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Survey Paper

Invariant representations of discrete-time periodic systems[☆]

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The correspondence between periodic and time-invariant systems can take a variety of forms. The purpose of this survey paper is to present a complete and clear picture on time-invariant representations for discrete-time periodic systems. The core of the treatment evolves around the concept of periodic transfer function.

Abstract

In this paper we overview, compare and elaborate on the invariant representations of periodic systems. Precisely, with reference to discrete-time systems, we first introduce the concept of periodic transfer function from which a notion of generalized frequency response can be worked out. Then we discuss the following four reformulations: (i) time lifted, (ii) cyclic, (iii) frequency lifted and (iv) Fourier. A number of interesting links will be established, and many theoretical aspects somewhat overlooked in the existing literature will be clarified. All reformulations are first worked out from the input–output description and then elaborated in a state-space formalism. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Ordinary differential equations with periodic coefficients have a long history in physics and mathematics going back to the contributions of the 19th century by Faraday (1831), Mathieu (1868), Floquet (1883), Raileigh (1883), Hill (1886), and Raileigh (1886). As an intermediate class of systems bridging the time-invariant realm to the time-varying one, periodic systems are often covered as a regular chapter in text books of differential equations or dynamical systems, such as Saaty and Bram (1981), Halanay (1966), D'Angelo (1970) and Nayfeh and Mook (1979). In the second-half of the present century, the development of systems and control theory together with the achievements of digital control and signal processing have set the stage for a renewed interest in the study of periodic systems, both in continuous and discrete time, see e.g. the books Marzollo (1972), Yakubovich and Starzhinskii (1975), Vaidyanathan (1993), Crochiere and Rabiner (1993), Gardner (1994), Feuer and Goodwin (1996) and the survey papers Bittanti (1986) and Bittanti and Colaneri (1996). This has been emphasized by specific application demands specially in the aerospace realm (Johnson, 1996; McKillip, 1991; Isniewski & Blanke, 1999), computer control of industrial processes, as outlined in Bittanti and Colaneri (1999), and communication systems, (Vaidyanathan, 1990; Crochiere & Rabiner, 1993; Tong, Xu, & Kailath, 1994; Xin, Kagiwada, Sano, Tsuj & Yoshimoto, 1997).

A main tool of analysis and design exploits the natural correspondence with time-invariant systems. In this respect, one could be tempted to believe that the theory of time-invariant systems can be trivially used to solve any problem formulated in the periodic realm. This is indeed an illusory argument, in that the reformulation of a periodic system always leads to a *particular class* of time invariant models. The characteristics of these models (structure, dimensions, etc.) cannot be neglected otherwise entering a mess of nonfeasibility/noncausality issues. For instance when facing the problem of designing a

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periodic controller for a periodic system, resorting to time-invariant design tools may be useful; however, if the controller so designed has a generic structure, the backward recovering of a periodic controller may become impossible.

The correspondence between periodic and time-invariant systems can take a variety of forms, each of which is useful in a specific context. The purpose of this paper is to present a complete and clear picture on time-invariant representations for discrete-time periodic systems. To achieve such an objective, however, many aspects previously unexplored in the literature need to be clarified and elaborated upon. Among these, the issue of introducing the notion of *periodic transfer function* on a solid mathematical ground is first encountered. From this, one can work out a frequency-domain characterization of a periodic system. This calls for the concept of *exponentially modulated periodic* (EMP) signal, which plays for periodic systems the same role played by exponential signals in the time-invariant case. Indeed, the frequency-domain description of periodic systems captures their behaviour in an EMP regime. Then, we will be in a position to present the so-called *time-lifted*, *cyclic* and *frequency-lifted* reformulations.

The overview will be carried on with the constant effort of relating each reformulation with the periodic transfer function. In so doing, the inter-relations among the reformulations will be enlightened.

The time-lifted reformulation is probably the most classical one. The operation of lifting consists in packing the values of a signal over one period in a new enlarged signal. The lifted reformulation relates the lifted external signals of the original system. It traces back to Krank (1957), Jury and Mullin (1959) and Meyer and Burrus (1975) and it has often used ever since. One can encounter it in the definitions of periodic zeros (Bolzern, Colaneri & Scattolini, 1986; Grasselli & Longhi 1988), in the parametrization of stabilizing controllers (Freudenberg & Grizzle, 1989), in optimal control design (Dahleh, Vulgaris & Valavani, 1992), in control performance improvement (Khargonekar, Poolla & Tannembaum, 1985) and so on and so forth.

The operation of cycling amounts to picking up a once-per-period sample of a signal whose position in an augmented vector shifts as time proceeds. This operation applied to the external variables of the periodic dynamical system (Park & Verriest, 1989; Flamm, 1991) brings to a time-invariant dynamics, which is useful in a variety of contexts, e.g. in the model-matching problem (Colaneri & Kucera, 1997).

More recently a frequency-domain characterization of lifting has been introduced leading to the so-called frequency-lifted reformulation. The idea stems from the fact that, in an EMP regime, the number of possible harmonics compatible with a periodic discrete-time system coincides with the system period. Collecting all harmonics of a signal in a vector is at the basis of the frequency-lifted reformulation, as explained in Zhang, Zhang and Furuta (1996). When considering an EMP regime in the state-space context, this reformulation corresponds to taking the Fourier expansion of the periodic system matrices. In this way it is possible to work out a Fourier representation, expressing the "gain" between the input–output harmonics of the EMP regime. The frequency-lifted representation has its roots in a long standing idea in the field, see e.g. D'Angelo (1970) for a classical reference, and has been recognized to be particularly useful in applications, as reported in Wereley and Hall (1990), Bittanti and Lovera (1996) and Arcara, Bittanti and Lovera (2000).

The paper is intended to provide a thorough survey covering all reformulations, both in an input–output and in a state-space viewpoint. However, many are the issues encountered along this road which can be considered as novel contributions. Among them the way in which the periodic transfer function determines the structure of all reformulations in the input–output framework. Moreover, to the best knowledge of the authors, the equivalence between the frequency-lifted and the Fourier representation is stated and proven here for the first time.

The paper is organized as follows. In Section 2, we first introduce the basic input-output and state-space models for periodic systems and then we define the periodic transfer function. The notion of EMP regime and the corresponding frequency-domain interpretation is provided in Section 3. Then the time-lifted, cyclic and frequency-lifted reformulations are the subject of Sections 4, 5 and 6, respectively. The paper ends with some final considerations.

2. State-space and input–output models for periodic systems

The state-space description of a periodic system in discrete time ($t \in Z$) is

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{1}$$

$$y(t) = C(t)x(t) + D(t)u(t), \tag{2}$$

where $x(t) \in R^n$ is the state variable, $y(t) \in R^p$ is the output signal and $u(t) \in R^m$ is the input vector. Matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are periodic of period T.

The stability of the system can be assessed by analyzing the so-called *monodromy matrix*. Precisely, let $\Phi_A(t,\tau)$ $(t \geq \tau)$ be the system state-transition matrix, i.e.

$$\Phi_A(t,\tau) = \begin{cases} A(t-1)A(t-2)\cdots A(\tau), & t > \tau \\ I, & t = \tau \end{cases}$$

The monodromy matrix is defined as the transition matrix over one period, e.g. $[\tau, \tau+T-1]$, and is denoted by $\Psi_A(\tau) = \Phi_A(\tau+T,\tau)$. Its eigenvalues do not depend upon τ (Bittanti, 1986) and are named *characteristic multipliers*. The system is stable iff the characteristic multipliers lie in the open unit disk.

In the same way as for time-invariant systems, the external properties of a periodic system can also be studied in a fully input–output context. The basic causal relationships supplies the output $y(t)$ as a linear combination of past values of the input up to time t:

$$y(t) = M_0(t)u(t) + M_1(t)u(t-1) + M_2(t)u(t-2) + M_3(t)u(t-3) + \cdots = \sum_{j=0}^{\infty} M_j(t)u(t-j). \quad (3)$$

The matrix coefficients $M_i(t)$, $i = 0, 1, \ldots$, are T-periodic functions, known as *periodic Markov coefficients*. This Markov parameters are linked to the impulsive response of the system in a way that can be easily assessed by making reference to the simple case where the input is a scalar variable ($m = 1$). Indeed, denoting by $\delta(t)$ the impulsive function, i.e.

$$\delta(t) = \begin{cases} 1, & t = 0, \\ 0, & \text{otherwise} \end{cases}$$

and by $y^{(i)}_{\mathrm{imp}}(t)$ the response of the system at the impulsive input $u(t) = \delta(t-i)$, it follows from (3) that

$$y^{(i)}_{\mathrm{imp}}(t) = M_{t-i}(t).$$

In the general multi-input case, the jth column of the Markov coefficients $M_{t-i}(t)$ represents the output response of the system to an impulse applied at time i to the jth component of the input vector.

The periodicity of the Markov coefficients entails that the output response of the system at a generic time instant $t = kT + s$, $s \in [0, T-1]$, can be written as finite sum of the output responses of T time invariant systems indexed in the integer s. As a matter of fact, consider again Eq. (3) and evaluate $y(t)$ in $t = kT + s$. It follows

$$y(kT+s) = \sum_{i=0}^{T-1} \hat{y}_{i,s}(k), \quad (4)$$

where

$$\hat{y}_{i,s}(k) = \sum_{j=0}^{\infty} M_{jT+i}(s)\hat{u}_{i,s}(k-j) \quad (5)$$

and

$$\hat{u}_{i,s}(k) = u(kT+s-i). \quad (6)$$

From these expressions, it is apparent that $\hat{y}_{i,s}(k)$ is the output of a time-invariant system having $M_i(s)$, $M_{i+T}(s)$, $M_{i+2T}(s), \ldots$, as Markov parameters. Note the role of the different indexes appearing in these expressions: s is the chosen tag time index for the output variable, $s-i$ is the tag time index for the input variable, and k is the sampled time current variable. For each $i \in [0, T-1]$ and $s \in [0, T-1]$, one can define

$$H_i(z,s) = \sum_{j=0}^{\infty} M_{jT+i}(s)z^{-j}. \quad (7)$$

This is the transfer function from $\hat{u}_{i,s}(k)$ to $\hat{y}_{i,s}(k)$ (both signals seen as function of k). By using z as the one-shift ahead operator in time k (namely, the T-shift operator in time t), and resorting to a mixed z/k notation, one can write

$$\begin{aligned} y(kT+s) = {} & H_0(z,s)u(kT+s) \\ & + H_1(z,s)u(kT+s-1) + \cdots \\ & + H_{T-2}(z,s)u(kT+s-T+2) \\ & + H_{T-1}(z,s)u(kT+s-T+1) \end{aligned} \quad (8)$$

or, equivalently

$$\begin{aligned} y(kT+s) = {} & H_0(z,s)u(kT+s) \\ & + H_{T-1}(z,s)z^{-1}u(kT+s+1) + \cdots \\ & + H_2(z,s)z^{-1}u(kT+s+T-2) \\ & + H_1(z,s)z^{-1}u(kT+s+T-1). \end{aligned} \quad (9)$$

The function $H_i(z,s)$ will be referred to as the *sampled transfer function* at tag time s with input–output delay i.

