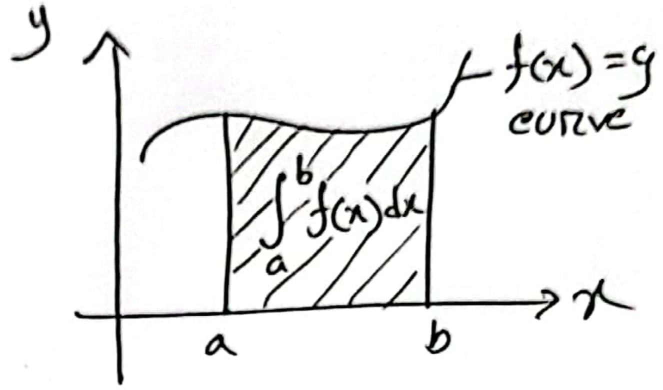
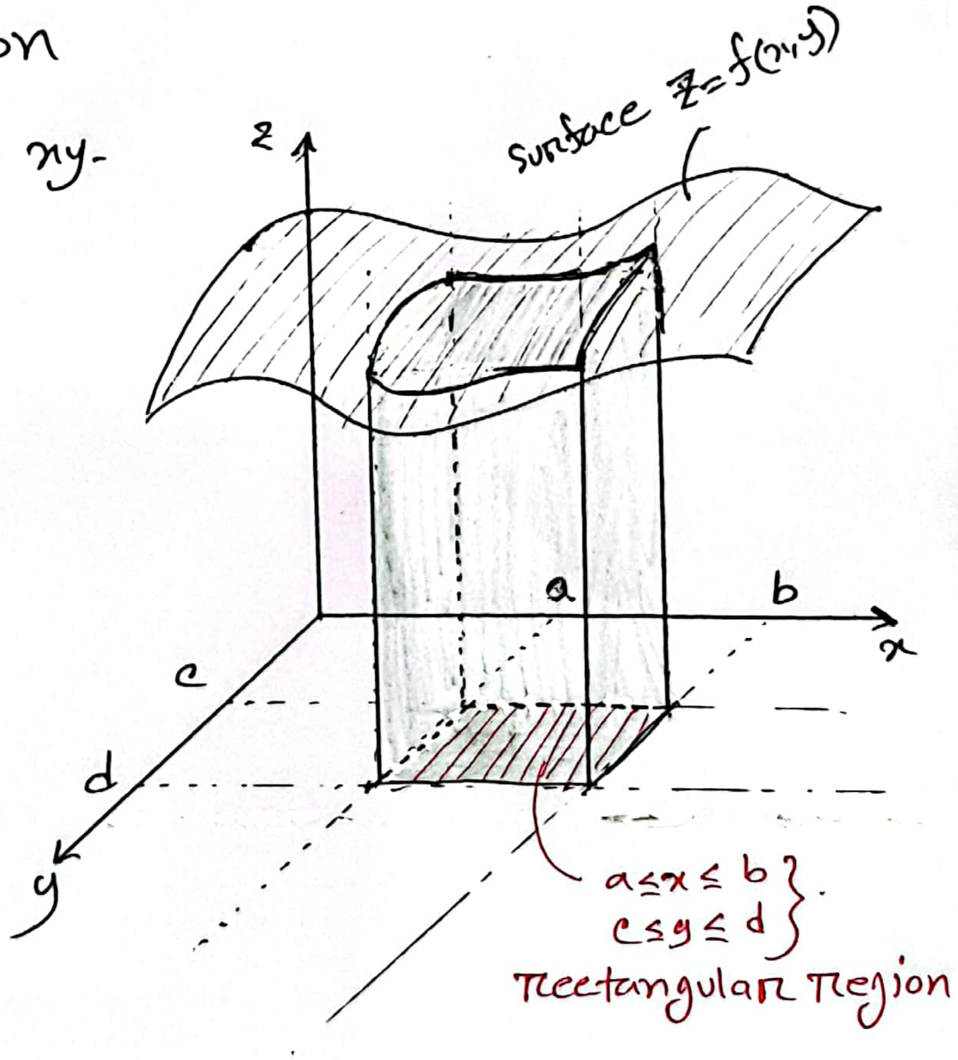
Double Integral

$\int_a^b f(x)\,dx$ (in 2D) is the area under the curve $y=f(x)$ and above the x-axis in the interval $[a,b]$



$\int_a^b\int_c^d f(x,y)\,dy\,dx$ / $\int_c^d\int_a^b f(x,y)\,dx\,dy$ is the volume under the surface $z=f(x,y)$ and above the region $a\le x\le b$, $c\le y\le d$ on xy-plane.



To evaluate the ~~inte~~ double integral start with the innermost integral and work your way outwards.

Example: Evaluate $\int_1^3\int_2^4 (40-2xy)\,dy\,dx$;

Solution:

Given $\int_1^3 \int_2^4 40-2xy\ dy\ dx$

$= \int_1^3 \left[40y - 2x\frac{y^2}{2}\right]_{y=2}^{4} dx$ [holding x fixed and integrating with respect to y]

$= \int_1^3 [160-16x-80-4x]\,dx$

$= \int_1^3 80-12x\ dx$

$= \left[80x - 12\frac{x^2}{2}\right]_1^3$

$= 112$

$\therefore \int_1^3\int_2^4 40-2xy\ dy\,dx = 112$

For Practice: (I) $\iint_R 1+4xy\ dA$; $R: 0\le x\le 1,\ 1\le y\le 3$

$\boxed{\text{Ans: } 10}$

(II) $\iint_R y^2x\ dA$; $R=\{(x,y): -3\le x\le 2, 0\le y\le 1\}$

$\boxed{\text{Ans: } -\frac{5}{6}}$

(III)

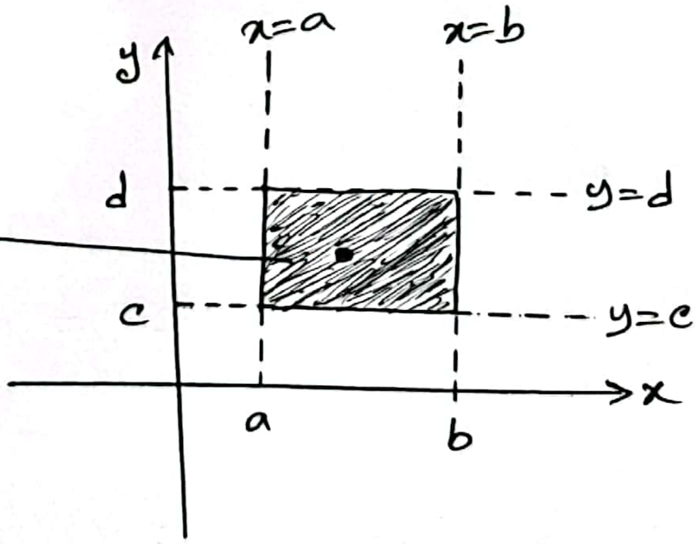
Fubini's Theorem: Let R be the rectangle define by the inequalities $a \le x \le$, $c \le y \le d$. If $f(x,y)$ is continuous on this rectangle, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Note: For rectangular region we can change the order of double integral. ($dy\,dx$, $dx\,dy$) with the limits.

Rectangle region:

The inequalities $a \le x \le b$ & $c \le y \le d$ presents a rectangle region.



Note: $\left.\begin{array}{l} a \le x \le b \\ c \le y \le d \end{array}\right\}$

~~Agn~~ Any points in this rectangle satisfied the inequalities.

$\boxed{\text{If the limits of } x \text{ and } y \text{ are constant that is a rectangle region.}}$

Example: Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4-x-y$ and below the rectangle $R = [0,1] \times [0\ 2]$

Solution: The volume given by

$$V = \iint_R z \, dA$$

$$= \int_0^2 \int_0^1 (4-x-y) \, dx \, dy$$

$$= \int_0^2 \left[4x - \frac{x^2}{2} - yx\right]_{x=0}^{1} dy$$

$$= \int_0^2 \left(\frac{7}{2} - y\right) dy$$

$$= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2$$

$$= 5$$

So the volume is 5 unit^3.

