Substitution in Formulae (10 pts)

1. For a given first-order formula ψ, $\psi[t/x]$ is a formula where the variable x is replaced by the term t according to the substitution rules discussed in the lectures. Let P, Q be relations and f, g be functions. Compute $\psi[t/x]$ for the first-order formulae ψ shown below (2.5 pts each):

1. $\exists x.(x < y)[x/y]$

∃x.(x < y) [x/y]
= ∃x.(x [x/y] < y [x/y])
= ∃x.(x < x)

2. $\forall x.(P(x) \implies Q(x, f(y)))[g(y)/x]$

∀x.(P(x) ==> Q(x, f(y))) [g(y)/x]
= ∀x.(P(x) [g(y)/x] ==> Q(x, f(y)) [g(y)/x])
= ∀x.(P(g(y)) ==> Q(x [g(y)/x], f(y) [g(y)/x]))
= ∀x.(P(g(y)) ==> Q(g(y), f(y [g(y)/x])))
= ∀x.(P(g(y)) ==> Q(g(y), f(y)))

3. $\forall x.(P(x) \implies Q(x, f(y)))[g(y)/y]$

∀x.(P(x) ==> Q(x, f(y))) [g(y)/y]
= ∀x.(P(x) [g(y)/y] ==> Q(x, f(y)) [g(y)/y])
= ∀x.(P(x [g(y)/y]) ==> Q(x [g(y)/y], f(y) [g(y)/y]))
= ∀x.(P(x) ==> Q(x, f(y [g(y)/y])))
= ∀x.(P(x) ==> Q(x, f(g(y))))

4. $\forall x.(P(x) \implies Q(x, f(y)))[g(x)/y]$

∀x.(P(x) ==> Q(x, f(y))) [g(x)/y]
= ∀x.(P(x) [g(x)/y] ==> Q(x, f(y)) [g(x)/y])
= ∀x.(P(x [g(x)/y]) ==> Q(x [g(x)/y], f(y) [g(x)/y]))
= ∀x.(P(x) ==> Q(x, f(y [g(x)/y])))
= ∀x.(P(x) ==> Q(x, f(g(x))))

Alpha Equivalence (15 pts)

2. Prove that the following first-order formulae are alpha equivalent (5 pts each):

1. $(x > 3 \wedge (\exists y.(\forall z.z \geq (y - x)) \vee (z \geq y))) \equiv^{\alpha} (x > 3 \wedge (\exists w.(\forall z.z \geq (w - x)) \vee (z \geq w)))$

(x > 3 ∧ (∃y.(∀z.z ≥ (y - x)) ∨ (z ≥ y))) =
(x > 3 ∧ (∃y.(∀z.z ≥ (y - x)) ∨ (z ≥ y))) [w/y] =
((x > 3) [w/y] ∧ (∃y.(∀z.z ≥ (y - x)) ∨ (z ≥ y)) [w/y]) =
((x [w/y] > 3 [w/y]) ∧ (∃w.((∀z.z ≥ (y - x)) ∨ (z ≥ y)) [w/y])) =
(x > 3 ∧ (∃w.(∀z.z ≥ (y - x)) [w/y] ∨ (z ≥ y) [w/y])) =
(x > 3 ∧ (∃w.(∀z.(z ≥ (y - x)) [w/y]) ∨ (z [w/y] ≥ y [w/y]))) =
(x > 3 ∧ (∃w.(∀z.z [w/y] ≥ (y - x) [w/y]) ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀z.z ≥ (y [w/y] - x [w/y])) ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀z.z ≥ (w - x)) ∨ (z ≥ w)))

2. $(x > 3 \wedge (\exists y.(\forall z.z \geq (y - x)) \vee (z \geq y))) \equiv^{\alpha} (x > 3 \wedge (\exists w.(\forall y.y \geq (w - x)) \vee (z \geq w)))$

(x > 3 ∧ (∃y.(∀z.z ≥ (y - x)) ∨ (z ≥ y))) =
(x > 3 ∧ (∃y.(∀z.z ≥ (y - x)) ∨ (z ≥ y))) [w/y] =
(x > 3 ∧ (∃w.(∀z.z ≥ (w - x)) ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀z.z ≥ (w - x)) [y/z] ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀y.(z ≥ (w - x)) [y/z]) ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀y.z [y/z] ≥ (w - x) [y/z]) ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀y.y ≥ (w [y/z] - x [y/z])) ∨ (z ≥ w))) =
(x > 3 ∧ (∃w.(∀y.y ≥ (w - x)) ∨ (z ≥ w)))

3. $(x > 3 \wedge (\exists y.(\forall y.y \geq (u-x)) \vee (y \geq w))) \equiv^{\alpha} (x > 3 \wedge (\exists z.(\forall w.w \geq (u-x)) \vee (z \geq w)))$

