Lecture 11

Absolute Extrema of Two Variables

Important notion

A region in $\mathbb{R}^2$ is called <u>closed</u> if it includes its boundary.

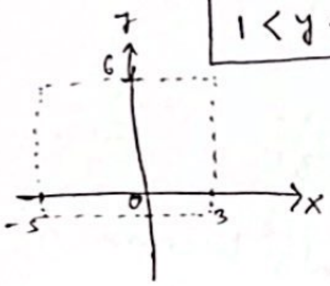
A region in $\mathbb{R}^2$ is called <u>open</u> if it doesn't include any of its boundary points.

A region in $\mathbb{R}^2$ is called bounded if it can be completely contained in a disk. In other words, a region will be bounded if it is finite.

ie..

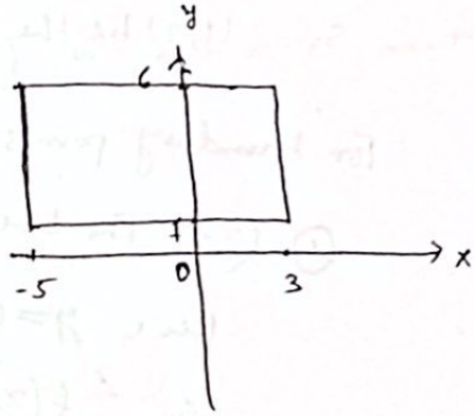
open

$$\boxed{\begin{matrix} -5 < x < 3 \\ 1 < y < 6 \end{matrix}}$$



Closed

$$\boxed{\begin{matrix} -5 \le x \le 3 \\ 1 \le y \le 6 \end{matrix}}$$



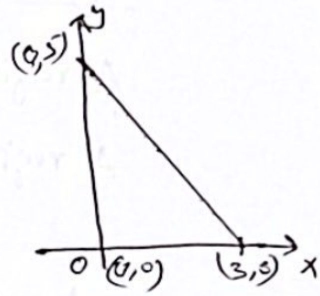
Finding Absolute Extrema

1. Find all the critical points of f that lie in the interior of $\mathbb{R}^2$.
2. Find all boundary points at which the absolute extrema can occur.
3. Evaluate $f(x,y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

__Example:__ Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region $\mathbb{R}^2$ with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



__Solⁿ__

Here,

$$f_x(x,y) = 3y - 6 \quad , \quad f_y(x,y) = 3x - 3$$

For critical points

$$3y - 6 = 0 \quad , \quad 3x - 3 = 0$$
$$y = 2 \qquad\qquad x = 1$$

So, $(1,2)$ be the only critical point.

For boundary points;

(i) ~~(0,0)~~ The line segment $(0,0)$ and $(3,0)$

Here $y = 0$

So, $f(x,0) = -6x + 7 \qquad 0 \le x \le 3$

So, the function has no critical points because $f'(x,0) = -6 \quad \forall x$.

So, only $(0,0)$ and $(3,0)$ may has occur ~~critical value~~ ~~some~~ extrema values.

(ii) Similarly, the line segment $(0,0)$ & $(0,5)$, $x = 0$

$$f(0,y) = -3y + 7 \qquad f'(0,y) = -3 \quad \forall y$$

(iii) the ~~line~~ line segmen ~~&~~ $(3,0)$ and $(0,5)$ we get

$$\frac{x-3}{3-0} = \frac{y-0}{0-5} \Rightarrow y = \frac{-5}{3}x + 5$$

$$f\left(x, -\frac{5}{3}x+5\right) = 3x\left(-\frac{5}{3}x+5\right) - 6x - 3\left(-\frac{5}{3}x+5\right) + 7$$

$$= -5x^2 + 15x - 6x + 5x - 15 + 7$$

$$= -5x^2 + 14x - 8$$

$$f'\left(x, -\frac{5}{3}x+5\right) = -10x + 14 = 0$$

$$x = \frac{14}{10} = \frac{7}{5}$$

$$y = -\frac{5}{3}\cdot\frac{7}{5} + 5 = -\frac{7}{3} + 5 = \frac{8}{3}$$

~~we get a critical point $(7/5, 8/3)$ for the line segment of (3,0) & (~~

~~(0,3) (3,0) and (0,5).~~

So, we have a critical point $\left(\frac{7}{5}, \frac{8}{3}\right)$ for the line segment of $(3,0)$ and $(0,5)$.

Therefore,

$$f(0,0) = 7$$

$$f(1,2) = 3.1.2 - 6.1 - 3.2 + 7 = 1$$

$$f(3,0) = -11$$

$$f(0,5) = -8$$

$$f\left(\frac{7}{5}, \frac{8}{3}\right) = \frac{9}{5}$$

Thus, Absolute maximum is 7 at $(0,0)$

Absolute minimum is -11 at $(3,0)$.

Extra Problem:

1. Find the absolute extrema of the ~~region~~ given function on the indicated closed and bounded set R.

ⓐ $f(x,y) = xy - x - 3y$; R is the triangular region with vertices $(0,0)$, $(0,4)$, and $(5,0)$.

ⓑ $f(x,y) = x^2 + 2y^2 - x$; R is the disk $x^2+y^2 \le 4$.