$K(\mu)$, $K_0(\mu)$, $K_{00}(\mu)$ = parameters of the minimum error function
$\kappa_n(t)$ = no energy storage gain computed from the director equations
$m(t)$ = control signal of the dynamic process
n = dummy index of summation
$q(t)$ = response signal of the dynamic process
$Q(t)$ = desired process response signal
$Q_n(t)$ = component of the desired response signal
t = present time
T = duration of pure-time delay
u = dummy index of summation
U = order of the dynamic process
ζ, μ, ξ, σ = dummy time variables
$\lambda(\sigma)$ = weighting factor
τ = interval where process response errors are weighted
τ' = interval where process response errors are weighted for case with a pure-time delay of duration T
Superscript (u) = uth derivative of function with respect to a time variable

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A Method for the Symbolic Representation and Analysis of Linear Periodic Feedback Systems

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LINEAR FEEDBACK systems with periodic sampling, finite pulsing, finite pulse clamping and similar variations have been a subject of increasing interest in recent years. Applications in such important areas as digital control and time-division multiplexing have motivated continued research and development. This is particularly true for certain types of systems where the theory is extensive and well developed. Unfortunately, there are many types of systems which have received little or no attention.

It is the purpose of this paper to present methods of symbolism and analysis which handle an extensive class of new problems and present an improved treatment of many problems previously considered.

The sampled-data system is the most studied and well-known system type. Here the periodic variation appears as instantaneous periodic sampling of signals. Mathematically, this sampling is represented by multiplying signals by a periodic impulse train. The literature is extensive and includes multirate and cyclic-rate systems.[1–5] The most important tools of analysis are the Z-transform and modified Z-transform.

Farmanfarma has considered finite pulsed systems where the periodic variation is signal multiplication by a train of periodic pulses with finite width and amplitude.[6–8] The P-transform is defined and used in his analysis. Analysis of closed-loop systems is considerably more complicated than that of open-loop systems. Approximate methods, based on sampled-data models have been proposed by Kranc,[9] Murphy and Kennedy,[10] and Tou.[3]

Another type of periodic variation is the operation of finite pulse clamping illustrated in Fig. 1, where Fig. 1(A) shows the signal input $e(t)$ and Fig. 1(B) shows the finite pulse-clamped signal $e_c(t)$. Mathematically,

$$\begin{aligned} e_c(t) &= e(t),\ nT<t\leqslant nT+t_1 \\ &= e(nT+t_1),\ nT+t_1<t\leqslant (n+1)T \end{aligned} \tag{1}$$

where T is the fundamental period of variation and n ranges over all integers. Analysis techniques for systems with finite pulse-clamped error signals have been developed.[11]

A periodic variation which includes finite pulsing as a special case is the piecewise constant variation of parameters. Such variation can be in gains and/or time constants of control elements. An example of a parameter with a 2-interval variation is the gain:

$$\begin{aligned} K(t) &= K_1,\ nT<t<nT+t_1 \\ &= K_2,\ nT+t_1<t<(n+1)T \end{aligned} \tag{2}$$

where T is the fundamental period. Finite pulsing obviously occurs when $K_2=0$. Unforced systems of this type are conveniently treated by the matrix method of Pipes.[12] The sinusoidal response of electric networks with piecewise constant variation of parameters was treated by Bennett[13] and extended by Desoer.[14] A still more complete theory of such systems has been developed by the author.[11]

All of these variations can be included in a single system. Fig. 2 shows an example of such a system where the error is finite pulse clamped, modified by a digital computer with clamper, and fed into a plant ($p=d/dt$) with periodic gain variation. The finite pulse-clamped error e_c is defined by equation 1 and the variable gain K by equation 2. The digital computer and clamper are described by the equation:

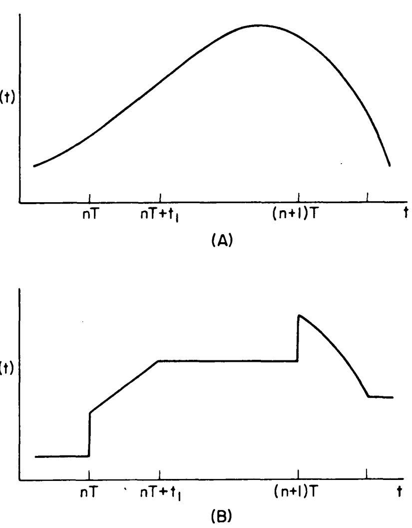

Fig. 1. Operation of finite pulse clamping

A—Continuous signal
B—Finite pulse-clamped signal

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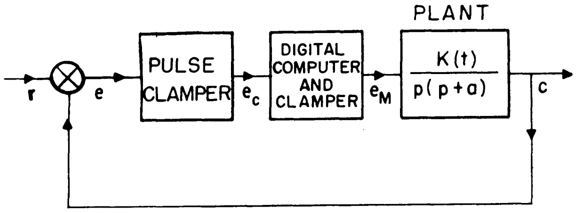

