7.5 Rational Function

<u>Rational function</u>: A function of the form $\frac{P(x)}{Q(x)}$ is known as rational function.

- Proper rational function if

  degree of the numerator < degree of denominator

  Viz; $\frac{x^5+7}{x^8+2}$

- Improper rational function if

  degree of the numerator > degree of denominator

  Viz; $\frac{x^8+2}{x^5+1}$

<u>Partial Fraction:</u>

$$\frac{2}{x-2}+\frac{3}{x+1}$$

$$= \frac{2(x+1)+3(x-2)}{(x-2)(x+1)} = \frac{2x+2+3x-6}{x^2+x-2x-2} = \frac{5x-4}{x^2-3x-2}$$

$$\therefore \quad \frac{5x-4}{x^2-3x-2} = \frac{2}{x-2}+\frac{3}{x+1} \qquad \text{———} (*)$$

The terms on the right side of (*) are called partial fraction of the expression on the left side.

Linear Factors

For any rational fonction $\frac{P(x)}{Q(x)}$. If all the factors of $Q(x)$ are linear, then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ can determine using following rule:

$$\frac{5x}{x^2+x-2} = \frac{5x}{x^2+2x-x-2} = \frac{5x}{x(x+2)-1(x+2)}$$

$$\therefore \frac{5x}{x^2+x-2} = \frac{5(x)}{(x+2)(x-1)}$$

Now

$$\frac{5x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} \quad \text{———} \quad (1)$$

$$\Rightarrow \frac{5x}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$\Rightarrow 5x = A(x-1)+B(x+2) \quad \text{———} \quad (ii)$$

Putting $x=1$ in (i)

$$5.1 = A.0 + B(\cancel{x}+2)$$

$$\therefore B = \frac{5}{3}$$

Putting $x=-2$ in (ii)

$$5(-2) = A(-2-1)+B.0$$

$$\Rightarrow A = \frac{10}{3}$$

(1) become,

$$\frac{5x}{(x+2)(x-1)} = \frac{10/3}{x+2} + \frac{5/3}{x-1}$$

Repeated Factors

If the factors of $Q(x)$ are repeated that is

$$\frac{2x+4}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

Quadratic Factors

Example of Quadratic Factors

$$\frac{2x+4}{(x^2-4)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{(x^2-4)}$$

For Example

(i) $$\frac{2x+1}{(x+2)^2(x^2+1)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1}$$

(ii) $$\frac{2x+1}{(x+1)(x-2)(x^2+1)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1}$$

(iii) $$\frac{2}{(x^2+5)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+5}$$

(iv) $$\frac{5}{x^2(x-1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$$

- Evaluate $\int \frac{dx}{x^2+x-2}$

<u>Solution:</u>

Here

$$\frac{1}{x^2+x-2} = \frac{1}{x^2+2x-x-2} = \frac{1}{x(x+2)-1(x+2)}$$

$$= \frac{1}{(x+2)(x-1)}$$

Now,

$$\frac{1}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} \quad \text{———— (1)}$$

$$\Rightarrow \frac{1}{(x+2)(x-1)} = \cancel{A(x} \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

$$\therefore \quad 1 = A(x-1)+B(x+2) \quad \text{———— (ii)}$$

putting $x=1$ in (ii)

$$1 = A\cdot 0 + B(1+2)$$

$$\boxed{\Rightarrow B = \tfrac{1}{3}}$$

putting $x=-2$ in (ii)

$$1 = A(-2-1)+B\cdot 0$$

$$\boxed{\Rightarrow A = -\tfrac{1}{3}}$$

$$\therefore \frac{1}{x^2+x-2} = \frac{-1/3}{x+2} + \frac{1/3}{x-1}$$

Now

$$\int \frac{dx}{x^2+x-2} = \int \left(-\frac{1}{3}\frac{1}{x+2} + \frac{1}{3}\frac{1}{x-1}\right) dx$$

$$= -\frac{1}{3}\int \frac{1}{x+2}\,dx + \frac{1}{3}\int \frac{1}{x-1}\,dx$$

$$\therefore \int \frac{dx}{x^2+x-2} = -\frac{1}{3}\ln(x+2) + \frac{1}{3}\ln(x-1) + c$$

A.

