Locally convex topologies and control theory

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Abstract

Using recent characterisations of topologies of spaces of vector fields for general regularity classes—e.g., Lipschitz, finitely differentiable, smooth, and real analytic—characterisations are provided of geometric control systems that utilise these topologies. These characterisations can be expressed as joint regularity properties of the system as a function of state and control. It is shown that the common characterisations of control systems in terms of their joint dependence on state and control are, in fact, representations of the fact that the natural mapping from the control set to the space of vector fields is continuous. The classes of control systems defined are new, even in the smooth category. However, in the real analytic category, the class of systems defined is new and deep. What are called "real analytic control systems" in this article incorporate the real analytic topology in a way that has hitherto been unexplored. Using this structure, it is proved, for example, that the trajectories of a real analytic control system corresponding to a fixed open-loop control depend on initial condition in a real analytic manner. It is also proved that control-affine systems always have the appropriate joint dependence on state and control. This shows, for example, that the trajectories of a control-affine system corresponding to a fixed open-loop control depend on initial condition in the manner prescribed by the regularity of the vector fields.

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1. Introduction

In nonlinear control theory, one considers systems of the form

$$\xi'(t) = F(\xi(t), \mu(t)),$$

where $t \mapsto \mu(t)$ is a curve taking values in a control set $\mathcal{C}$ and $t \mapsto \xi(t)$ is the corresponding trajectory, taking values in a differentiable manifold M. The control set is often taken to be a subset of some Euclidean space $\mathbb{R}^k$; this is especially meaningful for control-affine systems, where

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{m} u^a f_a(x)$$

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for vector fields $f_0, f_1, \dots, f_m$. In the general setting, it is more natural to work with something more general than subsets of Euclidean space, and a common assumption is that $\mathcal{C}$ is a separable metric space. (For a separable metric space, essentially bounded measurable controls can be approximated by piecewise constant controls [Sontag 1998, cf. Remark C.1.2].) In this paper, we will permit $\mathcal{C}$ to be an arbitrary topological space. In any case, to ensure useful properties of the system, one should place some properties on the map

$$\mathsf{M} \times \mathcal{C} \ni (x, u) \mapsto F(x, u) \in \mathsf{TM}$$

to ensure that the system is meaningful, e.g., that, if one substitutes a sufficiently nice control $t \mapsto \mu(t)$, the resulting differential equation has solutions. Let us denote $F^u(x) = F(x, u)$. A common sort of requirement is that the map $x \mapsto F^u(x)$ have some prescribed regularity, say smoothness, for each fixed u, and that the derivatives of F with respect to x be jointly continuous in (x, u). One might imagine, for example, that for smooth systems the condition that *all* derivatives with respect to x should be jointly continuous in (x, u) is somehow natural. However, in control theory one is often interested in real analytic systems, as these possess special properties that are of control theoretic significance [e.g., Sussmann 1990]. It is a natural question, therefore, to wonder what might be the appropriate joint conditions on (x, u), in the case that F^u is real analytic for every fixed u, to ensure that the system somehow "respects" the real analytic structure.

In this paper we address these questions by instead phrasing them in terms of continuity of the map $u \mapsto F^u$. Of course, continuity of this map requires topologies for both $\mathcal{C}$ and the set of vector fields. Therefore, a crucial ingredient in this approach is the prescription of appropriate topologies for spaces of vector fields. Such topologies are well-known in the smooth and finitely differentiable categories [Agrachev and Sachkov 2004, Michor 1980]. Recent work of Jafarpour and Lewis [2014] has provided, for the first time, a useable characterisation of the natural topology for the space of real analytic vector fields. This allows us to define what we can accurately call "real analytic control systems," and we provide equivalent characterisations of these in terms of maps $(x, u) \mapsto F(x, u)$ and $u \mapsto F^u$. We also provide the proper characterisations for Lipschitz, finitely differentiable, and smooth systems in order to show that our methodology is, in fact, a coherent framework for understanding the rôle of regularity in nonlinear control theory. However, it is for real analytic systems that one anticipates the greatest impact, since the structure we reveal will doubtless be important for investigations of local properties of systems, such as controllability. Indeed, one can imagine that, without the hypotheses we give for real analytic control systems, a complete characterisation of these sorts of properties is simply not possible, since the joint dependence on (x, u) would not be strong enough. These are, however, problems for future research.

1.1. Outline of the paper. The main results of the paper rely crucially on certain locally convex topologies for spaces of vector fields. In Section 2, following [Jafarpour and Lewis 2014], we review these topologies for the spaces of Lipschitz, finitely differentiable, smooth, holomorphic, and real analytic vector fields. (While we are not, per se, interested in holomorphic control systems, the treatment of real analytic systems is often made easier by considerations of holomorphic systems.) The presentation we give of these topologies is intended to be of the "user friendly" variety. That is to say, we simply present the seminorms we use to describe these topologies. A reader wishing to understand the topologies and their properties is encouraged to refer to [Jafarpour and Lewis 2014]. However, even a functional understanding of these topologies will require an understanding of locally convex topologies, and for this we refer to [Rudin 1991] as a gentle introduction and [Conway 1985, Groethendieck 1973, Horváth 1966, Jarchow 1981, Schaefer and Wolff 1999] as more advance treatments (which are certainly needed to understand the material in [Jafarpour and Lewis 2014]).

One of the nice features of our characterisations of control systems is that, upon substitution of an open-loop control, the resulting initial value problem has solutions depending on initial conditions in a manner consistent with the regularity of the system dependence on state. In Section 3 we review the material from [Jafarpour and Lewis 2014] regarding vector fields with measurable time dependence required to prove these results.

A control system, in a certain precise sense, is a parameterised family of vector fields, the parameter being control. In Section 4 we discuss vector fields parameterised by a parameter in a topological space. In particular, we are interested when the parameterised vector field depends continuously on control, where the topology on the space of vector fields is one of the topologies from Section 2. We characterise this continuous dependence by pointwise conditions on state and control. The characterisation in the real analytic case is novel, given that we are using the novel characterisation of the real analytic topology of Jafarpour and Lewis [2014]. We also characterise the real analytic case using holomorphic extensions, as this will likely be the easiest thing to do in practice.

Finally, in Section 5 we apply the results up to this point in the paper to control systems. We provide definitions of such systems, and show that, as mentioned above, the corresponding initial value problems have regular dependence on initial conditions. We also show that control-affine systems always have the continuous dependence on control as prescribed in Section 4.

1.2. Notation. Let us review the notation we shall use in the paper. There is a lot of machinery used, even in this fairly abbreviated treatment of the topologies for spaces of vector fields. We shall try to give as precise references as possible in the text to facilitate the reader acquiring the necessary background, if needed.

We shall use the slightly unconventional, but perfectly rational, notation of writing $A \subseteq B$ to denote set inclusion, and when we write $A \subset B$ we mean that $A \subseteq B$ and $A \neq B$. By id_A we denote the identity map on a set A. For a product $\prod_{i\in I} X_i$ of sets, $\mathrm{pr}_j\colon \prod_{i\in I} X_i \to X_j$ is the projection onto the jth component. For a subset $A \subseteq X$, we denote by χ_A the characteristic function of A, i.e.,

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

By $\mathfrak{S}_k$ we denote the symmetric group on k symbols. By $\mathbb{Z}$ we denote the set of integers, with $\mathbb{Z}_{\geq 0}$ denoting the set of nonnegative integers and $\mathbb{Z}_{>0}$ denoting the set of positive integers. We denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers. By $\mathbb{R}_{\geq 0}$ we denote the set of nonnegative real numbers and by $\mathbb{R}_{>0}$ the set of positive real numbers. By $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ we denote the extended nonnegative real numbers.

Elements of $\mathbb{F}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, are typically denoted with a bold font, e.g., "$\boldsymbol{x}$."

We shall use constructions from algebra and multilinear algebra, referring to [Hungerford 1980], [Bourbaki 1989, Chapter III], and [Bourbaki 1990, §IV.5]. If F is a field (for us, typically $\mathsf{F} \in \{\mathbb{R}, \mathbb{C}\}$) and if V is an F-vector space, we denote by $\mathsf{V}^* = \mathrm{Hom}_{\mathsf{F}}(\mathsf{V}; \mathsf{F})$ the algebraic dual. The k-fold tensor product of V with itself is denoted by $\mathrm{T}^k(\mathsf{V})$. Thus, if V is finite-dimensional, we identify $\mathrm{T}^k(\mathsf{V}^*)$ with the k-multilinear F-valued functions on V^k by

$$(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \dots, v_k) = \alpha^1(v_1) \cdots \alpha^k(v_k).$$

By $\mathrm{S}^k(\mathsf{V}^*)$ we denote the symmetric tensor algebra of degree k, which we identify with the symmetric k-multilinear F-valued functions on V^k, or polynomial functions of homogeneous degree k on V.

For a topological space $\mathcal{X}$ and $A \subseteq \mathcal{X}$, $\mathrm{int}(A)$ denotes the interior of A and $\mathrm{cl}(A)$ denotes the closure of A. Neighbourhoods will always be open sets.

By λ we denote Lebesgue measure. If $I \subseteq \mathbb{R}$ is an interval and if $A \subseteq \mathbb{R}$, by $\mathrm{L}^1(I; A)$ we denote the set of Lebesgue integrable A-valued functions on I. By $\mathrm{L}^1_{\mathrm{loc}}(I; A)$ we denote the A-valued locally integrable functions on I, i.e., those functions whose restrictions to compact subintervals are integrable. In like manner, we denote by $\mathrm{L}^\infty(I; A)$ and $\mathrm{L}^\infty_{\mathrm{loc}}(I; A)$ the essentially bounded A-valued functions and the locally essentially bounded A-valued functions, respectively.

For an interval I and a topological space $\mathcal{X}$, a curve $\gamma\colon I \to \mathcal{X}$ is ***measurable*** if $\gamma^{-1}(\mathcal{O})$ is Lebesgue measurable for every open $\mathcal{O} \subseteq \mathcal{X}$. By $\mathrm{L}^\infty(I; \mathcal{X})$ we denote the measurable curves $\gamma\colon I \to \mathcal{X}$ for which there exists a compact set $K \subseteq \mathcal{X}$ with

$$\lambda(\{t \in I \mid \gamma(t) \notin K\}) = 0,$$

i.e., $\mathrm{L}^\infty(I; \mathcal{X})$ is the set of ***essentially bounded*** curves. By $\mathrm{L}^\infty_{\mathrm{loc}}(I; \mathcal{X})$ we denote the ***locally essentially bounded*** curves, meaning those measurable curves whose restrictions to compact subintervals are essentially bounded.

Our differential geometric conventions mostly follow [Abraham, Marsden, and Ratiu 1988]. Whenever we write "manifold," we mean "second-countable Hausdorff manifold." This implies, in particular, that manifolds are assumed to be metrisable [Abraham, Marsden, and Ratiu 1988, Corollary 5.5.13]. If we use the letter "n" without mentioning what it is, it is the dimension of the connected component of the manifold M with which we are working at that time. The tangent bundle of a manifold M is denoted by $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ and the cotangent bundle by $\pi_{\mathsf{T}^*\mathsf{M}}\colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$. If $I \subseteq \mathbb{R}$ is an interval and if $\xi\colon I \to \mathsf{M}$ is a curve that is differentiable at $t \in I$, we denote the tangent vector field to the curve at t by $\xi'(t)$.

If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a vector bundle, we denote the fibre over $x \in \mathsf{M}$ by E_x and we sometimes denote by 0_x the zero vector in E_x. If $\mathsf{S} \subseteq \mathsf{M}$ is a submanifold, we denote by $\mathsf{E}|\mathsf{S}$ the restriction of E to S which we regard as a vector bundle over S. If $\mathbb{G}$ is a fibre metric on E, i.e., a smooth assignment of an inner product to each of the fibres of E, then $\|\cdot\|_{\mathbb{G}}$ denotes the norm associated with the inner product on fibres.

We will work in both the smooth and real analytic categories, with occasional forays into the holomorphic category. We will also work with finitely differentiable objects, i.e., objects of class C^r for $r \in \mathbb{Z}_{\geq 0}$. (We will also work with Lipschitz objects, but will develop the notation for these in the text.) A good reference for basic real analytic analysis is [Krantz and Parks 2002], but we will need ideas going beyond those from this text, or any other text. Relatively recent work of e.g., [Domański 2012], [Vogt 2013], and [Domański and Vogt 2000] has shed a great deal of light on real analytic analysis, and we shall take advantage of this work. An analytic manifold or mapping will be said to be of ***class*** $\mathbf{C}^{\boldsymbol{\omega}}$. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. The set of sections of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ of class C^r is denoted by $\Gamma^r(\mathsf{E})$. Thus, in particular, $\Gamma^r(\mathsf{TM})$ denotes the set of vector fields of class C^r. We shall think of $\Gamma^r(\mathsf{E})$ as a $\mathbb{R}$-vector space with the natural pointwise addition and scalar multiplication operations.

