

Distributed and time-varying primal-dual dynamics via contraction analysis

Pedro Cisneros-Velarde, Saber Jafarpour, *Member, IEEE*, Francesco Bullo, *Fellow, IEEE*

Abstract—In this note, we provide an overarching analysis of primal-dual dynamics associated to linear equality-constrained optimization problems using contraction analysis. For the well-known standard version of the problem: we establish convergence under convexity and the contracting rate under strong convexity. Then, for a canonical distributed optimization problem, we use partial contractivity to establish global exponential convergence of its primal-dual dynamics. As an application, we propose a new distributed solver for the least-squares problem with the same convergence guarantees. Finally, for time-varying versions of both centralized and distributed primal-dual dynamics, we exploit their contractive nature to establish bounds on their tracking error. To support our analyses, we introduce novel results on contraction theory.

Index Terms—primal-dual, time-varying optimization, contraction theory, distributed algorithms, least-squares

I. INTRODUCTION

Problem statement and motivation: Primal-dual (PD) dynamics are dynamical systems that solve constrained optimization problems. Their study can be traced back to many decades ago [4] and has regained interest since the last decade [11]. PD dynamics have been made popular due to their scalability and simplicity. They have been widely adopted in engineering applications such as resource allocation problems in power networks [25], frequency control in microgrids [19], solvers for linear equations [29], etc. In this note, we study optimization problems with linear equality constraints. In general, PD dynamics seek to find a saddle point of the associated Lagrangian function to the constrained problem, which is characterized by the equilibria of the dynamics. For a general treatise of asymptotic stability of saddle points, we refer to [6] and references therein. However, despite the long history of study and application, there are very recent studies on PD dynamics related to linear equality constraints further studying different dynamic properties such as: exponential convergence under different convexity assumptions [22], [5] and contractivity properties [20].

We are particularly interested in studying primal-dual dynamics in distributed and time-varying optimization

problems. We refer to the recent survey [30] for an overview of the long-standing interest on distributed optimization. Of particular interest is to provide strong convergence guarantees such as global (and exponential) convergence for the distributed solvers. We aim to provide them using contraction theory. Time-varying optimization has found applications in system identification, signal detection, robotics, traffic management, etc. [10], [26]. The goal is to employ a dynamical system able to track the time-varying optimal solution up to some bounded error in real time. Although different dynamics have been proposed to both time-varying centralized [10] and distributed problems [26], [23], to the best of our knowledge, there has not been a characterization of the PD dynamics in such application contexts. The importance of PD algorithms is their simplicity of implementation, i.e., they do not require more complex information structures like the inverse of the Hessian of the system at all times, as in [10] and [23] for the centralized and distributed cases respectively. However, simplicity may come with a possible trade-off in the tracking error.

Contraction is valuable in practice because it introduces strong stability and robustness guarantees. For example, it implies input-to-state stability for systems subject to state-independent disturbances. It also guarantees fast correction after transient perturbations to the trajectory of the solution, since initial conditions are forgotten. Moreover, a contractive system may be robust towards structural perturbations on the vector field, e.g., when a non-convex term is added to the objective function. Finally, contraction guarantees stable numerical discretizations with geometric convergence rates, an ideal situation for practical implementations. All these properties are transparent to whether the system is time-varying or not. All of this motivates a contraction analysis of PD algorithms in contrast to the prevalent Lyapunov or invariance analysis in the literature.

Literature review: The recent works [22], [20], [5] study convergence properties of PD dynamics under different assumptions on the objective function. In distributed optimization, solvers based on PD dynamics are fairly recent, e.g., [28], [8], [30]. An application of distributed optimization of current interest - as seen in the recent survey [29] - is the *distributed least-squares*

Pedro Cisneros-Velarde (pacisne@gmail.com), Saber Jafarpour and Francesco Bullo ({saber,bullo}@ucsb.edu) are with the University of California, Santa Barbara.

problem for solving an over-determined system of linear equations. To the best of our knowledge, solvers for this problem (in continuous-time) with exponential global convergence are still missing in the literature.

Finally, this note is related to contraction theory, a mathematical tool to analyze incremental stability [17], [27]. An introduction and survey can be found in [2]. A variant of contraction theory, *partial contraction* [21], [9], analyzes the convergence to linear subspaces and has been used in the synchronization analysis of diffusively-coupled network systems [21], [3]; however, its application to distributed algorithms is still missing, and our paper provides such contribution.

Contributions: In this note we consider the PD dynamics associated to optimization problems with a twice differentiable and strongly convex objective function and linear equality constraints. We use contraction theory to perform an overarching study of PD dynamics in a variety of implementations and applications; see Fig. 1. In particular:

(i) We introduce new theoretical results of how *non-expansiveness* and *partial contraction* can imply exponential convergence to a point in a subspace of equilibria.

(ii) For the standard and distributed PD dynamics, we prove: 1) convergence under non-expansiveness when the objective function is convex; 2) contraction for the standard problem and partial contraction for the distributed one in the strongly convex case, with closed-form exponential global convergence rates. The analysis in result 1) is novel, since it uses the new results introduced in (i). Compared to the work [20] that also shows contraction for the standard PD, our proof method provides an explicit closed-form expression of the system's contraction rate. Our exponential convergence rate is different from the one by [22] via Lyapunov analysis, and both rates cannot be compared without extra assumptions on the numerical relationships among various parameters associated to the objective function or constraints. Moreover, we propose using the *augmented Lagrangian* in order to achieve contraction when the objective function is only convex. In the case of distributed optimization, there exist other solvers that show exponential convergence, e.g., as in [13], [14], but none of these have contractivity.

(iii) We propose a new solver for the distributed least-squares problem based on PD dynamics, and use our results in (ii) to prove its convergence. Compared to the recent work [15], our new model exhibits global convergence; and compared to the recent work [16], ours exhibits exponential convergence and has a simpler structure.

(iv) We characterize the performance of PD dynamics associated to time-varying versions of both standard and distributed optimization problems in terms of the problems' parameters. In particular, we prove the track-

ing error to the time-varying solutions is uniformly ultimately bounded (UUB) in either case and that the bound decreases as the contraction rate increases — these results, to the best of our knowledge, are novel. Our analysis builds upon the contraction results in contribution (ii).

