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Set - F

Answer to the Question No. 1

$$f(x) = x^4 + 2x^3 - x^2 + \ln(\sin x)$$

$$f'(x) = \frac{d}{dx}\left(x^4 + 2x^3 - x^2 + \ln(\sin x)\right)$$

$$= \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^3) - \frac{d}{dx}(x^2) + \frac{d}{dx}(\ln(\sin x))$$

$$= 4x^3 + 2(3x^2) - 2x + \frac{1}{\sin x} \cdot \cos x$$

$$\therefore f'(x) = 4x^3 + 6x^2 - 2x + \cot x$$

(Ans.)

Answer to the Question No. 2

2. $f(x) = \dfrac{5x^3+8}{2x^3}$

$\Rightarrow f(x) = (5x^3+8) \cdot \dfrac{1}{2x^3}$

Let,

$g(x) = 5x^3+8$

$h(x) = \dfrac{1}{2x^3}$.

We know, Leibniz' Theorem $\rightarrow$

$$\frac{d^n}{dx^n}\left(g(x)\,h(x)\right) = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{d^{n-k} g}{dx^{n-k}} \frac{d^k h}{dx^k}$$

For $n=4$,

$$\frac{d^4}{dx^4}\left(g(x)\,h(x)\right) = \sum_{k=0}^{4} \binom{4}{k} \cdot \frac{d^{4-k} g}{dx^{4-k}} \cdot \frac{d^k h}{dx^k}$$

$$\Rightarrow \frac{d^4}{dx^4}\left(f(x)\right) = \binom{4}{0} \cdot \frac{d^{4-0} g}{dx^{4-0}} \cdot \frac{d^0 h}{dx^0} + \binom{4}{1} \cdot \frac{d^{4-1} g}{dx^{4-1}} \cdot \frac{d^1 h}{dx^1} + \binom{4}{2} \cdot \frac{d^{4-2} g}{dx^{4-2}} \cdot \frac{d^2 h}{dx^2} + \binom{4}{3} \cdot \frac{d^{4-3} g}{dx^{4-3}} \cdot \frac{d^3 h}{dx^3} + \binom{4}{4} \cdot \frac{d^{4-4} g}{dx^{4-4}} \cdot \frac{d^4 h}{dx^4}$$

$$\Rightarrow f^{(4)}(x) = 1 \cdot \frac{d^4 g}{dx^4} \cdot h + 4 \cdot \frac{d^3 g}{dx^3} \cdot \frac{dh}{dx} + 6 \cdot \frac{d^2 g}{dx^2} \cdot \frac{d^2 h}{dx^2} + 4 \cdot \frac{dg}{dx} \cdot \frac{d^3 h}{dx^3} + 1 \cdot g \cdot \frac{d^4 h}{dx^4}$$

$$\Rightarrow f^{(4)}(x) = h \cdot \frac{d^4 g}{dx^4} + 4 \cdot \frac{d^3 g}{dx^3} \cdot \frac{dh}{dx} + 6 \cdot \frac{d^2 g}{dx^2} \cdot \frac{d^2 h}{dx^2} + 4 \cdot \frac{dg}{dx} \cdot \frac{d^3 h}{dx^3} + g \cdot \frac{d^4 h}{dx^4}$$

Now,

$$g = 5x^3 + 8$$

$$\frac{dg}{dx} = 15x^2 + 0$$

$$= 15x^2$$

$$\frac{d^2g}{dx^2} = 30x$$

$$\frac{d^3g}{dx^3} = 30$$

$$\frac{d^4g}{dx^4} = 0$$

$$h = \frac{1}{2x^3}$$

$$\frac{dh}{dx} = -\frac{1}{(2x^3)^2} \cdot 6x^2$$

$$= -\frac{6x^2}{4x^6}$$

$$= -\frac{3}{2x^4}$$

$$\frac{d^2h}{dx^2} = \frac{3}{(2x^4)^2} \cdot (8x^3)$$

$$= \frac{24x^3}{4x^8}$$

$$= \frac{6}{x^5}$$

$$\frac{d^3h}{dx^3} = -\frac{6}{(x^5)^2} \cdot 5x^4$$

$$= \frac{-30x^4}{x^{10}} = \frac{-30}{x^6}$$

$$\frac{d^4h}{dx^4} = \frac{30}{(x^6)^2} \cdot 6x^5$$

$$= \frac{180x^5}{x^6} = \frac{180}{x}$$

So,

$$f^{(4)}(x) = \frac{1}{2x^3} \cdot 0 + 4 \cdot (30) \cdot \left(-\frac{3}{2x^4}\right) + 6 \cdot (30x) \cdot \left(\frac{6}{x^5}\right) + 4 \cdot (15x^2) \cdot \left(-\frac{30}{x^6}\right) + (5x^3+8) \cdot \left(\frac{180}{x}\right)$$