We also work with holomorphic, i.e., complex analytic, manifolds and associated geometric constructions; real analytic geometry, at some level, seems to unavoidably rely on holomorphic geometry. A nice overview of holomorphic geometry, and some of its connections to real analytic geometry, is given in the book of Cieliebak and Eliashberg [2012]. There are many specialised texts on the subject of holomorphic geometry, including [Demailly 2012, Fritzsche and Grauert 2002, Gunning and Rossi 1965, Hörmander 1966]. For our purposes, we shall just say the following things. By TM we denote the holomorphic tangent bundle of M. This is the object which, in complex differential geometry, is commonly denoted by $\mathsf{T}^{1,0}\mathsf{M}$. By $\Gamma^{\mathrm{hol}}(\mathsf{E})$ the space of holomorphic sections of an holomorphic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$. We shall use both the natural $\mathbb{C}$- and, by restriction, $\mathbb{R}$-vector space structures for $\Gamma^{\mathrm{hol}}(\mathsf{E})$.

We shall make use of locally convex topological vector spaces, and refer to [Conway 1985, Groethendieck 1973, Horváth 1966, Jarchow 1981, Rudin 1991, Schaefer and Wolff 1999] for details. In the proof of Theorem 4.10 we shall make use of the contemporary research literature on locally convex spaces, and will indicate this when required. We shall denote by $\mathrm{L}(\mathsf{U}; \mathsf{V})$ the set of continuous linear maps from a locally convex space U to a locally convex space V. We will break with the usual language one sees in the theory of locally convex spaces and call what are commonly called "inductive" limits, instead "direct" limits, in keeping with the rest of category theory. (The notion of a direct limit only occurs in the proof of Theorem 4.10, so readers not interesting in understanding this proof can forgo the rather difficult notion of direct limit topologies.)

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2. Topologies for spaces of vector fields

In this section we review the definitions of the topologies we use for spaces of Lipschitz, finitely differentiable, smooth, holomorphic, and real analytic vector fields. We will not work explicitly with holomorphic systems, but it is often easiest to describe real analytic attributes in terms of holomorphic extensions, particularly in practice, where one simply "replaces x with z."

While our interest in this paper is solely in vector fields, it is notationally simpler, and mathematically no more complicated, to work instead with general vector bundles much of the time. Thus, throughout this section we shall consider a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ that is either smooth, real analytic, or holomorphic, depending on our needs.

We comment that all topologies we define are locally convex topologies, of which the normed topologies are a special case. However, few of the topologies we define, and none of the interesting ones, are normable. So a reader who is not familiar with locally convex topologies will have to do some reading; we recommend [Rudin 1991] as a nice introduction.

2.1. Fibre norms for jet bundles. The classes of sections we consider are all characterised by their derivatives in some manner. The appropriate device for considering derivatives of sections is the theory of jet bundles, for which we refer to [Saunders 1989] and [Kolář, Michor, and Slovák 1993, §12]. By $\mathsf{J}^m\mathsf{E}$ we denote the vector bundle of m-jets of sections of E, with $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ denoting the projection. If ξ is a smooth section of E, we denote by $j_m\xi$ the corresponding smooth section of $\mathsf{J}^m\mathsf{E}$.

Sections of $\mathsf{J}^m\mathsf{E}$ should be thought of as sections of E along with their first m derivatives. In a local trivialisation of E, one has the local representatives of the derivatives, order-by-order. Such an order-by-order decomposition of derivatives is not possible globally, however. Nonetheless, following [Jafarpour and Lewis 2014, §2.1], we shall mimic this order-by-order decomposition globally using a linear connection ∇^0 on E and an affine connection ∇ on M. First note that ∇ defines a connection on $\mathsf{T}^*\mathsf{M}$ by duality. Also, ∇ and ∇^0 together define a connection ∇^m on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ by asking that the Leibniz Rule be satisfied for tensor product. Then, for a smooth section ξ of E, we denote

$$\nabla^{(m)}\xi = \nabla^m \cdots \nabla^1\nabla^0\xi,$$

which is a smooth section of $\mathrm{T}^{m+1}(\mathsf{T}^*\mathsf{M} \otimes \mathsf{E})$. By convention we take $\nabla^{(-1)}\xi = \xi$.

We then have a map

$$\begin{aligned} S^m_{\nabla,\nabla^0}\colon\ \mathsf{J}^m\mathsf{E} &\to \oplus_{j=0}^m(\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \\ j_m\xi(x) &\mapsto (\xi(x), \mathrm{Sym}_1 \otimes \mathrm{id}_{\mathsf{E}}(\nabla^0\xi)(x), \dots, \mathrm{Sym}_m \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(m-1)}\xi)(x)), \end{aligned} \tag{2.1}$$

which can be verified to be an isomorphism of vector bundles [Jafarpour and Lewis 2014, Lemma 2.1]. Here $\mathrm{Sym}_m\colon \mathrm{T}^m(\mathsf{V}) \to \mathrm{S}^m(\mathsf{V})$ is defined by

$$\mathrm{Sym}_m(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

Now we note that inner products on the components of a tensor product induce in a natural way inner products on the tensor product [Jafarpour and Lewis 2014, Lemma 2.3]. Thus, if we suppose that we have a fibre metric $\mathbb{G}_0$ on E and a Riemannian metric $\mathbb{G}$ on M, there is induced a natural fibre metric $\mathbb{G}_m$ on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ for each $m \in \mathbb{Z}_{\geq 0}$. We then define a fibre metric $\overline{\mathbb{G}}_m$ on $\mathsf{J}^m\mathsf{E}$ by

$$\overline{\mathbb{G}}_m(j_m\xi(x), j_m\eta(x)) = \sum_{j=0}^m \mathbb{G}_j\Big(\frac{1}{j!}\,\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(j-1)}\xi)(x), \frac{1}{j!}\,\mathrm{Sym}_j \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(j-1)}\eta)(x)\Big).$$

(The factorials are required to make things work out with the real analytic topology.) The corresponding fibre norm we denote by $\|\cdot\|_{\overline{\mathbb{G}}_m}$.

2.2. Seminorms for spaces of smooth vector fields. Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle. Using the fibre norms from the preceding section, it is a straightforward matter to define appropriate seminorms that define the locally convex topology for $\Gamma^\infty(\mathsf{E})$. For $K \subseteq \mathsf{M}$ compact and for $m \in \mathbb{Z}_{\geq 0}^m$, define a seminorm $p_{K,m}^\infty$ on $\Gamma^\infty(\mathsf{E})$ by

$$p_{K,m}^\infty(\xi) = \sup\{\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K\}.$$

The family of seminorms $p_{K,m}^\infty$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, is a locally convex topology, called the ***C$^\infty$-topology***,[1] with the following properties:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$ and $m \in \mathbb{Z}_{\geq 0}$, the sequence $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero.

In this paper we shall not make reference to other properties of the C$^\infty$-topology, but we mention that there are other properties that play an important rôle in the results in Section 3. For these details, and for references where the above properties are proved, we refer to [Jafarpour and Lewis 2014, §3.2].

2.3. Seminorms for spaces of finitely differentiable vector fields. We again take $\pi\colon \mathsf{E} \to \mathsf{M}$ to be a smooth vector bundle, and we fix $m \in \mathbb{Z}_{\geq 0}$. For the space $\Gamma^m(\mathsf{E})$ of m-times continuously differentiable sections, we define seminorms p_K^m, $K \subseteq \mathsf{M}$ compact, for $\Gamma^m(\mathsf{E})$ by

$$p_K^m(\xi) = \sup\{\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K\}.$$

The locally convex topology defined by the family of seminorms p_K^m, $K \subseteq \mathsf{M}$ compact, we call the ***C^m-topology***, and it has the following properties:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$, the sequence $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero;
4. if M is compact, then p_{M}^m is a norm that gives the C^m-topology.

As with the C$^\infty$-topology, we refer to [Jafarpour and Lewis 2014, §3.4] for details.

2.4. Seminorms for spaces of Lipschitz vector fields. In this section we again work with a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$. In defining the fibre metrics from Section 2.1, for the Lipschitz topologies the affine connection ∇ is required to be the Levi-Civita connection for the Riemannian metric $\mathbb{G}$ and the linear connection ∇^0 is required to be $\mathbb{G}_0$-orthogonal. While Lipschitz vector fields are often used, spaces of Lipschitz vector fields are not. Nonetheless, one may define seminorms for spaces of Lipschitz vector fields rather analogous to those defined above in the smooth and finitely differentiable cases. Let $m \in \mathbb{Z}_{\geq 0}$. By

[1] This is actually not a very good name. A better name, and the name used by Jafarpour and Lewis [2014], would be the "smooth compact-open topology." However, we wish to keep things simple here, and also use notation that is common between regularity classes.

$\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ we denote the space of sections of E that are m-times continuously differentiable and whose m-jet is locally Lipschitz. (One can think of this in coordinates, but Jafarpour and Lewis [2014] provide geometric definitions, if the reader is interested.) If a section ξ is of class $\mathrm{C}^{m+\mathrm{lip}}$, then, by Rademacher's Theorem [Federer 1969, Theorem 3.1.6], its $(m+1)$st derivative exists almost everywhere. Thus we define

$$\begin{aligned}\operatorname{dil} j_m\xi(x) = \inf\{\sup\{\|\nabla^{[m]}_{v_y} j_m\xi\|_{\overline{\mathbb{G}}_m} \mid\ & y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\\ & j_m\xi \text{ differentiable at } y\}\mid\ \mathcal{U} \text{ is a relatively compact neighbourhood of } x\},\end{aligned}$$

which is the ***local sectional dilatation*** of ξ. Here $\nabla^{[m]}$ is the connection in $\mathsf{J}^m\mathsf{E}$ defined by the decomposition (2.1). Let $K \subseteq \mathsf{M}$ be compact and define

$$\lambda^m_K(\xi) = \sup\{\operatorname{dil} j_m\xi(x) \mid\ x \in K\}$$

for $\xi \in \Gamma^{m+\mathrm{lip}}(\mathsf{E})$. We can then define a seminorm $p^{m+\mathrm{lip}}_K$ on $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ by

$$p^{m+\mathrm{lip}}_K(\xi) = \max\{\lambda^m_K(\xi), p^m_K(\xi)\}.$$

The family of seminorms $p^{m+\mathrm{lip}}_K$ defines a locally convex topology for $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$, which we call that ***$\mathbf{C}^{m+\mathrm{lip}}$-topology***, having the following attributes:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$, the sequence $(j_m\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero in both seminorms λ^m_K and p^m_K;
4. if M is compact, then $p^{m+\mathrm{lip}}_{\mathsf{M}}$ is a norm that gives the $\mathrm{C}^{m+\mathrm{lip}}$-topology.

We refer to [Jafarpour and Lewis 2014, §3.5] for details.

2.5. Seminorms for spaces of holomorphic vector fields. Now we consider a holomorphic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and denote by $\Gamma^{\mathrm{hol}}(\mathsf{E})$ the space of holomorphic sections of E. We let $\mathbb{G}$ be an Hermitian metric on the vector bundle and denote by $\|\cdot\|_{\mathbb{G}}$ the associated fibre norm. For $K \subseteq \mathsf{M}$ compact, denote by p^{hol}_K the seminorm

$$p^{\mathrm{hol}}_K(\xi) = \sup\{\|\xi(z)\|_{\mathbb{G}} \mid\ z \in K\}$$

on $\Gamma^{\mathrm{hol}}(\mathsf{E})$. The family of seminorms p^{hol}_K, $K \subseteq \mathsf{M}$ compact, define a locally convex topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ that we call the ***$\mathbf{C}^{\mathrm{hol}}$-topology***. This topology has the following properties:

1. it is Hausdorff, metrisable, and complete, i.e., it is a Fréchet topology;
2. it is separable;
3. it is characterised by the sequences converging to zero, which are the sequences $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ such that, for each $K \subseteq \mathsf{M}$, the sequence $(\xi_j|K)_{j\in\mathbb{Z}_{>0}}$ converges uniformly to zero;
4. if M is compact, then $p^{m+\mathrm{lip}}_{\mathsf{M}}$ is a norm that gives the $\mathrm{C}^{m+\mathrm{lip}}$-topology.

We refer to [Jafarpour and Lewis 2014, §4.2] and the references therein for details about the $\mathrm{C}^{\mathrm{hol}}$-topology.

We shall also require a result related to the classical Cauchy estimates from complex analysis. To state the result, denote by

$$\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}) = \{\xi \in \Gamma^{\mathrm{hol}}(\mathsf{E}) \mid \sup\{\|\xi(z)\|_{\mathbb{G}} \mid z \in \mathsf{M}\} < \infty\}$$

the subspace of bounded sections. This is a normed space with the norm

$$p^{\mathrm{hol}}_{\mathsf{M},\infty}(\xi) = \sup\{\|\xi(z)\|_{\mathbb{G}} \mid z \in \mathsf{M}\}.$$

We then have the following result.

