Taylor Series of Two Variable Functions :-

Let us recall the Taylor series expansion of a single variable function $f(x)$ at $x=a$

$$f(x) = f(a) + \left.\frac{df}{dx}\right|_{x=a} (x-a) + \frac{1}{2!} \left.\frac{d^2f}{dx^2}\right|_{x=a} (x-a)^2 + \cdots$$

For a function of two variables $f(x,y)$ the Taylor series expansion at (a,b) is

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$+ \frac{1}{2!}\Big(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b)$$
$$+ f_{yy}(a,b)(y-b)^2\Big) + \text{terms of degree 3 or more}$$

* When you are asked to calculate the polynomial you need not put the dot sign. Because the dot sign is used to notify the series expansion.

Example: Calculate the second degree Taylor polynomial of the function $f(x,y) = \sin 2x + \cos y$ near $(0,0)$.

Solution: Given that,

$$f(x,y) = \sin 2x + \cos y$$

$$f(0,0) = \sin 2(0) + \cos(0) = 1.$$

$$f_x(x,y) = 2\cos 2x, \qquad f_x(0,0) = 2.$$

$$f_y(x,y) = -\sin y, \qquad f_y(0,0) = 0.$$

$$f_{xx}(x,y) = -4\sin 2x \qquad f_{xx}(0,0) = 0.$$

$$f_{xy}(x,y) = 0 = f_{yx}(x,y) \qquad f_{xy}(0,0) = 0 = f_{yx}(0,0)$$

$$f_{yy}(x,y) = -\cos y \qquad f_{yy}(0,0) = -1.$$

Therefore the second degree Taylor polynomial of $f(x)$ is,

$$f(x) \simeq 1 + 2(x-0) + 0(y-b) + \frac{1}{2}\Big(0(x-0)^2 + 0(x-0)(y-0) + (-1)(y-0)^2\Big)$$

$$\Rightarrow \sin 2x + \cos y = 1 + 2x - \frac{y^2}{2}.$$

Ans.

Differentiation of Vectors :-

In three dimensional Cartesian coordinate system, a vector is represented mathematically as

$$\vec{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z.$$

with $\hat{i}$, $\hat{j}$. and $\hat{k}$ representing unit vectors along x, y and z axis respectively. A_x, A_y and A_z are the components of $\vec{A}$. If $\vec{A}$ is a function of some variable t,

$$\vec{A}(t) = \hat{i} A_x(t) + \hat{j} A_y(t) + \hat{k} A_z(t).$$

Differentiation with respect to t yields

$$\frac{d\vec{A}}{dt} = \hat{i}\frac{dA_x}{dt} + \hat{j}\frac{dA_y}{dt} + \hat{k}\frac{dA_z}{dt}.$$

Differentiation Rules :-

$$\frac{d}{dt}(a\vec{A}) = a\frac{d\vec{A}}{dt} + \frac{da}{dt}\vec{A} \quad [a - \text{Constant}]$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\vec{B}$$

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

Gradient :-

~~If a f is a~~

If f is a function of x and y, then the gradient of f is defined by

$$\vec{\nabla}f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}.$$

If f is a function of x, y and z, then the gradient of f is defined by

$$\vec{\nabla}f(x,y,z) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Here the operator $\vec{\nabla}$ (Nabla) is defined by

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

is defined as

Example :- If a function, $f(x, y) = x^2 e^y$ then find the gradient of f at $(-2, 0)$.

Solution :- Given that,

$$f(x, y) = x^2 e^y.$$

Gradient of f, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$= \frac{\partial}{\partial x}(x^2 e^y)\hat{i} + \frac{\partial}{\partial y}(x^2 e^y)\hat{j}$$

$$= 2x e^y \hat{i} + x^2 e^y \hat{j}$$

At $(-2, 0)$, $\vec{\nabla} f = 2\cdot(-2)e^0 \hat{i} + (-2)^2 e^0 \hat{j}$

$$= -4\hat{i} + 4\hat{j}.$$

Ans.

Divergence :- A vector $\vec{V}$ is defined as

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}.$$

Then the divergence of the vector $\vec{V}$ is defined as

$$\vec{\nabla}.\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(V_x) + \frac{\partial}{\partial y}(V_y) + \frac{\partial}{\partial z}(V_z).$$

In short, the divergence of a vector $\vec{V}$ is the dot product of nabla and the vector.