The input–output periodic model is said to be *rational* if all transfer functions (7) are indeed rational, i.e. they are transfer functions of finite-dimensional (time-invariant) systems. In this case, the periodic system can be given a state-space finite-dimensional realization, as shown in Colaneri and Longhi (1995). The problem of finding a minimal periodic realization in the form of difference equation is investigated in Bittanti, Bolzern and Guardabassi (1985) and Kuijper (1999).

By resorting to the unit delay operator σ^{-1}, expression (3) can be given a compact form

$$y(t) = G(\sigma,\cdot)|_t \; u(t),$$

where the *input–output transfer operator* $G(\sigma,\cdot)|_t$ is defined as

$$G(\sigma,\cdot)|_t = M_0(t) + M_1(t)\sigma^{-1} + M_2(t)\sigma^{-2} + M_3(t)\sigma^{-3} + \cdots. \quad (10)$$

The operator $G(\sigma,\cdot)|_t$ is periodic:

$$G(\sigma,\cdot)|_{t+T} = G(\sigma,\cdot)|_t, \quad \forall t.$$

Moreover, it enjoys a pseudo-commutative property with respect to the delays. Precisely,

$$\sigma^{-k}G(\sigma,\cdot)|_t = G(\sigma,\cdot)|_{t-k}\sigma^{-k}, \quad \forall t. \quad (11)$$

For instance, one can write

$$G(\sigma,\cdot)|_t = M_0(t) + \sigma^{-1}M_1(t+1) + \sigma^{-2}M_2(t+2) + \sigma^{-3}M_3(t+3) + \cdots$$

in place of (10). Hence, the operators σ^{-k} and $G(\sigma,\cdot)$ do commute if and only if the integer k is multiple of the period T.

If the transfer operator is evaluated in a specific time-point, say t, it results in a transfer function, henceforth denoted by $G(\sigma,t)$. In view of (7), such transfer function can be written as

$$G(\sigma,t)=\sum_{k=0}^{\infty}M_k(t)\sigma^{-k}=\sum_{i=0}^{T-1}H_i(\sigma^T,t)\sigma^{-i}. \tag{12}$$

We now introduce the symbol ϕ which will be often used in the paper:

$$\phi=\exp\left(\frac{2\pi \mathrm{i}}{T}\right)=\cos\left(\frac{2\pi}{T}\right)+\mathrm{i}\sin\left(\frac{2\pi}{T}\right). \tag{13}$$

Hence $1,\phi,\phi^2,\dots,\phi^{T-1}$ are the T-roots of the unit. Note that

$$\frac{1}{T}\sum_{k=0}^{T-1}\phi^{sk}=\begin{cases}1, & s=0,\pm T,\pm 2T,\dots,\\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

Then, it is easy to see that one can recover the sampled transfer functions $H_i(\sigma^T,t)$ from the transfer function $G(\sigma,t)$ as follows:

$$H_i(\sigma^T,t)=\frac{1}{T}\left[\sum_{k=0}^{T-1}G(\sigma\phi^k,t)\phi^{ki}\right]\sigma^i. \tag{15}$$

The converse formula is given by Eq. (12). The transfer operator plays a major role in the characterization of the properties of periodic systems, such as the notion of delay as discussed in Bittanti, Colaneri and Mongiovi (1998). A different pioneer attempt of characterizing periodic systems in an input–output operator form can be found in Verriest (1988). For general time-varying systems, see Kamen, Khargonekar and Poolla (1985).

If the original periodic system (1), (2) is indeed time-invariant (i.e. $T=1$ and hence $\sigma=z$), the transfer function operator $G(\sigma,t)$ does not depend on t and $G(\sigma,t)$ actually reduces to the transfer function $G(\sigma)$ of the time-invariant system, i.e.

$$G(\sigma,t)=G(\sigma),\quad \forall t.$$

Remark 1. If a state-space representation (1), (2) is available, it is possible to express the transfer functions $H_i(z,t)$ in terms of the system matrices. Indeed the Markov coefficients are determined as

$$M_0(t)=D(t),\qquad M_1(t)=C(t)B(t-1),$$

$$M_2(t)=C(t)A(t-1)B(t-2),$$

$$M_3(t)=C(t)A(t-1)A(t-2)B(t-3),\dots\ .$$

Owing to periodicity, the above expressions can be more concisely written as

$$\begin{aligned}&M_0(t)=D(t),\\ &\mathrm{M}_{jT+i}(t)=C(t)\Psi_A(t)^j\Phi_A(t,t-i+1)B(t-i),\end{aligned} \tag{16}$$

where $j=0,1,2,\dots$ and $i\in[1,T]$. Therefore, a simple computation shows that

$$H_0(z,t)=D(t)+C(t)(zI-\Psi_A(t))^{-1}\Phi_A(t,t-T+1)B(t), \tag{17}$$

$$H_i(z,t)=zC(t)(zI-\Psi_A(t))^{-1}\Phi_A(t,t-i+1)B(t-i),\quad i\in[1,T-1]. \tag{18}$$

Hence, all $H_i(z,t)$ are rational functions of z. From expressions (12), (17) and (18), the transfer function operator $G(\sigma,\cdot)$ can be given the compact form

$$G(\sigma,\cdot)|_t=D(t)+C(t)(\sigma^TI-\Psi_A(t))^{-1}\mathscr{B}(\sigma,\cdot)|_t, \tag{19}$$

$$\mathscr{B}(\sigma,\cdot)|_t=\sum_{j=0}^{T-1}\Phi_A(t,t+j-T+1)B(t+j)\sigma^j. \tag{20}$$

This expression is reminiscent of the classical formula of a transfer function of a time-invariant system, the only difference being that the input matrix is replaced by a polynomial operator of σ-powers up to $T-1$. The symbol $\mathscr{B}(\sigma,t)$ will be used to denote the associated polynomial matrix.

Example 2. Consider system (1), (2) with $T=2$, $D(0)=D(1)=0$ and

$$A(t)=\begin{cases}2, & t=0,\\ -5, & t=1,\end{cases}\qquad B(t)=\begin{cases}1, & t=0,\\ -2, & t=1,\end{cases}$$

$$C(t)=\begin{cases}0.5, & t=0,\\ 3, & t=1.\end{cases}$$

From (12), (17), (18), it follows that

$$H_0(z,0)=\frac{-2.5}{z+10},\qquad H_1(z,0)=\frac{-z}{z+10},$$

so that

$$G(\sigma,0)=\frac{-2.5-\sigma}{\sigma^2+10}.$$

Moreover,

$$H_0(z,1)=\frac{-12}{z+10},\qquad H_1(z,1)=\frac{3z}{z+10},$$

so that

$$G(\sigma,1)=\frac{-12+3\sigma}{\sigma^2+10}.$$

Alternatively $G(\sigma,t)$ can be computed on the basis of expression (19) by noting that, in view of (20)

$$\mathscr{B}(\sigma,t)=\begin{cases}-5-2\sigma, & t=0,\\ -4+\sigma, & t=1.\end{cases}$$

In conclusion, a periodic system can be described by means of two possible time-domain representations:

either state-space or input–output form (not to mention the recently introduced behavioral representation (Kuijper & Willems, 1997)). An alternative is to resort to the transfer function operator $G(\sigma,\cdot)$. The associated transfer function $G(\sigma,t)$ can be seen as the composition of the T transfer functions $H_i(z,t)$ of suitable time-invariant systems. If these transfer functions $H_i(z,t)$ are rational, then the periodic system is rational as well, and its input–output properties are specified by T rational functions, namely

$$\{G(\sigma,t),\ t=0,1,\dots,T-1\}.$$

Remark 3. In order to avoid ambiguous interpretations, it is useful to point out that the symbol σ is here used to represent both the one-shift ahead operator and a complex variable. Precisely, when σ is applied to operator $G(\sigma,\cdot)|_t$, then it takes an operator meaning, namely $\sigma\ G(\sigma,\cdot)|_t = G(\sigma,\cdot)|_{t+1}\sigma$. Conversely, when σ is applied to a transfer function, then its meaning as an operator is lost. This is why, in the above example, $\sigma G(\sigma,0) \neq G(\sigma,1)\sigma$. Precisely, when multiplied by a transfer function, σ must be interpreted as a complex number and one can legally write, e.g. $\sigma G(\sigma,0)=G(\sigma,0)\sigma$. We leave to the context the task of clarifying the role of σ in the various situations.

3. Spectral properties of the periodic transfer operator

The spectral properties of periodic systems can be worked out by making reference to the concept of *exponentially modulated periodic* (EMP) signal. A signal $v(\cdot)$ is said to be EMP if there exists a (complex) number $\lambda \neq 0$ such that

$$v(t+kT)=v(t)\lambda^{kT}, \qquad t\in[\tau,\tau+T-1].$$

Obviously, the class of periodic signals is a subset of the class of EMP signals since the above condition defines a periodic function when λ is set to 1 or to any Tth root of 1.

Notice that if $v(\cdot)$ is EMP relative to a complex number λ, $\lambda\neq 0$, then $\bar{v}(t)=v(t)\lambda^{-t}$ is T-periodic. Conversely, if $\bar{v}(\cdot)$ is T-periodic, then, for each nonnull λ, $v(t)=\bar{v}(t)\lambda^{t}$ is EMP. This last conclusion entails that any EMP signal can be written as

$$v(t)=\sum_{k=0}^{T-1}\bar{v}_k\phi^{kt}\lambda^{t}, \tag{21}$$

where $\bar{v}_k$ are the coefficients of the Fourier expansion of the periodic signal $\bar{v}(t)=v(t)\lambda^{-t}$. Expression (21) will be referred to as the *EMP Fourier expansion*.

Suppose to feed the system in the input–output representation (9) with the EMP input $u(t)$ characterized by the condition $u(t+kT)=u(t)\lambda^{kT}$, $t\in[0,T-1]$ and assume that all the sampled transfer functions $H_i(z,t)$ are well defined in $z=\lambda^T$ for each t. Note that $H_i(z,t)$ represents a time-invariant system operating over the sampled time-variable k. Over the k-axis, λ^{kT} is seen as the exponential μ^k with $\mu=\lambda^T$. On the other hand, an exponential signal μ^k, fed into a time-invariant system, leads, in an exponential regime, to an output obtainable by evaluating the transfer function in $z=\mu$. Thus,

$$H_i(z,t+r)u(kT+t+r)=H_i(\lambda^T,t)u(t+r)\lambda^{kT},$$
$$\forall r\in[0,T-1].$$

Consequently, from (9), for each $t=0,1,\dots,T-1$, it follows that

$$\begin{aligned}
y(t+kT) &= \sum_{i=0}^{T-1} H_i(z,t)u(t+kT-i)\\
&= H_0(z,t)u(t+kT)\\
&\quad + \left[\sum_{i=1}^{T-1} H_i(z,t)z^{-1}u(t+kT+T-i)\right]\\
&= H_0(\lambda^T,t)\lambda^{kT}u(t)\\
&\quad + \left[\sum_{i=1}^{T-1} H_i(\lambda^T,t)\lambda^{kT-T}u(t+T-i)\right]\\
&= \left[H_0(\lambda^T,t)+\left[\sum_{i=1}^{T-1} H_i(\lambda^T,t)\lambda^{-T}\sigma^{T-i}\right]u(t)\right]\lambda^{kT}.
\end{aligned}$$

In an EMP regime, it suffices to set $y(t+kT)=y(t)\lambda^{kT}$. Hence in such a regime, the values of the input and output over one period are related by

$$y(t)=G_\lambda(\sigma,\cdot)|_t u(t), \tag{22}$$

where

$$G_\lambda(\sigma,t)=H_0(\lambda^T,t)+\sum_{i=1}^{T-1}H_i(\lambda^T,t)\lambda^{-T}\sigma^{T-i}. \tag{23}$$

This defines an input–output periodic operator $G_\lambda(\sigma,\cdot)|_t$, mapping $u(i)$, $i\in[t,t+T-1]$ into $y(i)$, $i\in[t,t+T-1]$, which will be named *EMP operator*.

