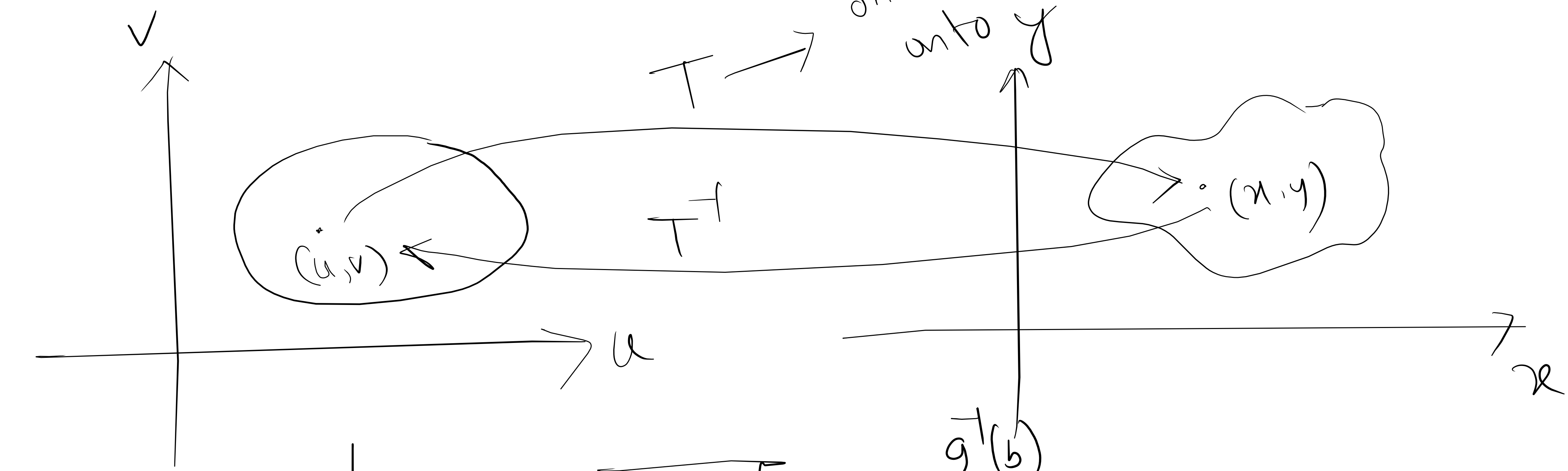
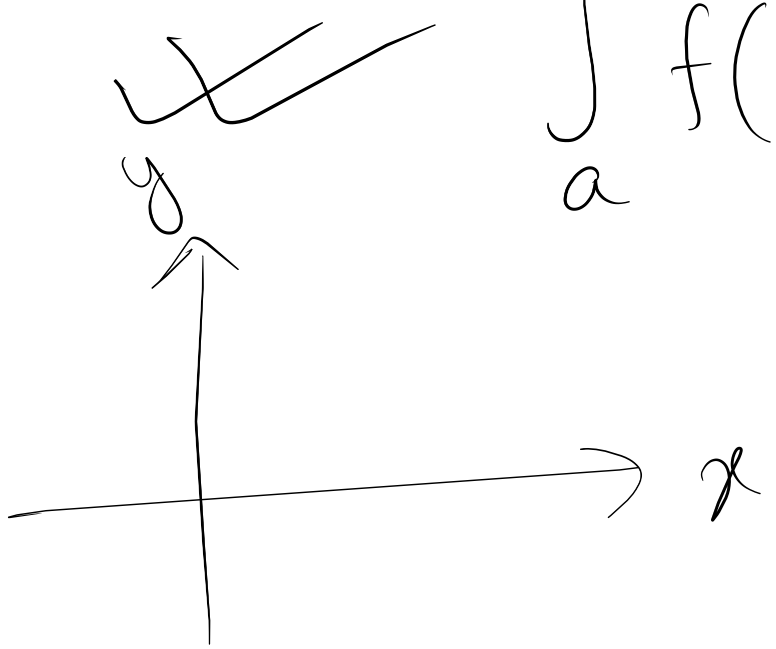
Jacobian :





$$\int_a^b f(g(x))\,\boxed{g'(x)\,dx} = \int_{g^{-1}(a)}^{g^{-1}(b)} f(u)\,du$$

let $g(x) = u$

$\Rightarrow g'(x)\,dx = du$

Jacobian Definition: If T is the transformation from uv plane to xy plane defined by the equations

