Lecture 4:

The chain Rule

~~Th~~ Here we will ~~menti~~ derive a formula that express the derivative of a composition $f \circ g$ in terms of the derivatives of f and g. This formula will enable us to differentiate complicated functions.

Theorem (Chain Rule):

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if $y = f(g(x))$ and $u = g(x)$ then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

So, we can write alternate way,

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x).$$

e.g. find $\frac{dy}{dx}$ where $y = \tan(4x^3+4x)$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan(4x^3+4x)\right]$$

$$= \sec^2(4x^3+4x) \cdot \frac{d}{dx}(4x^3+4x)$$

$$= \sec^2(4x^3+4x) \cdot (12x^2+4)$$

$$= (12x^2+4)\sec^2(4x^3+4x)$$

Example: ~~Find $\frac{dy}{dx}$ if $y=$~~

Find $\frac{d}{dx}\sin(\sqrt{1+\cos x})$.

$$u=\sqrt{1+\cos x}$$

$$\frac{du}{dx}=\frac{-\sin x}{2\sqrt{1+\cos x}}$$

$$\frac{d}{dx}\left(\sin\sqrt{1+\cos x}\right)=\frac{d}{dx}\sin u$$

$$= \cancel{\frac{d}{dx}(\sin u)}$$

$$=\cos u\,\frac{du}{dx}$$

$$=\cos\sqrt{1+\cos x}\cdot\frac{-\sin x}{2\sqrt{1+\cos x}}.$$

$$=-\frac{\sin x\cos\sqrt{1+\cos x}}{2\sqrt{1+\cos x}}$$

Example: Given that $f'(x)=\sqrt{3x+4}$ and $g(x)=x^2-1$, find $F'(x)$ if $F(x)=f(g(x))$.

Sol^n Since

$$\frac{d}{dx}F(x)=f'(g(x))\cdot g'(x)$$

$f'(x)=\sqrt{3x+4}$. and $g(x)=x^2-1$ so $g'(x)=2x$

$$\therefore\ F'(x)=2x\sqrt{3x^2+1}.$$

Note: If $f(x)=\sqrt{3x+4}$, $g(x)=x^2+1$, find $F'(x)$ where $F(x)=f(g(x))$.

Ans: $F'(x)=\left[f(g(x))\right]'=f'(g(x))\,g'(x)$ Ans: $F'(x)=\frac{3x}{\sqrt{3x^2+1}}$

what about if $F(x)=g(f(x))$?

Implicit Function

Definition: A given equation in x and y defines the function f is called implicitly if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.

Example: Use implicit differentiation to find $\frac{dy}{dx}$ if

$$5y^2 + \sin y = x^2.$$

Sol^n

$$\frac{d}{dx}\left[5y^2 + \sin y\right] = \frac{d}{dx}(x^2)$$

$$5\frac{d}{dx}y^2 + \frac{d}{dx}\sin y = 2x$$

$$5.2y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx}\left(10y + \cos y\right) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

Extra Problem:

· Find $\frac{dy}{dx}$ by implicit differentiation.

1. $x^2y + 3xy^3 - x = 3$.
2. $\sin(x^2y^2) = x$.
3. $x^2 = \frac{x+y}{x-y}$.
4. $x^3 + y^3 = 3xy$.
5. $y + \sin y = x$.

Logarithmic, ~~and~~ Exponential and Inverse Trigonometric Functions

Logarithmic

Formula: $f(x) = \ln x$ $\qquad \log_e x = \ln x$

So, ① ~~d~~ $f'(x) = \frac{d}{dx}[\ln(x)] = \frac{1}{x}$, $x>0$

② $\frac{d}{dx}[\log_b x] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$ $\qquad \log_b x = \frac{\log_e x}{\log_e b} = \frac{\ln x}{\ln b}$

$$= \frac{1}{x\ln(b)} , \quad x>0$$

Example: Find $\frac{d}{dx}[\ln(x^2+1)]$.

