Chapter – 1

LESSON -1 : Successive Differentiation

- In this lesson, the idea of differential coefficient of a function and its successive derivatives will be discussed. Also, the computation of n^{th} derivatives of some standard functions is presented through typical worked examples.

1.0 Introduction:- Differential calculus (DC) deals with problem of calculating rates of change.When we have a formula for the distance that a moving body covers as a function of time, DC gives us the formulas for calculating the body's **velocity** and **acceleration** at any instant.

- **Definition of derivative of a function y = f(x):-**

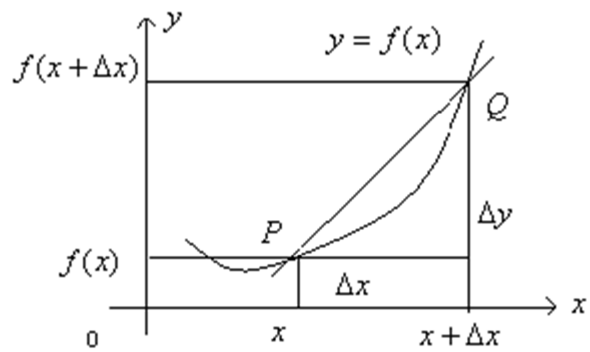


Fig.1. Slope of the line PQ is $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

The derivative of a function y = f(x) is the function $f'(x)$ whose value at each x is defined as

$$\frac{dy}{dx} = f'(x) = \text{Slope of the line PQ (See Fig.}\mathbf{1}\text{)}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \quad \text{-------- (1)}$$

$$= \lim_{\Delta x \to 0} \text{(Average rate change)}$$

$=$ Instantaneous rate of change of f at x provided the limit exists.

The instantaneous velocity and acceleration of a body (moving along a line) at any instant x is the derivative of its position co-ordinate $y = f(x)$ w.r.t x, i.e.,

$$\text{Velocity} = \frac{dy}{dx} = f'(x) \quad \text{--------- (2)}$$

And the corresponding acceleration is given by

$$\text{Acceleration} = \frac{d^2y}{dx^2} = f''(x) \quad \text{---------- (3)}$$

- **Session - 1**

1.1 Successive Differentiation:-

The process of differentiating a given function again and again is called as **Successive differentiation** and the results of such differentiation are called **successive derivatives.**

- The higher order differential coefficients will occur more frequently in spreading a function all fields of scientific and engineering applications.
- Notations:

i. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$,…….., nth order derivative: $\frac{d^ny}{dx^n}$

ii $f'(x)$, $f''(x)$, $f'''(x)$,……, nth order derivative: $f^n(x)$

iii Dy, D^2y, D^3y,………..., nth order derivative: D^ny

iv y', y'', y''',……, nth order derivative: $y^{(n)}$

v. y_1, y_2, y_3…, nth order derivative: y_n

- **Successive differentiation – A flow diagram**

Input function: $y = f(x)$ → Operation d/dx → Output function $y' = \frac{df}{dx} = f'(x)$ (first order derivative)

Input function $y' = f'(x)$ → Operation d/dx → Output function $y'' = \frac{d^2f}{dx^2} = f''(x)$ (second order derivative)

Input function $y'' = f''(x)$ → Operation d/dx → Output function $y''' = \frac{d^3f}{dx^3} = f'''(x)$ (third order derivative)

Input function $y^{n-1} = f^{n-1}(x)$ → Operation d/dx → Output function $y^n = \frac{d^nf}{dx^n} = f^n(x)$ (nth order derivative)

Animation Instruction
(Successive Differention-A flow diagram)
Output functions are to appear after operating

Operation d/dx

on Input functions, successively.

- **1.1 Solved Examples :**

1. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$

Solution: Differentiating $y = \sin(\sin x)$--------- (1) w.r.t.x, we get

$$y_1 = \frac{dy}{dx} = \cos(\sin x).\cos x \quad \text{-------------- (2)}$$

Again differentiating $y_1 = \frac{dy}{dx}$ w.r.t.x gives

$$y_2 = \frac{d^2y}{dx^2} = [\cos(\sin x)(-\sin x) + \cos x(-\sin(\sin x)\cos x] \text{ Using product rule}$$

$$y_2 = \frac{d^2y}{dx^2} = -[\sin x\cos(\sin x) + \cos^2 x\sin(\sin x)]$$

i.e. $$y_2 = -\left[\frac{\sin x}{\cos x}\cos x\cos(\sin x) + \cos^2 x\sin(\sin x)\right]$$

$$y_2 = -[\tan xy_1 + \cos^2 xy], \text{ using Eqs. (1) and (2)}$$

or $$y_2 + \tan xy_1 + \cos^2 xy = 0$$

or $$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0 \quad \blacksquare$$

2. If $y = \frac{(ax+b)}{(cx+d)}$, show that $2y_1y_3 = 3y_2^2$

Solution: We rewrite $y = \frac{(ax+b)}{(cx+d)}$, by actual division of $ax+b$ *by* c x+d, as

$$y = {}^{a}\!/_{c} + \left(b - \frac{ad}{c}\right)\frac{1}{cx+d} = \frac{a}{c} + k(cx+d)^{-1} \text{ ---------- (1)where } k = \left(b - \frac{ad}{c}\right)$$

Differentiating (1) successively thrice, we get

$$\frac{dy}{dx} = y_1 = -kc(cx+d)^{-2} \quad \text{---------- (2)}$$

$$\frac{d^2y}{dx^2} = y_2 = -2kc^2(cx+d)^{-3} \quad \text{---------- (3)}$$

$$\frac{d^3y}{dx^3} = y_3 = -6kc^3(cx+d)^{-4} \quad \text{---------- (4)}$$ From (2), (3) and (4) we get

$$2y_1y_3 = 2[\{-kc(cx+d)^{-2}\}\{-6kc^3(cx+d)^{-4}\}]$$

$$2y_1y_3 = 12k^2c^4(cx+d)^{-6}$$

$$2y_1y_3 = 3[-2kc^2(cx+d)^{-3}]^2$$

Therefore $2y_1y_3 = 3y_2^2$, as desired. $\blacksquare$

3. If $x = \sin t$, $y = \sin pt$, Prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$

Solution: Note that the function is given in terms a parameter *t*. So we find,

$$\frac{dy}{dt} = \cos t \quad \text{and} \quad \frac{dy}{dt} = p\cos pt \text{, so that}$$

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p\cos pt}{\cos t}$$. Squaring on both sides

$$(y_1)^2 = \frac{p^2\cos^2 pt}{\cos^2 t} = \frac{p^2(1-\sin^2 pt)}{1-\sin^2 t} = \frac{p^2(1-y^2)}{1-x^2} \quad \text{(by data)}$$

$$\therefore (1-x^2)(y_1)^2 = p^2(1-y^2).$$

Differentiating this equation w.r.t x, we get

$$(1-x^2)2y_1y_2 + (y_1)^2(-2x) = p^2(-2yy_1).$$

Canceling $2y_1$ throughout, this becomes

$$(1-x^2)y_2 - xy_1 = -p^2 y$$

or $(1-x^2)y_2 - xy_1 + p^2 y = 0$

i.e. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$ ■

4. If $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$, find $\frac{d^2y}{dx^2}$

Solution: $\frac{dy}{dt} = a(-\sin t + t\cos t + \sin t) = at\cos t$

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

$$\therefore \left(\frac{dy}{dx}\right) = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$$

Hence, $\frac{d^2y}{dx^2} = \sec^2 t\left(\frac{dt}{dx}\right) = \sec^2 t\left(\frac{1}{at\cos t}\right) = \frac{1}{at\cos^3 t}$ ■

5. If $y = a\cosh(x/a)$, prove that $a^2y_2^2 = 1 + y_1^2$

Solution: $y_1 = \frac{dy}{dx} = a\sinh(x/a)(1/a) = \sinh(x/a)$, and

$$y_2 = \frac{d^2y}{dx^2} = \cosh(x/a)(1/a)$$

$\therefore ay_2 = \cosh(x/a)$, so that $a^2y_2^2 = \cosh^2(x/a)$

i.e. $a^2y_2^2 = 1 + \sinh^2(x/a) = 1 + y_1^2$, as desired. ■

- **Problem Set No. 1.1 for practice.**

1. If $y = e^{ax}\sin bx$, prove that $y_2 - 2ay_1 + (a^2+b^2)y = 0$
2. If $ax^2 + 2hxy + by^2 = 1$, prove that $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$

3. If $y = Ae^{-kt}\cos(lt + c)$, show that $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + \left(k^2 + l^2\right)y = 0$

4. If $y = \log\left(x + \sqrt{1+x^2}\right)$, prove that $\left(1+x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

5. If $y = \tan^{-1}(\sinh x)$, prove that $\frac{d^2y}{dx^2} + \tan y\left(\frac{dy}{dx}\right)^2 = 0$

6. If $x = a\left(\cos t + \log\tan \tfrac{t}{2}\right)$, $y = a\sin t$, find $\frac{d^2y}{dx^2}$ **Ans:** $\sin t / a\cos^4 t$

7. Find $\frac{d^2y}{dx^2}$, when $x = a\cos^3\theta$, $y = b\sin^3\theta$ **Ans:** $\frac{b\,\mathrm{cosec}\,\theta\sec^4\theta}{3a^2}$

8. Find $\frac{d^3y}{dx^3}$, where $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ **Ans:** $\frac{1+2x^2}{\left(1-x^2\right)^{5/2}}$

9. If $x = 2\cos t - \cos 2t$, $x = 2\sin t - \sin 2t$, Find $\left(\frac{d^2y}{dx^2}\right)x = \frac{\pi}{2}$ **Ans**: -3/2

10. If $xy = e^x + be^{-x}$, prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$

- **Session -2**

1.2 Calculation of nth derivatives of some standard functions

- Below, we present a table of nth order derivatives of some standard functions for ready reference.

