On small-time local controllability

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Abstract

Consider a real analytic control system Σ which is small-time locally controllable from x_0 and, for small enough time, its reachable set from x_0 has a polynomial growth rate of order N with respect to time. We show that every other real analytic control system, with the property that its vector fields have the same Taylor polynomials of order N around x_0 as Taylor polynomials of the vector fields of Σ, is small-time locally controllable from x_0. In particular, this result connects two well-known conjectures about small-time local controllability of real analytic systems.

Keywords. Small-time local controllability, normal reachability, multi-valued maps, Brouwer fixed-point theorem, half-continuous maps.

1 Introduction

Consider a control system $\Sigma = \{X_1, X_2, \ldots, X_m\}$ which is defined by a finite number of real analytic vector fields on $\mathbb{R}^n$. Starting from a point x_0 in the state space $\mathbb{R}^n$, the reachable set of Σ at times less than T, which is denoted by $\mathrm{R}_\Sigma(< T, x_0)$, is defined as the set of all points on the state space which can be reached by starting from x_0 and traveling, in positive times, on the trajectories of these vector fields for times less than T. In other words, we have

$$\mathrm{R}_\Sigma(< T, x_0) = \{\phi_{t_1}^{X_{i_1}} \circ \phi_{t_2}^{X_{i_2}} \circ \ldots \circ \phi_{t_s}^{X_{i_s}}(x_0) \mid s \in \mathbb{N}, t_i \geq 0, \ X_{i_j} \in \Sigma, \ \sum_{i=1}^{s} t_i < T\}.$$

Using the reachable sets of a control system, one can define different notions of controllability for the system. In mathematical theory of control, the notions of "local accessibility" and "small time local controllability" are of fundamental importance. A control system Σ is called locally accessible from x_0, if, for small enough time t, $\mathrm{R}_\Sigma(< t, x_0)$ has a nonempty

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interior and a control system Σ is called small-time locally controllable from x_0 if, for small enough time t, $\mathrm{R}_\Sigma(<t, x_0)$ contains a neighbourhood of x_0.

In order to study controllability of a control system, one needs to study the reachable sets of the system. However, a complete analytic description of reachable sets of a control system requires solving a family of nonlinear differential equations, which is in the best case very difficult, if not impossible. In the past few decades, control theorists have developed methods and techniques for studying controllability of a control system using the family of vector fields of the system. In 1972, Sussmann and Jurdjevic showed that one can characterized local accessibility of a real analytic control system using the Lie brackets of the vector fields of the system [28, Corollary 4.7]. For small-time local controllability, many deep sufficient conditions (cf. [18], [25], [26], [27], [12]) as well as some necessary conditions (cf. [23], [27], [14], [17]) have been developed in the literature. However, as to our knowledge, small-time local controllability can only be characterized for some specific classes of systems (cf. [28], [21], [4]). Therefore, in general, the gap between the necessary and sufficient conditions for small-time local controllability of a system is huge.

One of the useful notions in studying small-time local controllability is the control variation. Control variations can be considered as suitable tools for approximating the reachable sets of control systems. The idea is to construct high-order tangent vectors using the families of control variations. It can be shown that the high-order tangent vectors are the admissible directions in the reachable set of a control system, i.e., for small enough time, one can travel in the reachable set of the system in those directions [15, Theorem 2.4]. Therefore, by constructing enough admissible control variations such that their associated high-order tangent vectors generate all the directions in $\mathbb{R}^n$, one can show, using a classical argument in topological degree theory, that the control system is small-time locally controllable [15, Corollary 2.5]. In the control literature, many different families of control variations are constructed for studying small-time local controllability of systems (cf. [19], [27], [9], [15], [5], [4]). Control variations can also be used to study the rate of growth of the reachable set of a control system with respect to time. In fact, the order of a control variation, which is defined as the order of its associated high-order tangent vector, gives us some information about how fast one can travel in the reachable set in that direction. More specifically, if one can get all direction in $\mathbb{R}^n$ using families of control variations of order less than N then, for small enough t, $\overline{\mathrm{B}}(x_0, t^N)$ is contained in $\mathrm{R}_\Sigma(<t, x_0)$ [15, Corollary 2.5].

As mentioned above, finding suitable families of control variations whose high-order tangent vectors generate the whole $\mathbb{R}^n$ guarantees small-time local controllability of the control system from x_0. However, for a small-time locally controllable system, it is interesting to investigate whether there exist appropriate families of control variations which can be used to prove small-time local controllability of the system. It is not surprising to see that this question has a close connection with the rate of growth of reachable sets of the system with respect to time. In [1], the following conjecture has been proposed.

Conjecture 1.1. Let Σ be a real analytic control system which is small-time locally con-

trollable from x_0. Then there exists $N \in \mathbb{N}$ such that, for small enough t, we have

$$\overline{\mathrm{B}}(x_0, t^N) \subseteq \mathrm{int}\left(\mathrm{R}_\Sigma(< t, x_0)\right).$$

One can easily check that if the answer to Conjecture 1.1 is positive then, for every small-time locally controllable system, one can find families of admissible control variations whose high-order tangent vectors generate all the directions in $\mathbb{R}^n$.

It is well-known that a real analytic control system Σ is locally accessible from x_0 if and only if $\mathrm{Lie}(\Sigma)(x_0) = T_{x_0}M$ [28]. However, in order to compute $\mathrm{Lie}(\Sigma)(x_0)$, one needs to know the vector fields of the system and all their derivatives at the point x_0. Since $T_{x_0}M$ is a finite-dimensional vector space, it is easy to see that, for a given real analytic control system, locally accessibility of the system Σ at the point x_0 can be checked using only "finite" number of differentiation of vector fields of the system at x_0. This raises an interesting question of whether small-time local controllability from the point x_0 of a given real analytic system can be characterized using finite number of differentiations of vector fields of the system at the point x_0. More precisely, one can formulate the following conjecture [1].

Conjecture 1.2. Given a real analytic control system Σ on $\mathbb{R}^n$ which is small-time locally controllable from x_0, there exists $N \in \mathbb{N}$ such that every other real analytic system with the same Taylor polynomials of order N around x_0 as Taylor polynomials of Σ is also small-time locally controllable from x_0.

As mentioned in [1], a positive answer to Conjecture 1.1 and Conjecture 1.2 for the case $n = 2$ follows from the results of [21]. However, as to our knowledge, these two conjectures are still open for the general case, when $n \geq 3$.

Suppose that one can prove small-time local controllability of a control system Σ using a family of control variations [15]. Then, by [15, Corollary 2.5], there exists an $N \in \mathbb{N}$ such that, for small enough t, we have

$$\overline{\mathrm{B}}(x_0, t^N) \subseteq \mathrm{int}\left(\mathrm{R}_\Sigma(< t, x_0)\right).$$

It is interesting to investigate the validity of Conjecture 1.2 for this control system. One can easily show that if the family of control variations used for proving the small-time local controllability of Σ are the classical ones constructed using the iterated Lie brackets of vector fields of the system (e.g. the control variations in [27]), then Conjecture 1.2 holds for the control system Σ. However, there exist control systems which are small-time locally controllable from x_0, but one cannot check small-time local controllability of them using classical control variations constructed by the iterated family of Lie brackets [15, Example 6.1]. Example 1.3 shows that one may need a more complicated family of variations to prove small-time local controllability of a control system.

