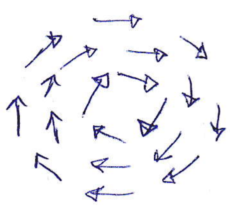
Week 8, Differential Calculus & Coordinate Geometry MAT110

Examples Vector Calculus,

$\boxed{1}$ a) Is $\vec{F} = y^3\hat{i} + xy\hat{j} - z\hat{k}$ an irrotational vector field?

b) Find divergence of $G(x,y) = \frac{4y}{x^2}\hat{i} + \sin y\,\hat{j} + 3\hat{k}$

a) Irrotational vector field implies $\text{Curl}\,\vec{F} = \nabla \times F = 0$



$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & xy & -z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-z) - \frac{\partial}{\partial z}(xy)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-z) - \frac{\partial}{\partial z}(y^3)\right) + \hat{k}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(y^3)\right)$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(y-3y^2)$$

$$= \hat{0} + \hat{0} + (y-3y^2)\hat{k} = y(1-3y)\hat{k}$$

An irrotational vector field is a vector field where curl is equal to zero everywhere. If the domain is simply connected (there are no discontinuities), the vector field will be conservative or equal to the gradient of a function (that is, it will have a scalar potential).

$\therefore$ $\text{Curl}\,\vec{F} \neq 0$

hence $\vec{F}$ is not an irrotational vector field.

b)

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}\left(\frac{4y}{x^2}\right) + \frac{\partial}{\partial y}(\sin y) + \frac{\partial}{\partial z}(3)$$

$$= 4y(-2x^{-3}) + \cos y + 0$$

$$= -\frac{8y}{x^3} + \cos y$$

$\boxed{2}$ Find the unit vector that has the same direction as $\vec{u} = 5\hat{i} - \hat{j} + 3k$

computing the magnitude:

$$|\vec{u}| = \sqrt{5^2 + (-1)^2 + 3^2}$$

$$= \sqrt{35}$$

Let $\hat{v}$ is the unit vector that has the same direction as $\vec{u}$

$$\hat{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{(5\hat{i} - \hat{j} + 3\hat{k})}{\sqrt{35}} = \frac{5}{\sqrt{35}}\hat{i} - \frac{1}{\sqrt{35}}\hat{j} + \frac{3}{\sqrt{35}}$$

$\boxed{3}$ Find the curl of the vector field $\vec{F}(x, y, z) = x^2\hat{i} + 2z\hat{j} - y\hat{k}$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2z & -y \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-y) - \frac{\partial}{\partial z}(2z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-y) - \frac{\partial}{\partial z}(x^2)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial y}(x^2)\right)$$

$$= \hat{i}(-1-2) - \hat{j}(0-0) + \hat{k}(0-0)$$

$$= -3\hat{i}$$

$\boxed{4}$ If $\vec{V} = xy\hat{i} + yz\hat{j} + zx\hat{k}$, then $\operatorname{div}\vec{V} = ?$

$$\operatorname{div}\vec{V} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$= y + z + x.$$

$\boxed{5}$ Find the directional derivative of $f(x,y,z) = x^2y - yz^3 + z$ at the point $(1,-2,0)$ in the direction of the vector $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$

$f_x(x,y,z) = 2xy$, $\quad f_y(x,y,z) = x^2 - z^3$, $\quad f_z = -3yz^2 + 1$

$f_x(1,-2,0) = -4$, $\quad f_y(1,-2,0) = 1$, $\quad f_z(1,-2,0) = 1$

$\because$ $a$ is not a unit vector, we normalize it:

$$u = \frac{a}{\|a\|} = \frac{1\,(2\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{9}}(2\hat{i} + \hat{j} - 2\hat{k})$$

$$= \underset{u_1}{\frac{2}{3}}\hat{i} + \underset{u_2}{\frac{1}{3}}\hat{j} - \underset{u_3}{\frac{2}{3}}\hat{k}$$

$$D_u f(1,-2,0) = f_x(1,-2,0)\,u_1 + f_y(1,-2,0)\,u_2 + f_z(1,-2,0)\,u_3$$

$$= (-4)\left(\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(-\frac{2}{3}\right)$$

$$= -3$$

$\boxed{6}$ Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ in the direction of the vector $u = 2\hat{i} - \hat{j} - 2\hat{k}$