Table : 1

Sl. No	y = f(x)	$y_n = \frac{d^n y}{dx^n} = D^n y$
1	e^{mx}	$m^n e^{mx}$
2	a^{mx}	$m^n (\log a)^n a^{mx}$
3	$(ax+b)^m$	i. $m(m-1)(m-2)....(m-n+1)a^n (ax+b)^{m-n}$ for all m. ii. 0 if $m < n$ iii. $(n!)$a^n if $m = n$ iv. $\frac{m!}{(m-n)!} x^{m-n}$ if $m < n$
4	$\frac{1}{(ax+b)}$	$\frac{(-1)^n n!}{(ax+b)^{n+1}} a^n$
5.	$\frac{1}{(ax+b)^m}$	$\frac{(-1)^n (m+n-1)!}{(m-1)!(ax+b)^{m+n}} a^n$
6.	$\log(ax+b)$	$\frac{(-1)^{n-1} (n-1)!}{(ax+b)^n} a^n$
7.	$\sin(ax+b)$	$a^n \sin(ax+b+n\pi/2)$
8.	$\cos(ax+b)$	$a^n \cos(ax+b+n\pi/2)$
9.	$e^{ax}\sin(bx+c)$	$r^n e^{ax}\sin(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta = \tan^{-1}(b/a)$
10.	$e^{ax}\cos(bx+c)$	$r^n e^{ax}\cos(bx+c+n\theta)$, $r=\sqrt{a^2+b^2}$ $\theta = \tan^{-1}(b/a)$

- We proceed to illustrate the proof of some of the above results, as only the above functions are able to produce a **sequential change** from one derivative to the other. Hence, in general we cannot obtain readymade formula for nth derivative of functions other than the above.

1. Consider e^{mx}. Let $y = e^{mx}$. Differentiating w.r.t x, we get

$y_1 = me^{mx}$. Again differentiating w.r.t x, we get

$y_2 = m(me^{mx}) = m^2 e^{mx}$

Similarly, we get

$y_3 = m^3 e^{mx}$

$y_4 = m^4 e^{mx}$

…………….

And hence we get

$$y_n = m^n e^{mx} \quad \therefore \frac{d^n}{dx^n}\left[e^{mx}\right] = m^n e^{mx}. \blacksquare$$

2. $(ax+b)^m$ **(See Sl. No-3 of Table-1)**

let $y = (ax+b)^m$ Differentiating w.r.t x,

$y_1 = m\,(ax+b)^{m-1}a$. Again differentiating w.r.t x, we get

$y_2 = m\,(m-1)\,(ax+b)^{m-2}a^2$

Similarly, we get

$y_3 = m\,(m-1)\,(m-2)\,(ax+b)^{m-3}a^3$

…………………………………

And hence we get

$y_n = m\,(m-1)\,(m-2)\ldots\ldots\ldots(m-n+1)\,(ax+b)^{m-n}a^n$ for all m.

Case (i) If $m = n$ (m-positive integer),then the above expression becomes

$$y_n = n\,(n-1)\,(n-2)\ldots\ldots\ldots 3.2.1\,(ax+b)^{n-n}a^n$$

i.e. $y_n = (n!)a^n$

Case (ii) If m<n,(i.e. if n>m) which means if we further differentiate the above expression, the right hand site yields zero. Thus $D^n\left[(ax+b)^m\right] = 0 \quad if\,(m < n)$

Case (iii) If m>n, then $y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)(ax+b)^{m-n}a^n$ becomes

$$= \frac{m(m-1)(m-2)\ldots\ldots(m-n+1)(m-n)!}{(m-n)!}(ax+b)^{m-n}a^n$$

. i.e $\quad y_n = \dfrac{m!}{(m-n)!}(ax+b)^{m-n}a^n \quad \blacksquare$

3. $\dfrac{1}{(ax+b)^m}$ **(See Sl. No-5 of Table-1)**

$$Let \quad y = \frac{1}{(ax+b)^m} = (ax+b)^{-m}$$

Differentiating w.r.t x

$y_1 = -m(ax+b)^{-m-1}a = (-1)m(ax+b)^{-(m+1)}a$

$y_2 = (-1)(m)\left[-(m+1)(ax+b)^{-(m+1)-1}a\right] = (-1)^2 m(m+1)(ax+b)^{-(m+2)}a^2$

Similarly, we get $\quad y_3 = (-1)^3 m(m+1)(m+2)(ax+b)^{-(m+3)}a^3$

$y_4 = (-1)^4 m(m+1)(m+2)(m+3)(ax+b)^{-(m+4)}a^4$

……………………………

$y_n = (-1)^n m(m+1)(m+2)\ldots\ldots(m+n-1)(ax+b)^{-(m+n)}a^n$

This may be rewritten as

$$y_n = \frac{(-1)^n(m+n-1)(m+n-2).....(m+1)m(m-1)!}{(m-1)!}(ax+b)^{-(m+n)}a^n$$

or $$y_n = \frac{(-1)^n(m+n-1)!}{(m-1)!(ax+b)^{m+n}}a^n \qquad \blacksquare$$

4. $\frac{1}{(ax+b)}$ **(See Sl. No-4 of Table-1)**

Putting $m=1$, in the result

$$D^n\left[\frac{1}{(ax+b)^m}\right] = \frac{(-1)^n(m+n-1)!}{(m-1)!(ax+b)^{m+n}}a^n$$

we get $$D^n\left[\frac{1}{(ax+b)}\right] = \frac{(-1)^n(1+n-1)!}{(1-1)!(ax+b)^{1+n}}a^n$$

or $$D^n\left[\frac{1}{(ax+b)}\right] = \frac{(-1)^n n!}{(ax+b)^{1+n}}a^n$$

- **1.2.1. Worked Examples:-**

In each of the following Questions find the n[th] derivative after reducing them into standard functions given in the table 1.2.1

1. (a) $\log(9x^2-1)$ (b) $\log\left[(4x+3)e^{5x+7}\right]$ (c) $\log_{10}\sqrt{\frac{(3x+5)^2(2-3x)}{(x+1)^6}}$

Solution: (a) Let $y = \log(9x^2-1) = \log\{(3x+1)(3x-1)\}$

$$y = \log(3x+1) + \log(3x-1) \qquad (\because \log(AB) = \log A + \log B)$$

$$\therefore\ y_n = \frac{dn}{dx^n}\{\log(3x+1)\} + \frac{dn}{dx^n}\{\log(3x-1)\}$$

i.e $$y_n = \frac{(-1)^{n-1}(n-1)!}{(3x+1)^n}(3)^n + \frac{(-1)^{n-1}(n-1)!}{(3x-1)^n}(3)^n \qquad \blacksquare$$

(b) Let $y = \log\left[(4x+3)e^{5x+7}\right] = \log(4x+3) + \log e^{5x+7}$

$$= \log(4x+3) + (5x+7)\log_e e \qquad (\because \log A^B = B\log A)$$

$$\therefore\ y = \log(4x+3) + (5x+7) \qquad (\because \log_e e = 1)$$

$$\therefore\ y_n = \frac{(-1)^{n-1}(n-1)!}{(4x+3)^n}(4)^n + 0 \qquad \left(\begin{array}{l}\because D(5x+6) = 5 \\ D^2(5x+6) = 0 \\ D^n(5x+1) = 0\ (n>1)\end{array}\right) \qquad \blacksquare$$

(c) Let $y = \log_{10}\sqrt{\dfrac{(3x+5)^2(2-3x)}{(x+1)^6}}$

$$= \frac{1}{\log_e 10}\left\{\sqrt{\frac{(3x+5)^2(2-3x)}{(x+1)^6}}\right\} \qquad \because \ \log_{10} X = \frac{\log_e X}{\log_e 10}$$

$$= \frac{1}{\log_e 10}\left\{\frac{1}{2}\log\left\{\frac{(3x+5)^2(2-3x)}{(x+1)^6}\right\}\right\} \qquad \because \ \log A^B = B\log A$$

$$\because \ \log\left(\frac{A}{B}\right) = \log A - \log B$$

$$= \frac{1}{2\log_e 10}\left\{\log(3x+5)^2 + \log(2-3x) - \log(x+1)^6\right\}$$

$$\therefore \quad y = \frac{1}{2\log_e 10}\left\{2\log(3x+5) + \log(2-3x) - 6\log(x+1)\right\}$$