Example 1.3. We consider the control system Σ defined by

$$
\begin{aligned}
\dot{x}_1 &= u(t), \\
\dot{x}_2 &= x_1, \\
\dot{x}_3 &= x_1^3, \\
\dot{x}_4 &= x_3^2 - x_2^7,
\end{aligned}
$$

where $u : \mathbb{R} \to [-\epsilon, \epsilon]$. We want to study small-time local controllability of Σ from $\mathbf{0} \in \mathbb{R}^4$. Using suitable families of control variations with finite number of switching, one can show that $\left\{ \pm \frac{\partial}{\partial x^1}, \pm \frac{\partial}{\partial x^2}, \pm \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4} \right\}$ are the admissible directions in the reachable set of the systems Σ [15], [27]. Therefore, in order to prove small-time local controllability of Σ, one needs to find a family of variations which generates the direction $-\frac{\partial}{\partial x^4}$. It can be shown that there is no family of control variations with finite number of switching which gives the direction $-\frac{\partial}{\partial x^4}$ [15, Claim 6.2]. More specifically, assume that N is the number of switching for the control variation u. If we have $x_1(T, u) = 0$ and $x_4(T, u) < 0$, then we have

$$
N > \epsilon^{\frac{-3}{28}} T^{\frac{-1}{2}}.
$$

However, by defining a family of control variations with an increasing number of switching, which is called fast switching variations, one can show that Σ is small-time locally controllable from $\mathbf{0} \in \mathbb{R}^4$ [15, Claim 6.1]. Let $u : [0, 4 + 2\sqrt{2}] \to [-1, 1]$ be defined as

$$
u(t) = \begin{cases} 1 & t \in [0, 1) \cup [2 + \sqrt{2}, 3 + 2\sqrt{2}), \\ -1 & \text{otherwise.} \end{cases}
$$

Then, for every $k \in \mathbb{N}$, we define a family of controls $u^{(k)} : [0, 2k(4 + 2\sqrt{2})] \to [-1, +1]$ inductively as

$$
u^{(k+1)}(t) = u * u^{(k)} * (-u).
$$

where $u * v$ is the concatenation of controls u and v. By setting $t_0 = \frac{\delta}{2kT}$, the fast switching control variations $u_{\epsilon, \delta}^{(k)} : [0, t_0] \to [-\epsilon, \epsilon]$ are defined as

$$
u_{\epsilon, \delta}^{(k)}(t) = \epsilon u^{(k)}\left(\frac{t}{\delta}\right), \qquad \forall \delta > 0.
$$

By choosing $k = C\epsilon^{-1} t_0^{-6}$, it can be shown that this family of control variations generates the direction $-\frac{\partial}{\partial x^4}$ [15]. Therefore, the system Σ is small-time locally controllable from x_0. Also, it can be shown that, for small enough t, we have

$$
\overline{\mathrm{B}}(\mathbf{0}, t^{58}) \subseteq \mathrm{int}\left(\mathrm{R}_\Sigma(< t, \mathbf{0})\right).
$$

Now consider the control system Σ' defined by

$$
\begin{aligned}
\dot{x}_1 &= u(t) + x_1^{60}, \\
\dot{x}_2 &= x_1, \\
\dot{x}_3 &= x_1^3, \\
\dot{x}_4 &= x_3^2 - x_2^7,
\end{aligned}
$$

where $u : \mathbb{R} \to [-\epsilon, \epsilon]$. Note that Σ and Σ' have the same Taylor polynomial of order 57 at $\mathbf{0} \in \mathbb{R}^4$. Using the classical finite switching control variations, it is easy to show that $\left\{ \pm \frac{\partial}{\partial x^1}, \pm \frac{\partial}{\partial x^2}, \pm \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4} \right\}$ are the admissible directions in the reachable set Σ'. However, using the same fast switching variations as for the system Σ, it is very complicated to study whether $-\frac{\partial}{\partial x^4}$ is an admissible direction for the reachable sets of the control system Σ' at point $\mathbf{0} \in \mathbb{R}^4$.

In general, the families of control variations that are used to prove small-time local controllability of a system may be even more complicated than the fast switching control variations in Example 1.3. Therefore, for small-time locally controllable systems, studying Conjecture 1.2 using a family of control variations for the system does not seem to be conclusive.

In this paper, we study a real analytic system Σ which is small-time locally controllable from x_0 and, for small enough t, $\overline{\mathrm{B}}(x_0, t^N)$ is contained in the interior of $\mathrm{R}_\Sigma(< t, x_0)$. Let Θ be another real analytic system with the property that the Taylor polynomial of order N around x_0 of its vector fields is the same as the Taylor polynomials of the vector fields of Σ. We show that Θ is also small-time locally controllable from x_0. In particular, our main theorem shows that a positive answer to Conjecture 1.1 implies a positive answer to Conjecture 1.2. In this paper, we only study small-time local controllability which is a local property of a system. Therefore, without loss of generality, we can assume that the state space of the control system Σ is an Euclidean space. Moreover, for the sake of simplicity, we define the control system Σ as finite family of real analytic vector fields $\Sigma = \{X_1, X_2, \ldots, X_m\}$. It is easy to see that the extension of the analysis in this paper for a control system defined by a parametrized family of real analytic vector fields $\{f_u\}_{u \in \mathcal{U}}$ on a real analytic manifold M is straightforward.

Let Σ be a real analytic control system such that, for small enough time t, the interior of $\mathrm{R}_\Sigma(< t, x_0)$ contains $\overline{\mathrm{B}}(x_0, t^N)$. Consider another control system Θ such that every vector field of Θ has the same Taylor polynomial of order N at point x_0 as the Taylor polynomial of a vector field of Σ. Then, using the normal reachability results in [10], for every small enough t, one can construct a multi-valued mapping $F_{\Sigma, \Theta}^t$ defined between reachable sets of the two control systems Σ and Θ. This mapping assigns to every point in the reachable set of Σ a finite collection of points in the reachable set of Θ. In order to prove small-time local controllability of the system Θ from x_0, one would like to show that, for small enough time t, the range of the multi-valued mapping $F_{\Sigma, \Theta}^t$ contains a neighbourhood of x_0 in $\mathbb{R}^n$.

For a single-valued differentiable mapping, one can study local openness of the range of the mapping using the derivatives of that mapping [11, Chapter 1, §4]. In the case that the mapping is only Lipschitz, one can use a topological argument, based on the Brouwer fixed-point theorem, to show the local openness of the range of the mapping [3, Lemma 12.4]. This topological argument plays an essential role in studying small-time local controllability of systems using a family of control variations (cf. [5], [4]).

Brouwer fixed-point theorem is one of the most fundamental theorems in topology.

Roughly speaking, the Brouwer fixed-point theorem says that a continuous map, from a compact convex subset of $\mathbb{R}^n$ into itself, has a fixed-point [11, Chapter 2, §2]. There are many different generalization of the Brouwer fixed-point theorem in the literature. In [6], the Brouwer fixed-point theorem has been generalized for discontinuous mappings which satisfy a condition called "half-continuity".

In order to study local openness of the range of the map $F_{\Sigma,\Theta}^t$, we focus on its single-valued selections. While, finding a continuous single-valued selection of the multi-valued mapping $F_{\Sigma,\Theta}^t$ seems elusive, one can show that $F_{\Sigma,\Theta}^t$ admits a "half-continuous" single-valued selection. Thus, using the generalized version of the Brouwer fixed-point theorem, it can be shown that the range of the mapping $F_{\Sigma,\Theta}^t$ contains an open neighbourhood of x_0 for all small enough t.

2   Mathematical background

In this section, we briefly review some of the mathematical notions essential for proving our main result.

2.1   Real analytic functions and vector fields

We first define the notations essential for studying partial derivatives of differentiable mappings on $\mathbb{R}^n$.

Definition 2.1. A **multi-index of order** m is an element $(r) = (r_1, r_2, \ldots, r_m) \in (\mathbb{Z}_{\geq 0})^m$. For all multi-indices (r) and (s) of order m, every $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$, and $f : \mathbb{R}^m \to \mathbb{R}^n$, we define

$$
\begin{aligned}
|(r)| &= r_1 + r_2 + \ldots + r_m, \\
(r)! &= (r_1!)(r_2!) \ldots (r_m!), \\
x^{(r)} &= x_1^{r_1} x_2^{r_2} \ldots x_m^{r_m}, \\
D^{(r)} f(x) &= \frac{\partial^{|r|} f}{\partial x_1^{r_1} \partial x_2^{r_2} \ldots \partial x_m^{r_m}}.
\end{aligned}
$$

The space of all decreasing sequences $\{a_i\}_{i \in \mathbb{N}}$ such that $a_i \in \mathbb{R}_{>0}$ and $\lim_{n \to \infty} a_n = 0$ is denoted by $\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$.

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}^{>0}$. Then we define

$$
\mathrm{B}(x, r) = \{ y \in \mathbb{R}^n \mid \|y - x\| < r \},
$$

and

$$
\overline{\mathrm{B}}(x, r) = \{ y \in \mathbb{R}^n \mid \|y - x\| \leq r \},
$$

Definition 2.2. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $x_0 \in \Omega$, and $f : \Omega \to \mathbb{R}^l$ be a C^∞-mapping at x_0. Then the **Taylor series** of f at x_0 is the power series

$$\sum_{(r) \in (\mathbb{Z}_{\geq 0})^m} \frac{1}{(r)!} \left[D^{(r)} f(x_0) \right] (x - x_0)^{(r)}. \tag{2.1}$$

A smooth mapping $f : \Omega \to \mathbb{R}^l$ is **real analytic** or **of class** C^ω if, for every $x_0 \in \Omega$, there exists $\rho > 0$ such that the Taylor series (2.1) of f at x_0 converges to $f(x)$ for all $\|x - x_0\| < \rho$. A mapping $f : \Omega \to R^l$ is **real analytic** on Ω if, for every $x \in \Omega$, it is real analytic at x.

