ModExp 4096 Definitions & Algorithms

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$$c = m^e \pmod{n}$$

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ModExp 4096

- These slides describe the architecture, input and output definitions, and algorithms, and analysis of a 4096-bit full ModExp code
- The 4096-bit numbers are organized as sw bits, where s is the number of words and w is the word length, such that sw = 4096
- We will take s = 128 and w = 32 bits in this implementation
- Additionally the exponent length k is available, as $k \le 4096$

ModExp 4096 Inputs and Output

- The **primary** input parameters are the basis *m* and the exponent *e*, each of which are *s*-word (*sw*-bits: 4096 bits) integers
- The **secondary** input parameters are n, n0', r and t, such that n is the s-word (sw-bit: 4096-bit) modulus, n0' is a 1-word (w-bit) integer, while r and t are s-word (sw-bit: 4096-bit) integers
- The output c is a s-word (sw-bit: 4096-bit) integer

Primary Input Parameter: m

• The code assumes that *m* is a 4096-bit **signed** integer:

$$m = (m_{4095}m_{4094}\cdots m_2m_1m_0)$$

such that $m_{4095}=0$ for m>0 due to 2s-complement representation, and organized as a s-word by w-bit number: sw=4096

• The LSB is on the top right and the MSB (the sign bit) is on the bottom left corner, for example, for s=4 and w=6, we have

	LSB: m ₍				
m ₅	m ₄	m ₃	m ₂	m ₁	m ₀
m ₁₁	m ₁₀	m ₉	m ₈	m ₇	m ₆
m ₁₇	m ₁₆	m ₁₅	m ₁₄	m ₁₃	m ₁₂
m ₂₃	m ₂₂	m ₂₁	m ₂₀	m ₁₉	m ₁₈

MSB: $m_{23} = 0$ for m > 0

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Primary Input Parameter: e

• The exponent e is k-bit **unsigned** binary number for $k \le 4096$

$$e=(e_{k-1}e_{k-2}\cdots e_2e_1e_0)$$

- Many practical exponents are fewer than 4096 bits; we assume e is k bits for k < 4096
- The LSB is on the top right and the MSB (the sign bit) is on the bottom left corner, for example, for s=4 and w=6, we have

	LSB: e ₍				
e ₅	e ₄	e ₃	e ₂	e ₁	e ₀
e ₁₁	e ₁₀	e ₉	e ₈	e ₇	e ₆
e ₁₇	e ₁₆	e ₁₅	e ₁₄	e ₁₃	e ₁₂
e ₂₃	e ₂₂	e ₂₁	e ₂₀	e ₁₉	e ₁₈

$$e_{23} = e_{22} = \dots = e_k = 0$$

• The MSBs $e_{23}, e_{22}, \ldots, e_k$ are zero

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Secondary Input Parameter: n

• The code assumes that n is a 4096-bit signed binary number:

$$n = (n_{4095}n_{4094}\cdots n_2n_1n_0)$$

such that $n_{4095} = 0$ for n > 0 due to 2s-complement representation

- We always have $n_{4095} = 0$ (n > 0) and $n_0 = 1$ (odd modulus)
- The LSB is on the top right and the MSB (the sign bit) is on the bottom left corner, for example, for s = 4 and w = 6, we have

		LSB:	n ₀ = 1		
n ₅	n ₄	n ₃	n ₂	n ₁	n ₀
n ₁₁	n ₁₀	n ₉	n ₈	n ₇	n ₆
n ₁₇	n ₁₆	n ₁₅	n ₁₄	n ₁₃	n ₁₂
n ₂₃	n ₂₂	n ₂₁	n ₂₀	n ₁₉	n ₁₈

MSB: $n_{23} = 0$

Note that $n_{23} = 0$ and $n_0 = 1$

Secondary Input Parameters: n0', r and t

• The parameters n0', r and t are derived from the modulus n:

$$n0' = -n^{-1} \pmod{2^w}$$

$$r = 2^{sw} \pmod{n}$$

$$t = 2^{2sw} \pmod{n}$$

- Since it is often the case that the modulus is fixed, these parameters are computed at once and saved in the memory
- The ModExp code assumes that these parameters are available at the same time with the modulus n
- We will later give code for computing these parameters outside the ModExp code

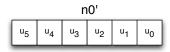
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Secondary Input Parameters: n0', r and t

- n0' is a 1-word (w-bit) unsigned integer: $n0' = (u_{w-1} \dots u_1 u_0)$
- r and t are s-word (sw-bit: 4096-bit) signed integers

r						
r ₅	r ₄	r ₃	r ₂	r ₁	r ₀	
r ₁₁	r ₁₀	r ₉	r ₈	r ₇	r ₆	
r ₁₇	^r 16	r ₁₅	r ₁₄	r ₁₃	r ₁₂	
r ₂₃	r ₂₂	r ₂₁	r ₂₀	r ₁₉	r ₁₈	

t						
t ₅	t ₄	t ₃	t ₂	t ₁	t ₀	
t ₁₁	t ₁₀	t ₉	t ₈	t ₇	t ₆	
t ₁₇	t ₁₆	t ₁₅	t ₁₄	t ₁₃	t ₁₂	
t ₂₃	t ₂₂	t ₂₁	t ₂₀	t ₁₉	t ₁₈	