Practice

14.1 → 1–16, 29–31

14.2 Double integral over non-rectangular region

Region I:

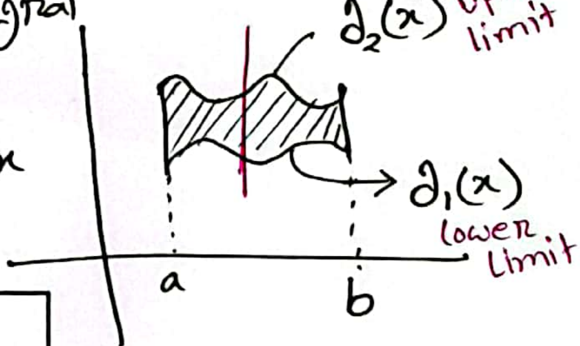
Bounded on

- Left and right by vertical lines $x = a$ and $x = b$
- below and above by continuous function $g_1(x)$ and $g_2(x)$ with $g_1(x) \le g_2(x)$

That is $a \le x \le b$, $g_1(x) \le y \le g_2(x)$

\# In this region we will evaluate the integral as

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$



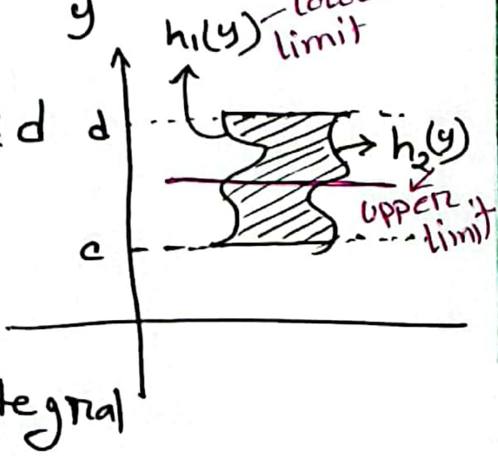
Limits of x are constant limits of y are function of x

Region-II:

Bounded on,

- Left and right by continuous function $h_1(y)$ and $h_2(y)$ with $h_1(y) \le h_2(y)$.
- below and above by horizontal line $y = c$ and $y = d$

That is $h_1(y) \le x \le h_2(y)$; $c \le y \le d$



limits of x are function of y limits of y are constant

\# In this region we will evaluate the integral as

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

$\boxed{\text{Note}}$ The variable which has constant limit will integrate later.

Example: Evaluate $\int_0^1 \int_{-x}^{x^2} y^2 x \, dy\, dx$

Solution! Given

$$\int_0^1 \int_{-x}^{x^2} y^2 x \, dy\, dx$$

$$= \int_0^1 x \left[\frac{y^3}{3}\right]_{y=-x}^{x^2} dx$$

$$= \int_0^1 x\left(\frac{(x^2)^3}{3} - \frac{(-x)^3}{3}\right) dx$$

$$= \int_0^1 \frac{x^7}{3} + \frac{x^4}{3}\, dx$$

$$= \left.\frac{x^8}{24} + \frac{x^5}{15}\right]_0^1$$

$$= \frac{13}{120}$$

$$\therefore \int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx = 13/120$$

Example: Evaluate $\int_0^{\pi/3}\int_0^{\cos y} x \sin y \, dx \, dy$

Solution:

$$\int_0^{\pi/3}\int_0^{\cos y} x \sin y \, dx \, dy$$

$$= \int_0^{\pi/3} \sin y \left[\frac{x^2}{2}\right]_0^{\cos y} dy$$

$$= \int_0^{\pi/3} \frac{1}{2} \sin y \cos^2 y \, dy$$

let
$u = \cos y$
$du = -\sin y \, dy$

$$= \frac{1}{2}\int - u^2 \, dy$$

$$= -\frac{1}{2}\frac{u^3}{3} + c$$

$$= -\frac{1}{6}\left[\cos y\right]_0^{\pi/3}$$

$$= 7/48$$

$\therefore \int_0^{\pi/3}\int_0^{\cos y} x \sin y \, dx \, dy = 7/48$

Example Evaluate $\iint_R (2x-y^2)\, dA$

over the triangular region R enclosed by the lines

$y=-x+1$, $y=x+1$ and $y=3$

Solution: The region enclosed by the lines $y=-x+1$

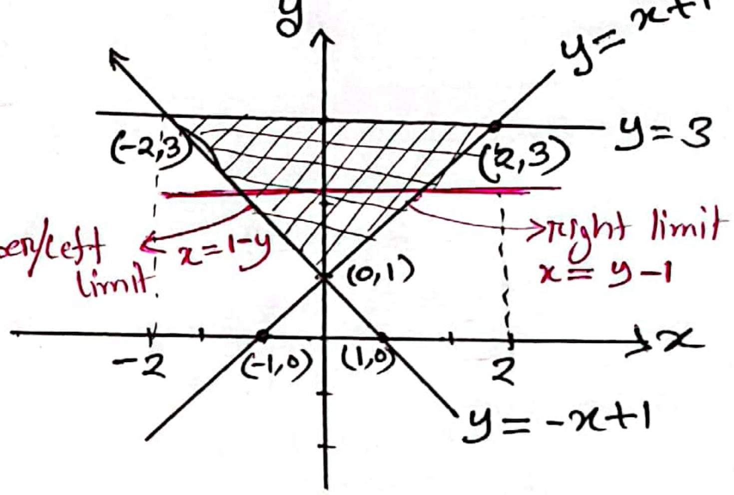
$y=x+1$ and $y=3$.

For, $y=-x+1$

$x=0,\ y=1 \quad \therefore (0,1)$

$x=1,\ y=0 \quad (1,0)$

For $y=x+1$; $(0,1), (-1,0)$



Now $\iint_R f(x,y)\, dA$

$$= \int_1^3 \int_{1-y}^{y-1} 2x - y^2 \, dx\, dy$$

$1 \le y \le 3$

$1-y \le x \le y-1$

Here we take limit of y as constant and limits of x as function of y. The left most function as lower limit $(1-y)$ and the right most function a upper limit $(y-1)$

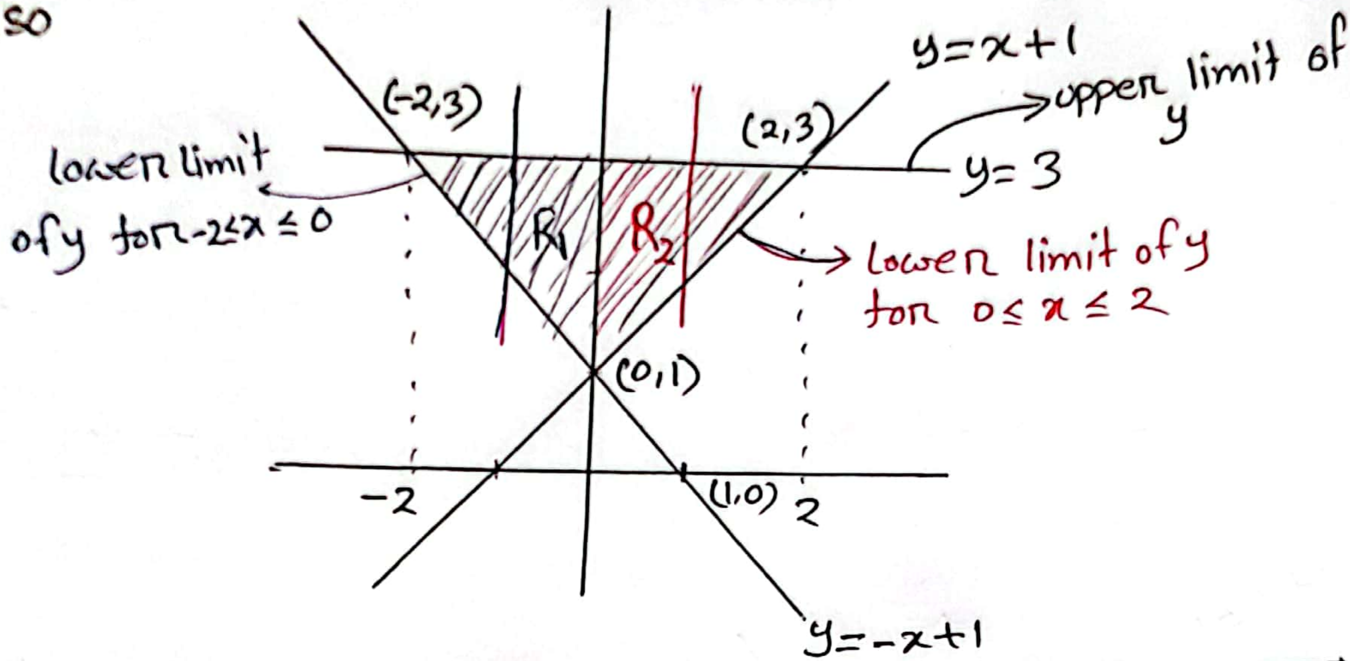
$$= \int_1^3 \left[x^2 - y^2 x \right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 2y^2 - 2y^3 \, dy$$

$$= \left[\frac{2y^3}{3} - \frac{2y^4}{4} \right]_1^3 = -\frac{68}{3}$$

Another Method

Also



$-2 \le x \le 0;\ -x+1 \le y \le 3$

$0 \le x \le 2;\ x+1 \le y \le 3$

$-2 \le x \le 2$; in this region y has two different function as lower limit that is for $-2 \le x \le 0$ the lower limit of y is $-x+1$ and for $0 \le x \le 2$ the lower limit of y is $x+1$. So it it should be divide into two region R_1 and R_2. The uper limit of y is $y = 3$

$$\iint_R 2x - y^2\, dA = \iint_{R_1} 2x - y^2\, dA + \iint_{R_2} 2x - y^2\, dA$$