The space of all real analytic functions on $\mathbb{R}^n$ is denoted by $C^\omega(R^n)$ and the space of all real analytic vector fields on $\mathbb{R}^n$ is denoted by $\Gamma^\omega(T\mathbb{R}^n)$. It is easy to see that both $C^\omega(\mathbb{R}^n)$ and $\Gamma^\omega(T\mathbb{R}^n)$ are vector spaces.

Definition 2.3. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Then a map $X : \mathbb{T} \times \mathbb{R}^n \to T\mathbb{R}^n$ is a **time-varying real analytic vector field** if, for every $t \in \mathbb{T}$, the map $X^t : \mathbb{R}^n \to T\mathbb{R}^n$ defined by

$$X^t(x) = X(t, x), \qquad \forall x \in \mathbb{R}^n,$$

is a real analytic vector field (i.e., $X^t \in \Gamma^\omega(T\mathbb{R}^n)$).

Let $X : \mathbb{T} \times \mathbb{R}^n \to T\mathbb{R}^n$ be a time-varying real analytic vector field. Then one can define the associated curve $\widehat{X} : \mathbb{T} \to \Gamma^\omega(T\mathbb{R}^n)$ as

$$\widehat{X}(t)(x) = X(t, x), \qquad \forall t \in \mathbb{T}, \ \forall x \in \mathbb{R}^n.$$

It is clear that this correspondence between time-varying C^ω-vector fields and curves on the space $\Gamma^\omega(T\mathbb{R}^n)$ is one-to-one.

In [2], time-varying vector fields and their flows has been studied using an operator approach called chronological calculus. In this framework, a vector field is considered as a derivation of the algebra $C^\infty(\mathbb{R}^n)$ and a diffeomorphism is considered as a unital algebra isomorphism on $C^\infty(\mathbb{R}^n)$. In [13], this framework has been extended by considering the algebra $C^\omega(\mathbb{R}^n)$ for the real analytic vector fields and real analytic mappings.

Definition 2.4. Let $X : \mathbb{R}^n \to T\mathbb{R}^n$ be a real analytic vector field. Then we define the map $\widehat{X} : C^\omega(\mathbb{R}^n) \to C^\omega(\mathbb{R}^n)$ by

$$\widehat{X}(f) = df(x)$$

It can be shown that $\widehat{X}$ is a derivation on the algebra $C^\omega(\mathbb{R}^n)$.

Definition 2.5. Let U be an open set in $\mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ be a real analytic mapping. Then we define the map $\widehat{\phi} : C^\omega(\mathbb{R}^n) \to C^\omega(U)$ by

$$\widehat{\phi}(f) = f \circ \phi.$$

It can be shown that $\widehat{\phi}$ is a unital algebra homomorphism between $C^\omega(\mathbb{R}^n)$ and $C^\omega(U)$.

One can consider real analytic vector fields and real analytic mapping on $\mathbb{R}^n$ as linear operators on $C^\omega(\mathbb{R}^n)$.

Definition 2.6. The space of linear mappings from $C^\omega(\mathbb{R}^n)$ to $C^\omega(\mathbb{R}^n)$ is denoted by $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$.

Thus, for every real analytic vector field $X : \mathbb{R}^n \to T\mathbb{R}^n$ and every real analytic mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, we have

$$\widehat{\phi}, \widehat{X} \in \mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n)).$$

For studying time-varying vector fields and their flows in this framework, we define suitable topologies on the vector spaces $C^\omega(\mathbb{R}^n)$, $\Gamma^\omega(\mathbb{R}^n)$, and $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$.

Definition 2.7. Let V be a vector space over $\mathbb{R}$ with addition $+ : V \times V \to \mathbb{R}$ and scalar multiplication $. : \mathbb{R} \times V \to V$. Let τ be a topology on V such that with respect to τ both addition and scalar multiplication are continuous. The pair (V, τ) is called a **topological vector space**.

Let (V, τ) be a topological vector space. A subset $B \subseteq V$ is **bounded** if, for every neighbourhood U of 0 in V, there exists $\alpha \in \mathbb{R}$ such that $B \subset \alpha U$.

A locally convex topological vector space (V, τ) is topological vector spaces whose topology τ is defined using a family of seminorms

Definition 2.8. A topological vector space (V, τ) is locally convex if there exists a family of seminorms $\{p_i\}_{i \in \Lambda}$ on V which generates the topology τ.

Now, using a family of seminorms, we define topologies on $C^\omega(R^n)$, $\Gamma^\omega(T\mathbb{R}^n)$, and $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$, which make them into locally convex spaces. The space of real analytic functions equipped with this topology has been throughly studied in [20], [8].

Definition 2.9. Let $K \subset \mathbb{R}^n$ be a compact set and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Then we define the seminorm $p_{K,\mathbf{a}}^\omega : C^\omega(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ as

$$p_{K,\mathbf{a}}^\omega(f) = \left\{ \frac{a_0 a_1 \ldots a_{|r|}}{|r|!} \left\| D^{(r)} f(x) \right\| \, \Big| \, x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}$$

The family of seminorms $p_{K,\mathbf{a}}^\omega$ defines a topology on $C^\omega(R^n)$. We denote this topology by C^ω-topology.

Let $K \subset \mathbb{R}^n$ be a compact set, $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and $f \in C^\omega(\mathbb{R}^n)$. Then we define the seminorm $p_{K,\mathbf{a},f}^\omega : \mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n)) \to \mathbb{R}_{\geq 0}$ as

$$p_{K,\mathbf{a},f}^\omega(X) = \left\{ \frac{a_0 a_1 \ldots a_{|r|}}{|r|!} \left\| D^{(r)} \left(Xf \right)(x) \right\| \, \Big| \, x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}$$

The family of seminorms $\left\{ p_{K,\mathbf{a},f}^\omega \right\}$ defines a topology on $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$. We denote this topology by C^ω-topology.

Note that $\Gamma^\omega(T\mathbb{R}^n) \subseteq \mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$. Therefore, the C^ω-topology on $\Gamma^\omega(T\mathbb{R}^n)$ is the subspace topology from $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$. Using the C^ω-topology on $\Gamma^\omega(T\mathbb{R}^n)$, one can define and study different properties of time-varying vector fields such as continuity, boundedness, and integrability.

Definition 2.10. Let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A curve $\lambda : \mathbb{T} \to \Gamma^\omega(T\mathbb{R}^n)$ is **essentially bounded** if, for every compact set $K \subset \mathbb{R}^n$, every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and every $f \in C^\omega(\mathbb{R}^n)$, there exists $M > 0$ such that

$$p_{K,\mathbf{a},f}^\omega(\lambda(t)) < M, \qquad \text{a.e. } t \in \mathbb{T}.$$

The set of essentially bounded curves with domain $\mathbb{T}$ on $\Gamma^\omega(T\mathbb{R}^n)$ is denoted by $\mathrm{L}^\infty(\mathbb{T}; \Gamma^\omega(T\mathbb{R}^n))$.

Let $S \subseteq \Gamma^\omega(T\mathbb{R}^n)$. We define the subset $\mathrm{L}^\infty(\mathbb{T}; S) \subseteq \mathrm{L}^\infty(\mathbb{T}; \Gamma^\omega(T\mathbb{R}^n))$ as

$$\mathrm{L}^\infty(\mathbb{T}; S) = \{\lambda \in \mathrm{L}^\infty(\mathbb{T}; \Gamma^\omega(T\mathbb{R}^n)) \mid \lambda(t) \in S, \text{ a.e. } t \in \mathbb{T}\}.$$

Let $X : [0, T] \times \mathbb{R}^n \to T\mathbb{R}^n$ be a time-varying real analytic vector field. By considering real analytic vector fields and real analytic maps as elements in $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$, we can translate the nonlinear differential equation governing the flow of X:

$$
\begin{aligned}
\dot{\phi}^X(t, x) &= X(t, \phi^X(t, x)), \qquad \text{a.e. } t \in [0, T] \\
\phi^X(0, x) &= x,
\end{aligned}
$$

into the following linear differential equation on the locally convex space $\mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$:

$$
\begin{aligned}
\frac{d\widehat{\phi}^X(t)}{dt} &= \widehat{\phi}^X(t) \circ \widehat{X}(t), \qquad \text{a.e. } t \in [0, T] \\
\widehat{\phi}^X(0) &= \mathrm{id}.
\end{aligned}
\tag{2.2}
$$

One can study the sequence of Picard iterations for the linear differential equation (2.2) [7, Chapter 1, §3].