$\phi_x = 2xyz + 4z^2$, $\quad \phi_y = x^2 z$, $\quad \phi_z = x^2 y + 8xz$

$\phi_x(1,-2,-1) = 2(1)(-2)(-1) + 4(-1)^2$, $\quad \phi_y(1,-2,-1)$ $\quad \phi_z(1,-2,-1)$

$= 4+4$ $\quad = (1)^2(-1)$ $\quad = 1^2(-2) + 8(-2)(-1)$

$= 8$ $\quad = -1$ $\quad = -2+16$

$= 14$

$\because \bar{u}$ is not a unit vector, we normalize it:

$$u = \frac{u}{\|u\|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{1}{\sqrt{9}}(2\hat{i} - \hat{j} - 2\hat{k})$$

$$= \underbrace{\frac{2}{3}}_{u_1}\hat{i} - \underbrace{\frac{1}{3}}_{u_2}\hat{j} - \underbrace{\frac{2}{3}}_{u_3}\hat{k}$$

$$D_u\phi(1,-2,-1) = \phi_x(1,-2,-1)u_1 + \phi_y(1,-2,-1)u_2 + \phi_z(1,-2,-1)u_3$$

$$= 8\left(\frac{2}{3}\right) + (-1)\left(-\frac{1}{3}\right) + (14)\left(-\frac{2}{3}\right)$$

$$= \frac{16}{3} + \frac{1}{3} - \frac{28}{3} = \frac{-11}{3}$$

$\boxed{7}$ Given $A = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$. Find curl A at the point $(1, -1, 1)$.

$$\text{Curl } A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2 & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(-2x^2) - \frac{\partial}{\partial y}(xz^3)\right)$$

$$= (2z^4 + 0)\hat{i} - (0 - 3xz^2)\hat{j} + (-4x - 0)\hat{k}$$

$$\text{Curl } A \Big|_{(1,-1,1)} = (2(1)^4)\hat{i} + (3(1)(1)^2)\hat{j} - (4(1))\hat{k}$$

$$= 2\hat{i} + 3\hat{j} - 4\hat{k}$$

$\boxed{8}$ Let $r = x\hat{i} + y\hat{j} + z\hat{k}$ and $\hat{r}$ be the unit vector in the direction of $r$. If $r = |r|$, what is $\nabla e^r$?

$$\hat{r} = \frac{r}{|r|}$$

unit vector

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|r| = \sqrt{x^2 + y^2 + z^2}$$

$r = |r|$ given

$$\nabla e^r = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)e^r$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(e^{\sqrt{x^2+y^2+z^2}}\right)$$

$$= e^{\sqrt{x^2+y^2+z^2}} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}\hat{i} + e^{\sqrt{x^2+y^2+z^2}} \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}}\hat{j}$$

$$+ e^{\sqrt{x^2+y^2+z^2}} \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{e^{|r|}}{|r|} r$$

$$= e^{|r|} \cdot \frac{r}{|r|}$$

$$= e^{r}\,\hat{r} \qquad \because \frac{r}{|r|} = \hat{r} \text{ \& given } |r| = r$$

9 If $W = x^2y^2 + xz$, what is the directional derivative of $W$ in the direction $3\hat{i}+4\hat{j}+6\hat{k}$ at $(1,2,0)$?

$W_x = 2xy^2 + z$

$W_x(1,2,0) = 2(1)(2)^2 + 0^2$
$= 8$

$W_y = 2x^2y,$

$W_y(1,2,0) = 2(1)^2(2)$
$= 4$

$W_z = x$

$W_z(1,2,0) = (1)$
$= 1$

Let $a = 3\hat{i}+4\hat{j}+6\hat{k}$

$\because$ $a$ is not a unit vector, we normalize it:

$$u = \frac{a}{\|a\|} = \frac{1\,(3\hat{i}+4\hat{j}+6\hat{k})}{\sqrt{3^2+4^2+6^2}} = \frac{1}{\sqrt{61}}(3\hat{i}+4\hat{j}+6\hat{k})$$

$$= \underbrace{\frac{3}{\sqrt{61}}}_{u_1}\hat{i} + \underbrace{\frac{4}{\sqrt{61}}}_{u_2}\hat{j} + \underbrace{\frac{6}{\sqrt{61}}}_{u_3}\hat{k}$$

$$D_uW(1,2,0) = W_x(1,2,0)\,u_1 + W_y(1,2,0)\,u_2 + W_z(1,2,0)\,u_3$$

$$= 8\left(\frac{3}{\sqrt{61}}\right) + 4\left(\frac{4}{\sqrt{61}}\right) + 1\left(\frac{6}{\sqrt{61}}\right)$$

$$= \frac{24}{\sqrt{61}} + \frac{16}{\sqrt{61}} + \frac{6}{\sqrt{61}}$$

$$= 5.89$$

10 What is the divergence of a vector field $F(x,y,z) = (x^2y, y^2, z^2x)$ at a point $(-1,2,3)$?

$$\operatorname{div} F = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2x)$$