Hence,

$$y_n = \frac{1}{2\log_e 10}\left\{2.\frac{(-1)^{n-1}(n-1)!}{(3x+5)^n}(3)^n + \frac{(-1)^{n-1}(n-1)!}{(2-3x)^n}(-3)^n - 6.\frac{(-1)^{n-1}(n-1)!}{(x+1)^n}(1)^n\right\}$$

$\blacksquare$

2. (a) $e^{2x+4} + 6^{2x+4}$ (b) $\cosh 4x + \cosh^2 4x$

(c) $e^{-x}\sinh 3x \cosh 2x$ (d) $\dfrac{1}{(4x+5)} + \dfrac{1}{(5x+4)^4} + (6x+8)^5$

Solution: (a) Let $y = e^{2x+4} + 6^{2x+4}$

$$= e^{2x}e^4 + 6^{2x}6^4$$

$$\therefore \quad y = e^4(e^{2x}) + 1296(6^{2x})$$

hence

$$y_n = e^4\frac{dn}{dx^n}(e^{2x}) + 1296\frac{dn}{dx^n}(6^{2x})$$

$$= e^4\left\{2^n e^{2x}\right\} + 1296\left\{2^n(\log 6)^n 6^{2x}\right\} \ \blacksquare$$

(b) Let $y = \cosh 4x + \cosh^2 4x$

$$= \left(\frac{e^{4x}+e^{-4x}}{2}\right) + \left(\frac{e^{4x}+e^{-4x}}{2}\right)^2$$

$$= \frac{1}{2}\left(e^{4x}+e^{-4x}\right) + \frac{1}{4}\left\{(e^{4x})^2 + (e^{-4x})^2 + 2(e^{4x})(e^{-4x})\right\}$$

$$y = \frac{1}{2}\left(e^{4x}+e^{-4x}\right) + \frac{1}{4}\left\{e^{8x} + e^{-8x} + 2\right\}$$

hence,

$$y_n = \frac{1}{2}\left[4^n e^{4x} + (-4)^n e^{-4x}\right] + \frac{1}{4}\left[8^n e^{8n} + (-8)^n e^{-8n} + 0\right] \ \blacksquare$$

(c) Let $y = e^{-x}\sinh 3x \cosh 2x$

$$= e^{-x}\left\{\frac{e^{3x} - e^{-3x}}{2}\right\}\left\{\frac{e^{2x} + e^{-2x}}{2}\right\}$$

$$= \frac{e^{-x}}{4}\left\{(e^{3x} - e^{-3x})(e^{2x} + e^{-2x})\right\}$$

$$= \frac{e^{-x}}{4}\left\{e^{5x} - e^{-x} + e^{x} - e^{-5x}\right\}$$

$$= \frac{1}{4}\left\{e^{4x} - e^{-2x} + 1 - e^{-6x}\right\}$$

$$y = \frac{1}{4}\left\{1 + e^{4x} - e^{-2x} - e^{-6x}\right\}$$

Hence, $$y_n = \frac{1}{4}\left\{0 + (4)^n e^{4x} - (-2)^n e^{-2x} - (-6)^n e^{-6x}\right\} \blacksquare$$

(d) Let $y = \dfrac{1}{(4x+5)} + \dfrac{1}{(5x+4)^4} + (6x+8)^5$

Hence, $$y_n = \frac{dn}{dx^n}\left\{\frac{1}{(4x+5)}\right\} + \frac{dn}{dx^n}\left\{\frac{1}{(5x+4)^4}\right\} + \frac{dn}{dx^n}\left\{(6x+8)^5\right\}$$

$$= \frac{(-1)^n n!}{(4x+5)^{n+1}}(4)^n + \frac{(-1)^n (4+n-1)!}{(4-1)!(5x+4)^{4+n}}(5)^n + 0$$

i.e $$y_n = \frac{(-1)^n n!}{(4x+5)^{n+1}}(4)^n + \frac{(-1)^n (3+n)!}{3!(5x+4)^{n+4}}(5)^n \quad \blacksquare$$

- **Session - 3**
- **1.2.2 Worked examples:-**

1. (i) $\dfrac{1}{x^2 - 6x + 8}$ (ii) $\dfrac{1}{1 - x - x^2 + x^3}$ (iii) $\dfrac{x^2}{2x^2 + 7x + 6}$

(iv) $\left(\dfrac{x+2}{x+1}\right) + \dfrac{1}{4x^2 + 12x + 9}$ (v) $\tan^{-1}\left(x/a\right)$ (vi) $\tan^{-1} x$ (vii) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

- In all the above problems, we use the method of partial fractions to reduce them into standard forms.

Solutions: (i) Let $y=\dfrac{1}{x^2-6x+8}$. The function can be rewritten as $y=\dfrac{1}{(x-4)(x-2)}$

- This is proper fraction containing two distinct linear factors in the denominator. So, it can be split into partial fractions as

$y=\dfrac{1}{(x-4)(x-2)}=\dfrac{A}{(x-4)}+\dfrac{B}{(x-2)}$ Where the constant A and B are found as given below.

$$\frac{1}{(x-4)(x-2)}=\frac{A(x-2)+B(x-4)}{(x-4)(x-2)}$$

$\therefore \quad 1=A(x-2)+B(x-4)$ -------------(*)

Putting x = 2 in (*), we get the value of B as $B=-\frac{1}{2}$

Similarly putting x = 4 in(*), we get the value of A as $A=\frac{1}{2}$

$\therefore\ y=\dfrac{1}{(x-4)(x-2)}=\dfrac{(1/2)}{x-4}+\dfrac{(-1/2)}{x-2}$ Hence

$$y_n=\frac{1}{2}\frac{d_n}{dx^n}\left(\frac{1}{x-4}\right)-\frac{1}{2}\frac{d_n}{dx^n}\left(\frac{1}{x-2}\right)$$

$$=\frac{1}{2}\left[\frac{(-1)^n n!}{(x-4)^{n+1}}(1)^n\right]-\frac{1}{2}\left[\frac{(-1)^n n!}{(x-2)^{n+1}}(1)^n\right]$$

$$=\frac{1}{2}(-1)^n n!\left[\frac{1}{(x-4)^{n+1}}-\frac{1}{(x-2)^{n+1}}\right]$$

■

(ii) Let $y=\dfrac{1}{1-x-x^2+x^3}=\dfrac{1}{(1-x)-x^2(1-x)}=\dfrac{1}{(1-x)(1-x^2)}$

ie $y=\dfrac{1}{(1-x)(1-x)(1+x)}=\dfrac{1}{(1-x)^2(1+x)}$

Though y is a proper fraction, it contains a repeated linear factor $(1-x)^2$ in its denominator. Hence, we write the function as

$y=\dfrac{A}{(1-x)}+\dfrac{B}{(1-x)^2}+\dfrac{C}{1+x}$ in terms of partial fractions. The constants *A, B, C* are found as follows:

$$y = \frac{1}{(1-x)^2(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{1+x}$$

ie $1 = A(1-x)(1+x) + B(1+x) + C(1-x)^2$ -------------(**)

Putting x = 1 in (**), we get B as $B = \frac{1}{2}$

Putting x = -1 in (**), we get C as $C = \frac{1}{4}$

Putting x = 0 in (**), we get $1 = A + B + C$

$\therefore\ A = 1 - B - C = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$\therefore\ A = \frac{1}{4}$

Hence, $y = \frac{(1/4)}{(1-x)} + \frac{(1/2)}{(1-x)^2} + \frac{(1/4)}{(1+x)}$

$$\therefore\ y_n = \frac{1}{4}\left[\frac{(-1)^n n!}{(1-x)^{n+1}}(1)^n\right] + \frac{1}{2}\left[\frac{(-1)^n (2+n-1)!}{(2-1)!(1-x)^{2+n}}(1)^n\right] + \frac{1}{4}\left[\frac{(-1)^n n!}{(1+x)^{n+1}}(1)^n\right]$$

$$= \frac{1}{4}(-1)^n n!\left[\frac{1}{(1-x)^{n+1}} + \frac{1}{(1+x)^{n+1}}\right] + \frac{1}{2}\left[\frac{(-1)^n (n+1)!}{(1-x)n+2}\right]$$ ■

(iii) Let $y = \frac{x^2}{2x^2 + 7x + 6}$ **(VTU July-05)**

This is an improper function. We make it proper fraction by actual division and later

spilt that into partial fractions.

i.e $x^2 \div (2x^2 + 7x + 6) = \frac{1}{2} + \frac{(-\frac{7}{2}x - 3)}{2x^2 - 7x + 6}$

$\therefore\ y = \frac{1}{2} + \frac{-\frac{7}{2}x - 3}{(2x+3)(x+2)}$ Resolving this proper fraction into partial fractions, we get

$y = \frac{1}{2} + \left[\frac{A}{(2x+3)} + \frac{B}{(x+2)}\right]$. Following the above examples for finding A & B, we get

$$y = \frac{1}{2} + \left[\frac{\frac{9}{2}}{2x+3} + \frac{(-4)}{x+2}\right]$$

Hence, $y_n = 0 + \frac{9}{2}\left[\frac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n\right] - 4\left[\frac{(-1)^n n!}{(x+2)^{n+1}}(1)^n\right]$

i.e $y_n = (-1)^n n!\left[\frac{\frac{9}{2}(2)^n}{(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}}\right]$ ■