2.1 Proposition: (Cauchy estimates for vector bundles) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle, let $K \subseteq \mathsf{M}$ be compact, and let $\mathcal{U}$ be a relatively compact neighbourhood of K. Then there exist $C, r \in \mathbb{R}_{>0}$ such that*

$$p^{\infty}_{K,m}(\xi) \leq C r^{-m} p^{\mathrm{hol}}_{\mathcal{U},\infty}(\xi)$$

for every $m \in \mathbb{Z}_{\geq 0}$ and $\xi \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$.

Proof: We refer to [Jafarpour and Lewis 2014, Proposition 4.2]. ■

2.6. Seminorms for spaces of real analytic vector fields. The topologies described above for spaces of smooth, finitely differentiable, Lipschitz, and holomorphic sections of a vector bundle are quite simple to understand in terms of their converging sequences. The topology one considers for real analytic sections does not have this attribute. There is a bit of a history to the characterisation of real analytic topologies, and we refer to [Jafarpour and Lewis 2014, §5] for *four* equivalent characterisations of the real analytic topology for the space of real analytic sections of a vector bundle. Here we will give the most elementary of these definitions to state, although it is probably not the most practical definition. In practice, it is probably best to somehow complexify and use the holomorphic topology; we give an instance of this in Theorem 4.10 below.

In this section we let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\Gamma^{\omega}(\mathsf{E})$ be the space of real analytic sections. One can show that there exist a real analytic linear connection ∇^0 on E, a real analytic affine connection ∇ on M, a real analytic fibre metric on E, and a real analytic Riemannian metric on M [Jafarpour and Lewis 2014, Lemma 2.4]. Thus we can define real analytic fibre metrics $\overline{\mathbb{G}}_m$ on the jet bundles $\mathsf{J}^m\mathsf{E}$ as in Section 2.1.

To define seminorms for $\Gamma^{\omega}(\mathsf{E})$, let $\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ denote the space of nonincreasing sequences in $\mathbb{R}_{>0}$, indexed by $\mathbb{Z}_{\geq 0}$, and converging to zero. We shall denote a typical element of $\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ by $\boldsymbol{a} = (a_j)_{j \in \mathbb{Z}_{\geq 0}}$. Now, for $K \subseteq \mathsf{M}$ and $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, we define a seminorm $p^{\omega}_{K,\boldsymbol{a}}$ for $\Gamma^{\omega}(\mathsf{E})$ by

$$p^{\omega}_{K,\boldsymbol{a}}(\xi) = \sup\{a_0 a_1 \cdots a_m \|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K,\ m \in \mathbb{Z}_{\geq 0}\}.$$

The family of seminorms $p^{\omega}_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, defines a locally convex topology on $\Gamma^{\omega}(\mathsf{E})$ that we call the ***$\mathbf{C}^{\boldsymbol{\omega}}$-topology***. This topology has the following attributes:

1. it is Hausdorff and complete;
2. it is not metrisable (and so it not a Fréchet topology);
3. it is separable.

We shall generally avoid dealing with the rather complicated structure of this topology, and shall be able to do what we need by just working with the seminorms. However, in the proof of the quite useful Theorem 4.10, we shall make reference to some of the more complicated characterisations of the C^ω-topology; we will make the appropriate references required in the course of that proof.

2.7. Summary and notation. In the real case, the degrees of regularity are ordered according to

$$\mathrm{C}^0 \supset \mathrm{C}^{\mathrm{lip}} \supset \mathrm{C}^1 \supset \cdots \supset \mathrm{C}^m \supset \mathrm{C}^{m+\mathrm{lip}} \supset \mathrm{C}^{m+1} \supset \cdots \supset \mathrm{C}^\infty \supset \mathrm{C}^\omega,$$

and in the complex case the ordering is the same, of course, but with an extra $\mathrm{C}^{\mathrm{hol}}$ on the right. Sometimes it will be convenient to write $\nu + \mathrm{lip}$ for $\nu \in \{\mathbb{Z}_{\geq 0}, \infty, \omega\}$, and in doing this we adopt the obvious convention that $\infty + \mathrm{lip} = \infty$ and $\omega + \mathrm{lip} = \omega$.

Where possible, we will state definitions and results for all regularity classes at once. To do this, we will let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, and consider the regularity classes $\nu \in \{m + m', \infty, \omega\}$. In such cases we shall require that the underlying manifold be of class "C^r, $r \in \{\infty, \omega\}$, as required." This has the obvious meaning, namely that we consider class C^ω if $\nu = \omega$ and class C^∞ otherwise. Proofs will typically break into the four cases $\nu = \infty$, $\nu = m$, $\nu = m + \mathrm{lip}$, and $\nu = \omega$. In some cases there is a structural similarity in the way arguments are carried out, so we will sometimes do all cases at once. In doing this, we will, for $K \subseteq \mathsf{M}$ be compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p_K = \begin{cases} p^\infty_{K,k}, & \nu = \infty, \\ p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m + \mathrm{lip}, \\ p^\omega_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases}$$

The convenience and brevity more than make up for the slight loss of preciseness in this approach.

3. Time-varying vector fields

One of the principle contributions of this paper is that, for the control systems we define in Section 5, if we substitute an open-loop control $\mu = \mu_*$ into a system

$$\xi'(t) = F(\xi(t), \mu(t))$$

to obtain a time-varying dynamical system

$$\xi'(t) = F(\xi(t), \mu_*(t)),$$

then the flow of this dynamical system depends on initial condition in a manner consistent with the regularity of F. We shall do this in Section 5 by showing that the time-varying vector field $t \mapsto F(x, \mu_*(t))$ falls into a class of vector fields having the appropriate regular dependence on initial condition. The body of work which characterises these classes of vector fields is, in and of itself, quite nontrivial, and we refer to [Jafarpour and Lewis 2014, §6] for details. Here we present the relevant definitions, and state the results from [Jafarpour and Lewis 2014] that we shall require.

The first four subsections below are thus dedicated to definitions, while the final subsection is dedicated to a summary of the required results. As in the preceding section, we give our definitions for sections of vector bundles rather than vector fields, since this is simpler notationally, and costs nothing in terms of complication.

3.1. Smooth time-varying vector fields. We will work with a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with a linear connection ∇^0 on E, an affine connection ∇ on M, a fibre metric $\mathbb{G}_0$ on E, and a Riemannian metric $\mathbb{G}$ on M. This defines the fibre norms $\|\cdot\|_{\overline{\mathbb{G}}_m}$ on $\mathsf{J}^m\mathsf{E}$ and seminorms $p^\infty_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, on $\Gamma^\infty(\mathsf{E})$ as in Section 2.2.

3.1 Definition: (Smooth Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A ***Carathéodory section of class*** $\mathbf{C}^\infty$ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t, x) \in \mathsf{E}_x$ for each $(t, x) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(x) = \xi(t, x)$ is of class C^∞;

(iii) for each $x \in \mathsf{M}$, the map $\xi^x\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^x(t) = \xi(t, x)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^\infty(\mathbb{T}; \mathsf{E})$ we denote the set of Carathéodory sections of class C^∞ of E. •

Note that the curve $t \mapsto \xi(t, x)$ is in the finite-dimensional vector space E_x, and so Lebesgue measurability of this is unambiguously defined, e.g., by choosing a basis and asking for Lebesgue measurability of the components with respect to this basis.

Now we put some conditions on the time dependence of the derivatives of the section.

3.2 Definition: (Locally integrally $\mathbf{C}^\infty$-bounded and locally essentially $\mathbf{C}^\infty$-bounded sections) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A Carathéodory section $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class C^∞ is

(i) ***locally integrally*** $\mathbf{C}^\infty$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $m \in \mathbb{Z}_{\geq 0}$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t, x) \in \mathbb{T} \times K,$$

and is

(ii) ***locally essentially*** $\mathbf{C}^\infty$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $m \in \mathbb{Z}_{\geq 0}$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t, x) \in \mathbb{T} \times K.$$

The set of locally integrally C^∞-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^\infty(\mathbb{T}, \mathsf{E})$ and the set of locally essentially C^∞-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^\infty(\mathbb{T}; \mathsf{E})$. •

3.2. Finitely differentiable and Lipschitz time-varying vector fields. In this section, so as to be consistent with our definition of Lipschitz norms in Section 2.4, we suppose that the affine connection ∇ on M is the Levi-Civita connection for the Riemannian metric $\mathbb{G}$ and that the vector bundle connection ∇^0 in E is $\mathbb{G}_0$-orthogonal.

3.3 Definition: (Finitely differentiable or Lipschitz Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $m \in \mathbb{Z}_{\geq 0}$ and let $m' \in \{0, \mathrm{lip}\}$. A ***Carathéodory section of class*** $\mathbf{C}^{m+m'}$ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t,x) \in \mathsf{E}_x$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(x) = \xi(t,x)$ is of class $\mathrm{C}^{m+m'}$;

(iii) for each $x \in \mathsf{M}$, the map $\xi^x\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^x(t) = \xi(t,x)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$ we denote the set of Carathéodory sections of class $\mathrm{C}^{m+m'}$ of E. •

Now we put some conditions on the time dependence of the derivatives of the section.

3.4 Definition: (Locally integrally $\mathbf{C}^{m+m'}$-bounded and locally essentially $\mathbf{C}^{m+m'}$-bounded sections) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $m \in \mathbb{Z}_{\geq 0}$ and let $m' \in \{0, \mathrm{lip}\}$. A Carathéodory section $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class $\mathrm{C}^{m+m'}$ is

(i) ***locally integrally*** $\mathbf{C}^{m+m'}$***-bounded*** if:

(a) $m' = 0$: for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K;$$

(b) $m' = \mathrm{lip}$: for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\operatorname{dil} j_m\xi_t(x), \|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K,$$

and is

(ii) ***locally essentially*** $\mathbf{C}^{m+m'}$***-bounded*** if:

(a) $m' = 0$: for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K;$$

(b) $m' = \mathrm{lip}$: for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\operatorname{dil} j_m\xi_t(x), \|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K.$$

The set of locally integrally $\mathrm{C}^{m+m'}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^{m+m'}(\mathbb{T},\mathsf{E})$ and the set of locally essentially $\mathrm{C}^{m+m'}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$. •

3.3. Holomorphic time-varying vector fields. We will consider an holomorphic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with an Hermitian fibre metric $\mathbb{G}$. This defines the seminorms p^{hol}_K, $K \subseteq \mathsf{M}$ compact, describing the $\mathrm{C}^{\mathrm{hol}}$-topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ as in Section 2.5.

Let us get started with the definitions.

3.5 Definition: (Holomorphic Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A ***Carathéodory section of class*** $\mathbf{C}^{\mathbf{hol}}$ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t,z) \in \mathsf{E}_z$ for each $(t,z) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(z)$ is of class $\mathrm{C}^{\mathrm{hol}}$;

(iii) for each $z \in \mathsf{M}$, the map $\xi^z\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^z(t) = \xi(t,z)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})$ we denote the set of Carathéodory sections of class $\mathrm{C}^{\mathrm{hol}}$ of E. •

The associated notions for time-dependent sections compatible with the $\mathrm{C}^{\mathrm{hol}}$-topology are as follows.

3.6 Definition: (Locally integrally $\mathbf{C}^{\mathbf{hol}}$-bounded and locally essentially $\mathbf{C}^{\mathbf{hol}}$-bounded sections) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A Carathéodory section $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class $\mathrm{C}^{\mathrm{hol}}$ is

(i) ***locally integrally*** $\mathbf{C}^{\mathbf{hol}}$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|\xi(t,z)\|_{\mathbb{G}} \leq g(t), \qquad (t,z) \in \mathbb{T} \times K$$

and is

(ii) ***locally essentially*** $\mathbf{C}^{\mathbf{hol}}$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|\xi(t,z)\|_{\mathbb{G}} \leq g(t), \qquad (t,z) \in \mathbb{T} \times K.$$

The set of locally integrally $\mathrm{C}^{\mathrm{hol}}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T},\mathsf{E})$ and the set of locally essentially $\mathrm{C}^{\mathrm{hol}}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})$. •

3.4. Real analytic time-varying vector fields.

We will consider a real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with ∇^0 a real analytic linear connection on E, ∇ a real analytic affine connection on M, $\mathbb{G}_0$ a real analytic fibre metric on E, and $\mathbb{G}$ a real analytic Riemannian metric on M. This defines the seminorms $p^\omega_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, describing the C^ω-topology as in Section 2.6.

3.7 Definition: (Real analytic Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A ***Carathéodory section of class*** $\mathbf{C}^{\boldsymbol{\omega}}$ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t,x) \in \mathsf{E}_x$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(x)$ is of class C^ω;

(iii) for each $x \in \mathsf{M}$, the map $\xi^x\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^x(t) = \xi(t,x)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^\omega(\mathbb{T};\mathsf{E})$ we denote the set of Carathéodory sections of class C^ω of E. •

Now we turn to placing restrictions on the time-dependence to allow us to do useful things.

