

# Local and global holomorphic extensions of time-varying real analytic vector fields

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2016/04/07

## Abstract

In this paper, we consider time-varying real analytic vector fields as curves on the space of real analytic vector fields. Using a suitable topology on the space of real analytic vector fields, we study and characterize different properties of time-varying real analytic vector fields. We study holomorphic extensions of time-varying real analytic vector fields and show that under suitable *integrability* conditions, a time-varying real analytic vector field on a manifold can be extended to a time-varying holomorphic vector field on a neighbourhood of that manifold. Moreover, we develop an operator setting, where the *nonlinear* differential equation governing the flow of a time-varying real analytic vector field can be considered as a *linear* differential equation on an infinite dimensional locally convex vector space. Using the holomorphic extension results, we show that the *integrability* of the time-varying vector field ensures the convergence of the sequence of Picard iterations for this linear differential equation. This gives us a series representation for the flow of an integrable time-varying real analytic vector field. We also define the exponential map between integrable time-varying real analytic vector fields and their flows. Using the holomorphic extensions of time-varying real analytic vector fields, we show that the exponential map is sequentially continuous.

**Keywords.** Space of real analytic vector fields, Time-varying vector field, Holomorphic extension, Linear differential equations on locally convex spaces.

## 1 Introduction

The early development of the notion of *real analyticity* in mathematics has a closed connection with the development of the notion of *function*. Prior to the nineteenth century, most of the functions used in mathematical analysis were constructed either by applying algebraic operators on elementary functions or by a power series except possibly at some

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singular points [5]. Therefore, mathematicians had difficulty understanding functions which are not real analytic. It is surprising to know that Lagrange and Hankel believed that the existence of all derivatives of a function implies the convergence of its Taylor series [5]. It was only in the late nineteenth century that mathematicians started to think more carefully about the natural question of which functions can be expanded in a Taylor series around a point. In 1823, Cauchy came up with a function which was  $C^\infty$  everywhere not real analytic at  $x = 0$  [7], [5]. In the modern terminology, this function can be expressed as

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & x \neq 0, \\ 0 & x = 0. \end{cases} \quad (1.1)$$

Starting from early twentieth century, with the advent of the more precise notion of function, mathematicians came up with other examples of smooth but not real-analytic functions whose singular points have completely different natures [5].

Roughly speaking, a map  $f$  is real analytic on a domain  $D$  if the Taylor series of  $f$  around every point  $x_0 \in D$  converges to  $f$  in a neighbourhood of  $x_0$ . By definition, for the Taylor series of  $f$  on  $D$  to exist, derivatives of  $f$  of any order should exist and be continuous at every point  $x_0 \in D$ . This means that all real analytic maps are of class  $C^\infty$ . As is shown by the function (1.1), the converse implication is not true. In fact, given an open connected set  $\Omega \subseteq \mathbb{R}^n$ , one can construct a family of nonzero smooth functions on  $\mathbb{R}^n$  which are zero on the set  $\Omega$ . However, by the identity theorem, every real analytic function which is zero on the set  $\Omega$  should be zero everywhere. This shows that the gap between real analytic functions and smooth functions is huge [18].

Real analytic vector fields on  $\mathbb{R}^n$  have a close connection with the holomorphic vector fields defined on neighbourhoods of  $\mathbb{R}^n$  in  $\mathbb{C}^n$ . It is well-known that every real analytic vector field  $f$  on  $\mathbb{R}^n$  can be extended to a holomorphic vector field defined on an appropriate domain in  $\mathbb{C}^n$ . However, it may not be possible to extend the real analytic vector field  $f$  to a holomorphic vector field on the whole domain  $\mathbb{C}^n$ . This observation suggests that one should consider a real analytic vector field as a germ of a holomorphic vector field. This perspective for real analytic vector fields motivates the definition of a natural topology on the space of real analytic vector fields. Unfortunately, there does not exist a single domain such that *every* real analytic vector field on  $\mathbb{R}^n$  can be extended to a holomorphic vector field on that domain. The following example shows this fact.

**Example 1.1.** For every  $n \in \mathbb{N}$ , consider the function  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$f_n(x) = \frac{1}{1 + n^2 x^2}, \quad \forall x \in \mathbb{R}.$$

It is easy to see that, for every  $n \in \mathbb{N}$ , the function  $f_n$  is real analytic on  $\mathbb{R}$ . We show that there does not exist a neighbourhood  $\Omega$  of  $\mathbb{R}$  in  $\mathbb{C}$  such that, for every  $n \in \mathbb{N}$ , the real analytic function  $f_n$  can be extended to a holomorphic function on  $\Omega$ . Suppose that such

an  $\Omega$  exists. Then there exists  $r > 0$  such that

$$\{x \in \mathbb{C} \mid \|x\| \leq r\} \subseteq \Omega.$$

Now let  $N \in \mathbb{N}$  be such that  $\frac{1}{N} < r$  and suppose that  $\bar{f}_N$  be the holomorphic extension of  $f_N$  to  $\Omega$ . Then, by the identity theorem, we have

$$\bar{f}_N(z) = \frac{1}{1 + N^2 z^2}, \quad \forall z \in \Omega.$$

By our choice of  $N$ , we have  $\frac{i}{N} \in \Omega$ , but  $\bar{f}_N$  is not defined at  $z = \frac{i}{N}$ . This is a contradiction and shows that such an  $\Omega$  does not exist.

Thus, the space of real analytic vector fields on  $\mathbb{R}^n$ , which we denote by  $\Gamma^\omega(\mathbb{R}^n)$ , can be considered as the *union* of the spaces of holomorphic vector fields defined on neighbourhoods of  $\mathbb{R}^n$  in  $\mathbb{C}^n$ . This process of taking union can be made precise using the mathematical notion of *inductive limit*. The space of holomorphic vector fields on an open set  $\Omega \subseteq \mathbb{C}^n$  has been studied in detail in the literature [19], [24]. One can show that the so-called “compact-open” topology on the space of holomorphic vector fields on  $\Omega$  is generated by a family of seminorms and thus is a locally convex topological vector space [19]. Therefore, we can represent the space of real analytic vector fields on  $\mathbb{R}$  as an inductive limit of a family of locally convex spaces. The locally convex *inductive limit topology* on  $\Gamma^\omega(\mathbb{R}^n)$  is defined as the finest locally convex topology which makes all the inclusions from the spaces of holomorphic vector fields to the space of real analytic vector fields continuous.

Inductive limits of locally convex spaces arise in many fields, including partial differential equations, Fourier analysis, distribution theory, and holomorphic calculus. Historically, locally convex inductive limits of locally convex spaces first appeared when mathematicians tried to define a suitable topology on the space of distributions. While there is little literature for inductive limit of arbitrary families of locally convex spaces, the countable inductive limit of locally convex spaces is rich in both theory and applications. The importance of the connecting maps in inductive limits of locally convex spaces was first realized by José Sebastião e Silva [30]. Motivated by studying the space of germs of holomorphic functions, Sebastião e Silva investigated inductive limit of locally convex spaces with compact connecting maps. Inductive limits with weakly compact connecting maps were studied later by Komatsu in [17], where he showed that weakly compact inductive limits share many nice properties with the compact inductive limits.

Unfortunately, the space of real analytic vector fields on  $\mathbb{R}^n$  is not the inductive limit of a countable family of locally convex spaces. However, it is possible to represent the space of germs of holomorphic vector fields around a *compact set* as the inductive limit of a countable family of locally convex spaces with compact connecting maps [19, Theorem 8.4]. Let  $\{K_i\}_{i \in \mathbb{N}}$  be a family of compact sets on  $\mathbb{R}^n$  such that  $\bigcup_{i=1}^\infty K_i = \mathbb{R}^n$  and

$$\text{cl}(K_i) \subseteq K_{i+1}, \quad \forall i \in \mathbb{N}.$$

It is interesting to note that the space of real analytic vector fields on  $\mathbb{R}^n$  can also be obtained by *gluing together* the vector spaces of germs of holomorphic vector fields on compact sets  $\{K_i\}_{i \in I}$ . The concept of *gluing together* mentioned above can be made precise using the notion of projective limit of vector spaces. The coarsest locally convex topology on  $\Gamma^\omega(\mathbb{R})$  which makes all the gluing maps continuous is called the *projective limit topology* on  $\Gamma^\omega(\mathbb{R}^n)$ . Having defined the *inductive limit topology* and *projective limit topology* on the space of real analytic vector fields on  $\mathbb{R}^n$ , it would be interesting to study the relation between these two topologies. As to our knowledge, the first paper that studied the relation between these two topologies on the space of real analytic vector fields is [22], where it is shown that these two topologies are identical. There has been a recent interest in this topology and its applications in the theory of partial differential equations [6], [20].

Time-varying vector fields and their flows arise naturally in studying physical problems. In particular, in some branches of applied sciences such as control theory, it is essential to work with time-varying vector fields whose dependence on time is only *measurable*. Existence and uniqueness of flows of time-varying vector field has been deeply studied in the literature [8, Chapter 2]. However, theory of time-varying vector fields with measurable dependence on time and their flows is not as well-developed as theory of time-invariant vector fields. In this paper, we study time-varying real analytic vector fields on a manifold  $M$  by considering them as curves on the vector space  $\Gamma^\omega(TM)$ . Using the  $C^\omega$ -topology on the space of real analytic vector fields, different properties of this curve can be studied and characterized. In particular, we can use the framework in [3] to define and characterize the *Bochner integrability* of curves on  $\Gamma^\omega(TM)$ .

It is well-known that every real analytic vector fields can be extended to a holomorphic vector field on a complex manifold. Consider a time-varying real analytic vector field on  $M$  with some regularity in time. It is interesting to study whether this time-varying real analytic vector field can be extended to a time-varying holomorphic vector field on a complex manifold containing  $M$ . Unfortunately this holomorphic extension is not generally possible. As the following example shows, a measurable time-varying real analytic vector field may not even have a local holomorphic extension to a complex manifold.

**Example 1.2.** Let  $X : \mathbb{R} \times \mathbb{R} \rightarrow T\mathbb{R}$  be a time-varying vector field defined as

$$X(t, x) = \begin{cases} \frac{t^2}{t^2+x^2} \frac{\partial}{\partial x} & x \neq 0 \text{ or } t \neq 0, \\ 0 & x, t = 0. \end{cases}$$

Then  $X$  is a time-varying vector field on  $\mathbb{R}$  which is locally integrally bounded with respect to  $t$  and real analytic with respect to  $x$ . However, there does not exist connected neighbourhood  $\bar{U}$  of  $x = 0$  in  $\mathbb{C}$  on which  $X$  can be extended to a holomorphic vector field. To see this, let  $\bar{U} \subseteq \mathbb{C}$  be a connected neighbourhood of  $x = 0$  and let  $\mathbb{T} \subseteq \mathbb{R}$  be a neighbourhood of  $t = 0$ . Let  $\bar{X} : \mathbb{T} \times \bar{U} \rightarrow T\mathbb{C}$  be a time-varying vector field which is measurable in time and

holomorphic in state such that

$$\overline{X}(t, x) = X(t, x) \quad \forall x \in \mathbb{R} \cap \overline{U}, \forall t \in \mathbb{T}.$$

Since  $0 \in \mathbb{T}$ , there exists  $t \in \mathbb{T}$  such that  $\text{cl}(D(0, t)) \subseteq \overline{U}$ . Let us fix this  $t$  and define the real analytic vector field  $X_t : \mathbb{R} \rightarrow T\mathbb{R}$  as

$$X_t(x) = \frac{t^2}{t^2 + x^2} \frac{\partial}{\partial x}, \quad \forall x \in \mathbb{R},$$

and the holomorphic vector field  $\overline{X}_t : \overline{U} \rightarrow T\mathbb{C}$  as

$$\overline{X}_t(z) = \overline{X}(t, z) \quad \forall z \in \overline{U},$$

Then it is clear that  $\overline{X}_t$  is a holomorphic extension of  $X_t$ . However, one can define another holomorphic vector field  $Y : D(0, t) \rightarrow T\mathbb{C}$  by

$$Y(z) = \frac{t^2}{t^2 + z^2} \frac{\partial}{\partial z}, \quad \forall z \in D(0, t),$$

It is easy to observe that  $Y$  is also a holomorphic extension of  $X_t$ . Thus, by the identity theorem, we should have  $Y(z) = \overline{X}_t(z)$ , for all  $z \in D(0, t)$ . Moreover, we should have  $\overline{U} \subseteq D(0, t)$ . However, this is a contradiction with the fact that  $\text{cl}(D(0, t)) \subseteq \overline{U}$ .

As the above example suggests, without any joint condition on time and space, it is impossible to prove any holomorphic extension of a time-varying real analytic vector field to a time-varying holomorphic vector field. It turns out that *local Bochner integrability* is the right joint condition for a time-varying real analytic vector field to ensure the existence of a holomorphic extension. Using the inductive limit characterization of the space of real analytic vector fields, we show that the *global* extension of *locally Bochner integrable* time-varying real analytic vector fields is possible. More specifically, we show that, for a locally Bochner integrable time-varying real analytic vector field  $X$  on  $M$ , there exists a locally Bochner integrable time-varying holomorphic vector field defined on a neighbourhood of  $M$  which agrees with  $X$  on  $M$ . We call this result a *global* extension since it proves the existence of the holomorphic extension of a time-varying vector field to a neighbourhood of its *whole* state domain.

In order to study the holomorphic extension of a *single* locally Bochner integrable time-varying real analytic vector field, the global extension result is a useful tool. However, this extension theorem is indecisive when it comes to questions about holomorphic extension of all elements of a family of locally Bochner integrable time-varying real analytic vector fields to a *single* domain. Using the projective limit characterization of space of real analytic vector fields, we show that one can *locally* extend every element of a bounded family of locally Bochner integrable time-varying real analytic vector fields to a locally Bochner integrable time-varying holomorphic vector field defined on a single domain.

The connection between time-varying vector fields and their flows is of fundamental importance in the theory of differential equations and mathematical control theory. The operator approach for studying time-varying vector fields and their flows in control theory started with the work of Agrachev and Gamkrelidze [1]. One can also find traces of this approach in the nilpotent Lie approximations for studying controllability of systems [31], [32]. In [1] a framework is proposed for studying complete time-varying vector fields and their flows. The cornerstone of this approach is the space  $C^\infty(M)$ , which is both an  $\mathbb{R}$ -algebra and a locally convex vector space. In this framework, a smooth vector field on  $M$  is considered as a derivation of  $C^\infty(M)$  and a smooth diffeomorphism on  $M$  is considered as a unital  $\mathbb{R}$ -algebra isomorphism of  $C^\infty(M)$ . Using a family of seminorms on  $C^\infty(M)$ , weak topologies on the space of derivations of  $C^\infty(M)$  and on the space of unital  $\mathbb{R}$ -algebra isomorphisms of  $C^\infty(M)$  are defined [1]. Then a time-varying vector field is considered as a curve on the space of derivations of  $C^\infty(M)$  and its flow is considered as a curve on the space of  $\mathbb{R}$ -algebra isomorphisms of  $C^\infty(M)$ . While this framework seems to be designed for smooth vector fields and their flows, in [1] and [2] the same framework is used for studying time-varying *real analytic* vector fields and their flows. In [1], using the characterizations of vector fields as derivations and their flows as unital algebra isomorphism, the *nonlinear* differential equation on  $\mathbb{R}^n$  for flows of a complete time-varying vector field is transformed into a *linear* differential equation on the infinite-dimensional locally convex space  $L(C^\infty(\mathbb{R}^n); C^\infty(\mathbb{R}^n))$ . While working with linear differential equations seems to be more desirable than working with their nonlinear counterparts, the fact that the underlying space of this linear differential equation is an infinite-dimensional locally convex spaces makes this study complicated. In fact, the theory of linear ordinary differential equations on a locally convex spaces is completely different from the classical theory of linear differential equations on  $\mathbb{R}^n$  or Banach spaces [21]. In [1] it has been shown that, if the vector field is integrable in time, real analytic in state, and has a bounded holomorphic extension to a neighbourhood of  $\mathbb{R}^n$ , the sequence of Picard iterations for the linear infinite-dimensional differential equation converges in  $L(C^\infty(\mathbb{R}^n); C^\infty(\mathbb{R}^n))$ . In this case, one can represent flows of a time-varying real analytic system as a series of iterated composition of the time-varying vector field.

In this paper, in order to study real analytic vector fields and their flows in a consistent way, we can extend the operator approach of [1] by replacing the locally convex space  $C^\infty(M)$  with  $C^\omega(M)$ . In particular, using the result of [12], we show that there is a one-to-one correspondence between real analytic vector fields on  $M$  and derivations of  $C^\omega(M)$ . Moreover, using the results of [23], we show that  $C^\omega$ -maps are in one-to-one correspondence with unital  $\mathbb{R}$ -algebra homomorphisms on  $C^\omega(M)$ . Thus, using the fact that time-varying real analytic vector fields and their flows are curves on  $L(C^\omega(M); C^\omega(M))$ , we translate the *nonlinear* differential equation governing the flow a time-varying real analytic vector field into a *linear* differential equation on  $L(C^\omega(M); C^\omega(M))$ . In the real analytic case, we show that a solution for the *linear* differential equation of a locally integrally

bounded time-varying real analytic vector field exists and is unique. In particular, using a family of generating seminorms on the space of real analytic functions, we show that the sequence of Picard iterations for our *linear* differential equation on the locally convex space  $L(C^\omega(M); C^\omega(M))$  converges. This will generalize the result of [1, Proposition 2.1] to the case of locally Bochner integrable time-varying real analytic vector fields.

Finally, we define the exponential map between locally integrally bounded time-varying real analytic vector fields and their flows. Using the sequence of Picard iteration for flows of time-varying vector fields, we show that the exponential map is sequentially continuous.

## 2 Mathematical Notations

In this section, we introduce the mathematical notations that we use in this paper.

Let  $r \in \mathbb{R}^{>0}$  and  $x_0 \in \mathbb{R}^n$ , we denote the disk of radius  $r$  with center  $x_0$  by  $D(x_0, r)$ . A **multi-index of order  $m$**  is an element  $(r) = (r_1, r_2, \dots, r_m) \in (\mathbb{Z}_{\geq 0})^m$ . For all multindices  $(r)$  and  $(s)$  of order  $m$ , every  $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$ , and every  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , we define

$$\begin{aligned} |(r)| &= r_1 + r_2 + \dots + r_m, \\ (r) + (s) &= (r_1 + s_1, r_2 + s_2, \dots, r_m + s_m), \\ (r)! &= r_1! r_2! \dots r_m!, \\ x^{(r)} &= x_1^{r_1} x_2^{r_2} \dots x_m^{r_m}, \\ D^{(r)} f(x) &= \frac{\partial^{|r|} f}{\partial x_1^{r_1} \partial x_2^{r_2} \dots \partial x_m^{r_m}}, \\ \binom{(r)}{(s)} &= \binom{r_1}{s_1} \binom{r_2}{s_2} \dots \binom{r_m}{s_m}. \end{aligned}$$

We denote the multi-index  $(0, 0, \dots, 1, \dots, 0) \in (\mathbb{Z}_{\geq 0})^m$ , where 1 is in the  $i$ -th place, by  $\widehat{(i)}$ . One can compare multindices  $(r), (s) \in (\mathbb{Z}_{\geq 0})^m$ . We say that  $(s) \leq (r)$  if, for every  $i \in \{1, 2, \dots, m\}$ , we have  $s_i \leq r_i$ .

The space of all decreasing sequences  $\{a_i\}_{i \in \mathbb{N}}$  such that  $a_i \in \mathbb{R}_{>0}$  and  $\lim_{n \rightarrow \infty} a_n = 0$  is denoted by  $\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ .

