MAT 110 Differential Calculus & Coordinate Geometry

Week 1 Topic: → Limits

→ Continuous and Discontinuous function

→ Rational Function and Asymptotes

→ Computing Limits

- Algebric Manipulation
- Squeezing or Squeeze Theorem
- Change of Variable
- L'Hôspital' Rule

Limit → What is Limit?

A limit looks at what happens to a function when the input approaches a certain value. Example $\lim_{x \to a} f(x)$. It denotes the limit of $f(x)$ as 'x' approaches 'a'.

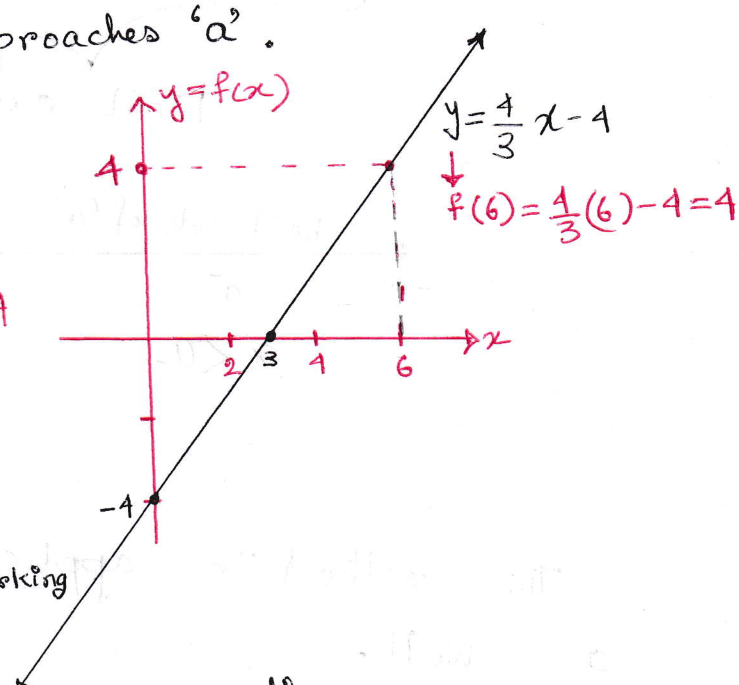
Consider $f(x) = \frac{4}{3}x - 4$

$y \to$ output
$x \to$ input
$y = f(x)$

if $x = 0 \Rightarrow f(x) = -4$

if $y = 0 \Rightarrow 0 = \frac{4}{3}x - 4$

$x = 3$



Limit Notation

The "lim" tells us we are looking for a limit value, not a function value

This tells us which function we are working with

$$\lim_{x \to 6} f(x) = 4$$

This this the value the function is approaching

This tells us what the variable is, and what it is approaching

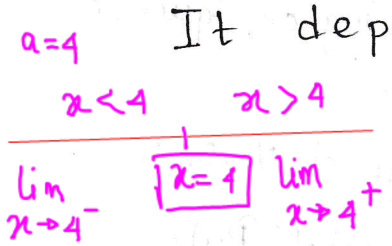
Existence of Limit

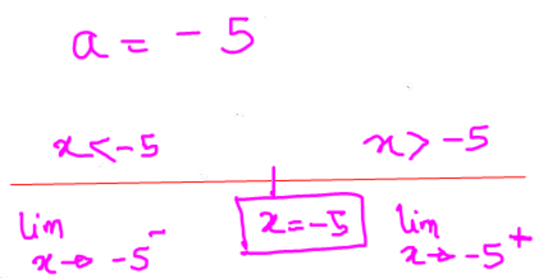
Limit of a function exist if L.H.L = R.H.L

Or, Limit exists if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

If we want to know $\lim\limits_{x \to a} f(x) = ?$

It depends on two cases:





$$\boxed{\lim_{x \to a} f(x) = ?}$$

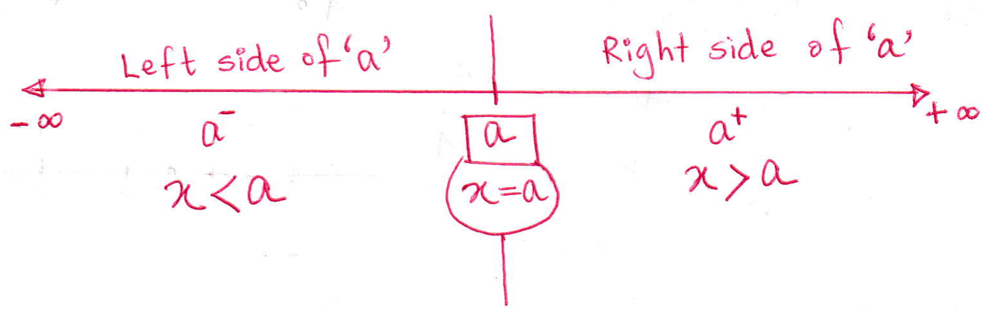
$$\boxed{\lim_{x \to a^-} f(x)}$$

Left hand limit (L.H.L)

$$\boxed{\lim_{x \to a^+} f(x)}$$

Right hand limit (R.H.L)

Limit exists if these two are equal.



This method is applicable for Continuous function as well.

Continuous Function:

A function $f(x)$ is said to be continuous if

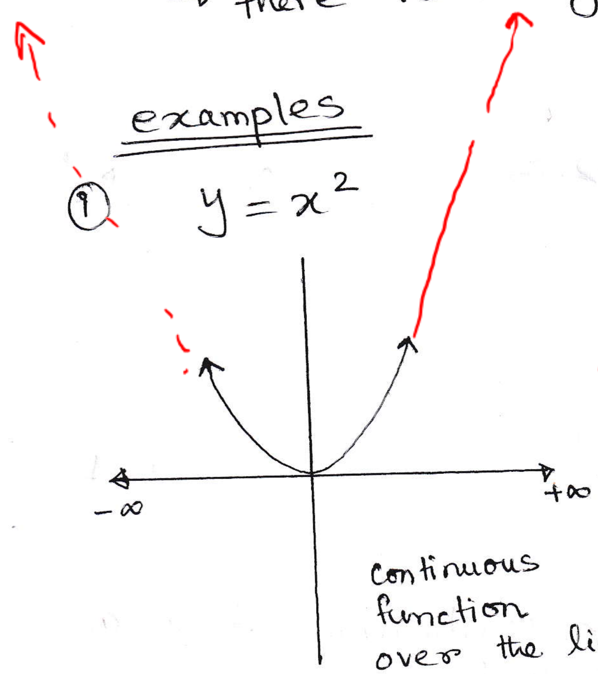
→ there is no discontinuity

OR

→ there is no gap in the graph

examples

(i) $y = x^2$



Continuous function over the limit $(-\infty, +\infty)$

$$\frac{n}{0} = \infty$$

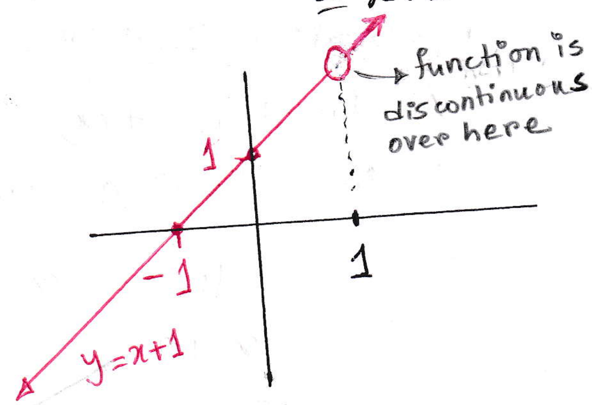
$$\frac{0}{n} = 0$$

(ii) $y = \dfrac{x^2-1}{x-1}$

$x \neq 1$ otherwise y will be undefined

Now $y = \dfrac{x^2-1}{x-1} = \dfrac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$= x+1$



Several Examples of Discontinuous functions:

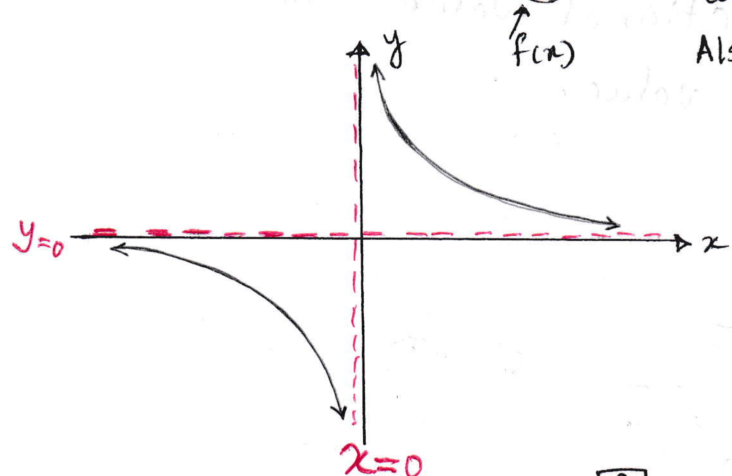
(a) $f(x) = \dfrac{1}{x}$ or $y = \dfrac{1}{x}$ → $x \neq 0$ otherwise function will be undefined.
$\underbrace{\text{function}}_{f(x)}$