$$= 0 - \frac{360}{2x^4} + \frac{1080x}{x^5} - \frac{1800x^2}{x^6} + \frac{900x^3 + 1440}{x}$$

$$= \frac{180}{x^4} + \frac{1080}{x^4} - \frac{1800}{x^4} + \frac{900x^3}{x} + \frac{1440}{x}$$

$$= \frac{180 + 1080 - 1800}{x^4} + 900x^2 + \frac{1440}{x}$$

$$= \frac{-540}{x^4} + 900x^2 + 1440x$$

$$\therefore f^{(4)}(x) = 1440x + 900x^2 - \frac{540}{x^4}$$

(Ans.)

Answer to the Question No. 3

$$f(x) = |x-1|$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0^-} \frac{x+h-1 - x+1}{h}$$

$$\boxed{\cancel{= \lim_{h \to 0^-} \frac{1 + h - 2}{h}}}$$

$$= \lim_{h \to 0^-} \frac{-h}{h} = -1 \quad [\text{When } h<0]$$

Again,

$$~~\text{RHS}~~ = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0^+} \frac{x+h-1 - x+1}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h}$$

$$= 1 \quad [\text{When } h > 0]$$

$\therefore$ LHL $\neq$ RHL

Therefore, the limit doesn't exist.

So, we can say that at $x = 1$, the function $f(x) = |x-1|$ is not differentiable.

(Showed.)

Answere to the Question No. 4

(a)

$$y = (2x^2 - x + 1)^3$$

Let,

$$y = u^3 \quad ; \quad u = 2x^2 - x + 1$$

Now,
$$\frac{dy}{du} = \frac{d}{du}(u^3)$$

$$\therefore \frac{dy}{du} = 3u^2$$

$$\frac{du}{dx} = \frac{d}{dx}(2x^2 - x + 1)$$

$$\Rightarrow \frac{du}{dx} = 4x - 1 + 0$$

$$\therefore \frac{du}{dx} = 4x - 1$$

Finally,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 3u^2 \cdot (4x-1)$$

$$= 3(2x^2-x+1)^2 \cdot (4x-1)$$

$$= 3(4x^4 - 4x^3 + 5x^2 - 2x + 1)(4x-1)$$

$$= 3(16x^5 - 20x^4 + 24x^3 - 13x^2 + 6x - 1)$$

$$= 48x^5 - 60x^4 + 72x^3 - 39x^2 + 18x - 3$$

(Ans.)

(b)

$$y = (\sin^{-1} x)^2$$

Let, $y = u^2$, $u = \sin^{-1} x$

Now, $\dfrac{dy}{du} = \dfrac{d}{du}(u^2)$

$\therefore \dfrac{dy}{du} = 2u$

$$\frac{du}{dx} = \frac{d}{dx}(\sin^{-1} x)$$

$$= \frac{1}{(\sin u)'}$$

$$= \frac{1}{\cos u}$$

$$= \frac{1}{\sqrt{1 - \sin^2 u}}$$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin x)}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Finally,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 2u \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= 2\sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

(Ans.)

Answere to the Question No. 5

Suppose we have, $y = f(g(x))$

Let, $y = f(u)$; $u = g(x)$

Now,

$$\frac{dy}{dx} = \frac{d}{dx}\{f(g(x))\}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \frac{f(u+k) - f(u)}{k} \cdot \frac{k}{h}$$

$$= \lim_{h \to 0} \frac{f(u+k) - f(u)}{k} \cdot \frac{u+k-u}{h}$$

$$= \lim_{h \to 0} \frac{f(u+k) - f(u)}{k} \cdot \lim_{h \to 0} \frac{u+k-u}{h}$$

If $h \to 0$, $g(x+h) \to g(x)$

So, $u+k \to u$, $k \to 0$

$$\therefore \frac{dy}{dx} = \lim_{k \to 0} \frac{f(u+k) - f(u)}{k} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{du}\{f(u)\} \cdot \frac{d}{dx}\{g(x)\}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

(Proved.)