Week 11
[PART 2]
Examples:

# MAT 110
## INTRO TO POLAR COORDIATES (CYLINDRICAL, SPHERICAL) [EXAMPLES

$\boxed{1.}$ Let $(r, \theta, z) = (4, \frac{\pi}{3}, -3) \rightarrow$ Cylindrical coordinate.
Evaluate the rectangular coordinate $(x, y, z)$.

$x = r\cos\theta = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2$

$y = r\sin\theta = 4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$z = -3$

$\therefore (x, y, z) = (2, 2\sqrt{3}, -3)$

$\boxed{2.}$ Given that $(\rho, \theta, \varphi) = (4, \frac{\pi}{3}, \frac{\pi}{4}) \rightarrow$ spherical coordinate.
Evaluate $(x, y, z)$, the rectangular coordinate

$x = \rho\sin\varphi\cos\theta = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = 4 \times \frac{1}{\sqrt{2}} \times \frac{1}{2} = \sqrt{2}$

$y = \rho\sin\varphi\sin\theta = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = 4\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) = \sqrt{2}\sqrt{3} = \sqrt{6}$

$z = \rho\cos\varphi = 4\cos\frac{\pi}{4} = 4\left(\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}$

$(x, y, z) = (\sqrt{2}, \sqrt{6}, 2\sqrt{2})$

$\boxed{3}$ Consider $x^2-y^2-z^2=0$. Transform this given equation into cylindrical coordinate system.

$$x^2-y^2-z^2=0$$

$$r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 0$$

$$r^2(\cos^2 - \sin^2\theta) - z^2 = 0$$

$$r^2 \cos 2\theta - z^2 = 0$$

$$z^2 = r^2 \cos 2\theta.$$

$\boxed{4}$ Consider $x^2-y^2-z^2=0$. Transform this given equation into spherical coordinate system.

$$x^2-y^2-z^2=0$$

$$\rho^2\sin^2\varphi\cos^2\theta - \rho^2\sin^2\varphi\sin^2\theta - \rho^2\cos^2\varphi = 0$$

$$\rho^2\sin^2\varphi(\cos^2\theta - \sin^2\theta) - \rho^2\cos^2\varphi = 0$$

$$\rho^2\sin^2\varphi\cos 2\theta - \rho^2\cos^2\varphi = 0$$

$$\rho^2\sin^2\varphi\cos 2\theta = \rho^2\cos^2\varphi$$

$$\sin^2\varphi\cos 2\theta = \cos^2\varphi$$

$$\cos 2\theta = \frac{\cos^2\varphi}{\sin^2\varphi}$$

$$\cos 2\theta = \cot^2\varphi$$

# Examples

**1** Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

Given $r = -4$, $\theta = \frac{2\pi}{3}$

$$x = r\cos\theta = -4\cos\frac{2\pi}{3} = (-4)\left(-\frac{1}{2}\right) = +2$$

$$y = r\sin\theta = -4\sin\frac{2\pi}{3} = (-4)\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

