Lecture 13

Taylor's polynomial for One-variable

~~We~~ Recall that Taylor's polynomial for one-variable is ~~at~~ $x = x_0$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Now, we introduce Taylor's polynomial for Two-variables. i.e. $f(x,y)$ so, we need partial derivative of x and y separately.

So, the Taylor's polynomial for two-variables is (2nd-order)

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{1}{2!}\Big[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\Big]$$

Example: Find the first and second degree Taylor polynomials for $f(x,y) = e^x \cos y$ at the point $(x_0, y_0) = (0,0)$.

Sol^n:

$f(x,y) = e^x \cos y$ $\qquad\qquad f(0,0) = e^0 \cos 0 = 1$

$\cancel{f_x(x,y) = e^x(\sin y) + e^x \cos y}$ $\qquad \cancel{f_x(0,0)}$

$\cancel{= e^x(\cos y - \sin y)}$

$f_x(x,y) = e^x \cos y$ $\qquad\qquad f_x(0,0) = 1$

$f_y(x,y) = -e^x \sin y$ $\qquad\qquad f_y(0,0) = 0$

$f_{xx}(x,y) = e^x \cos y$ $\qquad\qquad f_{xx}(0,0) = 1$

$$f_{yy}(x,y) = -e^x \cos y \qquad f_{yy}(0,0) = -1$$

$$f_{xy}(x,y) = -e^x \sin y \qquad f_{xy}(0,0) = 0$$

So, The Taylor's 1st and 2nd ~~order~~ degree polynomial,

$$f(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \frac{1}{2!}\Big[f_{xx}(0,0)(x-0)^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0)(y-0)^2\Big]$$

$$= 1 + 1\cdot(x-0) + 0\cdot(y-0) + \frac{1}{2!}\Big[1\cdot(x-0)^2 + 2\cdot 0\cdot(x-0)(y-0) + (-1)(y-0)^2\Big] + \cdots$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \cdots$$

Example 2:

Expand the function $\sin(xy)$ in power series of $(x-1)$ and $(y-\frac{\pi}{2})$ upto second degree terms.

Sol^n:

$$f(x,y) = \sin(xy) \qquad f(1,\pi/2) = \sin(\pi/2) = 1$$

$$f_x(x,y) = y\cos(xy) \qquad f_x(1,\pi/2) = \frac{\pi}{2}\cos(\pi/2) = 0$$

$$f_y(x,y) = x\cos(xy) \qquad f_y(1,\pi/2) = 0$$

$$f_{xx}(x,y) = -y^2\sin(xy) \qquad f_{xx}(1,\pi/2) = -\frac{\pi^2}{4}\cdot 1 = -\frac{\pi^2}{4}$$

$$f_{yy}(x,y) = -x^2\sin(xy) \qquad f_{yy}(1,\pi/2) = -1.$$

$$f_{xy}(x,y) = \cos(xy) - xy\sin(xy) \qquad f_{xy}(1,\pi/2) = \overbrace{\cos(\pi/2)}^{0} - \frac{\pi}{2}\underbrace{\sin(\pi/2)}_{1} = -\frac{\pi}{2}$$

By Taylor's polynomial,

$$f(x,y) = f(1,\pi/2) + f_x(1,\pi/2)(x-1) + f_y(1,\pi/2)(y-\pi/2)$$
$$+ \frac{1}{2!}\Big[f_{xx}(1,\pi/2)(x-1)^2 + 2f_{xy}(1,\pi/2)(x-1)(y-\pi/2)$$
$$+ f_{yy}(1,\pi/2)(y-\pi/2)^2\Big]$$
$$= 1 + 0\cdot(x-1) + 0(y-\pi/2) + \frac{1}{2!}\Big[-\frac{\pi^2}{4}(x-1)^2 - 2\frac{\pi}{2}(x-1)(y-\pi/2)$$
$$+ (-1)(y-\pi/2)^2\Big]$$
$$= 1 + \frac{1}{2!}\Big[(x-1)^2\Big(-\frac{\pi^2}{4}\Big) - \pi(x-1)(y-\pi/2) - (y-\pi/2)^2\Big] + \cdots$$

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Extra problem: Find the first & 2nd degree Taylor polynomials for ~~f(x,y)~~ the following functions:

1. $f(x,y) = e^{xy}$ at $(1,1)$.
2. $f(x,y) = e^x \log(1+y)$ at $(0,0)$
3. $f(x,y) = x^2y + 3y - 2$ at $(1,-2)$