$f(x)$

Also $y = \dfrac{1}{x} \Rightarrow xy = 1 \Rightarrow x = \dfrac{1}{y}$

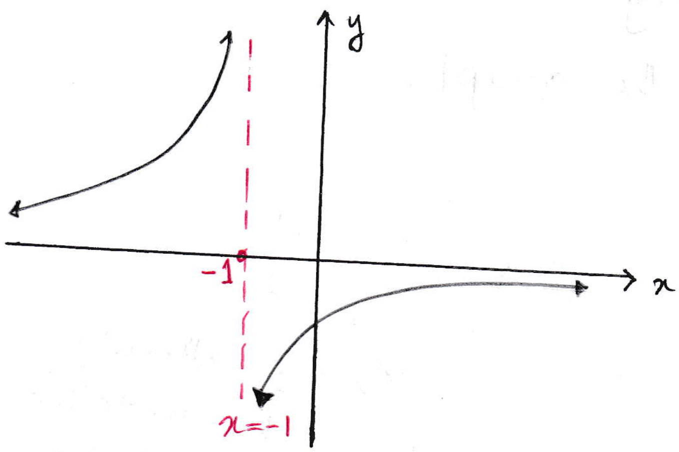
$f(y)$

$y \neq 0$ otherwise $f(y)$ will be undefined.



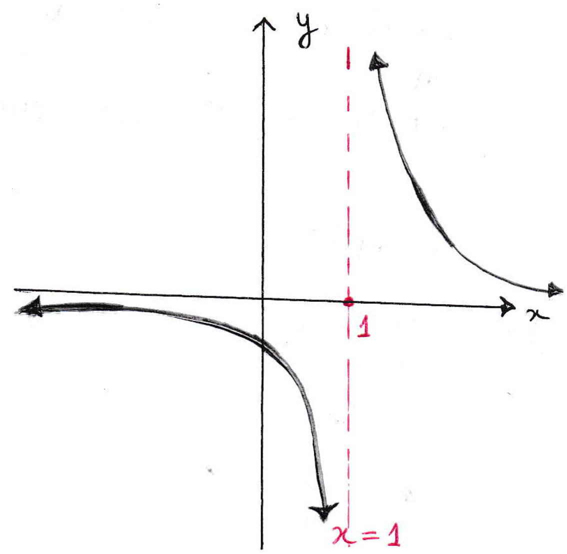
(b) $f(x) = \dfrac{x-4}{x+1}$

$x \neq -1$ o/w $f(x) = \infty$



Discontinuity at $x = -1$

(c) $f(x) = \dfrac{2}{x-1}$; $x \neq 1$



Discontinuity at $x = 1$

Mathematical Definition

A function $f(x)$ is said to be continuous at $x = a$ if the following condition is satisfied

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

Existance of Limit $= f(a)$

$\therefore$ Function is continuous if $L.H.L = R.H.L = f(a)$

So at $x = a$, the functional value should be equal to the limit value.

Rational Function:

Rational number: $\frac{p}{q}$, p & q both are integers and $q \neq 0$.

Rational function: $f(x) = \frac{p(x)}{q(x)} = \frac{\text{numerator}}{\text{denominator}}$

$= \frac{\text{constant or variable}}{\text{variable } (x\text{-type})}$

denominator must be independent variable type.