Paper organization: Section II has notation and preliminary concepts. Section III has results on contraction theory. Section IV analyzes contractive properties of the standard PD dynamics. The contractive analysis of distributed (with the least-squares problem application) and time-varying versions of PD dynamics are in Sections V and VI respectively. Section VII is the conclusion.

II. PRELIMINARIES AND NOTATION

A. Notation, definitions and useful results

Consider $A \in \mathbb{R}^{n \times n}$, then $\sigma_{\min}(A)$ denote its minimum singular value and $\sigma_{\max}(A)$ its maximum one. If A has only real eigenvalues, let $\lambda_{\max}(A)$ be its maximum eigenvalue. A is an orthogonal projection if it is symmetric and $A^2 = A$. Let $\|\cdot\|$ denote any norm, and $\|\cdot\|_p$ denote the ℓ_p -norm. When the argument of a norm is a matrix, we refer to its respective induced norm. The matrix measure associated to $\|\cdot\|$ is $\mu(A) = \lim_{h \rightarrow 0^+} \frac{\|I + hA\| - 1}{h}$; e.g., the one associated to the ℓ_2 -norm is $\mu_2(A) = \lambda_{\max}((A + A^\top)/2)$ [2]. Given invertible $Q \in \mathbb{R}^{n \times n}$, let $\|\cdot\|_{2,Q}$ be the weighted ℓ_2 -norm $\|x\|_{2,Q} = \|Qx\|_2$, $x \in \mathbb{R}^n$, and whose associated matrix measure is $\mu_{2,Q}(A) = \mu_2(QAQ^{-1})$ [2].

Let I_n be the $n \times n$ identity matrix, $\mathbf{1}_n$ and $\mathbf{0}_n$ be the all-ones and all-zeros column vector with n entries respectively. Let $\text{diag}(X_1, \dots, X_N) \in \mathbb{R}^{\sum_{i=1}^N n_i \times \sum_{i=1}^N n_i}$ be the block-diagonal matrix with elements $X_i \in \mathbb{R}^{n_i \times n_i}$. Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers. Given $x_i \in \mathbb{R}^{k_i}$, let $(x_1, \dots, x_N) = [x_1^\top \dots x_N^\top]$.

Consider a differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We say f is *Lipschitz smooth* with constant $K_1 > 0$ if $\|\nabla f(x) - \nabla f(y)\|_2 \leq K_1 \|x - y\|_2$ for any $x, y \in \mathbb{R}^n$; and *strongly convex* with constant $K_2 > 0$ if $K_2 \|x - y\|_2^2 \leq (\nabla f(x) - \nabla f(y))^\top (x - y)$ for any $x, y \in \mathbb{R}^n$. Assuming f is twice differentiable, these two conditions are equivalent to $\nabla^2 f(x) \preceq K_1 I_n$ and $K_2 I_n \preceq \nabla^2 f(x)$ for any $x \in \mathbb{R}^n$, respectively.

The proof of the next proposition is found in [7].

Proposition II.1. *For a full-row rank matrix $A \in \mathbb{R}^{m \times n}$, $B = B^\top \in \mathbb{R}^{n \times n}$, and $b_2 \geq b_1 > 0$ such that $b_2 I_n \succeq B \succeq b_1 I_n \succ 0$, the matrix $\begin{bmatrix} -B & -A^\top \\ A & \mathbf{0}_{m \times m} \end{bmatrix}$ is Hurwitz.*

B. Review of basic concepts on contraction theory

Consider the dynamical system $\dot{x} = f(x, t)$ with $x \in \mathbb{R}^n$. Let $t \mapsto \phi(t, t_0, x_0)$ be the trajectory of the system starting from $x_0 \in \mathbb{R}^n$ at time $t_0 \geq 0$. Consider

the system satisfies $\|\phi(t, t_0, x_0) - \phi(t, t_0, y_0)\| \leq \|x_0 - y_0\|e^{-c(t-t_0)}$, for any $x_0, y_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}_{\geq 0}$. We say it is *contractive* with respect to $\|\cdot\|$ when $c > 0$, and *non-expansive* when $c = 0$. A time-invariant contractive system has a unique equilibrium point. Now, assume the Jacobian of the system, i.e., $Df(x, t)$, satisfies: $\mu(Df(x, t)) \leq -c$ for any $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, with μ being the matrix measure associated to $\|\cdot\|$ and constant $c \geq 0$. Then, this system has contraction rate c with respect to $\|\cdot\|$. Now, assume the system has a flow-invariant linear subspace $\mathcal{M} = \{x \in \mathbb{R}^n \mid Vx = 0_k\}$ with $V \in \mathbb{R}^{k \times n}$ being full-row rank with orthonormal rows. Then the system is *partially contractive* with respect to $\|\cdot\|$ and \mathcal{M} if there exists $c > 0$ such that, for any $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}_{\geq 0}$, the system satisfies $\|V\phi(t, t_0, x_0)\| \leq \|Vx_0\|e^{-c(t-t_0)}$. When $c = 0$, the system is *partially non-expansive* with respect to \mathcal{M} [21]. Consequently, a partially contractive system has any of its trajectories approaching \mathcal{M} with exponential rate; and a partially non-expansive one has any of its trajectories at a non-increasing distance from \mathcal{M} .

Pick a symmetric positive-definite $P \in \mathbb{R}^{n \times n}$ and a scalar $c > 0$, then $\mu_{2, P^{1/2}}(Df(x, t)) \leq -c$ for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ is equivalent to f satisfying the *integral contractivity condition*, i.e., for every $x, y \in \mathbb{R}^n$ and $t \geq 0$, $(y - x)^\top P(f(x, t) - f(y, t)) \leq -c\|x - y\|_{2, P^{1/2}}^2$.

III. THEORETICAL CONTRACTION RESULTS

The next result will be used throughout the paper.

Theorem III.1 (Results on partial contraction). *Consider the system $\dot{x} = f(x, t)$, $x \in \mathbb{R}^n$, with a flow-invariant $\mathcal{M} = \{x \in \mathbb{R}^n \mid Vx = 0_k\}$ with $V \in \mathbb{R}^{k \times n}$ being a full-row rank matrix with orthonormal rows. Assume $\mu(VDf(x, t)V^\top) \leq -c$ for any $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, some constant $c \geq 0$ and some matrix measure μ .*

- (i) *If $c > 0$, then the system is partially contractive with respect to \mathcal{M} and every trajectory exponentially converges to the subspace \mathcal{M} with rate c .*
- (ii) *If $c = 0$ and $\mu(VDf((I_n - V^\top V)x, t)V^\top) < 0$ for any $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, then the system is partially non-expansive with respect to \mathcal{M} and every trajectory converges to the subspace \mathcal{M} .*

Moreover, assume that one of the conditions in parts (i) and (ii) holds and \mathcal{M} is a set of equilibrium points. If the system is non-expansive, then

- (iii) *every trajectory of the system converges to an equilibrium point, and if $c > 0$, then it does it with exponential rate c .*

Remark III.2. Statement (i) in Theorem III.1 was proved in [21]. To the best of our knowledge, statements (ii) and (iii) are novel. Due to space constraints, the proof is found in the online preprint [7].