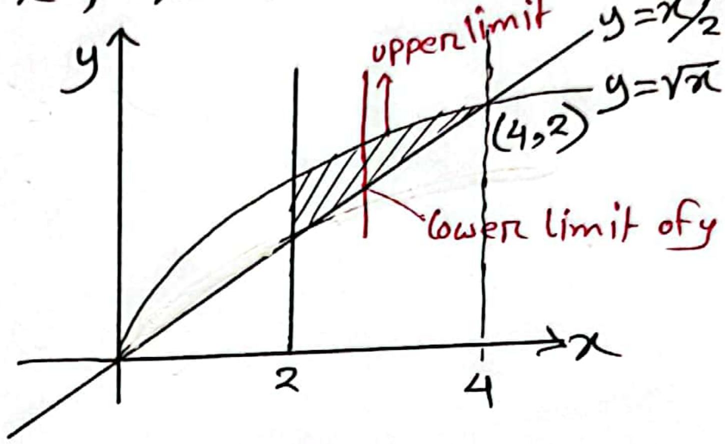
$$= \int_{-2}^{0}\int_{-x+1}^{3} 2x - y^2\, dy\, dx + \int_{0}^{2}\int_{x+1}^{3} 2x - y^2\, dy\, dx$$

[Evaluate by your wonself Answer will be the same as previous.

=

Example

Evaluate $\iint_R xy \, dA$ over the region R bounded by the curves $y = \frac{x}{2}$, $y = \sqrt{x}$, $x = 2$ and $x = 4$



$$\iint_R xy \, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy \, dy\,dx = \frac{11}{6}$$

Example:

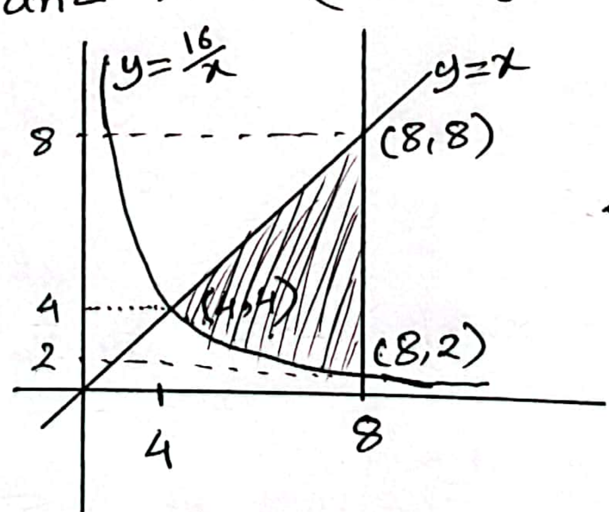
Evaluate $\iint_R x^2 \, dA$: R is the region bounded by $y = 16/x$, $y = x$ and $x = 8$ (Given graph)

y-as constant

$2 \le y \le 8$

$R_1 : 2 \le y \le 4$; $\frac{16}{y} \le x \le 8$

$R_2 : 4 \le y \le 8$; $y \le x \le 8$



x as constant

$$\begin{cases} 4 \le x \le 8 \\ \frac{16}{x} \le y \le x \end{cases}$$

Solution! x- has constant

$$\iint_R x^2 \, dA = \int_4^8 \int_{16/x}^{x} x^2 \, dy \, dx = 576$$

y- has constant limit

or,

$$\iint_R x^2\,dA = \int_2^4\int_{16/y}^{8} x^2\,dx\,dy + \int_4^8\int_{y}^{8} x^2\,dx\,dy$$

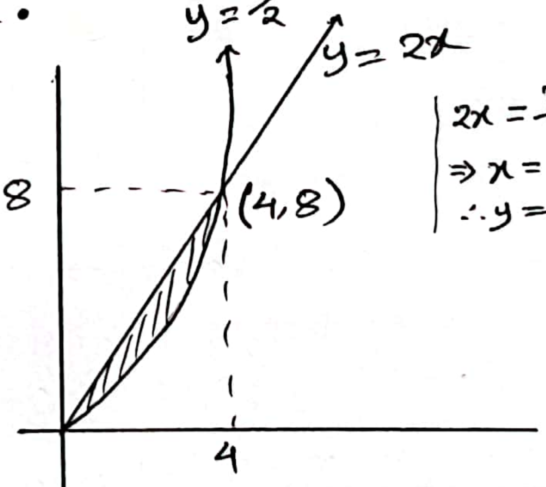
$$= \frac{640}{3} + \frac{1088}{3}$$

$$= 576$$

<u>Example</u>: Use a double integral to find the volome over the region R enclosed between the parabola $y = \frac{x^2}{2}$ and the line $y = 2x$.

$f(x,y) = 1$

~~Area of~~



$2x = \frac{x^2}{2}$

$\Rightarrow x = 4$

$\therefore y = 8$

volome $= \iint_R f(x,y)\,dA$

$$= \int_0^4\int_{x^2/2}^{2x} 1\,dy\,dx$$

$$= \int_0^4 y\Big]_{x^2/2}^{2x}\,dx$$

$$= \int_0^4 2x - \frac{x^2}{2}\,dx = \left[2\frac{x^2}{2} - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

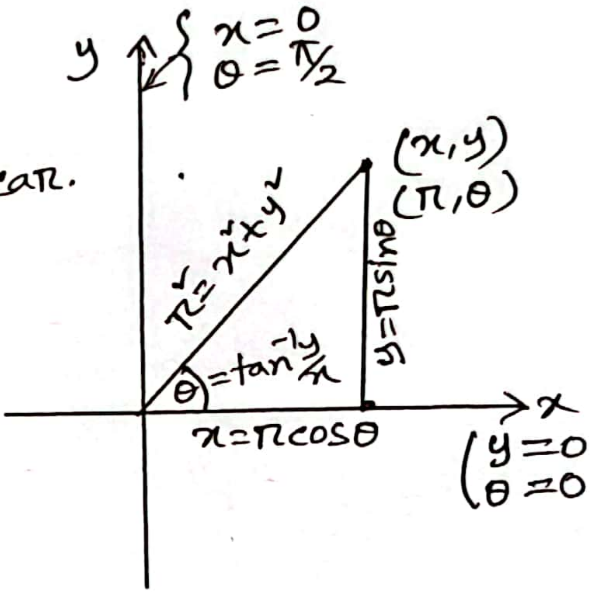
y-constant	x-constant
$0 \le y \le 8$	$0 \le x \le 4$
$y/2 \le x \le \sqrt{2y}$	$\frac{x^2}{2} \le y \le 2x$

or,

volome $= \int_0^8\int_{y/2}^{\sqrt{2y}} 1\,dx\,dy = \int_0^8 \sqrt{2y} - \frac{y}{2}\,dy = \frac{16}{3}$

14.3: ~~Polar~~ Double integral in polar system

Cartesian	polar
x, y	r, θ



relations between polar & car.

$x = r\cos\theta$

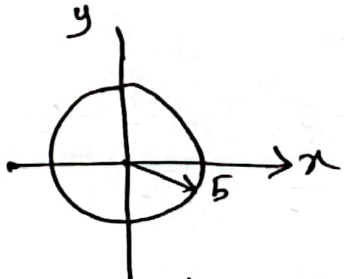
$y = r\sin\theta$

$r^2 = x^2 + y^2$

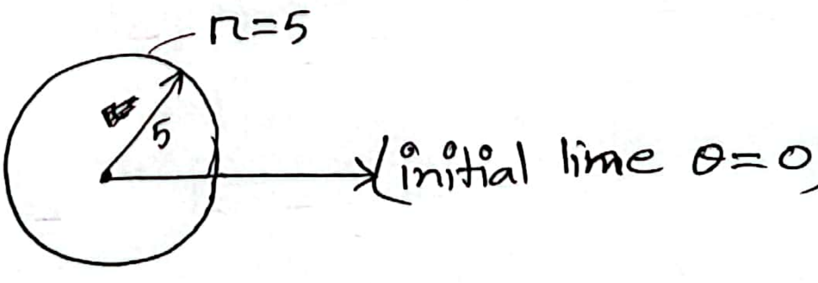
$\theta = \tan^{-1} y/x$

Some graph in polar region

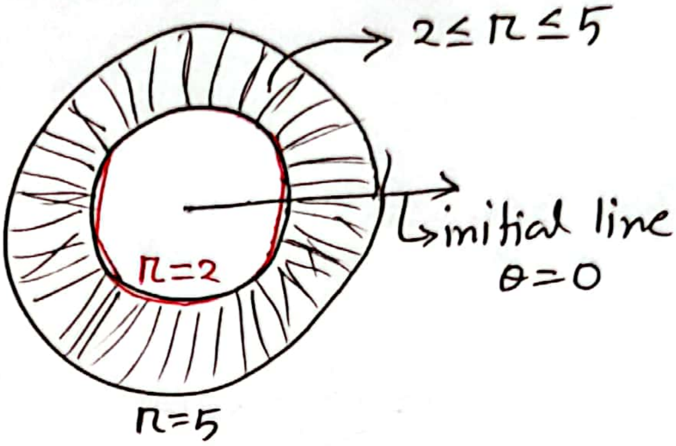
$x^2+y^2=25$ is a circle in cartesian coordinate system having centre at origin (0,0) and radius is 5.