(iv) Let $y = \underbrace{\frac{(x+2)}{(x+1)}}_{\text{(i)}} + \underbrace{\frac{x}{4x^2 + 12x + 9}}_{\text{(ii)}}$

Here (i) is improper & (ii) is proper function. So, by actual division (i) becomes $\left(\frac{x+2}{x+1}\right)=1+\left(\frac{1}{x+1}\right)$. Hence, y is given by

$$y=1+\left(\frac{1}{x+1}\right)+\frac{1}{(2x+3)^2} \qquad [\because\ (2x+3)^2=4x^2+12x+9\,]$$

Resolving the last proper fraction into partial fractions, we get

$\frac{x}{(2x+3)^2}=\frac{A}{(2x+3)}+\frac{B}{(2x+3)^2}$. Solving we get

$A=\frac{1}{2}$ and $B=-\frac{3}{2}$

$$\therefore\ y=1+\left(\frac{1}{1+x}\right)+\left[\frac{\frac{1}{2}}{(2x+3)}+\frac{-\frac{3}{2}}{(2x+3)^2}\right]$$

$$\therefore\ y_n=0+\left[\frac{(-1)^n n!}{(1+x)^n}(1)^n\right]+\frac{1}{2}\left[\frac{(-1)^n n!}{(2x+3)^{n+1}}(2)^n\right]-\frac{3}{2}\left[\frac{(-1)^n (n+1)!}{(2x+3)n+2}(2)^n\right] \quad \blacksquare$$

(v) $\tan^{-1}\left(\frac{x}{a}\right)$

Let $y=\tan^{-1}\left(\frac{x}{a}\right)$

$$\therefore\ y_1=\frac{1}{1+\left(\frac{x}{a}\right)^2}\left(\frac{1}{a}\right)=\frac{a}{x^2+a^2}$$

$$y_n=D^n y=D^{n-1}(y_1)=D^{n-1}\left(\frac{a}{x^2+a^2}\right)$$

Consider $\frac{a}{x^2+a^2}=\frac{a}{(x+ai)(x-ai)}$

$=\frac{A}{(x+ai)}+\frac{B}{(x-ai)}$, on resolving into partial fractions.

$=\frac{\left(-\frac{1}{2i}\right)}{(x+ai)}+\frac{\left(\frac{1}{2i}\right)}{(x-ai)}$, on solving for A & B.

$$\therefore\ D^{n-1}\left(\frac{a}{x^2+a^2}\right)=D^{n-1}\left(\frac{-\frac{1}{2i}}{x+ai}\right)+D^{n-1}\left(\frac{\frac{1}{2i}}{x-ai}\right)$$

$$=\left(-\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]+\left(\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}\right]\text{-----------}(*)$$

- Since above answer containing complex quantity i we rewrite the answer in terms of real quantity, We take transformation $x=r\cos\theta \quad a=r\sin\theta$ where $r=\sqrt{x^2+a^2}$, $\theta=\tan^{-1}\left(\frac{a}{x}\right)$

$$x+ai=r(\cos\theta+i\sin\theta)=re^{i\theta}$$
$$x-ai=r(\cos\theta-i\sin\theta)=re^{-i\theta}$$

$$\frac{1}{(x-ai)^n} = \frac{1}{r^n e^{-in\theta}} = \frac{e^{in\theta}}{r^n}, \quad \frac{1}{(x+ai)^n} = \frac{e^{-in\theta}}{r^n}$$

now(*) is $y_n = \dfrac{(-1)^{n-1}(n-1)!}{2\,i\,r^n}\left[e^{in\theta} - e^{-in\theta}\right]$

$$y_n = \frac{(-1)^{n-1}}{2\,i\,r^n}(2\,i\sin n\theta) \Rightarrow \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta \quad \blacksquare$$

(vi) Let $y = \tan^{-1} x$ .Putting a = 1 in Ex.(v) we get

y_n which is same as above with $r = \sqrt{x^2+1}$ $\theta = \tan^{-1}\left(1/x\right)$

$\theta = \cot^{-1}(x)$ *or* $x = \cot\theta$

$\therefore r = \sqrt{\cot^2\theta + 1} = \operatorname{cosec}\theta \Rightarrow \dfrac{1}{r^n} = \dfrac{1}{\operatorname{cosec}^n\theta} = \sin^n\theta$

$D^n\left(\tan^{-1} x\right) = (-1)^{n-1}(n-1)!\ \sin^n\theta \sin n\theta$ *where* $\theta = \cot^{-1} x$ $\blacksquare$

(vii) Let $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

put $x = \tan\theta$ $\theta = \tan^{-1} x$

$$\therefore \quad y = \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan(\pi/4 + \theta)\right] \quad \because \tan\left(\pi/4 + \theta\right) = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \pi/4 + \theta = \frac{\pi}{4} + \tan^{-1}(x)$$

$$y = \frac{\pi}{4} + \tan^{-1}(x)$$

$$y_n = 0 + D^n(\tan^{-1} x)$$

$$= \left(-\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right] + \left(\frac{1}{2i}\right)\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n}\right] \quad \blacksquare$$

Problem set No. 1.2.1 for practice

Find the n[th] derivative of the following functions**:**

1. $\dfrac{6x}{(x-1)(x^2-4)}$ 2. $\dfrac{x}{(x+2)(x^2-2x+1)}$ 3. $\dfrac{x^2+4x+1}{x^3+2x^2-x-2}$ 4. $\dfrac{x}{4x^2-x-3}$

5. $\dfrac{x^3}{x^2-3x+2}$ 6. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ 7. $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

• Session 4

1. $\sin(ax+b)$.**(See Sl. No-7 of Table-1)**

Let $y = \sin(ax+b)$. Differentiating w.r.t x,

$y_1 = \cos(ax+b).a$ As $\sin(X + \frac{\pi}{2}) = \cos X$

We can write

$y_1 = a\sin(ax+b+\pi/2).$

Again Differentiating w.r.t x,

$y_2 = a\cos(ax+b+\pi/2).a$ Again using $\sin(X+\frac{\pi}{2}) = \cos X$,we get y_2 as

$y_2 = a\sin(ax+b+\pi/2+\pi/2).a$

i.e. $y_2 = a^2\sin(ax+b+2\pi/2).$

Similarly, we get

$y_3 = a^3\sin(ax+b+3\pi/2).$

$y_4 = a^4\sin(ax+b+4\pi/2).$

……………………………

$y_n = a^n\sin(ax+b+n\pi/2).$ ■

2. $e^{ax}\sin(bx+c)$. **(See Sl. No-9 of Table-1)**

$Let\ y = e^{ax}\sin(bx+c)$.....(1)

Differentiating using product rule ,we get

$y_1 = e^{ax}\cos(bx+c)b + \sin(bx+c)ae^{ax}$

i.e. $y_1 = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$. For computation of higher order derivatives it is convenient to express the constants 'a' and 'b' in terms of the constants r and θ defined by $a = r\cos\theta$ & $b = r\sin\theta$,so t $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(b/a\right)$.thus,

y_1 can be rewritten as

$y_1 = e^{ax}[(r\cos\theta)\sin(bx+c) + (r\sin\theta)\cos(bx+c)]$

or $y_1 = e^{ax}[r\{\sin(bx+c)\cos\theta + \cos(bx+c)\cos\theta\}]$

i.e. $y_1 = re^{ax}[\sin(bx+c+\theta)]$..........(2)

Comparing expressions (1) and (2), we write y_2 as

$y_2 = r^2e^{ax}\sin(bx+c+2\theta)$

$y_3 = r^3e^{ax}\sin(bx+c+3\theta)$

Continuing in this way, we get

$y_4 = r^4e^{ax}\sin(bx+c+4\theta)$

$y_5 = r^5e^{ax}\sin(bx+c+5\theta)$

....

$y_n = r^ne^{ax}\sin(bx+c+n\theta)$

$\therefore D^n[e^{ax}\sin(bx+c)] = r^ne^{ax}\sin(bx+c+n\theta)$, where $r = \sqrt{a^2+b^2}$ & $\theta = \tan^{-1}\left(b/a\right)$ ■

- **1.2.3 Worked examples**

1. (i) $\sin^2 x + \cos^3 x$ (ii) $\sin^3\cos^3 x$ (iii) $\cos x\cos 2x\cos 3x$
   (iv) $\sin x\sin 2x\sin 3x$ (v) $e^{3x}\cos 2x$ (vi) $e^{2x}(\sin^2 x + \cos^3 x)$

- The following formulae are useful in solving some of the above problems.

