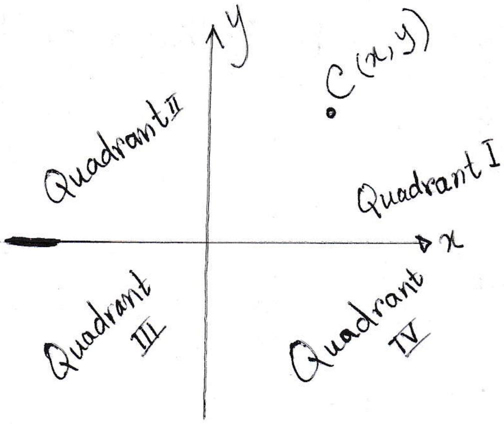
Week 11,

MAT 110 DIFFERENTIAL CALCULUS & COORDINATE GEOMETRY

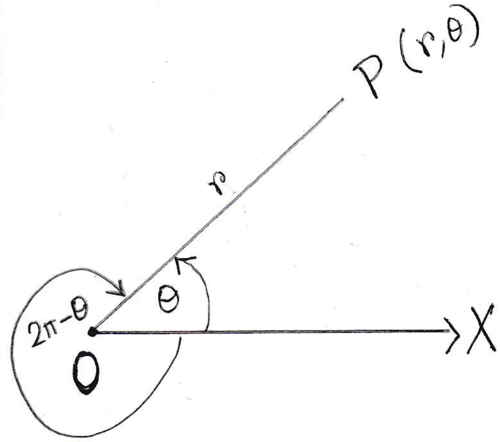
INTRODUCTION TO POLAR COORDINATES

Cartesian Coordinate



Coordinate point: $C(x,y)$

Polar Coordinate



O – pole

OX – polar axis

P – any point in the plane

OP ~ r ~ radius vector or radius

$\angle XOP = \theta$ Vectorial angle

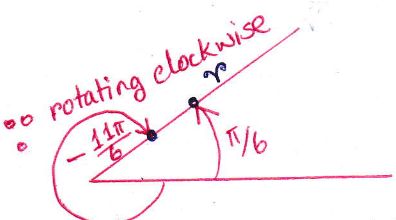
Polar coordinate point: $P(r,\theta)$

r → distance and always +ve.

We will get infinite number of polar coordinates for a specific "r" value such as:

$$\theta = \frac{\pi}{6} \pm 2n\pi$$

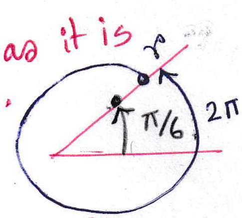
$n = 1, 2, 3, \ldots$ +ve integers



$$\left(r, \frac{\pi}{6}\right) = \left(r, -\frac{11\pi}{6}\right)$$

$2\pi - \frac{\pi}{6} = +\frac{11\pi}{6}$

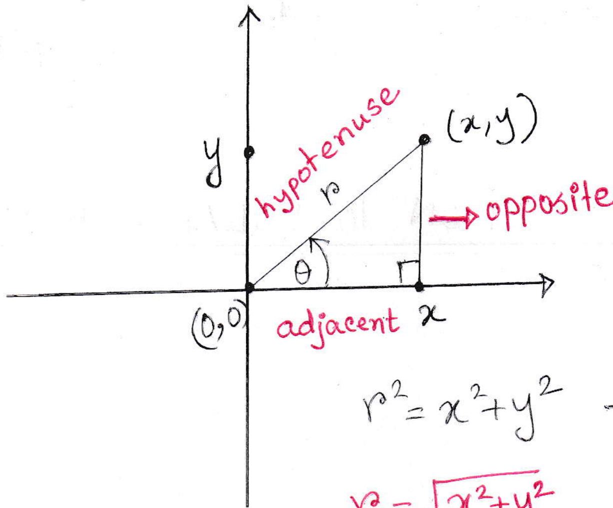
Consider as $-\frac{11\pi}{6}$ as it is rotating clockwise.