$x = x(u,v)$, $y = y(u,v)$, then

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = J$$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Using Jacobian in double integration: If the transformation $x = x(u,v)$, $y = y(u,v)$ maps the region S in the uv plane into the region R in the xy plane if $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, doesnot change sign on S, then

$$\iint_R f(x,y)\, dA_{xy} = \iint_S f\big(x(u,v), y(u,v)\big)\, |J|\, dA_{uv}$$

R → Region in xy plane

S - region in uv plane

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$dx\,dy = r\,dr\,d\theta = |J|\,dr\,d\theta$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$dx\,dy = |J|\,dr\,d\theta = r\,dr\,d\theta$$

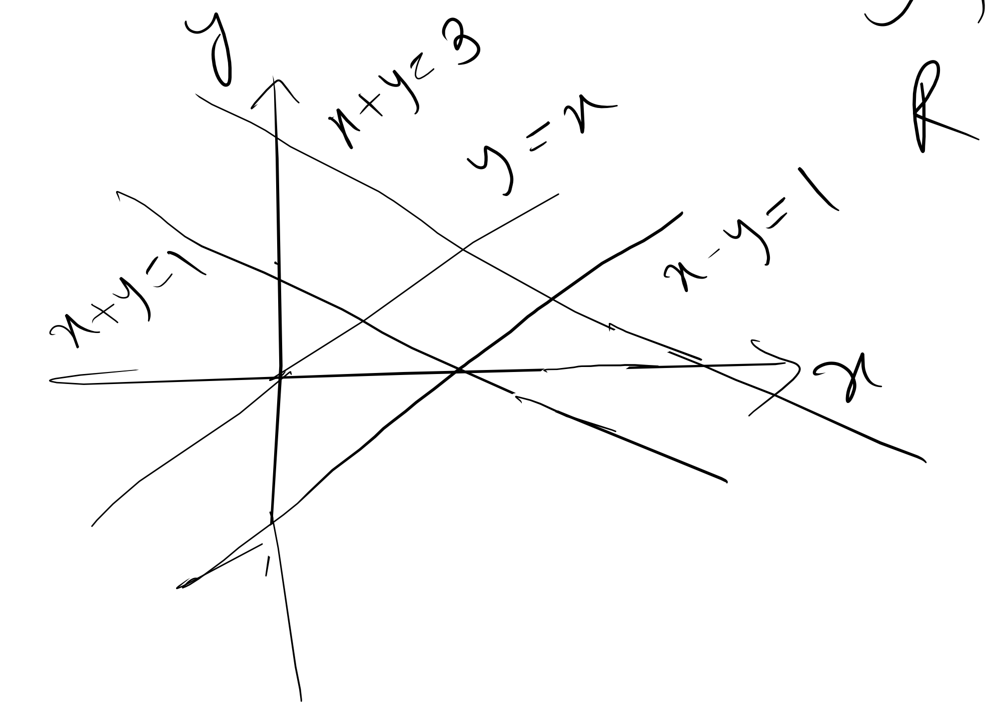
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r \geq 0$$

Ex: Evaluate $\displaystyle\iint_R \frac{x-y}{x+y}\, dA$, where R is the region enclosed by $\overset{v=0}{x-y=0}$, $\overset{v=1}{x-y=1}$, $\overset{u=1}{x+y=1}$, $\overset{u=3}{x+y=3}$.

