MAT 110, Differential Calculus & Coordinate Geometry

Week 9

→ Lagrange Multiplier

→ Optimization

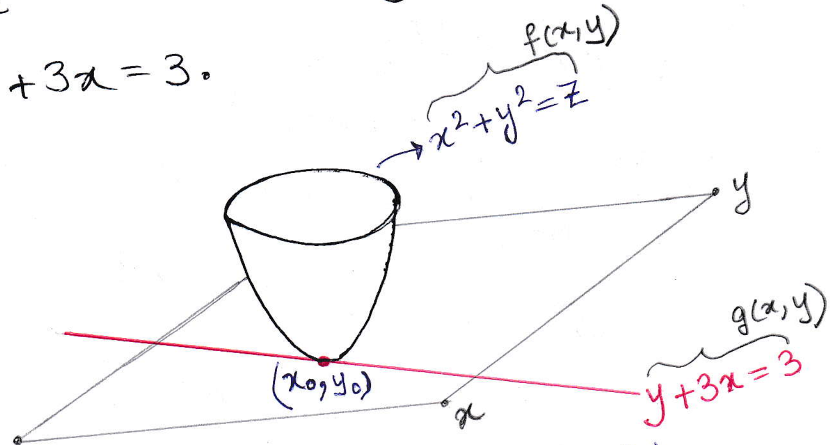
Lagrange Multiplier:

Find maxima / minima of $f(x,y)$ subject to

$\underbrace{\text{constraint}}_{\text{restriction/condition}}$ $g(x,y) = c$

For example we have $f(x,y) = x^2 + y^2$ and constraint

$g(x,y) = y + 3x = 3$.



$(x_0, y_0) \rightarrow$ critical point

We will decide whether $\underbrace{\text{extremum}}_{\text{maximum / minimum}}$ occurs at (x_0, y_0) or not.

specifically

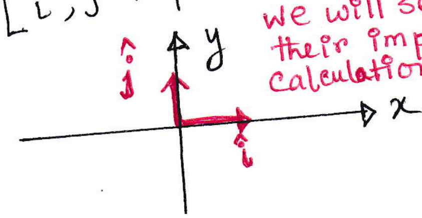
$\nabla g \neq 0$ $\quad \nabla \rightarrow$ slope / gradient on xy plane

$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ introducing λ (a scalar multiple)

[$\hat{i}, \hat{j}$ implies unit vectors]

while $\nabla f(x,y) = \frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j}$

$\nabla g(x,y) = \frac{dg}{dx}\hat{i} + \frac{dg}{dy}\hat{j}$



we will see their impact on calculation later.

Chapter 13.9 pg 996 Anton's Calculus 10^{th} Ed.

Exercise:

Use Lagrange multipliers to find the maximum & minimum values of f subject to the given Constraint. Also, find the pts at which these extreme values occur

(5) $f(x,y)=xy$; $\underbrace{4x^2+8y^2=16}_{g(x,y)}$

$\nabla f = y\hat{i} + x\hat{j}$ $\quad \nabla g = 8x\hat{i} + 16y\hat{j}$

At a constraint, for relative extremum we must have:

$\nabla f = \lambda \nabla g$

$\Rightarrow y\hat{i} + x\hat{j} = \lambda(8x\hat{i} + 16y\hat{j})$

Equating factors of Like terms such as (coefficients of $\hat{i}$ & coefficients of $\hat{j}$)

$x = \lambda 16y$ — ($\hat{j}$)

$y = \lambda 8x$ — ($\hat{i}$)

$\lambda = \dfrac{x}{16y}$

$\lambda = \dfrac{y}{8x}$

Compare

$$\frac{y}{8x} = \frac{x}{16y}$$

$$\frac{y}{x} = \frac{x}{2y}$$

$$x^2 = 2y^2 \Rightarrow \boxed{y^2 = \frac{1}{2}x^2} \text{ — (1)}$$

Substitute y^2 into $g(x,y)$

$$4x^2+8\left(\tfrac{1}{2}x^2\right)=16$$

$$4x^2+4x^2=16$$

$$8x^2=16$$

$$x^2=2$$

$$x=\pm\sqrt{2}$$

Substitute $x=\pm\sqrt{2}$ into (1)

$$y^2=\tfrac{1}{2}(\pm\sqrt{2})^2=\tfrac{1}{2}(2)=1$$

$$\therefore y=\sqrt{1}=\pm 1$$

All possible combination of (x,y):