(i) $\sin^2 x = \dfrac{1-\cos 2x}{2}$ $(ii)\cos^2 x = \dfrac{1+\cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$ $(iv)\cos 3x = 4\cos^3 x - 3\cos x$

(v) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$\left.\begin{array}{l}
\text{(vi) } 2\cos A\sin B = \sin(A+B) - \sin(A-B)\\
\text{(vii) } 2\cos A\cos B = \cos(A+B) + \cos(A-B)\\
\text{(viii) } 2\sin A\sin B = \cos(A-B) - \cos(A+B)
\end{array}\right\}$$

Solutions: (i) Let $y = \sin^2 x + \cos^3 x = \left(\dfrac{1-\cos 2x}{2}\right) + \dfrac{1}{4}(\cos 3x + 3\cos x)$

$$\therefore y_n = \frac{1}{2}\left[0-(2)^n \cos\left(2x+{}^{n\pi}\!/_2\right)\right]+\frac{1}{4}\left[(3)^n \cos\left(3x+{}^{n\pi}\!/_2\right)+3\cos\left(x+{}^{n\pi}\!/_2\right)\right] \blacksquare$$

(ii)Let $y = \sin^3 x \cos^3 x = \left(\frac{\sin 2x}{2}\right)^3 = \frac{\sin^3 2x}{8} = \frac{1}{8}\left[\frac{-\sin 6x + 3\sin 2x}{4}\right]$

$$= \frac{1}{32}\left[3\sin 2x - \sin 6x\right]$$

$$y_n = \frac{1}{32}\left[3.2^n \sin\left(2x+\frac{n\pi}{2}\right)-6^n \sin\left(6x+\frac{n\pi}{2}\right)\right] \blacksquare$$

(iii))Let y = $\cos 3x \cos x \cos 2x$

$$= \frac{1}{2}(\cos 4x + \cos 2x)\cos 2x = \frac{1}{2}\left[\cos 4x \cos 2x + \cos^2 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}(\cos 6x + \cos 2x) + \frac{1-\cos 4x}{2}\right]$$

$$= \frac{1}{4}\cos 6x + \frac{\cos 2x}{4} + \frac{1}{4}(1-\cos 4x)$$

$$\therefore y_n = \frac{1}{4}6^n \cos\left(6x+\frac{n\pi}{2}\right)+\frac{2^n \cos\left(2x+\frac{n\pi}{2}\right)}{4}-\frac{4^n \cos\left(4x+\frac{n\pi}{2}\right)}{4} \blacksquare$$

(iv))Let y = $\sin 3x \sin x \, sn 2x$

$$= \frac{1}{2}\left[\sin(2x) - \sin 4x\right]\sin 2x$$

$$= \frac{1}{2}\left[\sin^2 2x - \sin 4x \sin 2x\right]$$

$$= \frac{1}{2}\left[\frac{1-\cos 4x}{2} - \frac{1}{2}(\sin 2x - \sin 6x)\right]$$

$$= \left[\left(\frac{1-\cos 4x}{4}\right) - \frac{1}{4}(\sin 2x - \sin 6x)\right]$$

$$y_n = \frac{1}{4}\left[4^n \cos\left(4x+\frac{n\pi}{2}\right) - 2^n \sin\left(2x+\frac{n\pi}{2}\right) + 6^n \sin\left(6x+\frac{n\pi}{2}\right)\right] \blacksquare$$

(v) Let $y = e^{3x}\cos 2x$ (Refer Sl.No. 10 of Table 1)

$\therefore y_n = re^{3x}\cos(2x+n\theta)$ where

$r = \sqrt{3^2+2^2} = \sqrt{13}$ & $\theta = \tan^{-1}\left(\frac{2}{3}\right)$ $\blacksquare$

(vi) Let y = $e^{2x}\left(\sin^2 x + \cos^3 x\right)$

We know that $\left[\sin^2 x+\cos^3 x\right]=\dfrac{1-\cos 2x}{2}+\dfrac{1}{4}\left[\cos 3x+3\cos x\right]$

$\therefore$ y = $e^{2x}\left[\sin^2 x+\cos^3 x\right]=e^{2x}\left[\dfrac{1-\cos 2x}{2}\right]+\dfrac{e^{2x}}{4}\left[\cos 3x+3\cos x\right]$

$\therefore y=\dfrac{1}{2}\left[e^{2x}-e^{2x}\cos 2x\right]+\dfrac{1}{4}\left[e^{2x}\cos 3x+3e^{2x}\cos x\right]$

Hence, $y_n=\dfrac{1}{2}\left[2^n e^{2x}-r_1^n e^{2x}\cos(2x+n\theta_1)\right]+\dfrac{1}{4}\left[r_2^n e^{2x}\cos(3x+n\theta_2)+3r_3^n e^{2x}\cos(x+n\theta_3)\right]$

where $r_1=\sqrt{2^2+2^2}=\sqrt{8}$; $r_2=\sqrt{2^2+3^2}=\sqrt{13}$; $r_3=\sqrt{2^2+1^2}=\sqrt{5}$

$\theta_1=\tan^{-1}\left(\dfrac{2}{2}\right)$; $\theta_2=\tan^{-1}\left(\dfrac{3}{2}\right)$; $\theta_3=\tan^{-1}\left(\dfrac{1}{2}\right)$; ■

Problem set No. 1.2.2 for practice

Find n^th derivative of the following functions:

1. $(\sin^3 x+\cos^2 x)$ 2. $\sin 2x\cos 3x$ 3. $\cos 2x.\sin 3x$ 4. $\cos x\cos 2x$
5. $\sin x\sin 2x$ 6. $e^{3x}(\sin^3 x+\cos^2 x)$ 7. $e^x\cos 2x\cos 4x$ 8. $e^{-x}\sin^2 x\cos 2x$
.9. $e^{-3x}\cos^3 x$ **(VTU Jan-04)**

LESSON -2 : Leibnitz's Theorem

- **Session - 1**
- Leibnitz's theorem is useful in the calculation of n^th derivatives of product of two functions.
- **Statement of the theorem:**

If u and v are functions of x, then

$$D^n(uv)=D^n uv+{}^nC_1D^{n-1}uDv+{}^nC_2D^{n-2}uD^2v+....+{}^nC_rD^{n-r}uD^rv+...uD^nv,$$

where $D=\dfrac{d}{dx}$, ${}^nC_1=n$, ${}^nC_2=\dfrac{n(n-1)}{2}$,........,${}^nC_r=\dfrac{n!}{r!(n-r)!}$

- **Worked Examples**

1. If $x=\sin t, y=\sin pt$ *prove that*

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(p^2-n^2)y_n=0$ **(VTU July-05)**

Solution: Note that the function $y=f(x)$ is given in the parametric form with a parameter *t*.

So, we consider

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{p\cos pt}{\cos t} \qquad (p-\text{constant})$$

or $$\left(\frac{dy}{dx}\right)^2=\frac{p^2\cos^2 pt}{\cos^2 t}=\frac{p^2(1-\sin^2 pt)}{1-\sin^2 t}=\frac{p^2(1-y^2)}{1-x^2}$$

or $\left(1-x^2\right)y_1^2 = p^2\left(1-y^2\right)$

So that $\left(1-x^2\right)y_1^2 - p^2\left(1-y^2\right)$ Differentiating w.r.t. x,

$$\left[\left(1-x^2\right)\left(2y_1y_2\right)+y_1^2\left(-2x\right)\right]-p^2\left(-2yy_1\right)=0$$

$$\left(1-x^2\right)y_2 - xy_1 + p^2y = 0 \quad \text{--------------- (1)} \quad [\div 2y_1\text{, throughout}]$$

Equation (1) has second order derivative y_2 in it. We differentiate (1), n times, term wise,

using Leibnitz's theorem as follows.

$$D^n\left[\left(1-x^2\right)y_2 - xy_1 - p^2y\right]=0$$

i.e $D^n\left\{(1-x^2)y_2\right\} - D^n\left\{xy_1\right\} - D^n\left(p^2y\right)=0$ ---------- (2)

(a) (b) (c)

Consider the term (a):

$D^n\left[\left(1-x^2\right)y_2\right]$. Taking $u = y_2$ and $v=(1-x^2)$ and applying Leibnitz's theorem we get

$$D^n\left[uv\right] = D^nuv + {}^nC_1D^{n-1}uDv + {}^nC_2D^{n-2}D^2v + {}^nC_3D^{n-3}uD^3v + \ldots$$

i.e

$$D^n\left[y_2(1-x^2)\right] = D^n(y_2).(1-x^2) + {}^nC_1D^{n-1}(y_2).D(1-x^2) + {}^nC_2D^{n-2}(y_2)D^2(1-x^2) + {}^nC_3D^{n-3}(y_2)D^3(1-x^2)+\ldots$$

$$= y_{(n)+2} - x^2) + ny_{(n-1)+2}.(-2x) + \frac{n(n-1)}{2\,!}y_{(n-2)+2}.(-2) + \frac{n(n-1)(n-2)}{3\,!}.y_{(n-3)+2}.(0)+\ldots$$

$$D^n\left[\left(1-x^2\right)y_2\right] = \left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n \quad \text{----------- (3)}$$

Consider the term (b):