For the space  $\mathbb{R}^n$ , we define the Euclidean norm  $\|\cdot\|_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$\|\mathbf{v}\|_{\mathbb{R}^n} = (v_1^2 + v_2^2 + \dots + v_n^2)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbb{R}^n.$$

For the space  $\mathbb{C}^n$ , we define the norm  $\|\cdot\|_{\mathbb{C}^n} : \mathbb{C}^n \rightarrow \mathbb{R}$  as

$$\|\mathbf{v}\|_{\mathbb{C}^n} = (v_1 \bar{v}_1 + v_2 \bar{v}_2 + \dots + v_n \bar{v}_n)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbb{C}^n.$$

Let  $M$  be an  $n$ -dimensional  $C^\nu$ -manifold, where  $\nu \in \{\omega, \text{hol}\}$  and let  $(U, \phi)$  be a coordinate chart on  $M$ . Then we define  $\|\cdot\|_{(U, \phi)} : U \rightarrow \mathbb{R}$  as

$$\|x\|_{(U, \phi)} = \|\phi(x)\|_{\mathbb{R}^n}, \quad \forall x \in U.$$

Let  $M$  be an  $n$ -dimensional  $C^\nu$ -manifold, where  $\nu \in \{\omega, \text{hol}\}$ ,  $(U, \phi)$  be a coordinate chart on  $M$ , and  $f$  be a  $C^\nu$ -function on  $M$ . Then, for every multi-index  $(r)$ , we define  $\|D^{(r)}f(x)\|_{(U, \phi)}$  as

$$\|D^{(r)}f(x)\|_{(U, \phi)} = \|D^{(r)}(f \circ \phi)(\phi^{-1}(x))\|_{\mathbb{F}}, \quad \forall x \in U.$$

When the coordinate chart on  $M$  is understood from the context, we usually omit the subscript  $(U, \phi)$  in the norm.

For every  $C^\nu$ -vector field  $X$  and every multi-index  $(r)$ , we define  $\|D^{(r)}X(x)\|_{(U, \phi)}$  as

$$\|D^{(r)}X(x)\|_{(U, \phi)} = \|D^{(r)}(T\phi \circ X \circ \phi^{-1})(\phi(x))\|_{\mathbb{F}}, \quad \forall x \in U.$$

When the coordinate chart on  $M$  is understood from the context, we usually omit the subscript  $(U, \phi)$  in the norm.

In this paper, we only study holomorphic and real analytic regularity classes. We usually denote  $C^{\text{hol}}$  for the holomorphic regularity and  $C^\omega$  for the real analytic regularity. Let  $M$  be a real analytic manifold, we denote the space of real analytic functions on  $M$  by  $C^\omega(M)$  and the space of real analytic vector fields on  $M$  by  $\Gamma^\omega(TM)$ . Similarly, for a complex manifold  $M$ , we denote the space of holomorphic functions on  $M$  by  $C^{\text{hol}}(M)$  and the space of holomorphic vector fields on  $M$  by  $\Gamma^{\text{hol}}(TM)$ .

We denote the Lebesgue measure on  $\mathbb{R}$  by  $\mathbf{m}$ . Let  $\mathbb{T} \subseteq \mathbb{R}$  be an interval. Then we denote the space of integrable functions on  $\mathbb{T}$  by  $L^1(\mathbb{T})$ .

$$L^1(\mathbb{T}) = \left\{ f : \mathbb{T} \rightarrow \mathbb{R} \mid \int_{\mathbb{T}} |f| d\mathbf{m} < \infty \right\}.$$

The space of continuous functions on  $\mathbb{T}$  is denoted by  $C^0(\mathbb{T})$ .

Let  $V$  be a locally convex space on the field  $\mathbb{F}$ . Then the space of all linear continuous functionals from  $V$  to  $\mathbb{F}$  is the topological dual of  $V$  and is denoted by  $V'$ . We usually denote the space  $V'$  endowed with the weak topology by  $V'_\sigma$  and the space  $V'$  endowed with the strong topology by  $V'_\beta$ .

Let  $V$  and  $W$  be two locally convex spaces on the field  $\mathbb{F}$ . Then we denote their tensor product by  $V \otimes W$ . The projective tensor product of  $V$  and  $W$  is denoted by  $V \otimes_\pi W$  and the injective tensor product of  $V$  and  $W$  is denoted by  $V \otimes_\epsilon W$ . The completion of vector spaces  $V \otimes_\pi W$  and  $V \otimes_\epsilon W$  are denoted by  $V \widehat{\otimes}_\pi W$  and  $V \widehat{\otimes}_\epsilon W$ , respectively.

Let  $\Lambda$  be a set. A binary relation  $\succeq$  **directs**  $\Lambda$  if

1. for every  $i, j, k \in \Lambda$ ,  $i \succeq j$  and  $j \succeq k$  implies  $i \succeq k$ ,
2. for every  $i \in \Lambda$ , we have  $i \succeq i$ ,
3. for every  $i, j \in \Lambda$ , there exists  $m \in \Lambda$  such that  $m \succeq i$  and  $m \succeq j$ .

A **directed set** is a pair  $(\Lambda, \succeq)$  such that  $\succeq$  directs  $\Lambda$ .



Let  $\Lambda$  be a directed set and  $\{V_\alpha\}_{\alpha \in \Lambda}$  be a family of objects indexed by the elements in the set  $\Lambda$  and, for every  $\alpha, \beta \in \Lambda$  such that  $\alpha \succeq \beta$ , there exists a morphism  $f_{\alpha, \beta} : V_\alpha \rightarrow V_\beta$  such that

1.  $f_{\alpha, \alpha} = \text{id}$ , for every  $\alpha \in \Lambda$ , and
2.  $f_{\alpha, \gamma} = f_{\beta, \gamma} \circ f_{\alpha, \beta}$ , for every  $\alpha \succeq \beta \succeq \gamma$ .

Then, the pair  $(V_\alpha, \{f_{\alpha, \beta}\})$  is called an inductive family of objects.

Let  $(V_\alpha, \{f_{\alpha, \beta}\})$  be an inductive family of objects. Then we denote the inductive limit of  $(V_\alpha, \{f_{\alpha, \beta}\})$  by

$$\varinjlim V_\alpha$$

Let  $\Lambda$  be a directed set and  $\{V_\alpha\}_{\alpha \in \Lambda}$  be a family of objects indexed by the elements in the set  $\Lambda$  and, for every  $\alpha, \beta \in \Lambda$  such that  $\alpha \succeq \beta$ , there exists a morphism  $f_{\alpha, \beta} : V_\beta \rightarrow V_\alpha$  such that

1.  $f_{\alpha, \alpha} = \text{id}$ , for every  $\alpha \in \Lambda$ , and
2.  $f_{\alpha, \gamma} = f_{\alpha, \beta} \circ f_{\beta, \gamma}$ , for every  $\alpha \succeq \beta \succeq \gamma$ .

Then, the pair  $(V_\alpha, \{f_{\alpha, \beta}\})$  is called a projective family of objects.

Let  $(V_\alpha, \{f_{\alpha, \beta}\})$  be a projective family of objects. Then we denote the projective limit of  $(V_\alpha, \{f_{\alpha, \beta}\})$  by

$$\varprojlim V_\alpha$$

### 3 Holomorphic extension of real analytic mappings

In this section, we review some of the well-known results about extension of “time-invariant” real analytic functions and vector fields. Since every real analytic mapping is defined on a real analytic manifold, the first step for studying holomorphic extensions of such mappings is to extend the underlying real analytic manifold to a complex manifold. We start with definition of totally real submanifolds of complex manifolds.

**Definition 3.1.** Let  $M$  be a complex manifold with an almost complex structure  $J$ . A submanifold  $N$  of  $M$  is called a **totally real submanifold** if, for every  $p \in N$ , we have  $J(T_p N) \cap T_p N = \{0\}$ .

It can be shown that, for every real analytic manifold  $M$ , there exists a complex manifold  $M^\mathbb{C}$  which contains  $M$  as a totally real submanifold [34].

**Theorem 3.2.** *Let  $M$  be a real analytic manifold. There exists a complex manifold  $M^\mathbb{C}$  such that  $\dim_\mathbb{C} M^\mathbb{C} = \dim_\mathbb{R} M$  and  $M$  is a totally real submanifold of  $M^\mathbb{C}$ .*

The complex manifold  $M^{\mathbb{C}}$  is called a **complexification** of the real analytic manifold  $M$ .

Now that we can extend the real analytic manifolds to a complex manifold, it is time to study holomorphic extensions of real analytic mappings on the complexification of their domains. One can show that every real analytic function (vector field) on  $M$  can be extended to a holomorphic function (vector field) on some complexification of  $M$ .

**Theorem 3.3.** *Let  $M$  be a real analytic manifold and  $X : M \rightarrow TM$  be a real analytic vector field on  $M$ . Then there exists a complexification of  $M$  denoted by  $M^{\mathbb{C}}$  and a holomorphic vector field  $\overline{X} : M^{\mathbb{C}} \rightarrow TM^{\mathbb{C}}$  such that*

$$X(x) = \overline{X}(x), \quad \forall x \in M.$$

The vector field  $\overline{X}$  is called a **holomorphic extension** of the vector field  $X$ .

## 4 Real analytic vector fields as derivations on $C^{\omega}(M)$

In this section, we characterize real analytic vector fields as derivations on the  $\mathbb{R}$ -algebra  $C^{\omega}(M)$ . We will see that this characterization plays an important role in studying flows of time-varying vector fields.

Let  $M$  be a real analytic manifold and let  $X : M \rightarrow TM$  be a real analytic vector field on  $M$ . Then we define the corresponding linear map  $\hat{X} : C^{\omega}(M) \rightarrow C^{\omega}(M)$  as

$$\hat{X}(f) = df(X), \quad \forall f \in C^{\omega}(M).$$

Using the Leibniz rule, this linear map can be shown to be a derivation on the  $\mathbb{R}$ -algebra  $C^{\omega}(M)$ .

More interestingly, one can show there is a one-to-one correspondence between  $C^{\omega}$ -vector fields on  $M$  and derivations on the  $\mathbb{R}$ -algebra  $C^{\omega}(M)$ .

**Theorem 4.1.** *Let  $M$  be a real analytic manifold. If  $X$  is a real analytic vector field, then  $\hat{X}$  is a derivation on the  $\mathbb{R}$ -algebra  $C^{\omega}(M)$ . Moreover, for every derivation  $D : C^{\omega}(M) \rightarrow C^{\omega}(M)$ , there exists a  $C^{\omega}$ -vector field  $X$  such that  $\hat{X} = D$ .*

*Proof.* The sketch of proof is given in [12, Theorem 4.1] □

## 5 Real analytic maps as unital $\mathbb{R}$ -algebra homomorphism on $C^{\omega}(M)$

In this section, we characterize real analytic mappings as unital  $\mathbb{R}$ -algebra homomorphisms on  $C^{\omega}(M)$ .

Let  $\phi : M \rightarrow N$  be a real analytic map. Then we can define the associated map  $\widehat{\phi} : C^\omega(N) \rightarrow C^\omega(M)$  as

$$\widehat{\phi}(f) = f \circ \phi.$$

It is easy to see that  $\widehat{\phi}$  is an  $\mathbb{R}$ -algebra homomorphism. For every  $x \in M$ , one can define the unital  $\mathbb{R}$ -algebra homomorphism  $\text{ev}_x : C^\omega(M) \rightarrow \mathbb{R}$  as

$$\text{ev}_x(f) = f(x).$$

The map  $\text{ev}_x$  is called the evaluation map at  $x \in M$ . The evaluation map plays an essential role in characterizing unital  $\mathbb{F}$ -algebra homomorphisms. The following result is of significant importance.

**Theorem 5.1.** *Let  $M$  be a real analytic manifold. Let  $\phi : C^\omega(M) \rightarrow \mathbb{R}$  be a nonzero and unital  $\mathbb{R}$ -algebra homomorphism. Then there exists  $x \in M$  such that  $\phi = \text{ev}_x$ .*

*Proof.* For the case when  $M$  and  $N$  are open subsets of an Euclidean space, the proof for this theorem is given in [10, Theorem 2.1]. However, it seems that this proof cannot be generalized to include the general real analytic manifolds. Using the techniques and ideas in [23, Proposition 12.5], we present a proof of this theorem for the general case. Let  $\phi : C^\omega(M) \rightarrow \mathbb{R}$  be a unital  $\mathbb{R}$ -algebra homomorphism. It is easy to see that  $\text{Ker}(\phi)$  is a maximal ideal in  $C^\omega(M)$ . For every  $f \in C^\omega(M)$ , we define

$$Z(f) = \{x \in M \mid f(x) = 0\}.$$

**Lemma.** *Let  $n \in \mathbb{N}$  and  $f_1, f_2, \dots, f_n \in \text{Ker}(\phi)$ . Then we have*

$$\bigcap_{i=1}^n Z(f_i) \neq \emptyset.$$

*Proof.* Suppose that we have

$$\bigcap_{i=1}^n Z(f_i) = \emptyset.$$

Then we can define a function  $g \in C^\omega(M)$  as

$$g(x) = \frac{1}{(\sum_{i=1}^n (f_i(x))^2)}, \quad \forall x \in M.$$

Then it is clear that we have

$$\left( \sum_{i=1}^n (f_i)^2 \right) (g) = \mathbf{1},$$

where  $\mathbf{1} : C^\omega(M) \rightarrow \mathbb{F}$  is a unital  $\mathbb{F}$ -algebra homomorphism defined as

$$\mathbf{1}(f) = 1.$$

Since  $\text{Ker}(\phi)$  is an ideal in  $C^\omega(M)$ , we have  $\mathbf{1} \in \text{Ker}(\phi)$ . This implies that  $\phi = 0$ , which is a contradiction of  $\phi$  being unital.  $\square$

Since  $M$  is a real analytic manifold, there exists a  $C^\omega$ -embedding of  $M$  into some  $\mathbb{R}^N$  (one can use Grauert's embedding theorem with  $N = 4n + 2$ ). Let  $x_1, x_2, \dots, x_N$  be the standard coordinate functions on  $\mathbb{R}^N$  and  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$  be their restrictions to  $M$ . Now, for every  $i \in \{1, 2, \dots, N\}$ , consider the functions  $\hat{x}_i - \phi(\hat{x}_i)\mathbf{1} \in C^\omega(M)$ . It is easy to see that

$$\phi(\hat{x}_i - \phi(\hat{x}_i)\mathbf{1}) = \phi(\hat{x}_i) - \phi(\hat{x}_i)\phi(\mathbf{1}) = 0, \quad \forall i \in \{1, 2, \dots, N\}.$$

This implies that, for every  $i \in \{1, 2, \dots, N\}$ , we have  $\hat{x}_i - \phi(\hat{x}_i)\mathbf{1} \in \text{Ker}(\phi)$ . So, by the above Lemma, we get

$$\bigcap_{i=1}^N Z(\hat{x}_i - \phi(\hat{x}_i)\mathbf{1}) \neq \emptyset.$$

Since  $x_1, x_2, \dots, x_N$  are coordinate functions, it is easy to see that  $\bigcap_{i=1}^N Z(\hat{x}_i - \phi(\hat{x}_i)\mathbf{1})$  is just a one-point set. So we set  $\bigcap_{i=1}^N Z(\hat{x}_i - \phi(\hat{x}_i)\mathbf{1}) = \{x\}$ .

Now we proceed to prove the theorem. Note that, for every  $f \in \text{Ker}(\phi)$ , we have

$$Z(f) \cap \{x\} = Z(f) \bigcap \left( \bigcap_{i=1}^N Z(\hat{x}_i - \phi(\hat{x}_i)\mathbf{1}) \right).$$

So, by the above Lemma, we have

$$Z(f) \cap \{x\} \neq \emptyset, \quad \forall f \in \text{Ker}(\phi).$$

This implies that

$$\{x\} \subseteq Z(f), \quad \forall f \in \text{Ker}(\phi).$$

This means that

$$\{x\} \subseteq \bigcap_{f \in \text{Ker}(\phi)} Z(f).$$

This implies that  $\text{Ker}(\phi) \subseteq \text{Ker}(\text{ev}_x)$ . Since  $\text{Ker}(\text{ev}_x)$  and  $\text{Ker}(\phi)$  are both maximal ideals, we have

$$\text{Ker}(\text{ev}_x) = \text{Ker}(\phi).$$

Now let  $f \in C^\omega(M)$ , so we have  $f - f(x)\mathbf{1} \in \text{Ker}(\phi)$ . This implies that

$$0 = \phi(f - f(x)\mathbf{1}) = \phi(f) - f(x).$$

So, for every  $f \in C^\omega(M)$ ,

$$\phi(f) = f(x).$$

Therefore, we have  $\phi = \text{ev}_x$ . □

**Theorem 5.2.** *Let  $M$  and  $N$  be real analytic manifolds. Then, for every  $\mathbb{R}$ -algebra map  $A : C^\omega(M) \rightarrow C^\omega(N)$ , there exists a real analytic map  $\phi : N \rightarrow M$  such that*

$$\widehat{\phi} = A.$$

*Proof.* For every  $x \in N$ , consider the unital  $\mathbb{R}$ -algebra homomorphism  $\text{ev}_x \circ A : C^\omega(M) \rightarrow \mathbb{R}$ . By Theorem 5.1, there exists  $y_x \in M$  such that  $\text{ev}_x \circ A = \text{ev}_{y_x}$ . We define  $\phi : N \rightarrow M$  as

$$\phi(x) = y_x, \quad \forall x \in N.$$

Let  $(U, \eta = (x^1, x^2, \dots, x^m))$  be a coordinate neighbourhood on  $M$  around  $y_x$ . Then, by using the Grauert's embedding theorems, there exist functions  $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^m$  such that, for every  $i \in \{1, 2, \dots, m\}$ , we have

$$\begin{aligned} \tilde{x}^i &\in C^\omega(N), \\ \tilde{x}^i|_U &= x^i. \end{aligned}$$

Thus, for every  $x \in U$ , we have

$$y_x^i = \text{ev}_x \circ A(\tilde{x}^i) = A(\tilde{x}^i)(x), \quad \forall i \in \{1, 2, \dots, m\}.$$

However, for every  $i \in \{1, 2, \dots, m\}$ , we have  $A(\tilde{x}^i) \in C^\omega(N)$ . This implies that, for every  $i \in \{1, 2, \dots, m\}$ , the function  $y_x^i$  is real analytic with respect to  $x$  on the neighbourhood  $U$ . Therefore, the map  $\phi$  is real analytic. One can easily check that  $\widehat{\phi} = A$ .  $\square$

## 6 Inductive limit of topological vector spaces

In this section, we introduce two important classes of inductive limits of locally convex spaces. It turns out that these classes play an essential role in our analysis of extensions of time-varying real analytic vector fields

**Definition 6.1.** Let  $\{V_i, f_i\}_{i \in \mathbb{N}}$  be an inductive family of locally convex spaces and the pair  $(V, \{g_i\}_{i \in \mathbb{N}})$  be the locally convex inductive limit of  $\{V_i, f_i\}_{i \in \mathbb{N}}$ . The inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is **regular** if, for every bounded set  $B \subset V$ , there exists  $m \in \mathbb{N}$  and a bounded set  $B_m \subset V_m$  such that the restriction map  $g_m|_{B_m} : B_m \rightarrow V$  is a bijection onto  $B$ .

The inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is **boundedly retractive** if, for every bounded set  $B \subset V$ , there exists  $m \in \mathbb{N}$  and a bounded set  $B_m \subset V_m$  such that the restriction map  $g_m|_{B_m} : B_m \rightarrow V$  is a homeomorphism onto  $B$ .

While most of the well-known inductive family of locally convex spaces in mathematics are regular and/or boundedly retractive, checking whether an inductive family is regular or boundedly retractive using the definitions is very difficult. However, some properties of the connecting maps of the inductive family can ensure that the inductive limit is regular or boundedly retractive.

**Definition 6.2.** Let  $\{V_i\}_{i \in \mathbb{N}}$  be a family of locally convex topological vector spaces and let  $\{f_i\}_{i \in \mathbb{N}}$  be a family of continuous linear maps such that  $f_i : V_i \rightarrow V_{i+1}$ .

1. The inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is **compact** if, for every  $i \in \mathbb{N}$ , the map  $f_i : V_i \rightarrow V_{i+1}$  is compact.

2. The inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is **weakly compact** if, for every  $i \in \mathbb{N}$ , the map  $f_i : V_i \rightarrow V_{i+1}$  is weakly compact.

In order to study the compactness (weak compactness) of an inductive family of locally convex spaces  $\{V_i, f_i\}_{i \in \mathbb{N}}$ , it is essential that one can characterize the compact (weakly compact) subsets of locally convex vector spaces  $V_i$  for every  $i \in \mathbb{N}$ . For a metrizable topological vector space  $V$ , it is well-known that a set  $K \subseteq X$  is compact if and only if every sequence in  $K$  has a convergent subsequence. However, it is possible that the weak topology on  $V$  is not metrizable. Thus it would be interesting to see if the same characterization holds for weakly compact subsets of  $V$ . Eberlein–Smulian Theorem answers this question affirmatively for Banach spaces [29, Chapter IV, Corollary 2].