The same operator can be introduced starting from the state-space description. Indeed, consider the periodic system (1) and (2) fed by the EMP input function

$$u(t+kT)=u(t)\lambda^{kT}, \qquad t\in[\tau,\tau+T-1].$$

Assume that no characteristic multiplier of $A(\cdot)$ coincides with λ^T. Then the initial state

$$x_\lambda(\tau)=(\lambda^T I-\Psi_A(\tau))^{-1}\sum_{i=\tau}^{\tau+T-1}\Phi_A(\tau+T,i+1)B(i)u(i) \tag{24}$$

is such that both the state and the corresponding output are still EMP signals, i.e.

$$x(t+kT)=x(t)\lambda^{kT}, \qquad t\in[\tau,\tau+T-1],$$
$$y(t+kT)=y(t)\lambda^{kT}, \qquad t\in[\tau,\tau+T-1].$$

Note that, for any $t \in [\tau, \tau + T - 1]$, the EMP output $y(t)$ can be written as

$$y(t) = C(t)\Phi_A(t,\tau)x_\lambda(\tau) + C(t)\sum_{i=\tau}^{t-1} \Phi_A(t, i+1)B(i)u(i) + D(t)u(t).$$

Since $x_\lambda(\tau)$ is given by (24) and the following property holds true:

$$\Phi_A(t,\tau)(\lambda^T I - \Psi_A(\tau))^{-1} = (\lambda^T I - \Psi_A(t))^{-1}\Phi_A(t,\tau),$$

a cumbersome computation shows that $y(\cdot)$ can be equivalently rewritten as

$$y(t) = C(t)(\lambda^T I - \Psi_A(t))^{-1} \times \sum_{i=t}^{t+T-1} \Phi_A(t+T, i+1)B(i)u(i) + D(t)u(t).$$

This expression can be given the compact operator form (22) with

$$\begin{aligned} G_\lambda(\sigma,\cdot)|_t &= D(t) + C(t)(\lambda^T I - \Psi_A(t))^{-1} \times \sum_{i=0}^{T-1} \Phi_A(t+T, t+i+1)B(i+t)\sigma^i \\ &= D(t) + C(t)(\lambda^T I - \Psi_A(t))^{-1}\mathscr{B}(\sigma,\cdot)|_t. \end{aligned} \tag{25}$$

By taking into account (17) and (18), it is easy to see that expressions (23) and (25) coincide.

If the system is time invariant with transfer function $G(\sigma)$, by letting $T = 1$, it is easy to see that $G_\lambda(\sigma, t) = G(\lambda)$, which is indeed the gain between the input and output "amplitudes" $\bar{u}$-$\bar{y}$.

Along the same vain, it is worth pointing out the relationship between the transfer operator $G(\sigma,\cdot)|_t$ and the EMP operator $G_\lambda(\sigma,\cdot)|_t$. A direct inspection leads to the conclusion that

$$G(\sigma,\cdot)|_t = G_\sigma(\sigma,\cdot)|_t. \tag{26}$$

Notice that $G_\lambda(\sigma, t)$ when evaluated in $\lambda = \sigma$ becomes a causal rational function in σ.

Remark 4. The operator $G_\lambda(\sigma,\cdot)|_t$, above obtained following both state-space and input–output considerations, is noncausal. Indeed the value of the EMP regime output signal at time t depends upon the values taken by the EMP input function over the whole period. Correspondingly, $G_\lambda(\sigma, t)$ is a polynomial function in powers of σ up to $T - 1$. Since $u(\cdot)$ is EMP, and the system matrices are periodic, it is easy to see that the EMP output signal at time t can be written as a function of past inputs from time $t - T$ up to time t, i.e.

$$y(t) = P_\lambda(\sigma,\cdot)|_t u(t),$$

where

$$\begin{aligned} P_\lambda(\sigma,\cdot)|_t &= D(t) + C(t)(I - \Psi_A(t)\lambda^{-T})^{-1} \times \sum_{i=0}^{T-1} \Phi_A(t+T, t+i+1)B(i+t)\sigma^{i-T} \\ &= D(t) + C(t)(I - \Psi_A(t)\lambda^{-T})^{-1}\sigma^{-T}\mathscr{B}(\sigma,\cdot)|_t. \end{aligned} \tag{27}$$

This new operator is causal, since it is a polynomial function in the delays σ^{-i}, $i = 0, 1, \ldots, T$.

Remark 5. If one makes reference to operator $P_\lambda(\sigma,\cdot)$, then again it follows:

$$G(\sigma,\cdot)|_t = P_\sigma(\sigma,\cdot)|_t.$$

The EMP transfer operator plays in the periodic realm the role that the transfer function plays for time-invariant systems. Indeed, consider, for each $t \in [\tau, \tau + T - 1]$, the formal discrete-time series $\{u(t + kT)\}$ and $\{y(t + kT)\}$, obtained by uniformly sampling (with tag t) the input and output signals, and define the formal series

$$U(z,t) = \sum_{k=-\infty}^{\infty} u(t+kT)z^{-k},$$

$$Y(z,t) = \sum_{k=-\infty}^{\infty} y(t+kT)z^{-k}.$$

Note that this transformed signals are EMP signals in t, since $Y(z^T, t + kT) = Y(z^T, t)z^{kT}$, $t \in [\tau, \tau + T - 1]$ and analogously for $U(z^T, t)$. Hence, Eq. (22) can be used yielding for $t \in [0, T - 1]$

$$Y(z^T, t) = G_z(\sigma,\cdot)|_t U(z^T, t).$$

The above equation shows that $G_z(\sigma,\cdot)|_t$ is a proper generalization of the "transfer function at t" since maps the input z-transform $U(z^T, t)$ into the z-transform of the forced output $Y(z^T, t)$. Of course, when the system is time-invariant ($T = 1$) with transfer function $G(\sigma)$, it follows that $G_z(\sigma, t) = G(\sigma)$.

Example 6. Consider again the system of Example 2. By comparing Eq. (20) with (25) and (27) it easily follows:

$$G_\lambda(\sigma, 0) = \frac{-2.5 - \sigma}{\lambda^2 + 10}, \qquad G_\lambda(\sigma, 1) = \frac{-12 - 3\sigma}{\lambda^2 + 10},$$

and

$$P_\lambda(\sigma, 0) = \frac{-2.5\sigma^{-2} - \sigma^{-1}}{1 + 10\lambda^{-2}}, \quad P_\lambda(\sigma, 1) = \frac{-12\sigma^{-2} - 3\sigma^{-1}}{1 + \lambda^{-2}}.$$

4. Time-lifted reformulation

In this section we introduce the most classical time-invariant reformulation, namely the lifted reformulation.

The simplest way to explain the idea of lifting is to make reference to a (discrete-time) signal, say $v(\cdot)$. Associated with $v(\cdot)$ one can introduce the augmented signal

$$v_\tau(k) = [v(kT+\tau)'v(kT+\tau+1)' \dots v(kT+\tau+T-1)']', \tag{28}$$

where τ is a tag point, arbitrarily chosen. We will also use the symbol $v^{(i)}(k)$ for the components of vector $v_\tau(k)$, i.e. $v^{(i)}(k) = v(kT+\tau+i-1)$, $i = 1, 2, \dots, T-1$. As apparent, $v_\tau(k)$ is constituted by the samples of the original signal over the interval $[\tau, \tau+T-1]$, T being any positive integer. For a proper comprehension of the way in which signal lifting reflects into the input/output representation of a system, we start with by considering the effect of sampling in the z-domain. Then, we will pass to analyze the action of lifting on time-invariant and T-periodic systems, both in state-space and input/output forms.

4.1. Sampling in the z-domain

Again, consider a discrete-time signal $v(\cdot)$, whose z-transform is given by the celebrated formula (in the symbol σ)

$$V(\sigma) = \sum_{t=0}^{\infty} v(t)\sigma^{-t}.$$

Hence, the z-transform of the periodically sampled signal

$$v^{(i)}(k) = v(kT+i-1), \quad i = 1, 2, \dots, T$$

is given by

$$V^{(i)}(z) = \sum_{k=0}^{\infty} v(kT+i-1)z^{-k}.$$

As can be seen in the expressions above, we use different symbols for the complex variables entering the z-transforms: σ for time t and z for the sampled time k. In this way z corresponds to σ^T.

Now, evaluating the last expression in $z = \sigma^T$ and denoting by $\delta(t)$ the Kronecker symbol, i.e. $\delta(0) = 1$, $\delta(t) = 0$, $\forall t \neq 0$, it is easy to see that

$$\begin{aligned} V^{(i)}(\sigma^T) &= \sum_{k=0}^{\infty} v(kT+i-1)\sigma^{-kT} \\ &= \left[\sum_{k=0}^{\infty}\sum_{t=0}^{\infty} v(t)\delta(t-kT-i+1)\sigma^{-t}\right]\sigma^{i-1} \\ &= \left[\sum_{t=0}^{\infty} v(t)f_i(t)\sigma^{-t}\right]\sigma^{i-1}, \end{aligned} \tag{29}$$

where

$$f_i(t) = \sum_{k=0}^{\infty} \delta(t-kT-i+1) \tag{30}$$

is a T-periodic sequence of discrete pulses. Now, recall definition (13) of the T-roots of the unit and the associated property (14). It turns out that

$$f_i(t) = \frac{1}{T}\sum_{k=0}^{T-1} \phi^{k(i-1-t)}.$$

Note in passing that this expression is just the Fourier sum of the periodic function $f_i(t)$. From (29) and (30) it follows that

$$\begin{aligned} V^{(i)}(\sigma^T) &= \frac{1}{T}\left[\sum_{t=0}^{\infty} v(t)\sigma^{-t}\sum_{k=0}^{T-1}\phi^{k(i-1)}\phi^{-kt}\right]\sigma^{i-1} \\ &= \frac{1}{T}\left[\sum_{k=0}^{T-1}\phi^{k(i-1)}\sum_{t=0}^{\infty} v(t)(\sigma\phi^k)^{-t}\right]\sigma^{i-1}. \end{aligned}$$

