Work Sheet 7

Leibnitz's Theorem

Leibnitz's Theorem

1. $y = \tan^{-1}x$

$\Rightarrow y_1 = \frac{1}{1+x^2}$

$\Rightarrow y_1(1+x^2) - 1 = 0$

$\Rightarrow y_2(1+x^2) + 2xy_1 = 0$ ——— (1)

Now applying Leibnitz's theorem at eqⁿ (1)

$$\Rightarrow {}^nC_0\, y_{n+2}(1+x^2) + {}^nC_1\, y_{n+1}\cdot 2x + {}^nC_2\, y_n \cdot 2 + {}^nC_0\, y_{n+1}\, 2x + {}^nC_1\, y_n \cdot 2 = 0$$

$$\Rightarrow \frac{n!}{0!\,(n-0)!}\, y_{n+2}(1+x^2) + \frac{n!}{1!\,(n-1)!}\, y_{n+1}\cdot 2x + \frac{n!}{2!\,(n-2)!}\cdot y_n \cdot 2 + 1\cdot y_{n+1}\cdot 2x + \frac{n!}{1!\,(n-1)!}\, y_n\cdot 2 =$$

$$\Rightarrow 1\cdot y_{n+2}(1+x^2) + \frac{n(n-1)!}{(n-1)!}\cdot y_{n+1}\, 2x + \frac{n(n-1)(n-2)!}{2\,(n-2)!}\cdot y_n \cdot 2 + y_{n+1}\cdot 2x + ny_n\, 2 =$$

$$\Rightarrow y_{n+2}(1+x^2) + n\cdot y_{n+1}\, 2x + n(n-1)y_n + y_{n+1}\, 2x + ny_n\, 2 = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + 2(n+1)xy_{n+1} + n(n-1+2)y_n = 0$$

[Showed]

2. $y = \cot^{-1} x$

$\Rightarrow y_1 = \frac{-1}{1+x^2}$

$\Rightarrow y_1(1+x^2) + 1 = 0$

$\Rightarrow y_2(1+x^2) + y_1 2x = 0$ —— ①

Now applying Leibnitz's theorem at eqⁿ ①

$^nC_0\, y_{n+2}(1+x^2) + {}^nC_1 \cdot y_{n+1} \cdot 2x + {}^nC_2\, y_n \cdot 2 + {}^nC_0\, y_{n+1}\, 2x$
$+ {}^nC_1\, y_n \cdot 2 = 0$

$\Rightarrow 1 \cdot y_{n+2}(1+x^2) + \frac{n!}{1!(n-1)!} y_{n+1} \cdot 2x + \frac{n!}{2!(n-2)!} y_n \cdot 2 +$
$1 \cdot y_{n+1} \cdot 2x + \frac{n!}{1!(n-1)!} \cdot y_n \cdot 2 = 0$

$\Rightarrow y_{n+2}(1+x^2) + n \cdot y_{n+1} \cdot 2x + n(n-1) y_n + y_{n+1} 2x$
$+ n \cdot y_n \cdot 2 = 0$

$\Rightarrow y_{n+2}(1+x^2) + n\, y_{n+1} \cdot 2x + y_{n+1} 2x + y_n\, n(n-1) + n y_n \cdot 2 = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}(n+1)2x + y_n(n-1+2)n = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}(n+1)2x + y_n(n+1)n = 0$

(Showed)

3. $y\sqrt{1-x^2} = \sin^{-1}x$

$\Rightarrow y^2.(1-x^2) = (\sin^{-1}x)^2$ [squre]

$\Rightarrow 2y.y_1(1-x^2) + y^2(-2x) = 2\cdot\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow 2yy_1(1-x^2) - y^2 2x = 2\cdot y\cdot\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow 2y.y_1(1-x^2) - 2xy^2 = 2y$

$\Rightarrow y_1(1-x^2) - xy = 1$

$\Rightarrow y_1(1-x^2) - xy - 1 = 0$ —— (1)

now applying leibnitz theorem at (1)

${}^nC_0\, y_{n+1}(1-x^2) + {}^nC_1\, y_n(-2x) + {}^nC_2\, y_{n-1}(-2) - {}^nC_0\, y_n x$

$- {}^nC_1\, y_{n-1}\cdot 1 = 0$

$\Rightarrow 1\, y_{n+1}(1-x^2) + n\, y_n(-2x) - n(n-1)y_{n-1} - y_n x$

$- n.y_{n-1} = 0$

$\Rightarrow$

$\Rightarrow (1-x^2)y_{n+1} - x(2n+1)y_n - y_{n-1}(n-1+1)n = 0$

$\Rightarrow (1-x^2)y_{n+1} - x(2n+1)y_n - y_{n-1}(n)n = 0$

$\Rightarrow (1-x^2)y_{n+1} - x(2n+1)y_n - n^2 y_{n-1} = 0$

(Showed)

