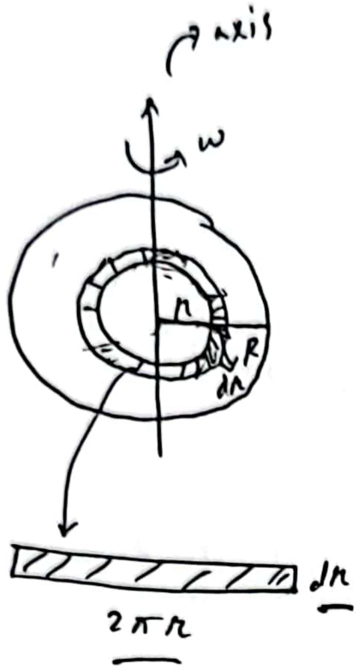


$\pi R^2 \longrightarrow M$

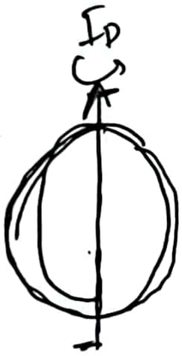
$dm = \frac{M}{\pi R^2} 2\pi r dr$

$I = \int r^2 dm$

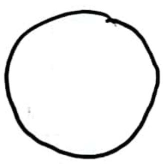
$2\pi r \, dr$

$$dm = \frac{M}{\pi R^2} 2\pi r dr$$
$$= 2\frac{M}{R^2} r dr$$

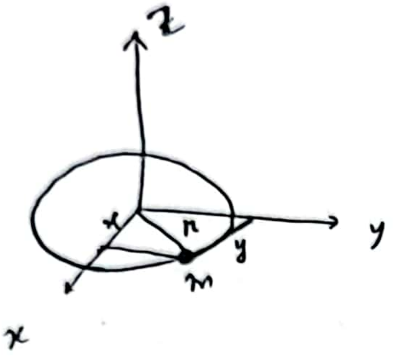
$$I = \int_0^R r^2 \left(\frac{2M}{R^2}\right) r dr$$
$$= \frac{2M}{R^2} \int_0^R r^3 dr$$
$$= \frac{2M}{R^2} \left[\frac{r^4}{4}\right]_0^R$$
$$= \frac{2M}{R^2} \times \frac{R^4}{4}$$
$$\boxed{I = \frac{1}{2} M R^2}$$



?



Perpendicular axis Theorem!



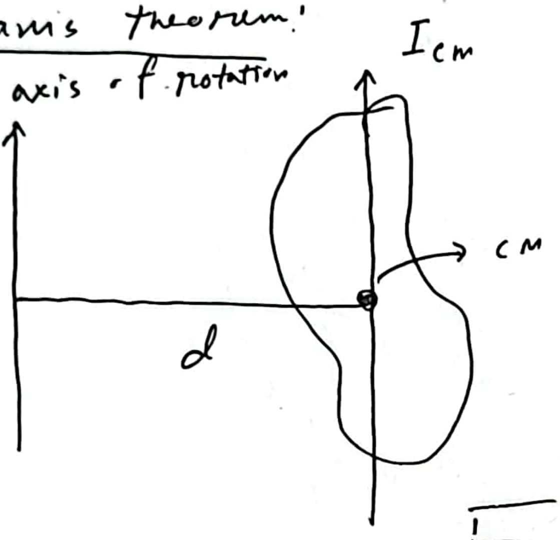
$I_z = mr^2$

$= m(x^2+y^2)$

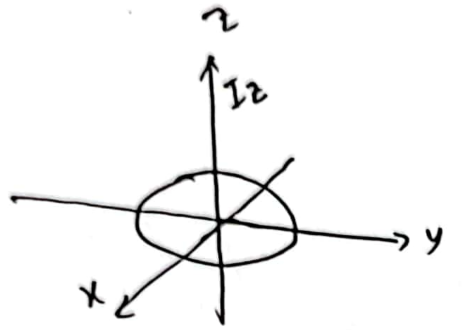
$= mx^2+my^2$

$\boxed{I_z = I_y + I_x}$

Parallel axis theorem!



