A Unified Analysis of Multirate and Periodically Time-Varying Digital Filters

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Abstract—**A unified approach to the analysis of periodically time-varying digital filters is introduced. It is shown that this method may be used to describe both time and frequency domain responses. By considering multirate filters as a special class of periodically time-varying filters we have characterized the frequency response of a general multirate filter. A time domain synthesis procedure for periodically varying difference equations is also presented.**

I. Introduction

THE USE of multirate digital filters has been proposed in several recent papers [1]–[6] with the principal applications being narrow-band filtering and modification of the sampling rate. The advantages of a multirate filter as compared with a single (higher) rate filter in these applications are improved processing efficiency and a reduction of coefficient word length. However, the analysis and design is complicated by the fact that multirate filters are periodically time-varying systems. In previous papers this problem has generally been circumvented by considering only those filters which may be realized as a cascade interconnection of time-invariant filters separated by discrete-time sampling switches. This restriction together with certain other simplifying assumptions has enabled the use of time-invariant analysis techniques for describing these multirate filters. Our approach is to examine general periodically time-varying discrete filters with multirate filters as a special case.

The analysis of periodically time-varying discrete-time systems was first performed in the context of sampled-data control systems. Jury and Mullin developed a technique for solving difference equations with periodically variable coefficients [7] and applied their results in an analysis of systems with a periodically varying sampling rate [8]. In an analysis of multirate feedback systems, Kranc [9] demonstrated a method for representing these systems by equivalent single rate systems. Friedland [10] observed that multirate systems belong to the class of general periodically time-varying discrete systems. In each of these papers a technique was introduced which converted a periodically varying system to a constant system. Time-invariant methods were used to analyze the constant system with emphasis on calculation of time domain response. Since each conversion technique was different, a unified theory of periodically time-varying discrete systems did not develop. As a consequence, the results of these earlier papers have not been applied to the types of periodically varying systems of current interest.

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In this paper we unify the previous results by introducing a single approach for describing linear periodically time-varying discrete systems. Our method is based on a block representation and is applicable to most systems of practical interest. It is shown that the impulse response of a periodically varying filter may be characterized as a periodic linear combination of geometric sequences. We describe a procedure based on Prony's method for designing a periodically varying discrete filter with a prescribed impulse response. A relation between the z-transforms of the input and output sequences is developed. For multirate filters, this relation leads to a well-defined concept of frequency response. We derive a general expression for the frequency response of a multirate filter which does not depend on a particular realization of the filter. This derivation provides new possibilities for the design of multirate filters.

II. Time Domain Representations

The purpose of this section is to develop a uniform representation of periodically time-varying discrete systems. By applying the concept of block processing [11], it is shown that a periodically varying discrete filter may be represented as a time-invariant block filter. This procedure is demonstrated for three common descriptions of discrete filters: 1) difference equations, 2) impulse response functions, and 3) state equations. The advantage of this representation is that conventional time-invariant systems analysis techniques are applicable to the block filter.

The difference equation representation for a linear periodically variable discrete time system may be written in the form

$$y(k) = \sum_{l=0}^{M} a_l(k)u(k-l) - \sum_{l=1}^{N} b_l(k)y(k-l). \quad (1)$$

Let P denote the period of the coefficient variation, thus

$$a_l(k+P) = a_l(k), \qquad \forall k$$
$$b_l(k+P) = b_l(k). \quad (2)$$

Equation (1) may be formulated as a matrix equation which is then simplified by partitioning into $L \times L$ submatrices and blocks of length L, where L is a multiple of the period P. This is illustrated below for the case $N = 3$, $M = 2$, and $L = P = 2$:

$$\begin{bmatrix} B_0 & 0 & 0 & \cdot \\ B_1 & B_0 & 0 & \cdot \\ B_2 & B_1 & B_0 & \cdot \\ 0 & B_2 & B_1 & \cdot \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_0 & 0 & 0 & \cdot & U_0 \\ A_1 & A_0 & 0 & \cdot & U_1 \\ 0 & A_1 & A_0 & \cdot & U_2 \\ 0 & 0 & A_1 & \cdot & U_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (3)$$

where

$$B_0 = \begin{bmatrix} 1 & 0 \\ b_1(1) & 1 \end{bmatrix} \qquad A_0 = \begin{bmatrix} a_0(0) & 0 \\ a_1(1) & a_0(1) \end{bmatrix}$$

$$B_1 = \begin{bmatrix} b_2(2) & b_1(2) \\ b_3(3) & b_2(3) \end{bmatrix} \qquad U_0 = \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$$

$$Y_0 = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}, \text{ etc.}$$

Equation (3) may be written as a constant coefficient vector difference equation (4) and considered as describing a linear time-invariant system with an input and output consisting of a sequence of blocks:

$$Y_k = \sum_{l=0}^{J} \hat{A}_l U_{k-l} - \sum_{l=1}^{K} \hat{B}_l Y_{k-l}. \tag{4}$$