Definition 2.11. Let $X \in \mathrm{L}^\infty([0, T]; \Gamma^\omega(T\mathbb{R}^n))$. We define $\widehat{\phi}_0^X : [0, T] \to \mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$ by

$$\widehat{\phi}_0^X(t) = \mathrm{id}, \qquad \forall t \in [0, T].$$

Then, for every $k \in \mathbb{N}$, we can define $\widehat{\phi}_k^X : [0, T] \to \mathrm{L}(C^\omega(\mathbb{R}^n); C^\omega(\mathbb{R}^n))$ inductively as

$$\widehat{\phi}_k^X(t) = \mathrm{id} + \int_0^t \widehat{\phi}_{k-1}^X(\tau) \circ X(\tau) d\tau, \qquad \forall t \in [0, T].$$

It can be shown that the sequence $\{\widehat{\phi}_k^X\}_{k \in \mathbb{N}}$ converges uniformly on $[0, T]$ to $\widehat{\phi}^X$ in C^ω-topology [13, Theorem 3.8.1].

Theorem 2.12. *Let B be a bounded set in $\Gamma^\omega(T\mathbb{R}^n)$. Then, for every compact set $K \subseteq \mathbb{R}^n$, every $f \in C^\omega(\mathbb{R}^n)$, and every $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exist positive constants $M > 0$ and $M_f > 0$ such that*

$$p_{K,\mathbf{a},f}^\omega(\widehat{\phi}_{k+1}^X(t) - \widehat{\phi}_k^X(t)) \leq (Mt)^{k+1} M_f, \quad \forall X \in \mathrm{L}^\infty([0,T]; B).$$

Moreover, if $T' < T$ is such that $MT' < 1$, then the sequence $\{\widehat{\phi}_k^X\}_{k \in \mathbb{N}}$ converges uniformly on $[0, T']$ to $\widehat{\phi}^X$ in C^ω-topology.

In order to work with concatenation of vector fields it is convenient to introduce the following notation.

Definition 2.13. Let $\Sigma = \{X_0, X_1, \ldots, X_m\}$ be a family of real analytic vector fields. Then, for every $p \in N$, every $i_1, i_2, \ldots, i_p \in \{0, 1, \ldots, m\}$, and every $t_0, t_1, \ldots, t_p \in \mathbb{R}$, we define the vector field $X_{t_1, t_2, \ldots, t_p}^{i_1, i_2, \ldots, i_p}$ as

$$X_{t_1, t_2, \ldots, t_p}^{i_1, i_2, \ldots, i_p} = \begin{cases} X_{i_p}, & t \in [0, t_p], \\ X_{i_{p-1}}, & t \in (t_p, t_{p-1} + t_p], \\ \vdots & \vdots \\ X_{i_1}, & t \in (t_2 + \ldots, t_p, t_1 + \ldots + t_p]. \end{cases}$$

It clear that we have

$$\phi^{X_{t_1, t_2, \ldots, t_p}^{i_1, i_2, \ldots, i_p}}(t_1 + \ldots + t_p) = \phi_{t_1}^{X_{i_1}} \circ \phi_{t_2}^{X_{i_2}} \circ \ldots \phi_{t_p}^{X_{i_p}}.$$

Using Theorem 2.12, one can show the following result.

Theorem 2.14. *Let B be a bounded set in $\Gamma^\omega(T\mathbb{R}^n)$. Then, for every $f \in C^\omega(\mathbb{R}^n)$, there exist positive real numbers $M > 0$ and $M_f > 0$ such that*

$$\left\| \mathrm{ev}_{x_0}(\widehat{\phi}^{X_{t_1, t_2, \ldots, t_p}^{i_1, i_2, \ldots, i_p}}(t)(f)) - \mathrm{ev}_{x_0}(\widehat{\phi}_k^{X_{t_1, t_2, \ldots, t_p}^{i_1, i_2, \ldots, i_p}}(t)(f)) \right\| \leq \frac{(Mt)^{k+1}}{1 - Mt} M_f,$$

for every $k \in N$, every $t \in [0, T]$ such that $Mt < 1$, every family of real analytic vector fields $\{X_1, X_2, \ldots, X_m\} \subseteq B$, every $p \in \mathbb{N}$, every $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, n\}$, and every $t_1, t_2, \ldots, t_p \in \mathbb{R}_{>0}$ such that $\sum_{i=1}^p t_i \leq t$.

2.2 Real analytic control systems

In this section, we define the notion of real analytic control system. In the geometric control literature, a control system is usually defined as "parametrized" family of vector fields on a manifold M, where the parameter is the control and the manifold M is the state space of the system. However, in this paper, for the sake of simplicity, we define a control system as a finite family of vector fields on $\mathbb{R}^n$.

Definition 2.15. By a **real analytic control system on** $\mathbb{R}^n$ we mean a family of real analytic vector fields $\Sigma = \{X_1, X_2, \ldots, X_m\}$, such that $X_i \in \Gamma^\omega(T\mathbb{R}^n)$, for every $i \in \{1, 2, \ldots, m\}$.

Let $T > 0$ and $x_0 \in \mathbb{R}^n$. The **reachable sets of** Σ **from** x_0 **in times less than** T is the set $\mathrm{R}_\Sigma(< T, x_0)$ defined as

$$\mathrm{R}_\Sigma(< T, x_0) = \{\phi_{t_1}^{X_{i_1}} \circ \phi_{t_2}^{X_{i_2}} \circ \ldots \circ \phi_{t_s}^{X_{i_s}}(x_0) \mid$$
$$s \in \mathbb{N}, \ t_i \geq 0, \ X_{i_j} \in \Sigma, \ t_1 + t_2 + \ldots + t_s < T\}.$$

A control system Σ is **small-time locally controllable from** x_0 if there exists $T > 0$ such that, for every $t \in (0, T]$, we have

$$x_0 \in \mathrm{int}\,(\mathrm{R}_\Sigma(< t, x_0))\,.$$

Let $N \in \mathbb{Z}_{>0}$ be a positive integer. Then the control system Σ which is small-time locally controllable from x_0 satisfies **growth condition of order** N if there exists $T > 0$ such that, for every $t \in (0, T]$, we have

$$\overline{\mathrm{B}}(x_0, t^N) \subset \mathrm{int}\,(\mathrm{R}_\Sigma(< t, x_0))\,.$$

Another important notion relevant to small-time local controllability of systems is normal reachability [10]. Normal reachability has been first introduced and studied by Sussmann in [24].

Definition 2.16. Let $\Sigma = \{X_0, X_1, \ldots, X_m\}$ be a real analytic control system. The point $x \in \mathbb{R}^n$ is **normally reachable in time less than** T from x_0, if there exists $p \in \mathbb{N}$, $F_1, F_2, \ldots, F_p \in \{X_0, X_1, \ldots, X_m\}$ and $(s_1, s_2, \ldots, s_p) \in \mathbb{R}_{>0}^p$ such that $s_1 + s_2 + \ldots + s_p < T$,

$$\phi_{s_1}^{F_1} \circ \phi_{s_2}^{F_2} \circ \ldots \circ \phi_{s_p}^{F_p}(x_0) = x,$$

and there exists an open neighbourhood of $(s_1, s_2, \ldots, s_p)$ in $\mathbb{R}_{>0}^p$ such that the map

$$(t_1, t_2, \ldots, t_p) \mapsto \phi_{t_1}^{F_1} \circ \phi_{t_2}^{F_2} \circ \ldots \circ \phi_{t_p}^{F_p}(x_0)$$

is C^1 and of rank n on this neighbourhood.

The connection between small-time local controllability and normal reachability has been studied in [10]. It is clear that if, for every $T > 0$, the point x_0 is normally reachable from x_0 in times less than T, then the system is small-time locally controllable from x_0. In [10], it has been shown that for real analytic control systems, small-time local controllability from x_0 implies that, for every time T, every point in the interior of the reachable set from x_0 in times less than T is normally reachable from x_0 [10, Theorem 5.5 and Corollary 4.15].

Theorem 2.17. *Let* $\Sigma = \{X_0, X_1, \ldots, X_m\}$ *be a real analytic control system. If* Σ *is small-time locally controllable from* x_0 *then, for every* $T > 0$, *every point in the set* $\mathrm{int}\,(\mathrm{R}_\Sigma(< T, x_0))$ *is normally reachable in time less than* T *form* x_0.