In polar system $r=5$ also present a circle centre at origin ~~and~~ with radius 5.



Polar region:

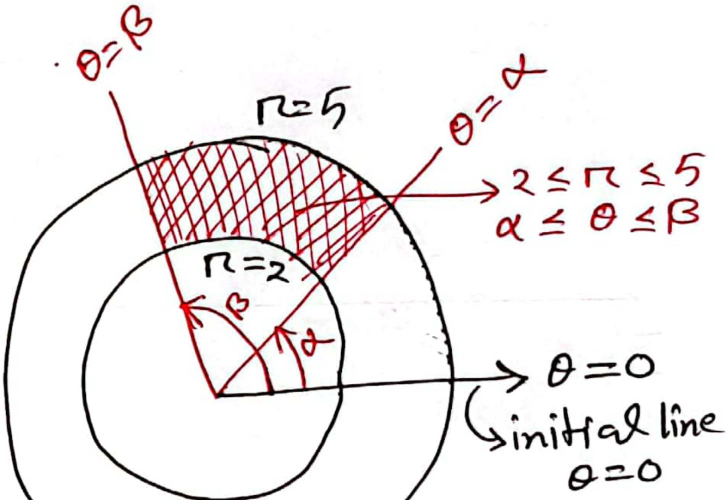


\# $2 \le r \le 5$ is the ring shape region that is outside the circle $r=2$ and inside the region $r=5$.

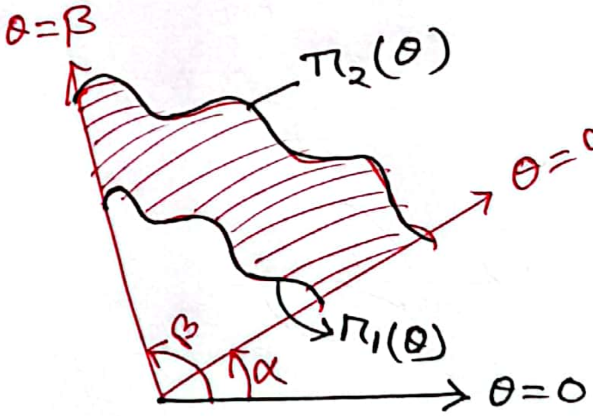
\# Polar rectangle

$$\left.\begin{array}{l} 2 \le r \le 5 \\ \alpha \le \theta \le \beta \end{array}\right\}$$

These inequalities presents the polar rectangle



Polar simple region:



A region define by the inequalities

$$\left.\begin{array}{l} r_1(\theta) \le r \le r_2(\theta) \\ \alpha \le \theta \le \beta \end{array}\right\} R$$

Is called simple polar region

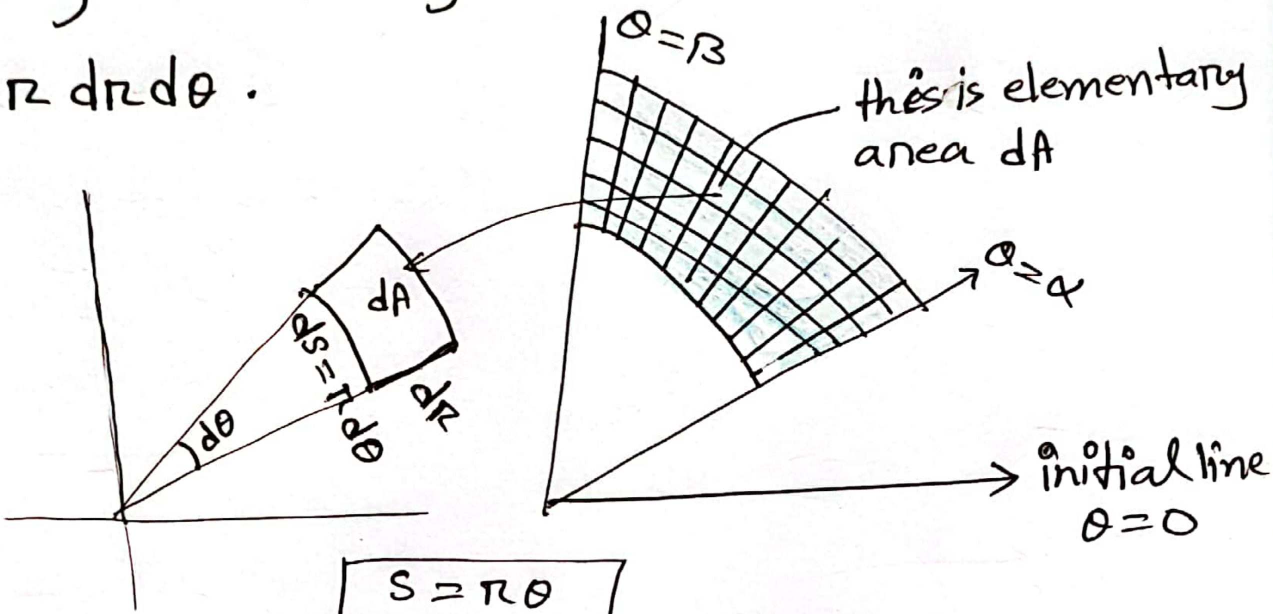
Evaluating Integral In polar region

$$\iint_R f(r,\theta)\, dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, \underline{r}\, dr\, d\theta$$

Note that in cartesian system we use

$dA = dx\,dy$ or $dA = dy\,dx$ but in polar system

$dA = r\,dr\,d\theta$.



$\boxed{S = r\theta}$

So the elementary area $dA = dr \cdot ds$

$= r\,dr\,d\theta$

In polar system we will integrate the function $f(r,\theta)$ with respect to θ later that is $\iint_{\theta\; r} f(r,\theta)\, r\,dr\,d\theta$

Example: Evaluate $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr \, d\theta$

$$= \int_0^{\pi} \left[\frac{r^3}{3}\right]_{r=0}^{a(1-\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \left[a^3(1-\cos\theta)^3 - 0\right] \sin\theta \, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} \sin\theta \, (1-\cos\theta)^3 \, d\theta$$

Let $u = 1-\cos\theta$

$du = \sin\theta \, d\theta$

Limit

$\theta \rightarrow$	u
0	0
π	2

$$= \frac{a^3}{3}\int_0^{2} u^3 \, du$$

$$= \frac{a^3}{3} \left.\frac{u^4}{4}\right]_0^2$$

$$= \frac{4}{3}a^3$$

$$\therefore \int_0^{\pi}\int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr \, d\theta = \frac{4}{3}a^3$$

Example: Evaluate $\iint_R \cos\theta \, dA$ over the polar region $0 \le \theta \le \pi/2$ & $0 \le r \le \sin\theta$.