Since B_0 is always nonsingular, $\hat{A}_l = B_0^{-1} A_l$ and $\hat{B}_l = B_0^{-1} B_l$.

An alternate representation for a periodically varying discrete filter is in terms of superposition:

$$y(k) = \sum_{l=0}^{k} h(k,l) u(l). \tag{5}$$

Thus $h(k,l)$ is the output at time k when the input was a unit pulse at time l, and for a periodically varying system

$$h(k + P, l + P) = h(k,l). \tag{6}$$

Equation (5) may also be expressed as a matrix equation in which the matrix relating input and output sequences is called the transmission matrix. If the transmission matrix is partitioned into $L \times L$ submatrices, H_k, called impulse response blocks, and the input and output sequences are partitioned into blocks of length L, then as a result of (6) and for L a multiple of P, (5) may be written as a time-invariant block convolution,

$$Y_k = \sum_{l=0}^{k} H_{k-l} U_l. \tag{7}$$

The idea of partitioning the transmission matrix to obtain a time-invariant description (7) was first employed by Friedland [10]. For block processing of time-invariant filters, Burrus [11] has derived (4) and (7).

A third representation of periodically varying discrete filters is a set of state equations

$$x(k+1) = A(k)x(k) + b(k)u(k)$$

$$y(k) = c^T(k)x(k) + d(k)u(k). \tag{8}$$

Recalling that the order of the system in (1) is N, then $x(k)$ is an N vector; $A(k)$ is $N \times N$; $b(k)$ and $c(k)$ are $N \times 1$; $d(k)$, $u(k)$, and $y(k)$ are scalars. For a periodically varying filter the state equation parameters have the property

$$A(k + P) = A(k) \qquad b(k + P) = b(k), \qquad \forall k$$

$$c(k + P) = c(k) \qquad d(k + P) = d(k). \tag{9}$$

The objective of the derivation which follows is to determine a state equation which describes the system of (8) in terms of blocks of input and output. As in the two previous representations, if the block length is a multiple of the period, then the block state equation is time-invariant.

Define the state vector $q(m)$ as a time-sampled version of the state vector $x(k)$; thus

$$q(m) = x(mL). \tag{10}$$

In order to write a state equation in terms of $q(m)$ and input blocks, we must determine how $q(m + 1)$ may be calculated from $q(m)$ and U_m. Recall that the input block U_m is an L component vector defined by

$$U_m = \{u_{mL}, u_{mL+1}, \cdots, u_{mL+L-1}\}^T. \tag{11}$$

Also note that, if L is a multiple of P, then (9) implies that $A(mL + k) = A(k)$. We then write

$$n \triangleq m + 1$$

$$q(n) = x(nL) = A(nL - 1)x(nL - 1) + b(nL - 1)u(nL - 1)$$

$$q(n) = A(L - 1)x(nL - 1) + b(L - 1)u(nL - 1).$$

If $x(nL - 1)$ is then written in terms of $x(nL - 2)$ and $u(nL - 2)$, we get

$$q(n) = A(L-1)[A(L-2)x(nL-2) + b(L-2)u(nL-2)] + b(L-1)u(nL-1).$$

This process is continued until we reach $x(nL - L)$:

$$q(n) = A(L-1)A(L-2)\cdots A(0)x(nL-L) + A(L-1)\cdots A(1)b(0)u(nL-L) + \cdots + b(L-1)u(nL-1). \tag{12}$$