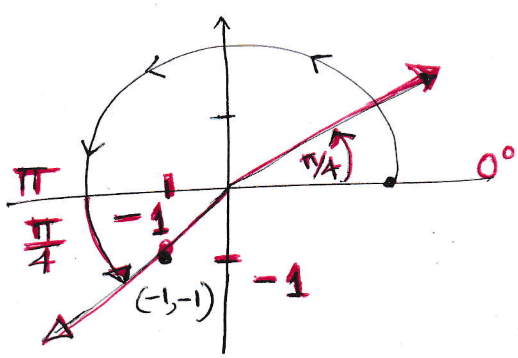
$$(x, y) = (+2, -2\sqrt{3})$$

**2** Convert $(-1, -1)$ into Polar coordinates.

$x = -1$, $y = -1$

$$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

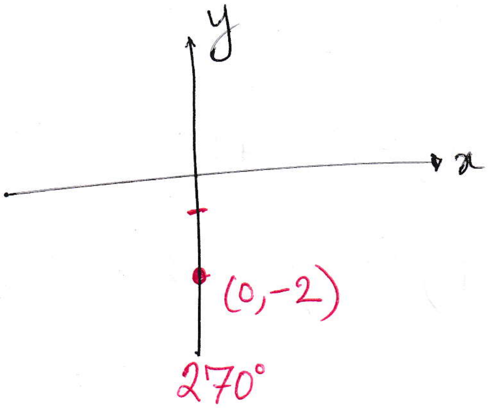



$$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

$$\therefore (r, \theta) = \left(\sqrt{2}, \frac{5\pi}{4}\right)$$

3. Convert into polar coordinates: $(0,-2)$.

$x = 0, \quad y = -2$




$$r = \sqrt{x^2+y^2}$$
$$= \sqrt{0^2+(-2)^2} = \sqrt{4} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{-2}{0}\right)$$
$$= \tan^{-1}(\infty)$$
$$= 90^\circ \text{ or } 270^\circ$$
$$\theta = 270^\circ \text{ or } \frac{3\pi}{2}$$

$$(r, \theta) = \left(2, \frac{3\pi}{2}\right)$$

4. Change the equations to cartesian coordinates:

(i) $r = a\sin\theta$ ; (ii) $\sqrt{r} = \sqrt{a}\cos\frac{\theta}{2}$

(i) $r = a\sin\theta$

$r^2 = ar\sin\theta$ (multiply by $r$)

$x^2+y^2 = ay$

(ii) $\sqrt{r} = \sqrt{a} \cos\frac{\theta}{2}$

$r = a\cos^2\frac{\theta}{2}$ Square both sides

$r = a\left[\frac{1+\cos 2\left(\frac{\theta}{2}\right)}{2}\right]$ $\left\{\cos^2\theta = \frac{1+\cos 2\theta}{2}\right\}$

$r = \frac{a}{2}(1+\cos\theta)$

$\frac{2r}{a} = 1+\cos\theta$

$\frac{2r^2}{a} = r + r\cos\theta$ (multiply by $r$)

$\frac{2(x^2+y^2)}{a} = \sqrt{x^2+y^2} + x$

$\boxed{5}$ Convert the equation $r = -8\cos\theta$ into cartesian coordinates.

$r = -8\cos\theta$

$r^2 = -8r\cos\theta$

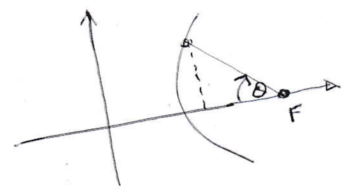
$x^2+y^2 = -8x$

$\boxed{6}$ Determine k and e from $r = \dfrac{2}{1-\cos\theta}$

Focus directrix eqn:

$$r = \frac{ke}{1+e\cos\theta} \quad -(a) \quad ; \quad r = \frac{ke}{1-e\cos\theta} \quad -(b)$$

Given $r = \dfrac{2}{1-\cos\theta}$ — (ii)




Comparing (b) & (ii)

$$\frac{ke}{1-e\cos\theta} = \frac{2}{1-\cos\theta}$$

$\therefore$ $ke = 2$ & $1 - e\cos\theta = 1 - \cos\theta$

$\Rightarrow e = 1$

$k(1) = 2$

$k = 2$

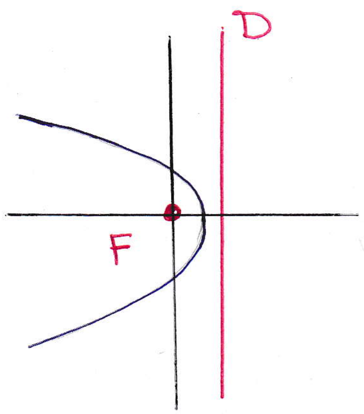
$\therefore$ $e = 1$, $k = 2$

$\boxed{7}$ The graph of $r = \dfrac{2}{1+2\sin\theta}$ is a/an _____.