let $x+y=u$, $x-y=v$

Solution:



$$\iint_R \left(\frac{x-y}{x+y}\right) dA_{xy} = \iint_S \frac{v}{u}\, |J|\, dA_{uv}$$

$x = x(u,v)$

$x+y=u \cdots (i)$

$x-y=v \cdots (ii)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(i)+(ii)

$$\begin{aligned} x + \cancel{y} &= u \\ x - \cancel{y} &= v \\ \hline 2x &= u+v \\ \Rightarrow x &= \frac{u+v}{2} \end{aligned}$$

putting $x = \frac{u+v}{2}$ in (i)

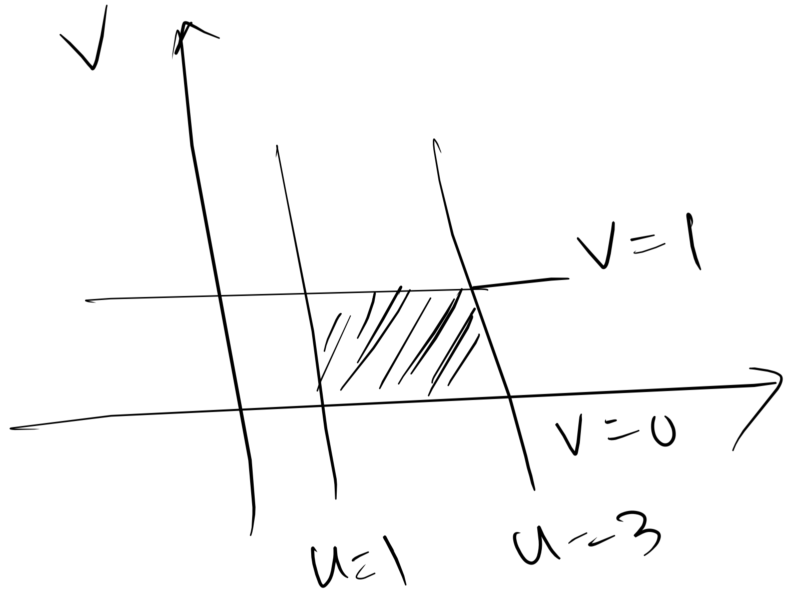
$$\frac{u+v}{2} + y = u$$

$$\Rightarrow y = u - \frac{u+v}{2} = \frac{2u-u-v}{2}$$

$$= \frac{u-v}{2}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial\left(\frac{u+v}{2}\right)}{\partial u} & \frac{1}{2}\frac{\partial(u+v)}{\partial v} \\ \frac{1}{2}\frac{\partial(u-v)}{\partial u} & \frac{1}{2}\frac{\partial(u-v)}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}$$