Note that $x(nL - L) = q(n - 1) = q(m)$, and $u(nL - L) = u(mL)$. Define

$$\bar{A} = A(L - 1)\cdots A(0)$$

and

$$\bar{B} = [A(L-1)\cdots A(1)b(0) | \cdots | A(L-1)b(L-2) | b(L-1)].$$

Then (12) becomes

$$q(m + 1) = \bar{A} q(m) + \bar{B} U_m. \tag{13}$$

To calculate the output block Y_m we write an equation for each component and combine these equations in matrix form:

$$y(mL) = c^T(mL)x(mL) + d(mL)u(mL)$$

$$y(mL) = c^T(0)q(m) + d(0)u(mL)$$

$$y(mL + 1) = c^T(1)[A(0)q(m) + b(0)u(mL)] + d(1)u(mL + 1) \tag{14}$$

etc. Define

$$\bar{C} = \begin{bmatrix} c^T(0) \\ \hline c^T(1)A(0) \\ \hline \vdots \\ \hline c^T(L-1)A(L-2)\cdots A(0) \end{bmatrix}$$

and $\bar{D} = \{d_{ij}\}$, an $L \times L$ matrix, where

$$d_{ij} = \begin{cases} 0, & i < j \\ d(i-1), & i = j \\ c^T(i-1)b(j-1), & i = j+1 \\ c^T(i-1)A(i-2)\cdots A(j)b(j-1), & i > j+1. \end{cases}$$

Then (14) becomes

$$Y_m = \bar{C}q(m) + \bar{D}U_m. \tag{15}$$

Equations (13) and (15) are the time-invariant block state equations which we wished to determine.

III. Time Domain Properties

As shown in the previous section, a periodically time-varying discrete system may be written in block form

$$\begin{aligned} q(m+1) &= \bar{A}q(m) + \bar{B}U_m \\ Y_m &= \bar{C}q(m) + \bar{D}U_m. \end{aligned} \tag{16}$$

Another interpretation of the system described by (16) is that it represents a time-invariant multiinput, multioutput system. Therefore, the properties of such systems may be employed to characterize periodically varying systems. Using this approach we describe the time domain behavior of periodically time-varying systems.

An important characteristic of a system is stability. It is known that (16) is asymptotically stable if and only if the eigenvalues of $\bar{A}$ have magnitude less than one. Since (16) is also a representation of the periodically varying system in (8), the stability of (8) is similarly determined. Evans [12] and Davis [13] have obtained similar results.

The impulse response function for (8) is easily characterized using known properties of (16). In particular, the impulse response blocks H_m may be calculated using the relation

$$\begin{aligned} H_0 &= \bar{D} \\ H_m &= \bar{C}\bar{A}^{m-1}\bar{B}, \qquad m \geq 1. \end{aligned} \tag{17}$$

As a consequence of (17), it can be shown that, if (18) is the characteristic polynomial of $\bar{A}$,

$$\alpha_N x^N + \cdots + \alpha_1 x + \alpha_0 \tag{18}$$

then H_m satisfies the matrix equation

$$\alpha_N H_{m+N} + \cdots + \alpha_1 H_{m+1} + \alpha_0 H_m = 0, \qquad \forall m \geq 1. \tag{19}$$

In terms of the impulse response function $h(k,l)$, (19) may be written as

$$\alpha_N h(k+NP, l) + \cdots + \alpha_1 h(k+P, l) + \alpha_0 h(k,l) = 0. \tag{20}$$

As a result of (19), it would appear that (20) is valid for $0 \leq l < k$ and $k \geq P$. However, if the entries of the H_0 matrix are examined individually, it can be shown that (20) is valid for $0 \leq l < k$. In order to keep the notation as simple as possible, we shall now assume that the eigenvalues of $\bar{A}$ are distinct and denote them by λ_i. For the case of repeated eigenvalues our result generalizes in the usual manner. Equation (20) implies that $h(k+nP, l)$ may be written as

$$h(k+nP, l) = \sum_{i=1}^{N} \gamma_i(k,l)\lambda_i^n,$$
$$\text{for } 0 \leq l < k < P, \qquad n \geq 0. \tag{21}$$

Let $m = k + nP$ and define

$$\begin{aligned} \phi_i &= \lambda_i^{1/P} \\ \beta_i(m,l) &= \gamma_i(k,l)\phi_i^{-k}. \end{aligned} \tag{22}$$

Then $h(m,l)$ may be written as

$$h(m,l) = \sum_{i=1}^{N} \beta_i(m,l)\phi_i^m,$$
$$\text{for } 0 \leq l < P, \qquad m > l \tag{23}$$

where $\beta_i(m+P, l) = \beta_i(m,l)$ as a result of (22). The interpretation of (23) is the following: the impulse response $h(m,l)$ for $m > l \geq 0$, of a periodically time-varying discrete filter may be written as a periodically varying linear combination of geometric sequences.