$\boxed{I = I_{cm} + Md^2}$



$$I_z = I_x + I_y$$

$$\Rightarrow \frac{1}{2} M R^2 = I_d + I_d$$

$$\Rightarrow \boxed{I_d = \frac{1}{4} M R^2}$$

See: $\boxed{\text{Table } 10-2}$

* Newton's 2nd Law For Rotation:

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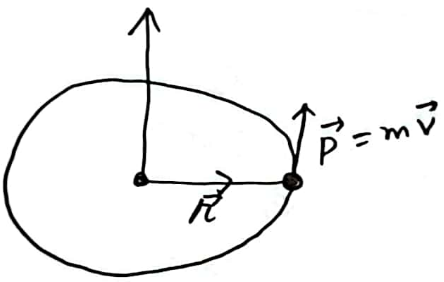


$$\vec{L} = \vec{r} \times \vec{P}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{P})$$

$$= \frac{d\vec{r}}{dt} \times \vec{P} + \vec{r} \times \frac{d\vec{P}}{dt}$$

$$= \cancel{\vec{v} \times m\vec{v}}^{\,0} + \vec{r} \times \vec{F}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

$$\therefore \vec{\tau} = \frac{d\vec{L}}{dt}$$

If, $\vec{\tau} = 0$, $\frac{d\vec{L}}{dt} = 0$

$\vec{L}$ = constant

$\vec{L_i} = \vec{L_f}$ [conservation of Angular Momentum]

* *

$\vec{r} \wedge \vec{p} = 90^\circ$

$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}$

$= \vec{r} \times \vec{F}$

$\Rightarrow |\vec{\tau}| = |\vec{r}| |\vec{F}| \sin 90^\circ$

$\Rightarrow \tau = r m a$

$= r m \alpha r$

$= m r^2 \alpha$

$\boxed{\tau = I \alpha} \longleftrightarrow \boxed{F = ma}$ (1D)

*

$W = \int_{x_1}^{x_2} F\, dx \longleftrightarrow W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$

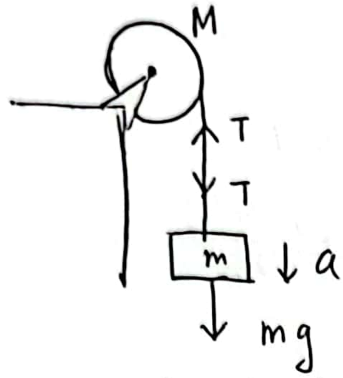
$K = \frac{1}{2} m v^2 \longleftrightarrow K = \frac{1}{2} I \omega^2$

$W = \Delta K = K_f - K_i \longleftrightarrow W = \Delta K = K_f - K_i$

$= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \quad = \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2$

$P = \vec{F} \cdot \vec{v} \rightarrow P = \tau \theta$

Sample problem - 10.10



$M = 2.5\ kg$

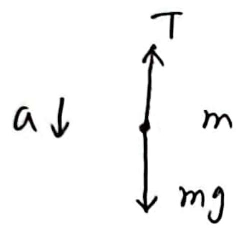
$R = 20 \times 10^{-2} m$

$m = 1.2\ kg$

Acceleration of the falling object, $a = ?$

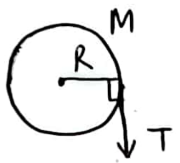
Angular acceleration of the disk, $\alpha = ?$

Tension of the cord, $T = ?$



Here Newton's 2nd Law

$$mg - T = ma \quad \text{——(1)}$$



Here Torque, $\vec{\tau} = \vec{r} \times \vec{F}$

$|\vec{r}| = R$, $\quad \therefore |\vec{\tau}| = |\vec{r}|\,|\vec{F}| \sin 90^\circ$

$|\vec{F}| = T$ $\quad = RT$

$\vec{r} \wedge \vec{F} = 90^\circ$

Applying Newton's 2nd Law for rotation,

$$\tau = I\alpha$$

$$\Rightarrow RT = \frac{1}{2} MR^2 \frac{a}{R} \qquad [\because \alpha R = a]$$

$$\Rightarrow \cancel{R} T = \frac{1}{2} M\cancel{R} a$$

$$\Rightarrow T = \frac{1}{2} Ma \quad \text{— (2)}$$

Putting $T = \frac{1}{2} Ma$ in equation (1),

$$mg - \frac{1}{2} Ma = ma$$

$$\Rightarrow mg = ma + \frac{1}{2} Ma$$

$$= a\left(m + \frac{M}{2}\right)$$

$$\Rightarrow mg = a\left(\frac{2m+M}{2}\right)$$

$$\Rightarrow a = \frac{2m}{2m+M} g$$

$$= \left(\frac{2 \times 1.2}{2 \times 1.2 + 2.5} \times 9.8\right) m/s^2$$

$$= \boxed{4.8\ m/s^2}$$

Angular acceleration of the disk, $\alpha = \frac{a}{R}$

$$= \frac{4.8}{20 \times 10^{-2}} \ rad/sec^2$$

$$= 24\ rad/sec^2$$

Tension of the cord, $T = \frac{1}{2} Ma$

$$= \left(\frac{1}{2} \times 2.5 \times 4.8\right) N$$

$$= \boxed{6\ N}$$

$\boxed{14}$

$\omega_0 = 10$ rev/s

$\theta - \theta_0 = 60$ rev

$\omega = 15$ rev/s , (a) $\alpha = ?$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

$$\Rightarrow \alpha = \frac{\omega^2 - \omega_0^2}{2(\theta-\theta_0)}$$

$$= \frac{15^2 - 10^2}{2 \times 60} \text{ rev/sec}^2$$

$$= \boxed{1.04 \text{ rev/sec}^2} \quad \text{Ans.(a)}$$

(b) $t = ?$ $\quad \omega = \omega_0 + \alpha t$

$$\Rightarrow t = \frac{\omega - \omega_0}{\alpha} = \left(\frac{15 - 10}{1.04}\right) s = \boxed{4.8\ s}$$