$(x > 3 \wedge (\exists y.(\forall y.y \geq (u - x)) \vee (y \geq w))) =$
$(x > 3 \wedge (\exists y.(\forall y.y \geq (u - x)) [w/y] \vee (y \geq w))) =$
$(x > 3 \wedge (\exists y.(\forall w.(y \geq (u - x)) [w/y]) \vee (y \geq w))) =$
$(x > 3 \wedge (\exists y.(\forall w.y [w/y] \geq (u - x) [w/y]) \vee (y \geq w))) =$
$(x > 3 \wedge (\exists y.(\forall w.w \geq (u [w/y] - x [w/y])) \vee (y \geq w))) =$
$(x > 3 \wedge (\exists y.(\forall w.w \geq (u - x)) \vee (y \geq w))) =$
$(x > 3 \wedge (\exists y.(\forall w.w \geq (u - x)) \vee (y \geq w))) [z/y] =$
$((x > 3) [z/y] \wedge (\exists y.(\forall w.w \geq (u - x)) \vee (y \geq w)) [z/y]) =$
$(x [z/y] > 3 [z/y] \wedge (\exists z.((\forall w.w \geq (u - x)) \vee (y \geq w)) [z/y])) =$
$(x > 3 \wedge (\exists z.(\forall w.w \geq (u - x)) [z/y] \vee (y \geq w) [z/y])) =$
$(x > 3 \wedge (\exists z.(\forall w.(w \geq (u - x)) [z/y]) \vee (y [z/y] \geq w [z/y]))) =$
$(x > 3 \wedge (\exists z.(\forall w.w [z/y] \geq (u - x) [z/y]) \vee (z \geq w))) =$
$(x > 3 \wedge (\exists z.(\forall w.w \geq (u [z/y] - x [z/y])) \vee (z \geq w))) =$
$(x > 3 \wedge (\exists z.(\forall w.w \geq (u - x)) \vee (z \geq w)))$

Semantic Argument

$$\frac{a \models^S \forall x.\, \psi}{a[x \mapsto v] \models^S \psi \;\; for\ any\ v \in D}$$ (universal q.)

$$\frac{a \models^S \exists x.\, \psi}{a\,[x \mapsto v] \models^S \psi \;\; \text{for a new } v \in D}$$ (existential q.)

$$\frac{a \not\models^S \exists x.\, \psi}{a[x \mapsto v] \not\models^S \psi \qquad \text{for any } v \in D}$$ negation of existential q.

$$\frac{a \not\models^S \forall x.\, \psi}{a[x \mapsto v] \not\models^S \psi \qquad \text{for a new } v \in D}$$ negation of universal q.

$$\frac{a_A \models^S r(x_1, \ldots, x_n) \;\; a_B \not\models^S r(y_1, \ldots, y_n) \quad \text{for } i = 1, \ldots, n,\ a_A[x_i] = a_B[y_i]}{\models^S F}$$ Contradiction

Semantic Argument Method (20 pts)

3. Let P, Q be any relations. For each of the following formulas, either use the semantic argument method to prove validity or provide a counterexample for which the formula does not hold. (5 pts each)

1. $(\exists x.P(x)) \implies (\forall y.P(y))$

Suppose this statement is Invalid. Then we must show:

1. $I \models \neg((\exists x.P(x)) \Longrightarrow (\forall y.P(y)))$ (Assumption)
2. $I \models (\exists x.P(x)) \wedge \neg(\forall y.P(y))$ (By 1. and semantics of ==>)

Let D = {0, 1}
$P_I = \{0\}$
Let P(x) be true iff $x \in P_I$

This means P(0) is True, and P(1) is False

Then $\exists x.P(x)$ and $\neg(\forall y.P(y))$ evaluate to true under I, so:
$I \models (\exists x.P(x)) \wedge \neg(\forall y.P(y))$

This means $(\exists x.P(x)) \Longrightarrow (\forall y.P(y))$ is Invalid. Interpretation I is a falsifying interpretation.