In conclusion, it turns out that

$$V^{(i)}(\sigma^T) = \frac{1}{T}\left[\sum_{k=0}^{T-1} V(\sigma\phi^k)\phi^{k(i-1)}\right]\sigma^{i-1}. \tag{31}$$

Conversely, the z-transform $V(\sigma)$ of the original signal $v(t)$ can be recovered from the z-transform $V^{(i)}(z)$ of the sampled signals $v(kT+i)$. Indeed

$$\begin{aligned} V(\sigma) &= \sum_{t=0}^{\infty} v(t)\sigma^{-t} \\ &= \sum_{i=0}^{\infty}\sum_{k=0}^{T-1} v(iT+k)\sigma^{-iT-k} \\ &= \sum_{k=0}^{T-1}\sigma^{-k}\sum_{i=0}^{\infty} v(iT+k)\sigma^{-iT}. \end{aligned}$$

Therefore the converse relation is

$$V(\sigma) = \sum_{k=1}^{T} V^{(k)}(\sigma^T)\sigma^{-k+1}. \tag{32}$$

4.2. Lifting a time-invariant system

Consider now a time-invariant system with transfer function $G(\sigma)$:

$$Y(\sigma) = G(\sigma)U(\sigma).$$

If one performs a uniform sampling of the input and output signals with sampling interval T, the relationship between the z-transform $U^{(j)}(z)$ of the sampled input and the z-transform $Y^{(i)}(z)$ of the sampled output can be worked out by referring to expressions (31) and (32). It follows that

$$\begin{aligned} Y^{(i)}(\sigma^T) &= \frac{1}{T}\left[\sum_{k=0}^{T-1} G(\sigma\phi^k)U(\sigma\phi^k)\phi^{k(i-1)}\right]\sigma^{i-1} \\ &= \frac{1}{T}\left[\sum_{k=0}^{T-1} G(\sigma\phi^k)\phi^{k(i-1)}\sum_{r=1}^{T} U^{(r)}(\sigma^T)(\sigma\phi^k)^{-r+1}\right]\sigma^{i-1} \\ &= \sum_{r=1}^{T}\left[\frac{1}{T}\sum_{k=0}^{T-1} G(\sigma\phi^k)\phi^{k(i-r)}\sigma^{i-r}\right]U^{(r)}(\sigma^T). \end{aligned} \tag{33}$$

A moment's reflection shows that the term in the square brackets in this expression is a rational function in the variable σ^T and depends on the difference $i-r$ only. In view of (33), the transfer function from the rth input channel to the ith output channel turns out to be $G^{(i-r)}(\sigma)$, where

$$G^{(s)}(\sigma^T) = \frac{1}{T}\left[\sum_{k=0}^{T-1} G(\sigma\phi^k)\phi^{ks}\right]\sigma^s. \tag{34}$$

Note that

$$G^{(s+T)}(\sigma^T) = \sigma^T G^{(s)}(\sigma^T). \tag{35}$$

By taking in Eq. (33) i and r spanning the interval $[1, T]$, one can work out the transfer function from the lifted input signal $u_0(k)$ to the lifted output signal $y_0(k)$. This will be called the *lifted transfer function* at time 0 and will be denoted by $W_0(z)$. In view of expression (33), it is apparent that the (i,j) block of $W_0(z)$ depends on the difference $j-i$ only. Therefore, taking also into account Eq. (35), it follows that

$$W_0(z) =$$

$$\begin{bmatrix} G^{(0)}(z) & z^{-1}G^{(T-1)}(z) & z^{-1}G^{(T-2)}(z) & \cdots & z^{-1}G^{(1)}(z) \\ G^{(1)}(z) & G^{(0)}(z) & z^{-1}G^{(T-1)}(z) & \cdots & z^{-1}G^{(2)}(z) \\ G^{(2)}(z) & G^{(1)}(z) & G^{(0)}(z) & \cdots & z^{-1}G^{(3)}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G^{(T-1)}(z) & G^{(T-2)}(z) & G^{(T-2)}(z) & \cdots & G^{(0)}(z) \end{bmatrix}. \tag{36}$$

Interestingly enough, the link between $u_\tau(k)$ and $y_\tau(k)$ coincides with that between $u_0(k)$ and $y_0(k)$, for any integer τ. In other words, the transfer function $W_\tau(z)$ from $u_\tau(k)$ to $y_\tau(k)$ is such that

$$W_\tau(z) = W_0(z), \quad \forall \tau. \tag{37}$$

This can be easily assessed by observing that, due to the time invariance of the system, the transfer function from $u(kT+\tau+i)$ to $y(kT+\tau+j)$ depends on the time lag $j-i$ only, and, as such, it is not affected by the specific value of τ.

The lifting procedure outlined above leads to an enlarged transfer function which presents a peculiar structure, as can be seen from (36). The original transfer function $G(\sigma)$ can be recovered from the T sampled transfer functions by means of the inverse formula

$$G(\sigma) = \sum_{j=0}^{T-1} G^{(j)}(\sigma^T)\sigma^{-j}. \tag{38}$$

In the telecommunication realm, the transfer functions $G^{(s)}(\sigma)$ are named polyphase components of $G(\sigma)$ and expression (38) is referred to as the polyphase decomposition, see e.g. Crochiere and Rabiner (1993) and Vaidyanathan (1990).

4.3. Lifting a periodic system in the input–output framework

The lifted reformulation of a periodic system is based on the obvious consideration that, as already seen in Section 2, a periodic system has an invariant behavior with respect to time shifts multiple of the period. Therefore if one performs a uniform sampling of the input at time $i, i+T, i+2T$, etc., and of the output signals at $j, j+T, j+2T$, etc., the dynamics underlying these sampled signals is time invariant. Of course, such a dynamics depends on the particular pair of time instant i,j. The lifted reformulation is just the collection of all these time-invariant dynamics for i,j ranging in the period. If one considers the generic tag instant τ, the *lifted transfer function* at τ has the form

$$W_\tau(z) = \begin{bmatrix} [W_\tau(z)]_{1,1} & [W_\tau(z)]_{1,2} & \cdots & [W_\tau(z)]_{1,T} \\ [W_\tau(z)]_{2,1} & [W_\tau(z)]_{2,2} & \cdots & [W_\tau(z)]_{2,T} \\ \cdots & \cdots & \ddots & \cdots \\ [W_\tau(z)]_{T,1} & [W_\tau(z)]_{T,2} & \cdots & [W_\tau(z)]_{T,T} \end{bmatrix},$$

where the generic block entry $[W_\tau(z)]_{i,j}$ is the transfer function from the sampled input $u(kT+\tau+i-1)$ to the sampled output $y(kT+\tau+j-1)$. Obviously matrix $W_\tau(z)$ is of dimension $pT \times mT$.

It is important to point out that there is a simple recursive formula to pass from $W_\tau(z)$ to $W_{\tau+1}(z)$. This can be seen, for instance, by the very definition of the packed input $u_\tau(k)$ and packed output $y_\tau(k)$. Indeed, resorting again to the symbol z as a one shift ahead operator in the variable k (one period shift ahead operator in the variable t), it follows that

$$\begin{bmatrix} u(kT+\tau) \\ u(kT+\tau+1) \\ u(kT+\tau+2) \\ \vdots \\ u(kT+\tau+T-1) \end{bmatrix} =$$

$$\times \begin{bmatrix} 0 & \cdots & 0 & 0 & z^{-1}I_m \\ I_m & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_m & 0 & 0 \\ 0 & \cdots & 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} u(kT+\tau+1) \\ u(kT+\tau+2) \\ u(kT+\tau+3) \\ \vdots \\ u(kT+\tau+T) \end{bmatrix},$$

$$\begin{bmatrix} y(kT+\tau+1) \\ y(kT+\tau+2) \\ y(kT+\tau+3) \\ \vdots \\ y(kT+\tau+T) \end{bmatrix} =$$

$$\times \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ zI_p & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y(kT+\tau) \\ y(kT+\tau+1) \\ y(kT+\tau+2) \\ \vdots \\ y(kT+\tau+T-1) \end{bmatrix}.$$

Thus, defining

$$\Delta_k(z) = \begin{bmatrix} 0 & z^{-1}I_k \\ I_{k(T-1)} & 0 \end{bmatrix}, \tag{39}$$

the above relationships can be rewritten as

$$u_\tau(k) = \Delta_m(z)u_{\tau+1}(k), \qquad y_{\tau+1}(k) = \Delta_p(z^{-1})' y_\tau(k).$$

Therefore, considering that $y_\tau(k) = W_\tau(z)u_\tau(k)$ and $y_{\tau+1}(k) = W_{\tau+1}(z)u_{\tau+1}(k)$, it is possible to work out a recursion for the lifted transfer functions as

$$W_{\tau+1}(z) = \Delta_p(z^{-1})' W_\tau(z)\Delta_m(z). \tag{40}$$

Notice that the matrix $\Delta_k(z)$ is inner, i.e.

$$\Delta_k(z^{-1})'\Delta_k(z) = I_{kT}.$$

We now express the lifted transfer function in terms of the sampled transfer functions $H_i(z,t)$ introduced in Section 2. Indeed, consider expressions (8), (9) and take s ranging from τ to $\tau+T-1$. It is apparent that, for any $i,j = 1,2,\dots,T$:

$$[W_\tau(z)]_{i,j} = \begin{cases} H_{T-j+i}(z, i-1+\tau)z^{-1}, & i<j, \\ H_{i-j}(z, i-1+\tau), & i \ge j, \end{cases} \tag{41}$$

so that

$$W_\tau(z) = \begin{bmatrix} H_0(z,\tau) & H_{T-1}(z,\tau)z^{-1} & \cdots & H_1(z,\tau)z^{-1} \\ H_1(z,\tau+1) & H_0(z,\tau+1) & \cdots & H_2(z,\tau+1)z^{-1} \\ \cdots & \cdots & \ddots & \cdots \\ H_{T-1}(z,\tau+T-1) & H_{T-2}(z,\tau+T-1) & \cdots & H_0(z,\tau+T-1) \end{bmatrix}. \tag{42}$$

Expression (41) can be given a converse formulation, enabling one to recover the sampled transfer functions from the lifted matrix. Precisely, for $i = 1,2,\dots,T$:

$$H_k(z, i-1+\tau) = \begin{cases} [W_\tau(z)]_{i,i-k}, & k = 0,1,\dots,i-1, \\ z[W_\tau(z)]_{i,i+T-k}, & k = i, i+1, \dots, T-1. \end{cases} \tag{43}$$

It can be seen from (42) that an important property of the lifted transfer function $W_\tau(z)$ is that it has a lower block-triangular structure at infinity, i.e.

$$\lim_{z\to\infty} [W_\tau(z)]_{i,j} = 0, \quad i<j.$$

The root of this property stems from the causality of the system, which implies the absence of direct feed-through between past output and future inputs.