(Ans b)

(c) $\omega_0' = 0$ rev/s

$\omega' = 10$ rev/s

$\alpha = 1.04$ rev/s^2

$t' = ?$

$\theta' - \theta_0' = ?$

$$\omega' = \omega_0' + \alpha t'$$

$$\Rightarrow t' = \frac{\omega' - \omega_0'}{\alpha}$$

$$= \left(\frac{10 - 0}{1.04}\right) s$$

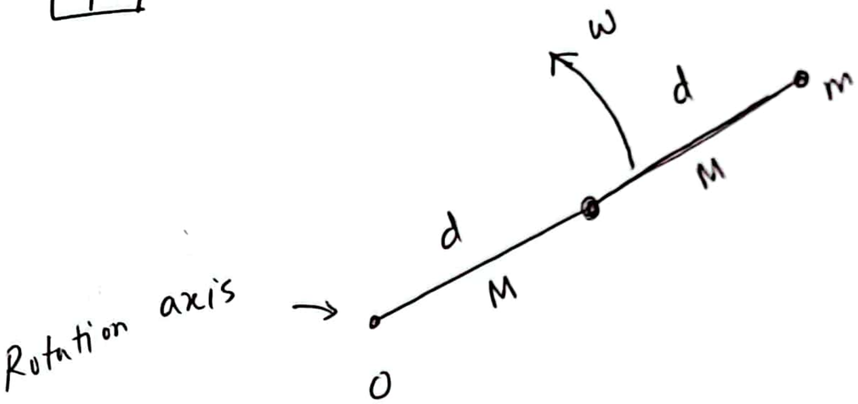
$$= \boxed{9.6\ s} \quad \text{(Ans c)}$$

(d) $\theta' - \theta_0' = \omega_0' t' + \frac{1}{2}\alpha t'^2$

$= 0 + \frac{1}{2} \times 1.04 \times (9.6)^2$ rev

$= \boxed{48 \text{ rev}}$ Ans (d)

$\boxed{41}$



$m = 0.85$ kg

$d = 5.6$ cm $= 5.6 \times 10^{-2}$ m

$M = 1.2$ kg

$\omega = 0.3$ rad/s

(a)

$I = \frac{1}{3} M d^2 + m d^2 + \left[M\left(d + \frac{d}{2}\right)^2 + \frac{1}{12} M d^2\right] + m(2d)^2$

$= \frac{1}{3} M d^2 + 5 m d^2 + \frac{9M}{4} d^2 + \frac{1}{12} M d^2$

$= \left(\frac{1}{3} + \frac{9}{4} + \frac{1}{12}\right) M d^2 + 5 m d^2$

$= \left(\frac{4 + 27 + 1}{12}\right) M d^2 + 5 m d^2$

$= \frac{\cancel{32}\,8}{\cancel{12}\,3} M d^2 + 5 m d^2$

$= \frac{8}{3} M d^2 + 5 m d^2 = \left[\frac{8}{3} \times 1.2 \times (0.056)^2 + 5 \times 0.85 \times (0.056)^2\right]$ kgm^2

$= \boxed{0.023 \text{ kg.m}^2}$ (Ans a) k)

Kinetic Energy, $K = \frac{1}{2} I \omega^2$

$= \frac{1}{2} \times 0.023 \times (0.3)^2$ J

$= \boxed{1.1 \times 10^{-3} \text{ J}}$ (Ans. b)

$\boxed{3}$

$\boxed{5}$

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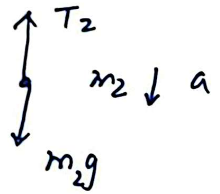
$m_1 = 460\,g$, $m_2 = 500\,g$, $R = 5\,cm$

(a) $h = \frac{1}{2} a t^2$

$\Rightarrow 75 \times 10^{-2}\,m = \frac{1}{2} \times a \times 5^2$

$\Rightarrow a = 6 \times 10^{-2}\,m/s$

(b)



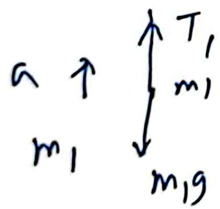
$m_2 g - T_2 = m_2 a$

$\Rightarrow T_2 = m_2 (g - a)$

$= 500 \times 10^{-2} \times (9.8 - 6 \times 10^{-2})\,N$

$= 4.87\,N$

(c)