(x,y)	$(-\sqrt{2},-1)$	$(\sqrt{2},1)$	$(-\sqrt{2},1)$	$(\sqrt{2},-1)$
$f(x,y)$ $=xy$	$(-\sqrt{2})(-1)$ $=\sqrt{2}$	$(\sqrt{2})(1)$ $=\sqrt{2}$	$(-\sqrt{2})(1)$ $=-\sqrt{2}$	$(\sqrt{2})(-1)$ $=-\sqrt{2}$
max/min	max	max	min	min

→ $f(x,y)$ has maxima at $(-\sqrt{2},-1)$ and $(\sqrt{2},1)$

The maximum $=\sqrt{2}$

→ $f(x)$ has minima at $(-\sqrt{2},1)$ and $(\sqrt{2},-1)$

The minimum $=-\sqrt{2}$

⑥ $f(x,y) = x^2 - y^2$; $g(x,y) = x^2 + y^2 - 25 = 0$

$\nabla f = 2x\hat{i} - 2y\hat{j}$ $\nabla g = 2x\hat{i} + 2y\hat{j}$

$\nabla f = \lambda \nabla g$

$2x\hat{i} - 2y\hat{j} = \lambda(2x\hat{i} + 2y\hat{j})$

Equating factors of like terms:

$-2y = \lambda 2y$ — ⓙ

$2x = \lambda 2x$ — ⓘ

$0 = 2y\lambda + 2y$

$2x - \lambda 2x = 0$

$2y(\lambda + 1) = 0$

$2x(1 - \lambda) = 0$

$2y = 0$

$2x = 0$

$y = 0$

$x = 0$

Substitute either $x = 0$ or $y = 0$ into $g(x,y)$

Substituting $y = 0$ into $g(x,y)$

$x^2 + 0^2 - 25 = 0$

$x^2 = 25$

$x = \pm 5$

All possible combination of (x,y):

(x,y)	$(5,0)$	$(-5,0)$
$f(x,y) = x^2 - y^2$	$5^2 - 0^2 = 25$	$(-5)^2 - 0^2 = 25$
max/min	max	max

$f(x,y)$ has maxima at $(5,0)$, $(-5,0)$

The maximum $= 25$

(7) $f(x,y) = 4x^3+y^2$; $\underbrace{2x^2+y^2-1=0}_{g(x,y)}$

$\nabla f = 12x^2\hat{i}+2y\hat{j}$ $\qquad$ $\nabla g = 4x\hat{i}+2y\hat{j}$

$12x^2\hat{i}+2y\hat{j} = \lambda(4x\hat{i}+2y\hat{j})$

Equating factors of like terms:

$12x^2 = 4x\lambda$ — (i) $\qquad$ $2y = \lambda 2y$ — (j)

$3x^2 = x\lambda$ $\qquad$ $2y - \lambda 2y = 0$

$3x^2 - x\lambda = 0$ $\qquad$ $2y(1-\lambda) = 0$

$2y = 0$

$y = 0$

substitute $y=0$ into $g(x,y)$

$2x^2+0^2-1=0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

All possible combination of (x,y):

(x,y)	$\left(\frac{1}{\sqrt{2}}, 0\right)$	$\left(-\frac{1}{\sqrt{2}}, 0\right)$
$f(x,y) = 4x^3+y^2$	$4\left(\frac{1}{\sqrt{2}}\right)^3+0^2 = \sqrt{2}$	$4\left(-\frac{1}{\sqrt{2}}\right)^3+0^2 = -\sqrt{2}$
max/min	max	min

$f(x,y)$ has maximum at $\left(\frac{1}{\sqrt{2}}, 0\right)$

The maximum $= \sqrt{2}$

$f(x,y)$ has minimum at $\left(-\frac{1}{\sqrt{2}}, 0\right)$.