3.8 Definition: (Locally integrally C^ω-bounded and locally essentially C^ω-bounded sections) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A Carathéodory section $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class C^ω is

(i) ***locally integrally C^ω-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K,\ m \in \mathbb{Z}_{\geq 0},$$

and is

(ii) ***locally essentially C^ω-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K,\ m \in \mathbb{Z}_{\geq 0}.$$

The set of locally integrally C^ω-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^\omega(\mathbb{T}, \mathsf{E})$ and the set of locally essentially C^ω-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^\omega(\mathbb{T}; \mathsf{E})$. •

3.5. Topological characterisations and regularity of flows. In this section we shall state two results, one giving topological characterisations of the preceding definitions and one giving regular dependence of flows on initial conditions. We shall state the results in such a way that all regularity cases are dealt with at once, recalling our notation from Section 2.7.

The topological characterisations we give associated with the above definitions have to do with measurability, integrability, and boundedness of the curve $t \mapsto \xi_t$ in the space of sections. In general, these notions require some care in their formulation for arbitrary locally convex topological vector spaces. However, the locally convex spaces we consider here are highly structured, and so many of the generally inequivalent definitions for measurability and integrability agree for our spaces.

Let us give the definitions for measurability, integrability, and boundedness we shall use for an arbitrary locally convex space V.

1. A curve $\gamma\colon \mathbb{T} \to \mathsf{V}$ is ***measurable*** if the preimage of every open set is Lebesgue measurable.
2. The notion of integral we use is known as the ***Bochner integral***. It permits a construction highly reminiscent of that of the Lebesgue integral. This is well understood for Banach spaces [Diestel and Uhl, Jr. 1977] and is often mentioned in an offhand manner as being "the same" for locally convex spaces [e.g., Schaefer and Wolff 1999, page 96]. A detailed textbook treatment does not appear to exist, but fortunately this has been worked out in the note of Beckmann and Deitmar [Beckmann and Deitmar 2011], to which we shall refer for details as needed. One has a notion of simple functions, meaning functions that are finite linear combinations, with coefficients in V, of characteristic functions of measurable sets. The ***integral*** of a simple function $\sigma = \sum_{j=1}^k v_j \chi_{A_j}$ is

$$\int_{\mathbb{T}} \sigma \,\mathrm{d}\mu = \sum_{j=1}^k \mu(A_j) v_j,$$

in the usual manner. A measurable curve γ is ***Bochner approximable*** if it can be approximated with respect to any continuous seminorm by a net of simple functions.

A Bochner approximable function γ is ***Bochner integrable*** if there is a net of simple functions approximating γ whose integrals converge in V to a unique value, which is called the ***integral*** of γ. If V is separable and complete, as will be the case for us in this paper, then a measurable curve $\gamma\colon \mathbb{T} \to \mathsf{V}$ is Bochner integrable if and only if

$$\int_{\mathbb{T}} p \circ \gamma \, \mathrm{d}\mu < \infty$$

for every continuous seminorm p on V [Beckmann and Deitmar 2011, Theorems 3.2 and 3.3]. The curve γ is ***locally Bochner integrable*** if $\gamma|\mathbb{T}'$ is Bochner integrable for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$. If $A \subseteq \mathsf{V}$, by $\mathrm{L}^1(\mathbb{T}; A)$ we denote the A-valued Bochner integrable mappings and by $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; A)$ we denote the A-valued locally Bochner integrable mappings.

3. A subset $\mathcal{B} \subseteq \mathsf{V}$ is ***von Neumann bounded*** if $p|\mathcal{B}$ is bounded for every continuous seminorm p for V. The curve $\gamma\colon \mathbb{T} \to \mathsf{V}$ is ***essentially von Neumann bounded*** if there exists a bounded set $\mathcal{B} \subseteq \mathsf{V}$ such that $\gamma(t) \in \mathcal{B}$ for almost every $t \in \mathbb{T}$, and is ***locally essentially von Neumann bounded*** if $\gamma|\mathbb{T}'$ is essentially von Neumann bounded for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$. We note that, if V is a normed vector space, then von Neumann bounded is not the same as norm bounded.[2]

With these definitions, we now have the following result.

3.9 Theorem: (Topological characterisations of time-varying vector fields) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. For a C^r-vector bundle and for a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ satisfying $\xi(t, x) \in \mathsf{E}_x$ for each $(t, x) \in \mathbb{T} \times \mathsf{M}$, the following two statements are equivalent:*

(i) $\xi \in \mathrm{CF}\Gamma^\nu(\mathbb{T}; \mathsf{E})$;

(ii) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\nu(\mathsf{E})$ *is measurable,*

the following two statements are equivalent:

(iii) $\xi \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{E})$;

(iv) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\nu(\mathsf{E})$ *is measurable and locally Bochner integrable,*

and the following two statements are equivalent:

(v) $\xi \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; \mathsf{E})$;

(vi) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\nu(\mathsf{E})$ *is measurable and locally essentially von Neumann bounded.*

Proof: We refer to Theorems 6.3, 6.9, and 6.21 of [Jafarpour and Lewis 2014]. ■

Next we state our result concerning regularity of flows of time-varying vector fields. In the statement of the result, we use the notation

$$|a, b| = \begin{cases} [a, b], & a \leq b, \\ [b, a], & b < a. \end{cases}$$

[2]There is a potential confusion about "boundedness" in this paper. In Section 1.2 we have defined a notion of "essentially bounded" that is different, in general, from the notion of "essentially von Neumann bounded" that we use here. We will not quite encroach on areas where this confusion causes problems, but it is something to bear in mind. Jafarpour and Lewis [2014] are a little more careful about this, explicitly making use of "bornologies."

In the following result, we do not provide the comprehensive list of properties of the flow, but only those required to make sense of its regularity with respect to initial conditions.

3.10 Theorem: (Flows of time-varying vector fields) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold, let $\mathbb{T}$ be an interval, and let $X \in \mathrm{LI}\Gamma^{\nu+\mathrm{lip}}(\mathbb{T}; \mathsf{TM})$. Then there exist a subset $D_X \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ and a map $\Phi^X : D_X \to \mathsf{M}$ with the following properties for each $(t_0, x_0) \in \mathbb{T} \times \mathsf{M}$:*

(i) the set

$$\mathbb{T}_X(t_0, x_0) = \{t \in \mathbb{T} \mid (t, t_0, x_0) \in D_X\}$$

is an interval;

(ii) there exists an absolutely continuous curve $t \mapsto \xi(t)$ satisfying

$$\xi'(t) = X(t, \xi(t)), \quad \xi(t_0) = x_0,$$

for almost all $t \in |t_0, t_1|$ if and only if $t_1 \in \mathbb{T}_X(t_0, x_0)$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t, t_0, x_0) = X(t, \Phi^X(t, t_0, x_0))$ for almost all $t \in \mathbb{T}_X(t_0, x_0)$;

(iv) for each $t \in \mathbb{T}$ for which $(t, t_0, x_0) \in D_X$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the mapping $x \mapsto \Phi^X(t, t_0, x)$ is defined and of class C^ν on $\mathcal{U}$.

Proof: We refer to Theorems 6.6, 6.11, and 6.26 of [Jafarpour and Lewis 2014]. ■

4. Parameterised vector fields

One can think of a control system as a family of vector fields parameterised by control. It is the exact nature of this dependence on the parameter that we discuss in this section. Specifically, we give pointwise characterisations that are equivalent to continuity of the natural map from the parameter space into the space of sections, using the topologies from Section 2.

As we have been doing thus far, we shall consider sections of general vector bundles rather than vector fields to simplify the notation.

4.1. The smooth case.

We begin by discussing parameter dependent smooth sections. Throughout this section we will work with a smooth vector bundle $\pi : \mathsf{E} \to \mathsf{M}$ with a linear connection ∇^0 on E, an affine connection ∇ on M, a fibre metric $\mathbb{G}_0$ on E, and a Riemannian metric $\mathbb{G}$ on M. These define the fibre metrics $\|\cdot\|_{\overline{\mathbb{G}}_m}$ and the seminorms $p^\infty_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, on $\Gamma^\infty(\mathsf{E})$ as in Sections 2.1 and 2.2.

4.1 Definition: (Sections of parameterised class $\mathbf{C}^\infty$) Let $\pi : \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi : \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(x, p) \in \mathsf{E}_x$ for every $(x, p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class $\mathbf{C}^\infty$*** if

(a) for each $x \in \mathsf{M}$, the map $\xi_x : \mathcal{P} \to \mathsf{E}$ defined by $\xi_x(p) = \xi(x, p)$ is continuous and

(b) for each $p \in \mathcal{P}$, the map $\xi^p : \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(x) = \xi(x, p)$ is of class C^∞,

and

(ii) is a ***jointly parameterised section of class*** $\mathbf{C}^\infty$ if it is a separately parameterised section of class C^∞ and if the map $(x,p) \mapsto j_m\xi^p(x)$ is continuous for every $m \in \mathbb{Z}_{\geq 0}$.

By $\mathrm{SP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ we denote the set of separately parameterised sections of E of class C^∞ and by $\mathrm{JP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ we denote the set of jointly parameterised sections of E of class C^∞. •

It is possible to give purely topological characterisations of this class of sections.

4.2 Proposition: (Characterisation of jointly parameterised sections of class $\mathbf{C}^\infty$) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi\colon \mathsf{M}\times\mathcal{P} \to \mathsf{E}$ satisfy $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M}\times\mathcal{P}$. Then $\xi \in \mathrm{JP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ if and only if the map $p \mapsto \xi^p \in \Gamma^\infty(\mathsf{E})$ is continuous, where $\Gamma^\infty(\mathsf{E})$ has the C^∞-topology.*

Proof: Given $\xi\colon \mathsf{M}\times\mathcal{P} \to \mathsf{E}$ we let $\xi_m\colon \mathsf{M}\times\mathcal{P} \to \mathsf{J}^m\mathsf{E}$ be the map $\xi_m(x,p) = j_m\xi^p(x)$. We also denote by $\sigma_\xi\colon \mathcal{P} \to \Gamma^\infty(\mathsf{E})$ the map given by $\sigma_\xi(p) = \xi^p$.

First suppose that ξ_m is continuous for every $m \in \mathbb{Z}_{\geq 0}$. Let $K \subseteq \mathsf{M}$ be compact, let $m \in \mathbb{Z}_{\geq 0}$, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $x \in K$ and let $\mathcal{W}_x$ be a neighbourhood of $\xi_m(x,p_0)$ in $\mathsf{J}^m\mathsf{E}$ for which

$$\mathcal{W}_x \subseteq \{j_m\eta(x') \in \mathsf{J}^m\mathsf{E} \mid \ \|j_m\eta(x') - \xi_m(x',p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon\}.$$

By continuity of ξ_m, there exist a neighbourhood $\mathcal{U}_x \subseteq \mathsf{M}$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{P}$ of p_0 such that $\xi_m(\mathcal{U}_x \times \mathcal{O}_x) \subseteq \mathcal{W}_x$. Now let $x_1,\dots,x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then, if $p \in \mathcal{O}$ and $x \in K$, we have $x \in \mathcal{U}_{x_j}$ for some $j \in \{1,\dots,k\}$. Thus $\xi_m(x,p) \in \mathcal{W}_{x_j}$. Thus

$$\|\xi_m(x,p) - \xi_m(x,p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon.$$

Therefore, taking supremums over $x \in K$, $p^\infty_{K,m}(\sigma_\xi(p) - \sigma_\xi(p_0)) \leq \epsilon$. As this can be done for every compact $K \subseteq \mathsf{M}$ and every $m \in \mathbb{Z}_{\geq 0}$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous and let $m \in \mathbb{Z}_{\geq 0}$. Let $(x_0,p_0) \in \mathsf{M}\times\mathcal{P}$ and let $\mathcal{W} \subseteq \mathsf{J}^m\mathsf{E}$ be a neighbourhood of $\xi_m(x_0,p_0)$. Let $\mathcal{U} \subseteq \mathsf{M}$ be a relatively compact neighbourhood of x_0 and let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\pi_m^{-1}(\mathcal{U}) \cap \{j_m\eta(x) \in \mathsf{J}^m\mathsf{E} \mid \ \|j_m\eta(x) - \xi_m(x,p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon\} \subseteq \mathcal{W},$$

where $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ is the projection. By continuity of σ_ξ, let $\mathcal{O} \subseteq \mathcal{P}$ be a neighbourhood of p_0 such that $p^\infty_{\mathrm{cl}(\mathcal{U}),m}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon$ for $p \in \mathcal{O}$. Therefore,

$$\|j_m\sigma_\xi(p)(x) - j_m\sigma_\xi(p_0)(x)\|_{\overline{\mathbb{G}}_m} < \epsilon, \qquad (x,p) \in \mathrm{cl}(\mathcal{U}) \times \mathcal{O}.$$

Therefore, if $(x,p) \in \mathcal{U}\times\mathcal{O}$, then $\pi_m(\xi_m(x,p)) = x \in \mathcal{U}$ and so $\xi_m(x,p) \in \mathcal{W}$, showing that ξ_m is continuous at (x_0,p_0). ■

4.2. The finitely differentiable or Lipschitz case. The preceding development in the smooth case is easily extended to the finitely differentiable and Lipschitz cases, and we quickly give the results and definitions here. In this section, when considering the Lipschitz case, we assume that ∇ is the Levi-Civita connection associated to $\mathbb{G}$ and we assume that ∇^0 is $\mathbb{G}_0$-orthogonal.