Fig. 2 (above). Example feedback system

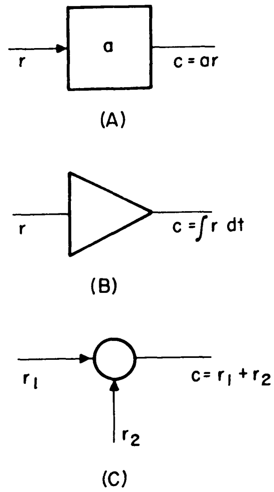

Fig. 3 (right). Basic symbolic elements

A—Multiplication by a constant
B—Integration
C—Summation

$$e_M(t)=\sum_{i=0}^{3} a_i e_c[t_2+(n-i-1)T]-\sum_{j=1}^{2} b_j e_M[t_2+(n-j)T]$$

$$nT<t\leqslant(n+1)T,\ 0<t_1<t_2<T \quad (3)$$

representing instantaneous sampling and clamping.

A typical problem in analysis would be to determine the step response of this system.

The methods of symbolism and analysis presented in this paper treat systems with any combination of the described periodic variations. The formulation is based on a state vector that completely defines the system and input behavior at all times. This state vector is defined by a series of constant coefficient differential equations and transition equations. The equations are readily obtained, from a block diagram consisting of basic symbolic elements. State vectors have been used with success in other applications by Kalman and Bertram,[15,16] Bashkow,[17] and others. The block diagram methods are in some ways similar to those of Bertram.[16]

Solution of the equations is accomplished by matrix methods. The Z-transformation is used to obtain the solution at the fundamental time intervals nT. The continuous solution in any fundamental period $nT<t\leqslant(n+1)T$ is also obtained. The methods are illustrated with examples.

System Description

All the periodic operations already described are easily represented in block diagram form by five basic operations: multiplication by a constant, integration, summation, switching, and sampling. When every system component is described by symbols representing the basic operations, the mathematical representation in terms of a state vector readily follows.

Invariant Components

Consider first components described by transfer functions with constant coefficients. In this case switching or sampling operations are not necessary and the component can be represented by the block diagram symbols of Fig. 3. The decomposition which results is by no means unique.

To illustrate several decompositions, consider the transfer function:

$$\frac{c}{r}=\frac{p^2+3p+2}{p^3+4.5p^2+2p}=\frac{(p+1)(p+2)}{p(p+0.5)(p+4)} \quad (4)$$

A direct decomposition is shown in Fig. 4. The correctness of this block diagram is made apparent by writing c as:

$$c=(p^{-1}+3p^{-2}+2p^{-3})\bar{c}$$

where

$$\bar{c}=\frac{1}{1+4.5p^{-1}+2p^{-2}}r$$

The state vector is defined by the components y_1, y_2, and y_3, equal in number to the system order. The differential equations for the state variables are readily obtained by inspection. For example, the time rate of change of y_3 equals the sum of inputs to the y_3 integrator, or:

$$\dot{y}_3=-4.5y_3-2y\ +r$$

Similarly, $\dot{y}_2=y_3$, and $\dot{y}_1=y_2$.

The desired output $c(t)$ is given by a linear combination of the state variables. Thus:

$$c(t)=2y_1+3y_2+y_3$$

By writing the transfer function 4 as the partial fraction expansion

$$\frac{c}{r}=\frac{1}{p}+\frac{-3/7}{p+0.5}+\frac{3/7}{p+4}$$

the block diagram of Fig. 5 results. The state variables are again the integrator outputs defined by the equations:

$$\dot{y}_1=r$$

$$\dot{y}_2=-0.5y\ +r$$

$$\dot{y}_3=-4y_3+r$$

The output c is given by:

$$c=y_1-\frac{3}{7}y_2+\frac{3}{7}y_3$$

Still another decomposition is obtained by cascading elements of the form $1/(p+a)$ and $(p+b)/(p+c)$. One such form for transfer function 4 is shown in Fig. 6. In this case the state variables are defined by:

$$\dot{y}_1=(1-0.5)y_2+(2-4)y_3+r$$

$$\dot{y}_2=-0.5y_2+(2-4)y_3+r$$

$$\dot{y}_3=-4y_3+r$$

and $c=y_1$.