$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$

$$\left(r, \frac{\pi}{6}\right) = \left(r, \frac{13\pi}{6}\right)$$

Adding or subtracting $2n\pi$ does not change our polar coordinate pt from the original "θ"



$r^2 = x^2 + y^2$ → by Pythagorean Thm

$r = \sqrt{x^2+y^2}$

$\sin\theta = \frac{y}{r}$ → $\mathrm{Cosec}\,\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ → $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ → $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$

$x = r\cos\theta$ (i) , $y = r\sin\theta$ (ii)

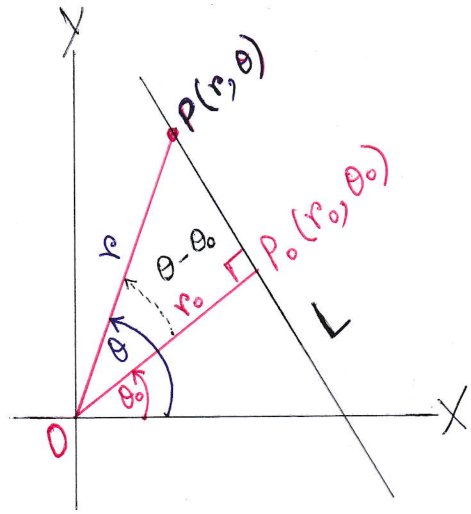
Sq eqn (i) → $x^2 = r^2\cos^2\theta$

Sq eqn (ii) → $y^2 = r^2\sin^2\theta$

Add → $x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$r^2 = x^2 + y^2$

POLAR EQUATIONS FOR CONIC Sections



OP, OP_0, PP_0 form a triangle

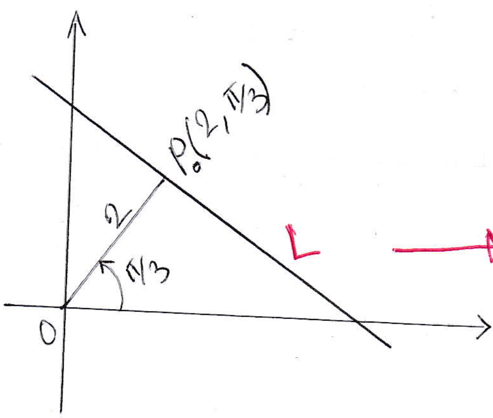
$\left.\begin{array}{l} OP = r \\ OP_0 = r_0 \\ PP_0 \end{array}\right\}$ Arms of the triangle

ΔOPP_0 is a right angle triangle

$\cos(\theta - \theta_0) = \dfrac{r_0}{r}$ $\left[\cos\theta = \dfrac{x}{r}\right]$ Standard

$\boxed{r\cos(\theta-\theta_0) = r_0}$ $\longrightarrow$ Eqn of Line in Polar coordinates

θ_0, r_0 are fixed



$\longrightarrow$ Given $r_0 = 2$, $\theta_0 = \dfrac{\pi}{3}$

Eqn of L is:

$r\cos(\theta - \theta_0) = r_0$

$r\cos\left(\theta - \dfrac{\pi}{3}\right) = 2$

We know,

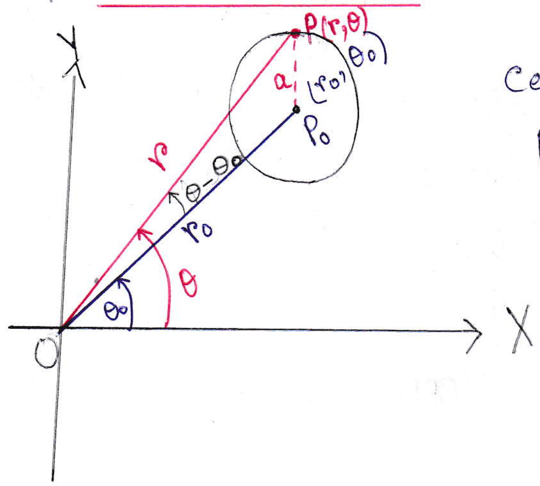
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\therefore\ r\cos\left(\theta - \frac{\pi}{3}\right) = \underbrace{r\cos\theta}_{x}\ \underbrace{\cos\frac{\pi}{3}}_{1/2} + \underbrace{r\sin\theta}_{y}\ \underbrace{\sin\frac{\pi}{3}}_{\sqrt{3}/2} = 2$$

$$\Rightarrow x\left(\tfrac{1}{2}\right) + y\left(\tfrac{\sqrt{3}}{2}\right) = 2$$

$$\Rightarrow x + \sqrt{3}\,y = 4$$ $\longrightarrow$ Eqn of Line in cartesian coordinate for the given line in Polar coordinate

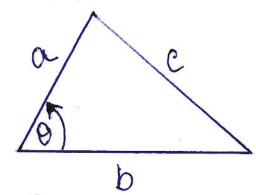
CIRCLE



Center: $(r_0, \theta_0) \rightarrow P_0$; radius $= a$

P is any pt on the circle with coordinate (r,θ)

Cosine formula for triangle:

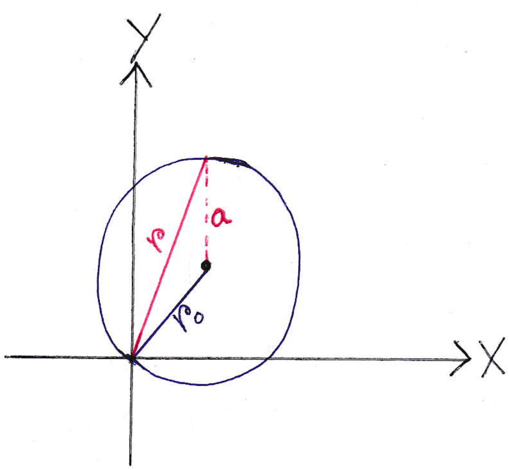


$$c = \sqrt{a^2+b^2-2ab\cdot\cos\theta}$$

For the ΔOPP_0

$$\boxed{a^2 = r^2 + r_0^2 - 2rr_0\cos(\theta-\theta_0)} \quad \text{(i)}$$

Eqn of circle in Polar coordinate system



If the circle passes through the origin then $r_0 = a$

$a \rightarrow$ radius of the circle

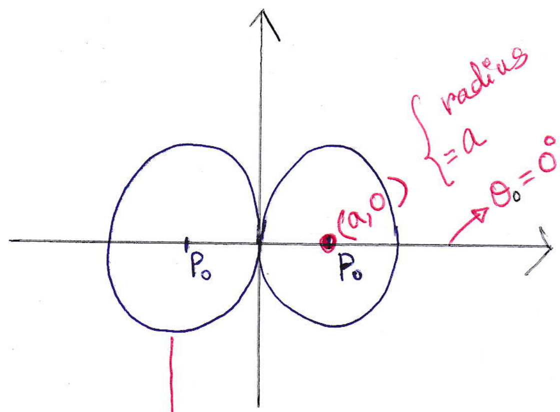
$\therefore$ Eqn of circle in polar coordinate systems:

$$a^2 = r^2 + a^2 - 2ra\cos(\theta-\theta_0)$$

$$0 = r^2 - 2ar\cos(\theta-\theta_0)$$

$$r^2 = 2ar\cos(\theta-\theta_0)$$

$$\boxed{r = 2a\cos(\theta-\theta_0)} \quad \text{(ii)}$$



Center of circle P_0 is on "x" axis

It passes through the origin

It is on the right side of "x" axis

Eqn of circle:

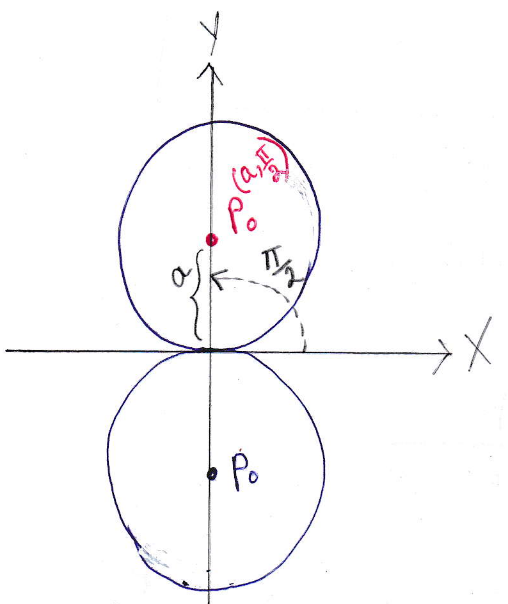
$r = 2a\cos(\theta - \theta_0)$ → Ref (ii) ∴ circle passes through the origin

$\boxed{r = 2a\cos\theta}$ — (iii) ∴ $\theta_0 = 0$

As "x" axis on the negative direction

$-r = 2a\cos\theta$

$\boxed{r = -2a\cos\theta}$ — (iii)



The circle is on the +ve "y" axis

radius $= a$

Angular coordinate $= \frac{\pi}{2}$

Eqn of circle:

$r = 2a\cos\left(\theta - \frac{\pi}{2}\right)$ → Refer (ii)