$$= -1/4 - 1/4 = -1/2$$



$$\iint_R \left(\frac{x-y}{x+y}\right) dA = \iint_S \frac{v}{u} |J| \, dA_{uv}$$

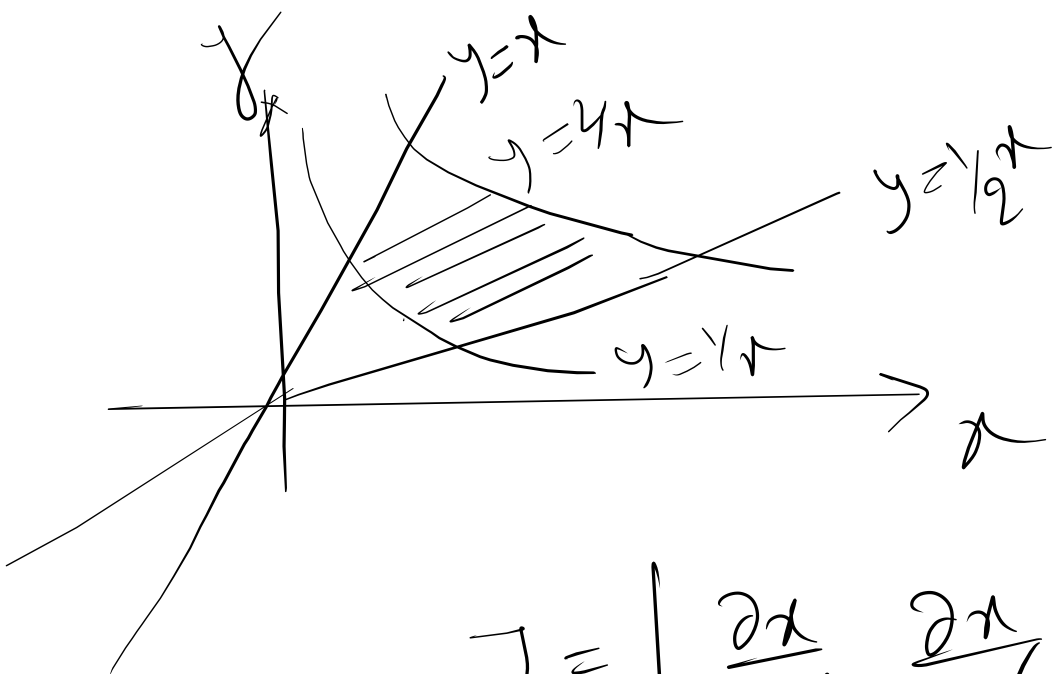
$$= \int_0^1 \int_1^3 \frac{v}{u} \, \frac{1}{2} \, du \, dv$$

$$= \frac{1}{2} \int_0^1 v \left[\ln u\right]_1^3 dv$$

$$= \frac{\ln 3}{2} \cdot \frac{1}{2} = \frac{\ln 3}{4}.$$

14.7 1-12, 21-24, 35-37

Evaluate $\iint_R e^{xy}\, dA$, where R is the region enclosed by $y = \frac{1}{2}x$ & $y = x$ & the hyperbola $y = 1/x$ & $y = 2/x$



let $y/x = u$, $xy = v$

$$\frac{y}{x} = \frac{1}{2} \qquad y/x = 1$$

$$\Rightarrow u = \frac{1}{2} \qquad u = 1$$

$xy = 1$, $xy = 2$

$\Rightarrow v = 1$, $v = 2$

$$x = x(u,v), \quad y = y(u,v)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$y/x = u$