Remark 7. As seen above, the lifted transfer function $W_\tau(z)$ is composed by T^2 blocks of dimensions $p\times m$. However, it is worthy pointing out that T transfer matrices of dimensions $p \times m$ suffice to work out the entire matrix $W_\tau(z)$. Indeed recall the definition (10) of transfer operator introduced in Section 2, and its link with the sampled transfer functions $H_i(z,t)$ given in (12). In view of (43) one obtains

$$G(\sigma,\tau) = \left[\sum_{j=0}^{T-1} [W_\tau(\sigma^T)]_{1,j+1}\right]\sigma^j, \quad \tau = 0,1,\dots,T-1. \tag{44}$$

Conversely, from (15) and (41):

$$[W_\tau(\sigma^T)]_{i,j} = \frac{1}{T}\left[\sum_{k=0}^{T-1} G(\sigma\phi^k, i-1+\tau)\phi^{k(i-j)}\right]\sigma^{i-j}, \quad i,j = 1,2,\dots,T. \tag{45}$$

Expression (45) points out that all block entries of $W_\tau(z)$ can be built from the T generating functions $G(\sigma,t)$, $t = 0,1,\dots,T-1$, defined in (10).

4.4. Lifting a periodic system in the state-space framework

Turn now to the state-space description (1) and (2) of a periodic system. A state-space realization of the lifted transfer function $W_\tau(z)$ in Section 4.3 can be recovered by making reference to the sampled state

$$x^{(\tau)}(k) = x(kT + \tau).$$

Indeed, in such a way, the state $x^{(\tau)}(k+1) = x((k+1)T + \tau)$ is determined by $x^{(\tau)}(k) = x(kT + \tau)$ and the packed input segment $u_\tau(k)$. As for the packed output segment $y_\tau(k)$, it can be obtained from the sampled state $x^{(\tau)}(k)$ and the packed input $u_\tau(k)$. More precisely, define $F_\tau \in R^{n \times n}$, $G_\tau \in R^{n \times mT}$, $H_\tau \in R^{pT \times n}$, $E_\tau \in R^{pT \times mT}$ as

$$F_\tau = \Psi_A(\tau), \tag{46}$$

$$G_\tau = [\Phi_A(\tau + T, \tau + 1)B(\tau)\Phi_A(\tau + T, \tau + 2)B(\tau + 1) \ldots B(\tau + T - 1)], \tag{47}$$

$$H_\tau = [C(\tau)'\Phi_A(\tau + 1, \tau)'C(\tau + 1)' \ldots \Phi_A(\tau + T - 1, \tau)'C(\tau + T - 1)']', \tag{48}$$

$$E_\tau = \{(E_\tau)_{ij}\}, \quad i,j = 1,2,\ldots,T, \tag{49}$$

$$(E_\tau)_{ij} = \begin{cases} 0, & i < j, \\ D(\tau + i - 1), & i = j, \\ C(\tau + i - 1)\Psi_A(\tau + i - 1, \tau + j)B(\tau + j - 1), & i > j. \end{cases} \tag{50}$$

Thanks to these definitions, the state-space version of the lifted reformulation takes the form

$$x^{(\tau)}(k+1) = F_\tau x^{(\tau)}(k) + G_\tau u_\tau(k), \tag{51}$$

$$y_\tau(k) = H_\tau x^{(\tau)}(k) + E_\tau u_\tau(k). \tag{52}$$

In this way one obtains a time-invariant system (51), (52) which is a state-sampled representation of the original system (1) and (2) fed by T input *channels* and producing T output *channels*.

Obviously, the transfer function of system (51), (52) coincides with $W_\tau(z)$ so that one can write

$$W_\tau(z) = H_\tau(zI - F_\tau)^{-1}G_\tau + E_\tau. \tag{53}$$

Remark 8. The dynamic matrix of the lifted system (51), (52) coincides with the monodromy matrix of $A(\cdot)$. Therefore the periodic system is stable iff the time-invariant lifted system is stable. This analogy extends to the structural properties. In particular, system (51), (52) is controllable (reconstructable, detectable, stabilizable) if and only if system (1), (2) is controllable (reconstructable, detectable, stabilizable, respectively). As for reachability and observability, some care must be deserved since the dimensions of the reachability and observability subspaces of a periodic system may be time varying. However, for a given τ, it is still true that system (51), (52) is reachable (observable) if and only if system (1), (2) is reachable (observable) at time τ. For more discussion on the structural properties see Bittanti (1986).

Example 9. Making again reference to the system considered in Examples 2 and 6, the lifted transfer function can be deduced from the transfer function operator $G(\sigma, t)$ by exploiting Eq. (45). Precisely,

$$W_0(z) = \frac{1}{z+10}\begin{bmatrix} -2.5 & -1 \\ 3z & -12 \end{bmatrix},$$

$$W_1(z) = \frac{1}{z+10}\begin{bmatrix} -12 & 3 \\ -z & -2.5 \end{bmatrix}.$$

The state-space lifted system can be worked out from (46)–(50) so obtaining

$$F_0 = -10, \qquad G_0 = [-5 \;\; -2], \qquad H_0 = \begin{bmatrix} 0.5 \\ 6 \end{bmatrix}, \qquad E_0 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix},$$

and

$$F_1 = -10, \qquad G_1 = [-4 \;\; 1], \qquad H_1 = \begin{bmatrix} 3 \\ -2.5 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

5. Cyclic reformulation

To explain the basic idea underlying the cyclic reformulation, we start again by considering a discrete-time signal $v(\cdot)$, of arbitrary dimension, say r. Having chosen an arbitrary tag τ, select a uniformly distributed grid $\mathscr{T}_\tau(i)$ of time points:

$$\mathscr{T}_\tau(i) = \{\tau + i - 1 + kT \mid k = 0, 1, \ldots,\}$$

and define a new signal $\hat{v}_\tau^{(i+1)}(t)$ which is the restriction of the original one over the grid:

$$\hat{v}_\tau^{(i)}(t) = v(t), \quad t \in \mathscr{T}_\tau(i).$$

It is now possible to define the *cycled signal* $\hat{v}_\tau(t)$ as

$$\hat{v}_\tau(t) = [\hat{v}_\tau^{(1)}(t)' \quad \hat{v}_\tau^{(2)}(t)' \quad \cdots \quad \hat{v}_\tau^{T}(t)']'. \tag{54}$$

Notice that this augmented vector, of dimension Tr, is peculiar in that each block-component is defined over a different set of time points. However, what really matters is that there is a one-to-one correspondence between the augmented signal $\hat{v}_\tau(\cdot)$ and the original one $v(\cdot)$.

Remark 10. To avoid the oddity of having a vector with undefined components at certain time points, one can alternatively define the cyclic vector as follows:

$$\hat{v}_\tau^{(i)}(t) = \begin{cases} v(t), & t = \tau + i - 1 + kT, \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

In other words, at each time point, the vector $\hat{v}_\tau(t)$ has a unique nonzero sub-vector, which cyclically shifts along the column blocks

$$\hat{v}_\tau(\tau) = \begin{bmatrix} v(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \hat{v}_\tau(\tau+1) = \begin{bmatrix} 0 \\ v(\tau+1) \\ \vdots \\ 0 \end{bmatrix}, \ldots,$$

$$\hat{v}_\tau(\tau+T-1) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v(\tau+T-1) \end{bmatrix}.$$

The introduction of the slack values is redundant, but has the advantage of leading to signals well defined for any time points.

5.1. Cycled signal in the z-domain

It is possible to establish a link between the z-transform $\hat{V}_\tau(\sigma)$ of the cycled signal $\hat{v}_\tau(t)$, the z-transform $V_\tau(z)$ of the lifted signal $v_\tau(k)$, and the z-transform $V(\sigma)$ of the original signal $v(t)$. Indeed, notice that

$$\hat{V}_\tau^{(i)}(\sigma) = \sum_{t=0}^{\infty} \hat{v}_\tau^{(i)}(t)\sigma^{-t} = \sum_{t \in \mathscr{T}_\tau(i)} v(t)\sigma^{-t}.$$

Hence

$$\hat{V}_\tau^{(i)}(\sigma) = \sum_{k=0}^{\infty} v(\tau + kT + i - 1)\sigma^{-kT}\sigma^{-\tau-i+1}$$

$$= \sum_{k=0}^{\infty} v_\tau^{(i)}(k)\sigma^{-kT}\sigma^{-\tau-i+1},$$

so that

$$\hat{V}_\tau^{(i)}(\sigma) = V_\tau^{(i)}(\sigma^T)\sigma^{-\tau-i+1}. \tag{56}$$

This expression brings into light a simple relation between the z-transform of the augmented lifted and cycled signals. Precisely, letting r be the dimension of the vector $v(\cdot)$, it follows:

$$\hat{V}_\tau(\sigma) = \nabla_r(\sigma)V_\tau(\sigma^T)\sigma^{-\tau}, \tag{57}$$

where $\nabla_r(\sigma)$ is the block-diagonal matrix

$$\nabla_r(\sigma) = \begin{bmatrix} I_r & 0 & \cdots & 0 \\ 0 & \sigma^{-1}I_r & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \sigma^{-T+1}I_r \end{bmatrix}. \tag{58}$$

Taking into account Eq. (31), one can also express the z-transform of any block entry of the cycled signal in terms of the z-transform of the original signal

$$\hat{V}_\tau^{(i)}(\sigma) = \frac{1}{T}\left[\sum_{k=0}^{T-1} V(\sigma\phi^k)\phi^{k(i-1)}\right]\sigma^{-\tau}. \tag{59}$$

Conversely, the z-transform of the original signal is easily obtained from (32)

$$V(\sigma) = \left[\sum_{k=1}^{T} \hat{V}_\tau^{(k)}(\sigma)\right]\sigma^{\tau}. \tag{60}$$

5.2. Cycling a time-invariant system

As in the lifting case it is advisable to first study the action of cycling for a time-invariant system with transfer function $G(\sigma)$. This is made easy by applying relation (57) between the lifted and cycled signals both to the input and output of the system. Then, the *cyclic transfer function* $\hat{W}_\tau(\sigma)$ from $\hat{u}_\tau(t)$ to $\hat{y}_\tau(t)$ can be computed from the lifted one $W_\tau(z)$ as follows:

$$\hat{W}_\tau(\sigma) = \nabla_p(\sigma)W_\tau(\sigma^T)\nabla_m(\sigma)^{-1},$$

so that, in view of the formula of the lifted transfer function (36), and Eqs. (34) and (37), it follows that

$$\hat{W}_\tau(\sigma) = \hat{W}_0(\sigma) =$$

$$\begin{bmatrix} \hat{G}^{(0)}(\sigma) & \hat{G}^{(T-1)}(\sigma) & \hat{G}^{(T-2)}(\sigma) & \cdots & \hat{G}^{(1)}(\sigma) \\ \hat{G}^{(1)}(\sigma) & \hat{G}^{(0)}(\sigma) & \hat{G}^{(T-1)}(\sigma) & \cdots & \hat{G}^{(2)}(\sigma) \\ \hat{G}^{(2)}(\sigma) & \hat{G}^{(1)}(\sigma) & \hat{G}^{(0)}(\sigma) & \cdots & \hat{G}^{(3)}(\sigma) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{G}^{(T-1)}(\sigma) & \hat{G}^{(T-2)}(\sigma) & \hat{G}^{(T-2)}(\sigma) & \cdots & \hat{G}^{(0)}(\sigma) \end{bmatrix}, \forall\tau, \tag{61}$$

where

$$\hat{G}^{(s)}(\sigma) = \frac{1}{T}\sum_{k=0}^{T-1} G(\sigma\phi^k)\phi^{ks}. \tag{62}$$

Noting that $\hat{G}^{(s)}(\sigma) = \sigma^{-s}G^{(s)}(\sigma^T)$. Then, it is possible to exploit Eq. (38) in order to recover the transfer function

$G(\sigma)$ of the time-invariant system from the T-functions

$\hat{G}^{(s)}(\sigma)$, $s = 0, 1, \ldots, T-1$:

$$G(\sigma) = \sum_{j=0}^{T-1} \hat{G}^{(j)}(\sigma). \tag{63}$$

Therefore, the transfer function of the original time-invariant system can be computed from the cyclic transfer function by summing the block entries of any row (or any column).