Solution:

$$\iint_R \cos\theta \, dA = \int_0^{\pi/2}\int_0^{\sin\theta} \cos\theta \; r \, dr \, d\theta$$

$[\because dA = r\,dr\,d\theta]$

$$= \int_0^{\pi/2} \cos\theta \left.\frac{r^2}{2}\right]_{r=0}^{\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos\theta \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{1} u^2 \, du$$

Let
$\sin\theta = u$
$\cos\theta \, d\theta = du$

limit

$\theta \rightarrow$	u
0	0
$\pi/2$	1

$$= \frac{1}{2} \left.\frac{u^3}{3}\right|_0^1$$

$$= \frac{1}{6}$$

$$\therefore \iint_R \cos\theta \, dA = \frac{1}{6} \quad \underline{\text{Ans.}}$$

Example

Evaluate $\iint_R r\cos\theta \, dA$ over the polar region

$0 \le \theta \le \pi$ & $0 \le r \le 1-\sin\theta$

Solution:

$$\iint_R r\cos\theta \, dA = \int_0^{\pi}\int_0^{1-\sin\theta} r\cos\theta \; r \, dr \, d\theta$$

$$= \int_0^{\pi}\int_0^{1-\sin\theta} \cos\theta \, r^2 \, dr \, d\theta$$

$$= \int_0^{\pi} \cos\theta \left.\frac{r^3}{3}\right|_{r=0}^{1-\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \cos\theta \,(1-\sin\theta)^3 \, d\theta$$

$$= \frac{1}{3}\int -u^3 \, du$$

let
$u = 1-\sin\theta$
$du = -\cos\theta \, d\theta$

$$= -\frac{1}{3}\frac{u^4}{4} + c$$

$$= -\frac{1}{12}(1-\sin\theta)^4\Big]_0^{\pi}$$

$$= -\frac{1}{12}\left((1-0)^4 - (1-0)^4\right)$$

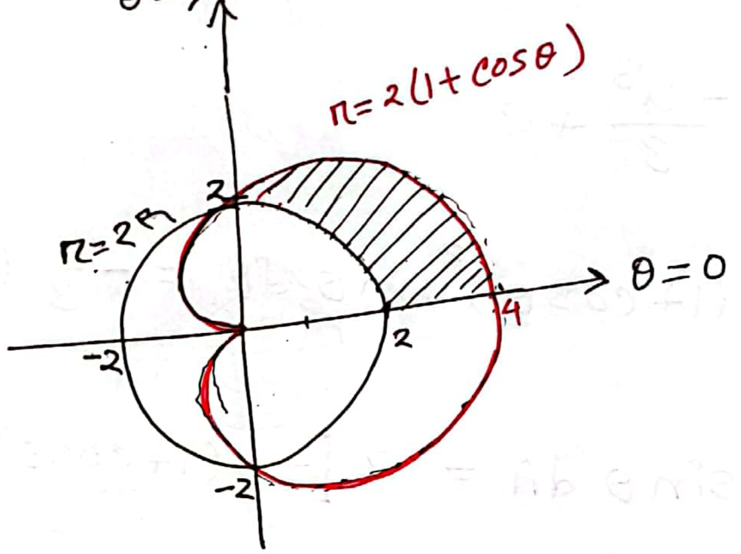
$$= 0$$

$\therefore \iint_R r\cos\theta \, dA = 0$ Ans.

Example

Evaluate $\iint_R \sin\theta \, dA$ where R is the region in the first quadrant that is outside the circle $r=2$ and inside the cardioid. $r = 2(1+\cos\theta)$

Solution:



$$\iint_R \sin\theta \, dA = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} \sin\theta \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\sin\theta \, \frac{r^2}{2}\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin\theta \{2(1+\cos\theta)\}^2 - \sin\theta \,(2)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4\sin\theta\,(1+\cos\theta)^2 - 4\sin\theta \, d\theta$$

$$= \frac{1}{2}\cdot 4 \int_0^{\pi/2} \sin\theta\,(1+\cos\theta)^2 - \sin\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin\theta\,(1+\cos\theta)^2 d\theta - 2\int_0^{\pi/2} \sin\theta \, d\theta$$

Now

$$\int (1+\cos\theta)^2 \sin\theta \, d\theta$$

$u = 1+\cos\theta$

$du = -\sin\theta \, d\theta$

$$= -\int u^2 \, du$$

$$= \frac{-u^3}{3} + c$$

$$\therefore \int (1+\cos\theta)^2 \sin\theta \, d\theta = -\frac{1}{3}(1+\cos\theta)^3 + c$$

$$\therefore \iint_R \sin\theta \, dA = 2\left[-\frac{1}{3}(1+\cos\theta)^3\right]_0^{\pi/2} + 2\left[\cos\theta\right]_0^{\pi/2}$$

$$= 2\left[-\frac{1}{3}(1+0)^3 + \frac{1}{3}(1+1)^3\right] + 2\left[0-1\right]$$

$$= 2\left[-\frac{1}{3} + \frac{8}{3} - 1\right]$$

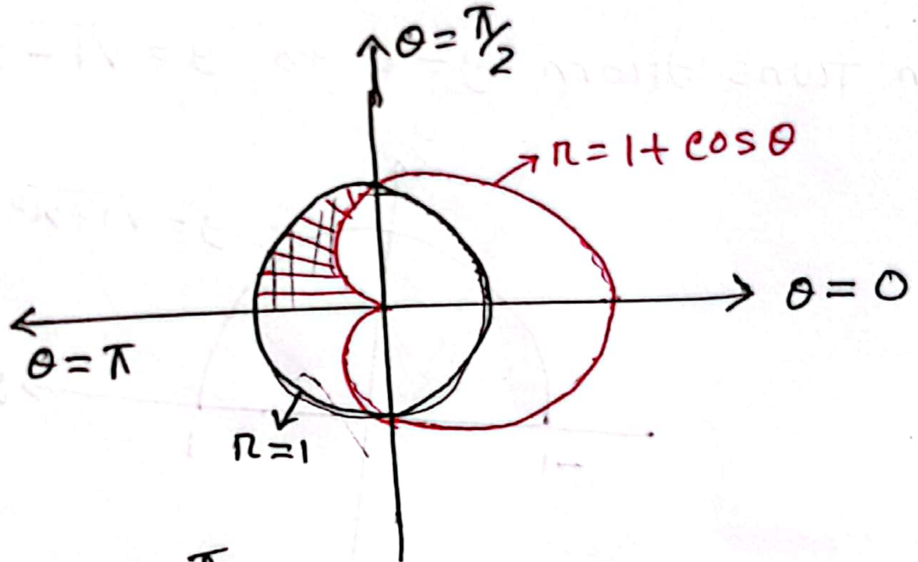
$$= 2 \, \frac{-1+8-3}{3}$$

$$\therefore \iint_R \sin\theta \, dA = \frac{8}{3}$$

Example #:

Evaluate $\iint_R dA$ where the region R in the second quadrant that is inside the circle $r=1$ and outside the cardioid $r = 1+\cos\theta$.

Solution:



$$\therefore \iint_R dA = \int_{\pi/2}^{\pi}\int_{1+\cos\theta}^{1} r\,dr\,d\theta$$

$$= \int_{\pi/2}^{\pi} \left.\frac{r^2}{2}\right]_{1+\cos\theta}^{1} d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{1}{2} - \frac{1}{2}(1+\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} 1 - 1 - 2\cos\theta - \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} -2\cos\theta - \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[-2\sin\theta - \frac{1}{2}\theta - \frac{1}{2}\frac{\sin 2\theta}{2}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{2}\left[\left(0 - \frac{\pi}{2} - 0\right) - \left(-2 - \frac{\pi}{4} - 0\right)\right]$$

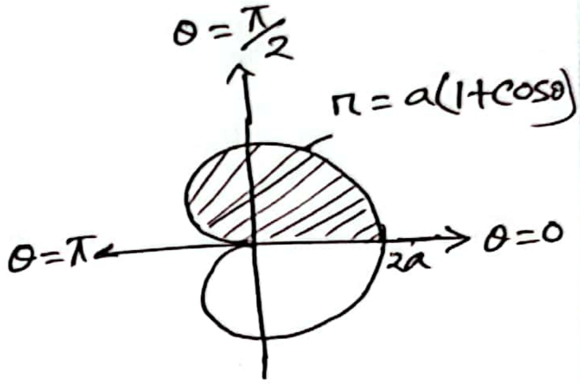
$$= \frac{1}{2}\left(-\frac{\pi}{2} + 2 + \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(2 - \frac{\pi}{4}\right)$$

$$= 1 - \frac{\pi}{8}$$

$$\therefore \iint_R dA = 1 - \frac{\pi}{8}$$

Example: Evaluate $\iint_R r \sin\theta \, dr \, d\theta$ over the polar region above the initial line and under the cardioid $r = a(1+\cos\theta)$.



Solution:

Here the angle θ varies from 0 to π and r varies from 0 to $a(1+\cos\theta)$.

$$\therefore \iint_R r\sin\theta \, dr\, d\theta = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \left.\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta\left[(a+a\cos\theta)^2 - 0\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin\theta\,(1+\cos\theta)^2 \, d\theta$$

Let,
$1+\cos\theta = u$
$-\sin\theta \, d\theta = du$

Limit

θ	$\to u$
0	2
π	0

$$= -\frac{a^2}{2}\int_2^0 u^2 \, du$$

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

$$= \frac{a^2}{2}\int_0^2 u^2 \, du$$

$$= \frac{a^2}{2}\left.\frac{u^3}{3}\right]_0^2 = \frac{4a^2}{3}$$

$$\therefore \iint_R r\sin\theta \, dr\, d\theta = \frac{4a^2}{3}$$

Change of variables in double integral

Using the relations between cartesian coordinate and polar coordinate we can change the integrals from polar to cartesian and vice versa.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2, \quad \theta = \tan^{-1} y/x$$

Example:

$\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$. Evaluate using change of variable in polar form.

