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① $f(x,y) = 3x^3y^2$

ⓐ $f_x(x,y) = 9x^2y^2$

ⓑ $f_y(x,y) = 6x^3y$

ⓒ $f_x(x,1) = 9x^2$

ⓓ $f_y(1,y) = \cancel{9 \cdot 2^2 \cdot 1 = 36}\ 6 \cdot 1 \cdot y = 6y$

ⓕ $f_x(1,2) = 9 \cdot 2^2 \cdot 1 = 36$

ⓖ $f_y(1,2) = \cancel{9 \cdot 2^2 \cdot 1 = 36}\ 6 \cdot 1^3 \cdot 2 = 12$

② ⓐ $f(x,y) = xe^{-y} + 5y$

$f_x(x,y) = e^{-y}$

$f_x(3,0) = e^{-0} = 1$

$f_x(4,2) = e^{-2}$

$\boxed{f(x,y) = \sqrt{3x+2y}}$

$f_x(x,y) = \dfrac{3}{2\sqrt{3x+2y}}$

$f_x(3,0) = \dfrac{1}{2}$

$f_x(4,2) = \dfrac{3}{8}$

(b) $f(x,y) = xe^{-y} + 5y$

$f_y(x,y) = -xe^{-y} + 5$

$$\boxed{f_y(3,0) = -3 \cdot e^{-0} + 5 = -3+5 = 2}$$

$$f_y(4,2) = -4 \cdot e^{-2} + 5 = \frac{-4}{e^2} + 5$$

$$= \frac{5e^2 - 4}{e^2}$$

$f(x,y) = \sqrt{3x+2y}$

$$f_y(x,y) = \frac{2}{2\sqrt{3x+2y}} = \frac{1}{\sqrt{3x+2y}}$$

$$\boxed{f_y(3,0) = \frac{1}{3}}$$

$$\boxed{f_y(4,2) = \frac{1}{\sqrt{12+4}} = \frac{1}{4}}$$

(3)

$$f(x,y) = 4x^2 - 2y + 7x^4y^5$$

$$f_x = 8x + 28x^3y^5$$

$$\boxed{f_{xx} = 8 + 84x^2y^5}$$

$$f_y = -2 + 35x^4y^4$$

$$\boxed{f_{yy} = 140x^4y^3}$$

$$\boxed{f_{xy} = 140x^3y^4}$$

$$\boxed{f_{yx} = 140x^3y^4}$$

(4) $z = \sin(y^2 - 4x)$

$$z_x = \cos(y^2+4x)\cdot(0 - 4)$$

$$= -4\cos(y^2 - 4x)$$

$$z_x(2,1) = -4\cos(1 - 8) = \boxed{-4\cos 7}$$

$$z_x(-2,4) = \cancel{4\cos(16-8) =}$$

$$= -4\cos(16+8) = \boxed{-4\cos 24}$$

$$z_x = (x+y)^{-1}$$

$$z_x = \frac{-1}{(x+y)^2}$$

$$z_x(2,1) = \boxed{\frac{-1}{9}}$$

$$z(-2,4) = \boxed{\frac{-1}{4}}$$

(b) $z = \sin(y^2 - 4x)$

$z_y = \cos(y^2 - 4x) \cdot 2y$

$= 2y \cos(y^2 - 4x)$

$z_y(2,1) = 2\cos(1-8) = \boxed{2\cos 7}$

$z_y(-2,4) = 8\cos(16-8) = 8\cos 8$

$z = (x+y)^{-1}$

$z_y = \dfrac{-1}{(x+y)^2}$

$f_y(2,1) = \dfrac{-1}{9}$

$f_y(-2,4) = \boxed{\dfrac{-1}{4}}$

(5)