$\Rightarrow y = xu \quad \ldots (1)$

$\Rightarrow y = \sqrt{\frac{v}{u}}\, u = \sqrt{uv}$

$xy = v$

$\Rightarrow x \, xu = v$

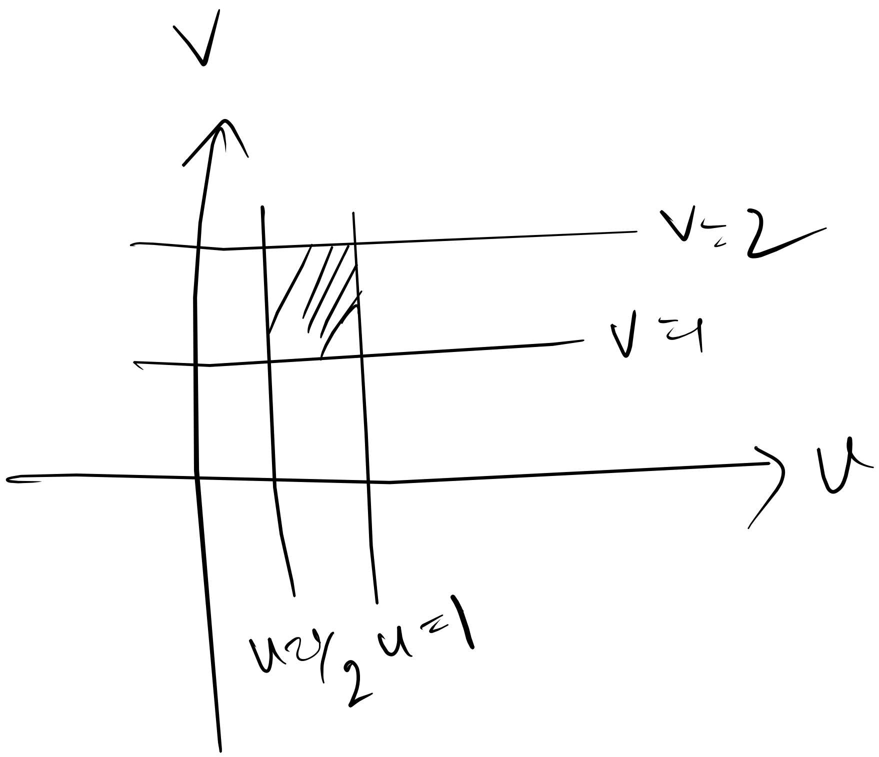
$\Rightarrow x^2 = v/u$

$\Rightarrow x = \sqrt{\frac{v}{u}}$

$$J = \begin{vmatrix} \frac{\partial}{\partial u}\left(\sqrt{\frac{v}{u}}\right) & \frac{\partial}{\partial v}\left(\sqrt{\frac{v}{u}}\right) \\ \frac{\partial}{\partial u}\left(\sqrt{uv}\right) & \frac{\partial}{\partial v}\left(\sqrt{uv}\right) \end{vmatrix} = \begin{vmatrix} -\frac{1}{2u}\sqrt{\frac{v}{u}} & \frac{1}{2\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$u = 1/2$, $u = 1$, $v = 1$, $v = 2$

$$= -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u}$$



$$\iint_R e^{xy}\, dA = \iint_S e^v\, |J|\, dA_{uv}$$

$$= \int_1^2 \int_{1/2}^1 e^v \frac{1}{2u}\, du\, dv$$

H.W.

Ex:

2) Use the transformation $u = x - 2y$, $v = 2x + y$ to find

$$\iint_R \left(\frac{x-2y}{2x+y}\right) dA$$, where R is the rectangle enclosed

by $x - 2y = 1$, $x - 2y = 4$, $2x + y = 1$, $2x + y = 3$.

$u = 1$ $\quad$ $u = 4$ $\quad$ $v = 1$ $\quad$ $v = 3$

$$\iint_R \left(\frac{x-2y}{x+2y}\right) dA = \int_1^3 \int_1^4 \frac{u}{v} |J| \, du\, dv = \int_1^3 \int_1^4 \frac{u}{v} \frac{3}{25} \, du\, dv$$

$$\boxed{\begin{aligned} x - 2y &= u \quad \ldots (1) \\ 2x + y &= v \quad \ldots (11) \end{aligned}}$$

$2 \times (11) + (1) \Rightarrow \quad 4x + 2y = 2v$

$x - 2y = u$

$x = \frac{u + 2v}{5}$

$$y = v - \frac{2u + 4v}{5}$$

$$= \frac{5v - 2u - 4v}{5} = \frac{v - 2u}{5}$$

$$J = \begin{vmatrix} \frac{1}{5}\frac{\partial}{\partial u}(u+2v) & \frac{1}{5}\frac{\partial}{\partial v}(u+2v) \\ \frac{1}{5}\frac{\partial}{\partial u}(v-2u) & \frac{1}{5}\frac{\partial}{\partial v}(v-2u) \end{vmatrix} = \begin{vmatrix} 1/5 & 2/5 \\ -2/5 & 1/5 \end{vmatrix} = \frac{1}{25} + \frac{4}{25}$$

$$= \frac{5}{25} = 1/5$$

$$\iint_R \left(\frac{x-2y}{2x+y}\right) dA = \int_1^3 \int_1^4 \frac{u}{v} \, 1/5 \, du\, dv$$

H.W ??