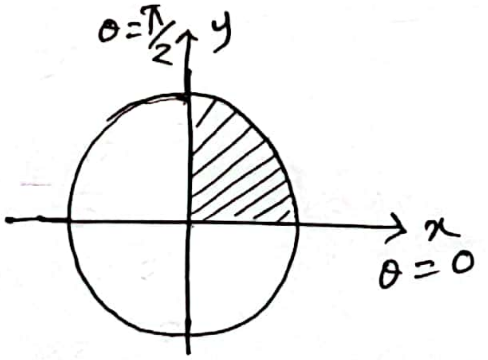
solution:

The limits of the integrals are

$x = 0$ to $x = 1$

$y = 0$ to $y = \sqrt{1-x^2} \Rightarrow x^2+y^2 = 1$

(circle centre at $(0,0)$ with radius 1)



So the limits in polar coordinates are

$$0 \le \theta \le \pi/2, \qquad 0 \le r \le 1$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = \iint_R dA = \int_0^{\pi/2} \int_0^1 r\, dr\, d\theta$$

$[\because dA = r\,dr\,d\theta]$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{\pi/2} (1-0)\, d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta$$

$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$

$$\therefore \iint_R dA = \frac{\pi}{4} \quad \underline{\text{Ans.}}$$

Example: Use polar co-ordinates to evaluate

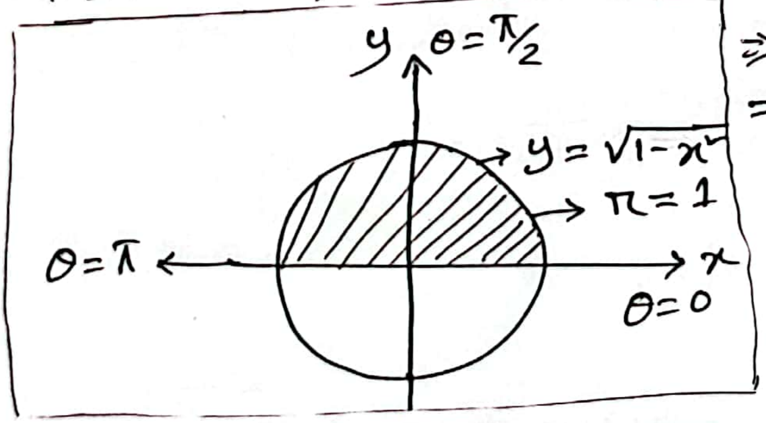
$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$

Solution: Limits of integration,

$-1 \le x \le 1$, $\quad y = 0$ to $y = \sqrt{1-x^2}$

$\Rightarrow y^2 + x^2 = 1$

$\Rightarrow r = 1$



So the limits in polar coordinats

$$0 \le \theta \le \pi, \quad 0 \le r \le 1$$

$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx = \iint_R f(r,\theta)\, dA = \int_0^{\pi}\int_0^1 (r^2)^{3/2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^1 r^4\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^5}{5}\right]_0^1 d\theta = \frac{1}{5}\int_0^{\pi} d\theta$$

$$= \frac{\pi}{5} \quad \text{(Ans.)}$$

Example: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \dfrac{x}{\sqrt{x^2+y^2}}\, dy\, dx$ by changing to polar co-ordinate.

Solution:

Limits of integration are

$x = 0$ to $x = 2$

$y = 0$ to $y = \sqrt{2x - x^2}$

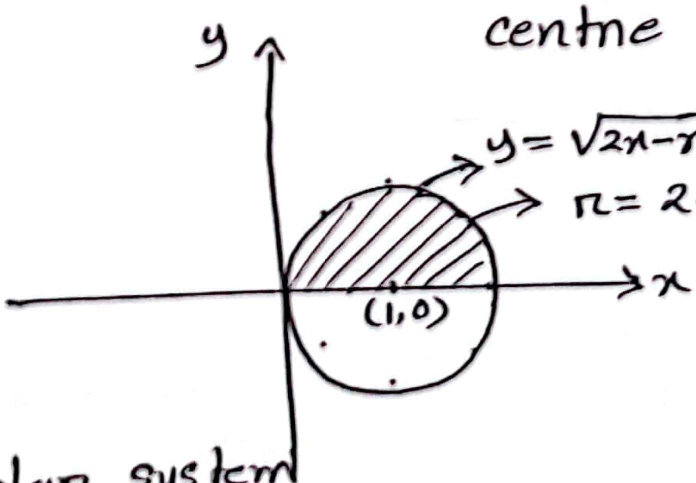
$\Rightarrow y^2 = 2x - x^2$

$\Rightarrow y^2 + x^2 - 2x = 0$

$\Rightarrow x^2 - 2.x.1 + 1^2 + y^2 - 1 = 0$

$\Rightarrow (x-1)^2 + y^2 = 1$

a circle with radius 1 centre at $(1, 0)$



$y = \sqrt{2x - x^2}$ (in cartesina)

$r = 2\cos\theta$ (in polar)

Limits in polar system

$y = \sqrt{2x - x^2}$

$\Rightarrow y^2 + x^2 = 2x$

$\Rightarrow r^2 = 2r\cos\theta$ $\quad [r^2 = x^2 + y^2,\ x = r\cos\theta]$

$\Rightarrow r = 2\cos\theta$

$\therefore \quad 0 \le \theta \le \pi/2$ and $0 \le r \le 2\cos\theta$

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx = \iint_R f(r,\theta)\, dA$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} \frac{r\cos\theta}{r}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left.\frac{r^2}{2}\right]_0^{2\cos\theta} \cos\theta\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} (2\cos\theta)^2 \cos\theta\, d\theta$$

$$= 2 \int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= 2 \int_0^{\pi/2} \cos\theta\, (1-\sin^2\theta)\, d\theta$$

$$= 2 \int_0^1 1-u^2\, du$$

let
$\sin\theta = u$
$\cos\theta\, d\theta = du$

$$= 2\left[u - \frac{u^3}{3}\right]_0^1$$

limit

θ	$\rightarrow$	u
0		0
$\pi/2$		1

$$= 2\left[\left(1-\frac{1}{3}\right) - 0\right]$$

$$= 2 \cdot \frac{2}{3}$$

$$\therefore \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2-y^2}}\, dy\, dx = 4/3 \quad \text{Ans.}$$

14.5 Triple integral in Cartesian coordinate

Triple Integral:

Let $f(x,y,z)$ be a continuous function at every point of a finite volume V of three dimensional space. Then triple integral is

$$\iiint_V f(x,y,z)\,dV = \int_{x=a}^{b}\int_{y=h_1(x)}^{h_2(x)}\int_{z=\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

where V is the volume bounded by the surfaces

$x=a$ to $x=b$ $\boxed{\text{limit of } x \text{ — constant}}$

$y=h_1(x)$ to $y=h_2(x)$ $\boxed{\text{Limit of } y \text{ is f}^n \text{ of } x}$

$z=\phi_1(x,y)$ to $z=\phi_2(x,y)$ $\boxed{\text{Limit of } z \text{ is f}^n \text{ of } x,y}$

Example 01: Evaluate $\iiint_V (x+y+z)\,dx\,dy\,dz$ where

$V: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le x \le 3$

Solution:

$$\iiint_V (x+y+z)\,dx\,dy\,dz = \int_2^3\int_1^2\int_0^1 x+y+z\,dx\,dy\,dz$$

$$= \int_0^1\int_1^2\int_2^3 x+y+z\,dz\,dy\,dx$$

$$= \int_0^1 \int_1^2 \left[xz + yz + \frac{z^2}{2}\right]_{z=2}^{3} dy\, dx$$

$$= \int_0^1 \int_1^2 \left(3x + 3y + \frac{9}{2} - 2x - 2y - 2\right) dy\, dx$$

$$= \int_0^1 \int_1^2 \left(x + y + \frac{5}{2}\right) dy\, dx$$

$$= \int_0^1 \left. xy + \frac{y^2}{2} + \frac{5}{2}y \right]_{y=1}^{2} dx$$

$$= \int_0^1 (2x + 2 + 5) - \left(x + \frac{1}{2} + \frac{5}{2}\right) dx$$

$$= \int_0^1 x + 4 \; dx$$

$$= \left. \frac{x^2}{2} + 4x \right]_0^1$$

$$= \frac{1}{2} + 4 - 0$$

$$= 9/2$$

$$\therefore \iiint_V x + y + z \; dV = 9/2$$

Note that if all the limits of x, y and z are constant then we can use any of ~~a~~ integration order $dx\,dy\,dz$, $dy\,dx\,dz$, $dz\,dx\,dy$, ...