5.3. Cycling a periodic system in the input–output framework

The very definition of the time-grids $\mathscr{T}_\tau(k)$, $k = 1, 2, \ldots, T$, obviously entails that the dynamics relating the input of a periodic system over $\mathscr{T}_\tau(j)$ to its output over $\mathscr{T}_\tau(i)$ is indeed time invariant. The associated transfer function will be denoted by $[\hat{W}_\tau(\sigma)]_{i,j}$. Hence, it is possible to define the *cyclic transfer function* at τ of a periodic system as

$$\hat{W}_\tau(\sigma) = \begin{bmatrix} [\hat{W}_\tau(\sigma)]_{1,1} & [\hat{W}_\tau(\sigma)]_{1,2} & \cdots & [\hat{W}_\tau(\sigma)]_{1,T} \\ [\hat{W}_\tau(\sigma)]_{2,1} & [\hat{W}_\tau(\sigma)]_{2,2} & \cdots & [\hat{W}_\tau(\sigma)]_{2,T} \\ \cdots & \cdots & \ddots & \cdots \\ [\hat{W}_\tau(\sigma)]_{T,1} & [\hat{W}_\tau(\sigma)]_{T,2} & \cdots & [\hat{W}_\tau(\sigma)]_{T,T} \end{bmatrix}.$$

Obviously matrix $\hat{W}_\tau(\sigma)$ is of dimension $pT \times mT$. Note that this reformulation is sometimes referred to as *twisted reformulation*, see Kuijper (1999).

There is a simple recursive formula to pass from $\hat{W}_\tau(\sigma)$ to $\hat{W}_{\tau+1}(\sigma)$. Indeed, it is apparent that

$$\mathscr{T}_{\tau+1}(i) = \begin{cases} \mathscr{T}_\tau(i+1), & i = 0, 1, \ldots, T-1, \\ \mathscr{T}_\tau(1), & i = T. \end{cases}$$

This implies that the z-transform $\hat{U}_{\tau+1}(\sigma)$ of $\hat{u}_{\tau+1}(t)$ is a permutation of the z-transform $\hat{U}_\tau(\sigma)$ of $\hat{u}_\tau(t)$:

$$\hat{U}_{\tau+1}(\sigma) = \hat{\Delta}_m \hat{U}_\tau(\sigma),$$

where

$$\hat{\Delta}_k = \begin{bmatrix} 0 & I_k \\ I_{k(T-1)} & 0 \end{bmatrix}. \tag{64}$$

As can be seen, the matrix Δ_k is orthogonal i.e. $\Delta_k' \Delta_k = I_{kT}$. By applying the same considerations to the output signal, one obtains the following recursion for the cyclic transfer function:

$$\hat{W}_{\tau+1}(\sigma) = \hat{\Delta}_p' \hat{W}_\tau(\sigma) \hat{\Delta}_m. \tag{65}$$

Owing again to (57), (58), it is apparent that the relation

$$\hat{W}_\tau(\sigma) = \nabla_m(\sigma) W_\tau(\sigma^T) \nabla_p(\sigma)^{-1} \tag{66}$$

holds, so that the single entries are simply related as

$$[\hat{W}_\tau(\sigma)]_{i,j} = [W_\tau(\sigma^T)]_{i,j} \sigma^{j-i}. \tag{67}$$

As for the link between the cyclic transfer function and the sampled transfer functions $H_i(z,t)$ of Section 3, it can be easily assessed from (41), (43), (67):

$$[\hat{W}_\tau(\sigma)]_{i,j} = \begin{cases} H_{T-j+i}(\sigma^T, \tau+i-1)\sigma^{j-i-T}, & i < j, \\ H_{i-j}(\sigma^T, i-1+\tau)\sigma^{j-i}, & i \geq j, \end{cases} \tag{68}$$

$$H_k(\sigma^T, i-1+\tau) = \begin{cases} \sigma^k [\hat{W}_\tau(\sigma)]_{i,i-k}, & k = 0,1, \ldots i-1, \\ \sigma^k [\hat{W}_\tau(\sigma)]_{i,i+T-k}, & k = i, i+1, \ldots, T-1. \end{cases} \tag{69}$$

Consequently,

$$\hat{W}_\tau(\sigma) = \begin{bmatrix} H_0(\sigma^T,\tau) & H_{T-1}(\sigma^T,\tau)\sigma^{-T+1} & \cdots & \bowtie_1(\sigma^T,\tau)\sigma^{-1} \\ \sigma^{-1} H_1(\sigma^T,\tau+1) & H_0(\sigma^T,\tau+1) & \cdots & H_2(\sigma^T,\tau+1)\sigma^{-2} \\ \cdots & \cdots & \ddots & \cdots \\ \sigma^{-T+1} H_{T-1}(\sigma^T,\tau+T-1) & \sigma^{-T+2} H_{T-2}(\sigma^T,\tau+T-1) & \cdots & H_0(\sigma^T,\tau+T-1) \end{bmatrix}.$$

This expression entails that the cyclic transfer function $\hat{W}_\tau(\sigma)$ is block-diagonal at infinity, i.e.

$$\lim_{z\to\infty} [\hat{W}_\tau(z)]_{i,j} = 0, \quad i \neq j.$$

The reason of this property stems from the fact that $\mathscr{T}_\tau(i)$ and $\mathscr{T}_\tau(j)$ have empty intersection for $i \neq j$.

Remark 11. As seen in Remark 7, the lifted transfer function can be constructed starting from T transfer matrices of dimensions $p \times m$ only. The same property holds for the cycled transfer function. Indeed from (44) and (67) one obtains

$$G(\sigma,\tau) = \sum_{j=0}^{T-1} [\hat{W}_\tau(\sigma)]_{1,j+1}, \quad \tau = 0, 1, \ldots, T-1. \tag{70}$$

The converse expression can be derived from (45) and (67):

$$[\hat{W}_\tau(\sigma)]_{i,j} = \frac{1}{T} \sum_{k=0}^{T-1} G(\sigma\phi^k,\tau)\phi^{k(i-j)}, \quad i,j = 1, 2, \ldots, T. \tag{71}$$

This last formula shows that the knowledge of $G(\sigma,t)$, $t = 0, 1, \ldots, T-1$, suffices to build the entire cycled transfer function.

5.4. Cycling a periodic system in the state-space framework

The cycled reformulation of a periodic system in the state-space form (1), (2) is just the state-space relation among the cycled state, input and output signals $\hat{x}_\tau(t)$, $\hat{u}_\tau(t)$, and $\hat{y}_\tau(t)$. In this regard, note that the block-component $\hat{x}_\tau^{(i)}(t+1)$ is defined for $t \in \mathscr{T}_\tau(i-1)$ and, as such, can be obtained from $\hat{x}_\tau^{(i-1)}(t)$ and $\hat{u}_\tau^{(i-1)}(t)$ (obviously if $i=1$, $\hat{x}_\tau^{(i)}(t+1)$ will depend on $\hat{x}_\tau^{(T)}(t)$ and $\hat{u}_\tau^{(T)}(t)$). In this way the obtained cycled reformulation would be defined component-wise over different grids of time points. In order to come up with a dynamical system defined for all time instant, one should rather refer to expression (55), where slack zero components are fictitiously introduced in the definition of cycled vectors (recall Remark 10). Adopting for simplicity this last viewpoint, the cycled reformulation at τ in state space is

$$\hat{x}_\tau(t+1) = \hat{F}_\tau \hat{x}_\tau(t) + \hat{G}_\tau \hat{u}_\tau(t), \tag{72}$$

$$\hat{y}_\tau(t) = \hat{H}_\tau \hat{x}_\tau(t) + \hat{E}_\tau \hat{u}_\tau(t), \tag{73}$$

where

$$\hat{F}_\tau = \begin{bmatrix} 0 & 0 & \cdots & 0 & A(\tau+T-1) \\ A(\tau) & 0 & \cdots & 0 & 0 \\ 0 & A(\tau+1) & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & A(\tau+T-2) & 0 \end{bmatrix}, \tag{74}$$

$$\hat{G}_\tau = \begin{bmatrix} 0 & 0 & \cdots & 0 & B(\tau+T-1) \\ B(\tau) & 0 & \cdots & 0 & 0 \\ 0 & B(\tau+1) & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & B(\tau+T-2) & 0 \end{bmatrix}, \tag{75}$$

$$\hat{H}_\tau = \begin{bmatrix} C(\tau) & 0 & \cdots & 0 \\ 0 & C(\tau+1) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & C(\tau+T-1) \end{bmatrix}, \tag{76}$$

$$\hat{E}_\tau = \begin{bmatrix} D(\tau) & 0 & \cdots & 0 \\ 0 & D(\tau+1) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & D(\tau+T-1) \end{bmatrix}. \tag{77}$$

The dimensions of the state, input and output spaces of the cyclic reformulation are those of the original periodic systems multiplied by T.

Obviously, the transfer function of the cyclic reformulation at τ is given by

$$\hat{W}_\tau(\sigma) = \hat{H}_\tau(\sigma I - \hat{F}_\tau)^{-1}\hat{G}_\tau + \hat{E}_\tau. \tag{78}$$

Remark 12. The peculiar structure of the dynamical matrix (74) is such that its eigenvalues coincides with all the Tth roots of the characteristic multipliers of $A(\cdot)$. Hence, the periodic system is stable if and only if its cycled reformulation is stable. The structural properties of the cyclic reformulation are determined by the corresponding structural properties of the original system too. However, there are some slight but remarkable differences with respect to the lifted case, due to the extended dimension of the state space. For example, if system (1), (2) is reachable at time τ, the cycled system (72),(73) is not necessarily reachable. The appropriate statement is that system (1), (2) is reachable (observable) at each time if and only if system (72), (73) is reachable (observable) (at any arbitrary time τ, which parametrizes Eqs. (72), (73)). This reflects the fact that if system (72), (73) is reachable for a value of τ, it is reachable for any τ. As for the remaining structural properties, one can recognize that system (1), (2) is controllable (reconstructable, detectable, stabilizable) if and only if system (72), (73) is controllable (reconstructable, detectable, stabilizable, respectively).