The characterization of the impulse response in (23) is analogous to a result of Floquet theory [14] for periodically varying continuous time systems. Therefore, we should expect a Floquet-type representation of (8) to exist. Theorem 1 provides a precise statement and proof of this result. We have included this result for theoretical completeness, rather than for its applicability as an analysis tool, since the representation in (24) is computationally more difficult to achieve than the block form of (16).

Theorem 1

For a periodically varying discrete system (8) with $A(k)$ nonsingular for all k, there exists a nonsingular linear transformation $T(k)$ such that if $w(k) = T(k)x(k)$, then

$$\begin{aligned} w(k+1) &= \hat{A}w(k) + \hat{b}(k)u(k) \\ y(k) &= \hat{c}^T(k)w(k) + \hat{d}(k)u(k) \end{aligned} \tag{24}$$

where $\hat{b}(k)$, $\hat{c}^T(k)$, and $\hat{d}(k)$ are periodic with period P.

Proof: Define $\hat{A}$ by

$$\hat{A}^P = \bar{A} = A(P-1)A(P-2)\cdots A(1)A(0).$$

The Pth root of a nonsingular matrix is always defined (see [15, pp. 231–234] for a constructive proof). Define

$$T(k) = \hat{A}^k A(P-1)A(P-2)\cdots A(k), \qquad 0 \leq k < P$$

and

$$T(k+mP) = T(k).$$

Substituting $w(k) = T(k)x(k)$ into (8) gives

$$\begin{aligned} w(k+1) &= T(k+1)A(k)T^{-1}(k)w(k) + T(k+1)b(k)u(k) \\ y(k) &= c^T(k)T^{-1}(k)w(k) + d(k)u(k). \end{aligned} \tag{25}$$

It is easily verified that $T(k+1)A(k)T^{-1}(k) = \hat{A}$; letting $\hat{b}(k) = T(k+1)b(k)$, $\hat{c}^T(k) = c^T(k)T^{-1}(k)$, and $\hat{d}(k) = d(k)$, (25) reduces to (24).

IV. Time Domain Synthesis

The problem of determining a difference equation which produces a desired time domain response is examined. Two techniques will be presented which include both exact and approximate solutions. The fundamental ideas are based on Prony's method and follow those of Burrus and Parks [16] for the similar problem with constant coefficient difference equations.

The time domain response of a discrete time system may be characterized by an impulse response function. For periodically varying discrete time systems, there are essentially P distinct functions, $h(k,j)$ for $j = 0,1,2,\cdots, P-1$. We assume that it is desired to find a difference equation description of a discrete time system with an impulse response specified by $h(k,j)$ for $0 \le k < K_j$, $0 \le j \le P-1$. For each j, $h(k,j)$ must satisfy a form of (1):

$$h(k,j) = a_{k-j}(k) - \sum_{l=1}^{N} b_l(k)h(k-l,j), \qquad j \le k \le j + M \tag{26}$$

and

$$h(k,j) = -\sum_{l=1}^{N} b_l(k)h(k-l,j), \quad j + M < k < K_j. \tag{27}$$

If the $b_l(k)$ are known, then (26) may be used to solve directly for the $a_l(k)$. Therefore, we concentrate our efforts on finding the $b_l(k)$ such that (27) is satisfied. Since we require $b_l(k) = b_l(k+P)$, (27) is equivalent to the following P matrix equations:

$$\begin{bmatrix} h(k,0) & h(k-1,0) & \cdots & h(k-N,0) \\ h(k+P,0) & & & h(k+P-N,0) \\ \vdots & & & \vdots \\ h(k+m_0P,0) & & & h(k+m_0P-N,0) \\ h(k+P,1) & & & h(k+P-N,1) \\ \vdots & & & \\ \text{etc.} & & & \\ \vdots & & & \\ h(k+Pm_{P-1},P-1) & & & h(k+Pm_{P-1}-N,P-1) \end{bmatrix} \begin{bmatrix} 1 \\ b_1(k) \\ \vdots \\ b_N(k) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{28}$$

where

1) $h(i,j) = 0$, if $i < j$;
2) $k + m_jP < K_j$; and
3) $k = M+1, M+2, \cdots, M+P$.