But if not all are constant then use : $dz\,dy\,dx$

Example: Evaluate $\iiint_V (x-2y+z)\,dx\,dy\,dz$ where

$V: 0 \le x \le 1, \quad 0 \le y \le x^2, \quad 0 \le z \le x+y$

Solution:

$$\int_0^1 \int_0^{x^2} \int_0^{x+y} x-2y+z \;dz\,dy\,dx$$

$$= \int_0^1 \int_0^{x^2} zx-2yz+\frac{z^2}{2}\Big|_{z=0}^{x+y} dy\,dx$$

$$= \int_0^1 \int_0^{x^2} x(x+y)-2y(x+y)+\frac{1}{2}(x+y)^2 \;dy\,dx$$

$$= \int_0^1 \int_0^{x^2} x^2+\cancel{xy}-\cancel{2xy}-2y^2+\frac{1}{2}x^2+\cancel{xy}+\frac{1}{2}y^2 \;dy\,dx$$

$$= \int_0^1 \int_0^{x^2} \frac{3}{2}x^2-\frac{3}{2}y^2 \;dy\,dx$$

$$= \int_0^1 \frac{3x^2}{2}y-\frac{3}{2}\frac{y^3}{3}\Big]_{y=0}^{x^2} dx$$

$$= \int_0^1 \left(\frac{3}{2}x^4-\frac{1}{2}x^6\right)-0 \;dx$$

$$= \frac{3x^5}{10}-\frac{x^7}{14}\Big]_0^1$$

$$= \frac{3}{10}-\frac{1}{14}$$

$$= \frac{8}{35} \quad \text{Ans.}$$

Example:

Evaluate $\iiint_V xyz \, dv$; $V: 0 \le x \le 1, 0 \le y \le \sqrt{1-x^2}$

and $0 \le z \le \sqrt{1-x^2-y^2}$

solution :

$$\iiint_V xyz \, dv = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left. \frac{z^2}{2} \right]_{z=0}^{\sqrt{1-x^2-y^2}} dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy - x^3 y - xy^3 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4} \right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2} \int_0^1 \frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-x^2)^2 \, dx$$

$$= \frac{1}{2} \int_0^1 \frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4} + \frac{x^3}{2} - \frac{x^5}{4} \, dx$$

$$= \frac{1}{2} \int_0^1 \frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4} \, dx$$

$$= \frac{1}{2} \left[\frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24} \right]_0^1 = \frac{1}{2}\left(\frac{1}{8} - \frac{1}{8} + \frac{1}{24}\right)$$

$$= \frac{1}{48} \text{ Ans.}$$

14.6 Triple integral in cylindrical & Spherical coordinates

Change of cartesian coordinates into cylindrical Coordinates :

- Polar coordinate deal with two dimensional space $(x, y) \rightarrow (r, \theta)$
- In cylindrical co-ordinate system we deal with three dimensional space

$$(x, y, z) \rightarrow (r, \theta, z)$$

Relations between cartesian and cylindrical co-ordi-nate system

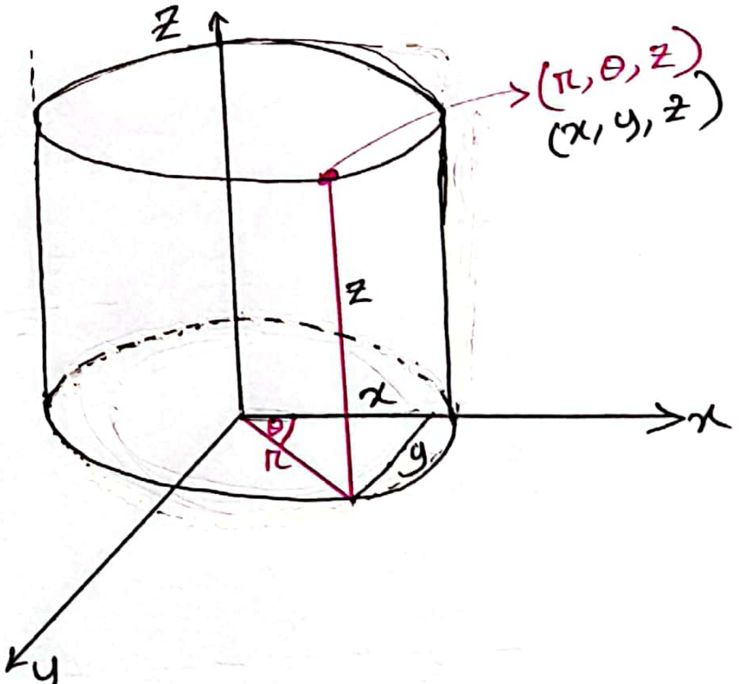
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z \text{ (height)}$$

$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1} y/x$$

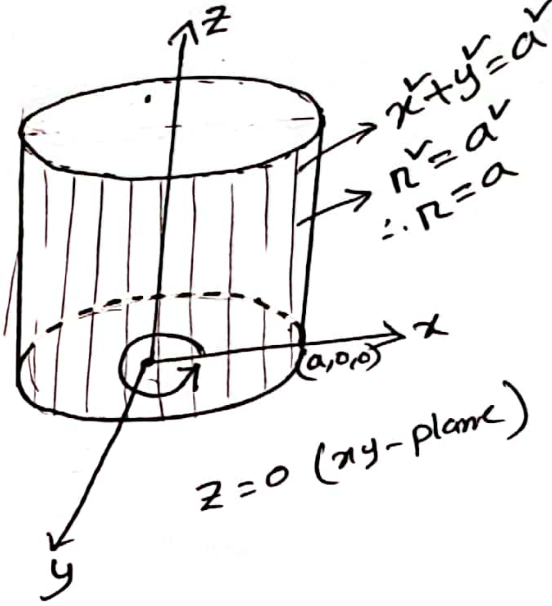


Evaluating triple integral in ~~polar~~ cylindrical coordinate system

$$\iiint_V f(x,y,z)\, dv = \int_z \int_\theta \int_r f(r,\theta,z)\, r\, dr\, d\theta\, dz$$

Note $dx\,dy\,dz = dv = r\,dr\,d\theta\,dz$

<u>Example</u>: Evaluate $\iiint_V z(x^2+y^2+z^2)\,dx\,dy\,dz$, over the volume of the cylinder $x^2+y^2=a^2$ intercepted by the planes $z=0$ and $z=b$



<u>Solution</u>:

The limits of integration are

$$0 \le \theta \le 2\pi$$

$$0 \le r \le a$$

$$0 \le z \le b$$

putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and

$dx\,dy\,dz = r\,dr\,d\theta\,dz$ we get

$$\iiint_V z(x^2+y^2+z^2)\,dv = \int_0^b\int_0^{2\pi}\int_0^a z(r^2+z^2)\,r\,dr\,d\theta\,dz$$

$$= \int_0^b\int_0^{2\pi}\int_0^a zr^3 + z^3 r\,dr\,d\theta\,dz$$

$$= \int_0^b\int_0^{2\pi} \left[z\frac{r^4}{4} + z^3\frac{r^2}{2} \right]_{r=0}^{a} d\theta\,dz$$

$$= \int_0^b\int_0^{2\pi} \frac{a^4}{4}z + \frac{a^2}{2}z^3\,d\theta\,dz$$

$$= \int_0^b \left(\frac{a^4}{4} z + \frac{a^2}{2} z^3\right) [\theta]_0^{2\pi} \, dz$$

$$= 2\pi \int_0^b \frac{a^4}{4} z + \frac{a^2}{2} z^3 \, dz$$

$$= 2\pi \left[\frac{a^4}{4} \frac{z^2}{2} + \frac{a^2}{2} \frac{z^4}{4} \right]_0^b$$

$$= 2\pi \left(\frac{a^4 b^2}{8} + \frac{a^2 b^4}{8} \right)$$

$$= (a^4 b^2 + a^2 b^4) \pi/4$$

$$= \pi/4 \, a^2 b^2 (a^2 + b^2) \quad \text{Ans.}$$

<u>Example</u>: use cylindrical co-ordinate to evaluate

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x^2 \, dz \, dy \, dx$$

<u>solution</u>:

Limits of the integrals in cartesian coordinates.