4.3 Definition: (Sections of parameterised class $\mathrm{C}^{m+m'}$) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class*** $\mathbf{C}^{m+m'}$ if

  (a) for each $x \in \mathsf{M}$, the map $\xi_x\colon \mathcal{P} \to \mathsf{E}$ defined by $\xi_x(p) = \xi(x,p)$ is continuous and

  (b) for each $p \in \mathcal{P}$, the map $\xi^p\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(x) = \xi(x,p)$ is of class $\mathrm{C}^{m+m'}$,

and

(ii) is a ***jointly parameterised section of class*** $\mathbf{C}^{m+m'}$ if it is a separately parameterised section of class $\mathrm{C}^{m+m'}$ and

  (a) $m' = 0$: the map $(x,p) \mapsto j_m\xi^p(x)$ is continuous;

  (b) $m' = \mathrm{lip}$: the map $(x,p) \mapsto j_m\xi^p(x)$ is continuous and, for each $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$ and each $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$j_m\xi(\mathcal{U} \times \mathcal{O}) \subseteq \{j_m\eta(x) \in \mathsf{J}^m\mathsf{E} \mid \operatorname{dil}(j_m\eta - j_m\xi^{p_0})(x) < \epsilon\},$$

  where, of course, $j_m\xi(x,p) = j_m\xi^p(x)$.

By $\mathrm{SP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$ we denote the set of separately parameterised sections of E of class $\mathrm{C}^{m+m'}$ and by $\mathrm{JP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$ we denote the set of jointly parameterised sections of E of class $\mathrm{C}^{m+m'}$. •

Let us give the purely topological characterisation of this class of sections.

4.4 Proposition: (Characterisation of jointly parameterised sections of class $\mathbf{C}^{m+m'}$) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ satisfy $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$. Then $\xi \in \mathrm{JP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$ if and only if the map $p \mapsto \xi^p \in \Gamma^{m+m'}(\mathsf{E})$ is continuous, where $\Gamma^{m+m'}(\mathsf{E})$ has the $\mathrm{C}^{m+m'}$-topology.*

Proof: We will prove the result only in the case that $m = 0$ and $m' = \mathrm{lip}$, as the general case follows by combining this case with the computations from the proof of Proposition 4.2. We denote $\sigma_\xi(p) = \xi(x,p)$.

Suppose that $(x,p) \mapsto \xi(x,p)$ is continuous and that, for every $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$ and for every $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that, if $(x,p) \in \mathcal{U} \times \mathcal{O}$, then $\operatorname{dil}(\xi^p - \xi^{p_0})(x) < \epsilon$. Let $K \subseteq \mathsf{M}$ be compact, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $x \in K$. By hypothesis, there exist a neighbourhood $\mathcal{U}_x \subseteq \mathsf{M}$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{P}$ of p_0 such that

$$\xi(\mathcal{U}_x \times \mathcal{O}_x) \subseteq \{\eta(x') \in \mathsf{J}^m\mathsf{E} \mid \operatorname{dil}(\eta - \xi^{p_0})(x') < \epsilon\}.$$

Now let $x_1,\dots,x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then, if $p \in \mathcal{O}$ and $x \in K$, we have $x \in \mathcal{U}_{x_j}$ for some $j \in \{1,\dots,k\}$. Thus

$$\operatorname{dil}(\xi(x,p) - \xi(x,p_0))_{\overline{\mathbb{G}}_m} < \epsilon.$$

Therefore, taking supremums over $x \in K$, we have $\lambda_K(\sigma_\xi(p) - \sigma_\xi(p_0)) \leq \epsilon$. By choosing $\mathcal{O}$ to be possibly smaller, the argument of Proposition 4.2 ensures that $p_K^0(\sigma_\xi(p) - \sigma_\xi(p_0)) \leq \epsilon$, and so $p^{\mathrm{lip}}_K(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon$ for $p \in \mathcal{O}$. As this can be done for every compact $K \subseteq \mathsf{M}$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous, let $(x_0, p_0) \in \mathsf{M} \times \mathcal{P}$, and let $\epsilon \in \mathbb{R}_{>0}$. Let $\mathcal{U}$ be a relatively compact neighbourhood of x_0. Since σ_ξ is continuous, let $\mathcal{O}$ be a neighbourhood of p_0 such that

$$p^{\mathrm{lip}}_{\mathrm{cl}(\mathcal{U})}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon, \qquad p \in \mathcal{O}.$$

Thus, for every $(x, p) \in \mathcal{U} \times \mathcal{O}$, $\mathrm{dil}\,(\xi^p - \xi^{p_0})(x) < \epsilon$. Following the argument of Proposition 4.2 one also shows that ξ is continuous at (x_0, p_0), which shows that $\xi \in \mathrm{JP}\Gamma^{\mathrm{lip}}(\mathcal{P}; \mathsf{E})$. ■

4.3. The holomorphic case. As with time-varying vector fields, we are not really interested, *per se*, in holomorphic control systems, and in fact we will not even define the notion. However, it is possible, and possibly sometimes easier, to verify that a control system satisfies our rather technical criterion of being a "real analytic control system" by verifying that it possesses an holomorphic extension. Thus, in this section, we present the required holomorphic definitions. We will consider an holomorphic vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$ with an Hermitian fibre metric $\mathbb{G}$. This defines the seminorms p^{hol}_K, $K \subseteq \mathsf{M}$ compact, describing the $\mathrm{C}^{\mathrm{hol}}$-topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ as in Section 2.5.

4.5 Definition: (Sections of parameterised class $\mathbf{C}^{\mathbf{hol}}$) Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi \colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(z, p) \in \mathsf{E}_z$ for every $(z, p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class*** $\mathbf{C}^{\mathbf{hol}}$ if

  (a) for each $z \in \mathsf{M}$, the map $\xi_z \colon \mathcal{P} \to \mathsf{E}$ defined by $\xi_z(p) = \xi(z, p)$ is continuous and

  (b) for each $p \in \mathcal{P}$, the map $\xi^p \colon \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(z) = \xi(z, p)$ is of class $\mathrm{C}^{\mathrm{hol}}$,

and

(ii) is a ***jointly parameterised section of class*** $\mathbf{C}^{\mathbf{hol}}$ if it is a separately parameterised section of class $\mathrm{C}^{\mathrm{hol}}$ and if the map $(z, p) \mapsto \xi^p(z)$ is continuous.

By $\mathrm{SP}\Gamma^{\mathrm{hol}}(\mathcal{P}; \mathsf{E})$ we denote the set of separately parameterised sections of E of class $\mathrm{C}^{\mathrm{hol}}$ and by $\mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P}; \mathsf{E})$ we denote the set of jointly parameterised sections of E of class $\mathrm{C}^{\mathrm{hol}}$. •

As in the smooth case, it is possible to give purely topological characterisations of these classes of sections.

4.6 Proposition: (Characterisation of jointly parameterised sections of class $\mathbf{C}^{\mathbf{hol}}$) *Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi \colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ satisfy $\xi(z, p) \in \mathsf{E}_z$ for every $(z, p) \in \mathsf{M} \times \mathcal{P}$. Then $\xi \in \mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P}; \mathsf{E})$ if and only if the map $p \mapsto \xi^p \in \Gamma^{\mathrm{hol}}(\mathsf{E})$ is continuous, where $\Gamma^{\mathrm{hol}}(\mathsf{E})$ has the $\mathrm{C}^{\mathrm{hol}}$-topology.*

Proof: We define $\sigma_\xi \colon \mathcal{P} \to \Gamma^{\mathrm{hol}}(\mathsf{E})$ by $\sigma_\xi(p) = \xi^p$.

First suppose that ξ is continuous. Let $K \subseteq \mathsf{M}$ be compact, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $z \in K$ and let $\mathcal{W}_z \subseteq \mathsf{E}$ be a neighbourhood of $\xi(z, p_0)$ for which

$$\mathcal{W}_z \subseteq \{\eta(z') \in \mathsf{E} \mid \ \|\eta(z') - \xi(z', p_0)\|_{\overline{\mathbb{G}}} < \epsilon\}.$$

By continuity of ξ, there exist a neighbourhood $\mathcal{U}_z \subseteq \mathsf{M}$ of z and a neighbourhood $\mathcal{O}_z \subseteq \mathcal{P}$ of p_0 such that $\xi(\mathcal{U}_z \times \mathcal{O}_z) \subseteq \mathcal{W}_z$. Now let $z_1, \dots, z_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{z_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{z_j}$. Then, if $p \in \mathcal{O}$ and $z \in K$, we have $z \in \mathcal{U}_{z_j}$ for some $j \in \{1, \dots, k\}$. Thus $\xi(z,p) \in \mathcal{W}_{z_j}$. Thus $\|\xi(z,p) - \xi(z,p_0)\|_{\overline{\mathbb{G}}} < \epsilon$. Therefore, taking supremums over $z \in K$, $p_K^{\mathrm{hol}}(\sigma_\xi(p) - \sigma_\xi(p_0)) \le \epsilon$. As this can be done for every compact $K \subseteq \mathsf{M}$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous. Let $(z_0, p_0) \in \mathsf{M} \times \mathcal{P}$ and let $\mathcal{W} \subseteq \mathsf{E}$ be a neighbourhood of $\xi(z_0, p_0)$. Let $\mathcal{U} \subseteq \mathsf{M}$ be a relatively compact neighbourhood of z_0 and let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\pi^{-1}(\mathcal{U}) \cap \{\eta(z) \in \mathsf{E} \mid \ \|\eta(z) - \xi(z,p_0)\|_{\overline{\mathbb{G}}} < \epsilon\} \subseteq \mathcal{W}.$$

By continuity of σ_ξ, let $\mathcal{O} \subseteq \mathcal{P}$ be a neighbourhood of p_0 such that $p_{\mathrm{cl}(\mathcal{U})}^{\mathrm{hol}}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon$ for $p \in \mathcal{O}$. Therefore,

$$\|\sigma_\xi(p)(z) - \sigma_\xi(p_0)(z)\|_{\overline{\mathbb{G}}} < \epsilon, \qquad (z,p) \in \mathrm{cl}(\mathcal{U}) \times \mathcal{O}.$$

Therefore, if $(z,p) \in \mathcal{U} \times \mathcal{O}$, we have $\xi(z,p) \in \mathcal{W}$, showing that ξ is continuous at (z_0, p_0). ■

4.4. The real analytic case. Now we repeat the procedure above for real analytic sections. We thus will consider a real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with ∇^0 a real analytic linear connection on E, ∇ a real analytic affine connection on M, $\mathbb{G}_0$ a real analytic fibre metric on E, and $\mathbb{G}$ a real analytic Riemannian metric on M. This defines the seminorms $p_{K,\boldsymbol{a}}^{\omega}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0}; \mathbb{R}_{>0})$, describing the C^ω-topology as in Section 2.6.

4.7 Definition: (Sections of parameterised class C^ω) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class C^ω*** if

  (a) for each $x \in \mathsf{M}$, the map $\xi_x\colon \mathcal{P} \to \mathsf{E}$ defined by $\xi_x(p) = \xi(x,p)$ is continuous and

  (b) for each $p \in \mathcal{P}$, the map $\xi^p\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(x) = \xi(x,p)$ is of class C^ω,

and

(ii) is a ***jointly parameterised section of class C^ω*** if it is a separately parameterised section of class C^∞ and if, for each $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$, for each $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0}, \mathbb{R}_{>0})$, and for each $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$j_m\xi(\mathcal{U} \times \mathcal{O}) \subseteq \{j_m\eta(x) \in \mathsf{J}^m\mathsf{E} \mid \ a_0 a_1 \cdots a_m \|j_m\eta(x) - j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m} < \epsilon\}$$

for every $m \in \mathbb{Z}_{\ge 0}$, where, of course, $j_m\xi(x,p) = j_m\xi^p(x)$.