2.3 Multi-valued mappings and fixed-point theorems

Definition 2.18. Let X and Y be two topological spaces. Then a **multi-valued map** is an assignment $F : X \rightrightarrows Y$ such that, for every $x \in X$, we have $F(x) \subseteq Y$.

It is sometimes useful to "select" a single-valued mapping from a multi-valued mapping.

Definition 2.19. Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ be a multi-valued mapping. Then a single-valued mapping $f : X \to Y$ is a **selection** of F if, for every $x \in X$, we have $f(x) \in F(x)$.

Given a multi-valued mapping, one would like to see if there exists a single-valued selection of that mapping with suitable regularity condition. This has been studied in theory of selections of multi-valued mapping [22].

The Brouwer fixed-point theorem is considered as one of the most important existence theorems in mathematics [11, Chapter 2, §2].

Theorem 2.20 (Brouwer fixed-point theorem)**.** *Let $C \subseteq \mathbb{R}^n$ be convex and compact and $f : C \to C$ be a continuous map. Then there exists $x \in C$ such that $x = f(x)$.*

In this section, we study an extension of the Brouwer fixed-point theorem using the notion of half-continuity [6, Definition 2.1].

Definition 2.21. Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ be a map. Then f is **half-continuous** if, for every $x \in X$ such that $x \neq f(x)$, there exist $v \in \mathbb{R}^n$ and $\epsilon > 0$ with the property that

$$v.\,(f(y) - y) < 0, \qquad \forall y \in \mathrm{B}(x, \epsilon) \cap X.$$

where for vectors $v = (v^1, v^2, \ldots, v^n)$ and $w = (w^1, w^2, \ldots, w^n)$, the scalar $v.w$ is defined by

$$v.w = \sum_{i=1}^{n} v^i w^i.$$

The following theorem connects the notion of half-continuity and the notion of lower(upper) semi-continuity [6, Proposition 2.4].

Theorem 2.22. *Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ be a map. If, for every $x \in X$ such that $x \neq f(x)$, there exists $v \in \mathbb{R}^n$ such that one of the following conditions holds*

1. $v.\,(f(x) - x) > 0$ and $v.f(x)$ is lower semi-continuous at x.

2. $v.\,(f(x) - x) < 0$ and $v.f(x)$ is upper semi-continuous at x.

then f is half-continuous.

Using the notion of half-continuity, one can generalize the Brouwer fixed-point theorem [6, Theorem 3.1]

Theorem 2.23. *Let $C \subseteq \mathbb{R}^n$ be a compact and convex set and $f : C \to C$ be a half-continuous mapping. Then there exists $x \in C$ such that $x = f(x)$.*

3 The main theorem

Now, we are in the position to prove the main result of this paper.

Theorem 3.1. *Let $\Sigma = \{X_0, X_1, \ldots, X_m\}$ be a real analytic control system on $\mathbb{R}^n$. Assume that Σ is small-time locally controllable from $x_0 \in \mathbb{R}^n$ and assume that Σ satisfies the growth condition of order N. Consider the real analytic control system $\Theta = \{Y_0, Y_1, \ldots, Y_m\}$ on $\mathbb{R}^n$ with the property that, for every $i \in \{0, 1, \ldots, m\}$ and every multi-index (k) with $|(k)| \leq N$, we have*

$$D^{(k)} X_i(x_0) = D^{(k)} Y_i(x_0).$$

Then Θ is small-time locally controllable from x_0.

Proof. Since Σ is small-time locally controllable from x_0 and satisfies the growth condition of order N, there exists T', small enough, such that

$$\overline{B}(x_0, t^N) \in \text{int} \left(R_\Sigma(< t, x_0) \right), \quad \forall t \in (0, T'].$$

We set

$$B = \Sigma \cup \Theta.$$

Since Σ and Θ are finite sets, B is also a finite set and therefore it is bounded in $\Gamma^\omega(T\mathbb{R}^n)$. Let $x^1, x^2, \ldots, x^n$ be the standard coordinate functions on $\mathbb{R}^n$. Then, by Theorem 2.14, there exist positive constants $M > 0$ and $L > 0$ such that

$$\left\| \text{ev}_{x_0} \big(\widehat{\phi}^{Z^{i_1,i_2,\ldots,i_p}_{t_1,t_2,\ldots,t_p}}(t)(x^i) \big) - \text{ev}_{x_0} \big(\widehat{\phi}^{X^{i_1,i_2,\ldots,i_p}_{t_1,t_2,\ldots,t_p}}_k(t)(x^i) \big) \right\|$$

$$\leq \frac{(Mt)^{k+1}}{1 - Mt} L, \quad \forall i \in \{1, 2, \ldots, n\},$$

for every $k \in \mathbb{Z}_{\geq 0}$, every $t \in (0, T']$ such that $Mt < 1$, every family of real analytic vector fields $\{Z_1, Z_2, \ldots, Z_m\} \subseteq B$, every $p \in \mathbb{N}$, every $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, m\}$, and every $t_1, t_2, \ldots, t_p \in \mathbb{R}_{>0}$ such that $\sum_{i=1}^{p} t_i \leq t$.

Suppose that $T'' \in (0, T']$ is such that $MT'' < \frac{1}{2}$. This implies that

$$MT < \frac{1}{2}, \quad \forall t \in (0, T'']$$

We set $T_{\min} = \min\{T', T'', \sqrt{n} M^{N+1} L, \frac{1}{\sqrt{n} M^{N+1} L}\}$.

Let us fix $T \in (0, T_{\min}]$. Since Σ is small-time locally controllable from x_0, by Theorem 2.17, for every $x \in \text{int} \left(R_\Sigma(< T, x_0) \right)$, there exist $p_x \in \mathbb{N}$, $s_1, s_2, \ldots, s_{p_x} \in \mathbb{R}_{>0}$, and $j_1, j_2, \ldots, j_{p_x} \in \{0, 1, \ldots, m\}$ such that $s_1 + s_2 + \ldots + s_{p_x} < T$ and

$$\phi^{X_{j_1}}_{s_1} \circ \phi^{X_{j_2}}_{s_2} \circ \ldots \circ \phi^{X_{j_{p_x}}}_{s_{p_x}}(x_0) = x,$$

Moreover, there exists an open neighbourhood V of $(s_1, s_2, \ldots, s_{p_x})$ in $\mathbb{R}^{p_x}_{>0}$ such that the map $\xi^x_\Sigma : V \to \mathbb{R}^n$, defined by

$$\xi^x_\Sigma(t_1, t_2, \ldots, t_{p_x}) = \phi^{X_{j_1}}_{t_1} \circ \phi^{X_{j_2}}_{t_2} \circ \ldots \circ \phi^{X_{j_{p_x}}}_{t_{p_x}}(x_0)$$

is C^1 and of rank n on V. Without loss of generality we can assume that, for every $(t_1, t_2, \ldots, t_{p_x}) \in V$, we have

$$t_1 + t_2 + \ldots + t_{p_x} < T.$$

By [16, Lemma 2.2], there exists a submanifold M_x of V containing $(s_1, s_2, \ldots, s_{p_x})$ such that $\xi_\Sigma^x(M_x)$ is an open neighbourhood of x in $\mathbb{R}^n$ and $\xi_\Sigma^x |_{M_x}$ is a C^1-diffeomorphism. Let S_x be an open neighbourhood of $(s_1, s_2, \ldots, s_{p_x})$ in M_x such that $\overline{S}_x \subseteq M_x$. Since $\overline{B}(x_0, T^N)$ is compact and $\xi_\Sigma^x(S_{x_i})$ are open in $\mathbb{R}^n$, there exists $x_1, x_2, \ldots, x_r \in \overline{B}(x_0, T^N)$ such that

$$\overline{B}(x_0, T^N) \subseteq \bigcup_{i=1}^r \xi_\Sigma^{x_i}(S_{x_i}).$$

Now let us define $p = p_{x_1} + p_{x_2} + \ldots + p_{x_r}$ and let $\{i_1, i_2, \ldots, i_p\}$ be the ordered set obtained by concatenation of sets $\{j_1, j_2, \ldots, j_{p_{x_k}}\}$ for $1 \leq k \leq r$. Since, $\mathbb{R}^{p_{x_i}} \subseteq \mathbb{R}^p$, for every $i \in \{1, 2, \ldots, r\}$, one can consider S_{x_i} as a submanfiold of $\mathbb{R}^p$. We define the multi-valued map $\eta_\Sigma^T : \overline{B}(x_0, T^N) \rightrightarrows \bigcup_{i=1}^r \overline{S}_{x_i}$ as

$$\eta_\Sigma^T(x) = \bigcup_{i \in \{1,2,\ldots,r\}} \left\{ (\xi_\Sigma^{x_i})^{-1}(x) \mid x \in \xi_\Sigma^{x_i}(\overline{S}_{x_i}) \right\}, \qquad \forall x \in \overline{B}(x_0, T^N).$$

Note that, for every $x \in \overline{B}(x_0, T^N)$, the number of elements in $\eta_\Sigma^T(x)$ is at most r. Now, for the real analytic system $\Theta = \{Y_0, Y_1, \ldots, Y_m\}$, we define the map $\xi_\Theta^T : \bigcup_{i=1}^r \overline{S}_{x_i} \to \mathbb{R}^n$ by

$$\xi_\Theta^T(t_1, t_2, \ldots, t_p) = \phi_{t_1}^{Y_{i_1}} \circ \phi_{t_2}^{Y_{i_2}} \circ \ldots \circ \phi_{t_p}^{Y_{i_p}}(x_0)$$

and the multi-valued map $F_{\Sigma, \Theta}^T : \overline{B}(x_0, T^N) \rightrightarrows \mathbb{R}^n$ by

$$F_{\Sigma, \Theta}^T = \xi_\Theta^T \circ \eta_\Sigma^T.$$

The multi-valued mapping $F_{\Sigma, \Theta}^T$ is finite-valued and has the following property.

