# 7.3 Integrating trigonometric function

① $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ / $\cos 2x = 1-2\sin^2 x$

② $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ / $\cos 2x = 2\cos^2 x - 1$

③ $2\sin A\cos A = \sin 2A$

④ $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

⑤ $2\cos A\sin B = \sin(B-A) + \sin(A+B)$

⑥ $2\cos A\cos B = \cos(A-B) + \cos(A+B)$

Reduction formula:

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} dx$$

$$\int \cos^n x \, dx = \frac{1}{n} \cancel{\sin^n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx] + c$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx] + c$$

Table 7.3.1

To evaluate $\int \sin^m x \cos^n x \, dx$

a) $n$: odd then, $\cos^2 x = 1 - \sin^2 x$

b) $m$: odd then, $\sin^2 x = 1 - \cos^2 x$

c) $m, n$ even, $\begin{cases} 2\sin^2 x = 1 - \cos 2x \\ 2\cos^2 x = 1 + \cos 2x \end{cases}$

Example:

a) $\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \, (\cos^2 x)^2 \cos x \, dx$

$$= \int \sin^4 x \, (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int \sin^4 x \, (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$$

$$= \int (\sin^4 x - 2\sin^6 x + \sin^8 x) \cos x \, dx$$

Let

$u = \sin x$

$du = \cos x \, dx$

$$\therefore \int \sin^4 x \cos^5 x \, dx = \int (u^4 - 2u^6 + u^8) \, du$$

$$= \frac{u^5}{5} + 2\frac{u^7}{7} + \frac{u^9}{9} + c$$

$$= \frac{1}{5}\sin^5 x + \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x \, \cancel{dx} + c$$

Ans.

Example : $\int \sin^4 x \cos^4 x \, dx$

Solution :

$$\int \sin^4 x \cos^4 x \, dx = \frac{1}{16}\int 16 (\sin x \cos x)^4 \, dx$$

$$= \frac{1}{16}\int (2 \sin x \cos x)^4 \, dx$$

$$= \frac{1}{16}\int (\sin 2x)^4 \, dx$$

$$= \frac{1}{16}\int \sin^4 2x \, dx$$

$$= \frac{1}{32}\int \sin^4 u \, du$$

Let

$u = 2x$

$du = 2dx$

$$= \frac{1}{32}\left[-\frac{1}{4}\sin^3 u \cos u + \frac{3}{4}\int \sin^2 u \, du\right]$$

$$= \frac{1}{32}\left[-\frac{1}{4}\sin^3 u \cos u + \frac{3}{4}\cdot\frac{1}{2}\int 2\sin^2 u \, du\right]$$

$$= \frac{1}{32}\left[-\frac{1}{4}\sin^3 u \cos u + \frac{3}{8}\int 1 - \cos 2u \, du\right]$$

$$= \frac{1}{34}\left[-\frac{1}{4}\sin^3 u \cos u + \frac{3}{8}\left(u - \frac{\sin 2u}{2}\right)\right] + C$$

$$\therefore \int \sin^4 x \cos^4 x \, dx = \frac{1}{34}\left[-\frac{1}{4}\sin^3 2x \cdot \cos 2x + \frac{3}{16}\sin 4x + \frac{6x}{8}\right] + C$$

Ans.

Evaluate $\int \sin 2x \cos 3x \, dx$

Solution:

$$\int \sin 2x \cos 3x \, dx = \frac{1}{2}\int 2 \sin 2x \cos 3x \, dx$$

$$= \frac{1}{2}\int \sin(2x+3x) + \sin(2x-3x) \, dx$$

$$= \frac{1}{2}\int \sin 5x + \sin(-x) \, dx$$

$$= \frac{1}{2}\int \sin 5x - \sin x \, dx$$

$$= \frac{1}{2}\left(\frac{-\cos 5x}{5} - (-\cos x)\right) + c$$

$$\therefore \int \sin 2x \cos 3x \, dx = -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + c$$ Ans.