**Theorem 6.3.** *Let  $V$  be a Banach space and  $A \subseteq V$ . Then the following statements are equivalent:*

- (i) *The weak closure of  $A$  is weakly compact,*
- (ii) *each sequence of elements of  $A$  has a subsequence that is weakly convergent.*

One can get a partial generalization of the Eberlein–Smulian Theorem for complete locally convex spaces [29, Chapter IV, Theorem 11.2].

**Theorem 6.4.** *Let  $V$  be a complete locally convex space and  $A \subseteq V$ . If every sequence of elements of  $A$  has a subsequence that is weakly convergent, then the weak closure of  $A$  is weakly compact.*

The next theorem shows that an inductive family of locally convex spaces with compact (weakly compact) connecting maps is boundedly retractive (regular).

**Theorem 6.5.** *Let  $\{V_i\}_{i \in \mathbb{N}}$  be a family of locally convex topological vector spaces and let  $\{f_i\}_{i \in \mathbb{N}}$  be a family of linear continuous maps such that  $f_i : V_i \rightarrow V_{i+1}$ . Then*

- 1. *if the inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is weakly compact, then it is regular, and*
- 2. *if the inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is compact, then it is boundedly retractive.*

*Proof.* The first part of this theorem has been proved in [17, Theorem 6] and the second part in [17, Theorem 6'] □

However, one can find boundedly retractive inductive families which are not compact [4]. In [26], Retakh studied an important condition on inductive families of locally convex spaces called condition  $(M)$ .

**Definition 6.6.** Let  $\{V_i\}_{i \in \mathbb{N}}$  be a family of locally convex topological vector spaces and let  $\{f_i\}_{i \in \mathbb{N}}$  be a family of linear continuous maps such that  $f_i : V_i \rightarrow V_{i+1}$ . The inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  satisfies **condition  $(M)$**  if there exists a sequence of absolutely convex neighbourhoods  $\{U_i\}_{i \in \mathbb{N}}$  of 0 such that, for every  $i \in \mathbb{N}$ , we have  $U_i \subseteq V_i$  and,

1. for every  $i \in \mathbb{N}$ , we have  $U_i \subseteq f_i^{-1}(U_{i+1})$ , and
2. for every  $i \in \mathbb{N}$ , there exists  $M_i > 0$  such that, for every  $j > M_i$ , the topologies induced from  $V_j$  on  $U_i$  are all the same.

It can be shown that condition (M) has close connection with regularity of inductive families of locally convex spaces [4].

**Theorem 6.7.** *Let  $\{V_i\}_{i \in \mathbb{N}}$  be a family of normed vector spaces and let  $\{f_i\}_{i \in \mathbb{N}}$  be a family of continuous linear maps such that  $f_i : V_i \rightarrow V_{i+1}$ . Suppose that the inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is regular. Then inductive family  $\{V_i, f_i\}_{i \in \mathbb{N}}$  is boundedly retractive if and only if it satisfies condition (M).*

*Proof.* This theorem is proved in [4, Proposition 9(d)]. □

## 7 Time-varying vector fields and their flows

In this section, we define and study time-varying  $C^\nu$ -vector field.

**Definition 7.1.** Let  $M$  be a  $C^\nu$ -manifold and  $\mathbb{T} \subseteq \mathbb{R}$  be an interval. Then a map  $X : \mathbb{T} \times M \rightarrow TM$  is a **time-varying  $C^\nu$ -vector field** if, for every  $t \in \mathbb{T}$ , the map  $X^t : M \rightarrow TM$  defined as

$$X^t(x) = X(t, x), \quad \forall x \in M,$$

is a  $C^\nu$ -vector field.

Associated to every time-varying  $C^\nu$ -vector field  $X : \mathbb{T} \times M \rightarrow TM$ , one can define a curve  $\widehat{X} : \mathbb{T} \rightarrow \Gamma^\nu(TM)$  such that

$$\widehat{X}(t)(x) = X(t, x), \quad \forall t \in \mathbb{T}, \forall x \in M.$$

It is clear that this correspondence between time-varying  $C^\nu$ -vector fields and curves on the space  $\Gamma^\nu(TM)$  is one-to-one.

In order to study properties of time-varying  $C^\nu$ -vector fields, we need to define a topology on the space  $\Gamma^\nu(TM)$ . In the holomorphic case, the natural topology on the space  $\Gamma^{\text{hol}}(TM)$  is the so-called “compact-open” topology, which has been thoroughly studied in the literature [19, §8].

**Definition 7.2.** Let  $K \subseteq M$  be a compact set. Then we define the seminorm  $p_K^{\text{hol}}$  on  $\Gamma^{\text{hol}}(TM)$  by

$$p_K^{\text{hol}}(X) = \{\|X(x)\| \mid x \in K\}$$

The family of seminorms  $\{p_K^{\text{hol}}\}$  define a locally convex topology on  $\Gamma^{\text{hol}}(TM)$  called the  **$C^{\text{hol}}$ -topology**.

Properties of  $C^{\text{hol}}$ -topology on  $\Gamma^{\text{hol}}(TM)$  has been investigated in [19, §]. The following theorem has been proved in [19, §8.4].

**Theorem 7.3.** *The vector space  $\Gamma^{\text{hol}}(TM)$  equipped with the  $C^{\text{hol}}$ -topology is a Hausdorff, separable, complete, metrizable, and nuclear locally convex space.*

In the real analytic case, it is natural to equip  $\Gamma^{\omega}(TM)$  with the subspace topology from  $\Gamma^{\infty}(TM)$ . However, it can be shown that this topology on  $\Gamma^{\omega}(TM)$  is not complete [15, Chapter 5]. Another topology on  $\Gamma^{\omega}(TM)$  can be defined using the fact that, every real analytic vector field is the germ of a holomorphic vector field, defined on a suitable domain. We will see that this topology on  $\Gamma^{\omega}(TM)$  makes it into a complete, separable, and nuclear space. Each of these properties is essential for validity of our extension results. In [22], using the so-called compact-open topology on space of holomorphic functions, two characterization for a topology on the space of real analytic functions has been developed. This topology on the space  $C^{\omega}(M)$  has been further studied in [9]. In this section, using the same setting as in [22], we define a topology on the space of real analytic functions.

While Two different characterization of this topology has been studied in.

Let  $M$  be a real analytic manifold and  $M^{\mathbb{C}}$  be a complexification of  $M$ . We denote the set of all holomorphic vector fields on  $\bar{U}$  by  $\Gamma^{\text{hol}}(T\bar{U})$ . We define  $\Gamma^{\text{hol},\mathbb{R}}(T\bar{U}) \subseteq \Gamma^{\text{hol}}(T\bar{U})$  as

$$\Gamma^{\text{hol},\mathbb{R}}(T\bar{U}) = \{X \in \Gamma^{\text{hol}}(T\bar{U}) \mid X(x) \in T_x M, \forall x \in M\}$$

Then, for every neighbourhood  $\bar{U} \subseteq M^{\mathbb{C}}$  containing  $M$ , we define the map  $i_{\bar{U}}^{\mathbb{R}} : \Gamma^{\text{hol},\mathbb{R}}(T\bar{U}) \rightarrow \Gamma^{\omega}(TM)$  as

$$i_{\bar{U}}^{\mathbb{R}}(X) = X|_M.$$

If we denote the set of all the neighbourhoods  $\bar{U} \in M^{\mathbb{C}}$  of  $M$  by  $\mathcal{N}_M$ . Then we can define the inductive limit topology on  $\Gamma^{\omega}(TM)$ .

**Definition 7.4.** The **inductive topology** on  $\Gamma^{\omega}(TM)$  is defined as the finest locally convex topology which makes all the maps  $\{i_{\bar{U}}^{\mathbb{R}}\}_{\bar{U} \in \mathcal{N}_M}$  continuous.

Although the definition of inductive topology on  $\Gamma^{\omega}(TM)$  is natural, characterization of properties of  $\Gamma^{\omega}(TM)$  using this topology is not easy. The main reason is that, for non-compact  $M$ , the inductive limit  $\varinjlim_{\bar{U} \in \mathcal{N}_M} \Gamma^{\text{hol},\mathbb{R}}(T\bar{U}) = \Gamma^{\omega}(TM)$  is not countable [9, Fact 14]. However, one can define another topology on the space of real analytic sections which is representable by countable inductive and projective limits [22].

Let  $K \subseteq M$  be a compact set and  $\mathcal{N}_K$  be the set of all neighbourhoods of  $K$  in  $M^{\mathbb{C}}$ . Then we denote the space of germs of holomorphic vector fields around  $K$  by  $\mathcal{G}_K^{\text{hol}}$ . In other words, we have

$$\varinjlim \Gamma^{\text{hol}}(T\bar{U}) = \mathcal{G}_K^{\text{hol}},$$

where the inductive limit is on the directed set  $\mathcal{N}_K$ . One can equip the space  $\mathcal{G}_K^{\text{hol}}$  with the locally convex topology defined using the above inductive limit.



It turns out that  $\mathcal{G}_K^{\text{hol}}$  can also be expressed as a inductive limit of a countable family of Banach spaces [9]. Note that, for every compact set  $K \subseteq M$ , one can choose a sequence of open sets  $\{\bar{U}_n\}_{n \in \mathbb{N}}$  in  $M^{\mathbb{C}}$  such that, for every  $n \in \mathbb{N}$ , we have

$$\text{cl}(\bar{U}_{n+1}) \subseteq \bar{U}_n,$$

and  $\bigcap_{i=1}^{\infty} \bar{U}_i = K$ . Then we have  $\varinjlim_{n \rightarrow \infty} \Gamma^{\text{hol}}(T\bar{U}_n) = \mathcal{G}_K^{\text{hol}}$ .

**Definition 7.5.** Let  $\bar{U}$  be an open set in  $M^{\mathbb{C}}$ . We define the map  $p_{\bar{U}} : \Gamma^{\text{hol}}(T\bar{U}) \rightarrow [0, \infty]$  by

$$p_{\bar{U}}(X) = \sup\{\|X(x)\| \mid x \in \bar{U}\}, \quad \forall X \in \Gamma^{\text{hol}}(\bar{U}).$$

Then  $\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$  is a subspace of  $\Gamma^{\text{hol}}(T\bar{U})$  defined as

$$\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}) = \{X \in \Gamma^{\text{hol}}(T\bar{U}) \mid p_{\bar{U}}(X) < \infty\}.$$

We equip  $\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$  with the norm  $p_{\bar{U}}$  and define the inclusion  $\rho_{\bar{U}} : \Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}) \rightarrow \Gamma^{\text{hol}}(T\bar{U})$  as

$$\rho_{\bar{U}}(X) = X, \quad \forall X \in \Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}).$$

**Theorem 7.6.** *The space  $(\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}), p_{\bar{U}})$  is a Banach space and the map  $\rho_{\bar{U}} : \Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}) \rightarrow \Gamma^{\text{hol}}(T\bar{U})$  is a compact continuous map.*

*Proof.* Let  $K$  be a compact subset of  $M \cap \bar{U}$ . Then, for every  $X \in \Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$ , we have  $p_K^{\text{hol}}(\rho_{\bar{U}}(X)) = p_K^{\text{hol}}(X) \leq p_{\bar{U}}(X)$ , which implies that  $\rho_{\bar{U}}$  is continuous. Now consider the open set  $p_{\bar{U}}^{-1}([0, 1))$  in  $\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$ . The set  $p_{\bar{U}}^{-1}([0, 1))$  is bounded and  $\rho_{\bar{U}}$  is continuous. So

$$\rho_{\bar{U}}\left(p_{\bar{U}}^{-1}([0, 1))\right),$$

is bounded in  $\Gamma^{\text{hol}}(T\bar{U})$ . Since  $\Gamma^{\text{hol}}(T\bar{U})$  is nuclear, it satisfies the Heine–Borel property [29, Chapter III, §7]. Thus, the bounded set  $\rho_{\bar{U}}\left(p_{\bar{U}}^{-1}([0, 1))\right)$  is relatively compact in  $\Gamma^{\text{hol}}(T\bar{U})$ . So  $\rho_{\bar{U}}$  is compact.

Now we show that  $(\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}), p_{\bar{U}})$  is a Banach space. Let  $\{X_n\}_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$ . It suffices to show that there exists  $X \in \Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$  such that  $\lim_{n \rightarrow \infty} X_n = X$  in the topology induced by  $p_{\bar{U}}$  on  $\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$ . Since  $\rho_{\bar{U}}$  is continuous, the sequence  $\{X_n\}_{n \in \mathbb{N}}$  is Cauchy in  $\Gamma^{\text{hol}}(T\bar{U})$ . Since  $\Gamma^{\text{hol}}(T\bar{U})$  is complete, there exists  $X \in \Gamma^{\text{hol}}(T\bar{U})$  such that  $\lim_{n \rightarrow \infty} X_n = X$  in the  $C^{\text{hol}}$ -topology. Now we show that  $\lim_{n \rightarrow \infty} X_n = X$  in the topology of  $(\Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U}), p_{\bar{U}})$  and  $X \in \Gamma_{\text{bdd}}^{\text{hol}}(T\bar{U})$ . Let  $\epsilon > 0$ . Then there exists  $N \in \mathbb{N}$  such that, for every  $n, m > N$ , we have

$$p_{\bar{U}}(X_n - X_m) < \frac{\epsilon}{2}.$$

This implies that, for every  $z \in U$  and every  $n, m > N$ , we have

$$\|X_n(z) - X_m(z)\| < \frac{\epsilon}{2}.$$

So, for every  $z \in U$  and every  $n > N$ , we choose  $m_z > N$  such that

$$\|X_m(z) - X(z)\| < \frac{\epsilon}{2}, \quad \forall m \geq m_z.$$

This implies that, for every  $z \in U$ , we have

$$\|X(z) - X_n(z)\| < \|X_n(z) - X_{m_z}(z)\| + \|X_{m_z}(z) - X(z)\| < \epsilon.$$

So, for every  $n > N$ , we have

$$p_{\overline{U}}(X_n - X) < \epsilon.$$

This completes the proof.  $\square$

**Theorem 7.7.** *Let  $K$  be a compact set and  $\{\overline{U}_n\}_{n \in \mathbb{N}}$  be a sequence of open, relatively compact neighbourhoods of  $K$  in  $M^{\mathbb{C}}$  such that*

$$\text{cl}(\overline{U}_{n+1}) \subseteq \overline{U}_n, \quad \forall n \in \mathbb{N},$$

*and  $\bigcap_{n \in \mathbb{N}} \overline{U}_n = K$ . Then we have  $\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(T\overline{U}_n) = \mathcal{G}_K^{\text{hol}}$ . Moreover, the inductive limit is compact.*

*Proof.* For every  $n \in \mathbb{N}$ , we define  $r_n : \Gamma^{\text{hol}}(T\overline{U}_n) \rightarrow \Gamma_{\text{bdd}}^{\text{hol}}(T\overline{U}_{n+1})$  as

$$r_n(X) = X|_{\overline{U}_{n+1}}, \quad \forall X \in \Gamma^{\text{hol}}(T\overline{U}_n).$$

For every compact set  $C$  with  $\overline{U}_{n+1} \subseteq C \subseteq \overline{U}_n$ , we have  $p_{\overline{U}_{n+1}}(X) \leq p_C^{\text{hol}}(X)$ . This implies that the map  $r_n$  is continuous and we have the following diagram:

$$\Gamma_{\text{bdd}}^{\text{hol}}(T\overline{U}_n) \xrightarrow{\rho_{U_n}} \Gamma^{\text{hol}}(T\overline{U}_n) \xrightarrow{r_n} \Gamma_{\text{bdd}}^{\text{hol}}(T\overline{U}_{n+1}) \xrightarrow{\rho_{U_{n+1}}} \Gamma^{\text{hol}}(T\overline{U}_{n+1}).$$

Since all maps in the above diagram are linear and continuous, by the universal property of the inductive limit of locally convex spaces, we have

$$\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(T\overline{U}_n) = \varinjlim_{n \rightarrow \infty} \Gamma^{\text{hol}}(T\overline{U}_n) = \mathcal{G}_K^{\text{hol}}.$$

Moreover, for every  $n \in \mathbb{N}$ , the map  $\rho_{\overline{U}_n}$  is compact and  $r_n$  is continuous. So the composition  $r_n \circ \rho_{\overline{U}_n}$  is also compact [16, §17.1, Proposition 1]. This implies that the direct limit

$$\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(T\overline{U}_n) = \mathcal{G}_K^{\text{hol}}$$

is compact.  $\square$

One can define the subspace  $\mathcal{G}_K^{\text{hol}, \mathbb{R}} \subseteq \mathcal{G}_K^{\text{hol}}$  as

$$\mathcal{G}_K^{\text{hol}, \mathbb{R}} = \{[X]_K \mid \exists \overline{U} \in \mathcal{N}_K, X \in \Gamma^{\text{hol}, \mathbb{R}}(T\overline{U})\}$$

Let  $\{K_n\}_{n \in \mathbb{N}}$  be a compact exhaustion for  $M$ . Then we have

$$\varprojlim \mathcal{G}_{K_n}^{\text{hol}, \mathbb{R}} = \Gamma^{\omega}(TM).$$

Using this projective limit, one can define another topology on space of real analytic vector fields.

**Definition 7.8.** Let  $\{K_n\}_{n \in \mathbb{N}}$  be a compact exhaustion for  $M$ . Then we define the **projective limit topology** on  $\Gamma^\omega(TM)$  as the projective limit topology defined using the following projective family of locally convex spaces:

$$\varprojlim \mathcal{G}_{K_n}^{\text{hol}, \mathbb{R}} = \Gamma^\omega(TM).$$

It is easy to show that the projective limit topology on  $\Gamma^\omega(TM)$  does not depend on a specific choice of the compact exhaustion  $\{K_n\}_{n \in \mathbb{N}}$  for  $M$ .

It is a deep theorem of Martineau that the projective limit topology and inductive limit topology on  $\Gamma^\omega(TM)$  coincide [22]. We denote this topology on  $\Gamma^\omega(TM)$  by the  $C^\omega$ -topology. One can show that this topology has nice properties [15, §5.3]

**Theorem 7.9.** *The vector space  $\Gamma^\omega(TM)$  equipped with the  $C^\omega$ -topology is a Hausdorff, separable, complete, and nuclear locally convex space.*

As is shown in Theorem 4.1 the real analytic vector fields are exactly the derivations of the  $\mathbb{R}$ -algebra  $C^\omega(M)$ . Since derivations of  $C^\omega(M)$  are linear mappings from  $C^\omega(M)$  to  $C^\omega(M)$ , it would be interesting to study the more general space of linear mapping from  $C^\omega(N)$  to  $C^\omega(M)$ .

**Definition 7.10.** Let  $M$  and  $N$  be real analytic manifolds. The space of linear mapping from  $C^\omega(N)$  to  $C^\omega(M)$  is denoted by  $L(C^\omega(N); C^\omega(M))$ .

One can define different topologies on  $L(C^\omega(N); C^\omega(M))$ , using the  $C^\omega$ -topologies on the spaces  $C^\omega(M)$  and  $C^\omega(N)$ . In this section, we focus on the topology of pointwise convergence on  $L(C^\omega(N); C^\omega(M))$ . We will see that this topology is consistent with the  $C^\omega$ -topology on  $\Gamma^\omega(TM)$ .

**Definition 7.11.** For  $f \in C^\omega(M)$ , we define the map  $\mathcal{L}_f : L(C^\omega(N); C^\omega(M)) \rightarrow C^\omega(N)$  as

$$\mathcal{L}_f(X) = X(f).$$

The **topology of pointwise convergence** on  $L(C^\omega(N); C^\omega(M))$  is the projective topology with respect to the family  $\{C^\omega(N), \mathcal{L}_f\}_{f \in C^\omega(M)}$ .

It can be shown that  $L(C^\omega(N); C^\omega(M))$  equipped with the topology of pointwise convergence has many nice properties.