Denote the above matrix by $\mathscr{H}(k)$, its first column by $\boldsymbol{h}_1(k)$, and the remaining columns by $\mathscr{H}_1(k)$; let $\boldsymbol{b}(k)$ denote the vector $[b_1(k)b_2(k)\cdots b_N(k)]^T$. Then (28) may be written as

$$\mathscr{H}_1(k)\boldsymbol{b}(k) = -\boldsymbol{h}_1(k). \tag{29}$$

A solution of (29) exists if and only if rank $\mathscr{H}_1(k) =$ rank $\mathscr{H}(k)$. If in addition, rank $\mathscr{H}_1(k) = N$, then the solution is unique and is given by

$$\boldsymbol{b}(k) = -[\mathscr{H}_1^T(k)\mathscr{H}_1(k)]^{-1}\mathscr{H}_1^T(k)\boldsymbol{h}_1(k). \tag{30}$$

For the overall synthesis problem to have an exact solution, there must exist a solution to (29) for the P values of k. In this case, (26) is then used to determine the $a_l(k)$, thus providing a complete description of the desired system in terms of a difference equation with periodically varying coefficients.

If (29) does not have a solution for one or more values of k, then an approximate solution must be sought. In the simpler problem involving constant coefficient difference equations, it is known [16] that an attempt to approximate the solution by minimization of the squared error between the desired and actual responses leads to nonlinear equations which must be solved iteratively. The difficulties of such an approach would be compounded by an extension to the more complex problem involving periodically varying parameters. The method proposed here is a simple extension of the classical method of Prony [17] to our problem.

The approximation error is defined by

$$\boldsymbol{e}(k) = \mathscr{H}_1(k)\boldsymbol{b}(k) + \boldsymbol{h}_1(k). \tag{31}$$

If the rank of $\mathscr{H}_1(k)$ is N, the unique solution which minimizes $\boldsymbol{e}^T(k)\boldsymbol{e}(k)$ is given by

$$\boldsymbol{b}(k) = -[\mathscr{H}_1^T(k)\mathscr{H}_1(k)]^{-1}\mathscr{H}_1^T(k)\boldsymbol{h}_1(k). \tag{32}$$

Notice that (32) and (30) are identical and both give unique solutions. If the rank of $\mathscr{H}(k)$ is also N, then as discussed above, the solution is exact and $\boldsymbol{e}^T(k)\boldsymbol{e}(k) = 0$. When the rank of $\mathscr{H}(k)$ is greater than N, $\boldsymbol{e}^T(k)\boldsymbol{e}(k) > 0$, but it is a minimum. If the rank of $\mathscr{H}_1(k)$ is less than N, then there are several solutions with the same value of $\boldsymbol{e}^T(k)\boldsymbol{e}(k)$.

In the event that no exact solution exists, there are several techniques which could be used to solve for the $a_l(k)$. Of course (26) can always be solved exactly. Other methods allow a combination of exact interpolation and minimization of the squared error between desired and actual impulse responses. These methods were explored in [16] and may be extended to this problem.

V. Frequency Domain Analysis

Analysis of time-invariant discrete systems in terms of frequency response is an important aid in understanding system behavior and in developing design methods. Since frequency response is in general not a well-defined concept for time-varying systems, the purpose of this section is to show that a frequency domain analysis of periodically time-varying discrete systems is possible. For certain classes of periodically varying filters the concept of frequency response is defined so that it characterizes the filter response in a manner similar to the time-invariant case.

We assume the system is described by the state equation (33) in which the parameters are periodic with period P:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A(k)\boldsymbol{x}(k) + b(k)u(k) \\ y(k) &= c^T(k)\boldsymbol{x}(k) + d(k)u(k). \end{aligned} \tag{33}$$

This system can also be represented in block form (34) where the block length is equal to the period:

$$\begin{aligned} \boldsymbol{q}(m+1) &= \bar{A}\boldsymbol{q}(m) + \bar{B}\boldsymbol{U}_m \\ \boldsymbol{Y}_m &= \bar{C}\boldsymbol{q}(m) + \bar{D}\boldsymbol{U}_m. \end{aligned} \tag{34}$$

