A function f of two variables, x and y, is a rule that assigns a unique real number $f(x,y)$ to each point (x,y) in some set D in the xy-plane.

A function f of three variables, x, y and z, is a rule that assigns a unique real number $f(x,y,z)$ to each point (x,y,z) in some set D in the three dimensional space.

Example :- Let $f(x,y) = \sqrt{y+1} + \ln(x^2-y)$. Find $f(e,0)$, $f(x,0)$ and $f(e,y)$.

Solution :- Given that,

$$f(x,y) = \sqrt{y+1} + \ln(x^2-y)$$

$$f(e,0) = \sqrt{0+1} + \ln(e^2-0)$$

$$= 1 + \ln e^2$$

$$= 1 + 2\ln e$$

$$= 1 + 2$$

$$= 3.$$

$f(x,0) = \sqrt{0+1} + \ln(x^2 - 0) = 1 + 2\ln x$

$f(e,y) = \sqrt{y+1} + \ln(e^2 - y)$

Problem: Find the domain of the function

$$f(x,y) = xe^{-\sqrt{y+2}}.$$

Solution: Given that,

$$f(x,y) = xe^{-\sqrt{y+2}}.$$

There is no value of x for which the function $f(x)$ is undefined, so $x \in (-\infty, \infty)$. For y, we set

$$y+2 \geq 0.$$

$$\Rightarrow y \geq -2$$

Thereforce $y \in [-2, +\infty)$

So the domain of the function is all points above or on the line $y = -2$.

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Partial Derivative Notation :-

If $z = f(x,y)$, then the partial derivatives f_x and f_y are also denoted by the symbols

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x,y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0,\, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)},$$

$$\frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

Example :- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for

i) $z = x^4 \sin(xy^3)$

ii) $z = x^2 + xy + y^3$

Solution :- i) Given that

$$z = x^4 \sin(xy^3)$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^4 \sin(xy^3)\right)$$

$$= 4x^3 \sin(xy^3) + x^4 \cos(xy^3) \cdot y^3$$

$$= 4x^3 \sin(xy^3) + x^4y^3 \cos(xy^3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^4 \sin(xy^3)\right)$$

$$= x^4 \cos(xy^3) \cdot 3xy^2$$

$$= 3x^5y^2 \cos(xy^3)$$

ii) Given that,

$$z = x^2 + xy + y^3.$$

$$\frac{\partial z}{\partial x} = 2x + y$$

$$\frac{\partial z}{\partial y} = x + 3y^2.$$

Problem:- Evaluate f_x and f_y for

a) $f(x,y) = x^2 + xy + y^4$

b) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

c) $f(x,y) = 3x^3y^2 + 2y^2 + 4x$.

Higher Order Partial Derivatives:-

Differentiate twice with respect to x:-

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to y:-

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}.$$

Differentiate first with respect to x and then with respect to y.

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(f_x) = f_{xy}$$

Differentiate ~~with~~ first with respect to y and then with respect to x:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(f_y) = f_{yx} .$$

Problem: Evaluate f_{xx}, f_{yy}, f_{xy} and f_{yx} for the previous problem.

Practice Problem:

13.1 → 1-4, 27

13.3 → 1-10, 25-40, 81, 82, 83, 84 .

Example:- If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$ then evaluate f_x and f_y.

Solution:- Given that

$$f(x,y) = \sin\left(\frac{x}{1+y}\right)$$

$$f_x = \frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$= \frac{1}{1+y} \cos\left(\frac{x}{1+y}\right)$$

$$f_y = \frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \left(\frac{-x}{(1+y)^2}\right)$$

$$= -\frac{x}{(1+y)^2} \cos\left(\frac{x}{1+y}\right).$$

Example:- If $f(x,y,z) = \sin(3x+yz)$ then evaluate f_{xxyz}.

Solution:- Given that

$$f(x,y,z) = \sin(3x+yz)$$

$$f_x = 3\cos(3x+yz)$$

$f_{xx} = -9\sin(3x+yz)$

$f_{xzy} = -9\cos(3x+yz) \cdot z$

$= -9z\cos(3x+yz)$

$f_{xzyz} = +9z\sin(3x+yz) \cdot y - 9\cos(3x+yz)$

$= 9zy\sin(3x+yz) - 9\cos(3x+yz)$

Example:- Show that $u(x,y) = e^x \sin y$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$.

Solution:- Given that

$u(x,y) = e^x \sin y$

$u_x = e^x \sin y$

$u_{xx} = e^x \sin y$

$u_y = e^x \cos y$

$u_{yy} = -e^x \sin y$

L.H.S. $= u_{xx} + u_{yy}$

$= e^x \sin y - e^x \sin y$

$= 0 =$ R.H.S.

[Showed]