The minimum $= -\sqrt{2}$

⑧ $f(x,y) = x - 3y$; $\underbrace{x^2+3y^2-16=0}_{g(x,y)}$

$\nabla f = \hat{i} - 3\hat{j}$ $\qquad$ $\nabla g = 2x\hat{i} + 6y\hat{j}$

$\nabla f = \lambda \nabla g$

$\hat{i} - 3\hat{j} = \lambda(2x\hat{i} + 6y\hat{j})$

Equating Factor of Like terms:

$1 = \lambda 2x$ — $\hat{i}$ $\qquad$ $-3 = \lambda 6y$ — $\hat{j}$

$\frac{1}{2x} = \lambda$ $\qquad$ $-\frac{1}{2y} = \lambda$

Compare

$$\frac{1}{2x} = -\frac{1}{2y}$$

$$x = -y$$

or $y = -x$

substitute either $x = -y$ or $y = -x$ into $g(x,y)$

Substituting $y = -x$ into $g(x,y) \Rightarrow x^2 + 3(-x)^2 = 16$

$4x^2 = 16$

$x^2 = 4$

$x = \pm 2$

Sub $x = +2$ into $y = -x \Rightarrow y = -(2) = -2$

$\therefore$ the critical pt is $(2,-2)$

Sub $x = -2$ into $y = -x \Rightarrow y = -(-2) = 2$

$\therefore$ the critical pt is $(-2,2)$

(x,y)	$(-2,2)$	$(2,-2)$
$f(x,y) = x - 3y$	$-2-3(2) = -8$	$2-3(-2) = 8$
max/min	min	max

$f(x,y)$ has maximum 8 at $(2,-2)$

$f(x,y)$ has minimum -8 at $(-2,2)$

(9) $f(x,y,z) = 2x + y - 2z \; ; \; \underbrace{x^2+y^2+z^2 = 4}_{g(x,y,z)}$

$\nabla f = 2\hat{i} + \hat{j} - k \qquad \nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla f = \lambda \nabla g$

$2\hat{i} + \hat{j} - 2k = \lambda(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

Equating Factors of Like terms:

$2 = \lambda 2x$ — (i) $\qquad 1 = \lambda 2y$ — (j) $\qquad -2 = \lambda 2z$ — (k)

$\lambda = \frac{1}{x} \qquad \lambda = \frac{1}{2y} \qquad \lambda = \frac{-2}{2z} = -\frac{1}{z}$

compare (i), (j)

$\frac{1}{x} = \frac{1}{2y}$

$x = 2y$

$\boxed{y = \frac{1}{2}x}$

compare (i), (k)

$\frac{1}{x} = -\frac{1}{z}$

$x = -z$

$\boxed{z = -x}$

Substitute "y" and "z" into $g(x,y)$

$x^2 + \left(\frac{1}{2}x\right)^2 + (-x)^2 = 4$

$x^2 + \frac{x^2}{4} + x^2 = 4$

$4x^2 + x^2 + 4x^2 = 16$

$9x^2 = 16$

$x^2 = \frac{16}{9} \Rightarrow x = \pm\frac{4}{3}$

$x = \frac{4}{3} \Rightarrow y = \frac{1}{2}\left(\frac{4}{3}\right) = \frac{2}{3} \;\&\; z = -\left(\frac{4}{3}\right) = -\frac{4}{3}$

$\therefore (x,y,z) = (4/3, 2/3, -4/3)$

$x = -\frac{4}{3} \Rightarrow y = \frac{1}{2}(-4/3) = -2/3 \;\&\; z = -(-4/3) = 4/3$

$\therefore (x,y,z) = (-4/3, -2/3, 4/3)$

(x,y,z)	$(4/3, 2/3, -4/3)$	$(-4/3, -2/3, 4/3)$
$f(x,y,z)$ $= 2x+y-2z$	$2(4/3)+2/3-2(-4/3)$ $= 8/3+2/3+8/3$ $= 18/3 = 6$	$2(-4/3)+(-2/3)-2(4/3)$ $= -8/3 - 2/3 - 8/3$ $= -18/3 = -6$
max/min	max	min

