Lecture :

Rolle's Theorem and Mean-Value Theorem

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If

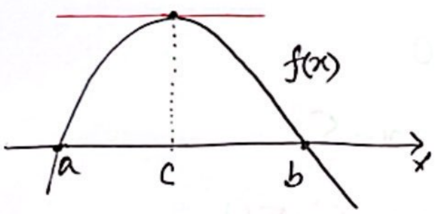
$$f(a) = 0 \text{ and } f(b) = 0$$

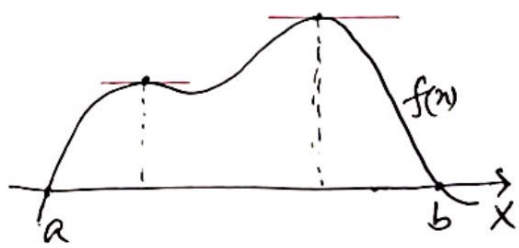
then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

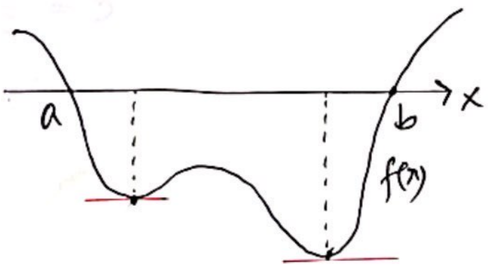
■

Meaning (Geometrically)

This theorem states the geometrically obvious fact that if the graph of a differentiable function intersects the x-axis at two points, a and b, then somewhere between a and b there must be at least one place where the tangent line is horizontal.







~~**Example:** Find the two x-intercepts of the function $f(x) = x^2 - 5x + 4$ and confirm that $f'(c) = 0$ at some point c between those intercepts.~~

Problem: Verify that the hypothesis of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

$$f(x) = x^2 - 5x + 4 \quad \text{on} \quad [1, 4].$$

Sol^n

$$f(1) = 1^2 - 5.1 + 4 = 0$$
$$f(4) = 4^2 - 5.4 + 4 = 0$$

Since $f(x)$ is a polynomial function on $[1,4]$ thus the function is continuous on $[1,4]$ and differentiable on $(1,4)$.

Now we will find out the values of c.

Rolle's theorem ~~guarantees that~~ guaranteed the existence of at least ~~of~~ one point c in $(1,4)$ s.t.

$$f'(c) = 0$$

$$f'(x) = 2x - 5$$

$$\therefore f'(c) = 2c - 5 = 0$$
$$c = \frac{5}{2} \in (1,4)$$

$\boxtimes$

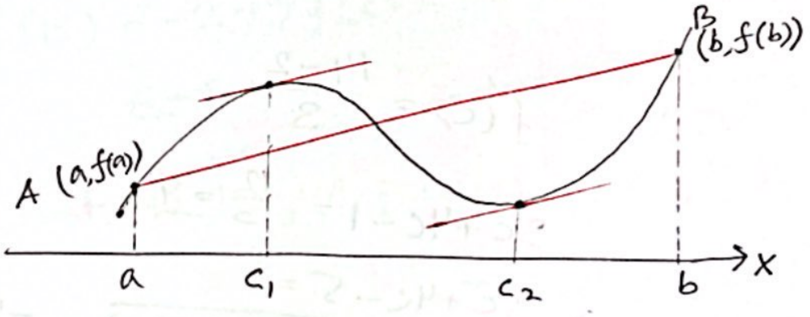
The Mean-Value Theorem:

Theorem: Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is at least one point c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$\square$

Geometric Meaning: This theorem states that between any two points $A(a, f(a))$ and $B(b, f(b))$ on the graph of differentiable function f, there is at least one place where the tangent line to the graph is parallel to the secant line joining A and B.



Problem: Suppose that we know that $f(x)$ is continuous and differentiable on $[6,15]$. Let's also suppose that we know that $f(6) = -2$, and that we know that $f'(x) \leq 10$. What is the largest possible value for $f(15)$?

Sol^n: By the Mean Value Th^m,

$$f'(c) = \frac{f(15) - f(6)}{15 - 6}$$

$$\cancel{f'(c) =} \; f(15) - f(6) = f'(c)(15-6) = f'(c)\,9$$

$$f(15) = 9f'(c) - 2.$$

Now we know that $f'(x) \leq 10$, so in particular we know that $f'(c) \leq 10$. Thus,

$$f(15) = -2 + 9f'(c) \leq -2 + 9 \cdot 10 = 88$$

This means that the largest possible value for $f(15)$ is 88. $\blacksquare$

Extra problem:

From Book: Page: 308

Problem: 1–8, 10(b),

Problem: A car travels from ~~city~~ Dhaka to Chittagong in 4 ~~5~~ hours. The total distance covered is 300 Km. ~~the~~ Speed limit on highway is 60 Km/h. ~~The~~ The car driver takes a ticket for over speed of this car. ~~why he has taken the ticket~~ what was the speed of his car?

Sol: The average velocity over $[0, 4]$ is given by

$$\frac{f(4) - f(0)}{4 - 0} = \frac{300 - 0}{4} = 75 \text{ Km/h}$$

(This represents the secant line slope).

Now find c such that $f'(c) = 75$

By MVT, there exists at least one point c in $(0, 4)$ where $f'(c) = 75$.

This means the car's instantaneous speed at $t = c$ (for some c between 0 and 4 hours) was exactly 75 Km/h.