$T_1 - m_1 g = m_1 a$

$\Rightarrow T_1 = m_1 (g + a)$

$= 400 \times 10^{-2} (9.8 + 6 \times 10^{-2})\,N$

$= 4.54\,N$

(d)

$a = \alpha R$

$\alpha = \dfrac{a}{R} = \dfrac{6 \times 10^{-2}\ m/s^2}{5 \times 10^{-2}\ m} = 1.2\ rad/s^2$

(e) ~~$\tau = I\alpha$~~

$\tau = I\alpha$

$= \frac{1}{2} M R^2 \times \alpha$

$= \frac{1}{2} \times$

$\vec{\tau} = \vec{r} \times \vec{F}$

$\tau = I\alpha = (T_2 - T_1) R$

$\Rightarrow I = \dfrac{(T_2 - T_1) R}{\alpha}$

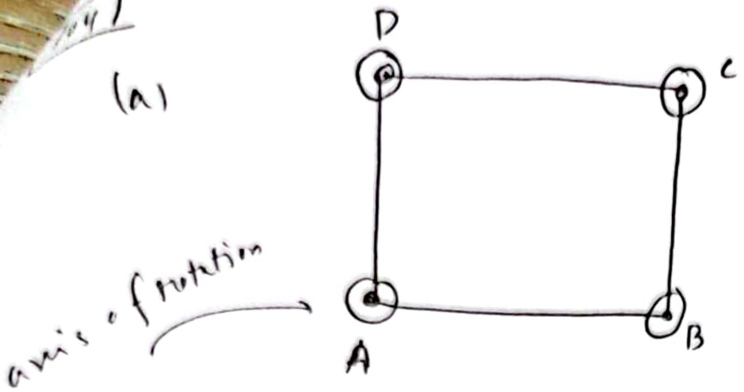
$= \dfrac{(4.87 - 4.54) \times 5 \times 10^{-2}}{1.2}\ kgm$

$\vec{r} \angle \vec{F} = 90°$

$r = R = 5 \times 10^{-2}\ m$

$F = T_2 - T_1 = (4.87 - 4.54) N$

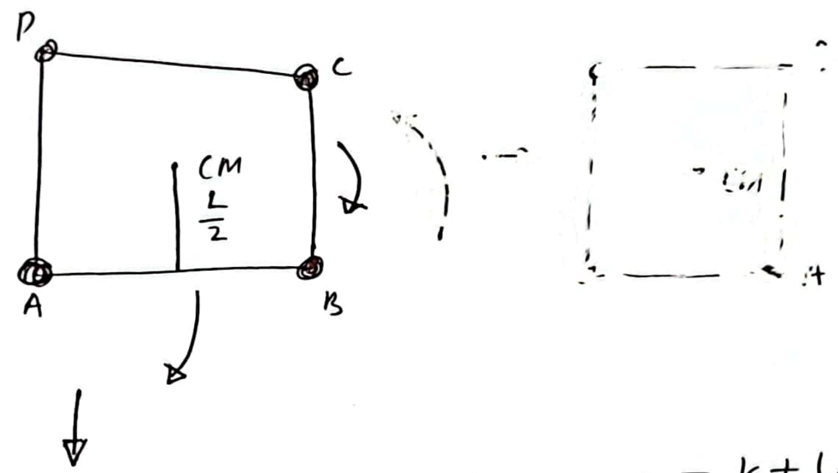
(a)

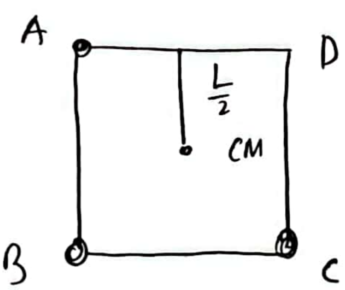


$m = 0.2\ kg$

$$I = \sum_i m_i r_i^2$$

$$= 0 + 0.2 \times (0.5)^2 + 0.2 \times (\sqrt{2} \times 0.5)^2 + 0.2 \times (0.5)^2 = \boxed{0.2\ kg\,m^2}$$





$$K_0 + U_0 = K + U$$

$$\Rightarrow 0 + (4m) g h = K + 0$$

$$\Rightarrow K = (4m) g L$$

$$\Rightarrow \frac{1}{2} I \omega^2 = 4 m g L$$

$$\Rightarrow \omega = \sqrt{\frac{2 \times 4 m g L}{I}}$$

$$= \sqrt{\frac{2 \times 4 \times 0.2 \times 9.8 \times 0.5}{0.2}}$$

$$\boxed{\omega = 6.3\ rad/s}$$