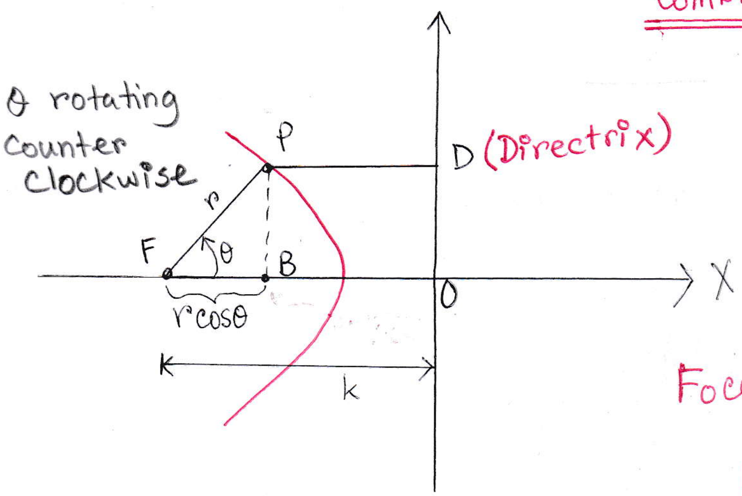
$r = 2a\sin\theta$

∴ $\boxed{r = 2a\sin\theta}$

The circle is on the -ve "y" axis

$\boxed{r = -2a\sin\theta}$

Combine Eqn of Parabola, Ellipse, Hyperbola



Focus directrix eqn:

$PF = e\,PD$

∵ $PF \propto PD$
the ratio of PF & PD is a constant "e"
e → eccentricity of the conic section

$r = e(OF - BF)$

$r = e(k - r\cos\theta)$

$r + er\cos\theta = ke$

$r(1 + e\cos\theta) = ke$

$$\boxed{r = \frac{ke}{1 + e\cos\theta}}$$

We can reach the eqn of parabola, ellipse, hyperbola from this general eqn above.

Parabola: $e = 1$, $r = \dfrac{k}{1 + \cos\theta}$

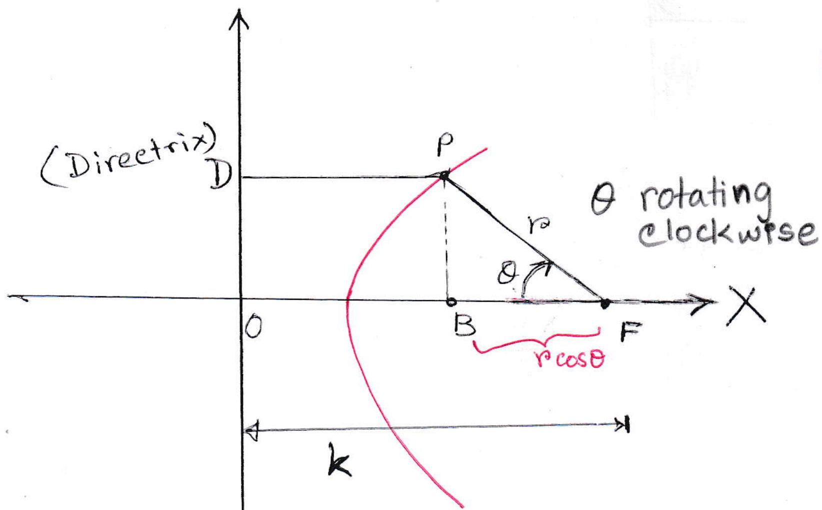
Ellipse: $0 < e < 1$, take $e = \frac{1}{2}$, $r = \dfrac{k/2}{1 + \frac{\cos\theta}{2}}$

$$r = \frac{k}{2 + \cos\theta}$$

Hyperbola: $e>1$

Let's take $e=2$

$$r = \frac{2k}{1+2\cos\theta}$$



Focus directrix eqn:

$$PF = e\,PD$$

$$r = e(OF - BF)$$

$$= e(k - (-r\cos\theta))$$

$$= e(k + r\cos\theta)$$

$$r - er\cos\theta = ke$$

$$r(1 - e\cos\theta) = ke$$

$$\boxed{r = \frac{ke}{1 - e\cos\theta}}$$

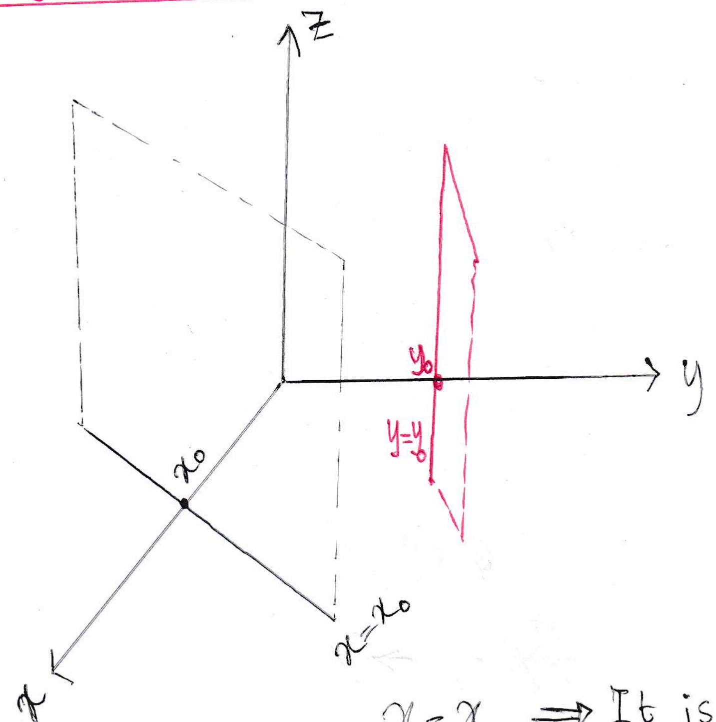
Parabola: $e=1$, $r = \dfrac{k}{1-\cos\theta}$