IV. THE STANDARD OPTIMIZATION PROBLEM

We consider the constrained optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad Ax = b \quad (1)$$

with the following standing assumptions: $A \in \mathbb{R}^{k \times n}$, $k < n$, $b \in \mathbb{R}^k$, A is full-row rank, and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex and twice differentiable.

Associated to the optimization problem (1) is the *Lagrangian function* $\mathcal{L}(x, \nu) = f(x) + \nu^\top (Ax - b)$ and the *primal-dual dynamics*

$$\begin{bmatrix} \dot{x} \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathcal{L}(x, \nu)}{\partial x} \\ \frac{\partial \mathcal{L}(x, \nu)}{\partial \nu} \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A^\top \nu \\ Ax - b \end{bmatrix}. \quad (2)$$

We introduce two possible sets of assumptions:

- (A1) the primal-dual dynamics (2) have an equilibrium (x^*, ν^*) and $\nabla^2 f(x^*) \succ 0_{n \times n}$;
- (A2) the function f is strongly convex with constant $\ell_{\inf} > 0$ and Lipschitz smooth with constant $\ell_{\sup} > 0$, and, for $0 < \epsilon < 1$, we define

$$\alpha_\epsilon := \frac{\epsilon \ell_{\inf}}{\sigma_{\max}^2(A) + \frac{3}{4} \sigma_{\max}(A) \sigma_{\min}^2(A) + \ell_{\sup}^2} > 0 \quad (3)$$

$$P := \begin{bmatrix} I_n & \alpha_\epsilon A^\top \\ \alpha_\epsilon A & I_k \end{bmatrix} \in \mathbb{R}^{(n+k) \times (n+k)}.$$

Theorem IV.1 (Contraction analysis of primal-dual dynamics). *Consider the constrained optimization problem (1), its standing assumptions, and its associated primal-dual dynamics (2).*

- (i) *The primal-dual dynamics is non-expansive with respect to $\|\cdot\|_2$ and, if Assumption (A1) holds, then (x^*, ν^*) is globally asymptotically stable.*
- (ii) *Under Assumption (A2),*
 - a) *the primal-dual dynamics are contractive with respect to $\|\cdot\|_{2, P^{1/2}}$ with contraction rate*

$$\alpha_\epsilon \frac{3}{4} \frac{\sigma_{\max}(A) \sigma_{\min}^2(A)}{\sigma_{\max}(A) + 1}, \quad \text{and} \quad (4)$$

- b) *there exists a unique globally exponentially stable equilibrium point (x^*, ν^*) , and x^* is the unique solution to the optimization problem (1).*

Proof. Let $(\dot{x}, \dot{\nu})^\top := F_{\text{PD}}(x, \nu)$. Then, $DF_{\text{PD}}(x, \nu) = \begin{bmatrix} -\nabla^2 f(x) & -A^\top \\ A & 0 \end{bmatrix}$, and so $\mu_2(DF_{\text{PD}}(x, \nu)) = \lambda_{\max}((DF_{\text{PD}}(x, \nu) + DF_{\text{PD}}(x, \nu)^\top)/2) = \lambda_{\max}(\text{diag}(-\nabla^2 f(x), 0_{k \times k})) = 0$ for any $(x, \nu) \in \mathbb{R}^n \times \mathbb{R}^m$, because of convexity $\nabla^2 f(x) \succeq 0$, which implies the system is non-expansive. For the second part of statement (i): Proposition II.1 implies $DF_{\text{PD}}(x^*, \nu^*)$ is Hurwitz since $\nabla^2 f(x^*) \succ 0$, and the proof follows from a simple generalization of [18, Lemma 6] (its proof can be found in [7]).

Now, we prove statement (ii). Define $P = \begin{bmatrix} I_n & \alpha A^\top \\ \alpha A & I_k \end{bmatrix}$ which is a positive-definite matrix when

$$0 < \alpha < \frac{1}{\sigma_{\max}(A)}. \quad (5)$$

We plan to use the integral contractivity condition to show that system (2) is contractive with respect to norm $\|\cdot\|_{2,P^{1/2}}$. Thus, we need to show

$$\begin{aligned} \eta := & \begin{bmatrix} x_1 - x_2 \\ \nu_1 - \nu_2 \end{bmatrix}^\top P (F_{\text{PD}}(x_1, \nu_1) - F_{\text{PD}}(x_2, \nu_2)) \\ & + c \begin{bmatrix} x_1 - x_2 \\ \nu_1 - \nu_2 \end{bmatrix}^\top P \begin{bmatrix} x_1 - x_2 \\ \nu_1 - \nu_2 \end{bmatrix} \leq 0 \end{aligned}$$

for any $x_1, x_2 \in \mathbb{R}^n$ and $\nu_1, \nu_2 \in \mathbb{R}^m$, and some constant $c > 0$ which will be the contraction rate. After completing squares, using the strong convexity and Lipschitz smoothness of f , along with $\sigma_{\min}^2(A)I_k \preceq AA^\top$ and $A^\top A \preceq \sigma_{\max}^2(A)I_n$, we obtain

$$\begin{aligned} \eta \leq & -((3\alpha/4)\sigma_{\min}^2(A) - c - c\alpha) \|\nu_1 - \nu_2\|_2^2 - (\ell_{\inf} \\ & - \alpha\sigma_{\max}^2(A) - c - \alpha\ell_{\sup}^2 - c\alpha\sigma_{\max}^2(A)) \|x_1 - x_2\|_2^2 \\ & - \alpha c \|(\nu_1 - \nu_2) - A(x_1 - x_2)\|_2^2. \end{aligned}$$

