Integration by Parts: (Quiz-1)

Reduction Formulae:

1. $I_n = \int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}, \; n \geq 2$

H.W.

2. $I_n = \int \cos^n x \, dx = -\frac{\cos^{n-1} x}{n} \sin x + \frac{n-1}{n} I_{n-2}, \; n \geq 2$

3. $I_n = \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}, \; n \geq 2$

4. $I_n = \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}, \; n \geq 2$

H.W.

Obtain the reduction formulae for $\int \sin^n x\,dx$ & evaluate $\int \sin^6 x\,dx$.

$\int u\,dv = uv - \int v\,du$ | $\frac{dx^n}{dx} = nx^{n-1}$

<u>Solution:</u> let $I_n = \int \sin^n x\,dx$

$= \int \underbrace{\sin^{n-1} x}_{u} \underbrace{\sin^1 x\,dx}_{dv}$

let

$u = \sin^{n-1} x$ $\quad \int dx = x + C$

$\Rightarrow du = (n-1)\sin^{n-2} x \cos x\,dx$

$dv = \sin x\,dx$

$\Rightarrow \int dv = \int \sin x\,dx$

$\Rightarrow v = -\cos x + K$

$= -\sin^{n-1} x \cos x + \int \cos x (n-1) \sin^{n-2} x \cos x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$

$I_{n-2} = \int \sin^{n-2} x\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\,dx$

$= -\sin^{n-1} x \cos x + (n-1)\left[\int \underbrace{\sin^{n-2} x\,dx}_{I_{n-2}} - \int \underbrace{\sin^n x\,dx}_{I_n}\right]$

$$I_n = -\sin^{n-1}x\cos x + (n-1)\left[I_{n-2} - I_n\right]$$

$$\Rightarrow I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} - \underline{(n-1)I_n}$$

$$\Rightarrow I_n + (n-1)I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} \qquad n \in \{0,1,2,3,4,5,\dots\}$$

$$\Rightarrow I_n = \frac{-\sin^{n-1}x\cos x}{n} + \frac{(n-1)}{n}I_{n-2}, \quad n \geq 2$$

$$I_6 = \int \sin^6 x\,dx$$

$$= \frac{-\sin^5 x\cos x}{6} + \frac{5}{6}I_4 = \frac{-\sin^5 x\cos x}{6} + \frac{5}{6}\left[\frac{-\sin^3 x\cos x}{4} + \frac{3}{4}I_2\right]$$

$$I_0 = \int \sin^0 x \, dx = \int dx = x$$

$$I_6 = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{15}{24} I_2$$

$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{\cancel{15}^{\,5}}{\cancel{24}_{\,8}}\left[-\frac{\sin x\cos x}{2} + \frac{1}{2} I_0\right]$$

$$= -\frac{1}{6}\sin^5 \cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + C$$

$2\sin x\cos x = \sin 2x$

$\sin^2 x = 1-\cos^2 x$, $\cos^2 x = 1-\sin^2 x$, $2\sin^2 x = 1-\cos 2x$

$2\cos^2 x = 1+\cos 2x$

$\int \sin^m x \cos^n x \, dx$

(1) if m/n is odd, split the odd powered term using $\underset{\sin x}{\sin^2 x} = (1-\cos^2 x)$ / $\underset{\cos x}{\cos^2 x} = (1-\sin^2 x)$ & opposite term substitution

Ex:

$\int \sin^3 x \cos^4 x \, dx$

$= \int \sin^2 x \sin x \cos^4 x \, dx$

$= \int (1-\cos^2 x) \sin x \cos^4 x \, dx$

(ii) If m & n are odd, split either of them using above formula.

$$\int \sin^3 x \cos^4 x \, dx$$

Let $\cos x = u$

$\Rightarrow -\sin x \, dx = du$

$\Rightarrow \sin x \, dx = -du$

$$= \int \sin^2 x \sin x \cos^4 x \, dx$$

$$= \int (1-\cos^2 x) \cos^4 x \sin x \, dx$$

$$= \int (1-u^2) u^4 (-du) = \int (u^2-1) u^4 \, du$$

$$= \int u^6 du - \int u^4 du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

Ex:

$$\int \sin^3 x \cos x \, dx$$

let $\cos x = u$

$$= \int \sin^2 x \sin x \cos x \, dx$$

$$= \int (1-\cos^2 x) \cos x \sin x \, dx$$

H.W.

(iii) If m & n are even, $2\sin^2 x = (1-\cos 2x)$, $2\cos^2 x = (1+\cos 2x)$

then reduction formula. $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\int \sin^6 x \cos^6 x \, dx \qquad \sin^2 2x = 1 - \cos^2 2x$$

$$= \int (\sin^2 x)^3 (\cos^2 x)^3 \, dx$$

$$= \int \left\{\tfrac{1}{2}(1-\cos 2x)\right\}^3 \left\{\tfrac{1}{2}(1+\cos 2x)\right\}^3 dx$$

$$= \tfrac{1}{64}\int \left\{(1-\cos 2x)(1+\cos 2x)\right\}^3 dx$$

$$= \tfrac{1}{64}\int (1-\cos^2 2x)^3 \, dx$$

$$= \frac{1}{64} \int (\sin^2 2x)^3 \, dx$$

$$= \frac{1}{64} \int \sin^6 2x \, dx$$

$$= \frac{1}{64} \cdot \frac{1}{2} \int \sin^6 u \, du$$

$$= \frac{1}{128} \int \sin^6 u \, du$$

H.W.

let

$\Rightarrow 2x = u$

$\Rightarrow dx = \frac{du}{2}$

$\int \sin^4 x \cos^5 x \, dx$, $\int \sin^4 x \cos^4 x \, dx$. H.W.