Ellipse: $0<e<1$, take $e=\frac{1}{2}$, $r = \dfrac{k/2}{1-\frac{\cos\theta}{2}} = \dfrac{k}{2+\cos\theta}$

Hyperbola: $e>1$

take $e=2$ $\quad r = \dfrac{2k}{1-2\cos\theta}$

Cylindrical Coordinates



3-D Space

$\underbrace{P(r, \theta, z)}$
Coordinate point

$x = x_0 \Rightarrow$ It is a plane through x-axis } 3-D

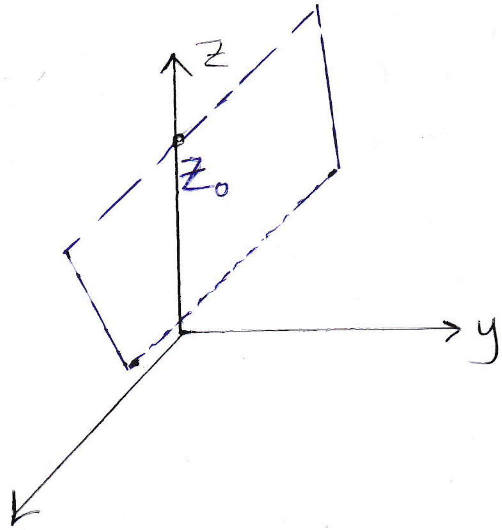
$x = x_0 \Rightarrow$ a $\underbrace{\text{straight}}_{\text{(vertical)}}$ line } 2-D

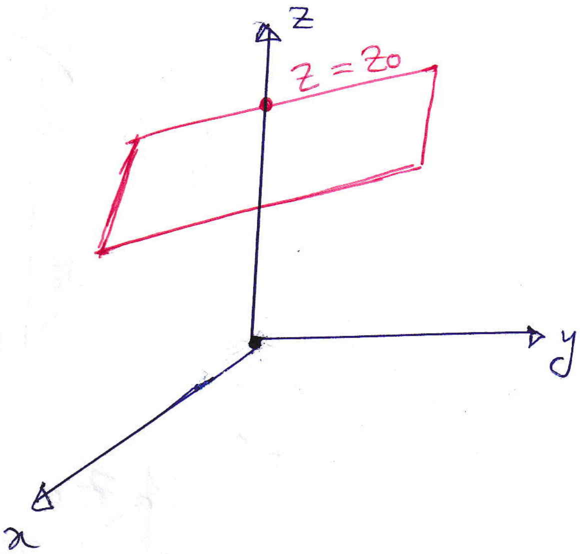
$x = x_0 \Rightarrow$ This plane can be extended infinitely in the direction of y-axis & in the direction of z-axis.

Similarly $y = y_0 \Rightarrow$ A plane through y-axis $\to$ 3-D
$\Rightarrow$ A horizonal line $\to$ 2-D

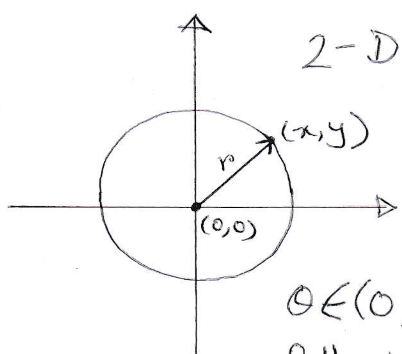
$y = y_0 \Rightarrow$ This plane can be extended infinitely in the direction of x-axis & in the direction of z-axis.

$Z = Z_0 \Rightarrow$ A plane parallel to the horizon or parallel to xy-plane.