$$f(x, y, z) = x^3y^5z^7 + xy^2 + y^3z$$

(a) $f_x = 3x^2y^5z^7 + y^2$

$\boxed{(a)\ f_{xy} = 15x^2y^4z^7 + 2y}$

$f_y = 5x^3y^4z^7 + 2xy + 3y^2z$

(b) $\boxed{f_{yz} = 35x^3y^4z^6 + 3y^2}$

(c) $\boxed{f_{xz} = 21x^2y^5z^6}$

(d) $f_z = 7x^3y^5z^6 + y^3$

$\boxed{f_{zz} = 42x^3y^5z^5}$

(e) $\cancel{f_{xyz}} =$

$\boxed{f_{zyy} = f_{yzy} = 140x^3y^3z^6 + 6y}$

(f) $\boxed{f_{zxy} = f_{xzy} = 105x^2y^4z^6}$

$\boxed{(g)\ f_{zyx} = f_{xzy} = \cancel{100} \ 105x^2y^4z^6}$

$\boxed{(h)\ f_{xxyz} = f_{zxyx} = 210xy^4z^6}$

⑧ $f(x, y, z) = \sqrt{xy} + \ln(x^2 z^3) - x\tan(z)$

$$f_x = \sqrt{y} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{x^2 z^3} \cdot (2x z^3) - \tan z$$

$$\boxed{f_x = \frac{\sqrt{y}}{2\sqrt{x}} + \frac{2}{x} - \tan z}$$

$$f_z = \frac{1}{x^2 z^3} \cdot (x^2 \cdot 3z^2) - x\sec^2 z$$

$$\boxed{f_z = \frac{3}{z} - x\sec^2 z}$$

$$\boxed{f_{xy} = \frac{1}{2\sqrt{y}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{xy}}}$$

$$\boxed{f_{xyz} = 0}$$

Maxima and minima of two variable function

Suppose that, the second partial derivatives of f are continuous ~~at~~ on a disk ~~at~~ with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$

Let,

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

ⓐ if $D > 0$ and $f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ is a local minimum.

ⓑ if $D > 0$, $f_{xx}(a,b) < 0 \Rightarrow$ $f(a,b)$ is a local maximum

If, $D < 0 \Rightarrow f(a,b)$ is not a local max/min and it becomes saddle point.

(9) $f(x,y) = y^2 + xy + 3y + 2x + 3$

$f_x(x,y) = y + 2$

$f_y(x,y) = 2y + x + 3$

$y + 2 = 0$ —— (I)

$y = -2$ —— (II)

$2y + x + 3 = 0$

$-4 + x + 3 = 0$

$x - 1 = 0$

$x = 1$

$\therefore (a,b) \equiv (1,-2)$

$f_{xx} = \cancel{1}\ 0$

$f_{xy} = 1$

$f_{yy} = 2$

$D \equiv 0 \cdot 1 - 2^2$

$= -4$

$\therefore$ No maximum or minimum.

And the saddle point is $(1,-2)$

go

(10) $f(x,y) = x^2 + xy - 2y - 2x + 1$

$f_x = 2x + y - 2$

$f_y = x - 2$

$f_{xx} = 2$ $\qquad f_{xy} = 1$

$f_{yy} = 1$

$2x + y - 2 = 0$ ——— (i)

$x - 2 = 0$ ——— (ii)

$x = 2$

$4 + y - 2 = 0$

$y + 2 = 0$

$y = -2$

Critical point $(2,-2)$

Again,

$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$= 2 \cdot 1 - 1^2$

$= -1$

$\therefore$ No maximum or minimum point.

Saddle point is : $(2,-2)$

(11)

$f(x,y) = x^2 + xy + y^2 - 3x$

$f_x = 2x + y - 3$

$f_y = x + 2y$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 1$

$2x + y - 3 = 0$ ——— (I)

$x + 2y = 0$ ——— (II)

$x = -2y$ ——— (III)

$2x + y - 3 = 0$

$2 \cdot (-2y) + y - 3 = 0$

$-4y + y - 3 = 0$

$-3y - 3 = 0$

$3y = -3$

$y = -1$

$x = -2y$

$x = -2(-1)$

$= 2$

$\therefore (a,b) \equiv (2,-1)$

Again,

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= 2 \cdot 2 - (1)^2$

$= 4 - 1 = 3$

And, $f_{xx} = 2 > 0$

$\therefore$ at $(2,-1) \rightarrow$ Relative minimam.