Solⁿ.
$$\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1}\frac{d}{dx}(x^2+1)$$
$$= \frac{2x}{x^2+1}$$

Example:
$$\frac{d}{dx}\left[\ln\left(\frac{x^2\sin x}{\sqrt{1+x}}\right)\right] = \frac{d}{dx}\left[\ln x^2 + \ln\sin x - \ln(\sqrt{1+x})\right]$$
$$= \frac{d}{dx}\left[2\ln x + \ln\sin x - \frac{1}{2}\ln(1+x)\right]$$
$$= \frac{2}{x} + \frac{1}{\sin x}\cdot\cos x - \frac{1}{2}\frac{1}{1+x}$$
$$= \frac{2}{x} + \cot x - \frac{1}{2(1+x)}$$

Exponential Function

① $\frac{d}{dx}[e^{mx}] = e^{mx} \cdot \frac{d}{dx}(mx) = m e^{mx}$

② $\frac{d}{dx}[b^{x}] = b^{x} \ln b.$

Example: 1. $\frac{d}{dx}[2^{x}] = 2^{x} \ln(2).$

2. $\frac{d}{dx}[e^{x^3}] = e^{x^3} \frac{d}{dx}(x^3) = 3x^2 e^{x^3}$

Example: $\frac{d}{dx}(x^2+1)^{\sin x}$

Say, $y = (x^2+1)^{\sin x}$

$\ln y = \ln (x^2+1)^{\sin x} = \sin x \ln(x^2+1)$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}[\sin x \ln(x^2+1)]$

$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{d}{dx}[\ln(x^2+1)] + \ln(x^2+1) \frac{d}{dx}[\sin x]$

$= \sin x \cdot \frac{1}{x^2+1} \cdot 2x + \cos x \ln(x^2+1)$

$\frac{dy}{dx} = y\left[\frac{2x \sin x}{x^2+1} + \cos x \ln(x^2+1)\right]$

$= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + \cos x \ln(x^2+1)\right]$

Lecture 5 $\boxed{\text{Continue...}}$

Inverse Trigonometric Function

Formulas :

1. $\frac{d}{dx}[\sin^{-1}u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, $\frac{d}{dx}[\cos^{-1}u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

2. $\frac{d}{dx}[\tan^{-1}u] = \frac{1}{1+u^2}\frac{du}{dx}$, $\frac{d}{dx}[\cot^{-1}u] = -\frac{1}{1+u^2}\frac{du}{dx}$

3. $\frac{d}{dx}[\sec^{-1}u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$, $\frac{d}{dx}[\csc^{-1}u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

e.g.

$$y = \sin^{-1}(x^3)$$

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}(x^3)$$

$$= \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{3x^2}{\sqrt{1-x^6}} \quad \divideontimes$$

Extra Problem: Find $\frac{dy}{dx}$.

1. $y = \ln(1-xe^{-x})$
2. $y = \pi^{\sin x}$
3. $y = x^{\sin x}$
4. $y = 4^{3\sin x - e^x}$
5. $y = 2^{\cos^{-1}x^2 + \ln x}$
6. $y = \tan^{-1}(x^3)$
7. $y = e^x \sec^{-1}x$
8. $y = x^2(\sin^{-1}x)^3$

Le - masculine

La - feminine

L'Hôpital's Rule

Earlier we ~~were~~ discussed only able to conjecture using numerical or graphical evidence for establishing the limits.

Now we use one theorem which is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

Theorem (L'Hôpital's Rule for $\frac{0}{0}$ form)

Suppose that f and g are differentiable functions on an open interval containing $x=a$, except possibly at $x=a$, and that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0.$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$\boxed{\text{Same for } \frac{\infty}{\infty} \text{ form}}$ $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$.

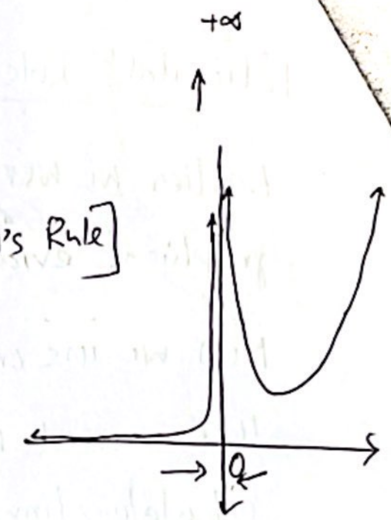
If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example: $\lim_{x \to 0} \frac{e^x - 1}{x^3}$ $\quad \frac{0}{0}$