$f(x,y)$ has maximum at $(\frac{4}{3}, 2/3, -4/3)$

The maximum $= 6$

$f(x,y)$ has minimum at $(-4/3, -2/3, 4/3)$

The minimum $= 6$

(10) $f(x,y,z) = 3x+6y+2z$; $g(x,y,z) = 2x^2+4y^2+z^2 = 70$

$\nabla f = 3\hat{i} + 6\hat{j} + 2\hat{k}$ $\qquad$ $\nabla g = 4x\hat{i} + 8y\hat{j} + 2z\hat{k}$

$\nabla f = \lambda \nabla g$

$3\hat{i} + 6\hat{j} + 2\hat{k} = \lambda(4x\hat{i} + 8y\hat{j} + 2z\hat{k})$

Equating factors of Like terms:

$3 = 4x\lambda$ — (i) $\qquad$ $6 = 8y\lambda$ — (j) $\qquad$ $2 = 2z\lambda$ — (k)

$\lambda = \frac{3}{4x}$ $\qquad$ $\lambda = \frac{6}{8y} = \frac{3}{4y}$ $\qquad$ $\lambda = \frac{1}{z}$

compare (i), (j)

$\frac{3}{4x} = \frac{3}{4y}$

$\frac{1}{x} = \frac{1}{y}$

$\boxed{y = x}$

compare (i), (k)

$\frac{3}{4x} = \frac{1}{z}$

$3z = 4x$

$\boxed{z = \frac{4x}{3}}$

substitute "y" and "z" into $g(x,y)$

$$2x^2+4(x)^2+\left(\frac{4x}{3}\right)^2=70$$

$$2x^2+4x^2+\frac{16x^2}{9}=70$$

$$18x^2+36x^2+16x^2=630$$

$$70x^2=630$$

$$x^2=9$$

$$x=\pm 3$$

Sub $x=3 \Rightarrow y=x=3$ & $z=\frac{4x}{3}=\frac{4(3)}{3}=4$

$\therefore (x,y,z)=(3,3,4)$

Sub $x=-3 \Rightarrow y=x=-3$ & $z=\frac{4x}{3}=\frac{4(-3)}{3}=-4$

$\therefore (x,y,z)=(-3,-3,-4)$

(x,y,z)	$(3,3,4)$	$(-3,-3,-4)$
$f(x,y,z)=3x+6y+2z$	$3(3)+6(3)+2(4)=35$	$3(-3)+6(-3)+2(-4)=-35$
max/min	max	min

$f(x,y,z)$ has maximum at $(3,3,4)$

The maximum $=35$

$f(x,y,z)$ has minimum at $(-3,-3,-4)$

The minimum $=-35$

(11) $f(x,y,z) = xyz$, $g(x,y,z) = x^2+y^2+z^2 = 1$

$\nabla f = yz\hat{i} + xz\hat{j} + xy\hat{k}$ $\quad \nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla f = \lambda \nabla g$

$yz\hat{i} + xz\hat{j} + xy\hat{k} = \lambda(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

Equating factors of Like terms

$yz = \lambda 2x$ — (i) $\quad xz = 2y\lambda$ — (j) $\quad xy = 2z\lambda$ — (k)

$\lambda = \dfrac{yz}{2x}$ $\quad \lambda = \dfrac{xz}{2y}$ $\quad \lambda = \dfrac{xy}{2z}$

compare (i), (j), (k)

$$\frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z} \Rightarrow \frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}$$

(i), (j)

$$\frac{yz}{x} = \frac{xz}{y}$$

$$\frac{y}{x} = \frac{x}{y}$$

$$y^2 = x^2$$

(i), (k)

$$\frac{yz}{x} = \frac{xy}{z}$$

$$\frac{z}{x} = \frac{x}{z}$$

$$x^2 = z^2$$

$$z^2 = x^2$$

Substitute y^2, z^2 into $g(x,y)$:

$$x^2 + (x^2) + (x^2) = 1$$

$$3x^2 = 1$$

$$x = \pm\frac{1}{\sqrt{3}}$$

All possible combination of (x,y,z)