# $\int \sin x \cos \frac{x}{2} \, dx = \frac{1}{2}\int \sin\left(x+\frac{x}{2}\right) + \sin\left(x-\frac{x}{2}\right) dx$

$$= \frac{1}{2}\int \left(\sin\frac{3x}{2} + \sin\frac{x}{2}\right) dx$$

$$= \frac{1}{2}\left(\frac{-\cos\frac{3x}{2}}{3/2} + \frac{-\cos\frac{x}{2}}{1/2}\right) + c$$

$$\therefore \int \sin x \cos \frac{x}{2} \, dx = -\frac{1}{3}\cos\frac{3x}{2} - \cos\frac{x}{2} + c$$ Ans.

# $\int_0^{\pi/2} \sin^2\frac{x}{2}\cos^2\frac{x}{2} \, dx = \frac{1}{4}\int_0^{\pi/2}\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2 dx$

$$= \frac{1}{4}\int_0^{\pi/2}\left[\sin\left(2\cdot\frac{x}{2}\right)\right]^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2}\sin^2 x \, dx$$

$$= \frac{1}{4}\cdot\frac{1}{2}\int_0^{\pi/2} 2\sin^2 x \, dx$$

$$= \frac{1}{8}\int_0^{\pi/2} 1 - \cos 2x \, dx$$

$$= \frac{1}{8}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi/2}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{1}{8}\left(\frac{\pi}{2} - 0 - 0 + 0\right)$$

$$\therefore \int_0^{\pi/2} \sin^2\frac{x}{2}\cos^2\frac{x}{2}\,dx = \frac{\pi}{16} \text{ Ans.}$$

\# $\int_0^{\pi/3} \sin^4 3x \cos^3 3x \, dx$

$$= \int_0^{\pi/3} \sin^4 3x\,(1 - \sin^2 3x)\cos 3x \, dx$$

Let $u = \sin 3x$

$du = 3\cos 3x \, dx$

Limit

| $x$ | $u$ |
|---|---|
| $0$ | $0$ |
| $\pi/3$ | $0$ |

$$\therefore \int_0^{\pi/3} \sin^4 3x \cos^3 3x \, dx = \int_0^0 u^4(1-u^2)\frac{1}{3}\,du$$

$$= 0 \quad \left[\int_a^a f(x)\,dx = 0\right]$$

\# $\int \sec^2(2x-1)\,dx = \dfrac{\tan(2x-1)}{\frac{d}{dx}(2x-1)} = \frac{1}{2}\tan(2x-1) + c$

# Evaluate $\int e^{-x} \tan(e^{-x})\, dx$

Let $u = e^{-x}$

$du = -e^{-x}\, dx$

$$\therefore \int e^{-x} \tan(e^{-x})\, dx = -\int \tan u\, du$$

$$= -\ln|\sec u| + c$$

$$= -\ln|\sec e^{-x}| + c$$

# $\int \frac{\sec\sqrt{x}}{\sqrt{x}}\, dx$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\, dx$

$\therefore 2\, du = \frac{1}{\sqrt{x}}\, dx$

$$\int \frac{\sec\sqrt{x}}{\sqrt{x}}\, dx = 2\int \sec u\, du = 2\int \frac{\sec u(\tan u + \sec u)}{\tan u + \sec u}\, du$$

$$= 2\int \frac{\sec^2 u + \sec u \tan u}{\tan u + \sec u}\, du$$

$$= 2\ln|\tan u + \sec u| + c \qquad \left[\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c\right]$$

$$\int \frac{\sec\sqrt{x}}{\sqrt{x}}\, dx = 2\ln|\tan\sqrt{x} + \sec\sqrt{x}| + c$$

Evaluate $\int \sqrt{\tan x}\, \sec^4 x \, dx$

Solution:

$$\int \sqrt{\tan x}\,(1+\tan^2 x)\sec^2 x \, dx$$

Let

$u = \tan x$

$du = \sec^2 x \, dx$

$$\therefore \int \sqrt{\tan x}\,(1+\tan^2 x)\sec^2 x \, dx = \int u^{1/2}(1+u^2)\, du$$

$$= \int u^{1/2} + u^{1/2+2} \, du$$

$$= \int u^{1/2} + u^{5/2} \, du$$

$$= \frac{u^{1/2+1}}{1/2+1} + \frac{u^{5/2+1}}{5/2+1} + c$$

$$= \frac{2}{3} u^{3/2} + \frac{2}{7} u^{7/2} + c$$

$$\therefore \int \sqrt{\tan x}\, \sec^4 x \, dx = \frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + c$$ An.