$D^n\left[xy_1\right]$. Taking $u = y_1$ and $v = x$ and applying Leibnitz's theorem, we get

$$D^n\left[y_1(x)\right] = D^n(y_1).(x) + {}^nC_1D^{n-1}y_1.D(x) + {}^nC_2D^{n-2}(y_1).D^2(x)+\ldots.$$

$$= y_{(n)+1}.x + ny_{(n-1)+1} + \frac{n(n-1)}{2\,!}y_{(n-2)+2}(0)+\ldots.$$

$$D^n\left[xy_1\right] = xy_{n+1} + ny_n \quad \text{---------- (4)}$$

Consider the term (c):

$$D^n(p^2y) = p^2D^n(y) = p^2y_n \quad \text{--------- (5)}$$

Substituting these values (3), (4) and (5) in Eq (2) we get

$$\left\{\left(1-x^2\right)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n\right\} - \left\{xy_{n+1} + ny_n\right\} + \left\{p^2y_n\right\} = 0$$

ie $\left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n + ny_n - ny_n + p^2y_n = 0$

$\therefore \left(1-x^2\right)y_{n+2} - (2n+1)xy_{n+1} + \left(p^2-n^2\right)y_n = 0$ as desired. ■

2. If $\sin^{-1}y = 2\log(x+1)$ or

$y = \sin\left[2\log(x+1)\right]$ or $y = \sin\left[\log(x+1)^2\right]$

or $y = \sin\log(x^2 + 2x + 1)$, show that

$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$ **(VTU Jan-03)**

Out of the above four versions, we consider the function as

$\sin^{-1}(y) = 2\log(x+1)$

Differentiating w.r.t x, we get

$$\frac{1}{\sqrt{1-y^2}}(y_1) = \left(\frac{2}{x+1}\right) \text{ ie } (x+1)y_1 = 2\sqrt{1-y^2}$$

Squaring on both sides

$(x+1)^2 y_1^2 = 4(1-y^2)$

Again differentiating w.r.t x,

$(x+1)^2(2y_1y_2) + y_1^2(2(x+1)) = 4(-2yy_1)$

or $(x+1)^2 y_2 + (x+1)y_1 = -4y \quad (\div 2y_1)$

or $(x+1)^2 y_2 + (x+1)y_1 + 4y = 0$ -----------*

Differentiating * w.r.t x, n-times, using Leibnitz's theorem,

$$\left\{D^n y_2(x+1)^2 + nD^{n-1}(y_2)2(x+1) + \frac{n(n-1)}{2!}D^{n-2}(y_2)(2)\right\} + \left\{D^n(g_1)(x+1) + nD^{n-1}y_1(1)\right\} + 4D^n y = 0$$

On simplification, we get

$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$ ■

3. If $x = \tan(\log y)$, then find the value of

$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1}$ **(VTU July-04)**

Consider

$x = \tan(\log y)$

i.e. $\tan^{-1} x = \log y$ or $y = e^{\tan^{-1} x}$

Differentiating w.r.t x,

$$y_1 = e^{\tan^{-1} x}.\frac{1}{1+x^2} = \frac{y}{1+x^2}$$

$\therefore (1+x^2)y_1 = y \quad ie (1+x^2)y_1 - y = 0$ -----------*

We differentiate * n-times using Leibnitz's theorem,

We get

$D^n\left[(1+x^2)y_1\right] - D^n(y) = 0$

ie. $\left\{D^n(y_1)(1+x^2) + {}^nC_1 D^{n-1}(y_1)D(1+x^2) + {}^nC_2 D^{n-2}(y_1)D^2(1+x^2) + ...\right\} - \left\{D^n y\right\} = 0$

ie $\left\{y_{n+1}(1+x^2) + ny_n(2x) + \frac{n(n-1)}{2!}y_{n-1}(2) + 0 + ...\right\} - y_n = 0$

$(1+x^2)y_{n+1} + (2nx-1)y_n + n(n-1)y_{n-1} = 0$ ■

4. If $y^{1/m} + y^{-1/m} = 2x$, or $y = \left[x + \sqrt{x^2-1}\right]^m$ or $y = \left[x - \sqrt{x^2-1}\right]^m$

Show that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$ **(VTU Feb-02)**

Consider

$$y^{1/m} + y^{-1/m} = 2x \quad \Rightarrow \quad y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$\Rightarrow \left(y^{1/m}\right)^2 - 2x\left(y^{1/m}\right) + 1 = 0 \text{ Which is quadratic equation in } y^{1/m}$$

$$\therefore y^{1/m} = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)} = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2 - 1}}{2} = \left(x \pm \sqrt{x^2 - 1}\right) \Rightarrow y^{1/m} = \left(x \pm \sqrt{x^2 - 1}\right)$$

$\therefore y = \left(x \pm \sqrt{x^2 - 1}\right)^m$. So, we can consider

$$y = \left[x + \sqrt{x^2 - 1}\right]^m \quad \text{or} \quad y = \left[x - \sqrt{x^2 - 1}\right]^m$$

Let us take $y = \left[x + \sqrt{x^2 - 1}\right]^m$

$$\therefore y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left(1 + \frac{1}{2\sqrt{x^2 - 1}}(2x)\right)$$

$$y_1 = m\left(x + \sqrt{x^2 - 1}\right)^{m-1}\left(\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}\right)$$

or

$\left(\sqrt{x^2 - 1}\right)y_1 = my$. On squaring

$\left(x^2 - 1\right)y_1^2 = m^2 y^2$. Again differentiating w.r.t x,

$$\left(x^2 - 1\right)2y_1 y_2 + y_1^2(2x) = m^2(2yy_1)$$

or

$$\left(x^2 - 1\right)y_2 + xy_1 = m^2 y \quad (\div 2y_1)$$

or

$$\left(x^2 - 1\right)y_2 + xy_1 - m^2 y = 0 \qquad \text{------------(*)}$$

Differentiating (*) n- times using Leibnitz's theorem and simplifying, we get

$$\left(x^2 - 1\right)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2 - m^2\right)y_n = 0 \quad \blacksquare$$

Problem set 1.3.1

In each of the following, apply Leibnitz's theorem to get the results.

1. show that $\frac{d^n}{dx^n}\left[\frac{\log x}{x}\right] = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots\ldots\ldots \frac{1}{n}\right]$ Hint: Take $v = \log x; \; u = \frac{1}{x}$
2. If $y = (x^2 - 1)^n$, Show that y_n satisfies the equation $\left(1 - x^2\right)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ Hint : It is required to show that $\left(1 - x^2\right)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0$
3. If $y = a\cos(\log x) + b\sin(\log x)$,

$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

4. If $y = e^{m\sin^{-1}x}$, Prove That
   $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$
5. If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, Show that
   $x^2 y_{n+2} + (2n+1)xy_{n+1} + 2n^2 y_n = 0$
6. $y = \sin(m\sin^{-1}x)$, Prove That
   $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$
7. If $y_n = D^n(x^n \log x)$, Prove That
   (i) $y_n = ny_{n-1} + (n-1)!$ (ii) $y_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots + \frac{1}{n}\right]$
8. If $y = x^n \log x$, Show that $y_{n+1} = \frac{n!}{x}$

- **Summary**:- The idea of successive differentiation was presented. The computation of n[th] derivatives of a few standard functions and relevant problems were discussed. Also, the concept of successive differention was extended for special type of functions using Liebnitz's theorem.

Chapter – 2 : POLAR CURVES

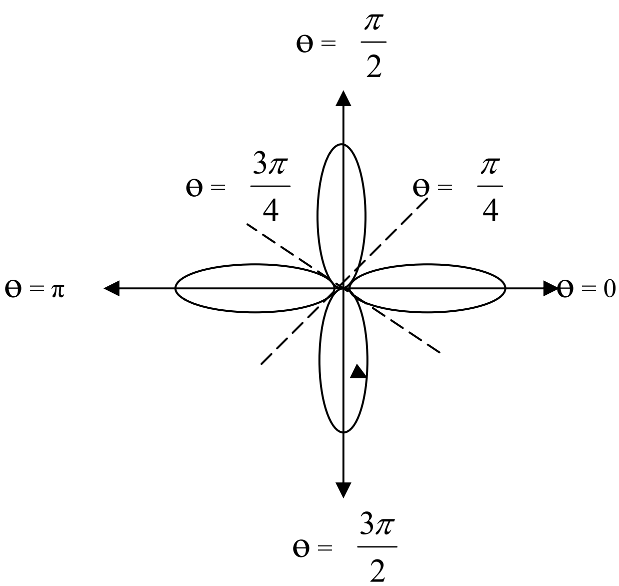
LESSON -1 : Angle between Polar Curves

- In this chapter we introduce a new coordinate system, where we can understand the idea of polar curves and their properties.

- **Session-1**

2.1.0 Introduction:- We are familiar with Cartesian coordinate system for specifying a point in the xy – plane. Another useful system for similar purpose is Polar coordinate system, and the curves specified by these coordinates are referred to as polar curves.

- **A polar curve by name "three-leaved rose" is displayed below:**



- Any point P can be located on a plane with co-ordinates (r,θ) called **polar co-ordinates** of P where r = **radius vector** $\overrightarrow{OP}$,(with pole 'O') ; θ = projection of $\overrightarrow{OP}$ on the **initial axis** OA.(See Fig.1)
- The equation $r = f(\theta)$ is known as a **polar curve**.
- Polar coordinates (r,θ) can be related with Cartesian coordinates (x, y) through the relations $x = r\cos\theta$ & $y = r\sin\theta$.