Set $c = D\alpha$ for some $D > 0$. Then, to ensure that $\eta \leq 0$, we need to ensure

$$\frac{3\alpha}{4}\sigma_{\min}^2(A) - D\alpha - D\alpha^2 \geq 0, \quad (6)$$

$$\ell_{\inf} - \alpha\sigma_{\max}^2(A) - D\alpha - \alpha\ell_{\sup}^2 - D\alpha^2\sigma_{\max}^2(A) \geq 0. \quad (7)$$

Now, to ensure inequality (6) holds, using the inequalities (5), it is easy to see that it suffices to ensure that

$$\frac{3\sigma_{\max}(A)\sigma_{\min}^2(A)}{4(\sigma_{\max}(A) + 1)} > D. \quad (8)$$

Now, using inequalities (5) and (8), we obtain: $\ell_{\inf} - \alpha\sigma_{\max}^2(A) - D\alpha - \alpha\ell_{\sup}^2 - D\alpha^2\sigma_{\max}^2(A) > \ell_{\inf} - \alpha(\sigma_{\max}^2(A) + \frac{3}{4}\sigma_{\max}(A)\sigma_{\min}^2(A) + \ell_{\sup}^2)$ and so, to ensure inequality (7) holds, it suffices that

$$\frac{\ell_{\inf}}{\sigma_{\max}^2(A) + \frac{3}{4}\sigma_{\max}(A)\sigma_{\min}^2(A) + \ell_{\sup}^2} > \alpha. \quad (9)$$

Now, the parameter α needs to satisfy inequalities (5) and (9); however, (9) implies (5) because the inequality $\pi_1^2 + \pi_2^2 \geq 2\pi_1\pi_2$ for $\pi_1, \pi_2 > 0$ let us conclude that $\frac{\ell_{\sup}}{\sigma_{\max}^2(A) + \ell_{\sup}^2} \leq \frac{1}{2\sigma_{\max}(A)}$. Finally, c must be less than the multiplication of the left-hand sides of the inequalities (8) and (9), which proves statement (ii)a.

Now, since the dynamics are contractive, there must exist a unique globally exponentially stable equilibrium point which also satisfies the (sufficient and necessary) KKT conditions of optimality for the optimization problem (1), thus proving statement (ii)b. \square

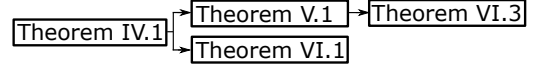


Fig. 1: Dependency relationship among the main theorems in this note. An arrow from A to B means A is used to prove B .

Remark IV.2. Theorem IV.1 is a fundamental building block for the rest of results in this note as seen in Fig. 1 and therefore, it was necessary to provide a comprehensive proof using the integral contractivity condition that could provide an explicit estimate of the contraction rate (as opposed to the different proof in [20]).

Throughout this note, the Lipschitz smoothness and strong convexity of f are used to prove contraction. However, the latter is relaxed in Corollary IV.3.

For the case of convex f , Theorem IV.1 does not state convergence — nor contraction — without additional assumptions; indeed, oscillations may appear and convergence to the saddle points is not guaranteed [11]. In order to still be able to use Theorem IV.1 in this case, we consider a modification to the Lagrangian, known as the *augmented Lagrangian* [24]: $\mathcal{L}_{\text{aug}}(x, \nu) = \mathcal{L}(x, \nu) + \frac{\rho}{2}\|Ax - b\|_2^2$ with gain $\rho > 0$. Its associated *augmented primal-dual dynamics* become

$$\begin{bmatrix} \dot{x} \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A^\top \nu - \rho A^\top Ax + \rho A^\top b \\ Ax - b \end{bmatrix} \quad (10)$$

and have the same equilibria as the original one in (2). We introduce two possible sets of assumptions:

- (A3) the primal-dual dynamics (2) have an equilibrium (x^*, ν^*) , $\nabla^2 f(x^*) \succeq 0_{n \times n}$, and $\text{Ker}(\nabla^2 f(x^*)) \cap \text{Ker}(A) = \{0_n\}$;
- (A4) $\text{Ker}(\nabla^2 f(x)) \cap \text{Ker}(A) = \{0_n\}$ for any $x \in \mathbb{R}^n$ and f is Lipschitz smooth with constant $\ell_{\sup} > 0$, and, for $0 < \epsilon < 1$, we define

$$\begin{aligned} \bar{\alpha}_\epsilon &:= \frac{\epsilon \rho \sigma_{\min}^2(A)}{(1 + \rho)\sigma_{\max}^2(A) + \frac{3}{4}\sigma_{\max}(A)\sigma_{\min}^2(A) + \ell_{\sup}^2} \\ \bar{P} &:= \begin{bmatrix} I_n & \bar{\alpha}_\epsilon A^\top \\ \bar{\alpha}_\epsilon A & I_k \end{bmatrix} \in \mathbb{R}^{(n+k) \times (n+k)}. \end{aligned}$$

Corollary IV.3 (Contraction analysis of the augmented primal-dual dynamics). *Consider the constrained optimization problem (1), its standing assumptions, and its associated augmented primal-dual dynamics (10) with $\rho > 0$.*

- (i) *Under Assumption (A3), the augmented primal-dual dynamics are non-expansive with respect to $\|\cdot\|_2$ and (x^*, ν^*) is globally asymptotically stable.*
- (ii) *Under Assumption (A4),*
 - a) *the augmented primal-dual dynamics are contractive with respect to $\|\cdot\|_{2,P^{1/2}}$ with contraction rate*

$$\bar{\alpha}_\epsilon \frac{3}{4} \frac{\sigma_{\max}(A)\sigma_{\min}^2(A)}{\sigma_{\max}(A) + 1}, \quad \text{and} \quad (11)$$

- b) there exists a unique globally exponentially stable equilibrium point (x^*, ν^*) for the augmented primal-dual dynamics and x^* is the unique solution to the constrained optimization problem (1).

