Weak and Semi-Contraction for Network Systems and Diffusively-Coupled Oscillators

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***Abstract*—We develop two generalizations of contraction theory, namely, semi-contraction and weak-contraction theory. First, using the notion of semi-norm, we propose a geometric framework for semi-contraction theory. We introduce matrix semi-measures and characterize their properties. We show that the spectral abscissa of a matrix is the infimum over weighted semi-measures. For dynamical systems, we use the semi-measure of their Jacobian to characterize the contractivity properties of their trajectories. Second, for weakly contracting systems, we prove a dichotomy for the asymptotic behavior of their trajectories and novel sufficient conditions for convergence to an equilibrium. Third, we show that every trajectory of a doubly-contracting system, i.e., a system that is both weakly and semi-contracting, converges to an equilibrium point. Finally, we apply our results to various important network systems including affine averaging and affine flow systems, continuous-time distributed primal-dual algorithms, and networks of diffusively-coupled dynamical systems. For diffusively-coupled systems, the semi-contraction theory leads to a sufficient condition for synchronization that is sharper, in general, than previously-known tests.**

***Index Terms*—contraction theory, stability analysis, synchronization**

Lohmiller and Slotine [5]. We refer to [6] for a historical review and to the surveys [7], [8] for recent developments and applications to consensus and synchronization in complex networks.

Several generalizations of contraction theory have been proposed in the literature. In [9], the notion of partial contraction is introduced to study convergence of system trajectories to a specific behavior or to a manifold. The idea is to impose contractivity only on a part of the states of the system. Partial contraction with respect to the ℓ_2-norm has been further developed in [10], [11] to study synchronization in complex networks. A similar notion is studied by [8] in the context of convergence to invariant subspaces. Extensions of contraction theory to non-Euclidean norms and metrics have been explored in the context of monotone dynamical systems. For compartmental systems, contractivity with respect to the ℓ_1-norm has been used to study convergence to the equilibrium point [12]. For monotone systems, [13] uses the contractivity of the so-called Hilbert metric to propose a nonlinear gen-