$= \lim_{x \to 0} \frac{e^x}{3x^2}$ [Apply L'Hopital's Rule]

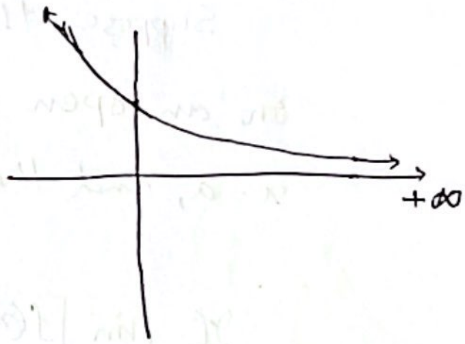
$= +\infty$



Example: $\lim_{x \to +\infty} \frac{x}{e^x}$ $\quad \left[\frac{\infty}{\infty}\right]$

$= \lim_{x \to +\infty} \frac{1}{e^x}$ [Applying L'Hopital's Rule]

$= 0$



* $0 \cdot \infty$ ~~also~~ in.

$0 \cdot \infty$ is also another indeterminate form. ~~But~~ ~~How~~ But it gives us sometime wrong result.

Since "zero times anything is zero. However, this is fallacious since $0 \cdot \infty$ is not a product of numbers, but rather a statement about limits.

So, when we get $0 \cdot \infty$, we will try to make it $\frac{\infty}{\infty}$ or $\frac{0}{0}$ form.

Example: $\lim_{x \to \pi/4} (1-\tan x) \cdot \sec 2x$ $\qquad 0 \cdot \infty$ form

So we make it like $\frac{0}{0}$

$$\therefore \lim_{x \to \pi/4} \cancel{\tan}(1-\tan x)\sec 2x = \lim_{x \to \pi/4} \frac{1-\tan x}{\frac{1}{\sec 2x}}$$

$$= \lim_{x \to \pi/4} \frac{1-\tan x}{\cos 2x} \qquad \frac{0}{0}$$

$$= \lim_{x \to \pi/4} \frac{-\sec^2 x}{-2\sin 2x}$$

$$= \frac{-2}{-2} = 1$$

Example: $\lim_{x \to 0} \left(\frac{1}{e^{1/2x}}\right)^x$; let $f(x) = \frac{1}{e^{1/2x}}$ $\qquad g(x) = x$

$\lim_{x \to 0} f(x) = 0 \qquad \lim_{x \to 0} g(x) = 0$

So, it has 0^0 form.

Now $y = \left(\frac{1}{e^{1/2x}}\right)^x$

$$\ln y = x \ln\left(\frac{1}{e^{1/2x}}\right) = \cancel{x \ln f} \; x\left[\ln 1 - \ln e^{1/2x}\right]$$

$$= x\left[0 - \frac{1}{2x}\ln e\right] = x\left(-\frac{1}{2x} \cdot 1\right)$$

$$= -\frac{1}{2}$$

$$\therefore \lim_{x \to 0} \ln y = \lim_{x \to 0} \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{2}$$

$$y = e^{-1/2}$$

Problem: $\lim_{x \to \infty} \left(\frac{2}{x}\right)^{e^{-x}}$

Problem: Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ $[\infty-\infty]$ form

[So we want to make it $\frac{0}{0}$ or $\frac{\infty}{\infty}$.]

Now, $\lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0^+} \frac{\sin x}{\cos x + \cos x - x\sin x}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{2\cos x - x\sin x}$$

$$\boxed{= \frac{0}{2}} = 0$$

Problem: Prove that $\lim_{x\to 0^+}(1+x)^{1/x} = e$

Sol^n: Since this limit form 1^∞ form.

So, let $y = (1+x)^{1/x}$

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1+x)}{x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{1/(1+x)}{1}$$

$$= \lim_{x\to 0} \frac{1}{1+x} = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$ i.e. $\lim_{x\to 0}(1+x)^{1/x} = e$

Some Problems:

Page: 227 Book: Reference book - H. Anton

Problems

Exercise: 3·6

Problems: 7, 27, 29, 34, 36, 37,