Example 13. The cyclic reformulation of the 2-periodic system of Example 2 can be determined from the lifted one (see Example 9)) by means of relation (66). Therefore

$$\hat{W}_0(\sigma) = \frac{1}{\sigma^2+10}\begin{bmatrix} -2.5 & -\sigma \\ 3\sigma & -12 \end{bmatrix},$$

$$\hat{W}_1(\sigma) = \frac{1}{\sigma^2+10}\begin{bmatrix} -12 & 3\sigma \\ -\sigma & -2.5 \end{bmatrix}$$

As for the state-space description, it can be worked out by using expressions (74)–(77). It turns out that

$$\hat{F}_0 = \begin{bmatrix} 0 & -5 \\ 2 & 0 \end{bmatrix}, \qquad \hat{G}_0 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix},$$

$$\hat{H}_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 3 \end{bmatrix},$$

$$\hat{F}_1 = \begin{bmatrix} 0 & 2 \\ -5 & 0 \end{bmatrix}, \qquad \hat{G}_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix},$$

$$\hat{H}_1 = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix},$$

whereas $\hat{E}_0$ and $\hat{E}_1$ are the null matrices.

6. The frequency-lifted reformulation

This section is devoted at presenting a reformulation which is naturally set in the z-domain directly. Precisely, with reference to a signal $v(t)$, with z-transform $V(\sigma)$, define the frequency-lifted vector $\tilde{V}(\sigma)$ as

$$\tilde{V}(\sigma) = \begin{bmatrix} V(\sigma) \\ V(\sigma\phi) \\ V(\sigma\phi^2) \\ \vdots \\ V(\sigma\phi^{T-1}) \end{bmatrix}, \tag{79}$$

where again $\phi = e^{j2\pi/T}$. Notice that the ith component of this vector is nothing but the z-transform of the complex signal obtained by multiplying signal $v(t)$ by the complex number $\phi^{(-i+1)t}$.

It is straightforward to link $\tilde{V}(\sigma)$ to the z-transform $V_0(z)$ of the lifted signal $v_0(k)$ introduced in Section 4. Indeed, consider the elements $v_0^{(i)}(k) = v(kT+i-1)$, $i = 0,1,\dots,T-1$, of $v_0(k)$ and their z-transform $V_0^{(i)}(z)$ which are the elements of $V_0(z)$. Now, consider Eq. (32), and note that $V^{(k)}(z) = V_0^{(k)}(z)$. Evaluating (32) in $\sigma\phi^i$, $i = 0,1,\dots,T-1$, it is easy to recognize that

$$\tilde{V}(\sigma) = M(\sigma)V_0(\sigma^T), \tag{80}$$

where

$$M(\sigma) = \begin{bmatrix} I & \sigma^{-1}I & \cdots & \sigma^{-T+1}I \\ I & (\sigma\phi)^{-1} & \cdots & (\sigma\phi)^{-T+1} \\ \cdots & \cdots & \cdots & \cdots \\ I & (\sigma\phi^{T-1})^{-1} & \cdots & (\sigma\phi^{T-1})^{-T+1} \end{bmatrix}. \tag{81}$$

6.1. Frequency lifting of a periodic system

The relationship between the frequency-lifted input signal $\tilde{U}(\sigma)$ and the frequency-lifted otput signal $\tilde{Y}(\sigma)$ of a periodic system can be worked out by exploiting the previous results on the lifted transfer function. Indeed, recalling the definition of $W_0(z)$, from Eq. (80) it follows that

$$\tilde{Y}(\sigma) = M(\sigma)W_0(\sigma^T)M(\sigma)^{-1}\tilde{U}(\sigma).$$

This leads to the definition of *frequency-lifted transfer function* of a periodic system as

$$\tilde{W}(\sigma) = M(\sigma)W_0(\sigma^T)M(\sigma)^{-1}. \tag{82}$$

This formula points out that, while $W_0(z)$ has real coefficients, $\tilde{W}(\sigma)$ is a proper rational matrix with complex coefficients. The reason is that, as above observed, $\tilde{W}(z)$ relates input–output time-domain signals with *complex coefficients*.

It is interesting to stress the relationships between $\tilde{W}(\sigma)$ and the generic transfer function $G(\sigma,t)$ introduced in Section 2. Indeed, from (32) it results

$$\begin{aligned} Y(\sigma\phi^q) &= \sum_{k=0}^{T-1} Y_k^{(1)}(\sigma^T)\sigma^{-k}\phi^{-qk} \\ &= \sum_{k=0}^{T-1}\sum_{i=0}^{T-1} [W_k(\sigma^T)]_{1,i+1} U_k^{(i+1)}(\sigma^T)\sigma^{-k}\phi^{-qk}, \end{aligned}$$

where (recall (31))

$$U_k^{(i+1)}(\sigma^T) = \frac{1}{T}\left[\sum_{p=0}^{T-1} U(\sigma\phi^p)\phi^{p(k+i)}\right]\sigma^{k+i}.$$

Therefore,

$$\begin{aligned} Y(\sigma\phi^q) &= \frac{1}{T}\sum_{k=0}^{T-1}\sum_{i=0}^{T-1} [W_k(\sigma^T)]_{1,i+1}(\sigma\phi^p)^i \\ &\quad\times \sum_{p=0}^{T-1} U(\sigma\phi^p)\phi^{k(p-q)}. \end{aligned}$$

Now, we can exploit expression (44) to work out the generic element of the frequency-lifted transfer function $\tilde{W}(\sigma)$ as a function of $G(\sigma,t)$, i.e.

$$[\tilde{W}(\sigma)]_{q+1,p+1} = \frac{1}{T}\sum_{k=0}^{T-1} G(\sigma\phi^p,k)\phi^{k(p-q)}. \tag{83}$$

Conversely, it is possible to obtain $G(\sigma,t)$ from $\tilde{W}(\sigma)$. Indeed, from (83) and recalling property (13) it follows that

$$\sum_{q=0}^{T-1} [\tilde{W}(\sigma)]_{q+1,p+1}\phi^{tq} = G(\sigma\phi^p,t)\phi^{pt},$$

so that

$$G(\sigma,t) = \sum_{q=0}^{T-1} [\tilde{W}(\sigma\phi^{-p})]_{q+1,p+1}\phi^{t(q-p)}. \tag{84}$$

Remark 14. For a time-invariant system with transfer function $G(\sigma)$, the periodic transfer operator is in fact constant, i.e. $G(\sigma,t) = G(\sigma)$, $\forall t$, so that, from (83) it follows:

$$\begin{aligned} [\tilde{W}(\sigma)]_{q+1,p+1} &= G(\sigma\phi^p)\frac{1}{T}\sum_{k=0}^{T-1}\phi^{k(p-q)} \\ &= \begin{cases} G(\sigma\phi^p), & p = q, \\ 0, & p \neq q, \end{cases} \\ &= \operatorname{diag}\{G(\sigma\phi^p),\ p = 0,\ 1,\dots,T-1\}. \end{aligned} \tag{85}$$

This expression points out that, in the time-invariant case, $\tilde{W}(\sigma)$ is block-diagonal. This conclusion could also be seen as a straightforward consequence of the very definition of frequency-lifted signals.

6.2. *Frequency lifting in state space: the Fourier reformulation*

The Fourier reformulation is obtained by considering the periodic system in an EMP regime. Then, one can express all signals (input, state, output) in the EMP Fourier expansion and the periodic system matrices in terms of their Fourier-sum coefficients. Recalling the general EMP Fourier formula (21), this amounts to writing

$$u(t) = \sum_{k=0}^{T-1} \bar{u}_k \phi^{kt} \sigma^t,$$

$$x(t) = \sum_{k=0}^{T-1} \bar{x}_k \phi^{kt} \sigma^t,$$

$$y(t) = \sum_{k=0}^{T-1} \bar{y}_k \phi^{kt} \sigma^t.$$

As for the periodic system matrices, the Fourier sum is

$$A(t) = \sum_{k=0}^{T-1} A_k \phi^{kt}$$

and similarly for $B(t)$, $C(t)$ and $D(t)$. Now, plugging the above expansion into the system equations and equating all terms at the same frequency, one obtains the following matrix equation:

$$\sigma \mathcal{N} \tilde{x} = \mathcal{A} \tilde{x} + \mathcal{B} \tilde{u},$$

$$\tilde{y} = \mathcal{C} \tilde{x} + \mathcal{D} \tilde{u},$$

where $\tilde{x}$, $\tilde{u}$ and $\tilde{y}$ are vectors formed with the harmonics of x, u and y, respectively, organized in the following fashion:

$$\tilde{x}' = [\bar{x}'_0 \ \bar{x}'_1 \cdots \bar{x}'_{T-1}]'$$

and similarly for $\tilde{u}$ and $\tilde{y}$. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are block Toeplitz matrices formed with the harmonics of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$, respectively, as

$$\mathcal{A} = \begin{bmatrix} A_0 & A_{T-1} & A_{T-2} & \cdots & A_1 \\ A_1 & A_0 & A_{T-1} & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{T-2} & A_{T-3} & A_{T-4} & \cdots & A_{T-1} \\ A_{T-1} & A_{T-2} & A_{T-3} & \cdots & A_0 \end{bmatrix}$$

and similarly for $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. As for matrix $\mathcal{N}$, it is the block diagonal matrix

$$\mathcal{N} = \text{blkdiag}\{\phi^k I\}, \quad k = 0, 1, \ldots, T-1.$$

Then, one can define the *harmonic transfer function* as the operator:

$$\hat{\mathcal{G}}(\sigma) = \mathcal{C}[\sigma \mathcal{N} I - \mathcal{A}]^{-1} \mathcal{B} + \mathcal{D}.$$

Such an operator provides a most useful connection between the input harmonics and the output harmonics (organized in the vectors $\tilde{u}$ and $\tilde{y}$, respectively). In particular, if one takes $\sigma = 1$ (so considering the truly periodic regimes), this expression enables one to compute the amplitudes and phases of the harmonics constituting the output signal $y(\cdot)$ in a periodic regime.