Proof. The proof follows directly from Theorem IV.1. For statement (i), note that $\text{Ker}(\nabla^2 f(x^*)) \cap \text{Ker}(A) = \{0_n\}$ implies that $\nabla^2 f(x^*) + \rho A^\top A \succ 0_{n \times n}$ for the Jacobian of the system $\begin{bmatrix} -\nabla^2 f(x) - \rho A^\top A & -A^\top \\ A & 0 \end{bmatrix}$. For statement (ii), note that $\text{Ker}(\nabla^2 f(x)) \cap \text{Ker}(A) = \{0_n\}$ for any $x \in \mathbb{R}^n$ implies that $x \mapsto f(x) + \frac{\rho}{2} x^\top A^\top A x$ is Lipschitz smooth with constant $\ell_{\sup}^2 + \rho \sigma_{\max}^2(A) > 0$ and strongly convex with constant $\rho \sigma_{\min}^2(A) > 0$. \square

Remark IV.4 (Augmented Lagrangian and contraction). The benefit of using the augmented Lagrangian is that, unlike the conditions in Theorem IV.1, the resulting primal-dual dynamics may be contractive despite f being only convex.

V. DISTRIBUTED ALGORITHMS

We study a popular distributed implementation for solving an unconstrained optimization problem [30]. We want to solve the problem $\min_{x \in \mathbb{R}^n} f(x) = \sum_{i=1}^N f_i(x)$ with $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ convex. Let \mathcal{G} be an undirected connected interaction graph between N distinct agents. Let \mathcal{N}_i be the neighborhood of node i and L be the Laplacian matrix of \mathcal{G} . Let $x^i \in \mathbb{R}^n$ be the state associated to agent i , and let $\mathbf{x} = (x^1, \dots, x^N)^\top$. Then, the problem becomes:

$$\min_{\mathbf{x} \in \mathbb{R}^{nN}} \sum_{i=1}^N f_i(x^i) \quad (L \otimes I_n) \mathbf{x} = 0_{nN} \quad (12)$$

The associated distributed primal-dual dynamics are

$$\begin{aligned} \dot{x}^i &= -\nabla_{x^i} f_i(x^i) - \sum_{j \in \mathcal{N}_i} (\nu^j - \nu^i) \\ \dot{\nu}^i &= \sum_{j \in \mathcal{N}_i} (x^j - x^i) \end{aligned} \quad (13)$$

for $i \in \{1, \dots, N\}$. In system (13), any agent only uses information from herself and the set of her neighbors.

To study this system, we introduce two possible sets of assumptions:

- (A5) $\min_{x \in \mathbb{R}^n} f(x)$ has a solution x^* and $\nabla^2 f_i(x^*) \succ 0_{n \times n}$ for any $i \in \{1, \dots, N\}$;
 (A6) $\min_{x \in \mathbb{R}^n} f(x)$ has a solution x^* and the function f_i is strongly convex with constant $\ell_{\inf, i} > 0$ and Lipschitz smooth with constant $\ell_{\sup, i} > 0$ for any $i \in \{1, \dots, N\}$, with $\ell_{\inf} = (\ell_{\inf, 1}, \dots, \ell_{\inf, N})$ and $\ell_{\sup} = (\ell_{\sup, 1}, \dots, \ell_{\sup, N})$.

With either assumption, note that we cannot apply Theorem IV.1 directly since the linear equality constraint

in (12) is not full-row rank. However, if we instead consider partial contraction, then Theorem IV.1 can be used to prove the next result — whose proof is found in the online preprint [7] due to space constraints.

Theorem V.1 (Contraction analysis of distributed primal-dual dynamics). Consider the distributed primal-dual dynamics (13).

- (i) The distributed primal-dual dynamics are non-expansive with respect to $\|\cdot\|_2$, and
 (ii) under Assumption (A5), for any $(x^i(0), \nu^i(0)) \in \mathbb{R}^n \times \mathbb{R}^n$, $\lim_{t \rightarrow \infty} x^i(t) = x^*$ and $\lim_{t \rightarrow \infty} \nu^i(t) = \nu_i^*$, for some ν_i^* such that $\sum_{k=1}^N \nu_k^* = \sum_{k=1}^N \nu^k(0)$.
 (iii) Under Assumption (A6), the convergence results in statement (ii) hold and, for $0 < \epsilon < 1$, the convergence of $(\mathbf{x}(t), \nu(t))^\top$ has exponential rate

$$\frac{3\epsilon}{4} \frac{\lambda_N \lambda_2^2}{\lambda_N + 1} \frac{\min_{i \in \{1, \dots, N\}} \ell_{\inf, i}}{\lambda_N^2 + \frac{3}{4} \lambda_N \lambda_2^2 + \|\ell_{\sup}\|_\infty^2}, \quad (14)$$

where λ_2 and λ_N are the smallest non-zero and the largest eigenvalues of L , respectively.

For the case of convex f_i , Theorem V.1 does not state convergence — nor partial contraction — without additional assumptions. Similar to the analysis in Section IV, we present an example where augmenting the Lagrangian let us use Theorem V.1. We consider the popular distributed least-squares problem [29]. Given a full-column rank matrix $H \in \mathbb{R}^{N \times n}$, $n < N$, it is known that $x^* = (H^\top H)^{-1} H^\top z$ is the unique solution to the least-squares problem $\min_{x \in \mathbb{R}^n} \|z - Hx\|_2^2$, for $z \in \mathbb{R}^N$. An equivalent distributed version is

$$\min_{\mathbf{x} \in \mathbb{R}^{nN}} \sum_{i=1}^N (h_i^\top x^i - z_i)^2 \quad (L \otimes I_n) \mathbf{x} = 0_{nN} \quad (15)$$

with $h_i^\top \in \mathbb{R}^{1 \times n}$ being the i th row of the matrix H , $\mathbf{x} = (x^1, \dots, x^N)^\top$ and $z = (z_1, \dots, z_N)^\top$. Notice that $f(\mathbf{x}) = \sum_{i=1}^N |h_i^\top x^i - z_i|^2$ is convex, since $\nabla^2 f(\mathbf{x}) = \text{diag}(h_1 h_1^\top, \dots, h_N h_N^\top) \succeq 0_{nN \times nN}$. We propose to augment the Lagrangian with the quadratic term $\frac{\rho}{2} \mathbf{x}^\top (L \otimes I_n) \mathbf{x}$ with $\rho > 0$ (which does not alter the original saddle points) and obtain

$$\begin{aligned} \dot{x}^i &= -(h_i^\top x^i - z_i) h_i - \rho \sum_{j \in \mathcal{N}_i} (x^j - x^i) \\ &\quad - \sum_{j \in \mathcal{N}_i} (\nu^j - \nu^i) \\ \dot{\nu}^i &= \sum_{j \in \mathcal{N}_i} (x^j - x^i) \end{aligned} \quad (16)$$

for $i \in \{1, \dots, N\}$. The new algorithm is distributed.