4. $y = e^{\tan^{-1}x}$

$\Rightarrow \ln y = \tan^{-1}x \ \ln(e)$

$\Rightarrow \ln y = \tan^{-1}x \cdot 1$

$\Rightarrow \frac{1}{y} \cdot y_1 = \frac{1}{1+x^2}$

$\Rightarrow y_1(1+x^2) = y$

$\Rightarrow y_1(1+x^2) - y = 0$

$\Rightarrow y_2(1+x^2) + y_1 2x - y_1 = 0 \Rightarrow y_2(1+x^2) + y_1(2x-1)$

now applying leibnitz theorem at eq^n 1

${}^nC_0\, y_{n+2}(1+x^2) + {}^nC_1\, y_{n+1} \cdot 2x + {}^nC_2\, y_n \cdot 2 + {}^nC_0\, y_{n+1}(2x-1) +$

${}^nC_1\, y_n \cdot 2 \ \cancel{{}^nC_0\, y_{n+1}} = 0$

$\Rightarrow y_{n+2}(1+x^2) + n2x\, y_{n+1} + n(n-1)y_n + y_{n+1}(2x-1) + n\, y_n 2 = 0$

$\Rightarrow (1+x^2)y_{n+2} + y_{n+1}(2xn+2x-1) + y_n(n+1)n = 0$

(showed)

5. $y = e^{m\sin^{-1}x}$

$\Rightarrow \ln y = m\sin^{-1}x \cdot \ln(e)$

$\Rightarrow \ln y = m\sin^{-1}x$

$\Rightarrow \frac{1}{y} \cdot y_1 = m \cdot \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y_1\left(\sqrt{1-x^2}\right) = my$

$\Rightarrow y_1^2(1-x^2) = m^2y^2$

$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) = m^2\, 2yy_1$

$\Rightarrow 2y_1y_2(1-x^2) - 2xy_1^2 = 2y\, m^2y_1$

$\Rightarrow y_2(1-x^2) - xy_1 = ym^2$

$\Rightarrow y_2(1-x^2) - xy_1 - m^2y = 0$

Now applying leibnitz theorem

$${}^nC_0\, y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - {}^nC_0\, y_{n+1}\, x$$
$$- {}^nC_1\, y_n \cdot 1 - {}^nC_0\, y_n\, m^2 = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - n \cdot y_{n+1} \cdot 2x - n(n-1)y_n - x\,y_{n+1}$$
$$- n y_n - y_n m^2 = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - y_{n+1}\,(2n+1)x - y_n\,(n^2-n+n+$$

$$\Rightarrow (1-x^2)\,y_{n+2} - y_{n+1}\,(2n+1)x - y_n\,(m^2+n^2)=0$$

(5

6.

$$y = (\sin^{-1}x)^2$$

$$\Rightarrow y_1 = 2\cdot\sin^{-1}x\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_1^2(1-x^2) = 4(\sin^{-1}x)^2$$

$$\Rightarrow y_1^2(1-x^2) = 4y$$

$$\Rightarrow y_1^2(1-x^2) - 4y = 0$$

$$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) - 4y_1 = 0$$

$$\Rightarrow y_2(1-x^2) - xy_1 - 2 = 0$$

Now applying libnitz theorem

$$^nC_0\,y_{n+2}(1-x^2) + {}^nC_1\,y_{n+1}(-2x) + {}^nC_2\,y_n(-2) - {}^nC_0\,y_{n+1}\cdot x$$
$$- {}^nC_1\,y_n\cdot 1 = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - n\,y_{n+1}(2x) - n(n-1)y_n - xy_{n+1}$$
$$- ny_n = 0$$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}(2n+1)x - y_n(n-1+1)n = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}(2n+1)x - y_n n^2 = 0$

(Showed)

7. $\log_e y = a\sin^{-1}x$

$\Rightarrow \frac{1}{y}\cdot y_1 = a\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y_1\sqrt{1-x^2} = ay$

$\Rightarrow y_1^2(1-x^2) = a^2y^2$

$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) = a^2 2yy_1$

$\Rightarrow y_2(1-x^2) - xy_1 - a^2y = 0$

now applying leibnitz theorem,

$^nC_0\, y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - {}^nC_0\, y_{n+1}x$

$- {}^nC_1\, y_n\cdot 1 - {}^nC_0\, y_n a^2 = 0$

$\Rightarrow y_{n+2}(1-x^2) - 2xn\, y_{n+1} - n(n-1)y_n - y_{n+1}x$

$- ny_n - y_n a^2 = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}x(2n+1) - y_n(n^2-n+n+a^2) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}x(2n+1) - y_n(n^2+a^2) = 0$

(Showed)