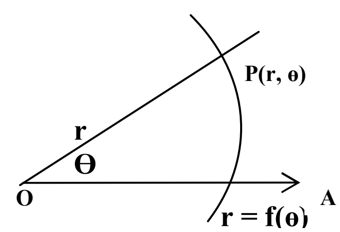
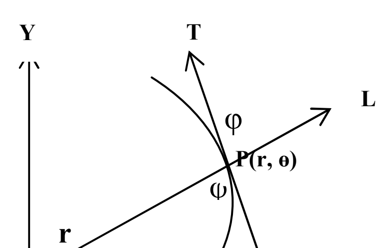


Fig.1. Polar coordinate system

2.1.1 Important results



- **<u>Theorem 1</u>: Angle between the radius vector and the tangent.:**

  i.e. With usual notation prove that $\tan\phi = r\dfrac{d\theta}{dr}$

- **Proof:-** Let "ϕ" be the angle between the radius vector OPL and the tangent TPT^1 at the point \`P\` on the polar curve $r = f(\theta)$. (See fig.2)

  Fig.2. Angle between radius vector and the tangent

  From Fig.2,

  $$\psi = \theta + \phi$$

  $$\tan\psi = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

  i.e. $$\frac{dy}{dx} = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} \text{................} (1)$$

  On the other hand, we have $x = r\cos\theta$; $y = r\sin\theta$ differentiating these, w.r.t θ,

  $$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right) \;\&\; \frac{dy}{d\theta} = r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)$$ **//NOTE//**

  $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r(\cos\theta) + \sin\theta\left(\frac{dr}{d\theta}\right)}{r(-\sin\theta) + \cos\theta\left(\frac{dr}{d\theta}\right)} \quad \textit{dividing the Nr \& Dr by } \frac{dr}{d\theta}\cos\theta$$

  $$\frac{dy}{dx} = \frac{r\left(d\theta/dr\right) + \tan\theta}{-\left(rd\theta/dr\right)\tan\theta + 1}$$

  i.e. $$\frac{dy}{dx} = \frac{\tan\theta + \left(r\,d\theta/dr\right)}{1-\tan\theta\left(rd\theta/dr\right)} \text{.....................} (2)$$

  Comparing equations (1) and (2)

  we get $\tan\phi = r\,d\theta/dr$

- **Note that** $\cot\phi = \left(\dfrac{1}{r}\dfrac{dr}{d\theta}\right)$

- **<u>A Note on Angle of intersection of two polar curves</u>**:-

  If ϕ_1 and ϕ_2 are the angles between the common radius vector and the tangents at the point of intersection of two curves $r = f_1(\theta)$ and $r = f_2(\theta)$ then the angle intersection of the curves is given by $|\phi_1 - \phi_2|$

- **<u>Theorem 2</u>: The length "p" of perpendicular from pole to the tangent in a polar curve i.e.**

  (i) $p = r\sin\phi$ **or (ii)** $\dfrac{1}{p^2} = \dfrac{1}{r^2} = \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

- **Proof:-** In the Fig.3, note that ON = p, the length of the perpendicular from the pole to the tangent at p on $r = f(\theta)$.from the right angled triangle OPN,

$$\sin\phi = \frac{ON}{OP} \quad \Rightarrow ON = (OP)\sin\phi$$

i.e. $p = r\sin\phi$..............(i)

Consider $\frac{1}{p} = \frac{1}{r\sin\phi} = \frac{1}{r}\operatorname{cosec}\phi$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot 2\phi)$$

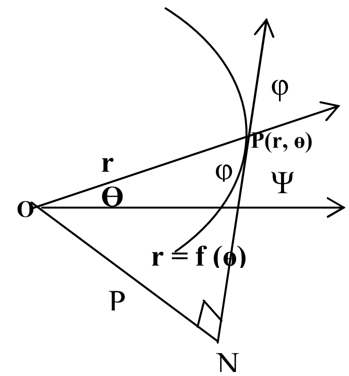


Fig.3 Length of the perpendicular from the pole to the tangent

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \text{............}(ii)$$

- **Note**:-If $u = \frac{1}{r}$, we get $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$
- **Session-2**
- In this session, we solve few problems on angle of intersection of polar curves and pedal equations.

2.1.2 Worked examples:-

- Find the acute angle between the following polar curves
  1. $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$ **(VTU-July-2003)**
  2. $r = (\sin\theta + \cos\theta)$ and $r = 2\sin\theta$ **(VTU-July-2004)**
  3. $r = 16\sec^2\left(\theta/2\right)$ and $r = 25\operatorname{cosec}^2\left(\theta/2\right)$
  4. $r = a\log\theta$ and $r = a/\log\theta$ **(VTU-July-2005)**
  5. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Solutions:

1. Consider
   $r = a(1+\cos\theta)$
   Diff w.r.t θ

Consider
$r = b(1-\cos\theta)$
Diff w.r.t θ

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$\tan\phi_1 = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= -\cot\theta/2$$

i.e $\tan\phi_1 = \tan(\pi/2 + \theta/2) \Rightarrow \phi_1 = (\pi/2 + \theta/2)$

$$\frac{dr}{d\theta} = b\sin\theta$$

$$r\frac{d\theta}{dr} = \frac{b(1-\cos\theta)}{b\sin\theta}$$

$$\tan\phi_1 = -\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= \tan\theta/2$$

$\tan\phi_1 = \tan\theta/2 \Rightarrow \phi_1 = \phi_2$

Angle between the curves

$$|\phi_1 - \phi_2| = \left|(\pi/2 + \theta/2) - \theta/2\right| = \pi/2$$

Hence ,the given curves intersect orthogonally $\blacksquare$

2. Consider

$$r = (\sin\theta + \cos\theta)$$

Diff w.r.t θ

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$r\frac{d\theta}{dr} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\tan\phi_1 = \frac{\tan\theta + 1}{1 - \tan\theta} \quad (\div \text{ Nr \& Dr} \cos\theta)$$

i.e $\tan\phi_1 = \dfrac{\tan\theta + 1}{1-\tan\theta} = \tan(\pi/4 + \theta)$

$$\Rightarrow \phi_1 = \pi/4 + \theta$$

Consider

$$r = 2\sin\theta$$

Diff w.r.t θ

$$\frac{dr}{d\theta} = 2\cos\theta$$

$$r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta}$$

$$\tan\phi_2 = \tan\theta$$

$$\Rightarrow \phi_2 = \theta$$

$\therefore$ Angle between the curves $= |\phi_1 - \phi_2| = \left|(\pi/4 + \theta) - \theta\right| = \pi/4$ $\blacksquare$

3. Consider

$$r = 16\sec^2(\theta/2)$$

Diff w.r.t θ

$$\frac{dr}{d\theta} = 32\sec^2(\theta/2)\tan(\theta/2).1/2$$

$$= 16\sec(\theta/2)\tan(\theta/2)$$

$$r\frac{d\theta}{dr} = \frac{16\sec^2(\theta/2)}{16\sec^2(\theta/2)\tan(\theta/2)}$$

$$\tan\phi_1 = \cot\theta/2 = \tan(\pi/2 - \theta/2)$$

Consider

$$r = 25\cos ec^2(\theta/2)$$

Diff w.r.t θ

$$\frac{dr}{d\theta} = -50\cos ec^2(\theta/2)\cot(\theta/2).1/2$$

$$= -25\cos ec^2(\theta/2)\cot(\theta/2)$$

$$r\frac{d\theta}{dr} = \frac{25\cos ec^2(\theta/2)}{-25\cos ec^2(\theta/2)\cot(\theta/2)}$$

$$\tan\phi_2 = -\tan\theta/2 = \tan(-\theta/2)$$

$$\Rightarrow \phi_1 = \left(\pi/2 - \theta/2\right) \qquad \Rightarrow \phi_2 = -\theta/2$$

Angle of intersection of the curves = $|\phi_1 - \phi_2| = \left|\left(\pi/2 - \theta/2\right) - \left(\theta/2\right)\right|$

$$= \pi/2 \quad \blacksquare$$

4. Consider

$r = a\log\theta$

Diff w.r.t θ

$$\frac{dr}{d\theta} = a/\theta$$

$$r\frac{d\theta}{dr} = a\log\theta\left(\theta/a\right)$$

$$\tan\phi_1 = \theta\log\theta..........(i)$$

Consider

$r = a/\log\theta$

Diff w.r.t θ

$$\frac{dr}{d\theta} = -a/(\log\theta)^2 \cdot 1/\theta$$

$$r\frac{d\theta}{dr} = -\left(a/\log\theta\right)\left(\frac{(\log\theta)^2\,\theta}{a}\right)$$

$$\tan\phi_2 = -\theta\log\theta..........(ii)$$

We know that

$$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1\,\tan\phi_2}$$

$$= \frac{\theta\log\theta - (-\theta\log\theta)}{1 + (\vartheta\log\theta)(-\theta\log\theta)}$$

i.e $\tan(\phi_1 - \phi_2) = \dfrac{2\theta\log\theta}{1 - (\theta\log\theta)^2}............(iii)$

From the data: $a\log\theta = r = a/\log\theta \Rightarrow (\log\theta)^2 = 1 \quad or\,\log\theta = \pm 1$