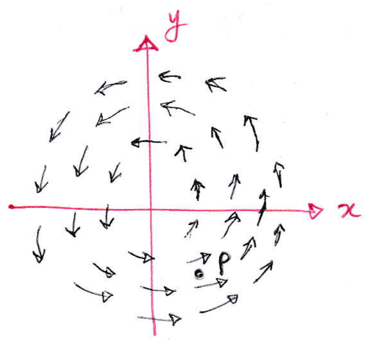
$$= 2xy + 2y + 2zx$$

$$\operatorname{div} F \Big|_{(-1,2,3)} = 2(-1)(2) + 2(2) + 2(3)(-1)$$

$$= -4 + 4 - 6$$

$$= -6$$

11




What is the sign of divergence of the vector field at P?

$\rightarrow$ positive (counter clockwise)

12 Let f and g be scalar functions of $x, y,$ and $z$. Let $\mathbf{F}(x,y,z)$ be a vector field. Which of the following is a sensible quantity?

✓ a) $\nabla \cdot f$

b) $f + \nabla g$

c) $F \cdot \nabla g$

d) $F \cdot \nabla g + F \times \nabla f$

$\boxed{13}$ Calculate the divergence of $\vec{V} = x^2\hat{i} + 3xz^2\hat{j} - 2xz\hat{k}$

$$\text{div } \vec{V} = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} 3xz^2 + \frac{\partial}{\partial z}(-2xz)$$

$$= 2x + 0 - 2x$$

$$= 0$$

$\boxed{14}$ Given that $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$. Find the gradient of $f(x,y,z)$ where $f(x,y,z)$ is the magnitude of $\vec{V}$ in terms of $x$, $y$, and $z$.

$$|\vec{V}| = \sqrt{x^2+y^2+z^2} = f(x,y,z)$$

$$\nabla f = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\,\hat{i} + \frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2}\,\hat{j} + \frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}\,\hat{k}$$

$$= \frac{1\,(2x)}{2\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{1\,(2y)}{2\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{1\,(2z)}{2\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \underbrace{\frac{1}{\sqrt{x^2+y^2+z^2}}(x\hat{i} + y\hat{j} + z\hat{k})}$$

This is the unit vector in the direction of $\vec{V}$.

unit vector

$$\hat{V} = \frac{V}{|V|}$$

$\boxed{15}$ Find the gradient $f(x,y,z) = e^x \sin(y) \ln(z)$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= e^x \sin y \ln z \,\hat{i} + e^x \cos y \ln z + \frac{e^x \sin y}{z}\hat{k}$$

$\boxed{16}$ Find the gradient of $f(x,y,z) = x^2y^3z^4$

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= 2xy^3z^4\hat{i} + 3x^2y^2z^4\hat{j} + 4x^2y^3z^3\hat{k}$$

$\boxed{17}$ Calculate the curl of $\vec{V} = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy+z^2 & 2yz \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(2xy+z^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(y^2)\right) + \hat{k}\left(\frac{\partial}{\partial x}(2xy+z^2) - \frac{\partial}{\partial y}(y^2)\right)$$

$$= \hat{i}(2z - 2z) - \hat{j}(0 - 0) + \hat{k}(2y - 2y)$$

$$= 0$$

$\boxed{18}$ Calculate the divergence of $\vec{V} = y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}$

$$\text{div } \vec{V} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= 0 + 2x + 2y$$

$$= 2x + 2y$$

$\boxed{19}$ Let $\vec{F}(x,y,z) = x^2y\hat{i} + 2y^3z\hat{j} + 3z\hat{k}$. Find the divergence of the vector field $\vec{F}$, at the point $(1, 2, 1)$.

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$$

$$= 2xy + 6y^2z + 3$$

$$\text{div } \vec{F}\Big|_{(1,2,1)} = 2(1)(2) + 6(2)^2(1) + 3$$

$$= 4 + 24 + 3$$

$$= 31$$