**Theorem 7.12.** *The vector space  $L(C^\omega(N); C^\omega(M))$  with the topology of pointwise convergence is a Hausdorff, separable, complete, and nuclear locally convex space.*

*Proof.* We show that  $L(C^\omega(N); C^\omega(M))$  is a closed subspace of  $C^\omega(N)^{C^\omega(M)}$ , if we equip the latter space with its natural topology of pointwise convergence. Let  $\{X_\alpha\}_{\alpha \in \Lambda}$  be a converging net in  $L(C^\omega(N); C^\omega(M))$  with the limit  $X \in C^\omega(N)^{C^\omega(M)}$ . We show that  $X$  is linear. Let  $f, g \in C^\omega(M)$  and  $c \in \mathbb{F}$ . Then we have

$$X_\alpha(f + cg) = X_\alpha(f) + cX_\alpha(g), \quad \forall \alpha \in \Lambda.$$

By taking limit on  $\alpha$ , we get

$$X(f + cg) = X(f) + cX(g).$$

This implies that  $X$  is linear and therefore  $L(C^\omega(N); C^\omega(M))$  is a closed subspace of  $C^\omega(N)^{C^\omega(M)}$ .

Since  $C^\omega(N)$  is Hausdorff, it is clear that  $C^\omega(N)^{C^\omega(M)}$  is Hausdorff. This implies that  $L(C^\omega(M); C^\omega(N)) \subseteq C^\omega(N)^{C^\omega(M)}$  is Hausdorff. Let  $\mathfrak{c}$  be the cardinality of the continuum. Note that  $C^\omega(M) \subseteq C^0(M)$  and  $M$  is second countable and hence separable. This implies that the cardinality of  $C^0(M)$  is  $\mathfrak{c}$  [13, Chapter 5, Theorem 2.6(a)]. Therefore, the cardinality of  $C^\omega(M)$  is at most  $\mathfrak{c}$ . The product of  $\mathfrak{c}$  separable spaces is separable [35, Theorem 16.4(c)]. This implies that  $C^\omega(N)^{C^\omega(M)}$  is separable. Since  $L(C^\omega(M); C^\omega(N))$  is a closed subspace of  $C^\omega(N)^{C^\omega(M)}$ , it is separable [35, Theorem 16.4]. Note that  $C^\omega(N)$  is complete. This implies that  $C^\omega(N)^{C^\omega(M)}$  is complete [29, Chapter II, §5.3]. Since  $L(C^\omega(M); C^\omega(N))$  is a closed subspace of  $C^\omega(N)^{C^\omega(M)}$ , it is complete. The product of any arbitrary family of nuclear locally convex vector spaces is nuclear [29, Chapter III, §7.4]. This implies that  $C^\omega(N)^{C^\omega(M)}$  is nuclear. Since every closed subspace of nuclear space is nuclear [29, Chapter III, §7.4],  $L(C^\omega(M); C^\omega(N))$  is also nuclear.  $\square$

We have already mentioned that real analytic vector fields on  $M$  are exactly derivations on  $\Gamma^\omega(TM)$ . Thus, we have

$$\Gamma^\omega(TM) \subseteq L(C^\omega(N); C^\omega(M)).$$

Therefore, the topology of pointwise convergence on  $L(C^\omega(N); C^\omega(M))$  will induce a subspace topology on  $\Gamma^\omega(TM)$ . It is interesting to note that this subspace topology on  $\Gamma^\omega(TM)$  and the  $C^\omega$ -topology on  $\Gamma^\omega(TM)$  are the same [15, Theorem 5.8].

**Theorem 7.13.** *The  $C^\omega$ -topology on  $\Gamma^\omega(TM)$  coincides with the subspace topology from  $L(C^\omega(N); C^\omega(M))$ .*

Thus, it is reasonable to denote the topology of pointwise convergence on  $L(C^\omega(N); C^\omega(M))$  by the  $C^\omega$ -topology.

It is well-known every locally convex topology can be characterized using a family of seminorms [28, Theorem 1.37]. Since the vector space  $\Gamma^\omega(TM)$  equipped with the  $C^\omega$ -topology is a locally convex space, it would be interesting to provide an explicit family of seminorm for the locally convex space  $\Gamma^\omega(TM)$ . As to our knowledge, the first characterization of the space of germs of holomorphic functions on compact subsets of  $\mathbb{C}^n$  using an explicit family of seminorms has been developed in [25]. In the notes [9], a family of seminorm on  $\Gamma^\omega(TM)$  has been introduced and it has been mentioned that the  $C^\omega$ -topology on  $C^\omega(M)$  is generated by this family of seminorms. For the case  $M = \mathbb{R}$ , the complete proof of the fact that this family of seminorms generates the  $C^\omega$ -topology on  $C^\omega(\mathbb{R})$  has been given in [33]. Using the idea of the proof in [33], a complete characterization of the locally

convex space  $\Gamma^\omega(TM)$  using a family of seminorm has been given in [15]. In this section, we provide a family of seminorms for the  $C^\omega$ -topology on the space  $L(C^\omega(M); C^\omega(N))$ . Since  $\Gamma^\omega(TM)$  can be considered as a subspace of  $L(C^\omega(M); C^\omega(M))$ , this family of seminorms also gives a family of generating seminorms for the  $C^\omega$ -topology on  $\Gamma^\omega(TM)$ .

**Definition 7.14.** Let  $\mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$  denote the set of all decreasing sequences  $\{a_n\}_{n \in \mathbb{Z}_{\geq 0}}$  such that, for every  $n \in \mathbb{Z}_{\geq 0}$ , we have  $0 < a_n \leq d$  and

$$\lim_{n \rightarrow \infty} a_n = 0.$$

**Definition 7.15.** Let  $U$  be a coordinate chart on  $N$ ,  $K \subseteq U$  be a compact set,  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , and  $f \in C^\omega(M)$ . Then, for every  $X \in L(C^\omega(M); C^\omega(N))$ , we define

$$p_{K, \mathbf{a}, f}^\omega(X) = \left\{ \frac{a_0 a_1 \cdots a_{|r|}}{|(r)|!} \left\| D^{(r)} X f(x) \right\| \mid |(r)| \in \mathbb{Z}_{\geq 0}, x \in K \right\}$$

Using [15, Theorem 5.5], we have

**Theorem 7.16.** *The family of seminorms  $\{p_{K, \mathbf{a}, f}^\omega\}$  generates the  $C^\omega$ -topology on  $L(C^\omega(M); C^\omega(N))$*

Now, we prove a specific approximation for the seminorms on  $\Gamma^\omega(M)$ . In section 10, we will see that this approximation is useful in studying flows of time-varying real analytic vector fields. Let  $d > 0$  be a positive real number and  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ . For every  $n \in \mathbb{N}$ , we define the sequence  $\mathbf{a}_n = (a_{n,0}, a_{n,1}, \dots, a_{n,m}, \dots)$  as

$$a_{n,m} = \begin{cases} \left(\frac{m+1}{m}\right)^n a_m, & m > n, \\ \left(\frac{m+1}{m}\right)^m a_m, & m \leq n. \end{cases}$$

Associated to every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , we define the sequence  $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$  as

$$b_{n,m} = \begin{cases} a_{n,m}, & m = 0, m = 1, \\ \left(\frac{(m+1)(m+2)}{(m-1)(m)}\right) a_{n,m}, & m > 1. \end{cases}$$

**Lemma 7.17.** *Let  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ . Then, for every  $n \in \mathbb{Z}_{\geq 0}$ , we have  $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$  and, for every  $m, n \in \mathbb{Z}_{\geq 0}$ , we have*

$$\begin{aligned} a_{n,m} &\leq e a_m, \\ \frac{(m+1)}{(n+1)} &\leq \frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})}, \end{aligned}$$

where  $e$  is the Euler constant. Moreover, for every  $n \in \mathbb{Z}_{\geq 0}$  we have  $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$  and, for every  $m > 1$ , we have

$$\begin{aligned} b_{n,m} &\leq 6e a_m, \\ \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,m})}{(m-2)!} &= \frac{(b_{n,0})(b_{n,1}) \cdots (b_{n,m})}{m!}. \end{aligned}$$

*Proof.* Let  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ . Then by definition of  $\mathbf{a}_n$ , for  $n < m$ , we have

$$a_{n,m} = \left(\frac{m+1}{m}\right)^n a_m \leq \left(\frac{m+1}{m}\right)^m a_m \leq e a_m$$

For  $n \geq m$ , we have

$$a_{n,m} = \left(\frac{m+1}{m}\right)^m a_m \leq e a_m.$$

This implies that  $\lim_{m \rightarrow \infty} a_{n,m} = 0$ . Moreover, for every  $m, n \in \mathbb{Z}_{\geq 0}$ , we have

$$a_{n,m} \leq e a_m \leq e d.$$

So we have  $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$ . Let  $m, n \in \mathbb{Z}_{\geq 0}$  be such that  $n+1 > m+1$ . Then we have

$$\frac{a_{n+1,m+1}}{a_{n,m+1}} = 1.$$

So we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} \geq 1.$$

Since we have  $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$ , we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} \geq 1 \geq \frac{m+1}{n+1}.$$

Now suppose that  $m, n \in \mathbb{Z}_{\geq 0}$  are such that  $n+1 \leq m+1$ . Then we have

$$\frac{a_{n+1,m+1}}{a_{n,m+1}} = \left(\frac{m+1}{m}\right).$$

Therefore, we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} = \left(\frac{n+2}{n+1}\right) \left(\frac{n+3}{n+2}\right) \cdots \left(\frac{m+2}{m+1}\right) = \frac{m+2}{n+1} > \frac{m+1}{n+1}.$$

Since we have  $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$ , we get

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m})} \geq \frac{m+1}{n+1}.$$

So, for all  $m, n \in \mathbb{Z}_{\geq 0}$ , we have

$$\frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,m+1})}{(a_{n,0})(a_{n,1}) \cdots (a_{n,m+1})} \geq \frac{m+1}{n+1}.$$

Finally, since  $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, ed)$  and we have  $\frac{(m+2)(m+1)}{m(m-1)} \leq 6$ , for all  $m > 1$ , we get

$$b_{n,m} = \frac{(m+2)(m+1)}{m(m-1)} a_{n,m} \leq 6 a_{n,m}.$$

So we have  $\lim_{m \rightarrow \infty} b_{n,m} = 6 \lim_{m \rightarrow \infty} a_{n,m} = 0$ . Moreover, we have

$$b_{n,m} \leq 6 a_{n,m} \leq 6 e a_m \leq 6 e d.$$

Thus we get  $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$ . This completes the proof of the lemma.  $\square$

**Theorem 7.18.** *Let  $M$  be a real analytic manifold of dimension  $N$ ,  $X \in \Gamma^\omega(TM)$ , and  $f \in C^\omega(M)$ . Let  $U$  be a coordinate neighbourhood in  $M$  and  $K \subseteq U$  be compact. For every  $d > 0$ , every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}, d)$ , and every  $n \in \mathbb{Z}_{\geq 0}$ , we have*

$$p_{K, \mathbf{a}_n}^\omega(X(f)) \leq 4N(n+1) \max_i \{p_{K, \mathbf{b}_n}^\omega(X^i)\} p_{K, \mathbf{a}_{n+1}}^\omega(f). \quad (7.1)$$

*Proof.* Let  $(U, \phi = (x^1, x^2, \dots, x^N))$  be a coordinate chart on  $M$ . We first prove that, for every  $f, g \in C^\omega(M)$ , every multi-index  $(r)$  and every  $x \in U$ , we have

$$\begin{aligned} \left\| D^{(r)}(fg)(x) \right\| &\leq \\ &\sum_{j=0}^{|r|} \binom{|r|}{j} \sup \left\{ \left\| (D^{(l)} f(x)) \right\| \mid |l| = j \right\} \sup \left\{ \left\| (D^{(l)} g(x)) \right\| \mid |l| = |r| - j \right\}. \end{aligned}$$

We prove this by induction on  $|r|$ . If  $|r| = 1$ , then it is clear that, for every  $x \in U$ , we have

$$\left\| \frac{\partial}{\partial x^i}(fg)(x) \right\| = \left\| \frac{\partial f}{\partial x^i}(x)g(x) + \frac{\partial g}{\partial x^i}(x)f(x) \right\| \leq \left\| \frac{\partial f}{\partial x^i}(x)g(x) \right\| + \left\| \frac{\partial g}{\partial x^i}(x)f(x) \right\|.$$

Now suppose that, for every  $x \in U$  and for every  $(r)$  such that  $|r| \in \{1, 2, \dots, k\}$ , we have

$$\begin{aligned} \left\| D^{(r)}(fg)(x) \right\| &\leq \\ &\sum_{j=0}^{|r|} \binom{|r|}{j} \sup \left\{ \left\| (D^{(l)} f(x)) \right\| \mid |l| = j \right\} \sup \left\{ \left\| (D^{(l)} g(x)) \right\| \mid |l| = |r| - j \right\}. \end{aligned}$$

Let  $(l)$  be a multi-index with  $|l| = k+1$ . Then there exists  $i \in \{1, 2, \dots, N\}$  and  $(r)$  with  $|r| = k$  such that  $(l) = (r) + (\hat{i})$ . So, for every  $x \in U$ , we have

$$\begin{aligned} \left\| D^{(l)}(fg)(x) \right\| &= \left\| D^{(r)} \left( \frac{\partial}{\partial x^i}(fg) \right)(x) \right\| \\ &\leq \left\| D^{(r)} \left( \frac{\partial f}{\partial x^i} g \right)(x) \right\| + \left\| D^{(r)} \left( \frac{\partial g}{\partial x^i} f \right)(x) \right\| \\ &\leq \sum_{j=0}^{|r|} \binom{|r|}{j} \sup \left\{ \left\| (D^{(l)} \frac{\partial f}{\partial x^i}(x)) \right\| \mid |l| = j \right\} \sup \left\{ \left\| (D^{(l)} g(x)) \right\| \mid |l| = |r| - j \right\} + \\ &\quad \binom{|r|}{j} \sup \left\{ \left\| (D^{(l)} f(x)) \right\| \mid |l| = j \right\} \sup \left\{ \left\| (D^{(l)} \frac{\partial g}{\partial x^i}(x)) \right\| \mid |l| = |r| - j \right\} \\ &= \sum_{j=0}^{|r|} \left( \binom{|r|}{j-1} + \binom{|r|}{j} \right) \sup \left\{ \left\| (D^{(l)} f(x)) \right\| \mid |l| = j \right\} \times \\ &\quad \sup \left\{ \left\| (D^{(l)} g(x)) \right\| \mid |l| = |r| - j + 1 \right\} \\ &= \sum_{j=0}^{|r|} \binom{|r|+1}{j} \sup \left\{ \left\| (D^{(l)} f(x)) \right\| \mid |l| = j \right\} \sup \left\{ \left\| (D^{(l)} g(x)) \right\| \mid |l| = |r| - j + 1 \right\}. \end{aligned}$$

This completes the induction. Note that in the coordinate neighbourhood  $U$ , we have

$$X(f) = \sum_{i=1}^N X(x^i) \frac{\partial f}{\partial x^i}.$$

Thus, for every  $x \in U$ , we get

$$\begin{aligned} & \left\| D^{(r)}(X(f))(x) \right\| \leq \\ & \sum_{j=0}^{|r|} \sum_{i=1}^N \binom{|r|}{j} \sup \left\{ \left\| (D^{(l)} X^i(x)) \right\| \mid |l| = |r| - j \right\} \sup \left\{ \left\| D^{(l)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |l| = j \right\}. \end{aligned} \quad (7.2)$$

Now let  $d > 0$  and  $\mathbf{a} \in \mathbf{c}_0^\perp(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ . Multiplying both sides of equation (7.2) by  $\frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!}$ , we get

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \left\| D^{(r)}(X(f))(x) \right\| \leq \\ & \sum_{i=1}^N \sum_{l=0}^{|r|} \left( \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,l+1})}{l!} \sup \left\{ \left\| D^{(s)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |s| = l \right\} \right) \\ & \times \left( \frac{(a_{n,l+2})(a_{n,l+3}) \cdots (a_{n,|r|})}{(|r| - l)!} \sup \left\{ \left\| D^{(s)} X^i(x) \right\| \mid |s| = |r| - l \right\} \right), \quad \forall x \in U. \end{aligned}$$

Since the sequence  $\mathbf{a}_n$  is decreasing, we have

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \left\| D^{(r)}(X(f))(x) \right\| \\ & \leq \sum_{i=1}^N \sum_{l=0}^{|r|} \left( \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,l+1})}{l!} \sup \left\{ \left\| D^{(s)} \frac{\partial f}{\partial x^i}(x) \right\| \mid |s| = l \right\} \right) \\ & \times \left( \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|-l-2})}{(|r| - l)!} \sup \left\{ \left\| D^{(s)} X^i(x) \right\| \mid |s| = |r| - l \right\} \right), \quad \forall x \in U. \end{aligned}$$

Using the above lemma, we have

$$\begin{aligned} \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,l+1})}{(l)!} & \leq (n+1) \frac{(a_{n+1,0})(a_{n+1,1} \cdots (a_{n+1,l+1}))}{(l+1)!}, \\ \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|-l-2})}{(|r| - l - 2)!} & = \frac{(b_{n+1,0})(b_{n+1,1} \cdots (b_{n+1,|r|-l}))}{(|r| - l)!} \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \left\| D^{(r)}(X(f))(x) \right\| \\ & \leq \sum_{i=1}^N \sum_{l=0}^{|r|} \frac{(n+1)}{(|r| - l)(|r| - l - 1)} \left( \frac{(a_{n+1,0})(a_{n+1,1}) \cdots (a_{n+1,l+1})}{(l+1)!} \sup \left\{ \left\| D^{(s)} f(x) \right\| \mid |s| = l+1 \right\} \right) \\ & \quad \left( \frac{(b_{n,0})(b_{n,1}) \cdots (b_{n,|r|-l})}{(|r| - l)!} \sup \left\{ \left\| D^{(s)} X^i(x) \right\| \mid |s| = |r| - l \right\} \right), \quad \forall x \in U. \end{aligned}$$



Thus, by taking the supremum over  $l \in \mathbb{Z}_{\geq 0}$  and  $x \in K$  of the two term in the right hand side of the above inequality, we get

$$\begin{aligned} & \frac{(a_{n,0})(a_{n,1}) \cdots (a_{n,|r|})}{|r|!} \left\| D^{(r)}(X(f))(x) \right\| \\ & \leq N(n+1) p_{K, \mathbf{a}_{n+1}}^\omega(f) p_{K, \mathbf{b}_n}^\omega(X^i) \sum_{l=0}^{|r|} \frac{1}{(|r|-l)(|r|-l-1)} \\ & \leq 4N(n+1) p_{K, \mathbf{a}_{n+1}}^\omega(f) p_{K, \mathbf{b}_n}^\omega(X^i), \quad \forall x \in U. \end{aligned}$$

By taking the supremum of the left hand side of the above inequality over  $|r| \in \mathbb{N}$  and  $x \in K$ , for every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0}, d)$ , we get

$$p_{K, \mathbf{a}_n}^\omega(X(f)) \leq 4N(n+1) \max_i \{p_{K, \mathbf{b}_n}^\omega(X^i)\} p_{K, \mathbf{a}_{n+1}}^\omega(f).$$

□

Using the  $C^\omega$ -topology on the space  $\Gamma^\omega(TM)$ , one can study different properties of time-varying real analytic vector fields as curves on  $\Gamma^\omega(TM)$ . In this part, we introduce the notions of integrability and absolute continuity for curves on locally convex spaces.

**Definition 7.19.** Let  $V$  be a locally convex space with a family of generating seminorms  $\{p_i\}_{i \in \Lambda}$  and let  $\mathbb{T} \subseteq \mathbb{R}$  be an interval. A curve  $f : \mathbb{T} \rightarrow V$  is **integrally bounded** if, for every  $i \in \mathbb{N}$ , we have

$$\int_{\mathbb{T}} p_i(f(\tau)) d\tau < \infty.$$

A function  $s : \mathbb{T} \rightarrow V$  is a **simple function** if there exist  $n \in \mathbb{N}$ , measurable sets  $A_1, A_2, \dots, A_n \subseteq \mathbb{T}$ , and  $v_1, v_2, \dots, v_n \in V$  such that  $\mathbf{m}(A_i) < \infty$  for every  $i \in \{1, 2, \dots, n\}$  and

$$s = \sum_{i=1}^n \chi_{A_i} v_i.$$

The set of all simple functions from the interval  $\mathbb{T}$  to the vector space  $V$  is denoted by  $S(\mathbb{T}; V)$ .