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$f(x,y) = xy - x^3 - y^2$

$f_x = y - 3x^2$

$f_y = x - 2y$

$f_{xx} = -6x$

$f_{yy} = -2$

$f_{xy} = 1$

$x - 2y = 0$

$x = 2y$ ——— ①

$y - 3x^2 = 0$

$y - 3(2y)^2 = 0$

$y - 3 \cdot 4y^2 = 0$

$y - 12y^2 = 0$

$\cancel{y - 12}$

$y(1 - 12y) = 0$

$y = 0$; $1 - 12y = 0$

$12y = 1$

$y = \frac{1}{12}$

$y = 0$; $x = 2 \cdot 0 = 0$

$y = \frac{1}{12}$; $x = 2 \cdot \frac{1}{12} = \frac{1}{6}$

$a, b \equiv (0,0)$ & $\left(\frac{1}{6}, \frac{1}{12}\right)$

$D \equiv \cancel{(0-0)^2 \cdot (}\ (-6 \cdot 0) \cdot (-2) - 1 = -1$

∴ Saddle point $\equiv (0,0)$

$D_2 = \left(-6 \cdot \frac{1}{6}\right)(-2) - 1 = 2 - 1 = 1$

∴ It has either a local max or min.

$f_{xx} \equiv -6x = -6 \cdot \frac{1}{6} = -1$

∴ at $(1/6, 1/12)$ There is a local maximam.

(13) $f(x,y) = x^2+y^2+\dfrac{2}{xy}$

$f_x(x,y) = 2x + \dfrac{2}{y}\cdot\dfrac{-1}{x^2}$

$= 2x - \dfrac{2}{x^2y}$

$= \dfrac{2x\cdot x^2y - 2}{x^2y} = \dfrac{2x^3y-2}{x^2y}$

$f_y(x,y) = 2y - \dfrac{2}{x}\cdot\dfrac{1}{y^2}$

$= 2y - \dfrac{2}{xy^2}$

$= \dfrac{2y\cdot xy^2 - 2}{xy^2} = \dfrac{2xy^3-2}{xy^2}$

$\dfrac{2x^3y-2}{x^2y} = 0$	$2xy^3-2=0$
$x^3y-1=0$	$xy^3=1$
$x^3y=1$	$x\cdot\left(\dfrac{1}{x^3}\right)^3 = 1$
$y=\dfrac{1}{x^3}$	$x\cdot\dfrac{1}{x^9}=1$
	$\dfrac{1}{x^8}=1$
	$x^8=1$
	$x=\pm 1$

$x=1;\ y=1$

$x=-1;\ y=-1$ $\therefore$ Critical point $\Rightarrow (1,1)\ (-1,-1)$

$f_{x} = 2x - \frac{2}{x^2 y}$

$f_{xx} = 2 - \frac{2}{y} \cdot \frac{-2}{x^3}$

$= 2 + \frac{4}{x^3 y}$

$f_{xy} = -\frac{2}{x^2} \cdot \frac{-1}{y^2} = \frac{2}{x^2 y^2}$

$f_{y} = 2y - \frac{2}{xy^2}$

$f_{yy} = 2 - \frac{2}{x} \cdot \frac{-2}{y^3}$

$= 2 + \frac{4}{xy^3}$

$\therefore D_{(1,1)} = \left(2 + \frac{4}{1}\right)\left(2 + \frac{4}{1}\right) - \frac{2}{1}$

$= 34$

$\therefore f_{xx} = 6$; at $(1,1)$ there is a local minima.