# Evaluate $\int \tan x \sec^{3/2} x \, dx$

$$= \int \tan x \sec x \sec^{1/2} x \, dx$$

Let $u = \sec x$

$du = \sec x \tan x \, dx$

$$\therefore \int \tan x \sec x \sec^{1/2} x \, dx = \int u^{1/2} du = \frac{u^{3/2}}{3/2} + c$$

$$= \frac{2}{3}\sec^{3/2} x + c$$

A.

Evaluate: $\int_0^{\pi/8} \tan^2 2x \, dx$

$$= \int_0^{\pi/8} \sec^2 2x - 1 \, dx$$

$$= \int_0^{\pi/8} \sec^2 2x \, dx - \int_0^{\pi/8} 1 \, dx$$

$$= \left.\frac{\tan 2x}{2}\right]_0^{\pi/8} - \left.x\right]_0^{\pi/8}$$

$$\therefore \int_0^{\pi/8} \tan^2 2x \, dx = \frac{1}{2} - \frac{\pi}{8}$$

Evaluate $\int_0^{\pi/2} \tan^5 \frac{x}{2} \, dx$

Try yourself....

Ans. $-\frac{1}{2} - 2\ln\left(\frac{1}{\sqrt{2}}\right)$

# 7.4 Trigonometric Substitution

**Table**

$\sqrt{a^2-x^2}$ ; $x = a\sin\theta$

$\sqrt{x^2+a^2}$ ; $x = a\tan\theta$

$\sqrt{x^2-a^2}$ ; $x = a\sec\theta$

Firstly convert an algebraic function to an trigonometric function by substitution the solve it using trigonometric formulas as we do in 7.3

**Evaluate:**

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx$$

Let

$x = 4\sin\theta \quad \therefore \theta = \sin^{-1}\frac{x}{4}$

$\Rightarrow \frac{dx}{d\theta} = 4\cos\theta$

$\therefore dx = 4\cos\theta\,d\theta$

Now $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}}\,4\cos\theta\,d\theta$

$$= 64\int \frac{\sin^2\theta\cos\theta}{4\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= 16\int \frac{\sin^2\theta\cos\theta}{\cos\theta}\,d\theta$$

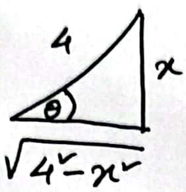
$$= 16\int \sin^2\theta\,d\theta$$

$$= 16\int \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= 8\left(\theta - \frac{\sin 2\theta}{2}\right) + c$$

$$= 8\theta - 4.2\sin\theta\cos\theta + c$$

Now, $\theta = \sin^{-1}\frac{x}{4}$




Note

$$\sin\theta = \frac{\text{Perpendicular}}{\text{hypotenous}}$$

$$\Rightarrow \theta = \sin^{-1}\frac{\text{Per}}{\text{hyp}}$$

$$\therefore \int \frac{x^2}{\sqrt{16-x^2}}\, dx = 8\sin^{-1}\frac{x}{4} - 8\sin\left(\sin^{-1}\frac{x}{4}\right).\cos\left(\cos^{-1}\frac{\sqrt{16-x^2}}{4}\right) + c$$

$$= 8\sin^{-1}\frac{x}{4} - 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + c$$

$$\therefore \int \frac{x^2}{\sqrt{16-x^2}}\, dx = 8\sin^{-1}\frac{x}{4} - 2\,\frac{1}{2}\, x\sqrt{16-x^2} + c$$ Ans.