By $\mathrm{SP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ we denote the set of separately parameterised sections of E of class C^ω and by $\mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ we denote the set of jointly parameterised sections of E of class C^ω. •

4.8 Remark: (Jointly parameterised sections of class C^ω) The condition that $\xi \in \mathrm{JP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ can be restated like this: for each $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$, for each $m \in \mathbb{Z}_{\ge 0}$, and for each $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$j_m\xi(\mathcal{U} \times \mathcal{O}) \subseteq \{j_m\eta(x) \in \mathsf{J}^m\mathsf{E} \mid \ \|j_m\eta(x) - j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m} < \epsilon\};$$

that this is so is, more or less, the idea of the proof of Proposition 4.2. Phrased this way, one sees clearly the grammatical similarity between the smooth and real analytic definitions. Indeed, the grammatical transformation from the smooth to the real analytic definition is, *put a factor of* $a_0a_1\cdots a_m$ *before the norm, precede the condition with "for every* $\boldsymbol{a}\in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$*", and move the "for every* $m\in\mathbb{Z}_{\geq 0}$*" from before the condition to after.* This was also seen in the definitions of locally integrally bounded and locally essentially bounded sections in Section 3. Indeed, the grammatical similarity is often encountered when using our locally convex topologies, and contributes to the unification of the analysis of the varying degrees of regularity. •

The following result records topological characterisations of jointly parameterised sections in the real analytic case.

4.9 Proposition: (Characterisation of jointly parameterised sections of class C^ω) *Let* $\pi\colon \mathsf{E}\to\mathsf{M}$ *be a real analytic vector bundle, let* $\mathcal{P}$ *be a topological space, and let* $\xi\colon\mathsf{M}\times\mathcal{P}\to\mathsf{E}$ *satisfy* $\xi(x,p)\in\mathsf{E}_x$ *for every* $(x,p)\in\mathsf{M}\times\mathcal{P}$*. Then* $\xi\in\mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ *if and only if the map* $p\mapsto\xi^p\in\Gamma^\omega(\mathsf{E})$ *is continuous, where* $\Gamma^\omega(\mathsf{E})$ *has the* C^ω*-topology.*

Proof: For $\boldsymbol{a}\in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ and $m\in\mathbb{Z}_{\geq 0}$, given $\xi\colon\mathsf{M}\times\mathcal{P}\to\mathsf{E}$ satisfying $\xi^p\in\Gamma^\omega(\mathsf{E})$, we let $\xi_{\boldsymbol{a},m}\colon\mathsf{M}\times\mathcal{P}\to\mathsf{J}^m\mathsf{E}$ be the map

$$\xi_{\boldsymbol{a},m}(x,p)=a_0a_1\cdots a_m j_m\xi^p(x).$$

We also denote by $\sigma_\xi\colon\mathcal{P}\to\Gamma^\omega(\mathsf{E})$ the map given by $\sigma_\xi(p)=\xi^p$.

Suppose that, for every $(x_0,p_0)\in\mathsf{M}\times\mathcal{P}$, for every $\boldsymbol{a}\in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, and for every $\epsilon\in\mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U}\subseteq\mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O}\subseteq\mathcal{P}$ of p_0 such that, if $(x,p)\in\mathcal{U}\times\mathcal{O}$, then

$$\|\xi_{\boldsymbol{a},m}(x,p)-\xi_{\boldsymbol{a},m}(x,p_0)\|_{\overline{\mathbb{G}}_m}<\epsilon,\qquad m\in\mathbb{Z}_{\geq 0}.$$

Let $K\subseteq\mathsf{M}$ be compact, let $\boldsymbol{a}\in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, let $\epsilon\in\mathbb{R}_{>0}$, and let $p_0\in\mathcal{P}$. Let $x\in K$. By hypothesis, there exist a neighbourhood $\mathcal{U}_x\subseteq\mathsf{M}$ of x and a neighbourhood $\mathcal{O}_x\subseteq\mathcal{P}$ of p_0 such that

$$\xi_{\boldsymbol{a},m}(\mathcal{U}_x\times\mathcal{O}_x)\subseteq\{j_m\eta(x')\in\mathsf{J}^m\mathsf{E}\mid \ \|a_0a_1\cdots a_m j_m\eta(x')-\xi_{\boldsymbol{a},m}(x',p_0)\|_{\overline{\mathbb{G}}_m}<\epsilon\},$$

for each $m\in\mathbb{Z}_{\geq 0}$. Now let $x_1,\dots,x_k\in K$ be such that $K\subseteq\cup_{j=1}^k\mathcal{U}_{x_j}$ and let $\mathcal{O}=\cap_{j=1}^k\mathcal{O}_{x_j}$. Then, if $p\in\mathcal{O}$ and $x\in K$, we have $x\in\mathcal{U}_{x_j}$ for some $j\in\{1,\dots,k\}$. Thus

$$\|\xi_{\boldsymbol{a},m}(x,p)-\xi_{\boldsymbol{a},m}(x,p_0)\|_{\overline{\mathbb{G}}_m}<\epsilon,\qquad m\in\mathbb{Z}_{\geq 0}.$$

Therefore, taking supremums over $x\in K$ and $m\in\mathbb{Z}_{\geq 0}$, we have $p^\omega_{K,\boldsymbol{a}}(\sigma_\xi(p)-\sigma_\xi(p_0))\leq\epsilon$. As this can be done for every compact $K\subseteq\mathsf{M}$ and every $\boldsymbol{a}\in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous, let $(x_0,p_0)\in\mathsf{M}\times\mathcal{P}$, let $\boldsymbol{a}\in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, and let $\epsilon\in\mathbb{R}_{>0}$. Let $\mathcal{U}$ be a relatively compact neighbourhood of x_0. Since σ_ξ is continuous, let $\mathcal{O}$ be a neighbourhood of p_0 such that

$$p^\omega_{\mathrm{cl}(\mathcal{U}),\boldsymbol{a}}(\sigma_\xi(p)-\sigma_\xi(p_0))<\epsilon,\qquad p\in\mathcal{O}.$$

Thus, for every $(x,p)\in\mathcal{U}\times\mathcal{O}$,

$$a_0a_1\cdots a_m\|j_m\xi(x,p)-j_m\xi(x,p_0)\|_{\overline{\mathbb{G}}_m}<\epsilon,\qquad m\in\mathbb{Z}_{\geq 0},$$

which shows that $\xi\in\mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$. ■

One can wonder about the relationship between sections of jointly parameterised class C^{ω} and sections that are real restrictions of sections of jointly parameterised class $\mathrm{C}^{\mathrm{hol}}$. We address this with a result and an example. First the result. The result here is a nontrivial one, and is the only place in this paper where we call upon the deeper properties of the real analytic topology. A reader wishing to comprehend all of the details of the proof will probably need to consult [Jafarpour and Lewis 2014, §5].

4.10 Theorem: (Jointly parameterised real analytic sections as restrictions of jointly parameterised holomorphic sections) *Let $\pi\colon \mathsf{E}\to\mathsf{M}$ be a real analytic vector bundle with complexification*[3] *$\overline{\pi}\colon \overline{\mathsf{E}}\to\overline{\mathsf{M}}$ and let $\mathcal{P}$ be a topological space. For a map $\xi\colon \mathsf{M}\times\mathcal{P}\to\mathsf{E}$ satisfying $\xi(x,p)\in\mathsf{E}_x$ for all $(x,p)\in\mathsf{M}\times\mathcal{P}$, the following statements hold:*

(i) if $\xi\in\mathrm{JP}\Gamma^{\omega}(\mathcal{P};\mathsf{E})$ and if $\mathcal{P}$ is locally compact and Hausdorff, then, for each $(x_0,p_0)\in\mathsf{M}\times\mathcal{P}$, there exist a neighbourhood $\overline{\mathcal{U}}\subseteq\overline{\mathsf{M}}$ of x_0, a neighbourhood $\mathcal{O}\subseteq\mathcal{P}$ of p_0, and $\overline{\xi}\in\mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{O};\overline{\mathsf{E}}|\overline{\mathcal{U}})$ such that $\xi(x,p)=\overline{\xi}(x,p)$ for all $(x,p)\in(\mathsf{M}\cap\overline{\mathcal{U}})\times\mathcal{O}$;

(ii) if there exists a section $\overline{\xi}\in\mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P};\overline{\mathsf{E}})$ such that $\xi(x,p)=\overline{\xi}(x,p)$ for every $(x,p)\in\mathsf{M}\times\mathcal{P}$, then $\xi\in\mathrm{JP}\Gamma^{\omega}(\mathcal{P};\mathsf{E})$.

Proof: (i) Let $p_0\in\mathcal{P}$ and let $\mathcal{O}$ be a relatively compact neighbourhood of p_0, this being possible since $\mathcal{P}$ is locally compact. Let $x_0\in\mathsf{M}$, let $\mathcal{U}$ be a relatively compact neighbourhood of x_0, and let $(\overline{\mathcal{U}}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of neighbourhoods of $\mathrm{cl}(\mathcal{U})$ in $\overline{\mathsf{M}}$ with the properties that $\mathrm{cl}(\overline{\mathcal{U}}_{j+1})\subseteq\overline{\mathcal{U}}_j$ and that $\cap_{j\in\mathbb{Z}_{>0}}\overline{\mathcal{U}}_j=\mathrm{cl}(\mathcal{U})$. Let $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ be the set of germs of those holomorphic sections of $\overline{\mathsf{E}}$ about $\mathrm{cl}(\mathcal{O})$ that, when restricted to M, are real. We recall from [Jafarpour and Lewis 2014, §5.2] that $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is the direct limit of the directed system $(\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))_{j\in\mathbb{Z}_{>0}}$. For $\overline{\xi}\in\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ for some $j\in\mathbb{Z}_{>0}$, let $[\overline{\xi}]_{\mathrm{cl}(\mathcal{U})}$ be the germ of $\overline{\xi}$. We note that

$$\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}})\simeq\mathrm{C}^0(\mathrm{cl}(\mathcal{O}))\check{\otimes}_e\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$$

and

$$\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))\simeq\mathrm{C}^0(\mathrm{cl}(\mathcal{O}))\check{\otimes}_e\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j),$$

with $\check{\otimes}_e$ denoting the completed injective tensor product; see [Jarchow 1981, Chapter 16] for the injective tensor product for locally convex spaces and [Diestel, Fourie, and Swart 2008, Theorem 1.1.10] for the preceding isomorphisms for Banach spaces (the constructions apply more or less verbatim to locally convex spaces [Bierstedt 2007, Proposition 5.4]). Note that since $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ and $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$, $j\in\mathbb{Z}_{>0}$, are nuclear, the injective tensor product can be swapped with the projective tensor product in the above constructions [Pietsch 1969, Proposition 5.4.2].

We claim that, with these identifications, $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}})$ is the direct limit of the directed system $(\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)))_{j\in\mathbb{Z}_{>0}}$ with the associated mappings $\mathrm{id}\,\widehat{\otimes}_{\pi}r_{\mathrm{cl}(\mathcal{U}),j}$, $j\in\mathbb{Z}_{>0}$, where

$$\begin{aligned}r_{\mathrm{cl}(\mathcal{O}),j}\colon\ \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{\mathrm{cl}(\mathcal{O}),j})&\to\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{O}),\overline{\mathsf{E}}}\\ \overline{\xi}&\mapsto[\overline{\xi}]_{\mathrm{cl}(\mathcal{O})}.\end{aligned}$$

We, moreover, claim that the direct limit topology is boundedly retractive, meaning that bounded sets in the direct limit are contained in and bounded in a single component of the

[3]Such complexifications exist, as shown in [Jafarpour and Lewis 2014, §5.1.1]

directed system and, moreover, the topology on the bounded set induced by the component is the same as that induced by the direct limit.

Results of this sort have been the subject of research in the area of locally convex topologies, with the aim being to deduce conditions on the structure of the spaces comprising the directed system, and on the corresponding mappings (for us, the inclusion mappings and their tensor products with the identity on $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R})$), that ensure that direct limits commute with tensor product, and that the associated direct limit topology is boundedly retractive. We shall make principal use of the results given by Mangino [1997]. To state the arguments with at least a little context, let us reproduce two conditions used by Mangino.

Condition (M) of Retakh [1970]: Let $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ be a directed system of locally convex spaces with strict direct limit V. The direct limit topology of V satisfies ***condition (M)*** if there exists a sequence $(\mathcal{O}_j)_{j\in\mathbb{Z}_{>0}}$ for which

(i) $\mathcal{O}_j$ is a balanced convex neighbourhood of $0 \in \mathsf{V}_j$,

(ii) $\mathcal{O}_j \subseteq \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}_{>0}$, and

(iii) for every $j \in \mathbb{Z}_{>0}$, there exists $k \geq j$ such that the topology induced on $\mathcal{O}_j$ by its inclusion in V_k and its inclusion in V agree. •

Condition (MO) of Mangino [1997]: Let $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ be a directed system of metrisable locally convex spaces with strict direct limit V. Let $i_{j,k}\colon \mathsf{V}_j \to \mathsf{V}_k$ be the inclusion for $k \geq j$ and let $i_j\colon \mathsf{V}_j \to \mathsf{V}$ be the induced map into the direct limit.