$D_{(-1,-1)} = (2+4)(2+4) - 2$

$= 36 - 2 = 34$

$f_{xx_{(-1,-1)}} = \cancel{2-4=-2}\ 2+4 = 6$

$\therefore$ $(-1,-1)$ there is a locat ~~maxima~~ minima

(14) $f(x,y) = xe^y$

$f_x = e^y$

$f_y = xe^y$

$f_{xx} = 0$

$f_{yy} = xe^y + y\,xe^y$

$e^y = 0$ [undefined]

$xe^y = 0$

$x = 0$ | $e^y = 0$ undefined

$\therefore f_x = e^y = 0$ is totall impossible. And that's why it has no critical point.

(15) $f(x,y) = x^2 + y - e^y$

$f_x = 2x$

$f_y = 1 - e^y$

$f_{xx} = 2$

$f_{yy} = -e^y$

$f_{xy} = 0$

$2x = 0$

$x = 0$

and,

$e^y = 1$

$y = \ln(1) = 0$

$(0,0)$

$\therefore D \equiv f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy})^2$

$= 2 \cdot (-e^0) - 0^2$

$= -2$

$\therefore$ saddle point $(0,0)$

(16) $f(x,y) = xy + \frac{2}{x} + \frac{4}{y}$

$f_x = y - \frac{2}{x^2}$

$f_{xx} = \frac{2 \cdot 2}{x^3} = \frac{4}{x^3}$

$f_y = x - \frac{4}{y^2}$

$f_{yy} = \frac{4 \cdot 2}{y^3} = \frac{8}{y^3}$

$f_{xy} = 1$

$y = \frac{2}{x^2}$

$x - \frac{4}{y^2} = 0$

$x - \frac{4}{\frac{4}{x^4}} = 0$

$x - 4 \cdot \frac{x^4}{4} = 0$

~~$x - \frac{x^4}{2} = 0$~~

~~$2x - x^4 = 0$~~

~~$x = 0$~~ ~~$2 - x^3 = 0$~~

~~$x^3 = 2$~~

~~$x = \sqrt[3]{2}$~~

$x - x^4 = 0$

$x = 0$ or,

$1 - x^3 = 0$

$x = 1$

~~$\therefore (a,b) = (0,1)$~~ ~~$(1,8)$~~

$y = 2$

$\therefore$ critical point $(1,2)$

Now,

$D = \frac{4}{1} \cdot \left(\frac{8}{8}\right) - 1$

$= 4 \cdot (1) - 1 = 3$

~~$\therefore$ No max or min~~

~~Saddle point : $(1,2)$~~

$\therefore f_{xx} = \frac{4}{x^3} = \frac{4}{1^3} = 4$

$\therefore$ at $(1,2) \exists$ a local minima

(17) $f(x,y) = e^x \sin y$

$f_x(x,y) = e^x \sin y$

$f_y(x,y) = e^x \cos y$

$e^x \sin y = 0$

$e^x \cos y = 0$

$\therefore \sin y = 0$

$y = \sin^{-1}(0) = 0$

$y = \pm n\pi$

$\cancel{e^x \cos y = 0}$

$\cancel{\cos y = 0}$

$\cancel{y = \cos^{-1}(0)} = \frac{\pi}{2}$

$e^x \cos y = 0$

$e^x . \cos 0 = 0$

$e^x = 0$ which is totally impossible because exponential function cannot be zero.

(18) $f(x,y) = y\sin x$

$f_x(x,y) = y\cos x$

$f_y(x,y) = \sin x$

Now,

$\sin x = 0$

$x = \sin^{-1}(0) = 0$

now,

$\sin x = 0$ if $x = n\pi =$

$0, \pm 1, \pm 2, \pm 3, \ldots$

$y\cos x = 0$

$y = 0$

$(0,0)$

$f_{xx}(x,y) = -y\sin x$

$f_{yy}(x,y) = 0$

$f_{xy} = \cos x$

$\therefore D = \{0 \cdot \sin 0\}\cdot\{0\} - (\cos 0)^2$

$= -1$

$\therefore D < -1$ so, $(n\pi, 0)$ $\cancel{(0,0)}$ is a saddle point.