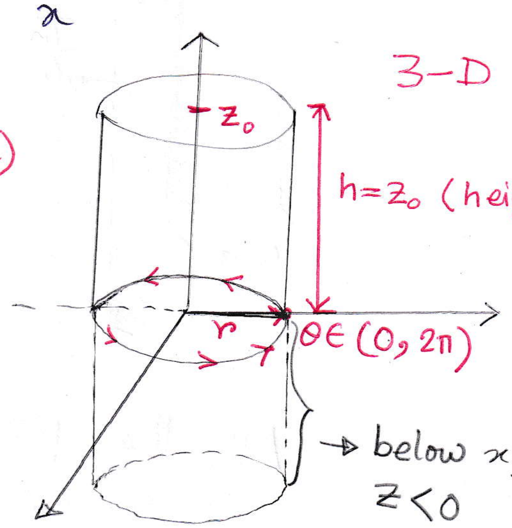


2-D (Circle)



$\theta \in (0, 2\pi)$
full circle

3-D (Cylinder)

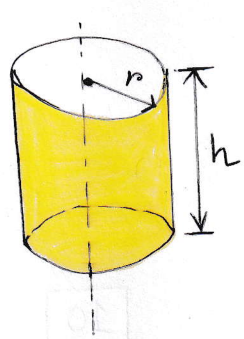


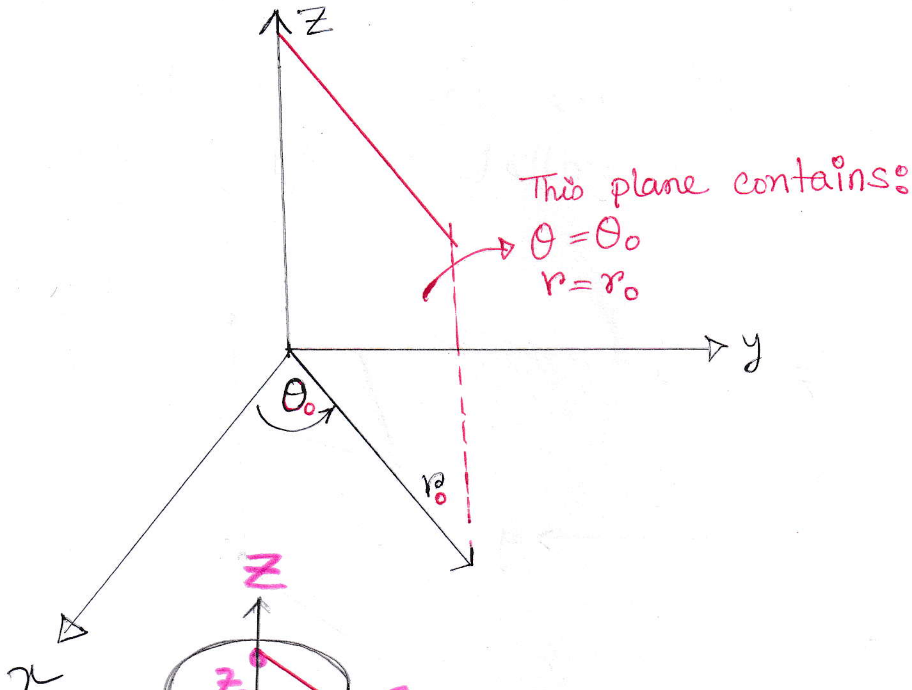
$h = Z_0$ (height)