- Evaluate $\int \frac{2x+4}{x^3-2x^2}\,dx$

<u>solution:</u>

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \quad \text{——— (I)}$$

$$\Rightarrow 2x+4 = Ax(x-2) + B(x-2) + cx^2 \quad \text{——— (II)}$$

Setting $x=0$ in (II)

$$0+4 = 0 + B(0-2) + c.0$$

$$\Rightarrow -2B = 4 \quad \therefore \boxed{B=-2}$$

Setting $x=2$ in (II)

$$4+4 = 0 + 0 + 4c$$

$$\Rightarrow 4c = 8 \quad \therefore \boxed{c=2}$$

putting $x=1$, $B=-2$, and $c=2$ in (ii)

$$2+4 = A\cdot 1\cdot (1-2) + (-2)(1-2) + 2\cdot 1^2$$

$$\therefore \boxed{A=-2}$$

$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = \int \frac{-2}{x}dx + \int \frac{-2}{x^2}dx + \int \frac{2}{x-2}dx$$

$$= -2\ln x - 2\frac{x^{-2+1}}{-2+1} + \ln|x-2| + c$$

$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = -2\ln x + \frac{2}{x} + \ln|x-2| + c$$

A

• Evaluate $\int \frac{x^2+x-2}{3x^3-x^2+3x-1}dx$

Solution: Here, $3x^3-x^2+3x-1$

$$\cancel{= x^2(3-x)+3(x}$$

$$= x^2(3x-1)+1(3x-1)$$

$$= (3x-1)(x^2+1)$$

Now $$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+c}{x^2+1} \quad \text{—— (1)}$$

$$= \frac{A(x^2+1)+(Bx+c)(3x-1)}{(3x-1)(x^2+1)}$$

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$$\Rightarrow x^2+x-2 = A(x^2+1)+(Bx+C)(3x-1) \quad \text{—— (ii)}$$

Putting $x=1/3$ in (ii)

$$\left(\tfrac{1}{3}\right)^2+\left(\tfrac{1}{3}\right)-2 = A\left(\tfrac{1}{9}+1\right)+0$$

$$\Rightarrow \frac{10}{9}A = \frac{1}{9}+\frac{1}{3}-2$$

$$\Rightarrow \frac{10}{9}A = \frac{1+3-18}{9}$$

$$\Rightarrow A = \frac{-14}{10} \qquad \therefore \boxed{A=-\tfrac{7}{5}}$$

Putting $x=0$, $A=-7/5$ in (ii)

$$0+0-2 = A-\tfrac{7}{5}(0+1)+(B.0+C)(0-1)$$

$$\Rightarrow -2 = -\tfrac{7}{5}-C$$

$$\Rightarrow C = -\tfrac{7}{5}+2 \qquad \therefore \boxed{C=\tfrac{3}{5}}$$

Again, substituting $x=1$, $A=-\tfrac{7}{5}$ and $C=\tfrac{3}{5}$ in (ii)

$$1+1-2 = -\tfrac{7}{5}(1+1)+\cancel{\tfrac{3}{5}\cdot 1\cdot}\ (B\cdot 1+\tfrac{3}{5})(3\cdot 1-1)$$

$$\Rightarrow 0 = -\frac{14}{5}+2\left(B+\frac{3}{5}\right)$$

$$\Rightarrow B+\frac{3}{5} = \frac{7}{5} \qquad \therefore \boxed{B=\tfrac{4}{5}}$$

$\therefore$ (1) $\Rightarrow$

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x+\frac{3}{5}}{x^2+1}$$

$$= -\frac{7}{5}\,\frac{1}{3x-1} + \frac{4}{5}\,\frac{x}{x^2+1} + \frac{3}{5}\,\frac{1}{x^2+1}$$

$$\therefore \int \frac{x^2+x-2}{(3x-1)(x^2+1)}\,dx = \int -\frac{7}{5}\,\frac{1}{3x-1}\,dx + \int \frac{4}{5}\,\frac{x}{x^2+1}\,dx$$

$$+ \int \frac{3}{5}\,\frac{1}{x^2+1}\,dx$$

$$= -\frac{7}{5}\cdot\frac{1}{3}\int \frac{3}{3x-1}\,dx + \frac{2}{5}\int \frac{2x}{x^2+1}\,dx$$

$$+ \frac{3}{5}\int \frac{1}{1+x^2}\,dx$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln|x^2+1| + \frac{3}{5}\tan^{-1}x + c$$

<u>Practice</u>

Solve (i) $\displaystyle\int \frac{2x^2-2x-1}{x^3-x^2}\,dx$

(ii) $\displaystyle\int \frac{2x^2+3x+3}{(x+1)^3}\,dx$

(iii) $\displaystyle\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$

(iv) $\displaystyle\int \frac{2x^2-10x+4}{(x+1)(x-3)^2}\,dx$

(v) $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$

(vi) $\displaystyle\int \frac{dx}{x^3+2x}$