Target Device & Properties

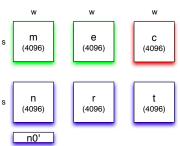
- The target device is Altera Cyclone II EP2C50
- This device has 129 4k memory blocks
- Each 4k block can be addressed at different configurations: $4k \times 1$, $2k \times 2$, up to 128×32
- The memory supports byte writes with port data width of 1, 2, 4, 8, 16, 32, 36 bits
- The device has more than sufficient memory of the input and output data, as well as for temporary values
- This allow us to implement several different optimizations on the exponentiation level: 1-bit, 2-bit and 4-bit window sizes
- Some configuration decisions will be made after the coding of the MonPro block is completed



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Input and Output

- The device assumes the primary inputs m and e are written into selected memory blocks (GREEN)
- The device also assumes the secondary input values n, n0', r, and t are already written into selected memory blocks (BLUE)
- It is often the case that the primary values will change after each computation, while the secondary values will remain in place
- The device computes c and writes into a memory block (RED)



The ModExp Algorithm

- The algorithm of choice is the Montgomery-transformed exponentiation using s-word integers with a word size of w bits and the Montgomery constant as $r=2^{sw}$
- We will describe the 1-bit (binary) and 2-bit (quaternary)
 exponentiation, however, we are experimenting with the 4-bit (hex)
 method, and make our final choice at the last phase of the project
- At the multiplication level, we are using the MonPro CIOS algorithm, whose details are given later on
- Primary Inputs: m and e
- Secondary Inputs: n, n0', r, and t
- Output: c
- Functions: MonPro



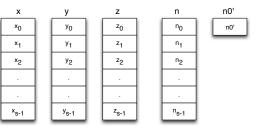
The 1-bit ModExp Algorithm

```
Find the index k < 4095 of the leftmost 1 in e
\bar{m} = \mathsf{MonPro}(m,t)
\bar{c} = r
for i = k - 1 down to 0
\bar{c} = \mathsf{MonPro}(\bar{c},\bar{c})
if e_i = 1 then \bar{c} = \mathsf{MonPro}(\bar{c},\bar{m})
c = \mathsf{MonPro}(\bar{c},1)
```

The 2-bit ModExp Algorithm

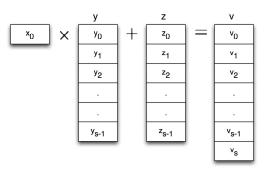
```
Express e in 2-bit blocks e = (E_{2047} \cdots E_1 E_0) with E_i \in \{0, 1, 2, 3\}
Find the index k < 2047 of the leftmost nonzero in e
\bar{m}1 = \mathsf{MonPro}(m, t)
\bar{m}^2 = \mathsf{MonPro}(\bar{m}^1, \bar{m}^1)
\bar{m}3 = \mathsf{MonPro}(\bar{m}2, \bar{m}1)
\bar{c} = MonPro(1, t)
for i = k down to 0
        \bar{c} = \mathsf{MonPro}(\bar{c}, \bar{c})
        \bar{c} = MonPro(\bar{c}, \bar{c})
        if E_i = 1 then \bar{c} = \text{MonPro}(\bar{c}, \bar{m}1)
            else if E_i = 2 then \bar{c} = \text{MonPro}(\bar{c}, \bar{m}2)
               else if E_i = 3 then \bar{c} = \text{MonPro}(\bar{c}, \bar{m}3)
c = \mathsf{MonPro}(\bar{c}, 1)
```

- The MonPro algorithm takes two inputs: x and y, and computes the output z such that each variable holds an s-word (sw-bit: 4096-bit) signed integer: $x=(x_{s-1}x_{s-2}\cdots x_1x_0)$, $y=(y_{s-1}y_{s-2}\cdots y_1y_0)$, and $z=(z_{s-1}z_{s-2}\cdots z_1z_0)$ with x_i,y_i,z_i as w-bit for $s-1\geq i\geq 0$, and the initial value of z is zero
- The secondary inputs n, n0', r and t are also available; we particularly need n0' and n, which are 1-word and s-word integers