8. $y = e^{m\cos^{-1}x}$

$\Rightarrow \ln y = m\cos^{-1}x \cdot \ln(e)$

$\Rightarrow \frac{1}{y} \cdot y_1 = m \cdot \frac{-1}{\sqrt{1-x^2}}$

$\Rightarrow y_1(\sqrt{1-x^2}) = -my$

$\Rightarrow y_1^2 (1-x^2) = m^2y^2$

$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) = m^2 2yy_1$

$\Rightarrow y_2(1-x^2) - xy_1 - m^2y = 0$

now applying leibnitz theorem,

$\downarrow$ As same as no 5

Ans: As same as no 5

$(1-x^2)y_{n+2} - y_{n+1}(2n+1)x - y_n(m^2+n^2) = 0$ (Showed)

9. $\log_e y = \tan^{-1}x$

$$\Rightarrow \frac{1}{y}\cdot y_1 = \frac{1}{1+x^2}$$

$$\Rightarrow y_1(1+x^2) = y$$

$$\Rightarrow y_2(1+x^2) + y_1\, 2x = y_1$$

$$\Rightarrow y_2 (1+x^2) + y_1(2x-1) = 0$$

now applying Leibnitz theorem,

$${}^nC_0\, y_{n+2}(1+x^2) + {}^nC_1\, y_{n+1}\cdot 2x + {}^nC_2\, y_n\, 2 + {}^nC_0\, y_{n+1}(2x-1)$$

$$+ {}^nC_1\, y_n \cdot 2 = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + 2xn\, y_{n+1} + n(n-1)y_n + y_{n+1}(2x-1)$$

$$+ 2ny_n = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}(2nx+2x-1) + y_n(n-1+2)n = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}(2nx+2x-1) + y_n(n+1)n = 0$$

(Showed

10. $y = (\cos^{-1}x)^2$

$\Rightarrow y_1 = 2\cos^{-1}x \cdot \frac{-1}{\sqrt{1-x^2}}$

$\Rightarrow y_1 (\sqrt{1-x^2}) = -2\cos^{-1}x$

$\Rightarrow y_1^2 (1-x^2) = 4(\cos^{-1}x)^2$

$\Rightarrow 2y_1 y_2 (1-x^2) + y_1^2(-2x) = 4y_1$

[as $y = (\cos^{-1}x)^2$]

$\Rightarrow y_2 (1-x^2) - xy_1 - 2 = 0$

Now applying leibnitz theorem,

$\Rightarrow {}^nC_0\, y_{n+2} (1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - {}^nC_0\, y_{n+1} x$

$- {}^nC_1\, y_n \cdot 1 = 0$

~~$\Rightarrow y_{n+2} (1-x^2) - 2nx$~~

$\Rightarrow y_{n+2} (1-x^2) - 2nx\, y_{n+1} - n(n-1) y_n - y_{n+1} x$

$- n y_n = 0$

$\Rightarrow y_{n+2} (1-x^2) - y_{n+1}(2n+1)x - y_n(n^2 - n + n) = 0$

$\Rightarrow y_{n+2} (1-x^2) - y_{n+1}(2n+1)x - y_n n^2 = 0$

[Showed]

$\ln y = m\cos^{-1}x$

$\Rightarrow \frac{1}{y}\cdot y_1 = m\frac{-1}{\sqrt{1-x^2}}$

$\Rightarrow y_1(\sqrt{1-x^2}) = -my$

$\Rightarrow y_1^2(1-x^2) = m^2y^2$

$\Rightarrow 2y_1y_2(1-x^2) + y_1^2(-2x) = m^2 2yy_1$

$\Rightarrow y_2(1-x^2) - xy_1 - m^2y = 0$

now applying leibnitz theorem,

$\Rightarrow {}^nC_0\, y_{n+2}(1-x^2) + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2\, y_n(-2) - {}^nC_0\, y_{n+1}x$

$- {}^nC_1\, y_n\cdot 1 - {}^nC_0\, y_n\cdot m^2 = 0$

$\Rightarrow y_{n+2}(1-x^2) - n2x\,y_{n+1} - n(n-1)y_n - xy_{n+1}$

$- ny_n - y_n m^2 = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}(2n+1)x - y_n(n^2-n+n+m^2) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}(2n+1)x - y_n(n^2+m^2) = 0$

[Showed]

(12) $x = \tan(\ln y)$

$\Rightarrow \ln y = \tan^{-1} x$

$\Rightarrow \frac{1}{y} \cdot y_1 = \frac{1}{1+x^2}$

$\Rightarrow y_1(1+x^2) = y$.

$\Rightarrow y_1(1+x^2) - y = 0$

$\downarrow \quad \downarrow$

as same as no 4

Ans: as same as no 4

$(1+x^2)\, y_{n+2} + y_{n+1}(2xn + 2x - 1) + y_n(n+1)n = 0$

(Showed)