Week 4

Optimization

→ choosing variables
→ Drawing a diagram indicating these variables
→ Finding expressions that establish the relationship between these variables
→ Determining a formula for the quantity to be optimized
→ Determining the restrictions on the variables
constraint
→ Use calculus to determine the critical numbers

$f'(x)=0$ → ?

→ stationary pts

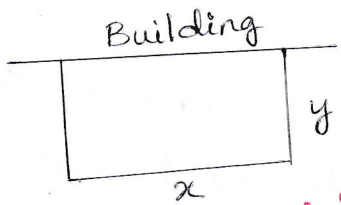
it is also been used

→ Determining the maxima or minima depending on the optimization problem.

Example $\boxed{1}$ We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Constraint

$x \times y$



Area: $A = xy$ → maximize

fencing : $500 = x + y + y$ ∵ [Building side won't need any fencing]

$\Rightarrow 500 = x + 2y$

Constraint → a condition of an optimization problem that the solution must satisfy

$$500 = x + 2y$$

$$\boxed{x = 500 - 2y} \qquad y = \frac{500 - x}{2}$$

We will avoid fraction

$\therefore A = xy$

$$A(y) = (500-2y)y = 500y - 2y^2$$

set $A(y) = 0 \longrightarrow$ to find the interval of y

$$(500-2y)y = 0$$

$$\boxed{y=0} \qquad \boxed{y = 250}$$

$$y \in [0, 250]$$

$$A'(y) = 500 - 4y$$

evaluate the critical point / stationary pt

$f'(x) = 0$

$$A'(y) = 0$$

$$500 - 4y = 0$$

$$y = 125$$

Substitute $y = 125$ into $A(y)$ (Area)

$$A(125) = (250)(125) = 31250\,ft^2$$

while

$$x = 500 - 2(125) = 250\ ft$$

$\therefore$ The dimensions of the field that will give the largest area, subject to the fact that we used exactly 500 ft of fencing material, are 250×125.

Example $\boxed{2}$ We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost \$10/ft^2 and the material used to build the sides cost \$6/ft^2. If the box must have a volume of 50 ft^3 determine the dimensions that will minimize the cost to build the box.

constraint

$\underbrace{\text{dimensions}}_{l \times w \times h}$

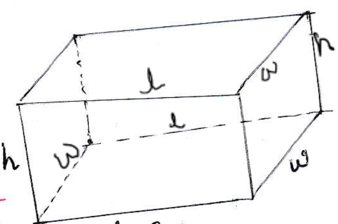
building the sides:

(left+right) sides: $wh \times 2 = 2wh$

(back+front) other sides: $lh \times 2 = 2lh$

Total 4 sides $= 2wh + 2lh$

cost = \$6



building the top & bottom:

Top area $= lw$

bottom area $= lw$

Total $= 2lw$

Cost \$10/ft^2

Cost: C

Minimize: $C = 10(2lw) + 6(2wh + 2lh)$

$C = 60w^2 + 48wh$

Constraint: $50 = lwh$

$\Rightarrow 50 = (3w)wh = 3w^2h$

$\therefore 50 = 3w^2h$

$h = \dfrac{50}{3w^2}$

We have $C = 60w^2 + 48wh$ → cost function

$\Rightarrow \boxed{C(w) = 60w^2 + 48w\left(\frac{50}{3w^2}\right)}$ → (*)

$= 60w^2 + \dfrac{800}{w}$

$C'(w) = 120w - 800w^{-2} = \dfrac{120w^3 - 800}{w^2}$

We need the C.N. | stationary pt for the cost function. $w=0$ is not a C.N. The derivative does not exist at $w=0$ & neither does the function. Note: values of w will only be C.N. if the function also exists at that point. Also, physicall the width of a box w can't be '0'.

$C'(w) = 0$

so $\Rightarrow \dfrac{120w^3 - 800}{w^2} = 0$ while $w \neq 0$

(physically width of a box can't be '0' $w \neq 0$)

$\Rightarrow 120w^3 - 800 = 0$

$w = \sqrt[3]{\dfrac{800}{120}} = 1.8821$

$\omega = 1.8821$

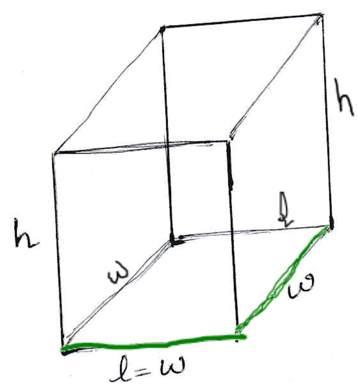
given $l = 3\omega = 3(1.8821) = 5.6463$