Examples

Rational Function

(a) $f(x) = \frac{x-12}{4x^2+x+1}$

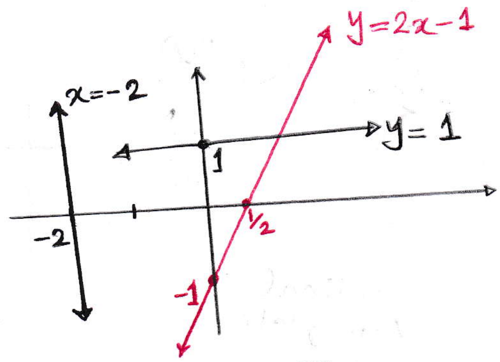
(b) $G(x) = \frac{8x^2-x+2}{4x^2-1}$

(c) $f(x) = \frac{2x^3+x^2-7x-3}{x^2-4}$

(d) $R(x) = \frac{1}{x^2-4}$

Not Rational Function

(e) $R(x) = \frac{x+1}{2} = \frac{1}{2}x + \frac{1}{2}$



Asymptotes

Slope	Lines	Equation	Examples
$m = \infty$	Vertical	$x = a$	$x=2$, $x=-4$, $x=0$, $x=\frac{1}{2}$ etc
$m = 0$	Horizontal	$y = b$	$y=-3$, $y=9$, $y=0$, $y=-\frac{1}{3}$ etc
$m\ (\pm)$	Oblique	$y = mx + c$	$y = 2x - 1$

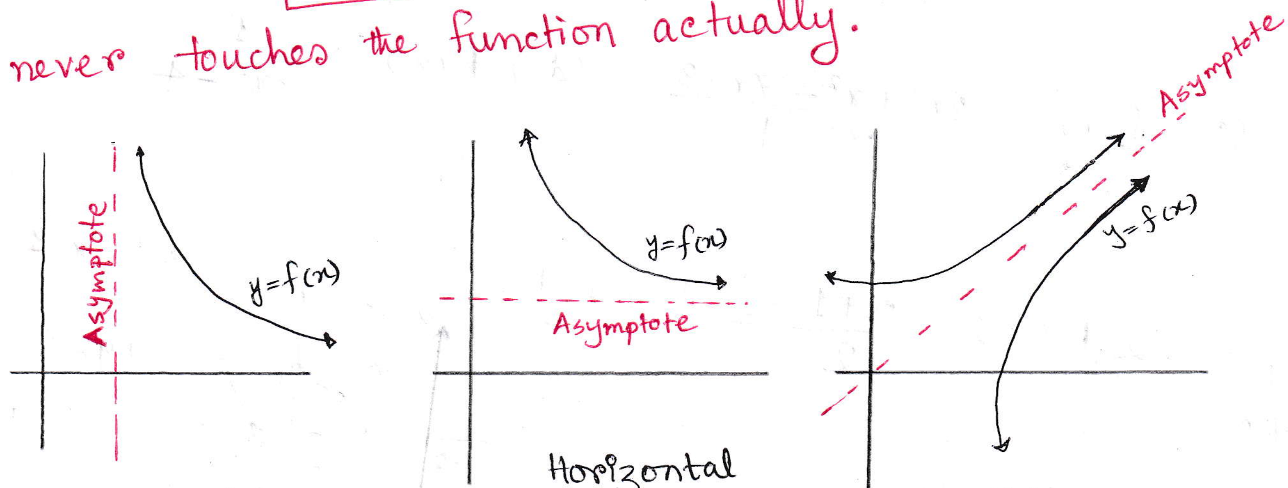
Classification of Asymptote

Asymptote

- Vertical Asymptote → $\boxed{x=a}$
- Horizontal Asymptote → $\boxed{y=b}$
- Oblique Asymptote → $\boxed{y=mx+c}$

Definition:

An asymptote is a straight line that seems to touch a function $\boxed{y=f(x)}$ at infinity $(+\infty / , -\infty)$, but never touches the function actually.



Vertical Asymptote

Horizontal Asymptote

Oblique Asymptote

We will need these TWO for MAT 110

Computing Limits:

Properties of Limit

Theorem: Let a be a real number, and suppose that

$$\lim_{x\to a} f(x) = L_1 \quad \text{and} \quad \lim_{x\to a} g(x) = L_2$$

That is, the limit exist and have values L_1 and L_2, respectively. Then:

(a) $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L_1 \pm L_2$

Addition OR Subtraction

(b) $\lim_{x\to a} [f(x)\, g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

Multiplication

(c) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, note that $L_2 \neq 0$

division

(e) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$; provided that $L_1 > 0$ if n is even

root

$(-2)^2 = +4$
$(+2)^2 = +4$

$(-2)^3 = -8$
$(+2)^3 = +8$

Ex: $\sqrt[4]{16} = 2$, $\sqrt[3]{8} = 2$
$\sqrt[4]{-16} = \infty$, $\sqrt[3]{-8} = -2$