(19)

$$f(x,y) = e^{-(x^2+y^2+2x)}$$

$$f_x(x,y) = e^{-(x^2+y^2+2x)} \cdot \{2x+2\}$$

$$f_y(x,y) = e^{-(x^2+y^2+2x)} \cdot (2y)$$

Now,
$$2x+2=0 \Rightarrow x=-1$$
$$2y=0 \Rightarrow y=0$$

$(-1,0)$

$$f_{xx}(x,y) = e^{-(x^2+y^2+2x)} \cdot (2x+2)(2x+2) + 2e^{-(x^2+y^2+2x)}$$

$$= e^{-(x^2+y^2+2x)}\{(2x+2)^2+2\}$$

$$f_{xx}(-1,0) = e^{-(1+0-2)} \cdot (2)$$

$$= 2e$$

$$f_{yy} = e^{-(x^2+y^2+2x)} \cdot (0+2y) \cdot (2y) + 2e^{-(x^2+y^2+2x)}$$

$$= 4y^2 e^{-(x^2+y^2+2x)} + 2e^{-(x^2+y^2+2x)}$$

$$= e^{-(x^2+y^2+2x)}(2+4y^2)$$

$$f_{yy}(-1,0) = e^{-(1-2)} \cdot (2+0) = 2e$$

$\cancel{D = 4e^2}$

$$f_{xy} = e^{-(2x+2)} \cdot 2y \qquad f_{xy}(-1,0) = 0$$

$D = 2e \cdot 2e - 0 = 4e^2 > 0$

Since, $f_{xx} = 2e$ which is greater than zero,

$\therefore (-1, 0)$ is a relative maximum

(20) $f(x,y) = xy + \frac{a^3}{\cancel{y}x} + \frac{b^3}{y}$

$f_x(x,y) = y + \frac{-a^3}{x^2} = y - \frac{a^3}{x^2}$

$f_y(x,y) = x - \frac{b^3}{y^2}$

Now, $y = \frac{a^3}{x^2}$

$\therefore \cancel{x - \frac{b^3}{\frac{b^6}{y^4}}}$

$x - \frac{b^3}{\frac{a^6}{x^4}} = 0$

$x - b^3 \cdot \frac{x^4}{a^6} = 0$

$x\left(1 - \frac{b^3 x^3}{a^6}\right) = 0$

$x = 0$

or,

$\frac{b^3 x^3}{a^6} = 1$

$x^3 = \frac{a^6}{b^3}$

$x = \frac{a^{6 \cdot \frac{1}{3}}}{b^{3 \cdot \frac{1}{3}}} = \frac{a^2}{b}$

$y = a^3 \cdot \frac{b^2}{a^4} = \frac{b^2}{a}$

$\left(\frac{a^2}{b}, \frac{b^2}{a}\right)$

$$f_{xx} = 0 - a^3 \cdot \frac{-2}{x^3} = \frac{2a^3}{x^3}$$

$$f_{xx}(a^2/b, b^2/a) = \frac{2a^3}{\frac{a^6}{b^3}} = 2a^3 \cdot \frac{b^3}{a^6} = \frac{2b^3}{a^3}$$

$$f_{yy} = 0 - \frac{b^3 \cdot -2}{y^3} = \frac{2b^3}{y^3}$$

$$\therefore \cancel{D = \frac{2b^3}{a^3} \quad 2b}$$

$$f_{yy}(a^2/b, b^2/a) = 2b^3 \cdot \frac{a^3}{b^6} = \frac{2a^3}{b^3}$$

$$\therefore \cancel{D = \frac{2b^3}{a^3} \cdot \frac{2a^3}{b^3} =}$$

$$f_{xy} = 1$$

$$\therefore D = \frac{2\cancel{b^3}}{\cancel{a^3}} \cdot \frac{2\cancel{a^3}}{\cancel{b^3}} - 1$$

$$= \frac{2 \cdot 2}{1} - 1 = 3 > 0$$

$\therefore$ $(a^2/b, b^2/a)$ is a stationary point.

Since $2b^3/a^3$ is greater than zero, so it has a relative maximum.