Gamma – Beta Function

Derivation of Gamma function

$$\int_0^\infty e^{-Ax}\,dx = \left.\frac{e^{-Ax}}{-A}\right]_0^\infty$$

$$= -\frac{1}{A}\left(e^{-\infty} - e^{0}\right)$$

$$= -\frac{1}{A}(0-1)$$

$$\therefore \int_0^\infty e^{-Ax}\,dx = \frac{1}{A} \quad \text{—— (I)}$$

Differentiate both sides with respect to A

$$\frac{d}{dA}\int_0^\infty e^{-Ax}\,dx = \frac{d}{dA}\left(\frac{1}{A}\right)$$

$$\Rightarrow \int_0^\infty \frac{d}{dA} e^{-Ax}\,dx = -\frac{1}{A^2}$$

$$\Rightarrow \int_0^\infty (-x)\, e^{-Ax}\,dx = -\frac{1}{A^2}$$

$$\Rightarrow \int_0^\infty x\, e^{-Ax}\,dx = \frac{1}{A^2} \quad \text{—— (II)}$$

Differentiate (II) w. r. to A

$$\int_0^\infty x^2 e^{-Ax}\,dx = \frac{2}{A^3} = \frac{2!}{A^{2+1}} \quad \text{—— (III)}$$

Similarly differentiate w.r.to A

$$\int_0^\infty x^3 e^{-Ax} dx = \frac{6}{A^4} = \frac{3!}{A^{3+1}} \quad \text{——— (iv)}$$

Similarly

$$\boxed{\int_0^\infty x^n e^{-Ax} dx = \frac{n!}{A^{n+1}}} \quad \text{——— (*)}$$

Putting A=1 in (*) we get,

$$\boxed{\int_0^\infty x^n e^{-x} dx = n!} \; ; n > 0 \text{ and } n \text{ is a integer.}$$

which is known the Euler integral formula of Second kind.

Now for any $n > 0$ (integer or fraction)

$$\int_0^\infty x^n e^{-x} dx = \Gamma(n+1)$$

$$\boxed{\Rightarrow \int_0^\infty x^{n-1} e^{-x} dx = \Gamma n}$$

which is known Gamma function / Euler integral formula of 2nd kind.

Formula

① $\Gamma n = (n-1)!$

② $\Gamma n = (n-1)\,\Gamma(n-1)$

~~③ $\Gamma 0$~~

③ $\Gamma \frac{1}{2} = \sqrt{\pi}$

④ $\Gamma 1 = 1$

⑤ $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Problem

Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[6]{x}}\, dx$ using Gamma function.

Solution:

Let $\sqrt[6]{x} = t \Rightarrow x = t^6$

$dx = 6t^5 dt$

Limit	
x	t
0	0
∞	∞

Now $$\int_0^\infty \sqrt{x}\, e^{-\sqrt[6]{x}}\, dx = \int_0^\infty \sqrt{t^6}\; e^{-t} \cdot 6t^5 dt$$

$$= \int_0^\infty t^3 e^{-t}\, 6t^5\, dt$$

$$= 6\int_0^\infty e^{-t} \cdot t^8\, dt$$

$$= 6 \times \Gamma 9$$

$$= 6 \times 8! \text{ Ans.}$$

Evaluate $\int_0^\infty e^{-x^2} dx$.

Solution:

Let $x^2 = z$

$\Rightarrow 2x\,dx = dz$

$\Rightarrow dx = \frac{1}{2x} dz$

$\therefore dx = \frac{1}{2\sqrt{z}} dz$

limit

x	z
0	0
∞	∞

Now $\int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-z} \frac{1}{2\sqrt{z}} dz$

$$= \frac{1}{2}\int_0^\infty z^{-1/2} e^{-z}\, dz$$

$$= \frac{1}{2} \Gamma\left(-\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2} \Gamma\tfrac{1}{2}$$

$$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \text{ Ans.}$$

Use Gamma function to evaluate $\int_0^1 \frac{1}{\sqrt{x \ln(1/x)}} dx$

solution:

$$\int_0^1 \frac{dx}{\sqrt{x \ln(1/x)}} = \int_0^1 \frac{dx}{\sqrt{x \ln(x^{-1})}} = \int_0^1 \frac{dx}{\sqrt{x(-\ln x)}}$$