$x = 3$ to $x = -3$

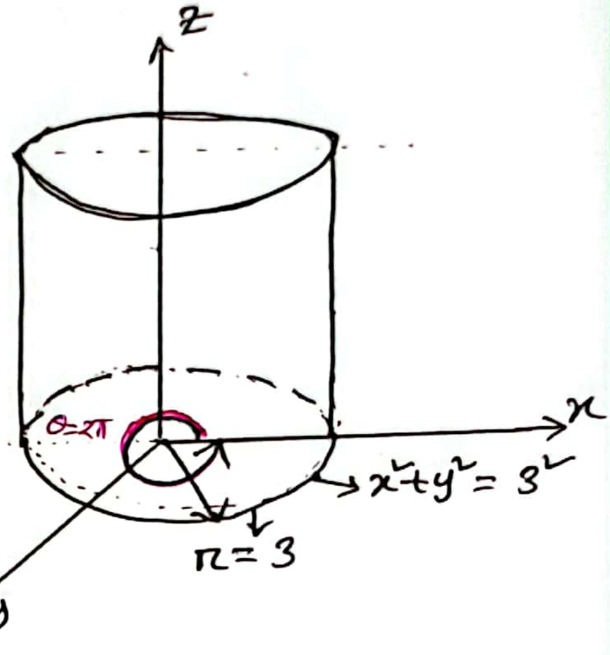
$y = -\sqrt{9-x^2}$ to $y = \sqrt{9-x^2}$ $\therefore y^2 = 9 - x^2$

$\Rightarrow y^2 + x^2 = 3^2$

$z = 0$ to $z = 9 - x^2 - y^2$

$\Rightarrow z = 9 - r^2$

Limits in polar coordinates



$0 \le \theta \le 2\pi$, $0 \le r \le 3$ and

$0 \le z \le 9 - r^2$

Limit of z is function of r
So we integrate with respect
to z first then r

$$\therefore \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9+x^2}} \int_{0}^{9-x^2-y^2} x^2 \, dz\,dy\,dx$$

$$= \iiint_V f(r,\theta,z)\, dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2\cos^2\theta \; r\,\cancel{dr\,d\theta}\; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos^2\theta \;\cancel{dr\,d\theta}\; dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 r^3\cos^2\theta \,[z]_0^{9-r^2}\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \cos^2\theta \; r^3(9-r^2)\, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \cos^2\theta \,(9r^3 - r^5)\, dr\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \left[\frac{9r^4}{4} - \frac{r^6}{6}\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{243}{4} \cos^2\theta \, d\theta$$

$$= \frac{243}{4} \int_0^{2\pi} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{243}{8} \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{243}{8} \times 2\pi$$

$$= \frac{243}{4}\pi \quad \text{Ans.}$$

Example: Use triple integration in cylindrical co-ordinate to find the volume of the solid G that is bounded the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane and laterally by the cylinder $x^2 + y^2 = 9$

Sol^n: Put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and

$dx\,dy\,dz = r\,dr\,d\theta\,dz$ then,

$$\int_0^{2\pi}\int_0^{3}\int_0^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r\,[z]_0^{\sqrt{25-r^2}}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r\sqrt{25-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int \sqrt{u} \left(-\frac{du}{2}\right) d\theta$$

$u = 25 - r^2$

$du = -2r\,dr$

$$= -\frac{1}{2} \int_0^{2\pi} \int \sqrt{u}\ du\ d\theta$$

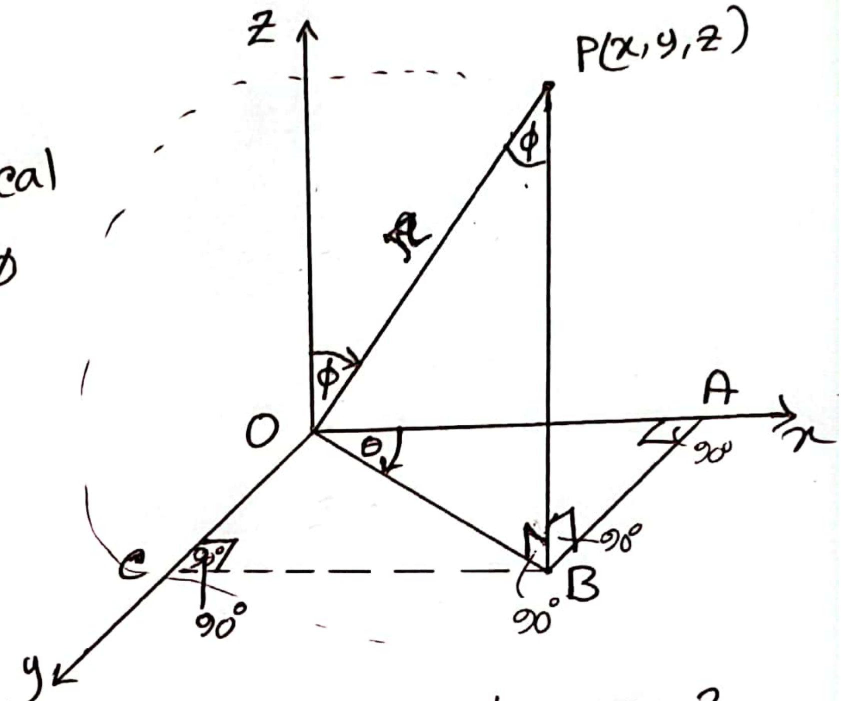
$$= -\frac{1}{2} \int_0^{2\pi} \frac{u^{3/2}}{3/2} + c\ d\theta$$

$$= -\frac{1}{2} \int_0^{2\pi} \frac{2}{3} \left(25 - r^2\right)^{3/2} \Big]_0^3\ d\theta$$

$$= -\frac{1}{3} \int_0^{2\pi} \frac{61}{3}\, d\theta$$

$$= \frac{122}{3}\pi \quad \underline{\text{Ans.}}$$

Change of cartesian co-ordinates to spherical co-ordinates



Cartesian spherical

x, y, z r, θ, φ

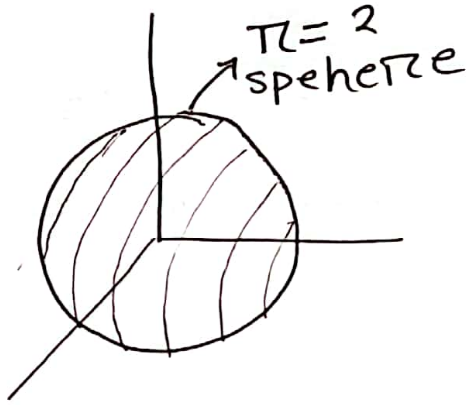
where

$x^2+y^2+z^2 = r^2$

$x = r \sin\phi \cos\theta$

$y = r \sin\theta \sin\phi$

$z = r \cos\phi$



2D

cartesian	polar
(x, y)	(r, θ)
$x^2+y^2=5^2$ (circle)	$r=5$ (circle)

3D

Cartesian	Spherical
(x, y, z)	(r, θ, φ)
$x^2+y^2+z^2=5^2$ sphere with radius 5	$r=5$ sphere with radius 5

r : distance from origin

θ : angle with the positive x-axis (in xy-plane)

φ : angle with z-axis

Evaluating integrations in speherical system

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint f(r,\theta,\phi)\, r^2\sin\phi \, dr\,d\theta \times d\phi$$

Example: Evaluate $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\; dz\,dy\,dx$

using speherical co-ordinate systems.

Solution:

Limits of integration

$x=-3$ to $x=3$

$y=-\sqrt{9-x^2}$ to $y=\sqrt{9-x^2}$

$\therefore\; y^2 = 9-x^2$

$\Rightarrow\; y^2+x^2 = 9$ [Cylinder]

And $z=0$ to $z=\sqrt{9-x^2-y^2}$

$\Rightarrow\; z^2 = 9-x^2-y^2$

$\Rightarrow\; z^2+x^2+y^2 = 9$

$\Rightarrow\; r^2 = 9$

$\therefore r = 3$ (Spehere centre at origin
radius 3)

$$\therefore \iiint_V \sqrt{x^2+y^2+z^2}\, dv = \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 \sqrt{r^2}\cdot r^2 \sin\phi \, dr\, d\theta \times d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 r^3 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi} \left.\frac{r^4}{4}\right]_0^3 \sin\phi \, d\theta\, d\phi$$

$$= \frac{81}{4}\int_0^{\pi/2}\int_0^{2\pi} \sin\phi \, d\theta\, d\phi$$

$$= \frac{81}{4}\int_0^{\pi/2} \sin\phi \, [\theta]_0^{2\pi}\, d\phi$$

$$= \frac{81}{4} 2\pi \int_0^{\pi/2} \sin\phi \, d\phi$$

$$= \frac{81\pi}{2}\left[-\cos\phi\right]_0^{\pi/2}$$

$$= \frac{81\pi}{2}$$ Ans.