As θ is acute, we take by $\theta = 1 \Rightarrow \theta = e \quad \|NOTE\|$

Substituting $\theta = e$ in (iii), we get

$$\tan(\phi_1 - \phi_2) = \frac{2e\log e}{1 - (e\log e)^2} = \left(\frac{2e}{1 - e^2}\right) \qquad \left(\because \log_e^e = 1\right)$$

$$\therefore |\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e}{1 - e^2}\right) \quad \blacksquare$$

5. Consider

$r = \dfrac{a\theta}{1+\theta}$ as

$$\frac{1}{r} = \frac{1+\theta}{a\theta} = \frac{1}{a}\left(1/\theta + 1\right)$$

Diff w.r.t θ

$$-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{1}{a}\left(-1/\theta^2\right)$$

Consider

$$r = \frac{a\theta}{1+\theta^2}$$

$$\therefore \left(1+\theta^2\right) = a/r$$

Diff w.r.t θ

$$2\theta = -a/r^2\,\frac{dr}{d\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{r}{a\theta^2}$$

$$r\frac{d\theta}{dr}=\frac{a\theta^2}{r}$$

$$\tan\phi_1=\frac{a\theta^2}{a\theta/(1+\theta)}$$

$$\therefore \tan\phi_1=\theta(1+\theta)$$

$$\frac{-2r\theta}{a}=\frac{1}{r}\frac{dr}{d\theta}$$

i.e $r\dfrac{d\theta}{dr}=\dfrac{-a}{2r\theta}$

$$\tan\phi_2=-\frac{a}{2\theta}\left(\frac{1+\theta^2}{a}\right)$$

$$\tan\phi_2=-\frac{1}{2\theta}\left(1+\theta^2\right)$$

Now, we have

$$\frac{a\theta}{1+\theta}=r=\frac{a}{1+\theta^2}\Rightarrow a\theta\left(1+\theta^2\right)=a(1+\theta)$$

$$\text{or } \theta+\theta^3=1+\theta\Rightarrow\theta^3=1 \text{ or } \theta=1$$

$$\therefore \tan\phi_1=2 \;\&\; \tan\phi_2=(-1)$$

Consider $\tan|(\phi_1-\phi_2)|=\left|\dfrac{\tan\phi_1-\tan\phi_2}{1+(\tan\phi_1)(\tan\phi_2)}\right|$

$$=\left|\frac{2-(-1)}{1+(2)(-1)}\right|=|-3|=3$$

$$\therefore|\phi_1-\phi_2|=\tan^{-1}(3)$$

Problem Set No. 2.1.1 for practice.

- Find the acute angle between the curves

1. $r^n=a^n(\cos n\theta+\sin n\theta)$ and $r^n=a^n\sin n\theta$ (ans: $\pi/4$)
2. $r^n\cos n\theta=a^n$ and $r^n\sin n\theta=b^n$ (ans: $\pi/2$)
3. $r=a\theta$ and $r=a/\theta$ (ans: $\pi/2$)
4. $r=a\cos\theta$ and $r=a/2$ (ans: $5\pi/6$)
5. $r^m=a^m\cos m\theta$ and $r^m=b^m\sin m\theta$ (ans: $\pi/2$)

LESSON -2 : Pedal Equations

- **Session - 1**

- **2.2.0 Pedal equations (p-r equations)**:- Any equation containing only **p** & **r** is known as pedal equation of a polar curve.

- **Working rules to find pedal equations:-**

(i) Eliminate r and ϕ from the Eqs.: (i) $r=f(\theta)$ & $p=r\sin\phi$

(ii) Eliminate only θ from the Eqs.: (i) $r=f(\theta)$ & $\therefore \dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

- **2.2.1 Worked Examples on pedal equations:-**

- Find the pedal equations for the polar curves:-
  1. $\frac{2a}{r} = 1 - \cos\theta$
  2. $r = e^{\theta c \cot\alpha}$
  3. $r^m = a^m \sin m\theta + b^m \cos m\theta$ **(VTU-Jan-2005)**
  4. $l/_r = 1 + e\cos\theta$

Solutions:

1. Consider $\frac{2a}{r} = 1 - \cos\theta$ ……….(i)

Diff. w.r.t θ

$$2a\left(-1/_{r^2}\right)\frac{dr}{d\theta} = \sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-r\sin\theta}{2a}$$

$$r\frac{d\theta}{dr} = -\frac{2a}{r}\frac{1}{\sin\theta}$$

$$\tan\phi = -\frac{(1-\cos\theta)}{\sin\theta} = -\frac{2\sin^2 \theta/_2}{2\sin\ \theta/_2 \cos\theta/_2} = -\tan\left(\theta/_2\right)$$

$$\tan\phi = \tan\left(-\theta/_2\right) \Rightarrow \phi = -\theta/_2$$

Using the value of ϕ is $p = r\sin\phi$, we get

$$p = r\sin\left(-\theta/_2\right) = -r\sin\theta/_2 \text{.............}(ii)$$

Eliminating "θ" between (i) and (ii)

$$p^2 = r^2\sin^2\theta/_2 = r^2\left(\frac{1-\cos\theta}{2}\right) = \frac{r^2}{2}\left(\frac{2a}{r}\right) \quad \text{[See eg: - (i)]}$$

$$p^2 = ar.$$

This eqn. is only in terms of p and r and hence it is the pedal equation of the polar curve. ∎

2. Consider $r = e^{\theta\cot\alpha}$

Diff. w.r.t θ

$$\frac{dr}{d\theta} = e^{\theta\cot\alpha}(\cot\alpha) = r\cot\alpha \quad \left(\because r = e^{\theta\cot\alpha}\right)$$

We use the equation

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}(r\cot\alpha)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\cot^2\alpha\right) = \frac{1}{r^2}\left(1+\cot^2\alpha\right) = \frac{1}{r^2}\cos ec^2\alpha$$

$$\frac{1}{p^2} = \frac{1}{r^2}\cos ec^2\alpha$$

$p^2 = r^2/\text{cosec}^2\alpha$ or $r^2 = p^2 \text{cosec}^2\alpha$ is the required pedal equation ■

3.Consider $r^m = a^m \sin m\theta + b^m \cos m\theta$

Diff. w.r.t θ

$$mr^{m-1}\frac{dr}{d\theta} = a^m(m\cos m\theta) + b^m(-m\sin m\theta)$$

$$\frac{r^m}{r}\frac{dr}{d\theta} = a^m \cos m\theta - b^m \sin m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}$$

$$\cot\phi = \frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}$$

Consider $p = r\sin\phi$, $\frac{1}{p} = \frac{1}{r}\text{cosec}\phi$

$$\frac{1}{p^2} = \frac{1}{r^2}\text{cosec}^2\phi$$

$$= \frac{1}{r^2}\left(1 + \cot^2\phi\right)$$

$$= \frac{1}{r^2}\left[1 + \left(\frac{a^m \cos m\theta - b^m \sin m\theta}{a^m \sin m\theta + b^m \cos m\theta}\right)^2\right]$$

$$= \frac{1}{r^2}\left[\frac{\left(a^m \sin m\theta + b^m \cos m\theta\right)^2 + \left(a^m \cos m\theta - b^m \sin m\theta\right)^2}{\left(a^m \sin m\theta + b^m \cos m\theta\right)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^{2m} + b^{2m}}{r^{2m}}\right] \quad \|Note\|$$

$\Rightarrow p^2 = \dfrac{r^{2(m+1)}}{a^{2m} + b^{2m}}$ is the required *p-r* equation ■

4. Consider $l/r = (1 + \cos\theta)$

Diff w.r.t θ

$$l\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right) = -e\sin\theta \Rightarrow l/r\left(1/r\,\frac{dr}{d\theta}\right) = e\sin\theta$$

$$l/r\left(\cot\phi\right) = e\sin\theta$$

$$\therefore \cot\phi = \left(r/l\right)e\sin\theta$$

We have $\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\phi\right)$ (see eg: 3 above) Now

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{l^2 + e^2r^2\sin^2\theta}{l^2}\right]$$

$$= \frac{1}{r^2}\left(1 + \frac{e^2r^2}{l^2}\sin^2\theta\right)$$

$$1 + e\cos\theta = \frac{l}{r}$$

$$e\cos\theta = \frac{l-r}{r}$$

$$\cos\theta = \left(\frac{l-r}{re}\right)$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \left(\frac{l-r}{re}\right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{l^2 + e^2r^2\left\{1 - \left(\frac{l-r}{re}\right)^2\right\}}{l^2}\right]$$

On simplification $\frac{1}{p^2} = \left(\frac{e^2-1}{e^2}\right) + \frac{2}{lr}$ $\blacksquare$

- **<u>Problem Set No. 2.2.1 for practice.</u>**

Find the pedal equations of the following polar curves

1. $r^n\cos n\theta = a^n$ and $r^n\sin n\theta = b^n$
2. $r = a\theta$ and $r = a/\theta$
3. $r = a\cos\theta$ and $r = a/2$
4. $r^m = a^m\cos m\theta$ and $r^m = b^m\sin m\theta$