Lemma 3.2. *For every* $x \in \overline{B}(x_0, T^N)$, *there exist a neighbourhood* W *containing* x, *a positive integer* $l \in \mathbb{N}$, *and continuous functions* $f^1, f^2, \ldots, f^l : W \to \mathbb{R}^n$ *such that*

$$F_{\Sigma, \Theta}^T(x) = \left\{ f^1(x), f^2(x), \ldots, f^l(x) \right\}$$

and

$$\left\{ f^1(y) \right\} \subseteq F_{\Sigma, \Theta}^T(y) \subseteq \left\{ f^1(y), f^2(y), \ldots, f^l(y) \right\}, \qquad \forall y \in W.$$

Proof. Let $x \in \overline{B}(x_0, T^N)$. Since $\overline{B}(x_0, T^N) \subseteq \bigcup_{i=1}^r \xi_\Sigma^{x_i}(S_{x_i})$, we have $x \in \bigcup_{i=1}^r \xi_\Sigma^{x_i}(S_{x_i})$. Without loss of generality, we can assume that

$$x \in \xi_\Sigma^{x_1}(S_{x_1}).$$

Note that $\xi_\Sigma^{x_1}(S_{x_1})$ is open in $\mathbb{R}^n$. Therefore, there exists a neighbourhood U of x such that $U \subseteq \xi_\Sigma^{x_1}(S_{x_1})$. On the other hand, since we have $\overline{B}(x_0, T^N) \subseteq \bigcup_{i=1}^r \xi_\Sigma^{x_i}(\overline{S}_{x_i})$, without loss

of generality, we can assume that there exists $l \in \{1, 2, \ldots, r\}$ such that

$$x \in \xi_\Sigma^{x_i}(\overline{S}_{x_i}), \qquad i \in \{1, 2, \ldots, l\},$$
$$x \notin \xi_\Sigma^{x_i}(\overline{S}_{x_i}), \qquad i \in \{l+1, l+2, \ldots, r\}.$$

For every $i \in \{l+1, l+2, \ldots, r\}$, the set $\xi_\Sigma^{x_i}(\overline{S}_{x_i})$ is closed in $\mathbb{R}^n$. Therefore,

$$\bigcup_{i=l+1}^{r} \xi_\Sigma^{x_i}(\overline{S}_{x_i})$$

is closed in $\mathbb{R}^n$. Moreover, we know that $x \notin \bigcup_{i=l+1}^{r} \xi_\Sigma^{x_i}(\overline{S}_{x_i})$. This implies that there exists a neighbourhood V of x such that

$$V \cap \left(\bigcup_{i=l+1}^{r} \xi_\Sigma^{x_i}(\overline{S}_{x_i}) \right) = \emptyset.$$

Note that, for every $i \in \{1, 2, \ldots, l\}$, $\overline{S}_{x_i} \subseteq M_{x_i}$. Therefore, for every $i \in \{1, 2, \ldots, l\}$, we have $\xi_\Sigma^{x_i}(\overline{S}_{x_i}) \subseteq \xi_\Sigma^{x_i}(M_{x_i})$. Since $x \in \xi_\Sigma^{x_i}(M_{x_i})$, for every $i \in \{1, 2, \ldots, l\}$, the set $\bigcap_{i=1}^{l} \xi_\Sigma^{x_i}(M_{x_i})$ is nonempty. We set

$$W = \left(\bigcap_{i=1}^{l} \xi_\Sigma^{x_i}(M_{x_i}) \right) \cap V \cap U$$

For every $i \in \{1, 2, \ldots, l\}$, we define the function $f^i : W \to \mathbb{R}^n$ as

$$f^i(y) = \xi_\Theta^T \circ (\xi_\Sigma^{x_i})^{-1}(y), \qquad \forall y \in W$$

Note that, for every $i \in \{1, 2, \ldots, l\}$, the map $\xi_\Sigma^{x_i}$ is a C^1-diffeomorphism on M_{x_i}. Therefore, for every $i \in \{1, 2, \ldots, l\}$, the map $f^i : W \to \mathbb{R}^n$ is continuous. Now, it is clear from the definition of $F_{\Sigma,\Theta}^T$ that we have

$$F_{\Sigma,\Theta}^T(x) = \left\{ f^1(x), f^2(x), \ldots, f^l(x) \right\}$$

Since $W \subseteq V$, and V is chosen such that

$$V \cap \left(\bigcup_{i=l+1}^{r} \xi_\Sigma^{x_i}(\overline{S}_{x_i}) \right) = \emptyset,$$

for every $i \in \{l+1, l+2, \ldots, r\}$ and every $y \in W$, we have

$$\xi_\Theta^T \circ (\xi_\Sigma^{x_i})^{-1}(y) \notin F_{\Sigma,\Theta}^T(y).$$

Thus, we have

$$F_{\Sigma,\Theta}^T(y) \subseteq \left\{ f^1(y), f^2(y), \ldots, f^l(y) \right\}, \qquad \forall y \in W.$$

Finally, since $W \subseteq U$, and U is chosen such that

$$U \subseteq \xi_\Sigma^{x_1}(S_{x_1})$$

This implies that, for every $y \in W$

$$\xi_\Theta^T \circ (\xi_\Sigma^{x_1})^{-1}(y) \in F_{\Sigma,\Theta}^T(y).$$

Therefore, for every $y \in W$, we have $f^1(y) \in F_{\Sigma,\Theta}^T(y)$. This completes the proof of the lemma. $\qquad \square$

Next, we prove the following Lemma.

Lemma 3.3. *Let $\Theta = \{Y_0, Y_1, \ldots, Y_m\}$ be a real analytic control system with the property that, for every $i \in \{0, 1, \ldots, m\}$ and every multi-index (k) with $|(k)| \leq N$, we have*

$$D^{(k)}X_i(x_0) = D^{(k)}Y_i(x_0).$$

Then there exists $\alpha \in \mathbb{R}$ such that, for every $v \in \overline{B}(x_0, T^N)$ we have

$$\|\xi - v\| \leq \alpha T^{N+1}, \qquad \forall \xi \in F_{\Sigma,\Theta}^T(v).$$

Proof. Note that, if $\xi \in F_{\Sigma,\Theta}^T(v)$, then there exist $i_1, i_2, \ldots, i_p \in \{1, 2, \ldots, m\}$ and $t_1, t_2, \ldots, t_p \in \mathbb{R}^{>0}$ such that $t_1 + t_2 + \ldots + t_p < T$ and

$$
\begin{aligned}
v &= \phi_{t_1}^{X_{i_1}} \circ \phi_{t_2}^{X_{i_2}} \circ \ldots \circ \phi_{t_p}^{X_{i_p}}(x_0), \\
\xi &= \phi_{t_1}^{Y_{i_1}} \circ \phi_{t_2}^{Y_{i_2}} \circ \ldots \circ \phi_{t_p}^{Y_{i_p}}(x_0).
\end{aligned}
$$

Therefore, by Theorem 2.14, for every $N \in \mathbb{N}$, we have

$$\left\| \mathrm{ev}_{x_0}(\phi^{Z_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i)) - \mathrm{ev}_{x_0}(\phi_N^{Z_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i)) \right\| \leq \frac{(MT)^{N+1}}{1-MT}L,$$

for every $t \in (0, T]$ and every family of real analytic vector fields $\{Z_1, Z_2, \ldots, Z_m\} \subset B$.