One can define **Bochner integral** of a simple function  $s = \sum_{i=1}^n \chi_{A_i} v_i$  as

$$\int_{\mathbb{T}} s(\tau) d\tau = \sum_{i=1}^n \mathbf{m}(A_i) v_i.$$

It is easy to show that the above expression does not depend on choice of  $A_1, A_2, \dots, A_n \subseteq \mathbb{T}$ .

A curve  $f : \mathbb{T} \rightarrow V$  is **Bochner approximable** if there exists a net  $\{f_\alpha\}_{\alpha \in \Lambda}$  of simple functions on  $V$  such that, for every seminorm  $p_i$ , we have

$$\lim_{\alpha} \int_{\mathbb{T}} p_i(f_\alpha(\tau) - f(\tau)) d\tau = 0.$$

The net of simple functions  $\{f_\alpha\}_{\alpha \in \Lambda}$  is an **approximating net** for the mapping  $f$ .

**Theorem 7.20** ([3]). *Let  $\{f_\alpha\}_{\alpha \in \Lambda}$  be an approximating net for the mapping  $f : \mathbb{T} \rightarrow V$ . Then  $\{\int_{\mathbb{T}} f_\alpha(\tau) d\tau\}_{\alpha \in \Lambda}$  is a Cauchy net.*

Let  $f : \mathbb{T} \rightarrow V$  be a mapping and let  $\{f_\alpha\}_{\alpha \in \Lambda}$  be an approximating net of simple functions for  $f$ . If the net  $\{\int_{\mathbb{T}} f_\alpha(\tau) d\tau\}_{\alpha \in \Lambda}$  converges, then we say that  $f$  is **Bochner integrable**. One can show that the limit of  $\{\int_{\mathbb{T}} f_\alpha(\tau) d\tau\}_{\alpha \in \Lambda}$  doesn't depend on the choice of approximating net and is called **Bochner integral of  $f$** . The set of all Bochner integrable curves from  $\mathbb{T}$  to  $V$  is denoted by  $L^1(\mathbb{T}; V)$ .

A curve  $f : \mathbb{T} \rightarrow V$  is **locally Bochner integrable** if for every compact set  $J \subseteq \mathbb{T}$ , the map  $f|_J$  is Bochner integrable. The set of all locally Bochner integrable curves from  $\mathbb{T}$  to  $V$  is denoted by  $L^1_{\text{loc}}(\mathbb{T}; V)$ .

**Theorem 7.21.** *Let  $V$  be a complete, separable locally convex space,  $\mathbb{T} \subseteq \mathbb{R}$  be an interval, and  $f : \mathbb{T} \rightarrow V$  be a curve on  $V$ . Then  $f$  is locally integrally bounded if and only if it is locally Bochner integrable.*

Using the  $C^\nu$ -topology on the space  $\Gamma^\nu(TM)$ , one can apply the Theorem 7.9 and Theorem 7.21 to get the following result.

**Theorem 7.22.** *Let  $X : \mathbb{T} \rightarrow \Gamma^\nu(TM)$  be a time-varying  $C^\nu$ -vector fields. Then  $X$  is locally integrally bounded if and only if it is locally Bochner integrable.*

We denoted the space of Bochner integrable curves from a compact interval  $\mathbb{T} \subseteq \mathbb{R}$  to a locally convex vector space  $V$  by  $L^1(\mathbb{T}; V)$ . One can show that  $L^1(\mathbb{T}; V)$  is a vector space. Let  $\{p_i\}_{i \in \Lambda}$  be a family of generating seminorms for  $V$ . Then, for every  $i \in \Lambda$ , one can define a seminorm  $p_{i, \mathbb{T}}$  on  $L^1(\mathbb{T}; V)$  by

$$p_{i, \mathbb{T}}(f) = \int_{\mathbb{T}} p_i(f(\tau)) d\tau.$$

Therefore, one can consider  $L^1(\mathbb{T}; V)$  as a locally convex space with the generating family of seminorms  $\{p_{i, \mathbb{T}}\}_{i \in \Lambda}$ .

It would be interesting to investigate whether this locally convex space can be characterized using the locally convex space  $V$  and the Banach space  $L^1(\mathbb{T})$ .

**Theorem 7.23** ([16]). *Let  $\mathbb{T} \subseteq \mathbb{R}$  and  $V$  be a complete locally convex space. Then there exists a linear homeomorphism between  $L^1(\mathbb{T}; V)$  and  $L^1(\mathbb{T}) \hat{\otimes}_\pi V$ .*

One can find the similar characterizations for the space of continuous mappings from  $\mathbb{T}$  to the locally convex space  $V$ .

**Theorem 7.24** ([16]). *Let  $\mathbb{T} \subseteq \mathbb{R}$  be a compact interval and  $V$  be a complete locally convex space. Then there exists a linear homeomorphism between  $C^0(\mathbb{T}; V)$  and  $C^0(\mathbb{T}) \hat{\otimes}_\epsilon V$ .*

It is possible to define different notions of absolute continuity for a curve on a locally convex space  $V$ . In this paper, we choose to use the following notion which turns out to be the most applicable one in our study of flows of time-varying vector fields.

**Definition 7.25.** A curve  $f : \mathbb{T} \rightarrow V$  is **absolutely continuous** if there exists a Bochner integrable curve  $g : \mathbb{T} \rightarrow V$  such that, for every  $t_0 \in \mathbb{T}$ , we have

$$f(t) = f(t_0) + \int_{t_0}^t g(\tau) d\tau, \quad \forall t \in \mathbb{T}.$$

The set of all absolutely continuous curves on  $V$  on the interval  $\mathbb{T}$  is denoted by  $\text{AC}(\mathbb{T}; V)$ .

**Theorem 7.26.** Let  $\xi : \mathbb{T} \rightarrow L(C^\nu(M); C^\nu(N))$  be a locally absolutely continuous curve on  $L(C^\nu(M); C^\nu(N))$ . Then  $\xi$  is differentiable for almost every  $t \in \mathbb{T}$ .

*Proof.* Without loss of generality, we assume that  $\mathbb{T}$  is compact. Then there exists  $\eta \in L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$  such that

$$\xi(t) = \xi(t_0) + \int_{t_0}^t \eta(\tau) d\tau, \quad \forall t \in \mathbb{T}.$$

Therefore, it suffice to show that, for almost every  $t_0 \in \mathbb{T}$ , we have

$$\limsup_{t \rightarrow t_0} \frac{1}{t - t_0} \int_{t_0}^t (\eta(\tau) - \eta(t_0)) d\tau = 0.$$

Since  $C^0(\mathbb{T})$  is dense in  $L^1(\mathbb{T})$ , the set  $C^0(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N))$  is dense in  $L^1(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N))$  [16, §15.2, Proposition 3(a)]. Since the locally convex space  $L(C^\nu(M); C^\nu(N))$  is complete, by Theorem 7.23 and Theorem 7.24, we have

$$\begin{aligned} C^0(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N)) &= C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N))), \\ L^1(\mathbb{T}) \widehat{\otimes}_\pi L(C^\nu(M); C^\nu(N)) &= L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N))). \end{aligned}$$

This implies that  $C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$  is dense in  $L^1(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$ . Let  $\{p_i\}_{i \in I}$  be a generating family of seminorms for  $L(C^\nu(M); C^\nu(N))$ . For  $\epsilon > 0$  and  $i \in I$ , there exists  $g \in C^0(\mathbb{T}; L(C^\nu(M); C^\nu(N)))$  such that

$$\int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau < \epsilon.$$

So we assume that  $t > t_0$  and we can write

$$\begin{aligned} \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau &\leq \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \\ &\quad + \frac{1}{t - t_0} \int_{t_0}^t p_i(g(\tau) - g(t_0)) d\tau + p_i(g(t_0) - \eta(t_0)). \end{aligned} \quad (7.3)$$

Since  $g$  is continuous, we get

$$\limsup_{t \rightarrow t_0} \frac{1}{t - t_0} \int_{t_0}^t p_i(g(\tau) - g(t_0)) d\tau = 0.$$

If we take limit supremum of both side of (7.3), we have

$$\begin{aligned} \limsup_{t \rightarrow t_0} \left( \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau \right) \\ \leq \limsup_{t \rightarrow t_0} \left( \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \right) + p_i(g(t_0) - \eta(t_0)). \end{aligned}$$

Now suppose that there exists a set  $A$  such that  $\mathbf{m}(A) \neq 0$  and we have

$$\limsup_{t \rightarrow t_0} \left( \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau \right) \neq 0, \quad \forall t_0 \in A.$$

This implies that, there exists  $\alpha > 0$  such that the set  $B$  defined as

$$B = \left\{ t_0 \in \mathbb{T} \mid \limsup_{t \rightarrow t_0} \left( \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - \eta(t_0)) d\tau \right) > \alpha \right\}.$$

has positive Lebesgue measure. However, we have

$$\int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau = \int_C p_i(g(\tau) - \eta(\tau)) d\tau + \int_D p_i(g(\tau) - \eta(\tau)) d\tau.$$

Where  $C, D \subseteq \mathbb{T}$  are defined as

$$\begin{aligned} C &= \{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) > \frac{\alpha}{2}\}, \\ D &= \{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) \leq \frac{\alpha}{2}\}. \end{aligned}$$

This implies that

$$\int_C p_i(g(\tau) - \eta(\tau)) d\tau \geq \mathbf{m}\{C\} \frac{\alpha}{2}.$$

Therefore we have

$$\int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau \geq \int_C p_i(g(\tau) - \eta(\tau)) d\tau \geq \mathbf{m}\{C\} \frac{\alpha}{2}.$$

This means that

$$\mathbf{m}\left\{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) > \frac{\alpha}{2}\right\} \leq \frac{2}{\alpha} \int_{\mathbb{T}} p_i(g(\tau) - \eta(\tau)) d\tau < \frac{2\epsilon}{\alpha}.$$

Also, by [11, Chapter 1, Theorem 4.3(a)], we have

$$\begin{aligned} \mathbf{m}\left\{t_0 \in \mathbb{T} \mid \limsup_{t \rightarrow t_0} \left( \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \right) > \frac{\alpha}{2}\right\} \\ \leq \frac{4}{\alpha} \int_{\mathbb{T}} p_i(g(\tau) - \xi(\tau)) d\tau < \frac{4\epsilon}{\alpha}. \end{aligned}$$

So this implies that

$$\begin{aligned} \mathbf{m}(B) &\leq \mathbf{m}\left\{t_0 \in \mathbb{T} \mid p_i(g(t_0) - \eta(t_0)) > \frac{\alpha}{2}\right\} \\ &\quad + \mathbf{m}\left\{t_0 \in \mathbb{T} \mid \limsup_{t \rightarrow t_0} \left( \frac{1}{t - t_0} \int_{t_0}^t p_i(\eta(\tau) - g(\tau)) d\tau \right) > \frac{\alpha}{2}\right\} \leq \frac{6\epsilon}{\alpha}. \end{aligned}$$

Since  $\epsilon$  can be chosen arbitrary small, this is a contradiction.  $\square$

It is easy to see that the space  $\text{AC}(\mathbb{T}; \text{L}(C^\omega(M); C^\omega(N)))$  is a vector space. Let  $\{p_{K,\mathbf{a},f}^\omega\}$  be the family of generating seminorms for the  $C^\omega$ -topology on  $\text{L}(C^\omega(M); C^\omega(N))$  and let  $\mathbb{T} \subseteq \mathbb{R}$  be an interval. For every compact subinterval  $\mathbb{I} \subseteq \mathbb{T}$ , we define the seminorm  $q_{K,\mathbf{a},f,\mathbb{I}}^\omega$  as

$$q_{K,\mathbf{a},f,\mathbb{I}}^\omega(X) = \int_{\mathbb{I}} p_{K,\mathbf{a},f}^\omega \left( \frac{dX}{d\tau}(\tau)(f) \right) d\tau.$$

The family of seminorms  $\{p_{K,\mathbf{a},f,\mathbb{I}}^\omega, q_{K,\mathbf{a},f,\mathbb{I}}^\omega\}$  generates a locally convex topology on the space  $\text{AC}(\mathbb{T}; \text{L}(C^\omega(M); C^\omega(N)))$ .

## 8 Global extension of real analytic vector fields

As mentioned in the introduction, not every time-varying real analytic vector field can be extended to a holomorphic one on a neighbourhood of its domain. However, by imposing some appropriate joint condition on time and state, one can show that such an extension exists. In this section, we show that every “locally integrally bounded” time-varying real analytic vector field on a real analytic manifold  $M$ , can be extended to a locally Bochner integrable, time-varying holomorphic vector field on a complex neighbourhood of  $M$ . Moreover, we show that if  $X$  is a continuous time-varying real analytic vector field, then its extension  $\bar{X}$  is a continuous time-varying holomorphic vector field.

We state the following lemma which turns out to be useful in studying extension of real analytic vector fields. The proof of the first lemma is given in [14, Corollary 1].

**Lemma 8.1.** *Let  $\Lambda$  be a directed set and  $(E_\alpha, \{i_{\alpha\beta}\}_{\beta \succeq \alpha})$  be an inductive family of locally convex spaces with locally convex inductive limit  $(E, \{i_\alpha\}_{\alpha \in \Lambda})$ . Let  $F$  be a subspace of  $E$  such that, for every  $\alpha \in \Lambda$ , we have*

$$E_\alpha = \text{cl}_{E_\alpha} (i_\alpha^{-1}(F)).$$

*Then  $F$  is a dense subset of  $E$ .*

Having a directed set  $\Lambda$  and an inductive family of locally convex spaces  $(E_\alpha, \{i_{\alpha\beta}\}_{\beta \succeq \alpha})$ , for every  $\beta \succeq \alpha$ , one can define  $\tilde{i}_{\alpha\beta} : \text{L}^1(\mathbb{T}; E_\alpha) \rightarrow \text{L}^1(\mathbb{T}; E_\beta)$  as

$$\tilde{i}_{\alpha\beta}(f)(t) = i_{\alpha\beta}(f(t)), \quad \forall t \in \mathbb{T}.$$

We can also define the map  $\tilde{i}_\alpha : \text{L}^1(\mathbb{T}; E_\alpha) \rightarrow \text{L}^1(\mathbb{T}; E)$  as

$$\tilde{i}_\alpha(f)(t) = i_\alpha(f(t)).$$

Then it is clear that  $(\text{L}^1(\mathbb{T}; E_\alpha), \{\tilde{i}_{\alpha\beta}\}_{\beta \succeq \alpha})$  is an inductive family of locally convex spaces.

**Lemma 8.2.** *Let  $\mathbb{T} \subseteq \mathbb{R}$  be a compact interval,  $\Lambda$  be a directed set, and  $(E_\alpha, \{i_{\alpha\beta}\}_{\beta, \alpha \in \Lambda})$  be an inductive family of locally convex spaces with locally convex inductive limit  $(E, \{i_\alpha\}_{\alpha \in \Lambda})$ . Then  $(\text{L}^1(\mathbb{T}; E_\alpha), \{\tilde{i}_{\alpha\beta}\}_{\beta, \alpha \in \Lambda})$  is an inductive family of locally convex spaces with locally convex inductive limit  $(\text{L}^1(\mathbb{T}; E), \{\tilde{i}_\alpha\}_{\alpha \in \Lambda})$ .*

*Proof.* Since  $L^1(\mathbb{T})$  is a normable space, by [16, Corollary 4, §15.5], we have  $\varinjlim_{\alpha} L^1(\mathbb{T}) \otimes_{\pi} E_{\alpha} = L^1(\mathbb{T}) \otimes_{\pi} E$ . Let  $F = L^1(\mathbb{T}) \otimes_{\pi} E$ . Then, for every  $\alpha \in \Lambda$ , we have

$$L^1(\mathbb{T}) \otimes_{\pi} E_{\alpha} \subseteq \tilde{i}_{\alpha}^{-1}(F).$$

This implies that

$$L^1(\mathbb{T}; E_{\alpha}) = \text{cl}(\tilde{i}_{\alpha}^{-1}(F)).$$

Then by using Lemma 8.1, we have that  $F$  is a dense subset of  $\varinjlim_{\alpha} L^1(\mathbb{T}; E_{\alpha})$ . This means that  $\varinjlim_{\alpha} L^1(\mathbb{T}; E_{\alpha}) = L^1(\mathbb{T}; E)$ .  $\square$

Using Lemmata 8.1 and 8.2, one can deduce the following result which we refer to as the global extension of real analytic vector fields.

**Theorem 8.3.** *Let  $M$  be a real analytic manifold and let  $\mathcal{N}_M$  be the family of all neighbourhoods of  $M$ . Then we have*

$$\varinjlim_{\bar{U}_M \in \mathcal{N}_M} L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\bar{U}_M)) = L^1(\mathbb{T}; \Gamma^{\omega}(TM)).$$

**Corollary 8.4.** *Let  $X \in L^1(\mathbb{T}; \Gamma^{\omega}(TM))$ . There exists a neighbourhood  $\bar{U}_M$  of  $M$  and a locally Bochner integrable time-varying holomorphic vector field  $\bar{X} \in L^1(\mathbb{T}; \Gamma^{\text{hol}}(\bar{U}_M))$  such that  $\bar{X}(t, x) = X(t, x)$ , for every  $t \in \mathbb{T}$  and every  $x \in M$ .*

Similarly, one can study the extension of continuous time-varying real analytic vector fields. While a continuous time-varying real analytic vector fields is locally Bochner integrable, it has a holomorphic extension to a suitable domain. However, this raises the question of whether the holomorphic extension of a “continuous” time-varying real analytic vector field is a “continuous” time-varying holomorphic vector field or not. Using the following lemma, we show that the answer to the above question is positive.

**Lemma 8.5.** *Let  $K$  be a compact topological space,  $\Lambda$  be a directed set, and  $(E_{\alpha}, \{i_{\alpha\beta}\}_{\beta \geq \alpha})$  be an inductive family of nuclear locally convex spaces with locally convex inductive limit  $(E, \{i_{\alpha}\}_{\alpha \in \Lambda})$ . Suppose that  $E$  is also a nuclear space. Then  $(C^0(K; E_{\alpha}), \{\hat{i}_{\alpha\beta}\}_{\beta \geq \alpha})$  is an inductive family of locally convex spaces with inductive limit  $(C^0(K; E), \{\hat{i}_{\alpha}\}_{\alpha \in \Lambda})$ .*

*Proof.* Since  $C^0(K)$  is a normable space, by [16, Corollary 4, §15.5], we have  $\varinjlim_{\alpha} C^0(K) \otimes_{\pi} E_{\alpha} = C^0(K) \otimes_{\pi} E$ . For every  $\alpha \in \Lambda$ , the space  $E_{\alpha}$  is nuclear. Therefore, by [16, §21.3, Theorem 1], we have

$$C^0(K) \otimes_{\pi} E_{\alpha} = C^0(K) \otimes_{\epsilon} E_{\alpha}, \quad \forall \alpha \in \Lambda.$$

Moreover, the space  $E$  is nuclear. So, again using [16, §21.3, Theorem 1], we have

$$C^0(K) \otimes_{\pi} E = C^0(K) \otimes_{\epsilon} E.$$

This implies that

$$\varinjlim_{\alpha} C^0(K) \otimes_{\epsilon} E_{\alpha} = C^0(K) \otimes_{\epsilon} E.$$

We set  $F = C^0(K) \otimes_{\epsilon} E$ . Then, for every  $\alpha \in \Lambda$ , we have

$$C^0(K) \otimes_{\epsilon} E_{\alpha} \subseteq \hat{i}_{\alpha}^{-1}(F).$$

This implies that

$$C^0(K; E_{\alpha}) \subseteq \text{cl} \left( \hat{i}_{\alpha}^{-1} F \right).$$

Then, by using Lemma 8.1, we have that  $F$  is a dense subset of  $\varinjlim_{\alpha} C^0(K; E_{\alpha})$ . This means that we have  $\varinjlim_{\alpha} C^0(K; E_{\alpha}) = C^0(K; E)$ .  $\square$

**Theorem 8.6.** *Let  $K$  be a compact topological space,  $M$  be a real analytic vector field and  $\mathcal{N}_M$  be the family of all neighbourhoods of  $M$ , which is a directed set under inclusion. Then we have*

$$\varinjlim_{\overline{U}_M \in \mathcal{N}_M} C^0(K; \Gamma^{\text{hol}}(\overline{U}_M)) = C^0(K; \Gamma^{\omega}(TM)).$$

*Proof.* Let  $\Lambda$  be a directed set and  $(E_{\alpha}, \{i_{\alpha\beta}\}_{\beta \succeq \alpha})$  be a directed system of locally convex spaces. Then, for every  $\beta \succeq \alpha$ , one can define  $\hat{i}_{\alpha\beta} : C^0(K; E_{\alpha}) \rightarrow C^0(K; E_{\beta})$  as

$$\hat{i}_{\alpha\beta}(f)(u) = i_{\alpha\beta}(f(u)), \quad \forall u \in K.$$

For every  $\alpha \in \Lambda$ , we can also define the map  $\hat{i}_{\alpha} : C^0(K; E_{\alpha}) \rightarrow C^0(K; E)$  as

$$\hat{i}_{\alpha}(f)(u) = i_{\alpha}(f(u)), \quad \forall u \in K.$$

Then it is clear that  $(C^0(K; E_{\alpha}), \{\hat{i}_{\alpha\beta}\}_{\beta \succeq \alpha})$  is an inductive family of locally convex spaces. The result follows from the above lemma.  $\square$

## 9 Local extension of real analytic vector fields

In the previous section, we proved that every locally Bochner integrable real analytic vector field on  $M$  has a holomorphic extension on a neighbourhood of  $M$ . However, this result is true for extending one vector field. It is natural to ask that, if we have a family of locally integrally bounded real analytic vector fields on  $M$ , can we extend every member of the family to holomorphic vector fields on one neighbourhood of  $M$ ? In order to answer this question, we need a finer result for the extension of real analytic vector fields. We will see that the projective limit representation of the space of real analytic vector fields helps us to get this extension result.