Let,

$u = -\ln x \Rightarrow -u = \ln x$

$\Rightarrow e^{-u} = x$

$\therefore x = e^{-u}$

$\Rightarrow dx = -e^{-u} du$

Limit

x	u
0	∞
1	0

Now $$\int_0^1 \frac{dx}{\sqrt{x(-\ln x)}} = \int_\infty^0 \frac{-e^{-u} du}{\sqrt{e^{-u} u}}$$

$$= -\int_\infty^0 \frac{e^{-u}}{e^{-u/2} . u^{1/2}} du$$

$$= \int_0^\infty e^{-u+u/2} . u^{-1/2} du$$

$$= \int_0^\infty e^{-u/2} . u^{-1/2} du$$

Again let,

$v = \frac{u}{2}$

$dv = \frac{1}{2} du$

limit

u	v
0	0
∞	∞

$$= \int_0^\infty e^{-v} (2v)^{-1/2} . 2 dv$$

$$= \int_0^\infty e^{-v} \cdot v^{-1/2} \frac{2}{\sqrt{2}} dv$$

$$= \sqrt{2} \int_0^\infty e^{-v} v^{\frac{1}{2}-1} dv$$

$$= \sqrt{2}\, \Gamma\tfrac{1}{2}$$

$$= \sqrt{2\pi} \text{ Ans.}$$

Beta function / ~~First Euler's Integral~~

Formula :

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\frac{p+1}{2} \, \Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Evaluate

$$\int_0^{\pi/6} \sin^2 6x \cos^4 3x \, dx$$

$$= \int_0^{\pi/6} (\sin 2\cdot 3x)^2 \cos^4 3x \, dx$$

$$= \int_0^{\pi/6} (2\sin 3x \cos 3x)^2 \cos^4 3x \, dx$$

$$= 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x \, dx$$

Let,

$\theta = 3x$		Limit	
$d\theta = 3dx$		x	θ
		0	0
$\therefore \frac{1}{3}d\theta = dx$		$\pi/6$	$\pi/2$

$$\therefore 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x \, dx = \frac{4}{3}\int_0^{\pi/2} \sin^2\theta \cos^6\theta \, d\theta$$

$$= \frac{4}{3} \frac{\Gamma\frac{2+1}{2} \, \Gamma\frac{6+1}{2}}{2\,\Gamma\frac{2+6+2}{2}}$$

$$= \frac{2}{3} \frac{\Gamma 3/2 \, \Gamma 7/2}{\Gamma 5}$$

$$= \frac{2}{3} \frac{1/2\,\Gamma 1/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2}{4!} = \frac{5\pi}{192}$$

Note

$$\Gamma 7/2 = (7/2 - 1)\,\Gamma 7/2 - 1$$

$$= 5/2\,\Gamma 5/2$$

$$= 5/2 \cdot 3/2 \cdot \Gamma 3/2$$

reduction formula

Beta function / First Euler formula

① $B(m,n) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx \;;\; m,n>0$

② Symmetric: $B(m,n) = B(n,m)$

③ Relation Between Gamma & Beta function

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

<u>Problem</u> Evaluate $\int_0^1 \left(1-\frac{1}{x}\right)^{1/3} dx$ using Euler's integral of first kind.

<u>Solution:</u>

Given $\int_0^1 \left(1-\frac{1}{x}\right)^{1/3} dx$

$$= \int_0^1 \left(\frac{x-1}{x}\right)^{1/3} dx$$

$$= \int_0^1 x^{-1/3}(x-1)^{1/3}\,dx$$

$$= -\int_0^1 x^{-1/3}(1-x)^{1/3}\,dx$$

$$= -\int_0^1 x^{(-\frac{1}{3}+1)-1}(1-x)^{(\frac{1}{3}+1)-1}\,dx$$

$$= -\int_0^1 x^{2/3-1}(1-x)^{4/3-1}\,dx$$

$$= -\int_0^1 x^{\frac{2}{3}-1} (1-x)^{\frac{4}{3}-1} \, dx$$

$$= -\beta\left(\tfrac{2}{3}, \tfrac{4}{3}\right)$$

$$= -\frac{\Gamma(2/3) \cdot \Gamma 4/3}{\Gamma\left(2/3 + 4/3\right)}$$

$$= -\frac{\Gamma 2/3 \cdot \Gamma 4/3}{\Gamma 2}$$

$$= -\Gamma 2/3 \cdot \Gamma 4/3 \quad \left[\because \Gamma 2 = 1! = 1\right]$$