Assume that $\Theta = \{Y_0, Y_1, \ldots, Y_m\}$ is a real analytic control system such that, for every $i \in \{0, 1, \ldots, m\}$, we have

$$D^{(k)}X_i(x_0) = D^{(k)}Y_i(x_0), \qquad \forall k \in \{0, 1, \ldots, N\}.$$

This implies that, for every $t \in [0, T]$, we have

$$\mathrm{ev}_{x_0}\left(\phi_N^{X_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) = \mathrm{ev}_{x_0}\left(\phi_N^{Y_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right), \qquad \forall i \in \{1, 2, \ldots, n\}.$$

Thus, for every $t \in [0, T]$, we have

$$
\begin{aligned}
&\left\| \mathrm{ev}_{x_0}\left(\widehat{\phi}^{X_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) - \mathrm{ev}_{x_0}\left(\widehat{\phi}^{Y_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) \right\| \\
&\qquad < \left\| \mathrm{ev}_{x_0}\left(\widehat{\phi}^{X_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) - \mathrm{ev}_{x_0}\left(\widehat{\phi}_N^{X_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) \right\| \\
&\qquad\quad + \left\| \mathrm{ev}_{x_0}\left(\widehat{\phi}^{Y_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) - \mathrm{ev}_{x_0}\left(\widehat{\phi}_N^{Y_{t_1,t_2,\ldots,t_p}^{i_1,i_2,\ldots,i_p}}(t)(x^i) \right) \right\|.
\end{aligned}
$$

However, $\{X_1, X_2, \ldots, X_m\} \subset B$ and $\{Y_1, Y_2, \ldots, Y_m\} \subseteq B$. Therefore, for every $t \in [0, T]$, we have

$$\left\| \mathrm{ev}_{x_0} \left(\widehat{\phi}^{X^{i_1, i_2, \ldots, i_p}}_{t_1, t_2, \ldots, t_p}(t)(x^i) \right) - \mathrm{ev}_{x_0} \left(\widehat{\phi}^{Y^{i_1, i_2, \ldots, i_p}}_{N \quad t_1, t_2, \ldots, t_p}(t)(x^i) \right) \right\| \leq \frac{(MT)^{N+1}}{1 - MT} L,$$

$$\left\| \mathrm{ev}_{x_0} \left(\widehat{\phi}^{Y^{i_1, i_2, \ldots, i_p}}_{t_1, t_2, \ldots, t_p}(t)(x^i) \right) - \mathrm{ev}_{x_0} \left(\widehat{\phi}^{Y^{i_1, i_2, \ldots, i_p}}_{N \quad t_1, t_2, \ldots, t_p}(t)(x^i) \right) \right\| \leq \frac{(MT)^{N+1}}{1 - MT} L.$$

Thus, for every $t \in [0, T]$, we have

$$\left\| \mathrm{ev}_{x_0} \left(\widehat{\phi}^{X^{i_1, i_2, \ldots, i_p}}_{t_1, t_2, \ldots, t_p}(t)(x^i) \right) - \mathrm{ev}_{x_0} \left(\widehat{\phi}^{Y^{i_1, i_2, \ldots, i_p}}_{t_1, t_2, \ldots, t_p}(t)(x^i) \right) \right\|$$

$$\leq 2 \frac{(MT)^{N+1}}{1 - MT} L, \qquad \forall i \in \{1, 2, \ldots, n\}.$$

Therefore, for every $i \in \{1, 2, \ldots, n\}$, we have

$$\left\| \phi^{Y_{i_1}}_{t_1} \circ \phi^{Y_{i_2}}_{t_2} \circ \ldots \circ \phi^{Y_{i_p}}_{t_p}(x_0) - \phi^{X_{i_1}}_{t_1} \circ \phi^{X_{i_2}}_{t_2} \circ \ldots \circ \phi^{X_{i_p}}_{t_p}(x_0) \right\| < 2\sqrt{n} \frac{(MT)^{N+1}}{1 - MT} L.$$

Note that, by our choice of $t_1, t_2, \ldots, t_p \in \mathbb{R}_{>0}$, we have

$$\xi = \phi^{Y_{i_1}}_{t_1} \circ \phi^{Y_{i_2}}_{t_2} \circ \ldots \circ \phi^{Y_{i_p}}_{t_p}(x_0),$$

$$v = \phi^{X_{i_1}}_{t_1} \circ \phi^{X_{i_2}}_{t_2} \circ \ldots \circ \phi^{X_{i_p}}_{t_p}(x_0).$$

Moreover, for every $T < T_{\min}$, we have $MT < \frac{1}{2}$. Therefore, by setting $\alpha = \sqrt{n} M^{N+1} L$, we get

$$\|\xi - v\| \leq \alpha T^{N+1}, \qquad \forall v \in \overline{\mathrm{B}}(x_0, T^N).$$

This completes the proof of the lemma. $\qquad \square$

Now we proceed to prove the main theorem. The real analytic control system $\Sigma = \{X_1, X_2, \ldots, X_m\}$ is small-time locally controllable from x_0 and satisfies the growth condition of order N. Since $T \in (0, T_{\min}]$, we have

$$\overline{\mathrm{B}}(x_0, T^N) \subseteq \mathrm{int}\left(\mathrm{R}_\Sigma(< T, x_0) \right).$$

For every $y \in \overline{\mathrm{B}}(x_0, \frac{T^N}{2})$, we define the multi-valued mapping $G^T_y : \overline{\mathrm{B}}(x_0, T^N) \rightrightarrows \mathbb{R}^n$ as

$$G^T_y(v) = v - F^T_{\Sigma, \Theta}(v) + y, \qquad \forall v \in \overline{\mathrm{B}}(x_0, T^N).$$

We first show that, there exists $v \in \overline{\mathrm{B}}(x_0, T^N)$ such that $v \in G^T_y(v)$. Suppose that such a v does not exist. Then, for every $v \in \overline{\mathrm{B}}(x_0, T^N)$, we have

$$v \notin G^T_y(v).$$

Using Lemma 3.2, for every $v \in \overline{\mathrm{B}}(x_0, T^N)$, there exist $l \in \mathbb{Z}_{\geq 0}$, a neighbourhood W of x, and continuous functions $g^1, g^2, \ldots, g^l : W \to \mathbb{R}^n$ such that

$$G_y^T(v) = \left\{ g^1(v), g^2(v), \ldots, g^l(v) \right\},$$

and

$$\left\{ g^1(x) \right\} \subseteq G_y^T(x) \subseteq \left\{ g^1(x), g^2(x), \ldots, g^l(x) \right\}, \qquad \forall x \in W.$$

Since $v \notin G_y^T(v)$, one can choose $p_v \in \mathbb{R}^n$ such that

$$p_v.(g^1(v) - v) > 0,$$

and

$$p_v.(g^i(v) - v) \neq 0, \qquad \forall i \in \{2, 3, \ldots, l\}.$$

Since, for every $i \in \{1, 2, \ldots, l\}$, the map g^i is continuous, one can assume that, there exists a neighbourhood U_v of v such that $\overline{U_v} \subseteq W$ and

$$p_v.(g^1(x) - x) > 0, \qquad \forall x \in \overline{U}_v,$$

and

$$p_v.(g^i(x) - x) \neq 0, \qquad \forall x \in \overline{U}_v, \ \forall i \in \{2, 3 \ldots, l\}.$$

Since $\overline{\mathrm{B}}(x_0, T^N)$ is compact, there exists $v_1, v_2, \ldots, v_k \in \overline{\mathrm{B}}(x_0, T^N)$ such that

$$\overline{\mathrm{B}}(x_0, T^N) \subseteq \bigcup_{i=1}^{k} U_{v_i}.$$

Now we define a multi-valued mapping $P : \overline{\mathrm{B}}(x_0, T^N) \rightrightarrows \mathbb{R}^n$ as

$$P(x) = \bigcup_{i=1}^{k} \left\{ p_{v_i} \mid x \in \overline{U}_{v_i} \right\}.$$

Lemma 3.4. *For every $x \in \overline{\mathrm{B}}(x_0, T^N)$, there exist $s \in \mathbb{N}$, a neighbourhood W of x, and $p_1, p_2, \ldots, p_s \in \mathbb{R}^n$ such that*

$$P(x) = \{p_1, p_2, \ldots, p_s\}.$$

Moreover, for every $y \in W$, we have

$$\{p_1\} \subseteq P(y) \subseteq \{p_1, p_2, \ldots, p_s\}.$$

Proof. The proof is similar to Lemma 3.2, so we omit it. $\qquad\square$

By Lemma 3.4 for the multi-valued P and the fact that G_y^T and P are finite-valued, one can show that, for every $v \in \overline{\mathrm{B}}(x_0, T^N)$

$$\operatorname{argmin} \left\{ p.\eta \mid \eta \in G_y^T(v), \ p \in P(v), \ \text{s.t.} \ p.(\eta - v) > 0 \right\} \neq \emptyset.$$

Therefore, one can choose a single-valued selection $g_y^T : \overline{\mathrm{B}}(x_0, T^N) \to \mathbb{R}^n$ of G_y^T such that

$$g_y^T(v) \in \operatorname{argmin} \left\{ p.\eta \mid \eta \in G_y^T(v), \ p \in P(v), \ \text{s.t.} \ p.(\eta - v) > 0 \right\}.$$

We show that such a g_y^T is half-continuous.