Observe that $\text{Ker}(\text{diag}(h_1 h_1^\top, \dots, h_N h_N^\top)) \cap \text{Ker}(L \otimes I_n) = \{0_{nN}\}$ implies $\ell_{\inf}^* I_{nN} \preceq$

$\text{diag}(h_1 h_1^\top, \dots, h_N h_N^\top) + (L \otimes I_n)$ for some constant $\ell_{\inf}^* > 0$. Then, the following follows from Theorem V.1.

Corollary V.2 (Contraction analysis of distributed least-squares). *Consider the system (16), and let x^* be the unique solution to the least-squares problem. Then, for any $(x^i(0), \nu^i(0)) \in \mathbb{R}^n \times \mathbb{R}^n$, $\lim_{t \rightarrow \infty} x^i(t) = x^*$ and $\lim_{t \rightarrow \infty} \nu^i(t) = \nu_i^*$ for some ν_i^* such that $\sum_{k=1}^N \nu_k^* = \sum_{k=1}^N \nu^k(0)$; and, for $0 < \epsilon < 1$, the convergence of $(x(t), \nu(t))$ has exponential rate*

$$\epsilon \frac{3}{4} \frac{\lambda_N \lambda_2^2}{\lambda_N + 1} \frac{\ell_{\inf}^*}{\lambda_N^2 + \frac{3}{4} \lambda_N \lambda_2^2 + (\lambda_N + \rho \max_i \|h_i\|_2^2)^2}$$

where λ_2 and λ_N are the smallest non-zero and the largest eigenvalues of L , respectively.

VI. TIME-VARYING OPTIMIZATION

A. Time-varying standard optimization

Our results in Section IV can be used to analyze the case where the associated optimization problem is time-varying. Consider

$$\min_{x \in \mathbb{R}^n} f(x, t) \quad \text{subject to} \quad Ax = b(t) \quad (17)$$

with the following standing assumptions: $A \in \mathbb{R}^{k \times n}$, $k < n$, $b \in \mathbb{R}^k$, A is full-row rank, and, for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$,

- (i) $x \mapsto f(x, t)$ is twice continuously differentiable, uniformly strongly convex with constant $\ell_{\inf} > 0$, i.e., $\nabla^2 f(x, t) \succeq \ell_{\inf} I_n$; and uniformly Lipschitz smooth with constant $\ell_{\sup} > 0$, i.e., $\nabla^2 f(x, t) \preceq \ell_{\sup} I_n$;
- (ii) $t \mapsto \nabla f(x, t)$ and $t \mapsto b(t)$ are continuously differentiable functions.

The associated time-varying primal-dual dynamics are

$$\begin{bmatrix} \dot{x} \\ \dot{\nu} \end{bmatrix} = \begin{bmatrix} -\nabla f(x, t) - A^\top \nu \\ Ax - b(t) \end{bmatrix}. \quad (18)$$

Given a fixed time t , let $x^*(t)$ be a solution to the program $\min_{x: Ax=b(t)} f(x, t)$ and $\nu^*(t)$ its associated Lagrange multiplier. From the standing assumptions and Theorem IV.1, for any fixed t , there exists a unique optimizer $(x^*(t), \nu^*(t))$. Then, $(x^*(t), \nu^*(t))_{t \geq 0}$ defines the *optimizer trajectory* of the optimization problem (17). The following result establishes the performance of the primal-dual dynamics in tracking the optimizer trajectory. The proof can be found in the online preprint [7] because of space constraints and because it is similar and simpler than the provided proof of Theorem VI.3.

Theorem VI.1 (Contraction analysis of time-varying primal-dual dynamics). *Consider the time-varying optimization problem (17), its standing assumptions, and its associated primal-dual dynamics (18).*

- (i) *The primal-dual dynamics are contractive with respect to $\|\cdot\|_{2, P^{1/2}}$ with contraction rate c , where P is the matrix defined in (3) and c is the same contraction rate as in (4) of Theorem IV.1.*

Assume that, for any $t \geq 0$, $\|\dot{b}(t)\|_2 \leq \beta_1$ and $\|\frac{\partial}{\partial t} \nabla f(x, t)\|_2 \leq \beta_2$ for some positive constants β_1, β_2 , and let $z(t) := (x(t), \nu(t))^\top$ and $z^*(t) := (x^*(t), \nu^*(t))^\top$.

- (ii) *Then,*

$$\begin{aligned} & \|z(t) - z^*(t)\|_{2, P^{1/2}} \\ & \leq \left(\|z(0) - z^*(0)\|_{2, P^{1/2}} - \frac{\rho}{c} \right) e^{-ct} + \frac{\rho}{c}, \end{aligned} \quad (19)$$

i.e., the tracking error is uniformly ultimately bounded by $\frac{\rho}{c}$ with

$$\begin{aligned} \rho = \lambda_{\max}(P) & \left(\frac{\beta_2}{\ell_{\inf}} + \left(\frac{\sigma_{\max}(A)}{\ell_{\inf}} + 1 \right) \right. \\ & \left. \frac{\ell_{\max}}{\sigma_{\min}^2(A)} \left(\beta_1 + \frac{\sigma_{\max}(A)}{\ell_{\inf}} \beta_2 \right) \right). \end{aligned}$$

Remark VI.2. *The bounds in the assumptions for statement (ii) in Theorem VI.1 ensure that the rate at which the time-varying optimization changes is bounded. Indeed, the right-hand side of equation (19) is consistent: the larger (lower) these bounds, the larger (lower) the asymptotic tracking error. Moreover, the tracking is better the larger the contraction rate.*

B. Time-varying distributed optimization

Our partial contraction analysis of Section V can be extended to obtain performance guarantees for the following time-varying distributed optimization problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^{nN}} \quad & \sum_{i=1}^N f_i(x^i, t) \\ & (L \otimes I_n) \mathbf{x} = 0_{nN}, \end{aligned} \quad (20)$$

where we consider a time-invariant connected undirected graph whose Laplacian matrix is L , and set $\mathbf{x} = (x^1, \dots, x^N)^\top$ with $x^i \in \mathbb{R}^n$, with the following standing assumptions: for every $(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, and for any $i \in \{1, \dots, N\}$