1: We take the LSW of x, namely x_0 and multiply by the s-word y, and add it to the s-word partial product z (which is now all zero) to obtain the (s+1)-word temporary result v as

$$v = x_0 \cdot \sum_{i=0}^{s-1} y_i 2^{wi} + \sum_{i=0}^{s-1} z_i 2^{wi}$$



1a: The computation in Step 1 is accomplished using a Multiply-Add block that multiplies two 1-word numbers $(x_0 \text{ and } y_0)$, adds the previous higher word (C_0) , and adds another 1-word number (z_0) , producing a 2-word number (C_1, S_0) ; the lower word word (S_0) is assigned to (v_0) , while the higher word (C_1) is kept for the next Multiply-Add step, as follows:

$$(C_1, S_0) = x_0 \cdot y_0 + C_0 + z_0$$

$$v_0 = S_0$$

$$(C_2, S_1) = x_0 \cdot y_1 + C_1 + z_1$$

$$v_1 = S_1$$

$$(C_3, S_2) = x_0 \cdot y_2 + C_2 + z_2$$

$$v_2 = S_2$$
...

such that the initial value $C_0 = 0$



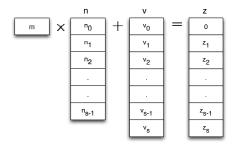
2: Then, we take the LSW of v, namely v_0 an multiply by the 1-word n0' modulo 2^w and obtain the 1-word integer m as

$$m = n0' \cdot v_0 \pmod{2^w}$$

$$\boxed{\begin{array}{c} \mathsf{v}_0 \\ \end{array}} \hspace{0.1cm} \hspace{0.1cm}$$

3: Then, we take the 1-word m and multiply by the s-word n, and add it to the (s+1)-word temporary value v to obtain the new partial product which is (s+1)-word

$$z = m \cdot \sum_{i=0}^{s-1} n_i 2^{wi} + \sum_{i=0}^{s} v_i 2^{wi}$$



3a: The computation in Step 3 is accomplished using a Multiply-Add block that multiplies two 1-word numbers $(m \text{ and } n_0)$, adds the previous higher word (C_0) , adds another 1-word number (v_0) , producing a 2-word number (C_1, S_0) ; assigning S_0 to z_0 , while keeping the higher word (C_1) for the next Multiply-Add step, as follows:

$$(C_{1}, S_{0}) = m \cdot n_{0} + C_{0} + v_{0}$$

$$z_{0} = S_{0}$$

$$(C_{2}, S_{1}) = m \cdot n_{1} + C_{1} + v_{1}$$

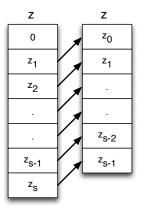
$$z_{1} = S_{1}$$

$$(C_{3}, S_{2}) = m \cdot n_{2} + C_{2} + v_{2}$$

$$z_{2} = S_{2}$$
...

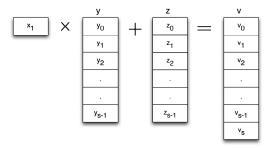
such that the initial value $C_0 = 0$

4: The resulting partial product z has its LSW as zero, due to the Montgomery property, and therefore, we shift up z to obtain the new s-word partial product



5: In the next step the next word of x, namely x_1 , is taken and multiplied with the s-word y, and added to the partial product z to obtain the new temporary result v as

$$v = x_1 \cdot \sum_{i=0}^{s-1} y_i 2^{wi} + \sum_{i=0}^{s-1} z_i 2^{wi}$$



6: This is followed up by computing the new *m* value

$$m = n0' \cdot v_0 \pmod{2^w}$$

and then the multiplication of m by n, and then the addition of the result to v to obtain the new (s+1)-word partial product z

$$z = m \cdot \sum_{i=0}^{s-1} n_i 2^{wi} + \sum_{i=0}^{s} v_i 2^{wi}$$

and finally shifting up z by one word (since $z_0 = 0$) to obtain the new s-word z value

7: Proceeding in this way, we multiply all x_i values by the multiplicand y and reduce it modulo n, for $i=0,1,2,\ldots,s-1$

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Resources and Components

- The ModExp requires implementation of several blocks
- A finite state machine to scan the bits of the exponent, 1-bit, 2-bit and 4-bit at a time
- A w-by-w bit integer multiplier producing 2w-bit products
- A w-by-w bit integer adder producing (w+1)-bit sums
- A finite state machine to control the flow of the MonPro algorithm from i=0 to i=s-1
- Several registers for the primary and secondary inputs, for the output, and for the temporary values

Interface

- Our analysis shows that w = 32 and s = 128 values will produce an implementation that will satisfy the specified requirements
- The target device Altera Cyclone II EP2C50 has 129 4k memory blocks which can be configured as 128×32
- At this configuration these memory blocks are not accessible in the true dual-port mode, however, we will not need that
- It is therefore assumed that the initiating processor uploads the 128-word primary inputs m and e, and 128-word secondary inputs n, r, and t, and 1-word secondary input n0' to selected memory blocks
- The ModExp code will run and produce the 128-word output c in a selected memory block

Interface

- The ModExp code will not destroy the input values and all of them can be reused for subsequent executions
- It is generally the case the primary inputs m and e will be refreshed by the initiating processor, leaving the secondary input values n, r, and t, and n0' intact for subsequent execution(s)