Suppose that, for each $j \in \mathbb{Z}_{>0}$, we have a sequence $(p_{j,l})_{l\in\mathbb{Z}_{>0}}$ of seminorms defining the topology of V_j such that $p_{j,l_1} \geq p_{j,l_2}$ if $l_1 \geq l_2$. Let

$$\mathsf{V}_{j,l} = \mathsf{V}_j/\{v \in \mathsf{V}_j \mid p_{j,l}(v) = 0\}$$

and denote by $\hat{p}_{j,l}$ the norm on $\mathsf{V}_{j,l}$ induced by $p_{j,l}$ [Schaefer and Wolff 1999, page 97]. Let $\pi_{j,l}\colon \mathsf{V}_j \to \mathsf{V}_{j,l}$ be the canonical projection. Let $\overline{\mathsf{V}}_{j,l}$ be the completion of $\mathsf{V}_{j,l}$. The family $(\overline{\mathsf{V}}_{j,l})_{j,l\in\mathbb{Z}_{>0}}$ is called a ***projective spectrum*** for V_j. Denote

$$\mathcal{O}_{j,l} = \{v \in \mathsf{V}_j \mid p_{j,l}(v) \leq 1\}.$$

The direct limit topology of V satisfies ***condition (MO)*** if there exists a sequence $(\mathcal{O}_j)_{j\in\mathbb{Z}_{>0}}$ and if, for every $j \in \mathbb{Z}_{>0}$, there exists a projective spectrum $(\overline{\mathsf{V}}_{j,l})_{j,l\in\mathbb{Z}_{>0}}$ for V_j for which

(i) $\mathcal{O}_j$ is a balanced convex neighbourhood of $0 \in \mathsf{V}_j$,

(ii) $\mathcal{O}_j \subseteq \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}_{>0}$, and

(iii) for every $j \in \mathbb{Z}_{>0}$, there exists $k \geq j$ such that, for every $l \in \mathbb{Z}_{>0}$, there exists $A \in \mathrm{L}(\mathsf{V};\overline{\mathsf{V}}_{k,l})$ satisfying

$$(\pi_{k,l} \circ i_{jk} - A \circ i_j)(\mathcal{O}_j) \subseteq \mathrm{cl}(\pi_{k,l}(\mathsf{O}_{k,l})),$$

the closure on the right being taken in the norm topology of $\overline{\mathsf{V}}_{k,l}$. •

With these concepts, we have the following statements. We let $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ be a directed system of metrisable locally convex spaces with strict direct limit V.

1. If the direct limit topology on V satisfies condition (MO), then, for any Banach space U, $\mathsf{U} \otimes_\pi \mathsf{V}$ ($\otimes_\pi$ is the uncompleted projective tensor product) is the direct limit of the directed system $(\mathsf{U} \otimes_\pi \mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$, and the direct limit topology on $\mathsf{U} \otimes_\pi \mathsf{V}$ satisfies condition (M) [Mangino 1997, Theorem 1.3].
2. If the inclusion of V_j in V_{j+1} is nuclear and if the direct limit topology on V is regular, then the direct limit topology on V satisfies condition (MO) [Mangino 1997, Theorem 1.3].
3. If the direct limit topology on V satisfies condition (M), then this direct limit topology is boundedly retractive [Wengenroth 1995].

Using these arguments we make the following conclusions.

4. The direct limit topology on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ satisfies condition (MO) (by virtue of assertion 2 above and by the properties of the direct limit topology enunciated in [Jafarpour and Lewis 2014, §5.3], specifically that the direct limit is a regular direct limit of nuclear Fréchet spaces).
5. The space $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is the direct limit of the directed sequence $(\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes_\pi \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))_{j\in\mathbb{Z}_{>0}}$ (by virtue of assertion 1 above).
6. The direct limit topology on $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ satisfies condition (M) (by virtue of assertion 1 above).
7. The direct limit topology on $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is boundedly retractive (by virtue of assertion 3 above).

We shall also need the following lemma.

1 Lemma: *Let $K \subseteq \mathsf{M}$ be compact. If $[\overline{\xi}]_K \in \mathrm{C}^0(\mathrm{cl}(\mathcal{O}); \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}})$ then there exists a sequence $([\overline{\xi}_k]_K)_{k\in\mathbb{Z}_{>0}}$ in $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ converging to $[\overline{\xi}]_K$ in the topology of $\mathrm{C}^0(\mathrm{cl}(\mathcal{O}); \mathscr{G}^{\mathrm{hol}}_{K,\overline{\mathsf{E}}})$.*

Proof: Since $\mathrm{C}^0(\mathrm{cl}(\mathcal{O}); \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}})$ is the completion of $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, there exists a *net* $([\overline{\xi}_i]_K)_{i\in I}$ converging to $[\overline{\xi}]$, so the conclusion here is that we can actually find a converging *sequence*. This can be proved, however, using the argument from the proof of [Diestel, Fourie, and Swart 2008, Theorem 1.1.10] (see top of page 15 of that reference) ▼

The remainder of the proof is straightforward. Since $\xi \in \mathrm{JP}\Gamma^\omega(\mathbb{T};\mathsf{E})$, the map

$$\mathrm{cl}(\mathcal{O}) \ni p \mapsto \xi^p \in \Gamma^\omega(\mathsf{E})$$

is an element of $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\Gamma^\omega(\mathsf{E}))$ by Theorem 3.9. Therefore, if $[\overline{\xi}]_{\mathrm{cl}(\mathcal{U})}$ is the image of ξ under the natural mapping from $\Gamma^\omega(\mathsf{E})$ to $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$, the map

$$\mathbb{T}' \ni t \mapsto [\overline{\xi(t)}]_{\mathrm{cl}(\mathcal{U})} \in \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$$

is an element of $\mathrm{C}^0(\mathrm{cl}(\mathcal{O}); \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}})$. Therefore, by the preceding lemma, there exists a sequence $([\overline{\xi}_k]_{\mathrm{cl}(\mathcal{U})})_{k\in\mathbb{Z}_{>0}}$ in $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ that converges to $[\overline{\xi}]_{\mathrm{cl}(\mathcal{U})}$. By our conclusion 5 above, the topology in which this convergence takes place is the completion of the direct limit topology associated to the directed system $(\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R}) \otimes_\pi \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))_{j\in\mathbb{Z}_{>0}}$.

The sequence $([\overline{\xi}_k]_{\mathrm{cl}(\mathcal{U})})_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence and so bounded. The direct limit topology on $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathbb{R})\otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is boundedly retractive by our conclusion 7 above. Thus, in particular, it is regular, and so there exists $j\in\mathbb{Z}_{>0}$ such that the sections $(\overline{\xi}_k)_{k\in\mathbb{Z}_{>0}}$ can be holomorphically extended to $\overline{\mathcal{U}}_j$. Bounded retractivity additionally implies that j can be chosen so that the sequence $(\overline{\xi}_k)_{k\in\mathbb{Z}_{>0}}$ is a Cauchy sequence in $\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))$ and so converges to a limit $\overline{\eta}$ satisfying $[\overline{\eta}]_{\mathrm{cl}(\mathcal{U})}=[\overline{\xi}]_{\mathrm{cl}(\mathcal{U})}$. Thus $\overline{\xi}$ can be holomorphically extended to $\overline{\mathcal{U}}_j$. This completes this part of the proof.

(ii) Let $(x_0,p_0)\in\mathsf{M}\times\mathcal{P}$, let $\boldsymbol{a}\in\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, and let $\epsilon\in\mathbb{R}_{>0}$. Let $\mathcal{U}\subseteq\mathsf{M}$ be a relatively compact neighbourhood of x_0 and let $\overline{\mathcal{U}}$ be a relatively compact neighbourhood of $\mathrm{cl}(\mathcal{U})$. By Proposition 2.1, there exist $C,r\in\mathbb{R}_{>0}$ such that

$$p^\infty_{\mathrm{cl}(\mathcal{U}),m}(\sigma_\xi(p)-\sigma_\xi(p_0))\leq Cr^{-m}\sup\{\|\overline{\xi}(z,p)-\overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}}\mid\ z\in\overline{\mathcal{U}}\}$$

for all $m\in\mathbb{Z}_{\geq 0}$ and $p\in\mathcal{P}$. Now let $N\in\mathbb{Z}_{\geq 0}$ be such that $a_{N+1}<r$ and let $\mathcal{O}$ be a neighbourhood of p_0 such that

$$\|\overline{\xi}(z,p)-\overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}}<\frac{\epsilon r^m}{Ca_0a_1\cdots a_m},\qquad m\in\{0,1,\dots,N\},$$

for $(z,p)\in\overline{\mathcal{U}}\times\mathcal{O}$. Then, if $m\in\{0,1,\dots,N\}$, we have

$$\begin{aligned}a_0a_1\cdots a_m&\|j_m\xi^p(x)-j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m}\\&\leq a_0a_1\cdots a_mCr^{-m}\sup\{\|\overline{\xi}(z,p)-\overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}_m}\mid\ z\in\overline{\mathcal{U}}\}<\epsilon,\end{aligned}$$

for $(x,p)\in\mathcal{U}\times\mathcal{O}$. If $m>N$ we also have

$$\begin{aligned}a_0a_1\cdots a_m&\|j_m\xi^p(x)-j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m}\\&\leq a_0a_1\cdots a_Nr^{-N}r^m\|j_m\xi^p(x)-j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m}\\&\leq a_0a_1\cdots a_Nr^{-N}r^mCr^{-m}\sup\{\|\overline{\xi}(z,p)-\overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}_m}\mid\ z\in\overline{\mathcal{U}}\}<\epsilon,\end{aligned}$$

for $(x,p)\in\mathcal{U}\times\mathcal{O}$, as desired. ■

The next example shows that the assumption of local compactness cannot be generally relaxed.

4.11 Example: (Jointly parameterised real analytic sections are not always restrictions of jointly parameterised holomorphic sections) Let $\mathsf{M}=\mathbb{R}$, let $\mathcal{P}=\mathrm{C}^\omega(\mathbb{R})$, and define $f\colon\mathbb{R}\times\mathcal{P}\to\mathbb{R}$ by $f(x,g)=g(x)$. Since $g\mapsto f^g$ is the identity map, we conclude from Proposition 4.9 that $f\in\mathrm{JPC}^\omega(\mathcal{P};\mathsf{M})$. Let $x_0\in\mathbb{R}$. We claim that, for any neighbourhood $\overline{\mathcal{U}}$ of x_0 in $\mathbb{C}$ and any neighbourhood $\mathcal{O}$ of $0\in\mathcal{P}$, there exists $g\in\mathcal{O}$ such that g, and therefore f^g, does not have an holomorphic extension to $\overline{\mathcal{U}}$. To see this, let $\sigma\in\mathbb{R}_{>0}$ be such that the closed disk $\overline{\mathsf{D}}(\sigma,x_0)$ of radius σ centred at x_0 in $\mathbb{C}$ is contained in $\overline{\mathcal{U}}$. Let $K_1,\dots,K_r\subseteq\mathbb{R}$ be compact, let $\boldsymbol{a}_1,\dots,\boldsymbol{a}_r\in\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, and let $\epsilon_1,\dots,\epsilon_r\in\mathbb{R}_{>0}$ be such that

$$\cap_{j=1}^r\{g\in\mathcal{P}\mid\ p_{K_j,\boldsymbol{a}_j}(g)\leq\epsilon_j\}\subseteq\mathcal{O}.$$

Now define

$$g(x)=\frac{\alpha}{1+((x-x_0)/\sigma)^2},\qquad x\in\mathbb{R},$$

with $\alpha \in \mathbb{R}_{>0}$ chosen sufficiently small that $p_{K_j,\boldsymbol{a}_j}(g) < \epsilon_j$, $j \in \{1, \dots, r\}$, and note that $g \in \mathcal{O}$ does not have an holomorphic extension to $\overline{\mathcal{U}}$. Indeed, suppose that such an holomorphic extension $\overline{g}$ exists. Then $\overline{g}(z)$ must be equal to $\frac{\alpha\sigma^2}{\sigma^2+(z-x_0)^2}$ for z in the open disk $\mathsf{D}(\sigma, x_0)$ by uniqueness of holomorphic extensions [Cieliebak and Eliashberg 2012, Lemma 5.40]. But this immediately prohibits $\overline{g}$ from being holomorphic on any neighbourhood of the closed disk $\overline{\mathsf{D}}(\sigma, x_0)$, giving our claim. •

4.5. Mixing regularity hypotheses. It is possible to consider parameterised sections with mixed regularity hypotheses. Indeed, the conditions of Definitions 4.1, 4.3, and 4.7 are joint on state and parameter. Thus we may consider the following situation. Let $m \in \mathbb{Z}_{\geq 0}$, $m' \in \{0, \mathrm{lip}\}$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and $r' \in \{0, \mathrm{lip}\}$. If $r + r' \geq m + m'$ (with the obvious convention that $\infty + \mathrm{lip} = \infty$ and $\omega + \mathrm{lip} = \omega$), we may then consider a parameterised section in

$$\mathrm{SP}\Gamma^{r+r'}(\mathcal{P}; \mathsf{E}) \cap \mathrm{JP}\Gamma^{m+m'}(\mathcal{P}; \mathsf{E})$$

As with time-varying vector fields, there is nothing wrong with this—indeed this is often done—as long as one remembers what is true and what is not in the case when $r + r' > m + m'$.