As noted previously (34) may be considered as a multiinput, multioutput, time-invariant system. Since the application of z-transform theory to this system is routine, we need only relate the transforms of the input and output sequences of (33) to the transforms of the vector sequences in (34). Before proceeding with this in Lemmas 1 and 2, we introduce the following notation. The usual z-transform of the sequence $y(k)$ is denoted by

$$Y(z) = \sum_{k=0}^{\infty} y(k)z^{-k}. \tag{35}$$

The transform of a sampled version of $y(k)$ is

$$Y(z,i) = \sum_{m=0}^{\infty} y(i+mP)z^{-m} \tag{36}$$

and the transform of the vector sequence $\boldsymbol{Y}_m$ is

$$\hat{\boldsymbol{Y}}(z) = [Y(z,0), Y(z,1), \cdots, Y(z, P-1)]^T.$$

A similar notation will be used for $u(k)$.

Lemma 1

$$Y(z) = \sum_{l=0}^{P-1} z^{-l}Y(z^P,l). \tag{37}$$

Proof: Expand $Y(z^P,l)$ as in (36) and notice that the resulting double summation is equivalent to (35):

$$\sum_{l=0}^{P-1} z^{-l}Y(z^P,l) = \sum_{l=0}^{P-1}\sum_{m=0}^{\infty} y(l+mP)z^{-mP-l}$$

$$\sum_{l=0}^{P-1} z^{-l}Y(z^P,l) = \sum_{k=0}^{\infty} y(k)z^{-k} = Y(z).$$

Lemma 2

$$U(z^P,l) = \frac{1}{P} z^l \sum_{i=0}^{P-1} \phi^{li}U(z\phi^i) \tag{38}$$

where $\phi = \exp(2\pi j/P)$.

Proof: By expanding $U(z\phi^i)$ and interchanging the order of summation we get

$$\sum_{i=0}^{P-1} \phi^{li}U(z\phi^i) = \sum_{k=0}^{\infty} u(k)z^{-k}\sum_{i=0}^{P-1}\phi^{li}\phi^{-ki}.$$

It is easily shown that

$$\sum_{i=0}^{P-1} \phi^{i(l-k)} = \begin{cases} P, & l-k = \pm mP \\ 0, & \text{otherwise} \end{cases}$$

and therefore

$$\sum_{i=0}^{P-1} \phi^{li}U(z\phi^i) = P\sum_{m=0}^{\infty} u(mP+l)z^{-(mP+l)}$$

from which the result follows.

Theorem 2

$$Y(z) = \frac{1}{P}G_0(z)U(z) + \frac{1}{P}\sum_{n=1}^{P-1} G_n(z)U(z\phi^n) \tag{39}$$

where $\phi = \exp(2\pi j/P)$.

Proof: The usual transform relations for (34) are

$$\hat{\boldsymbol{Y}}(z) = H(z)\hat{\boldsymbol{U}}(z) \tag{40}$$

where $H(z)$ is a $P \times P$ matrix of rational functions of z given by $H(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D}$. Let $H_{i,j}(z)$ denote the entry in row i, column j starting at row 0, column 0. We start from Lemma 1,

$$Y(z) = \sum_{l=0}^{P-1} z^{-l}Y(z^P,l)$$

$$Y(z) = \sum_{l=0}^{P-1} z^{-l}\sum_{i=0}^{P-1} H_{l,i}(z^P)U(z^P,i) \tag{41}$$

substitute (38) into (41) and interchange the order of summation to yield

$$Y(z) = \sum_{n=0}^{P-1} U(z\phi^n)\left[\frac{1}{P}\sum_{l=0}^{P-1}\sum_{i=0}^{P-1}\phi^{in}z^{i-l}H_{l,i}(z^P)\right].$$

Define

$$G_n(z) = \sum_{l=0}^{P-1}\sum_{i=0}^{P-1}\phi^{in}z^{i-l}H_{l,i}(z^P) \tag{42}$$

and thus

$$Y(z) = \frac{1}{P}G_0(z)U(z) + \frac{1}{P}\sum_{n=1}^{P-1} G_n(z)U(z\phi^n).$$

The relation in (39) may be considered as a generalized transfer function since it relates the z-transforms of the input and output sequences. However, the advantage of this particular formulation is its interpretation when z is evaluated along the unit circle in the complex plane. For $z = \exp(j\omega T)$ where T is the sampling interval, $Y(z)$ is the frequency spectrum of the output sequence, and $U(z\phi^n)$ is a frequency shifted version of the input signal spectrum. Equation (39) indicates that the output spectrum is a sum of shifted and shaped versions of the input spectrum. This may be compared to a time-invariant filter in which the output spectrum is simply a shaped version of the input spectrum, and the shaping function is called the frequency response. For general periodically time-varying filters it is difficult to define a frequency response function since several shaping functions and spectra are involved.