These statement are also true for one-sided limits as $x \to a^-$ or as $x \to a^+$

Example:

$$\lim_{x \to 1} \left(x^2 + \frac{x^3+1}{x^2+1} + 2x \sin\left(\pi \sqrt{3x^2+1}\right)\right)$$

$$= 1^2 + \frac{1^3+1}{1^2+1} + 2(1) \sin\left(\pi \sqrt{3(1)^2+1}\right)$$

$$= 1 + 1 + 2\sin 2\pi = 2$$

Various Methods to Evaluate Limits:

$\boxed{i}$ Algebric Manipulation

$$\lim_{x \to 3} \frac{x^2-9}{x-3}$$

$$= \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} x+3$$

$$= 3+3$$

$$= 6$$

$$\lim_{x \to -4} \frac{2x+8}{x^2+x-12}$$

$$= \lim_{x \to -4} \frac{2x+8}{x^2+4x-3x-12}$$

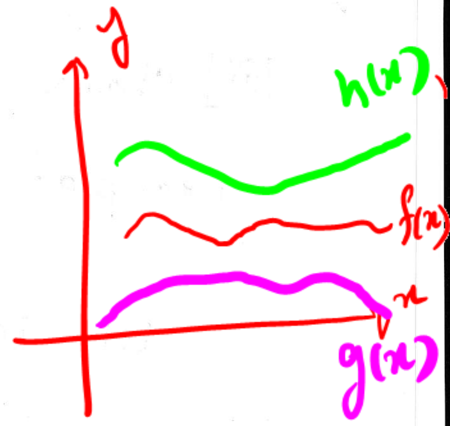
$$= \lim_{x \to -4} \frac{2x+8}{x(x+4)-3(x+4)}$$

$$= \lim_{x \to -4} \frac{2\cancel{(x+4)}}{\cancel{(x+4)}(x-3)}$$

$$= \lim_{x \to -4} \frac{2}{x-3}$$

$$= \frac{2}{-4-3} = -\frac{2}{7}$$

III Squeeze Theorem

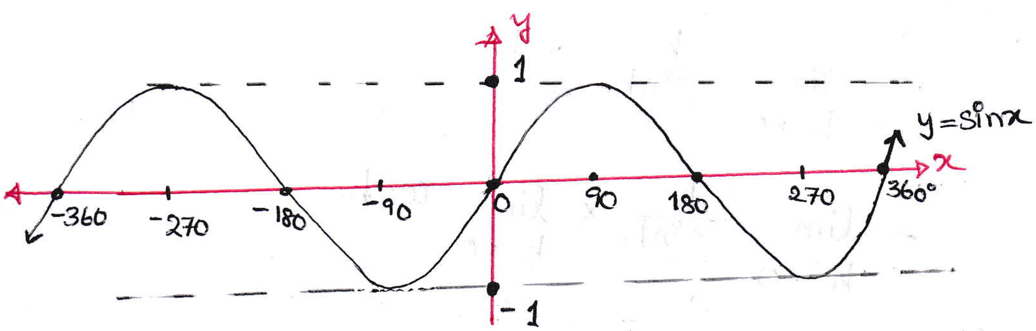


Suppose : $g(x) \leq f(x) \leq h(x)$

If $\lim_{x \to a} g(x) = L$ and $\lim_{x \to a} h(x) = L$

Then $\lim_{x \to a} f(x) = ? = L$

Example: $\lim_{x \to 0} x \sin \frac{1}{x} = ?$



We know, $-1 \leq \sin x \leq 1$

$\therefore -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$\Rightarrow -1(x) \leq (x) \sin\left(\frac{1}{x}\right) \leq 1(x)$ multiply by x in all sides to create the main (given) function

$\Rightarrow -x \leq x \sin\left(\frac{1}{x}\right) \leq x \longrightarrow g(x) \leq f(x) \leq h(x)$

comparing

$\lim_{x \to 0} (-x) = 0$

$\lim_{x \to 0} (x) = 0$

$\lim_{x \to 0} \left(x \sin \frac{1}{x}\right) = 0$ by Squeez Theorem.