Example :- If a vector is defined as $\vec{V} = 2x^3\hat{i} + 3yz^2\hat{j} + 3x^2\hat{k}$, ~~at (1,0,1)~~ then find the divergence of the vector at $(1,0,1)$.

Solution :- Given that

$$\vec{V} = 2x^3\hat{i} + 3yz^2\hat{j} + 3x^2\hat{k}.$$

$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(2x^3\hat{i} + 3yz^2\hat{j} + 3xz^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(2x^3) + \frac{\partial}{\partial y}(3yz^2) + \frac{\partial}{\partial z}(3xz^2)$$

$$= 6x^2 + 3z^2.$$

At $(1, 0, 1)$ $\quad \vec{\nabla}\cdot\vec{V} = 6(1)^2 + 3(1)^2$

$$= 6 + 3$$

$$= 9.$$

Curl :- If a vector $\vec{V}$ is defined as

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}.$$

Then the curl of the vector $\vec{V}$ is defined as.

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

In short, the curl of a vector $\vec{V}$ is the cross product of nabla and the vector.

Example:– If a vector is defined as $\vec{A} = 2x^3\hat{i} + 3yz^2\hat{j} + 4zx\hat{k}$ then find the curl of $\vec{A}$, at $(1, 2, -1)$

Solution:– Given that,

$$\vec{A} = 2x^3\hat{i} + 3yz^2\hat{j} + 4zx\hat{k}$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^3 & 3yz^2 & 4zx \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(4zx) - \frac{\partial}{\partial z}(3yz^2)\right) \quad \left(\frac{\partial}{\partial x}(3yz^2) - \frac{\partial}{\partial y}(2x^3)\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}(4zx) - \frac{\partial}{\partial z}(2x^3)\right) + \hat{k}\cancel{\left(\frac{\partial}{\partial y}(4zx) - \frac{\partial}{\partial x}(3yz^2)\right)}$$

$= -6yz\hat{i} - 4z\hat{j}$.

At $(1, 2, -1)$ $\quad \vec{\nabla} \times \vec{A} = 12\hat{i} + 4\hat{j}$

Ans.

Field :- Field refers to the physical quantity of interest and the region of space where it has a value.

Directional Derivative :- If the rate of change of any physical quantity depends on the direction we are moving, it is called the directional derivative.

The directional derivative represents the instantaneous rate of change of $f(x, y)$ with respect to distance in the direction of unit vector $\vec{u}$ at the point (x_0, y_0).

If a function $f(x,y)$ is of x and y and if $\vec{u} = u_1\hat{i} + u_2\hat{j}$ is a unit vector then the directional derivative of f in the direction of $\vec{u}$ at (x_0, y_0) is denoted by $D_u f(x_0, y_0)$ and is

defined by $$D_u f(x_0, y_0) = \vec{\nabla} f \cdot \vec{u} \Big|_{(x_0, y_0)}.$$

<u>Example:</u> Find the directional derivative of $f(x,y) = xy$ at $(1, 2)$ in the direction of $\vec{A} = \sqrt{3}\,\hat{i} + \hat{j}$.

<u>Solution:</u> The unit vector is defined as

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{\sqrt{3}\,\hat{i} + \hat{j}}{\sqrt{(\sqrt{3})^2 + (1)^2}} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}.$$

Gradient of f, $\vec{\nabla} f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right)(xy)$

$$= y\hat{i} + x\hat{j}.$$

Directional derivative, $D_u f = (\vec{\nabla} f \cdot \vec{u})$

$= \left(y\hat{i} + x\hat{j}\right) \cdot \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

$= \frac{\sqrt{3}}{2}y + \frac{1}{2}x$.

At $(1, 2)$ $\quad D_u f(1,2) = \frac{\sqrt{3}}{2} \cdot 2 + \frac{1}{2} \cdot 1$

$= \sqrt{3} + \frac{1}{2}$

Ans .

Practice Problem :-

13.5 - 17-29

13.6 — 1-18, 33-46

13.8 - 9-20 .