- (i) $x \mapsto f_i(x, t)$ is twice continuously differentiable, uniformly strongly convex with constant $\ell_{\inf, i} > 0$, i.e., $\nabla^2 f_i(x, t) \succeq \ell_{\inf, i} I_n$; and uniformly Lipschitz smooth with constant $\ell_{\sup, i} > 0$, i.e., $\nabla^2 f_i(x, t) \preceq \ell_{\sup, i} I_n$;
 - (ii) $t \mapsto \nabla f_i(x, t)$ is continuously differentiable.
- Then, the associated primal-dual dynamics are

$$\begin{aligned} \dot{x}^i &= -\nabla_{x^i} f_i(x^i, t) - \sum_{j \in \mathcal{N}_i} (\nu^j - \nu^i) \\ \dot{\nu}^i &= \sum_{j \in \mathcal{N}_i} (x^j - x^i) \end{aligned} \quad (21)$$

for $i \in \{1, \dots, N\}$. Given a fixed time t , let $\mathbf{x}^*(t) = \mathbb{1}_N \otimes x^*(t)$ with $x^*(t)$ being the unique solution to the program $\min_x \sum_{i=1}^N f_i(x, t)$. Then, $(x^*(t))_t$ is a unique trajectory; however, there may exist multiple trajectories of the dual variables associated to the constraint in (20). Let $\nu^*(t) = (\nu^1(t), \dots, \nu^{N^*}(t))^\top$ be any dual variable obtained by solving the problem (20) for a fixed t . Then, we define the *time-varying set of optimizers* as:

$$\mathcal{M}(t) = \{(\mathbf{x}, \nu) \in \mathbb{R}^{nN} \times \mathbb{R}^{nN} \mid V(\mathbf{x} - \mathbb{1}_N \otimes x^*(t), \nu - \nu^*(t))^\top = (\mathbb{0}_{nN}, \mathbb{0}_{n(N-1)})^\top\}$$

where $V = \text{diag}(I_{nN}, R \otimes I_n)$ with $R \in \mathbb{R}^{N-1 \times N}$ as in the proof of Theorem V.1. For convenience, let $\ell_{\inf} = (\ell_{\inf,1}, \dots, \ell_{\inf,N})$ and $\ell_{\sup} = (\ell_{\sup,1}, \dots, \ell_{\sup,N})$; and for $0 < \epsilon < 1$, we define

$$\begin{aligned} \tilde{\alpha}_\epsilon &:= \frac{\epsilon \min_{i \in \{1, \dots, N\}} \ell_{\inf,i}}{\lambda_N^2 + \frac{3}{4} \lambda_N \lambda_2^2 + \|\ell_{\sup}\|_\infty^2} > 0 \\ \tilde{P} &:= \begin{bmatrix} I_{nN} & \alpha_\epsilon \bar{A}^{*\top} \\ \alpha_\epsilon \bar{A}^* & I_{n(N-1)} \end{bmatrix} \in \mathbb{R}^{nN \times nN} \end{aligned} \quad (22)$$

where $\bar{A}^* = (\Lambda R) \otimes I_n$, with $\Lambda = \text{diag}(\lambda_2, \dots, \lambda_N)$ containing the nonzero eigenvalues of L in nondecreasing order. The following result establishes the performance of the primal-dual dynamics at tracking the time-varying set of optimizers.

Theorem VI.3 (Contraction analysis of time-varying distributed primal-dual dynamics). *Consider the time-varying optimization problem (20), its standing assumptions, and its associated primal-dual dynamics (21). Set $z(t) := V(\mathbf{x}(t), \nu(t))^\top$ and $z^*(t) := V(\mathbf{x}^*(t), \nu^*(t))^\top$.*

(i) *The system associated to \dot{z} is contractive with respect to $\|\cdot\|_{2, \tilde{P}^{1/2}}$ with rate $c := \alpha_\epsilon \frac{3}{4} \frac{\lambda_2^2}{\lambda_N + 1}$.*

Moreover, for any $t \geq 0$, if $\|\frac{\partial}{\partial t} \nabla f_i(x, t)\|_2 \leq \beta_{1,i}$ for some positive constant $\beta_{1,i}$ and any $i \in \{1, \dots, N\}$, then,

(ii)

$$\begin{aligned} \|z(t) - z^*(t)\|_{2, \tilde{P}^{1/2}} &\leq \left(\|z(0) - z^*(0)\|_{2, \tilde{P}^{1/2}} - \frac{\rho}{c} \right) e^{-ct} + \frac{\rho}{c}, \end{aligned} \quad (23)$$

i.e., the tracking error is asymptotically bounded by $\frac{\rho}{c}$ with

$$\begin{aligned} \rho &= \lambda_{\max}(P) \frac{\|\beta_1\|_1}{\|\ell_{\inf}\|_1} N \\ &\quad + \lambda_{\max}(P) \frac{\|\beta_1\|_1}{\lambda_2} \left(\frac{\|\ell_{\sup}\|_\infty}{\|\ell_{\inf}\|_1} + 1 \right). \end{aligned} \quad (24)$$

Proof. Define $f(\mathbf{x}(t), t) := \sum_{i=1}^N f_i(x^i(t), t)$; then

$$\dot{z} = \begin{bmatrix} -\nabla f(\mathbf{x}(t), t) - (L \otimes I_n) \nu(t) \\ (\Lambda R \otimes I_n) \mathbf{x}(t) \end{bmatrix}.$$

Then, decomposing $(\mathbf{x}(t), \nu(t))^\top = U(\mathbf{x}(t), \nu(t))^\top + V^\top z$ where $U = I_{n(2N-1)} - V^\top V$ is a projection matrix,

we use the chain rule and obtain that the Jacobian for this system is $\begin{bmatrix} -\nabla^2 f(\mathbf{x}(t), t) & -(R^\top \Lambda \otimes I_n) \\ (\Lambda R \otimes I_n) & \mathbb{0}_{n(N-1) \times n(N-1)} \end{bmatrix}$, so then, based on our standing assumptions, using Proposition II.1 and following a similar proof to Theorem V.1, we obtain that this system is contractive as in item (i).