- ellipse
- ✓ hyperbola
- parabola
- circle

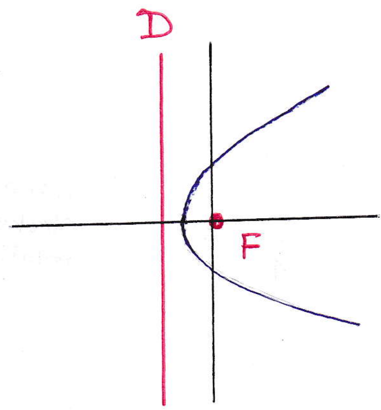
$\because e = 2$ while we have $r = \dfrac{2}{1+2\sin\theta}$

$$= \frac{ke}{1+e\sin\theta}.$$




$$r = \frac{ke}{1+e\cos\theta}$$

Directrix right of pole




$$r = \frac{ke}{1-e\cos\theta}$$

Directrix left of pole

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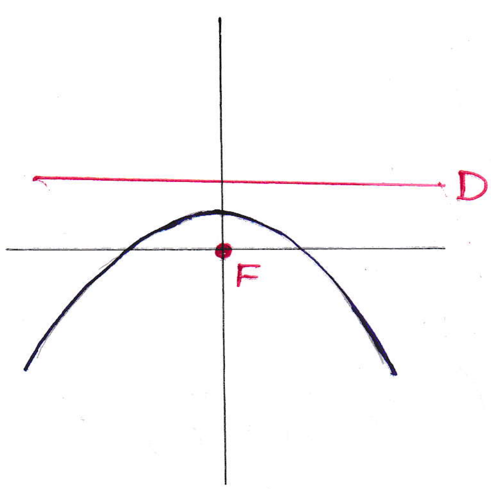
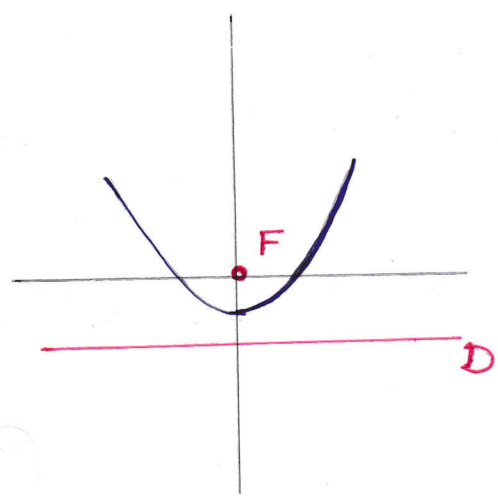



$$r = \frac{ke}{1+e\sin\theta}$$

Directrix above pole




$$r = \frac{ke}{1-e\sin\theta}$$

Directrix below pole

**8** Find the spherical coordinates of the point $(4, -4, 4\sqrt{6})$ which is in rectangular coordinates.

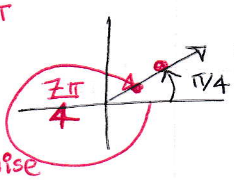
Given: $(x, y, z) = 4, -4, 4\sqrt{6})$

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4^2+(-4)^2+(4\sqrt{6})^2}$$

$$= \sqrt{16+16+96}$$

$$= \sqrt{128}$$

$\theta = \theta_0 \pm 2n\pi$




counter clockwise rotation

clockwise rotation

$$= \sqrt{64 \times 2} = 8\sqrt{2}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-4}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

But $\theta \in (0, 2\pi)$ $\therefore \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$$\cos\varphi = \frac{z}{\rho} = \frac{4\sqrt{6}}{\sqrt{128}}$$

$$\varphi = \cos^{-1}\left(\frac{4\sqrt{6}}{\sqrt{128}}\right)$$

$$= 30^\circ \text{ or } \frac{\pi}{6}$$

$$\therefore (\rho, \theta, \varphi) = \left(8\sqrt{2}, \frac{7\pi}{4}, \frac{\pi}{6}\right)$$