$h = \frac{50}{3\omega^2} = \frac{50}{3(1.8821)^2} = 4.7050$

All the dimensions are: $l \times \omega \times h = 5.6463 \times 1.8821 \times 4.7050$

The minimum cost is: $C(1.8821) = \$637.60$ → from eqn (*)

Example [3] We want to construct a box with a square base and we only have $10\ m^2$ of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that box can have



$10\ m^2$ of material to use in construction of the box

6 faces are there

↳ $lw + lw = 2lw$ (top, bottom)

$wh + wh = 2wh$ (left, right)

$lh + lh = 2lh$ (front, back)

Maximize: $V = lwh = \omega^2 h$

Constraint: $10 = 2lw + 2wh + 2\underbrace{lh}_{\rightarrow w}$

$10 = 2\omega^2 + 4\omega h$

$10 = 2\omega^2 + 4\omega h$

$4\omega h = 10 - 2\omega^2$

$h = \frac{10 - 2\omega^2}{4\omega} = \frac{5 - \omega^2}{2\omega}$

$V = \omega^2 h = \omega^2\left(\frac{5 - \omega^2}{2\omega}\right)$

$V(\omega) = \frac{\omega(5 - \omega^2)}{2} = \frac{1}{2}(5\omega - \omega^3)$ ⟶ (*)

$V'(w) = \frac{1}{2}[5 - 3w^2]$

for Critical numbers / stationary no. we solve $V'(w) = 0$

$\frac{1}{2}[5 - 3w^2] = 0$

$3w^2 = 5$

$w = \pm\sqrt{\frac{5}{3}} = \pm 1.2910$

we take $+1.2910$ as width is never $-ve$

Using eqn (*)

$\therefore V(1.2910) = \frac{1}{2}\left(5(1.2910) - (1.2910)^3\right) = 2.1517\, m^3$

$\therefore l = w$

$\therefore l = 1.2910$

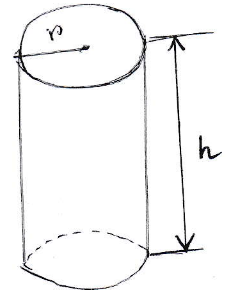
$h = \dfrac{5 - (1.2910)^2}{2(1.2910)} = 1.2910$

In this case we actually have a perfect cube.

Example $\boxed{4}$ A manufacturer needs to make a cylindrical can that will hold 1.5 Liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

1 Liter $= 1000\, cm^3$

$\therefore$ 1.5 " $= 1500\, cm^3$



$V = \pi r^2 h$

Minimize: $A = \underbrace{2\pi r h + 2\pi r^2}_{\text{Area of cylinder}}$

Constraint: $1500 = \pi r^2 h$

$1500 = \pi r^2 h$

$h = \frac{1500}{\pi r^2}$

$\therefore A(r) = 2\pi r\left(\frac{1500}{\pi r^2}\right) + 2\pi r^2 = 2\pi r^2 + \frac{3000}{r}$

$A'(r) = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3 - 300}{r^2}$

For C. No/ stationary pt $A'(r) = 0$

$$\frac{4\pi r^3 - 3000}{r^2} = 0 \; ; \quad r \neq 0$$

radius is not '0' physically besides if $r = 0$ then $A(r)$ will be undefined.

$$4\pi r^3 - 3000 = 0$$

$$r^3 = \frac{3000}{4\pi}$$

$$r = \sqrt[3]{\frac{750}{\pi}} = 6.2035$$

$$\therefore r = 6.2035$$

$$h = \frac{1500}{\pi (6.2035)^2} = 12.4070$$

$\therefore$ If the manufacturer makes the can with a radius of 6.2035 cm and a height of 12.4070 cm the least amount of material will be used to make the can.

buX question

Multiple Choice

1/1 point (graded)

A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

Let a and b be the length ans bredth of the rectangular field.

Then, $2 \cdot (a+b) = 100 \implies b = 50 - a$

$$Area(A) = a \cdot b = a \cdot (50 - a) = 50a - a^2$$

$$\frac{dA}{da} = 50 - 2a$$

For area to be maximum, $\dfrac{dA}{da} = 0$

$$\therefore 50 - 2a = 0$$

$$\therefore a = 25 \implies b = 25$$

$$\therefore Area = 625 ft^2$$