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.

Ex:
$$\int \sin 4x \sin 9x \, dx$$

$$\boxed{\cos(-\theta) = \cos\theta}$$

$$\int \cos ax \, dx = \frac{\sin ax}{a} + C$$

$$= \frac{1}{2}\int 2 \sin 4x \sin 9x \, dx$$

$$= \frac{1}{2}\int \cos(4x - 9x) \, dx - \frac{1}{2}\int \cos(4x + 9x) \, dx$$

$$= \frac{1}{2} \frac{\sin 5x}{5} - \frac{1}{2} \frac{\sin 13x}{13} + C.$$

$I_n = \int \sec^n x\,dx$, $\int \sec^5 x\,dx$ $\int u\,dv = uv - \int v\,du$

let $I_n = \int \sec^n x\,dx$

$$= \int \sec^{n-2} x \sec^2 x\,dx$$

let $u = \sec^{n-2} x$

$du = (n-2)\sec^{n-3} x \sec x \tan x\,dx$

$dv = \sec^2 x\,dx$

$\Rightarrow \int dv = \int \sec^2 x\,dx$

$\Rightarrow v = \tan x + K$

$$= \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-3} x \sec x \tan x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \sec^1 x \tan^2 x\,dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \left(\sec^2 x - 1\right) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \Big[\underbrace{\int \sec^n x\,dx}_{I_n} - \underbrace{\int \sec^{n-2} x\,dx}_{I_{n-2}} \Big]$$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_1 = \int \sec^1 x \, dx$$

$$= \ln|\sec x + \tan x| + C$$

$$\Rightarrow I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow (1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow (n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$\Rightarrow I_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{n-1} I_{n-2}, \quad \boxed{n \geq 2}$$

$$\int \sec^5 x \, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3$$

$$I_3 = \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

$$\int \sec x \, dx = \int \sec x \, \frac{(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

$$\int \frac{f'(x)}{f(x)} \, dx$$

$$= \ln|f(x)| + C.$$

$$= \int \frac{(\sec^2 x + \tan x \sec x)}{\sec x + \tan x} \, dx$$

$$= \ln|\sec x + \tan x| + C.$$

$\int \tan^m x \sec^n x \, dx$

(1) $\boxed{n \text{ is even}}$, $\boxed{\sec^{n-2} x \cdot \sec^2 x} \longrightarrow$ split.

$\boxed{\sec^2 x = 1 + \tan^2 x} \longrightarrow$ split; $\tan x = u$

$$\int \tan^3 x \sec^4 x \, dx = \int \tan^3 x \cdot \sec^2 x \cdot \underline{\sec^2 x \, dx}$$

$$= \int \tan^3 x \left(1 + \tan^2 x\right) \underline{\sec^2 x \, dx}$$

let $\tan x = u$

$\sec^2 x \, dx = du$

$$= \int u^3 (1 + u^2) \, du = \int u^3 du + \int u^5 du$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^5 x}{5} + C.$$

$$\int \tan^3 x \sec^6 x \, dx$$

let $\tan x = u$.

$$= \int \tan^3 x \sec^4 x \underline{\sec^2 x \, dx}$$

$$= \int \tan^3 x (\sec^2 x)^2 \sec^2 x \, dx$$

$$= \int \tan^3 x (1 + \tan^2 x)^2 \sec^2 x \, dx$$

H.W.

(ii) If m & n both are odd, $\boxed{\sec x \tan x} \longrightarrow$ split

$$\boxed{\tan^4 x = (\tan^2 x)^2 = (\sec^2 x - 1)^2},$$

let $\sec x = u$

$$\int \tan^3 x \sec^3 x \, dx$$

$$= \int \tan^4 x \sec^2 x \tan x \sec x \, dx$$

$$= \int (\tan^2 x)^2 \sec^2 x \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1)^2 \sec^2 x \tan x \sec x \, dx$$

$$= \int (u^2 - 1)^2 u^2 \, du$$

let $\sec x = u$

$\Rightarrow \sec x \tan x \, dx = du$

$$= \int u^2(u^4 - 2u^2 + 1)\, du$$

$$= \int (u^6 - 2u^4 + u^2)\, du = \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C.$$

(iii) If $\underline{m}$ is even & n is odd

We convert $\tan^2 x$ into $(\sec^2 x - 1)$ & then apply Reduction

$$\int \tan^4 x \sec x\, dx$$

$$= \int (\sec^2 x - 1)^2 \sec x\, dx = \int (\sec^4 x - 2\sec^2 x + 1)\sec x\, dx$$

$$= \int (\underline{\sec^5 x} - 2\underline{\sec^3 x} + \sec x)\, dx$$

Now Reduction Apply