(x,y,z)	$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$	$(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$	$(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$	$(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$
$f(x,y,z) = xyz$	$\frac{1}{3\sqrt{3}}$	$-\frac{1}{3\sqrt{3}}$	$-\frac{1}{3\sqrt{3}}$	$-\frac{1}{3\sqrt{3}}$
max/min	maximum	minimum	minimum	minimum
(x,y,z)	$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	$(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$	$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$
$f(x,y,z) = (x,y,z)$	$-\frac{1}{3\sqrt{3}}$	$\frac{1}{3\sqrt{3}}$	$\frac{1}{3\sqrt{3}}$	$\frac{1}{3\sqrt{3}}$
max/min	minimum	maximum	maximum	maximum

The maximum is $\frac{1}{3\sqrt{3}}$ and it occurs at

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

The minimum is $-\frac{1}{3\sqrt{3}}$ and it occurs at

$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

12 $f(x,y,z) = x^4+y^4+z^4$; $g(x,y,z) = x^2+y^2+z^2 = 1$

$\nabla f = 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}$ $\quad \nabla g = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla f = \lambda \nabla g$

$4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k} = \lambda(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

$4x^3 = \lambda 2x$ — (i) $\qquad 4y^3 = \lambda 2y$ — (j) $\qquad 4z^3 = \lambda 2z$ — (k)

$2x^2 = \lambda \qquad 2y^2 = \lambda \qquad 2z^2 = \lambda$

$4x^3 - \lambda 2x = 0 \qquad 4y^3 - \lambda 2y = 0 \qquad z = 0$ & $2z^2 = \lambda$ — (k)

$2x(2x^2 - \lambda) = 0 \qquad 2y(2y^2 - \lambda) = 0$

$2x = 0 \quad 2x^2 = \lambda$ — (i) $\qquad 2y = 0 \quad 2y^2 - \lambda = 0$

$x = 0 \qquad y = 0 \quad 2y^2 = \lambda$ — (j)

$(x,y,z) = (0,0,0)$ $\qquad$ comparing (i), (j), (k) $\Rightarrow 2x^2 = 2y^2 = 2z^2$

$\Rightarrow x^2 = y^2 = z^2$

substitute $y^2 = x^2$, $z^2 = x^2$ into $g(x,y,z)$:

$x^2 + (x^2) + (x^2) = 1$

$3x^2 = 1$

$x = \pm\frac{1}{\sqrt{3}}$

(x, y, z)	$f(x,y,z) = x^4+y^4+z^4$	max/min
$(0,0,0)$	0	—
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\left(\frac{1}{\sqrt{3}}\right)^4+\left(\frac{1}{\sqrt{3}}\right)^4+\left(\frac{1}{\sqrt{3}}\right)^4$ $=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{3}{9}=\frac{1}{3}$	maximum
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"
$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"
$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"
$\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"
$\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$	$\frac{1}{3}$	"

Optimization

[1] At what point or points on the circle $x^2+y^2=1$ does $f(x,y)=xy$ have an absolute maximum, and what is that maximum?

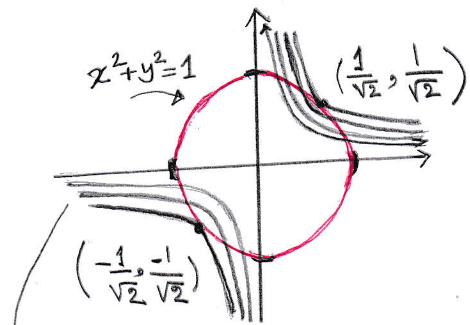
Given $f(x,y)=xy$ $\qquad$ $g(x,y)=x^2+y^2-1=0$

$\nabla g = 2x\hat{i}+2y\hat{j}$

$\nabla f = y\hat{i}+x\hat{j}$

$\nabla f = \lambda \nabla g$

$y\hat{i}+x\hat{j} = \lambda(2x\hat{i}+2y\hat{j})$



$f(x,y)=xy=c$
$\Rightarrow y=\frac{c}{x}$

Equating factors of like terms:

$y=\lambda 2x$ — (i) $\qquad$ $x=\lambda 2y$ — (j)

$\lambda = \frac{y}{2x}$ $\qquad$ $\lambda = \frac{x}{2y}$

compare

$$\frac{y}{2x} = \frac{x}{2y}$$

$$y^2 = x^2$$

substitute $y^2=x^2$ into "g"

$x^2+(x^2)-1=0$

$2x^2=1$

$x=\pm\frac{1}{\sqrt{2}}$

$x=\pm\frac{1}{2} \Rightarrow y^2=x^2 \Rightarrow y=\pm\frac{1}{\sqrt{2}}$

All possible outcomes of (x,y)

(x,y)	$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
xy	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

The function $f(x,y)=xy$ has an absolute maximum of $\frac{1}{2}$ occuring at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

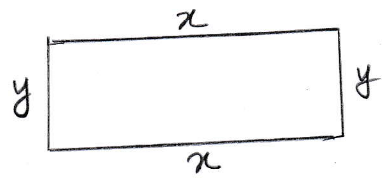
$\boxed{2}$ Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

let x = length of the rectangle
y = width of the rectangle
A = area of the rectangle

$\therefore A = xy$

maximize $A = xy$

perimeter of rectangle $P = x + x + y + y$
$= 2x + 2y$



$x, y > 0$

$\therefore f(x,y) = xy$

constraint (perimeter)

$g(x,y) = 2x + 2y = P$
$\Rightarrow 2x + 2y - p = 0$

$\nabla f = y\hat{i} + x\hat{j}$

$\nabla g = 2\hat{i} + 2\hat{j}$

$\nabla f = \lambda \nabla g$

$y\hat{i} + x\hat{j} = \lambda(2\hat{i} + 2\hat{j})$

Equating factor of like terms:

$y = \lambda 2$ — (i)

$x = \lambda 2$ — (j)

$\lambda = \frac{y}{2}$

$\lambda = \frac{x}{2}$

compare: $\frac{y}{2} = \frac{x}{2} \Rightarrow x = y$

Substitute $x = y$ into "g"

$2(y) + 2y = P$

$y = \frac{P}{4}$

$\Rightarrow x = \frac{P}{4}$

$\boxed{3}$ Find the points on the sphere $x^2+y^2+z^2=36$ that are $\underbrace{\text{closest}}$ to and $\underbrace{\text{farthest}}$ from the point $(1,2,2)$.

We will find points on the sphere that minimize and maximize the square of the distance to $(1,2,2)$

$\therefore$ we want to find the relative extrema of

$$f(x,y,z)=(x-1)^2+(y-2)^2+(z-2)^2$$

subject to the constraint $x^2+y^2+z^2=36$

$$\nabla f(x,y,z)=2(x-1)\hat{i}+2(y-2)\hat{j}+2(z-2)\hat{k}$$

$$\nabla g(x,y,z)=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$\nabla f=\lambda\nabla g$$

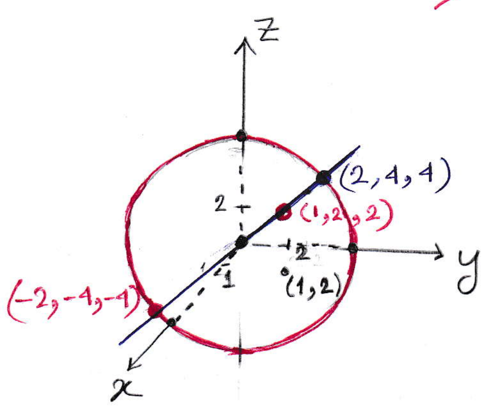
$$2(x-1)\hat{i}+2(y-2)\hat{j}+2(z-2)\hat{k}=\lambda(2x\hat{i}+2y\hat{j}+2z\hat{k})$$

Equating Factor of Like terms:

$2(x-1)=2\lambda x$ — (i) $\Rightarrow \lambda=\dfrac{x-1}{x}$

$2(y-2)=2\lambda y$ — (j) $\Rightarrow \lambda=\dfrac{y-2}{y}$

$2(z-2)=2\lambda z$ — (k) $\Rightarrow \lambda=\dfrac{z-2}{z}$



Compare (i), (j)

$$\frac{x-1}{x}=\frac{y-2}{y}$$

$$xy-2x=xy-y$$

$$xy-xy=2x-y$$

$$0=2x-y$$

$$y=2x$$

Compare (i), (k)

$$\frac{x-1}{x}=\frac{z-2}{z}$$

$$xz-z=xz-2x$$

$$z=2x$$

Substitute $y=2x$, $z=2x$ into "g"

$$x^2+(2x)^2+(2x)^2=36$$

$$x^2+4x^2+4x^2=36 \Rightarrow 9x^2=36$$

$$x^2=4 \Rightarrow x=\pm 2$$

$x = 2 \Rightarrow \quad y = 2x = 2(2) = 4$

$\qquad\qquad\quad z = 2x = 2(2) = 4$

$\therefore (x, y, z) = (2, 4, 4)$

$x = -2 \Rightarrow \quad y = 2(-2) = -4$

$\qquad\qquad\quad z = 2(-2) = -4$

$\therefore (x, y, z) = (-2, -2, -4)$

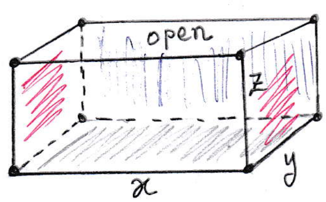
$f(2,4,4) = (2-1)^2 + (4-2)^2 + (4-2)^2 = 9$

$f(-2,-4,-4) = (-2-1)^2 + (-4-2)^2 + (-4-2)^2 = 81$

→ $(2,4,4)$ is the pt on the sphere closest to $(1,2,2)$

→ $(-2,-4,-4)$ is the pt on the sphere that is farthest to $(1,2,2)$.

④ Consider a rectangular box having volume of $32\ ft^3$. Minimize the surface area of the box. consider the box is open at the top



Volume

$V = xyz = 32$

Surface Area $S = xy + 2xz + 2yz$

Areas:

$xy \rightarrow$ bottom

$yz + yz \rightarrow$ sides

$xz + xz \rightarrow$ front back

Minimize "f" subject to "g"

$f = xy + 2xz + 2yz \qquad g = xyz = 32$

$\nabla f = (y+2z)\hat{i} + (x+2z)\hat{j} + (2a+2y)\hat{k} \quad \Big\} \quad \nabla g = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\nabla f = \lambda \nabla g$

$(y+2z)\hat{i} + (x+2z)\hat{j} + (2x+2y)\hat{k} = \lambda(yz\hat{i} + xz\hat{j} + xy\hat{k})$

Equating factor of like terms:

$y+2z=\lambda yz$ — (i)

$\lambda = \frac{y+2z}{yz}$

$\lambda = \frac{1}{z}+\frac{2}{y}$

$x+2z=\lambda xz$ — (j)

$\lambda = \frac{x+2z}{xz}$

$\lambda = \frac{1}{z}+\frac{2}{x}$

$2x+2y=\lambda xy$ — (k)

$\lambda = \frac{2x+2y}{xy}$

$\lambda = \frac{2}{y}+\frac{2}{x}$

compare (i), (j)

$\frac{1}{z}+\frac{2}{y}=\frac{1}{z}+\frac{2}{x}$

$\frac{2}{y}=\frac{2}{x}$

$\boxed{y=x}$

Compare (i), (k)

$\frac{1}{z}+\frac{2}{y}=\frac{2}{y}+\frac{2}{x}$

$\frac{1}{z}=\frac{2}{x}$

$\boxed{z=\frac{1}{2}x}$

substitute $y=x$, $z=\frac{1}{2}x$ into "g"

$xyz=32$

$x(x)(\frac{1}{2}x)=32$

$x^3=64$

$x=\sqrt[3]{64}=4$

$\because y=x \quad \therefore y=4$

$\because z=\frac{1}{2}x \quad \therefore z=\frac{1}{2}(4)=2$

To maximize the surface area of the box, the dimension should be

$x=4,\ y=4,\ z=2.$