5. Control systems

In this section we introduce classes of control systems that integrate the properties of the locally convex topologies from Section 2, noting that, of course, control systems are merely parameterised vector fields as in Section 4. We show that, for the classes of control systems we introduce, the regularity of the initial value problem one obtains upon substitution of a control matches the regularity of the system in a satisfying manner. As we have been doing all along so far, we consider the finitely differentiable, Lipschitz, smooth, and real analytic cases. We see that the manner in which we have characterised the topologies for spaces of vector fields allows for a unified way of treating control systems, across all sorts of regularity, in our framework, cf. the notation of Section 2.7.

5.1. Control systems with locally essentially bounded controls. With the notions of parameterised sections from the preceding section, we readily define what we mean by a control system.

5.1 Definition: (Control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***C$^\nu$-control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$, where

(i) M is a C^r-manifold whose elements are called ***states***,

(ii) $\mathcal{C}$ is a topological space called the ***control set***, and

(iii) $F \in \mathrm{JP}\Gamma^\nu(\mathcal{C}; \mathsf{TM})$. •

The governing equations for a control system are

$$\xi'(t) = F(\xi(t), \mu(t)),$$

for suitable functions $t \mapsto \mu(t) \in \mathcal{C}$ and $t \mapsto \xi(t) \in \mathsf{M}$. To ensure that these equations make sense, the differential equation should be shown to have the properties needed for existence and uniqueness of solutions, as well as appropriate dependence on initial conditions. We do this by allowing the controls for the system to be as general as reasonable.

5.2 Proposition: (Property of control system when the control is specified) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. If $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ then $F^\mu \in \mathrm{LB}\Gamma^\nu(\mathbb{T}, \mathsf{TM})$, where $F^\mu \colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ is defined by $F^\mu(t,x) = F(x, \mu(t))$.*

Proof: Let us define $\hat{F}^\mu \colon \mathbb{T} \to \Gamma^\nu(\mathsf{TM})$ by $\hat{F}^\mu(t) = F^\mu_t$. By Propositions 4.2, 4.4, and 4.9, the mapping $u \mapsto F^u$ is continuous. Since $\hat{F}^\mu$ is thus the composition of the measurable function μ and the continuous mapping $u \mapsto F^u$, it follows that $\hat{F}^\mu$ is measurable. It follows from Theorem 3.9 that F^μ is a Carathéodory vector field of class C^ν.

Let $\mathbb{T}' \subseteq \mathbb{T}$ be compact. Since μ is locally essentially bounded, there exists a compact set $K \subseteq \mathcal{C}$ such that

$$\lambda(\{t \in \mathbb{T}' \mid \mu(t) \notin K\}) = 0.$$

Since the mapping $u \mapsto F^u$ is continuous,

$$\{F^\mu_t \mid t \in \mathbb{T}'\}$$

is contained in a compact subset of $\Gamma^\nu(\mathsf{TM})$, i.e., F^μ is locally essentially bounded. ■

This then gives the following result, characterising the nature of flows associated with open-loop control systems.

5.3 Corollary: (Regularity of flows of open-loop control systems) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a $\mathrm{C}^{\nu+\mathrm{lip}}$-control system and let $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, with $F^\mu \in \mathrm{LB}\Gamma^\nu(\mathbb{T}, \mathsf{TM})$ defined by $F^\mu(t,x) = F(x, \mu(t))$. Then there exist a subset $D_{\Sigma,\mu} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ and a map $\Phi^X \colon D_{\Sigma,\mu} \to \mathsf{M}$ with the following properties for each $(t_0, x_0) \in \mathbb{T} \times \mathsf{M}$:*

(i) the set

$$\mathbb{T}_{\Sigma,\mu}(t_0, x_0) = \{t \in \mathbb{T} \mid (t, t_0, x_0) \in D_{\Sigma,\mu}\}$$

is an interval;

(ii) there exists an absolutely continuous curve $t \mapsto \xi(t)$ satisfying

$$\xi'(t) = X(t, \xi(t)), \quad \xi(t_0) = x_0,$$

for almost all $t \in |t_0, t_1|$ if and only if $t_1 \in \mathbb{T}_{\Sigma,\mu}(t_0, x_0)$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t, t_0, x_0) = X(t, \Phi^X(t, t_0, x_0))$ for almost all $t \in \mathbb{T}_{\Sigma,\mu}(t_0, x_0)$;

(iv) for each $t \in \mathbb{T}$ for which $(t, t_0, x_0) \in D_{\Sigma,\mu}$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the mapping $x \mapsto \Phi^X(t, t_0, x)$ is defined and of class C^ν on $\mathcal{U}$.

Proof: This follows from Proposition 5.2 and Theorem 3.10. ■

5.2. Control systems with locally integrable controls. In this section we specialise the discussion from the preceding section in one direction, while generalising it in another. To be precise, we now consider the case where our control set $\mathcal{C}$ is a subset of a locally convex topological vector space, and the system structure is such that the notion of integrability is preserved (in a way that will be made clear in Proposition 5.6 below).

5.4 Definition: (Sublinear control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \text{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***C$^\nu$-sublinear control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$, where

(i) M is a C^r-manifold whose elements are called ***states***,

(ii) $\mathcal{C}$ is a subset of a locally convex topological vector space V, $\mathcal{C}$ being called the ***control set***, and

(iii) F has the following property: for every continuous seminorm p for $\Gamma^\nu(\mathsf{TM})$, there exists a continuous seminorm q for V such that

$$p(F^{u_1} - F^{u_2}) \leq q(u_1 - u_2), \qquad u_1, u_2 \in \mathcal{C}. \qquad \bullet$$

Note that, by Propositions 4.2, 4.4, and 4.9, the sublinearity condition (iii) implies that a C$^\nu$-sublinear control system is a C$^\nu$-control system.

Let us demonstrate a class of sublinear control systems in which we will be particularly interested.

5.5 Example: (Control-linear systems and control-affine systems) We let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \text{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let V be a locally convex topological vector space, and let $\mathcal{C} \subseteq \mathsf{V}$. We suppose that we have a continuous linear map $\Lambda \in \mathrm{L}(\mathsf{V}; \Gamma^\nu(\mathsf{TM}))$ and we correspondingly define $F_\Lambda \colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ by $F_\Lambda(x, u) = \Lambda(u)(x)$. Continuity of Λ immediately gives that the control system $(\mathsf{M}, F_\Lambda, \mathcal{C})$ is sublinear, and we shall call a system such as this a ***C$^\nu$-control-linear system***.

Note that we can regard a control-affine system as a control-linear system as follows. For a control-affine system with $\mathcal{C} \subseteq \mathbb{R}^k$ and with

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{k} u^a f_a(x),$$

we let $\mathsf{V} = \mathbb{R}^{k+1} \simeq \mathbb{R} \oplus \mathbb{R}^k$ and take

$$\mathcal{C}' = \{(u^0, \boldsymbol{u}) \in \mathbb{R} \oplus \mathbb{R}^k \mid u^0 = 1, \ \boldsymbol{u} \in \mathcal{C}\}, \quad \Lambda(u^0, \boldsymbol{u}) = \sum_{a=0}^{k} u^a f_a.$$

Clearly we have $F(x, \boldsymbol{u}) = F_\Lambda(x, (1, \boldsymbol{u}))$ for every $\boldsymbol{u} \in \mathcal{C}$. Since linear maps from finite-dimensional locally convex spaces are continuous [Horváth 1966, Proposition 2.10.2], we conclude that control-affine systems are control-linear systems. Thus they are also control systems as per Definition 5.1. $\bullet$

One may want to regard the generalisation from the case where the control set is a subset of $\mathbb{R}^k$ to being a subset of a locally convex topological vector space to be mere fancy generalisation, but this is, actually, far from being the case. Indeed, this observation is the foundation for a general, control parameterisation-independent formulation for control theory that will be part of forthcoming work.

We also have a version of Proposition 5.2 for sublinear control systems.

5.6 Proposition: (Property of sublinear control system when the control is specified) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-sublinear control system for which $\mathcal{C}$ is a subset of a locally convex topological vector space V. If $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, then $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$, where $F^\mu \colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ is defined by $F^\mu(t, x) = F(x, \mu(t))$.*

Proof: The proof that F^μ is a Carathéodory vector field of class C^ν goes exactly as in Proposition 5.2.

To prove that $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$, let $K \subseteq \mathsf{M}$ be compact, let $k \in \mathbb{Z}_{\geq 0}$, let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and denote

$$p_K = \begin{cases} p^\infty_{K,k}, & \nu = \infty, \\ p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m + \mathrm{lip}, \\ p^\omega_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases}$$

Define $g \colon \mathbb{T} \to \mathbb{R}_{\geq 0}$ by $g(t) = p_K(F^\mu_t)$. We claim that $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$. From the first part of the proof of Proposition 5.2, $t \mapsto F^\mu_t(x)$ is measurable for every $x \in \mathsf{M}$. By Theorem 3.9, it follows that $t \mapsto F^\mu_t$ is measurable. Since p_K is a continuous function on $\Gamma^\nu(\mathsf{TM})$, it follows that $t \mapsto p_K(F^\mu_t)$ is measurable, as claimed. We claim that $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$. Note that $X \mapsto p_K(X)$ is a continuous seminorm on $\Gamma^\infty(\mathsf{TM})$. By hypothesis, there exists a continuous seminorm q for the locally convex topology for V such that

$$p_K(F^{u_1} - F^{u_2}) \leq q(u_1 - u_2)$$

for every $u_1, u_2 \in \mathcal{C}$. Therefore, if $\mathbb{T}' \subseteq \mathbb{T}$ is compact and if $u_0 \in \mathcal{C}$, we also have

$$\begin{aligned} \int_{\mathbb{T}'} g(t)\,\mathrm{d}t &= \int_{\mathbb{T}'} p_K(F^\mu_t) \\ &\leq \int_{\mathbb{T}'} p_K(F^\mu_t - F^{u_0})\,\mathrm{d}t + \int_{\mathbb{T}'} p_K(F^{u_0})\,\mathrm{d}t \\ &\leq \int_{\mathbb{T}'} q(\mu(t))\,\mathrm{d}t + (q(u_0) + p_K(F^{u_0}))\lambda(\mathbb{T}') < \infty, \end{aligned}$$

the last inequality by the characterisation of Bochner integrability from [Beckmann and Deitmar 2011, Theorems 3.2 and 3.3]. Thus g is locally integrable. It follows from Theorem 3.9 that $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$, as desired. ■

One also has the associated result regarding the regularity of flows of open-loop systems in this case.

5.7 Corollary: (Regularity of flows of open-loop sublinear control systems) *Let $m \in \mathbb{Z}_{\geq 0}$, let $\nu \in \{m, \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a $\mathrm{C}^{\nu+\mathrm{lip}}$-sublinear control system and let $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, with $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T}, \mathsf{TM})$ defined by $F^\mu(t, x) = F(x, \mu(t))$. Then there exist a subset $D_{\Sigma,\mu} \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ and a map $\Phi^X \colon D_{\Sigma,\mu} \to \mathsf{M}$ with the following properties for each $(t_0, x_0) \in \mathbb{T} \times \mathsf{M}$:*

(i) the set

$$\mathbb{T}_{\Sigma,\mu}(t_0, x_0) = \{t \in \mathbb{T} \mid (t, t_0, x_0) \in D_{\Sigma,\mu}\}$$

is an interval;

(ii) there exists an absolutely continuous curve $t \mapsto \xi(t)$ satisfying

$$\xi'(t) = X(t, \xi(t)), \quad \xi(t_0) = x_0,$$

for almost all $t \in |t_0, t_1|$ if and only if $t_1 \in \mathbb{T}_{\Sigma,\mu}(t_0, x_0)$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t, t_0, x_0) = X(t, \Phi^X(t, t_0, x_0))$ for almost all $t \in \mathbb{T}_{\Sigma,\mu}(t_0, x_0)$;

(iv) for each $t \in \mathbb{T}$ for which $(t, t_0, x_0) \in D_{\Sigma,\mu}$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the mapping $x \mapsto \Phi^X(t, t_0, x)$ is defined and of class C^ν on $\mathcal{U}$.

Proof: This follows from Proposition 5.6 and Theorem 3.10. ■

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