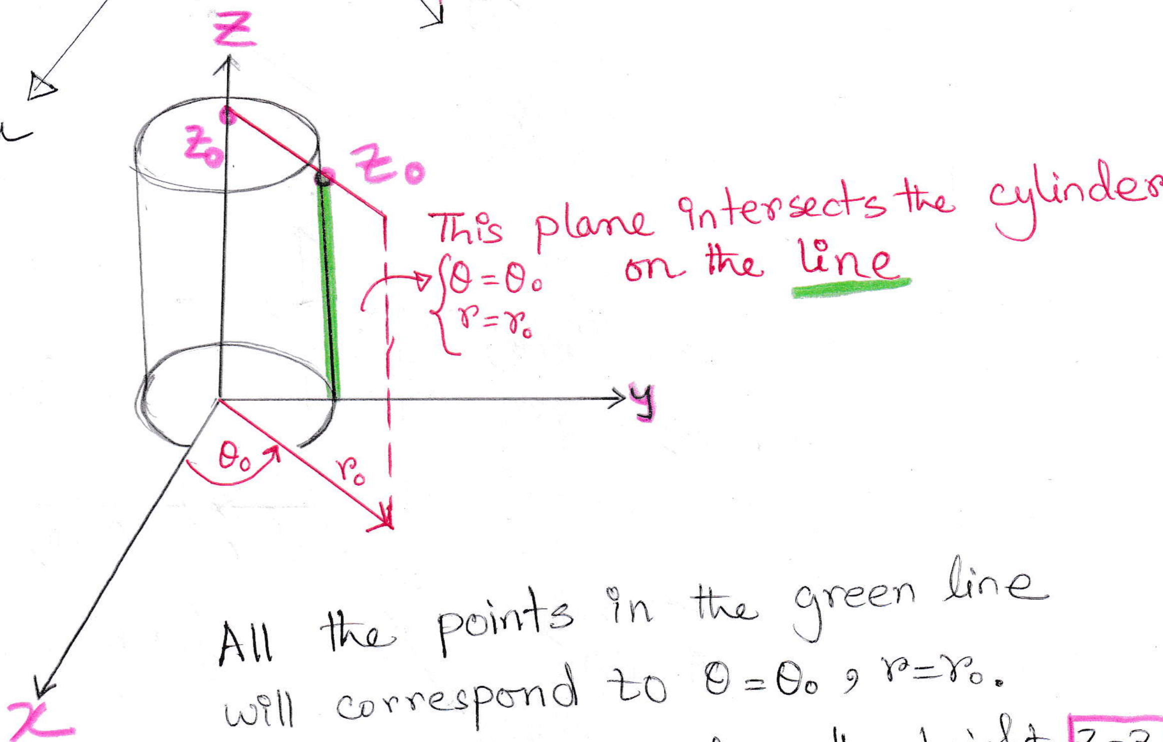
$\theta \in (0, 2\pi)$

$\rightarrow$ below xy-plane
$Z < 0$

A right circular cylinder is a cylinder that has a closed circular surface having two parallel bases on both ends and whose elements are perpendicular to its base. It is also called a right cylinder.



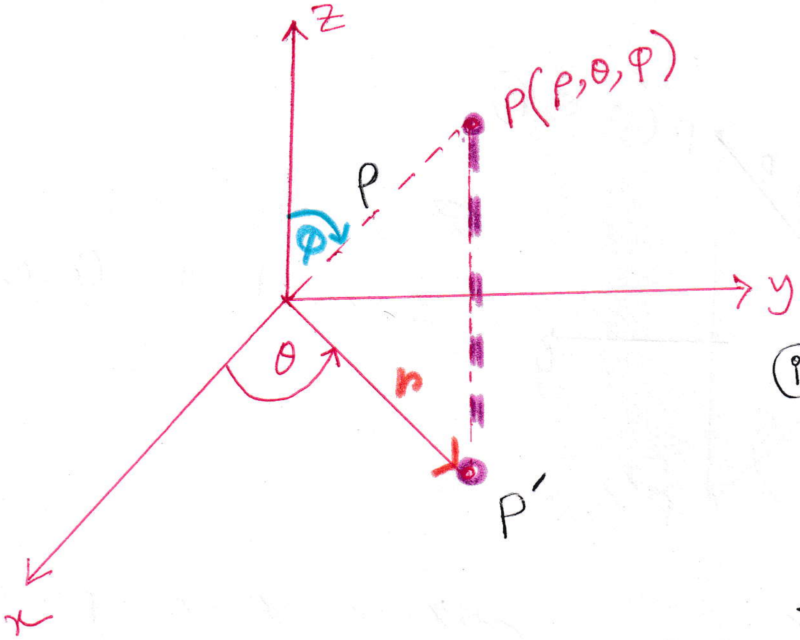




All the points in the green line will correspond to $\theta = \theta_0$, $r = r_0$.
Similarly if we introduce the height $\boxed{z = z_0}$, it will correspond to the green line as well.

Rectangular to Cylindrical	**Cylindrical to Rectangular**
$x = r\cos\theta$	$r = \sqrt{x^2 + y^2}$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

Spherical Coordinate



(i) $\rho \to$ rho

$\rho = |\vec{OP}| \to$ radius of sphere

$\rho \geq 0$

(ii) $\theta \to$ angle from x axis to the projection of the point P on xy-plane

$\to$ Vertical projection of P on xy-plane

(iii) $\to$ Let's call the projecting point on xy-plane P' from the point P.