Now we prove statement (ii). The KKT conditions that the optimizers $\mathbf{x}^*(t)$ and $\nu^*(t)$ must satisfy (i.e., the equilibrium equation of the system (21)), for any t , are

$$\mathbb{0}_{nN} = -\nabla f(\mathbf{x}^*(t), t) - (L \otimes I_n) \nu^*(t) \quad (25)$$

$$\mathbb{0}_{nN} = (L \otimes I_n) \mathbf{x}^*(t). \quad (26)$$

Now, observe that (26) and (25) $\implies \mathbf{x}^*(t) = \mathbb{1}_N \otimes x^*(t)$ with $x^*(t)$ being the first nN coordinates of any element of $\mathcal{M}(t)$. Moreover, by left multiplying (26) with $\mathbb{1}_N^\top \otimes I_n$, we obtain that $\mathbb{0}_n = \sum_{i=1}^N \nabla_{x^i} f_i(x^*(t), t)$. Then, the Implicit Function Theorem [1, Theorem 2.5.7] (akin to its use in the proof of Theorem VI.1) implies the curve $t \mapsto x^*(t)$ is continuously differentiable for any $t \in \mathbb{R}_{\geq 0}$.

Now, from (25) we obtain that $\mathbb{0}_{n(N-1)} = (R \otimes I_n) \nabla f(\mathbf{x}^*(t), t) + (\Lambda \otimes I_n) (R \otimes I_n) \nu^*(t)$. Defining $y^*(t) := (R \otimes I_n) \nu^*(t)$, we get $\mathbb{0}_{n(N-1)} = (R \otimes I_n) \nabla f(\mathbf{x}^*(t), t) + (\Lambda \otimes I_n) y^*(t)$. Again, an application of the Implicit Function Theorem let us conclude that the solution $(\mathbf{x}^*, t) \mapsto y^*(\mathbf{x}^*, t)$ is continuously differentiable for any $(\mathbf{x}^*, t) \in \mathbb{R}^{nN} \times \mathbb{R}_{\geq 0}$; however, since $t \mapsto \mathbf{x}^*(t)$ is continuously differentiable for any $t \in \mathbb{R}_{\geq 0}$, then $t \mapsto y^*(t)$ is continuously differentiable too.

Then, we can differentiate equation (25) and left multiply it by $(\mathbb{1}_N^\top \otimes I_n)$ to obtain

$$\dot{x}^*(t) = -g(x^*(t), t) \sum_{i=1}^N \frac{\partial}{\partial t} \nabla_{x^i} f_i(x^*(t), t))$$

with $g(x^*(t), t) := (\sum_{i=1}^N \nabla_{x^i}^2 f_i(x^*(t), t))^{-1}$. Recall that $RL = \Lambda R$. Then, since y^* is continuously differentiable, we differentiate equation (25) and left multiply it by $(R \otimes I_n)$ to obtain

$$\begin{aligned} \dot{y}^*(t) &= -(\Lambda^{-1} R \otimes I_n) (\nabla^2 f(\mathbf{x}^*(t), t) (\mathbb{1}_N \otimes h_1(x^*(t))) \\ &\quad + \frac{\partial}{\partial t} \nabla f(\mathbf{x}^*(t), t)). \end{aligned}$$

Therefore, observe that $\frac{\|\dot{x}^*(t)\|_2}{\sum_{i=1}^N \frac{1}{\ell_{\inf,i}} \sum_{i=1}^N \beta_{1,i}} \leq \frac{\|\beta_1\|_1}{\|\ell_{\inf}\|_1}$, and $\mathbf{x}^*(t) = \mathbb{1}_N \otimes x^*(t)$. Moreover, $\|y^*(t)\|_2 \leq \frac{1}{|\lambda_2|} (\|\ell_{\sup}\|_\infty \|\mathbf{x}^*(t)\|_2 + \|\beta_1\|_1)$, where we used: $\|\frac{\partial}{\partial t} \nabla f(\mathbf{x}^*(t), t)\|_2 \leq \sum_{i=1}^N \|\frac{\partial}{\partial t} \nabla_{x^i} f_i(x^*(t), t)\|_2$, and $\|(\Lambda^{-1} R) \otimes I_n\|_2 = \sqrt{\lambda_{\max}(\Lambda^{-2} \otimes I_n)} = \frac{1}{\lambda_2}$.

Now, for any t , let $(a_1(t), a_2(t)) \in \mathcal{M}(t)$. Note that, no matter which element of \mathcal{M} we choose, $a_1(t) = \mathbb{1}_N \otimes x^*(t)$ and so it is uniquely defined for any t and we also know is differentiable. Now, note that $a_2(t) = \gamma(t) + \mathbb{1}_N \otimes \alpha$, with $\alpha \in \mathbb{R}^n$ and some uniquely defined $\gamma(t)$; and note that $(R \otimes I_n) a_2(t) =$

$(R \otimes I_n)\gamma(t)$ for any t . Therefore $(R \otimes I_n)a_2(t)$ is uniquely defined for any t and we also know is differentiable. In conclusion, the trajectory $((a_1(t), (R \otimes I_n)a_2(t)))_{t \geq 0} = (V(a_1(t), a_2(t)))_{t \geq 0}^\top$ is unique and $t \mapsto V(a_1(t), a_2(t))^\top$ is continuously differentiable.

Since the system associated to \dot{z} is contractive and the curve, as we just proved above, $t \mapsto z^*(t) := V(\mathbf{x}^*(t), \nu^*(t))^\top$ is unique and differentiable, we set $\Delta(t) := \|z(t) - z^*(t)\|_{2, P^{1/2}}$ and use the result in item (i) and [20, Lemma 2] to obtain the differential inequality

$$\begin{aligned} \dot{\Delta}(t) &\leq -c\Delta(t) + \left\| \begin{bmatrix} \dot{\mathbf{x}}^*(t) \\ \frac{d}{dt}((R \otimes I_n)\nu^*(t)) \end{bmatrix} \right\|_{2, P^{1/2}} \\ &\leq -c\Delta(t) + \lambda_{\max}(P)(N\|\dot{x}^*(t)\|_2 + \|\dot{z}^*(t)\|_2). \end{aligned}$$

Finally, replacing our previous results and using the Comparison Lemma [12] conclude the proof for (ii). \square

Remark VI.4. As in Remark VI.2, there is consistency on the right-hand side of equation (24).

VII. CONCLUSION

Primal-dual (PD) dynamics associated to linear equality constrained optimization problems are studied in centralized, distributed and time-varying cases. Contraction theory provides an overarching analysis of the dynamical behavior and performance for all these cases of PD dynamics. As future work, we plan to design controllers that can improve the PD solver's tracking properties in the time-varying setting. We also plan to study distributed PD solvers for globally coupled linear equation constraints and PD solvers in nonsmooth domains.

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