SIAM REVIEW
Vol. 64, No. 1, pp. 000–000

Flow and Elastic Networks on the n-Torus: Geometry, Analysis, and Computation*

Saber Jafarpour†
Elizabeth Y. Huang†
Kevin D. Smith†
Francesco Bullo†

Abstract. Networks with phase-valued nodal variables are central in modeling several important societal and physical systems, including power grids, biological systems, and coupled oscillator networks. One of the distinctive features of phase-valued networks is the existence of multiple operating conditions corresponding to critical points of an energy function or feasible flows of a balance equation. For networks with phase-valued states it is not yet fully understood how many operating conditions exist, how to characterize them, and how to compute them efficiently. A deeper understanding of feasible operating conditions, including their dependence upon network structures, may lead to more reliable and efficient network systems.

This paper introduces flow and elastic network problems on the n-torus and provides a rigorous and comprehensive framework for their study. Based on a monotonicity assumption, this framework localizes the solutions, bounds their number, and leads to an algorithm to compute them. Our analysis is based on a novel *winding partition* of the n-torus into winding cells, induced by *Kirchhoff's angle law* for undirected graphs. The winding partition has several useful properties, e.g., each winding cell contains at most one solution. The proposed algorithm is based on a novel contraction mapping and is