Lemma 3.5. *Every single-valued mapping $g_y^T : \overline{B}(x_0, T^N) \to \mathbb{R}^n$ defined as above is half-continuous.*

Proof. Let $g_y^T : \overline{B}(x_0, T^N) \to \mathbb{R}^n$ be a single-valued mapping which satisfies the above conditions. For every $v \in \overline{B}(x_0, T^N)$, there exist $p_v \in P(v)$ such that $p_v.(g_y^T(v) - v) > 0$ and

$$p_v.g_y^T(v) = \min\left\{p.\eta \mid \eta \in G_y^T(v), \ p \in P(v), \ \text{s.t. } p.(\eta - v) > 0\right\}.$$

We define the map $\phi_{p_v} : \overline{B}(x_0, T^N) \to \mathbb{R}^n$ as

$$\phi_{p_v}(x) = p_v.g_y^T(x), \qquad \forall x \in \overline{B}(x_0, T^N)$$

We show that ϕ_{p_v} is lower semicontinuous at v. By Lemma 3.2, there exist a positive integer $l \in \mathbb{Z}_{\geq 0}$, a neighbourhood W of v and continuous mappings $g^1, g^2, \ldots, g^l : W \to \mathbb{R}^n$ such that

$$G_y^T(v) = \left\{g^1(v), g^2(v), \ldots, g^l(v)\right\},$$

and we have

$$\left\{g^1(x)\right\} \subseteq G_y^T(x) \subseteq \left\{g^1(x), g^2(x), \ldots, g^l(x)\right\}, \qquad \forall x \in W.$$

For every $x \in \overline{B}(x_0, T^N)$, we define

$$J(x) = \left\{\eta \in G_y^T(x) \mid \exists p \in P(x), \ \text{s.t. } p.(\eta - x) > 0\right\}.$$

Without loss of generality, one can assume that there exists $s \in \{1, 2, \ldots, l\}$ such that

$$g^1(v), g^2(v), \ldots, g^s(v) \in J(v),$$

and

$$g^{s+1}(v), g^{s+2}(v), \ldots, g^l(v) \notin J(v).$$

This means that, for $i \in \{s+1, s+2, \ldots, l\}$, for every $p \in P(v)$, we have

$$p.(g^i(v) - v) < 0.$$

Using Lemma 3.4, there exist a positive integer $j \in \mathbb{Z}_{>0}$, a neighbourhood U of v, and $p_1, p_2, \ldots, p_j \in \mathbb{R}^n$ such that

$$P(v) = \{p_1, p_2, \ldots, p_j\}.$$

Moreover, for every $x \in U$, we have

$$\{p_1\} \subseteq P(x) \subseteq \{p_1, p_2, \ldots, p_j\}.$$

Therefore, for every $\alpha \in \{1, 2, \ldots, j\}$, we have

$$p_\alpha.(g^i(v) - v) < 0, \qquad \forall i \in \{s+1, s+2, \ldots, l\}.$$

Since g^i are all continuous on W, there exists a neighbourhood $V \subseteq U \cap W$ such that, for every $x \in V$, we have

$$p_\alpha.(g^i(x) - x) < 0, \qquad \forall i \in \{s+1, s+2, \ldots, l\}.$$

Moreover, for every $x \in V$, we have $P(x) \subseteq \{p_1, p_2, \ldots, p_j\}$. This implies that, for every $x \in V$ and every $p \in P(x)$, we have

$$p.(g^i(x) - x) < 0, \qquad \forall i \in \{s+1, s+2, \ldots, l\}.$$

Therefore, for every $x \in V$, we have

$$g^i(x) \notin J(x), \qquad \forall i \in \{s+1, s+2, \ldots, l\}.$$

Thus, for every $x \in V$, we have

$$J(x) \subseteq \left\{ g^1(x), g^2(x), \ldots, g^l(x) \right\}.$$

Without loss of generality, one can assume that $g_y^T(v) = g^l(v)$. Since g^i are continuous, for every $\epsilon > 0$, there exists $\delta > 0$ such that $B(v, \delta) \subseteq V$ and we have

$$|p_v.(g^i(v) - g^i(x))| < \epsilon, \qquad \forall i \in \{1, 2, \ldots, l\}, \ \forall x \in B(v, \delta).$$

This implies that, for every $i \in \{1, 2, \ldots, l\}$,

$$p_v.g_y^T(v) = p_v.g^l(v) \leq p_v.g^i(v) \leq \epsilon + \inf\{p_v.g^i(x) \mid x \in B(v, \delta) - \{v\}\}.$$

Therefore, we have

$$p_v.g_y^T(v) \leq \epsilon + \inf\{p_v.g^i(x) \mid x \in B(v, \delta) - \{v\}, \ i \in \{1, 2, \ldots, l\}\}.$$

Since, for every $x \in B(v, \delta)$, we have

$$J(x) \subseteq \left\{ g^1(x), g^2(x), \ldots, g^l(x) \right\},$$

This implies that

$$\inf\{p_v.g^i(x) \mid x \in B(v, \delta) - \{v\}, \ i \in \{1, 2, \ldots, l\}\} \leq \inf\{p_v.\eta \mid \eta \in J(x), \ x \in B(v, \delta) - \{v\}\}.$$

However, it is clear that, for every $x \in B(v, \delta)$, we have $g_y^T(x) \in J(x)$. This implies that

$$\inf\{p_v.\eta \mid \eta \in J(x), \ x \in B(v, \delta) - \{v\}\} \leq \inf\{p_v.g_y^T(x) \mid x \in B(v, \delta) - \{v\}\}.$$

Therefore, we have

$$p_v.g_y^T(v) \leq \epsilon + \inf\{p_v.g_y^T(x) \mid x \in B(v, \delta) - \{v\}\}.$$

By taking limit of the both side of the above equality as δ goes to zero, we get

$$\phi_{p_v}(v) = p_v.g_y^T(v) \leq \lim_{\delta \to 0} \inf\{p_v.(g_y^T(x)) \mid x \in B(v, \delta) - \{v\}\} = \liminf_{x \to v} \phi_{p_v}(x)$$

Thus, ϕ_{p_v} is lower semicontinuous at v. Note that, since $g_y^T(v) = g^l(v) \in J(v)$, we have

$$p_v(g_y^T(v) - v) > 0.$$

Therefore, by Theorem 2.22, the single-valued selection g_y^T is half-continuous. This completes the proof of the lemma $\qquad\square$

Using Lemma 3.3, since $T \leq \frac{1}{2\alpha}$, we have

$$\|v - \eta\| < \alpha T^{N+1} \leq \frac{T^N}{2}, \qquad \forall \eta \in F_{\Sigma,\Theta}^T(v).$$

Thus, the map g_y^T is from $\overline{B}(x_0, T^N)$ into $\overline{B}(x_0, T^N)$. Therefore, by Theorem 2.23, there exists $v \in \overline{B}(x_0, T^N)$ such that

$$v \in G_y^T(v).$$

However, this is a contradiction with our assumption that, for every $v \in \overline{B}(x_0, T^N)$, $v \notin G_y^T(v)$. This implies that there exists $v \in \overline{B}(x_0, T^N)$ such that

$$v \in G_y^T(v).$$

Therefore, there exists $\eta \in F_{\Sigma,\Theta}^T(v)$ such that

$$v = v - \eta + y.$$

As a result, we have

$$y = \eta \in F_{\Sigma,\Theta}^T(v).$$

Since $y \in \overline{B}(x_0, \frac{T^N}{2})$ is arbitrary, this implies that $0 \in \text{int}(R_\Theta(< T, x_0))$. One can repeat the same argument for every $T \in (0, T_{\min}]$. Therefore, the system Θ is small-time locally controllable from x_0. $\qquad\square$

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