At this point we restrict our attention to specific types of periodically time-varying filters for which the concept of frequency response may be defined. As discussed in the introduction, multirate filters are an important class of periodically time-varying filters. The concept of frequency response may be defined for multirate filters in the following manner.

A multirate filter in which the input sampling rate is P times the output sampling rate may be represented in block form (34) where all components in the output vector Y_m, except one, are zero. Therefore, we may assume $H_{l,i}(z) = 0$ for all $l > 0$, and the output sequence may be reindexed as follows: $y(m) = y(k)$ for $k = mP$. Then (42) becomes

$$G_n(z) = \sum_{i=0}^{P-1} \phi^{in} z^i H_{0,i}(z^P)$$

and in particular

$$G_0(z) = \sum_{i=0}^{P-1} z^i H_{0,i}(z^P) \tag{43}$$

and

$$G_n(z) = G_0(z\phi^n).$$

Thus

$$Y(z) = \frac{1}{P} \sum_{n=0}^{P-1} G_0(z\phi^n) U(z\phi^n). \tag{44}$$

Since the output signal has a sampling interval PT, its (radian) frequency spectrum must be periodic with period $2\pi/PT$. This is confirmed by (44) which indicates that the output signal spectrum consists of a periodic repetition of $G_0(z)U(z)$ at intervals of $2\pi/PT$.

To interpret $G_0(z)$ for $z = \exp(j\omega T)$ as the frequency response, consider the response to a sampled sinusoid at the input. Let $u(k) = \sin(\omega kT)$; then the output sequence is

$$y(m) = |G_0| \sin(\omega mPT + \theta) \tag{45}$$

where $G_0(\exp[j\omega T]) = |G_0| \exp(j\theta)$. Equation (45) illustrates the interpretation of $G_0(\exp[j\omega T])$ as the multirate filter frequency response in a manner analogous to the time-invariant case. For most applications of a multirate filter the frequency response would ideally satisfy the conditions in (46). Since these conditions

$$G_0(\exp[j\omega T]) = \begin{cases} 1, & |\omega \pm 2\pi n/T| < \pi/PT \\ 0, & \text{otherwise} \end{cases} \tag{46}$$

cannot be satisfied exactly, the output sequence will contain signals with aliased frequencies when the input contains signals with frequencies in the interval $[\pi/PT, \pi/T]$. The degree to which these signals are present is determined by evaluation of $G_0(\exp[j\omega T])$ for $\omega \in [\pi/PT, \pi/T]$.

An important result of this derivation is the general expression (43) for the frequency response of a multirate filter. Notice that $G_0(z)$ is of a more general form than the frequency response for a time-invariant Nth-order filter. In particular, for an Nth-order filter, $G_0(z)$ has the form

$$G_0(zj) = \frac{a_{PN} z^{PN} + \cdots + a_1 z + a_0}{z^{PN} + b_{N-1} z^{(N-1)P} + \cdots + b_1 z^P + b_0}.$$

This suggests that new design methods should be developed for multirate filters.

A similar derivation could be made for multirate filters in which the output sampling rate is greater than the input rate. However, since such filters appear to be of less practical interest we will not consider them further. Another class of periodically time-varying filters for which a frequency response function may be defined is a class we shall call modulation filters. A modulation filter simply combines the operation of filtering and modulation (frequency translation) in a single operation. Kalet and Weinstein [18] have recently reported on an application of this type filter. Using (39), it is easy to see that if $G_n(z) = 0$ for all n except $n = n_1$ and $P - n_1$, then the input signals are filtered and translated in the frequency domain as if they had been modulated by a carrier signal of frequency $n_1 2\pi/PT$. The output spectrum is then

$$Y(z) = \frac{1}{P}[G_{n_1}(z)U(z\phi^{n_1}) + G_{P-n_1}(z)U(z\phi^{-n_1})]$$

where

$$G_{n_1}(z) = \tilde{G}(z\phi^{n_1})$$
$$G_{P-n_1}(z) = \tilde{G}(z\phi^{-n_1}).$$

The function $\tilde{G}(z)$ for $z = \exp(j\omega T)$ specifies the baseband filtering of the input signal prior to frequency translation. Hence $\tilde{G}(\exp[j\omega T])$ may be interpreted as the frequency response for the filter. Equation (42) can then be used to determine the $H_{l,i}(z)$ which result in a desired $\tilde{G}(z)$. It is evident that new techniques should be developed for the design of optimal modulation filters.