Interestingly the transfer function $\hat{\mathcal{G}}(\sigma)$ coincides with that of the frequency-lifted reformulation $\tilde{W}(\sigma)$. Indeed, write the state equation in terms of the Fourier expansion of the $A(\cdot)$ and $B(\cdot)$ matrices, i.e.

$$x(t+1) = \sum_{k=0}^{\infty} A_k \phi^{kt} x(t) + \sum_{k=0}^{\infty} B_k \phi^{kt} u(t).$$

By letting $X(\sigma)$ and $U(\sigma)$ be the z-transforms of $x(\cdot)$ and $u(\cdot)$, respectively, and neglecting the initial state, it then follows:

$$\sigma X(\sigma) = \sum_{k=0}^{\infty} A_k X(\sigma \phi^{-k}) + \sum_{k=0}^{\infty} B_k U(\sigma \phi^{-k}).$$

Proceeding in the same way for the z-transform $Y(\sigma)$ of the output signal, one also obtains

$$Y(\sigma) = \sum_{k=0}^{\infty} C_k X(\sigma \phi^{-k}) + \sum_{k=0}^{\infty} D_k U(\sigma \phi^{-k}).$$

Therefore, by introducing the frequency-lifted vectors $\tilde{X}(\sigma)$, $\tilde{U}(\sigma)$ and $\tilde{Y}(\sigma)$ (recall Eq. (79)), the two above equations can be rewritten as

$$\sigma \mathcal{N} \tilde{X}(\sigma) = \mathcal{A} \tilde{X}(\sigma) + \mathcal{B} U(\sigma),$$
$$\tilde{Y}(\sigma) = \mathcal{C} \tilde{X}(\sigma) + \mathcal{D} \tilde{U}(\sigma).$$

This shows that the relation between the harmonic transfer function coincides with the transfer function of the frequency-lifted reformulation, i.e.

$$\tilde{\mathcal{G}}(\sigma) = \tilde{W}(\sigma).$$

Example 15. The frequency-lifted transfer function relative to our leading example (Examples 2–13) can be derived in a number of equivalent ways. For instance, one can start from the lifted transfer function (Example 9) and then resort to expression (82) for the computation of $\tilde{W}(\sigma)$. Another possibility is to exploit Eq. (83) based on the transfer function operator $G(\sigma, t)$ (Example 2). In any case, one obtains

$$\tilde{W}(\sigma) = \frac{1}{\sigma^2 + 10} \begin{bmatrix} -7.25 + \sigma & 4.75 + 2\sigma \\ 4.75 - 2\sigma & -7.25 - \sigma \end{bmatrix}.$$

If one is interested in the state-space description, then one can consider the Fourier expansion of the system matrices

$$A_0 = \tfrac{1}{2}(A(0) + A(1)) = -1.5,$$

$$A_1 = \tfrac{1}{2}(A(0) - A(1)) = 3.5$$

and analogously

$$B_0 = -0.5, \qquad B_1 = 1.5, \qquad C_0 = 1.75,$$

$$C_1 = -1.25, \qquad D_0 = D_1 = 0.$$

Hence $\mathscr{D}$ is the null matrix and

$$\mathscr{A} = \begin{bmatrix} -1.5 & 3.5 \\ 3.5 & -1.5 \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} -0.5 & 1.5 \\ 1.5 & -0.5 \end{bmatrix},$$

$$\mathscr{C} = \begin{bmatrix} 1.75 & -1.25 \\ -1.25 & 1.75 \end{bmatrix}.$$

Taking into account that $\mathscr{N} = \mathrm{diag}(1, -1)$ the Fourier expansion is fully defined. It is a matter of a simple computation to show that the associated harmonic transfer function $\tilde{G}(\sigma) = \mathscr{C}(\sigma\mathscr{N} - \mathscr{A})^{-1}\mathscr{D}$ coincides with the frequency transfer function $\tilde{W}(\sigma)$.

7. Concluding remarks

A number of time-invariant reformulations have been used here and these in the literature of periodic systems and control. In this paper, we organize these reformulations in a coordinated picture, having as unifying theme the notion of periodic transfer function. In this way, the mutual relations among the various reformulations are clarified through closed-form expressions. A feature of this contribution is that it does not consider the usual state-space description as the only model to start with; the analysis heavily relies on input-output arguments as well. This constitutes a relatively novel point of view in a scenario traditionally dominated by the state-space formalism.

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References

Arcara, P., Bittanti, S., & Lovera, M. (2000). Periodic control of helicopters rotors for attenuation of vibrations in forward flight. *IEEE Transactions of Control Systems Technology*, to appear.

Bittanti, S. (1986). Deterministic and stochastic linear periodic systems, In: S. Bittani, *Time series and linear systems* (pp. 141–182). Berlin: Springer.

Bittanti, S., Bolzern, P., & Guardabassi, G. (1985). Some critical issues concerning the state-representation of time-varying ARMA models. *7th IFAC symposium on identification and system parameter estimation*, York, England, (pp. 1479–1484).

Bittanti, S., & Colaneri, P. (1996). Analysis of discrete-time linear periodic systems. In C.T. Leondes *Control and dynamic systems*, vol. 78. New York: Academic Press.

Bittanti, S., & Colaneri, P. (1999). *Periodic control. Encyclopedia of electrical engineering*. New York: Wiley.

Bittanti, S., Colaneri, P., & Mongiovi, M. F. (1998). From singular to nonsingular filtering for periodic systems: Filling the gap with the spectral interactor matrix. *IEEE Transactions on Automatic Control*, *44*, 222–227.

Bittanti, S., & Lovera, M. (1996). On the zero dynamics of helicopter rotors loads. *European Journal of Control*, *2*(1), 57–68.

Bolzern, P., Colaneri, P., & Scattolini, R. (1986). Zeros of discrete-time linear periodic systems.. *IEEE Transactions on Automatic Control*, *31*, 1057–1058.

Colaneri, P., & Kucera, V. (1997). The model matching problem for discrete-time periodic systems. *IEEE Transactions on Automatic Control*, *42*, 1997.

Colaneri, P., & Longhi, S. (1995). The minimal realization problem for discrete-time periodic systems. *Automatica*, *31*, 779–783.

Crochiere, R. E., & Rabiner, L. R. (1993). *Multirate digital signal processing*. Englewood Cliffs, NJ: Prentice Hall.

Dahleh, M. A., Vulgaris, P. G., & Valavani, L. S. (1992). Optimal and robust controllers for periodic and multirate systems. *IEEE Transactions on Automatic Control*, *37*, 90–99.

D'Angelo, H. (1970). *Linear time-varying systems: analysis and synthesis*. Boston: Allyn and Bacon.

Faraday, M. (1831). On a peculiar class of acoustical figures and on certain forms assumed by a group of particles upon vibrating elasting surfaces. *Philosophical Transactions of the Royal Society of London*, 299–318.

Feuer, A., & Goodwin, G. C. (1996). *Sampling in digital signal processing and control*. New York: Birkhauser.

Flamm, D. S. (1991). A new shift-invariant representation for periodic systems. *Systems and Control Letters*, *17*, 9–14.

Floquet, G. (1883). Sur les equations differentielles lineaires a coefficients periodicques. *Annales de l'Ecole Normale Superieur*, *12*, 47–89.

Freudenberg, J. S., & Grizzle, J. W. (1989). An observation on the parametrizazion of causal stabilizing controllers for lifted systems. *Control Theory and Advanced Technology*, *5*(3), 367–372.

Gardner, W.A. (Ed.). (1994). *Cyclostationarity in communications and signal processing*. New York: IEEE Press.

Grasselli, O. M., & Longhi, S. (1988). Zeros and poles of linear periodic multivariable discrete-time systems. *IEEE Transactions on Circuit, Systems and Signal Processing*, *7*, 361–380.

Halanay, A. (1966). *Differential equations — stability, oscillations, time-lags*. New York: Academic Press.

Hill, G. W. (1886). On the part of the lunar perigee which is a function of the mean motion of the sun and moon. *Acta Mathmatica*, *8*, 1–36.

Isniewski, R. W., & Blanke, M. (1999). Fully magnetic attitude control for spacecraft subject to gravity gradient. *Automatica*, *35*, 1201–1214.

Johnson, W. (1996). *Helicopter theory*. Princeton, NJ: Princeton University Press. 1981, rep. Dover Pub.

Jury, E. J., & Mullin, F. J. (1959). The analysis of sampled data control system with a periodically time varying sampling rate. *IRE Transactions on Automatic Control*, *5*, 15–21.

Kamen, E. W., Khargonekar, P. P., & Poolla, K. R. (1985). A transfer function approach to linear time-varying discrete-time system. *SIAM Journal Control and Optimization*, *23*(4), 550–565.

Khargonekar, P. P., Poolla, K., & Tannembaum, A. (1985). Robust control of linear time-invariant plants using periodic compensation. *IEEE Transactions on Automatic Control*, *30*, 1088–1096.

Krank, G. M. (1957). Input–output analysis of multirate feedback systems.. *IRE Transactions on Automatic Control*, *32*, 21–28.

Kuijper, M. (1999). A periodically time-varying minimal partial realization algorithm based on twisting. *Automatica*, *35*, 1543–1548.

Kuijper, M., & Willems, J. C. (1997). A behavioral framework for periodically time-varying systems. *IEEE conference of decision and control*, San Diego, USA (pp. 2013–2016).

Marzollo, A. (Ed.). (1972) Periodic optimization. Berlin: Springer.

Mathieu, E. (1868). Memoire sur le mouvement vibratoire d'une membrane de forme elliptique. *Journal of Mathematics*, *13*, 137–203.

McKillip, R. (1991). Periodic model following controller for the control-configured helicopter. *Journal of the American Helicopter Society*, *36*(3), 4–12.

Meyer, R. A., & Burrus, C. S. (1975). A unified analysis of multirate and periodically time-varying digital filters. *IEEE* Transactions on Circuits Systems, *22*, 162–167.

Nayfeh, A. H., & Mook, D. T. (1979). *Nonlinear oscillations*. New York: Wiley.

Park, B., & Verriest, E.I. (1989). Canonical forms for discrete-linear peiodically time-varying systems and a control application. In *Proceedings of 28th conference on decision and control*, Tampa USA, pp. 1220–1225.

Raileigh, J. S. (1883). On the crispation of fluid resting upon vibrating support. *Philosophical Magazine*, *16*, 50–58.

Raileigh, J.S. (1886). *The theory of sound*, vol. 2. New York.

Saaty, T.L., & Bram, J. (1981). *Nonlinear mathematics*. Toronto: General Publishing Company Ltd., 1964, reprinted New York Dover Pub.

Tong, L., Xu, G., & Kailath, T. (1994). Blind identification and equalization based on second-order statistics: A time domain approach.. *IEEE Transactions on Information Theory*, *40*(2), 340–349.

Vaidyanathan, P. P. (1990). Multirate digital filters, filter banks, polyphase networks, and applications: A tutorial. *Proceedings of the IEEE*, *78*(1), 56–93.

Vaidyanathan, P. P. (1993). *Multirate systems and filter-banks*. Englewood Cliffs, NJ: Prentice–Hall.

Verriest, E.I. (1988). The operational transfer function and parametrization of N-periodic systems. In Proceedings 27th Conference on Decision and Control, Austin USA, (pp. 1994–1999).

Wereley, N.M., & Hall, S.R. (1990). Frequency response of linear time periodic systems. In *Proceedings of 29th conference on decision and control*, Honululu, USA.

Xin, J., Kagiwada, H., Sano, A., Tsuj, H., & Yoshimoto, S. (1997). Regularization approach for detection of cyclostationary signals in antenna array processing. *IFAC Symposium on System Identification*, vol. 2 pp. 529–534.

Yakubovich, V. A., & Starzhinskii, V. M. (1975). *Linear differential equations with periodic coefficients*. New York: Wiley.

Zhang, C., Zhang, J., & Furuta, K. (1996). Performance analysis of periodically time-varying controllers. *13th IFAC World Congress*, San Francisco, USA, (pp. 207–212).

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