iii) Change of Variable

Example: $\lim_{x\to 0} \frac{\tan^{-1}x}{x} = ?$

Let $\tan^{-1}x = h$ introducing new variable 'h'

$\therefore x = \tan h$

when $x \to 0$ then $h \to 0$

Now $\lim_{x\to 0} \frac{\tan^{-1}x}{x} = \lim_{h\to 0} \frac{h}{\tan h}$

$= \lim_{h\to 0} \frac{h}{\frac{\sin h}{\cos h}}$

$= \lim_{h\to 0} \frac{h}{\sin h} \cdot \cos h$

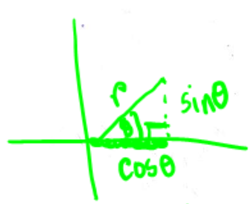
$\lim_{h\to 0} \frac{h}{\sin h} = ?$ $\quad = \lim_{h\to 0} \frac{h}{\sin h} = \frac{0}{0} \times \lim_{h\to 0} \cos h$

Consider $1 \le \frac{h}{\sin h} \le \frac{1}{\cos h}$ $= \underbrace{1 \times 1}_{\text{by Squeeze Theorem}} = 1$

$\lim_{h\to 0} 1 = 1$ $\quad \lim_{h\to 0} \frac{1}{\cos h} = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$

So, by squeeze theorem

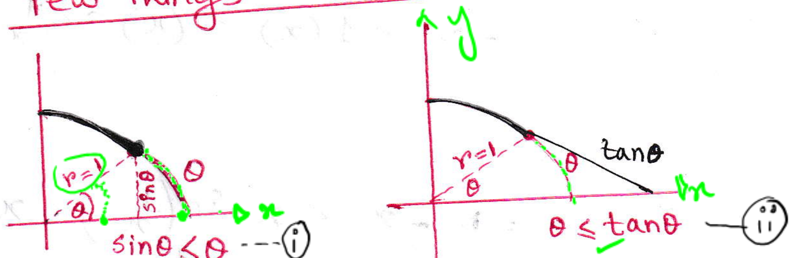
$\lim_{h\to 0} \frac{h}{\sin h} = 1$



$x = r\cos\theta$
$y = r\sin\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

Few things for Squeeze Theorem:



$\sin\theta \le \theta$ --- (i) $\quad \theta \le \tan\theta$ --- (ii)

by adjusting the inequalities:

$\sin\theta \le \theta \le \tan\theta$ → comparing (i) & (ii)

$\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$

$\frac{\sin\theta}{\sin\theta} \le \frac{\theta}{\sin\theta} \le \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sin\theta}$

$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$

Also $1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta$ OR $\cos\theta \le \frac{\sin\theta}{\theta} \le 1$

iv L'Hôspital's Rule:

We need to know all the rules of differentiation

If we have to evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ and

initially if we have $\lim_{x\to a} \frac{f(x)}{g(x)} = \underbrace{\frac{0}{0} \text{ or } \frac{\infty}{\infty}}$

This form is known as Indeterminate form

Note

✱ $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$

✱ If the limiting part becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ again, then differentiation will be continued.

Example:

(a) $\lim_{x\to 1} \frac{\ln x}{x-1}$ $\quad \frac{\ln 1}{1-1} = \frac{0}{0}$ form

$= \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)}$

$= \lim_{x\to 1} \frac{1/x}{1-0}$

$= \lim_{x\to 1} \frac{1}{x}$

$= \frac{1}{1} = 1$

(b) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x}$ $\quad \frac{1}{0} - \frac{1}{\sin 0^\circ} = \infty - \infty$ $= \infty$

$= \lim_{x\to 0} \frac{\sin x - x}{x \sin x}$ $\quad \frac{\sin 0^\circ - 0}{0 \sin 0^\circ} = \frac{0-0}{0} = \frac{0}{0}$ form

$= \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x - x)}{\frac{d}{dx}(x \sin x)}$

$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x}$ $\quad \frac{\cos 0^\circ - 1}{\sin 0^\circ + 0 \cdot \cos 0^\circ} = \frac{1-1}{0+0} = \frac{0}{0}$ form again

$= \lim_{x\to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx}(\sin x + x\cos x)}$

$= \lim_{x\to 0} \frac{-\sin x - 0}{\cos x + \cos x - x\sin x}$

$= \frac{0}{1+1-0} = \frac{0}{2} = 0.$