Analysis of periodically time-varying discrete systems using z-transform techniques has also been explored by Jury and Mullin [7], Friedland [10], Jury [19], Evans [12], and Davis [13]. In each of these papers, the approach taken is a variation of the basic ideas leading up to Theorem 2. However, the result of Theorem 2 and its application to specific filter types of interest, such as multirate filters, is new. A somewhat different approach was taken by Liu and Franaszek [20]. They were concerned with a more general class of time-varying filters which included periodically varying filters. Their results include an analysis of periodic filters with stochastic inputs. Previous papers [1]–[6] describing multirate filters have also included frequency domain analysis techniques. However, the multirate filters under consideration have generally been restricted to a particular realization and the analysis has assumed the ideal conditions of (46) hold.

VI. Conclusions

A block representation of periodically time-varying filters has been presented. Using this representation, we have studied the time and frequency domain responses of these filters. By considering multirate filters as a special class of periodically varying filters, we have derived an expression for the frequency response of general multirate filters. This derivation has shown that new design methods for multirate filters should be investigated. Since these results are independent of the realization chosen, we also see a need for research to determine the realizations best suited for particular applications.

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Low-Sensitivity Digital Ladder Filters

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***Abstract*—A class of low-sensitivity digital ladder filters may be realized in the voltage–current domain by direct analogy with the continuous resistively terminated *LC* ladder filter. The problem of unrealizability that is implied by delay-free loops in the discrete signal flow graph is overcome by using transformations that correspond to *LC* elements that exhibit finite Cube factors. The resultant deterioration in the passband of the transfer function is determined from the Blowstein *LC* ladder sensitivity theory and is, thereby, shown to be low valued at high sampling frequencies. The sensitivity properties of this class of digital ladder filters are directly analogous to the *LC* prototype because each *LC* element is replaced by a digital multiplier.**

Consequently, if maximum power transfer is approximately maintained throughout the passband of the *LC* prototype, then the first-order sensitivity of the corresponding digital transfer function to multiply a coefficient quantization is necessarily low valued.

I. Introduction

An outstanding practical problem that is encountered in the realization of digital filters is the requirement for highly accurate digital multipliers. The multiplier coefficients are limited in accuracy by the multiplier coefficient *wordlengths*, and from the hardware point of view, it is most important that these multipler coefficient wordlengths should be as small as possible. Consequently, there is much interest in the synthesis of digital filters where the discrete transfer function exhibits low valued sensitivities to the multiplier coefficients. The synthesis techniques that are proposed in this contribution achieve low sensitivity with respect to multiplier coefficients; these techniques allow digital ladder filters to be realized by direct analogy, in the voltage–current domain, with resistively terminated *LC* ladder filters, in such a way that the magnitude of the discrete transfer-function frequency response is highly insensitive to perturbations of the multiplier coefficients at frequencies throughout the passband of the filter. It is well known that resistively terminated *LC* filters with a "brick-wall" type of magnitude transfer-function frequency response are virtually insensitive to first-order perturbations of the *L* or *C* elements at frequencies within the passband, provided that the passband insertion loss is very close to zero [3]. This property is retained in the proposed digital filter synthesis techniques because each *L* and *C* element is directly analogous to the coefficient of a digital multiplier.

The realization of wave digital filters has been proposed by Fettweis [1], [2] and others [4]. Subsequent work [5], [6] has verified the original conjecture that wave digital filters exhibit exceptional insensitivity to digital multiplier coefficients. The basis of wave digital filter synthesis is the theory of unit elements and the analogy between a discrete (digital) filter and a transmission line equivalent. Perhaps the most essential feature leading to the development, by Fettweis, of wave concepts is the apparent failure of the conventional bilinear transformation

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