**Theorem 9.1.** *Let  $K \subseteq M$  be a compact set and  $\{\overline{U}_n\}_{n \in \mathbb{N}}$  be a sequence of neighbourhoods of  $M$  such that*

$$\text{cl}(\overline{U}_{n+1}) \subseteq \overline{U}_n, \quad \forall n \in \mathbb{N}.$$

and  $\bigcap_{n \in \mathbb{N}} \overline{U}_n = K$ . Then we have  $\varinjlim_{n \rightarrow \infty} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n)) = L^1(\mathbb{T}; \mathcal{G}_K^{\text{hol}, \mathbb{R}})$ . Moreover the direct limit is weakly compact and boundedly retractive.

*Proof.* We know that, by Theorem 7.6, for every  $n \in \mathbb{N}$ , the map  $\rho_{\overline{U}_n}^{\mathbb{R}} : \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\overline{U}_n) \rightarrow \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$  is a compact continuous map. Note that every  $n \in \mathbb{N}$ , the map  $\text{id} \otimes \rho_{\overline{U}_n}^{\mathbb{R}} : L^1(\mathbb{T}) \otimes_{\pi} \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\overline{U}_n) \rightarrow L^1(\mathbb{T}) \otimes_{\pi} \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$  is defined by

$$\text{id} \otimes \rho_{\overline{U}_n}^{\mathbb{R}}(\xi(t) \otimes \eta) = \xi(t) \otimes \rho_{\overline{U}_n}^{\mathbb{R}}(\eta).$$

Since  $L^1(\mathbb{T}) \otimes_{\pi} \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\overline{U}_n)$  is a dense subset of  $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\overline{U}_n))$ , one can extend the map  $\text{id} \otimes \rho_{\overline{U}_n}^{\mathbb{R}}$  into the map  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}} : L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\overline{U}_n)) \rightarrow L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n))$ . We show that  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}$  is weakly compact.

In order to show that  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}$  is weakly compact, it suffices to show that for a bounded set  $B \subset L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\overline{U}_n))$ , the set  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B)$  is relatively weakly compact in  $L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n))$ . Since  $L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n))$  is a complete locally convex space, by Theorem 6.4, the set

$$\text{cl} \left( \text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B) \right)$$

is weakly compact if it is weakly sequentially compact. Therefore, it suffices to show that  $\text{cl} \left( \text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B) \right)$  is weakly sequentially compact. Let  $\{f_n\}_{n=1}^{\infty}$  in  $\text{cl} \left( \text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B) \right)$ . Since  $\text{cl} \left( \text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B) \right)$  is bounded, for every seminorm  $p$  on  $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$ , there exists  $M > 0$  such that

$$p \left( \int_{\mathbb{T}} f_n(\tau) d\tau \right) \leq \int_{\mathbb{T}} p(f_n(\tau)) d\tau \leq M.$$

This implies that the sequence  $\left\{ \int_{\mathbb{T}} f_n(\tau) d\tau \right\}_{n=1}^{\infty}$  is bounded in  $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$ . Since  $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$  is a nuclear locally convex space, the sequence  $\left\{ \int_{\mathbb{T}} f_n(\tau) d\tau \right\}_{n=1}^{\infty}$  is relatively compact in  $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$ . Therefore, there is a subsequence  $\{f_{n_r}\}_{r=1}^{\infty}$  of  $\{f_n\}_{n=1}^{\infty}$  such that

$$\left\{ \int_{\mathbb{T}} f_{n_r}(\tau) d\tau \right\}_{r=1}^{\infty}$$

is Cauchy in  $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$ .

Note that the strong dual of  $L^1(\mathbb{T})$  is  $L^{\infty}(\mathbb{T})$  [27, Chapter 8]. We also know that  $\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n)$  is a nuclear complete metrizable space and  $L^1(\mathbb{T})$  is a Banach space. Therefore, using [29, Chapter IV, Theorem 9.9], the strong dual of  $L^1(\mathbb{T}; \Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n))$  is exactly  $L^{\infty}(\mathbb{T}) \widehat{\otimes}_{\pi} (\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n))'_{\beta}$ . We first show that, for every  $\xi \otimes \eta \in L^{\infty}(\mathbb{T}) \otimes (\Gamma^{\text{hol}, \mathbb{R}}(\overline{U}_n))'$ , the sequence

$$\{\xi \otimes \eta(f_{n_r})\}_{r=1}^{\infty}$$

is Cauchy in  $\mathbb{R}$ . Note that we have

$$\begin{aligned} \xi \otimes \eta(f_{n_r} - f_{n_s}) &= \int_{\mathbb{T}} \xi(t) \eta(f_{n_s}(t) - f_{n_r}(t)) dt \\ &\leq M \int_{\mathbb{T}} \eta(f_{n_s}(t) - f_{n_r}(t)) dt = M \eta \left( \int_{\mathbb{T}} (f_{n_s}(t) - f_{n_r}(t)) dt \right). \end{aligned}$$



Since the sequence  $\{\int_{\mathbb{T}} f_{n_r}(\tau) d\tau\}_{r=1}^{\infty}$  is Cauchy in  $\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n)$ , this implies that the sequence  $\{\xi \otimes \eta(f_{n_r})\}_{r=1}^{\infty}$  is Cauchy in  $\mathbb{R}$ . Now we show that, for every  $\lambda \in L^{\infty}(\mathbb{T}) \widehat{\otimes} (\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'$ , the sequence

$$\{\lambda(f_{n_r})\}_{r=1}^{\infty}$$

is Cauchy in  $\mathbb{R}$ . Note that  $L^{\infty}(\mathbb{T}) \otimes_{\pi} (\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'$  is a dense subset of  $L^{\infty}(\mathbb{T}) \widehat{\otimes}_{\pi} (\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'_{\beta}$ . So there exist a net  $\{\xi_{\alpha}\}_{\alpha \in \Lambda}$  in  $L^{\infty}(\mathbb{T})$  and a net  $\{\eta_{\alpha}\}_{\alpha \in \Lambda}$  in  $(\Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))'$  such that

$$\lim_{\alpha} \xi_{\alpha} \otimes \eta_{\alpha} = \lambda.$$

Thus, for every  $\epsilon > 0$ , there exists  $\theta \in \Lambda$  such that

$$\|\xi_{\theta} \otimes \eta_{\theta}(v) - \lambda(v)\| \leq \frac{\epsilon}{3}, \quad \forall v \in \text{cl}\left(\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}(B)\right).$$

Since the sequence  $\{\xi_{\theta} \otimes \eta_{\theta}(f_{n_r})\}_{r=1}^{\infty}$  is Cauchy in  $\mathbb{R}$ , for every  $\epsilon > 0$ , there exists  $\tilde{N} > 0$  such that

$$\|\xi_N \otimes \eta_N(f_{n_s} - f_{n_r})\| < \frac{\epsilon}{3}, \quad \forall r, s > \tilde{N}.$$

Thus, for every  $\epsilon > 0$ , there exists  $\tilde{N} > 0$  such that

$$\begin{aligned} \|\lambda(f_{n_s} - f_{n_r})\| &\leq \|\lambda(f_{n_s} - f_{n_r}) - \xi_{\theta} \otimes \eta_{\theta}(f_{n_s} - f_{n_r})\| + \|\xi_{\theta} \otimes \eta_{\theta}(f_{n_s} - f_{n_r})\| \\ &\leq \|\lambda(f_{n_s}) - \xi_{\theta} \otimes \eta_{\theta}(f_{n_s})\| + \|\lambda(f_{n_r}) - \xi_{\theta} \otimes \eta_{\theta}(f_{n_r})\| + \|\xi_{\theta} \otimes \eta_{\theta}(f_{n_s} - f_{n_r})\| < \epsilon. \end{aligned}$$

Therefore, the sequence  $\{f_{n_r}\}_{r=1}^{\infty}$  is weakly Cauchy in  $L^1(\mathbb{T}; \Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))$ . This completes the proof of weak compactness of the map  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}} : L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\overline{U}_n)) \rightarrow L^1(\mathbb{T}; \Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n))$ . Recall that in the proof of Theorem 7.7, for every  $n \in \mathbb{N}$ , we defined the continuous linear map  $r_n^{\mathbb{R}} : \Gamma^{\text{hol},\mathbb{R}}(\overline{U}_n) \rightarrow \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\overline{U}_{n+1})$  by

$$r_n^{\mathbb{R}}(X) = X|_{\overline{U}_{n+1}}.$$

Then we have the following diagram:

$$\Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(U_n) \xrightarrow{\rho_{\overline{U}_n}^{\mathbb{R}}} \Gamma^{\text{hol},\mathbb{R}}(U_n) \xrightarrow{r_n^{\mathbb{R}}} \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(U_{n+1}).$$

Therefore, we have the following diagram:

$$L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(U_n)) \xrightarrow{\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}} L^1(\mathbb{T}; \Gamma^{\text{hol}}(U_n)) \xrightarrow{\text{id} \widehat{\otimes} r_n^{\mathbb{R}}} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(U_{n+1})).$$

Since,  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}}$  is weakly compact, by [16, §17.2, Proposition 1], the composition  $\text{id} \widehat{\otimes} \rho_{\overline{U}_n}^{\mathbb{R}} \circ \text{id} \widehat{\otimes} r_n^{\mathbb{R}}$  is weakly compact. Therefore, the connecting maps in the inductive limit  $\varinjlim_{n \rightarrow \infty} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n)) = L^1(\mathbb{T}; \mathcal{G}_K^{\text{hol},\mathbb{R}})$  are weakly compact.

Using Theorem 6.7, if we can show that the direct limit satisfies condition (M), then it would be boundedly retractive. Since the inductive limit  $\varinjlim_{n \rightarrow \infty} \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n) = \mathcal{G}_K^{\text{hol},\mathbb{R}}$

is compact, by Theorem 6.5, it satisfies condition (M). This means that there exists a sequence  $\{V_n\}_{n \in \mathbb{N}}$  such that, for every  $n \in \mathbb{N}$ ,  $V_n$  is an absolutely convex neighbourhood of 0 in  $\Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n)$  and there exists  $M_n > 0$  such that, for every  $m > M_n$ , the topologies induced from  $\Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_m)$  on  $V_n$  are all the same. Now consider the sequence  $\{L^1(\mathbb{T}; V_n)\}_{n \in \mathbb{N}}$ . It is clear that, for every  $n \in \mathbb{N}$ ,  $L^1(\mathbb{T}; V_n)$  is an absolutely convex neighbourhood of 0 in  $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n))$ . For every seminorm  $p$  on  $\Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n)$  and every  $m > M_n$ , there exists a seminorm  $q_m$  on  $\Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_m)$  such that

$$p(v) \leq q_m(v), \quad \forall v \in V_n.$$

This implies that, for every  $X \in L^1(\mathbb{T}; V_n)$ , we have

$$\int_{\mathbb{T}} p(X(\tau)) d\tau \leq \int_{\mathbb{T}} q_m(X(\tau)) d\tau.$$

So, for every  $m > M_n$ , the topology induced on  $L^1(\mathbb{T}; V_n)$  from  $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_m))$  is the same as its original topology. Therefore, the inductive limit

$$\varinjlim_{n \rightarrow \infty} L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n)) = L^1(\mathbb{T}; \mathcal{G}_K^{\text{hol}, \mathbb{R}})$$

satisfies condition (M) and it is boundedly retractive.  $\square$

Using the local extension theorem developed here, we can state the following result, which can be considered as generalization of Corollary 8.4.

**Corollary 9.2.** *Let  $B \subseteq L^1(\mathbb{T}; \Gamma^\omega(TM))$  be a bounded set. Then, for every compact set  $K \subseteq M$ , there exists a neighbourhood  $\overline{U}_K$  of  $K$  and a bounded set  $\overline{B} \in L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}}(\overline{U}_n))$  such that, for every  $X \in B$ , there exists a  $\overline{X} \in \overline{B}$  such that*

$$\overline{X}(t, x) = X(t, x) \quad \forall t \in \mathbb{T}, \forall x \in K.$$

Let  $M$  be a real analytic manifold and let  $U \subseteq M$  be a relatively compact subset of  $M$ . Then, by the local extension theorem, for every  $f \in C^\omega(M)$ , there exists a neighbourhood  $\overline{V} \subseteq M^\mathbb{C}$  of  $U$  such that  $f$  can be extended to a bounded holomorphic function  $\overline{f} \in C_{\text{bdd}}^{\text{hol}}(\overline{V})$ . It is useful to study the relationship between the seminorms of  $f$  and the seminorms of its holomorphic extension  $\overline{f}$ .

**Theorem 9.3.** *Let  $M$  be a real analytic manifold and  $U$  be a relatively compact subset of  $M$ . Then, for every neighbourhood  $\overline{V} \subseteq M^\mathbb{C}$  of  $\text{cl}(U)$ , there exists  $d > 0$  such that, for every  $f \in C^\omega(M)$  with a holomorphic extension  $\overline{f} \in C_{\text{bdd}}^{\text{hol}}(\overline{V})$ , we have*

$$p_{K, \mathbf{a}}^\omega(f) \leq p_{\overline{V}}(\overline{f}), \quad \forall \mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d), \forall \text{ compact } K \subseteq U.$$

*Proof.* Since  $\overline{f}$  is a holomorphic extension of  $f$ , we have

$$\overline{f}(x) = f(x), \quad \forall x \in \text{cl}(U).$$

Since  $\text{cl}(U)$  is compact, one can choose  $d > 0$  such that, for every  $x \in \text{cl}(U)$ , we have  $D_{(d)}(x) \subseteq \bar{V}$ , where  $(d) = (d, d, \dots, d)$ . We set  $D = \bigcup_{x \in U} D_{(d)}(x)$ . Then we have  $D \subseteq \bar{V}$ . Using Cauchy's estimate, for every multi-index  $(r)$  and for every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , we have

$$\frac{a_0 a_1 \dots a_{|r|}}{(r)!} \|D^{(r)} f(x)\| \leq \frac{a_0}{d} \frac{a_1}{d} \dots \frac{a_{|r|}}{d} \sup \{ \|\bar{f}(x)\| \mid x \in D \} \leq p_{\bar{V}}(\bar{f}), \quad \forall x \in U.$$

This implies that, for every compact set  $K \subseteq U$  and every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , we have

$$p_{K, \mathbf{a}}^\omega(f) \leq p_{\bar{V}}(\bar{f}).$$

□

## 10 Series representation of flows of time-varying real analytic vector fields

In this section, using the holomorphic extension theorems, we study flows of time-varying real analytic vector fields. The operator framework that we use for this analysis (as to our knowledge) has been first introduced in [1]. As mentioned in the previous sections, a time-varying  $C^\omega$ -vector field can be considered as a curve on the locally convex space  $L(C^\omega(M); C^\omega(M))$ . Let  $X : \mathbb{T} \times M \rightarrow TM$  be a time-varying real analytic vector field. Then we define  $\hat{X} : \mathbb{T} \rightarrow L(C^\omega(M); C^\omega(M))$  as

$$\hat{X}(t)(f) = df(X(t)), \quad \forall t \in \mathbb{T}, \forall f \in C^\omega(M)$$

Following the analysis in [1], the flow of a time-varying  $C^\omega$ -vector field  $X$  can be considered as a curve  $\zeta : \mathbb{T} \rightarrow L(C^\omega(M); C^\omega(U))$  which satisfies the following initial value problem on the locally convex space  $L(C^\omega(M); C^\omega(U))$ :

$$\begin{aligned} \frac{d\zeta}{dt}(t) &= \zeta(t) \circ \hat{X}(t), & a.e. \ t \in \mathbb{T} \\ \zeta(0) &= \text{id}. \end{aligned} \tag{10.1}$$

Therefore, one can reduce the problem of studying the flow of a time-varying vector field to the problem of studying solutions of a linear differential equation on a locally convex vector space. The theory of ordinary differential equations on locally convex spaces is different in nature from the classical theory of ordinary differential equations on Banach spaces. In the theory of differential equations on Banach spaces, there are many general results about existence, uniqueness and properties of the flows of vector fields, which hold independently of the underlying Banach space. However, the theory of ordinary differential equations on locally convex spaces heavily depends on the nature of their underlying space. Many methods in the classical theory of ordinary differential equations in Banach spaces have no

counterpart in the theory of ordinary differential equations on locally convex spaces [21]. For instance, one can easily find counterexamples for Peano's existence theorem for linear differential equations on locally convex spaces [21].

In [1], the initial value problem (10.1) for both time-varying smooth vector fields and time-varying real analytic vector fields has been studied on  $L(C^\infty(\mathbb{R}^n); C^\infty(\mathbb{R}^n))$ . In the real analytic case,  $X$  is assumed to be a locally integrally bounded time-varying  $C^\omega$ -vector field on  $\mathbb{R}^n$  such that it can be extended to a bounded holomorphic vector field on a neighbourhood  $\Omega \subseteq \mathbb{C}^n$  of  $\mathbb{R}^n$ . Using the  $C^{\text{hol}}$ -topology on the space of holomorphic vector fields, it has been shown that the well-known sequence of Picard iterations for the initial value problem (10.1) converges and gives us the unique solution of (10.1) [1, §2, Proposition 2.1]. In the smooth case, the existence and uniqueness of solutions of (10.1) has been shown. However, for smooth but not real analytic vector fields, the sequence of Picard iterations associated to the initial value problem (10.1) does not converge [2, §2.4.4].

In this section, we study the initial value problem (10.1) for the real analytic cases on the locally convex space  $L(C^\omega(M); C^\omega(M))$ . Using the local extension theorem (8.4) and estimates for seminorms on the space of real analytic functions, we provide a direct method for proving and studying the convergence of sequence of Picard iterations. This method helps us to generalize the result of [1, §2, Proposition 2.1] to arbitrary locally integrally bounded time-varying real analytic vector fields.