$P' \to (r, \theta)$ $\quad r = d\{(0,0), P'\}$

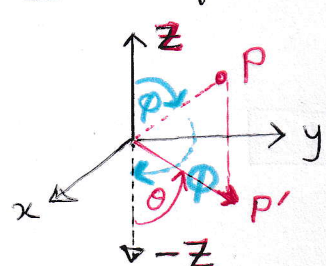
$\theta \in (0, 2\pi)$

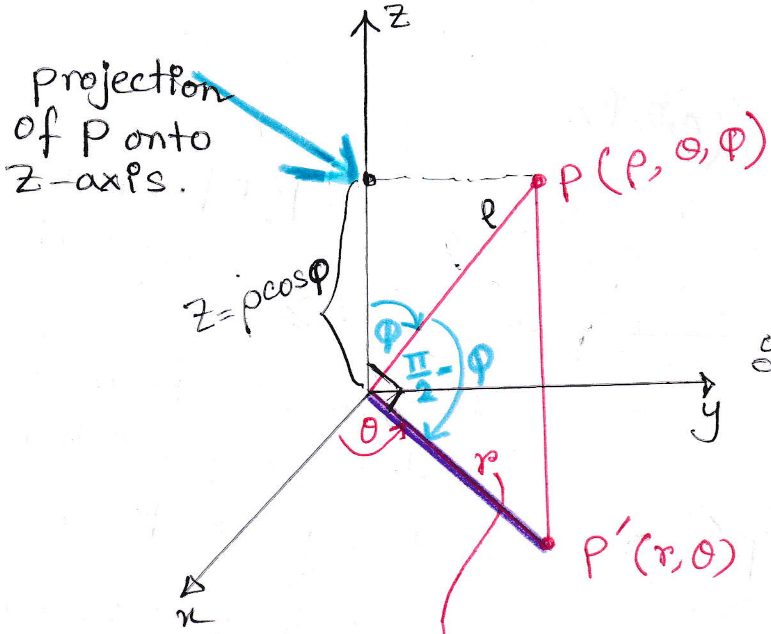
(iv) $\varphi \to$ angle between OP to z-axis.

It starts to rotate from z-axis towards x-y plane. $\therefore \varphi \in (0, \pi/2)$

Then it rotates from xy-plane towards the $-$ve side of z-axis. $\therefore \varphi \in (\pi/2, \pi)$

Hence the complete turn of φ is $(0, \pi)$.





$z \perp$ on xy-plane

$\therefore$ z axis to the line is 90°.

$r = \rho \sin\varphi$

$x = r\cos\theta = \rho \sin\varphi \cos\theta$

$y = r\sin\theta = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

Rectangular to Spherical:

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

Spherical to Rectangular

$\rho^2 = r^2 + z^2$

$\rho^2 = x^2 + y^2 + z^2$

$\tan\theta = \dfrac{y}{x}$

$\cos\varphi = \dfrac{z}{\rho} = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

Examples:

1. Let $(r, \theta, z) = (4, \frac{\pi}{3}, -3) \to$ Cylindrical coordinate.

Evaluate the rectangular coordinate (x, y, z).

$x = r\cos\theta = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2$

$y = r\sin\theta = 4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$z = -3$

$\therefore (x, y, z) = (2, 2\sqrt{3}, -3)$

2. Given that $(\rho, \theta, \varphi) = (4, \frac{\pi}{3}, \frac{\pi}{4}) \to$ spherical coordinate.

Evaluate (x, y, z), the rectangular coordinate

$x = \rho\sin\varphi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = 4 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \sqrt{2}$

$y = \rho\sin\varphi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) = \sqrt{2}\sqrt{3} = \sqrt{6}$

$z = \rho\cos\varphi = 4\cos\frac{\pi}{4} = 4\left(\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$

3. Consider $x^2-y^2-z^2=0$. Transform this given equation into cylindrical coordinate system.

$$x^2-y^2-z^2=0$$

$$r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 0$$

$$r^2(\cos^2 - \sin^2\theta) - z^2 = 0$$

$$r^2 \cos 2\theta - z^2 = 0$$

$$z^2 = r^2 \cos 2\theta.$$

4. Consider $x^2-y^2-z^2=0$. Transform this given equation into spherical coordinate system.

$$x^2-y^2-z^2=0$$

$$\rho^2 \sin^2\varphi \cos^2\theta - \rho^2 \sin^2\varphi \sin^2\theta - \rho^2\cos^2\varphi = 0$$

$$\rho^2 \sin^2\varphi \,(\cos^2\theta - \sin^2\theta) - \rho^2 \cos^2\varphi = 0$$

$$\rho^2 \sin^2\varphi \cos 2\theta - \rho^2 \cos^2\varphi = 0$$

$$\rho^2 \sin^2\varphi \cos 2\theta = \rho^2 \cos^2\varphi$$

$$\sin^2\varphi \cos 2\theta = \cos^2\varphi$$

$$\cos 2\theta = \frac{\cos^2\varphi}{\sin^2\varphi}$$

$$\cos 2\theta = \cot^2\varphi$$