# Evaluate $\int \frac{dx}{(4+x^2)^2}$

Solution:

$x = 2\tan\theta \quad \therefore \theta = \tan^{-1}\left(\frac{x}{2}\right)$

$dx = 2\sec^2\theta\, d\theta$

$$\therefore \int \frac{dx}{(4+x^2)^2} = \int \frac{2\sec^2\theta\, d\theta}{(4+4\tan^2\theta)^2}$$

$$= 2\int \frac{\sec^2\theta\, d\theta}{16(1+\tan^2\theta)^2} \cancel{d\theta}$$

$$= \frac{2}{16}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\, d\theta$$

$$= \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{8}\int \cos^2\theta \, d\theta$$

$$= \frac{1}{8}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\int 1+\cos 2\theta \, d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$$

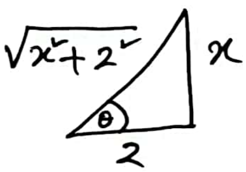
$$= \cancel{16}\, {}^{1}\!/_{16}\,\theta + \frac{1}{32}\sin 2\theta + C$$

$$= {}^{1}\!/_{16}\,\theta + \frac{1}{32}\, 2\sin\theta \cos\theta + C$$

$$= {}^{1}\!/_{16}\,\theta + \frac{1}{16}\sin\theta \cos\theta + C$$

Now

$$\theta = \tan^{-1}\frac{x}{2}$$




Note

$$\tan\theta = \frac{\text{Per.}}{\text{base}} \Big/ \frac{\text{opp}}{\text{adj}}$$

$$\therefore \theta = \sin^{-1}\left(\frac{x}{\sqrt{x^2+2^2}}\right)$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{x^2+2^2}}\right)$$

$$\therefore \int \frac{dx}{(4+x^2)^2} = \frac{1}{16}\tan^{-1}{}^{x}\!/_{2} + \frac{1}{16}\sin\left(\sin^{-1}\frac{x}{\sqrt{x^2+4}}\right)\cos\left(\cos^{-1}\frac{2}{\sqrt{x^2+4}}\right) + C$$

$$= \frac{1}{16}\tan^{-1}{}^{x}\!/_{2} + \frac{1}{16}\,\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}} + C$$

$$\therefore \int \frac{dx}{(4\cancel{x}+x^2)^2} = \frac{1}{16}\tan^{-1}{}^{x}\!/_{2} + \frac{1}{8}\,\frac{x}{x^2+4} + C$$

# Evaluate $\int \frac{\sqrt{x^2-9}}{x}\,dx$

<u>Solution:</u>

Let, $x = 3\sec\theta$ $\therefore \theta = \sec^{-1} x/3$

$\Rightarrow dx = 3\sec\theta\tan\theta\,d\theta$

$$\therefore \int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} \cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int 3\sqrt{\sec^2\theta - 1}\,\tan\theta\,d\theta$$

$$= 3\int \sqrt{\tan^2\theta}\,\tan\theta\,d\theta$$

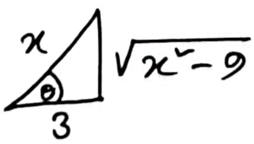
$$= 3\int \tan^2\theta\,d\theta$$

$$= 3\int \sec^2\theta - 1\,d\theta$$

$$= 3(\tan\theta - \theta) + c$$

Now,

$\theta = \cancel{\tan^{-1}} \sec^{-1} x/3$




$\cos\theta = \frac{adj}{hyp}$

$\sec\theta = \frac{hyp}{adj}$

$\therefore \theta = \tan^{-1} \frac{\sqrt{x^2-9}}{3}$

$$\therefore \int \frac{\sqrt{x^2-9}}{x}\,dx = 3\left(\tan\left(\tan^{-1}\frac{\sqrt{x^2-9}}{3}\right) - \sec^{-1} x/3\right) + c$$

$$= \sqrt{x^2-9} - 3\sec^{-1} x/3 + c$$