**Theorem 10.1.** *Let  $X : \mathbb{T} \rightarrow \Gamma^\omega(TM)$  be a locally integrally bounded time-varying vector field. Then, for every  $t_0 \in \mathbb{T}$  and every  $x_0 \in M$ , there exists an interval  $\mathbb{T}' \subseteq \mathbb{T}$  containing  $t_0$  and an open set  $U \subseteq M$  containing  $x_0$  such that there exists a unique locally absolutely continuous curve  $\zeta : \mathbb{T}' \rightarrow L(C^\omega(M); C^\omega(U))$  which satisfies the following initial value problem:*

$$\begin{aligned} \frac{d\zeta}{dt}(t) &= \zeta(t) \circ \widehat{X}(t), & \text{a.e. } t \in \mathbb{T}', \\ \zeta(t_0) &= \text{id}, \end{aligned} \tag{10.2}$$

and, for every  $t \in \mathbb{T}'$ , we have

$$\zeta(t)(fg) = \zeta(t)(f)\zeta(t)(g), \quad \forall f, g \in C^\omega(M). \tag{10.3}$$

*Proof.* Let  $N = \dim(M)$  and  $(V, (x^1, x^2, \dots, x^N))$  be a coordinate chart around  $x_0$ . Without loss of generality, we can assume that  $\mathbb{T}$  is a compact interval containing  $t_0$ . Let  $U$  be a relatively compact set such that  $\text{cl}(U) \subseteq V$ ,  $K \subseteq U$  be a compact set. For every  $k \in \mathbb{N}$ , we define  $\phi_k : \mathbb{T} \rightarrow L(C^\omega(M); C^\omega(U))$  inductively as

$$\begin{aligned} \phi_0(t)(f) &= f|_U, & \forall t \in [t_0, T], \\ \phi_k(t)(f) &= f|_U + \int_{t_0}^t \phi_{k-1}(\tau) \circ \widehat{X}(\tau)(f) d\tau, & \forall t \in \mathbb{T}. \end{aligned}$$

Let  $K \subseteq M$  be a compact set and  $\mathbf{a} \in \mathbf{c}_0^\perp(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$ . Then, we have the following lemma.

**Lemma.** *There exist a locally integrally bounded function  $m \in L^1_{\text{loc}}(\mathbb{T})$  such that, for every  $f \in C^\omega(M)$ , there exist constants  $M_f, \tilde{M}_f \in \mathbb{R}^{\geq 0}$*

$$\begin{aligned} p_{K,\mathbf{a},f}^\omega(\phi_n(t) - \phi_{n-1}(t)) &\leq (M(t))^n M_f, \quad \forall t \in \mathbb{T}, \forall n \in \mathbb{N}. \\ p_{K,\mathbf{a},f}^\omega\left((\phi_n(t) - \phi_{n-1}(t)) \circ \widehat{X}(t)\right) &\leq m(t)(M(t))^n \tilde{M}_f, \quad \forall t \in \mathbb{T}, \forall n \in \mathbb{N}. \end{aligned}$$

where  $M : \mathbb{T} \rightarrow \mathbb{R}$  is defined as

$$M(t) = \left| \int_{t_0}^t m(\tau) d\tau \right|, \quad \forall t \in \mathbb{T}.$$

*Proof.* Since  $X$  is locally Bochner integrable, by Corollary 9.2, there exist a neighbourhood  $\bar{V}$  of  $U$ , a locally Bochner integrable vector field  $\bar{X} \in L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{V}))$ , and a function  $\bar{f} \in C_{\text{bdd}}^{\text{hol},\mathbb{R}}(\bar{V})$  such that  $\bar{X}_t$  and  $\bar{f}$  are the holomorphic extension of  $X$  and  $f$  over  $\bar{V}$ , respectively. Then, by Theorem 9.3, there exists  $d > 0$  such that, for every compact set  $K \subseteq U$  and every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$ , we have

$$\begin{aligned} p_{K,\mathbf{a}}^\omega(f) &\leq p_{\bar{V}}(\bar{f}), \\ \max_i \{p_{K,\mathbf{a}}^\omega(X^i(t))\} &\leq \max_i \{p_{\bar{V}}(\bar{X}(t))\}, \quad \forall t \in \mathbb{T}, \end{aligned}$$

Since  $X$  is locally Bochner integrable, there exists  $m \in L^1(\mathbb{T})$  such that

$$4N \max_i \{p_{\bar{V}}(\bar{X}^i(t))\} \leq m(t), \quad \forall t \in \mathbb{T},$$

Then we define  $M : \mathbb{T} \rightarrow \mathbb{R}$  as

$$M(t) = \int_{t_0}^t m(\tau) d\tau.$$

Let  $K \subseteq U$  be a compact set and let  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ . We show by induction that, for every  $n \in \mathbb{N}$ , the function  $\phi_n \circ X$  is locally Bochner integrable and  $\phi_{n+1} \in \text{AC}(\mathbb{T}, L(C^\omega(M); C^\omega(U)))$ . Moreover, we have

$$p_{K,\mathbf{a},f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \leq (M(t))^{n+1} p_{K,\mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in \mathbb{T},$$

where, for every  $n \in \mathbb{N}$ , the sequence  $\mathbf{a}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0})$  is defined as in Lemma 7.17:

$$a_{n,m} = \begin{cases} \left(\frac{m+1}{m}\right)^n a_m & n < m, \\ \left(\frac{m+1}{m}\right)^m a_m & n \geq m. \end{cases}$$

First note that for  $n = 0$ , we have

$$\phi_0 \circ \widehat{X}(f) = \widehat{X}(f) \mid_U, \quad \forall f \in C^\omega(M),$$

Since  $X$  is locally Bochner integrable,  $\phi_0 \circ X$  is locally Bochner integrable. Therefore,  $\phi_1 \in \text{AC}([t_0, T], L(C^\omega(M); C^\omega(U)))$ . Moreover, we have

$$\phi_1(t) - \phi_0(t) = \int_{t_0}^t \widehat{X}(\tau) d\tau, \quad \forall t \in \mathbb{T}.$$

This implies that

$$p_{K,\mathbf{a},f}^\omega(\phi_1(t) - \phi_0(t)) \leq \int_{t_0}^t p_{K,\mathbf{a}}^\omega(\widehat{X}(\tau)f) d\tau, \quad \forall t \in \mathbb{T}.$$

By inequality (7.1), we have

$$p_{K,\mathbf{a}}^\omega(X(t)f) \leq 4N \max_i \{p_{K,\mathbf{b}_1}^\omega(X^i(t))\} p_{K,\mathbf{a}_1}^\omega(f), \quad \forall t \in \mathbb{T}.$$

Therefore we have

$$\begin{aligned} p_{K,\mathbf{a},f}^\omega(\phi_1(t) - \phi_0(t)) &\leq \int_{t_0}^t 4N \max_i \{p_{K,\mathbf{b}_1}^\omega(X^i(\tau))\} p_{K,\mathbf{a}_1}^\omega(f) d\tau \\ &\leq M(t) p_{K,\mathbf{a}_1}^\omega(f). \end{aligned}$$

Now suppose that, for every  $k \in \{1, 2, \dots, n-1\}$ ,  $\phi_k \circ X$  is locally Bochner integrable and we have

$$p_{K,\mathbf{a},f}^\omega(\phi_{k+1}(t) - \phi_k(t)) \leq (M(t))^{k+1} p_{K,\mathbf{a}_{k+1}}^\omega(f), \quad \forall t \in \mathbb{T}.$$

Now consider the following inequality:

$$\begin{aligned} p_{K,\mathbf{a},f}^\omega(\phi_{n-1}(t) \circ \widehat{X}(t)) &\leq p_{K,\mathbf{a},f}^\omega(\widehat{X}(t)) + \sum_{i=1}^{n-1} p_{K,\mathbf{a},f}^\omega((\phi_i(t) - \phi_{i-1}(t)) \circ \widehat{X}(t)) \\ &\leq p_{K,\mathbf{a},f}^\omega(\widehat{X}(t)) + \sum_{i=1}^{n-1} m(t)(M(t))^{i+1} \tilde{M}_f \leq m(t) \left( \sum_{i=0}^{n-1} (M(t))^i \right) \tilde{M}_f, \quad \forall t \in \mathbb{T}. \end{aligned}$$

The function  $g_n : [t_0, T] \rightarrow \mathbb{R}$  defined as

$$g_n(t) = m(t) \left( \sum_{i=0}^{n-1} M^i(t) \right), \quad \forall t \in \mathbb{T},$$

is locally integrable. Thus, by Theorem 7.22,  $\phi_{n-1} \circ \widehat{X}$  is locally Bochner integrable. So, by Definition 7.25,  $\phi_n$  is absolutely continuous.

On the other hand, we have

$$\phi_{n+1}(t) - \phi_n(t) = \int_{t_0}^t \left( \phi_n(\tau) \circ \widehat{X}(\tau) - \phi_{n-1}(\tau) \circ \widehat{X}(\tau) \right) d\tau, \quad \forall t \in \mathbb{T}.$$

Taking  $p_{K,\mathbf{a},f}^\omega$  of both side of the above equality, we have

$$\begin{aligned} p_{K,\mathbf{a},f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \\ \leq \int_{t_0}^t p_{K,\mathbf{a},f}^\omega \left( (\phi_n(\tau) - \phi_{n-1}(\tau)) \circ \widehat{X}(\tau) \right) d\tau, \quad \forall t \in \mathbb{T}. \end{aligned}$$

However, we know that by the induction hypothesis

$$p_{K,\mathbf{a},f}^\omega \left( (\phi_n(t) - \phi_{n-1}(t)) \circ \widehat{X}(t) \right) \leq (M(t))^n p_{K,\mathbf{a}_n}^\omega(\widehat{X}(t)f), \quad \forall t \in \mathbb{T}.$$

Moreover, by the inequality (7.1), we have

$$p_{K, \mathbf{a}_n}^\omega(\widehat{X}(t)f) \leq 4N(n+1) \max_i \{p_{K, \mathbf{b}_n}^\omega(X^i(t))\} p_{K, \mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in \mathbb{T}.$$

By Lemma 7.17, for every  $n \in \mathbb{N}$ , we have  $\mathbf{b}_n \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, 6ed)$ . This implies that, for every  $n \in \mathbb{N}$ , we have

$$\max_i \{p_{K, \mathbf{b}_n}^\omega(X^i(t))\} \leq \max_i \{p_{\overline{V}}(\overline{X}^i(t))\} < \frac{1}{4N} m(t), \quad \forall t \in \mathbb{T}.$$

Therefore, for every  $n \in \mathbb{N}$ , we have

$$p_{K, \mathbf{a}, f}^\omega \left( (\phi_n(t) - \phi_{n-1}(t)) \circ \widehat{X}(t) \right) \leq (n+1)m(t)M^n(t)p_{K, \mathbf{a}_{n+1}}^\omega(f).$$

Thus we get

$$\begin{aligned} p_{K, \mathbf{a}, f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \\ \leq \int_{t_0}^t (n+1)(M(\tau))^n m(\tau) p_{K, \mathbf{a}_{n+1}}^\omega(f) d\tau \\ = (M(t))^{n+1} p_{K, \mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in \mathbb{T}. \end{aligned}$$

This completes the induction. Note that by Lemma 7.17, for every  $m, n \in \mathbb{Z}_{\geq 0}$ , we have

$$a_{n,m} \leq ea_m \leq 6ed$$

This implies that, for every  $n \in \mathbb{N}$ , we have

$$p_{K, \mathbf{a}_n}^\omega(f) \leq p_{\overline{V}}(\overline{f}).$$

If we set  $M_f = p_{\overline{V}}(\overline{f})$  then, for every  $n \in \mathbb{N}$ , we have

$$p_{K, \mathbf{a}, f}^\omega(\phi_{n+1}(t) - \phi_n(t)) \leq (M(t))^{n+1} M_f, \quad \forall t \in \mathbb{T}.$$

Moreover, for every  $n \in \mathbb{N}$ , we have

$$p_{K, \mathbf{a}, f}^\omega \left( (\phi_n(t) - \phi_{n-1}(t)) \circ \widehat{X}(t) \right) \leq (M(t))^n p_{K, \mathbf{a}_n}^\omega(\widehat{X}(t)f), \quad \forall t \in \mathbb{T}.$$

However, by inequality (7.1), we have

$$p_{K, \mathbf{a}_n}^\omega(\widehat{X}(t)f) \leq 4N \max_i \{p_{K, \mathbf{b}_n}^\omega\} p_{K, \mathbf{a}_{n+1}}^\omega(f), \quad \forall t \in \mathbb{T}.$$

Noting that we have

$$\max_i \{p_{K, \mathbf{b}_n}^\omega(X^i(t))\} \leq \max_i \{p_{\overline{V}}(\overline{X}^i(t))\} < \frac{1}{4N} m(t), \quad \forall t \in \mathbb{T},$$

and

$$p_{K, \mathbf{a}_{n+1}}^\omega(f) \leq p_{\overline{V}}(\overline{f}), \quad \forall t \in \mathbb{T}.$$

Therefore, if we set  $\tilde{M}_f = p_{\overline{V}}(\overline{f})$ , we have

$$p_{K, \mathbf{a}, f}^\omega \left( (\phi_n(t) - \phi_{n-1}(t)) \circ \widehat{X}(t) \right) \leq m(t)(M(t))^n \tilde{M}_f, \quad \forall t \in \mathbb{T}.$$

This completes the proof of the lemma.  $\square$

Therefore, for every  $n \in \mathbb{N}$ , we have

$$p_{K,\mathbf{a},f}^\omega(\phi_n(t) - \phi_{n-1}(t)) \leq (M(T))^n M_f, \quad \forall t \in [t_0, T].$$

Since  $M$  is continuous, there exists  $T \in \mathbb{T}$  such that

$$M(t) < 1, \quad \forall t \in [t_0, T].$$

Since  $M(T) < 1$ , one can deduce that the sequence  $\{\phi_n\}_{n \in \mathbb{N}}$  converges uniformly on  $[t_0, T]$  in  $L(C^\omega(M); C^\omega(U))$ . Since uniform convergence implies  $L^1$ -convergence and the space  $L^1([t_0, T]; L(C^\omega(M); C^\omega(U)))$  is complete, there exists  $\phi \in L^1([t_0, T]; L(C^\omega(M); C^\omega(U)))$  such that

$$\lim_{n \rightarrow \infty} \phi_n = \phi,$$

where the limit is in  $L^1$ -topology on  $L^1([t_0, T]; L(C^\omega(M); C^\omega(U)))$ . We first show that  $\phi \circ X$  is locally Bochner integrable on  $[t_0, T]$ . Note that, by the above Lemma, for every  $n \in \mathbb{N}$ , we have

$$p_{K,\mathbf{a},f}^\omega(\phi(t) - \phi_n(t)) \leq \sum_{k=n+1}^{\infty} (M(t))^k M_f. \quad (10.4)$$

This implies that, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int_{t_0}^t p_{K,\mathbf{a},f}^\omega((\phi(\tau) - \phi_n(\tau)) \circ \widehat{X}(\tau)) d\tau &\leq \int_{t_0}^t \sum_{k=n+1}^{\infty} m(\tau) (M(\tau))^k \tilde{M}_f \\ &\leq N(T - t_0) \sum_{i=n+1}^{\infty} (M(T))^i \tilde{M}_f, \quad \forall t \in [t_0, T]. \end{aligned}$$

Therefore, we get

$$\begin{aligned} \int_{t_0}^t p_{K,\mathbf{a},f}^\omega(\phi(\tau) \circ \widehat{X}(\tau)) d\tau \\ \leq \int_{t_0}^t p_{K,\mathbf{a},f}^\omega(\phi_n(\tau) \circ \widehat{X}(\tau)) d\tau + \frac{\tilde{M}_f N(T - t_0) (M(T))^{n+1}}{1 - M(T)}, \quad \forall t \in [t_0, T]. \end{aligned}$$

However, from the proof of the above Lemma, we know that

$$\int_{t_0}^t p_{K,\mathbf{a},f}^\omega(\phi_n(\tau) \circ \widehat{X}(\tau)) d\tau \leq g_n(t) \tilde{M}_f, \quad \forall n \in \mathbb{N}, \quad \forall t \in [t_0, T],$$

where  $g_n : [t_0, T] \rightarrow \mathbb{R}$  is locally integrable. Therefore, we define the function  $h_n : [t_0, T] \rightarrow \mathbb{R}$  as

$$h_n(t) = g_n(t) \tilde{M}_f + \frac{\tilde{M}_f N(T - t_0) (M(T))^{n+1}}{1 - M(T)}, \quad \forall t \in [t_0, T].$$

It is clear that  $h_n$  is locally integrable and

$$\int_{t_0}^t p_{K,\mathbf{a},f}^\omega(\phi(\tau) \circ \widehat{X}(\tau)) d\tau \leq h_n(t).$$



This implies that  $\phi \circ \widehat{X}$  is locally Bochner integrable. Moreover, using equation (10.4), we get

$$\lim_{n \rightarrow \infty} \int_{t_0}^t \phi_n(\tau) \circ \widehat{X}(\tau) d\tau = \int_{t_0}^t \phi(\tau) \circ \widehat{X}(\tau) d\tau, \quad \forall t \in [t_0, T].$$

Therefore, we have

$$\phi(t) = \lim_{n \rightarrow \infty} \phi_n(t) = \lim_{n \rightarrow \infty} \int_{t_0}^t \phi_{n-1}(\tau) \circ \widehat{X}(\tau) d\tau = \int_{t_0}^t \phi(\tau) \circ \widehat{X}(\tau) d\tau.$$

This shows that  $\phi$  satisfies the initial value problem (10.2).

One can also show that the sequence  $\{\phi_n\}_{n \in \mathbb{N}}$  converges to  $\phi$  in  $\text{AC}([t_0, T]; \text{L}(C^\omega(M); C^\omega(U)))$ . In order to show this, it suffices to show that, for every compact set  $K \subseteq U$  and every  $f \in C^\omega(M)$ , we have

$$\lim_{n \rightarrow \infty} \int_{t_0}^t p_{K, \mathbf{a}, f}^\omega \left( \frac{d\phi_{n+1}}{dt} - \frac{d\phi_n}{dt} \right) = 0, \quad \forall t \in [t_0, T].$$

Note that, for every  $n \in \mathbb{N}$ , we have

$$\frac{d\phi_{n+1}}{dt} = \phi_n(t) \circ \widehat{X}(t), \quad \text{a.e., } t \in [t_0, T].$$

Therefore, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{t_0}^t p_{K, \mathbf{a}, f}^\omega (\phi_n(t) \circ \widehat{X}(t) - \phi_{n-1}(t) \circ \widehat{X}(t)) = 0, \quad \forall t \in [t_0, T].$$

But we know that, for every  $n \in \mathbb{N}$ , we have

$$p_{K, \mathbf{a}, f}^\omega (\phi_n(t) \circ \widehat{X}(t) - \phi_{n-1}(t) \circ \widehat{X}(t)) \leq m(t)(M(t))^n \tilde{M}_f \leq m(t)(M(t))^n \tilde{M}_f, \quad \forall t \in [t_0, T].$$

So we have

$$\begin{aligned} \int_{t_0}^t p_{K, \mathbf{a}, f}^\omega (\phi_n(t) \circ \widehat{X}(t) - \phi_{n-1}(t) \circ \widehat{X}(t)) &\leq \frac{d}{(n+1)N} (M(T))^{n+1} \tilde{M}_f \\ &\leq \frac{d}{(n+1)N} (M(T))^{n+1} \tilde{M}_f. \end{aligned}$$

This complete the proof of convergence of  $\{\phi_n\}_{n \in \mathbb{N}}$  in  $\text{AC}([t_0, T]; \text{L}(C^\omega(M); C^\omega(U)))$ .  $\square$

Using Theorem 5.2 and the multiplicative property (10.3) of the solution of the initial value problem (10.2), one can show that the solution  $\phi$  constructed in Theorem 10.1 is the flow of the time-varying real analytic vector field  $X$ .