2. $(\exists x.\forall y.P(x,y)) \implies (\forall y.\exists x.P(x,y))$

Suppose this statement is invalid. Then:

1. I $\nvDash$ ($\exists$x.$\forall$y.P(x, y)) ==> ($\forall$y.$\exists$x.P(x, y)) (Assumption)
2. I $\vDash$ $\exists$x.$\forall$y.P(x, y) (By 1. and semantics of $\nvDash$)
    a. I [x --> v_0] $\vDash$ $\forall$y.P(x, y) for some $v_0 \in$ D (By 2. and semantics of $\exists$)
    b. I [x --> v_0, y --> v_1] $\vDash$ P(x, y) for some $v_0 \in$ D and for all $v_1 \in$ D (By 2a. and semantics of $\forall$)
3. I $\nvDash$ $\forall$y.$\exists$x.P(x, y) (By 1. and semantics of $\nvDash$)
    a. I [y --> v_1] $\nvDash$ $\exists$x.P(x, y) for some $v_1 \in$ D (By 3. and semantics of $\forall$)
    b. I [x --> v_0, y --> v_1] $\nvDash$ P(x, y) for all $v_0 \in$ D and for some $v_1 \in$ D (By 3a. and semantics of $\exists$)
4. I $\vDash$ False (By 2b. and 3b.)

Because lines 2b. and 3b. are contradictory, ($\exists$x.$\forall$y.P(x, y)) ==> ($\forall$y.$\exists$x.P(x, y)) must be valid.

3. $(\exists x.P(x) \implies \forall x.Q(x)) \implies \forall x.(P(x) \implies Q(x))$

Suppose this statement is invalid. Then:

1. I ⊭ (∃x.P(x) ==> ∀x.Q(x)) ==> ∀x.(P(x) ==> Q(x)) (Assumption)
2. I ⊨ ∃x.P(x) ==> ∀x.Q(x) (By 1. and semantics of ==>)
   a. I ⊨ (∃x.P(x) ∧ ∀x.Q(x)) ∨ ¬(∃x.P(x)) (By 2. and semantics of ==>)
   b. I ⊨ (∃x.P(x) ∨ ¬(∃x.P(x)) ∧ (∀x.Q(x) ∨ ¬(∃x.P(x))
      (By 2a. and semantics of ∨)
   c. I ⊨ ∃x.P(x) ∨ ¬(∃x.P(x)) (By 2b. and semantics of ∧)
   d. I ⊨ ∀x.Q(x) ∨ ¬(∃x.P(x)) (By 2b. and semantics of ∧)
3. I ⊭ ∀x.(P(x) ==> Q(x)) (By 1. and semantics of ⊭)
4. I ⊨ ¬(∀x.(P(x) ==> Q(x))) (By 3. and semantics of ¬)

Let D = {0, 1}
P_I = {0}
Q_I = {0, 1}
Let P(x) be true iff x ∈ P_I, and Q(x) be true iff x ∈ Q_I.

This means P(0), Q(0), Q(1) are True, while P(1) is False.

Then ∃x.P(x) ∨ ¬(∃x.P(x)), ∀x.Q(x) ∨ ¬(∃x.P(x)), and ¬(∀x.(P(x) ==> Q(x))) all evaluate to true under I.

Thus:
I ⊨ (∃x.P(x) ∨ ¬(∃x.P(x))) ∧ (∀x.Q(x) ∨ ¬(∃x.P(x))) ∧ (¬(∀x.(P(x) ==> Q(x))))

This means (∃x.P(x) ==> ∀x.Q(x)) ==> ∀x.(P(x) ==> Q(x)) is Invalid. Interpretation I is a falsifying interpretation.

4. $(\exists x.P(x) \wedge \exists x.Q(x)) \implies (\exists x.(P(x) \wedge Q(x)))$

Suppose this statement is invalid. Then:

1. $I \not\models (\exists x.P(x) \wedge \exists x.Q(x)) \implies (\exists x.(P(x) \wedge Q(x)))$ (Assumption)
2. $I \models \exists x.P(x) \wedge \exists x.Q(x)$ (By 1. and semantics of ==>)
    a. $I \models \exists x.P(x)$ (By 2. and semantics of $\wedge$)
    b. $I \models \exists x.Q(x)$ (By 2. and semantics of $\wedge$)
3. $I \not\models \exists x.(P(x) \wedge Q(x))$ (By 1. and semantics of ==>)
4. $I \models \neg(\exists x.(P(x) \wedge Q(x)))$ (By 3. and semantics of $\neg$)

Let $D = \{0, 1\}$
$P_I = \{0\}$
$Q_I = \{1\}$
Let $P(x)$ be true iff $x \in P_I$, and $Q(x)$ be true iff $x \in Q_I$.

This means P(0) and Q(1) are True, while P(1) and Q(0) are False.

Then $(\exists x.P(x))$, $(\exists x.Q(x))$, and $(\neg(\exists x.(P(x) \wedge Q(x))))$ all evaluate to true under I.

Thus:
$I \models (\exists x.P(x)) \wedge (\exists x.Q(x)) \wedge (\neg(\exists x.(P(x) \wedge Q(x))))$

This means $(\exists x.P(x) \wedge \exists x.Q(x)) \implies (\exists x.(P(x) \wedge Q(x)))$ is invalid. Interpretation I is a falsifying interpretation.