**Corollary 10.2.** *Let  $X : \mathbb{T} \times M \rightarrow TM$  be a locally integrally bounded real analytic vector field. Let  $t_0 \in \mathbb{T}$ ,  $x_0 \in M$ , and  $\phi^X : \mathbb{T}' \times U \rightarrow M$  be the flow of  $X$  defined on a time interval*

$\mathbb{T}' \subseteq \mathbb{T}$  containing  $t_0$  and a state neighbourhood  $U \subseteq M$  containing  $x_0$ . We know that  $\phi^X$  satisfies the following initial value problem for every  $x \in U$ .

$$\begin{aligned}\dot{\phi}^X(t, x) &= X(t, \phi^X(t, x)), \quad \text{a.e. } t \in \mathbb{T}', \\ \phi^X(t_0, x) &= x.\end{aligned}\tag{10.5}$$

Then there exists a positive real number  $T \in \mathbb{T}'$  such that  $T > t_0$  and a neighbourhood  $V$  of  $x_0$  such that, for every  $t \in [t_0, T]$  and every  $x \in V$ , we have

$$\begin{aligned}f(\phi^X(t, x)) &= f(x) \\ &+ \sum_{i=1}^{\infty} \int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{i-1}} \widehat{X}(t_i) \circ \widehat{X}(t_{i-1}) \circ \dots \circ \widehat{X}(t_1)(f)(x) dt_i dt_{i-1} \dots dt_1.\end{aligned}$$

*Proof.* By Theorem 10.1, there exist  $T > 0$ , a neighbourhood  $V \subseteq U$  of  $x_0$ , and a locally absolutely continuous curve  $\xi : [t_0, T] \rightarrow L(C^\omega(M); C^\omega(U))$  such that

$$\xi(t)(fg) = \xi(t)(f)\xi(t)(g), \quad \text{a.e. } t \in [t_0, T].\tag{10.6}$$

and, for every  $t \in [t_0, T]$  and every  $x \in V$ , we have

$$\begin{aligned}\xi(t)(f)(x) &= f(x) \\ &+ \sum_{i=1}^{\infty} \int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{i-1}} \widehat{X}(t_i) \circ \widehat{X}(t_{i-1}) \circ \dots \circ \widehat{X}(t_1)(f)(x) dt_i dt_{i-1} \dots dt_1.\end{aligned}$$

Since  $\xi$  satisfies equation (10.6), by Theorem 5.2, there exists a map  $\phi : [t_0, T] \times V \rightarrow M$  such that

$$\widehat{\phi}(t) = \xi(t), \quad \text{a.e. } t \in [t_0, T].$$

This implies that, for almost every  $t \in [t_0, T]$  and every  $x \in V$ , we have

$$\begin{aligned}f(\phi(t, x)) &= \xi(t)(f)(x) \\ &= f(x) + \sum_{i=1}^{\infty} \int_{t_0}^t \int_{t_0}^{t_1} \dots \int_{t_0}^{t_{i-1}} \widehat{X}(t_i) \circ \widehat{X}(t_{i-1}) \circ \dots \circ \widehat{X}(t_1)(f)(x) dt_i dt_{i-1} \dots dt_1.\end{aligned}$$

Therefore, by the uniqueness of the solution of the differential equation 10.5, it suffice to show that  $\phi$  satisfies differential equations (10.5). Note that, for every  $t \in [t_0, T]$ , we have

$$\frac{d\widehat{\phi}(t)}{dt} = \lim_{h \rightarrow \infty} \frac{\widehat{\phi}(t+h) - \widehat{\phi}(t)}{h}.$$

By applying  $f \in C^\omega(M)$  and noting that the topology on  $L(C^\omega(M); C^\omega(V))$  is topology of pointwise convergence, for almost every  $t \in [t_0, T]$  and every  $x \in V$ , we have

$$\begin{aligned}\frac{d\widehat{\phi}(t)}{dt}(f)(x) &= \lim_{h \rightarrow \infty} \frac{\widehat{\phi}(t+h)(f) - \widehat{\phi}(t)(f)}{h}(x) \\ &= \lim_{h \rightarrow \infty} \frac{f(\phi(t+h, x)) - f(\phi(t, x))}{h} = \frac{d(f(\phi(t, x)))}{dt}, \quad \forall f \in C^\omega(M).\end{aligned}$$

On the other hand, for almost every  $t \in [t_0, T]$  and every  $x \in V$ , we have

$$\frac{d\widehat{\phi}(t)}{dt}(f)(x) = \widehat{\phi}(t) \circ \widehat{X}(t)(f)(x) = X(t, \phi(t, x))(f), \quad \forall f \in C^\omega(M).$$

Therefore, we have

$$\frac{d(f(\phi(t, x)))}{dt} = X(t, \phi(t, x))(f), \quad \forall f \in C^\omega(M), \text{ a.e. } t \in [t_0, T], \forall x \in V.$$

This implies that

$$\dot{\phi}(t, x) = X(t, \phi(t, x)), \text{ a.e. } t \in [t_0, T], \forall x \in V.$$

□

## 11 The exponential map

In this section, we study the relationship between locally integrally bounded time-varying real analytic vector fields and their flows. In order to define such a map connecting time-varying vector fields and their flows, one should note that there may not exist a fixed interval  $\mathbb{T} \subseteq \mathbb{R}$  containing  $t_0$  and a fixed open neighbourhood  $U \subseteq M$  of  $x_0$ , such that the flow of “every” locally integrally bounded time-varying vector field  $X \in L^1(\mathbb{R}, \Gamma^\omega(TM))$  is defined on time interval  $\mathbb{T}$  and on neighbourhood  $U$ . The following example shows this for a family of real analytic vector fields.

**Example 11.1.** Consider the family of vector fields  $\{X_n\}_{n \in \mathbb{N}}$ , where  $X_n : \mathbb{R} \times \mathbb{R} \rightarrow T\mathbb{R} \simeq \mathbb{R}^2$  is defined as

$$X_n(t, x) = (x, nx^2), \quad \forall t \in \mathbb{T}, \forall x \in \mathbb{R}.$$

Let  $\mathbb{T} = [-1, 1]$ . Then, for every  $n \in \mathbb{N}$ , the flow of  $X_n$  is defined as

$$\phi^{X_n}(t, x) = \frac{x}{1 - nxt}.$$

This implies that  $\phi^{X_n}$  is only defined for  $x \in [-\frac{1}{n}, \frac{1}{n}]$ . Therefore, there does not exist an open neighbourhood  $U$  of 0 such that, for every  $n \in \mathbb{N}$ ,  $\phi^{X_n}$  is defined on  $U$ .

The above example suggest that it is natural to define the connection between vector fields and their flows on their germs around  $t_0$  and  $x_0$ . Let  $\mathbb{T} \subseteq \mathbb{R}$  be a compact interval containing  $t_0 \in \mathbb{R}$  and  $U \subseteq M$  be an open set containing  $x_0 \in M$ . We define

$$L_{(t_0, x_0)}^{1, \omega} = \varinjlim L^1(\mathbb{T}; \Gamma^\omega(TM)),$$

and

$$AC_{(t_0, x_0)}^\omega = \varinjlim AC(\mathbb{T}; L(C^\omega(M); C^\omega(U))).$$

These direct limits are in the category of topological spaces. We define the exponential map  $\exp : L_{(t_0, x_0)}^{1, \omega} \rightarrow AC_{(t_0, x_0)}^\omega$  as

$$\exp([X]_{(t_0, x_0)}) = [\phi^X]_{(t_0, x_0)}, \quad \forall [X]_{(t_0, x_0)} \in L_{(t_0, x_0)}^{1, \omega}.$$

**Theorem 11.2.** *The exponential map is sequentially continuous.*

*Proof.* To show that  $\exp : L_{(t_0, x_0)}^{1, \omega} \rightarrow AC_{(t_0, x_0)}^\omega$  is a sequentially continuous map, it suffices to prove that, for every sequence  $\{X_n\}_{n \in \mathbb{N}}$  in  $L^1(\mathbb{T}; \Gamma^\omega(TM))$  which converges to  $X \in L^1(\mathbb{T}; \Gamma^\omega(TM))$ , the sequence  $\{[\phi^{X_n}]_{(t_0, x_0)}\}$  converges to  $[\phi^X]_{(t_0, x_0)}$  in  $AC_{(t_0, x_0)}^\omega$ . Since the sequence  $\{X_n\}_{n \in \mathbb{N}}$  is converging, it is bounded in  $L^1(\mathbb{T}; \Gamma^\omega(TM))$ . So, by Theorem 10.1, there exists  $T > t_0$  and a relatively compact coordinate neighbourhood  $U$  of  $x_0$  such that  $[t_0, T] \subseteq \mathbb{T}$  and, for every  $n \in \mathbb{N}$ , we have  $\phi^{X_n} \in AC([t_0, T]; L(C^\omega(M); C^\omega(U)))$ . Therefore, it suffices to show that, for the sequence  $\{X_n\}_{n \in \mathbb{N}}$  in  $L^1(\mathbb{T}; \Gamma^\omega(TM))$  converging to  $X \in L^1(\mathbb{T}; \Gamma^\omega(TM))$ , the sequence  $\{\phi^{X_n}\}$  converges to  $\phi^X$  in  $AC([t_0, T]; L(C^\omega(M); C^\omega(U)))$ .

Let  $f \in C^\omega(M)$  be a real analytic function and suppose that we have

$$\lim_{m \rightarrow \infty} X_m = X$$

in  $L^1(\mathbb{T}; \Gamma^\omega(U))$ . By Theorems 8.3 and 9.1, there exists a neighbourhood  $\bar{V} \subseteq M^\mathbb{C}$  of  $U$  such that the bounded sequence of locally integrally bounded real analytic vector fields  $\{X_m\}_{m \in \mathbb{N}}$ , the real analytic vector field  $X$ , and the real analytic function  $f$  can be extended to a converging sequence of locally integrally bounded holomorphic vector fields  $\{\bar{X}_m\}_{m \in \mathbb{N}}$ , a locally integrally bounded holomorphic vector field  $\bar{X}$ , and a holomorphic function  $\bar{f}$  respectively. Moreover, by Theorem 9.1, the inductive limit

$$\varinjlim L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{U}_n)) = L^1(\mathbb{T}; \Gamma^\omega(TM))$$

is boundedly retractive. Therefore, we have

$$\lim_{m \rightarrow \infty} \bar{X}_m = \bar{X}$$

in  $L^1(\mathbb{T}; \Gamma_{\text{bdd}}^{\text{hol}, \mathbb{R}}(\bar{V}))$ . Now, according to Theorem 9.3, there exists  $d > 0$ , such that for every compact set  $K \subseteq U$ , every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , and every  $t \in \mathbb{T}$ , we have

$$\begin{aligned} p_{K, \mathbf{a}}^\omega(f) &\leq p_{\bar{V}}(\bar{f}), \\ \max_i \{p_{K, \mathbf{a}}^\omega(X^i(t))\} &\leq \max_i \{p_{\bar{V}}(\bar{X}^i(t))\}, \\ \max_i \{p_{K, \mathbf{a}}^\omega(X^i(t) - X_m^i(t))\} &\leq \max_i \{p_{\bar{V}}(\bar{X}^i(t) - \bar{X}_m^i(t))\}. \end{aligned}$$

Since  $\bar{X}$  is locally integrally bounded, there exists  $g \in L^1(\mathbb{T})$  such that

$$\max_i \{p_{\bar{V}}(\bar{X}^i(t))\} < g(t), \quad \forall t \in \mathbb{T}.$$

This implies that, for every compact set  $K \subseteq U$  and every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , we have

$$\max_i \{p_{K, \mathbf{a}}^\omega(X^i(t))\} \leq \max_i \{p_{\bar{V}}(\bar{X}^i(t))\} < g(t), \quad \forall t \in \mathbb{T}.$$

This means that, for every  $\epsilon > 0$ , there exists  $C \in \mathbb{N}$  such that

$$\int_{t_0}^t \max_i \{p_{\bar{V}}(\bar{X}_m^i(\tau) - \bar{X}^i(\tau))\} d\tau < \epsilon, \quad \forall m > C, t \in \mathbb{T}.$$

Therefore, if  $m > C$ , we have

$$\max_i \left\{ p_{\overline{V}}(\overline{X}_m^i(t)) \right\} \leq \max_i \left\{ p_{\overline{V}}(\overline{X}^i(t)) \right\} + \epsilon \leq g(t) + \epsilon, \quad \forall t \in \mathbb{T}, \forall m > C.$$

We define  $m \in L^1(\mathbb{T})$  as

$$m(t) = g(t) + \epsilon, \quad \forall t \in \mathbb{T}.$$

We also define  $\tilde{m} \in C(\mathbb{T})$  as

$$\tilde{m}(t) = \int_{t_0}^t (4N)m(\tau)d\tau, \quad \forall t \in \mathbb{T}.$$

We choose  $T > t_0$  such that  $|\tilde{m}(T)| < \frac{1}{2}$ .

**Lemma.** *Let  $K \subseteq U$  be a compact set and  $\mathbf{a} \in \mathbf{c}_0^\perp(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ . Then, for every  $n \in \mathbb{N}$ , we have*

$$p_{K, \mathbf{a}, f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) \leq \left( \sum_{r=0}^{n-1} (r+1)(\tilde{m}(t))^r p_{K, \mathbf{a}_{r+1}}^\omega(f) \right) \times \int_{t_0}^t \max_i \left\{ p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) \right\} d\tau, \quad \forall t \in [t_0, T], \forall m > C,$$

where  $\mathbf{a}_k$  is as defined in Lemma 7.17.

*Proof.* We prove this lemma using induction on  $n \in \mathbb{N}$ . We first check the case  $n = 1$ . For  $n = 1$ , using Theorem 7.18, we have

$$\begin{aligned} p_{K, \mathbf{a}, f}^\omega(\phi_1^X(t) - \phi_1^{X_m}(t)) &= p_{K, \mathbf{a}, f}^\omega \left( \int_{t_0}^t \widehat{X}(\tau) - \widehat{X}_m(\tau) d\tau \right) \\ &\leq \int_{t_0}^t p_{K, \mathbf{a}, f}^\omega \left( \widehat{X}(\tau) - \widehat{X}_m(\tau) \right) d\tau \\ &\leq p_{K, \mathbf{a}_1}^\omega(f) \int_{t_0}^t \max_i \left\{ p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) \right\} d\tau, \quad \forall t \in [t_0, T], \forall m > C, \end{aligned}$$

Now assume that, for  $j \in \{1, 2, \dots, n\}$ , we have

$$p_{K, \mathbf{a}, f}^\omega(\phi_j^X(t) - \phi_j^{X_m}(t)) \leq \sum_{r=0}^{j-1} \left( (r+1)(\tilde{m}(t))^r p_{K, \mathbf{a}_{r+1}}^\omega(f) \right) \times \int_{t_0}^t \max_i \left\{ p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) \right\} d\tau, \quad \forall t \in [t_0, T], \forall m > C.$$

We want to show that

$$p_{K, \mathbf{a}, f}^\omega(\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t)) \leq \sum_{r=0}^n \left( (r+1)(\tilde{m}(t))^r p_{K, \mathbf{a}_{r+1}}^\omega(f) \right) \times \int_{t_0}^t \max_i \left\{ p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) \right\} d\tau, \quad \forall t \in [t_0, T], \forall m > C.$$

Note that one can write

$$\begin{aligned}
\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t) &= \int_{t_0}^t (\phi_n^X(\tau) \circ \widehat{X}(\tau) - \phi_n^{X_m} \circ \widehat{X}_m(\tau)) d\tau \\
&= \int_{t_0}^t (\phi_n^X(\tau) - \phi_n^{X_m}(\tau)) \circ \widehat{X}(\tau) d\tau \\
&\quad + \int_{t_0}^t \phi_n^{X_m}(\tau) \circ (\widehat{X}(\tau) - \widehat{X}_m(\tau)) d\tau \quad \forall t \in [t_0, T], \quad \forall m > C.
\end{aligned}$$

Therefore, for every compact set  $K \subseteq U$  and every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ , we have

$$\begin{aligned}
p_{K, \mathbf{a}, f}(\phi_n^X(t) - \phi_n^{X_m}(t)) &\leq \int_{t_0}^t p_{K, \mathbf{a}, f}^\omega \left( (\phi_n^X(\tau) - \phi_n^{X_m}(\tau)) \circ \widehat{X}(\tau) \right) d\tau \\
&\quad + \int_{t_0}^t p_{K, \mathbf{a}, f}^\omega \left( \phi_n^{X_m}(\tau) \circ (\widehat{X}(\tau) - \widehat{X}_m(\tau)) \right) d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C.
\end{aligned}$$

Note that, for every  $\widehat{X}, \widehat{Y} \in L^1([t_0, T]; \Gamma^\omega(TM))$ , we have

$$p_{K, \mathbf{a}, f}^\omega \left( \phi_n^X(t) \circ \widehat{Y}(t) \right) = p_{K, \mathbf{a}, f}^\omega(\widehat{Y}(t)) + \sum_{r=1}^n p_{K, \mathbf{a}, f}^\omega \left( (\phi_r^X(t) - \phi_{r-1}^X(t)) \circ \widehat{Y}(t) \right)$$

Since, for every  $r \in \mathbb{N}$ , we have

$$p_{K, \mathbf{a}, f}^\omega \left( \phi_r^X(t) - \phi_{r-1}^X(t) \right) \leq (\tilde{m}(t))^r p_{K, \mathbf{a}_r}^\omega(f), \quad \forall t \in [t_0, T]$$

for every  $\widehat{X}, \widehat{Y} \in L^1([t_0, T]; \Gamma^\omega(TM))$ , we have

$$p_{K, \mathbf{a}, f}^\omega \left( \phi_n^X(t) \circ \widehat{Y}(t) \right) \leq \sum_{r=0}^n (\tilde{m}(t))^r p_{K, \mathbf{a}_r, f}^\omega \left( \widehat{Y}(t) \right), \quad \forall t \in [t_0, T].$$

This implies that, for every  $t \in [t_0, T]$  and every  $m > C$ , we have

$$\begin{aligned}
p_{K, \mathbf{a}, f}^\omega \left( \phi_n^{X_m}(t) \circ (\widehat{X}(t) - \widehat{X}_m(t)) \right) &\leq \sum_{r=0}^n (\tilde{m}(t))^r p_{K, \mathbf{a}_r, f}^\omega \left( \widehat{X}(t) - \widehat{X}_m(t) \right) \\
&\leq \sum_{r=0}^n \left( (r+1)(\tilde{m}(t))^r p_{K, \mathbf{a}_{r+1}}^\omega(f) \right) \max_i \left\{ p_{\overline{V}} \left( \overline{X}^i(t) - \overline{X}_m^i(t) \right) \right\}.
\end{aligned}$$

Therefore, for every  $t \in [t_0, T]$  and every  $m > C$ , we get

$$\begin{aligned}
&p_{K, \mathbf{a}, f}^\omega(\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t)) \\
&\leq \int_{t_0}^t \sum_{r=0}^{n-1} \left( (r+1)(r+2)(\tilde{m}(\tau))^r m(\tau) p_{K, \mathbf{a}_{r+2}}^\omega(f) \right) \int_{t_0}^t \max_i \left\{ p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) \right\} d\tau \\
&\quad + \int_{t_0}^t \sum_{r=0}^n \left( (r+1)(\tilde{m}(\tau))^r p_{K, \mathbf{a}_{r+1}}^\omega(f) \right) \max_i \left\{ p_{\overline{V}}(\overline{X}^i(t) - \overline{X}_m^i(t)) \right\} d\tau.
\end{aligned}$$

Using integration by parts, we have

$$p_{K,\mathbf{a},f}^\omega(\phi_{n+1}^X(t) - \phi_{n+1}^{X_m}(t)) \leq \sum_{r=0}^n (r+1)(\tilde{m}(t))^r p_{K,\mathbf{a}_{r+1}}^\omega(f) \times \\ \int_{t_0}^t p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C.$$

This completes the proof of the lemma  $\square$

Thus, for every  $n \in \mathbb{N}$ , we have

$$p_{K,\mathbf{a},f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) \\ \leq \sum_{r=0}^{n-1} (r+1)(\tilde{m}(t))^r p_{K,\mathbf{a}_{r+1}}^\omega(f) \left( \int_{t_0}^t p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) d\tau \right), \quad \forall t \in [t_0, T], \quad \forall m > C.$$

Since, for every  $t \in [t_0, T]$ , we have

$$|\tilde{m}(t)| < \frac{1}{2},$$

the series

$$\sum_{r=0}^{\infty} (r+1)(\tilde{m}(t))^r p_{K,\mathbf{a}_{r+1}}^\omega(f)$$

converges to a function  $h(t)$ , for every  $t \in [t_0, T]$ . By Lebesgue's monotone convergence theorem,  $h$  is integrable. This implies that, for every  $n \in \mathbb{N}$  and every  $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}, \mathbb{R}_{>0}, d)$ ,

$$p_{K,\mathbf{a},f}^\omega(\phi_n^X(t) - \phi_n^{X_m}(t)) \leq h(t) \int_{t_0}^t p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C.$$

Therefore, by taking the limit as  $n$  goes to infinity of the left hand side of the inequality, we have

$$p_{K,\mathbf{a},f}^\omega(\phi^X(t) - \phi^{X_m}(t)) \leq h(t) \int_{t_0}^t p_{\overline{V}}(\overline{X}^i(\tau) - \overline{X}_m^i(\tau)) d\tau, \quad \forall t \in [t_0, T], \quad \forall m > C.$$

This completes the proof of sequential continuity of  $\exp$ .  $\square$

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