These and other possible decompositions illustrate the plurality of the symbolic representations. The appropriate decomposition and corresponding state vector are determined by a number of factors. If a transfer function is of high

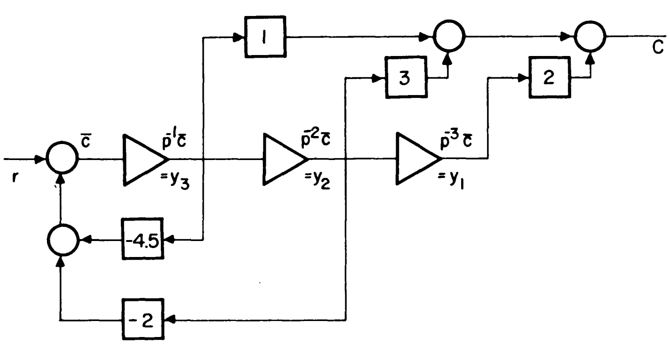

Fig. 4 (left). Direct decomposition

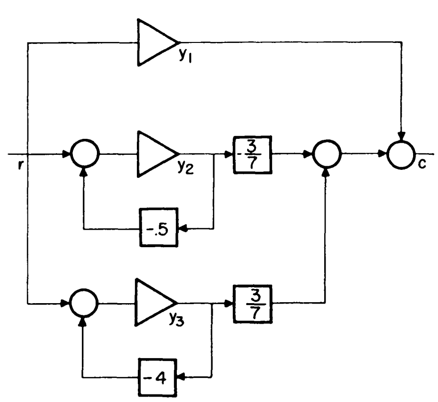

Fig. 5 (right). Partial fraction decomposition

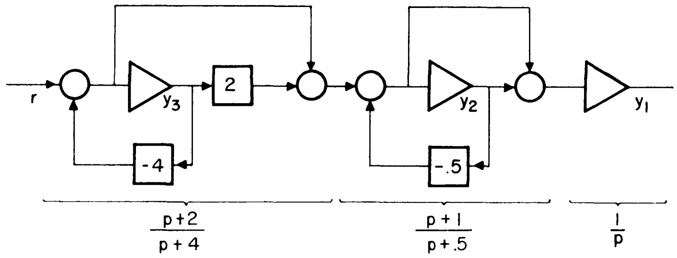

Fig. 6. Cascade decomposition

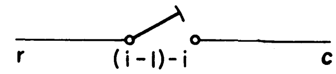

Fig. 7. Symbolic switching elements

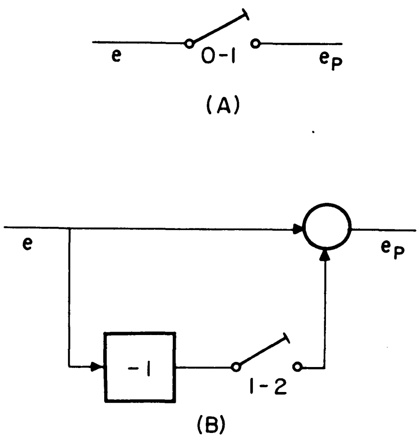

Fig. 8. Two representations of finite pulsing

order and unfactored, the direct decomposition method is most convenient. The partial fraction representation has a computational advantage in that the differential equation for each state variable is independent of the other state variables. Often it is desirable from the standpoint of synthesis and compensation to have the state variables correspond directly to physical quantities. In control systems, this can frequently be done by using the cascade representation.

In many cases the application of physical laws leads directly to a state vector. One example occurs in electrical network analysis when dependent variables are selected as the voltages across capacitors and the currents through inductors.[17] Another example is the linearized equations for an aircraft which are a set of first-order differential equations. In these cases, the state variables represent physical quantities and transfer function derivation is superfluous.

Switching and Sampling

Before describing the basic switching and sampling operations, the interval number and sampling number of a system will be defined. The interval number N is the number of intervals of time invariant behavior in the fundamental period T. These intervals of invariance are specified by $0<t-nT\leq t_1$, $t_1<t-nT\leq t_2$, $\ldots$, $t_{n-1}<t-nT\leq t_N=T$. The sampling number M is the number of instantaneous samples in the fundamental period T. Since sampling is a time variability, the sampling instants separate time invariant behavior and must occur at one or more of the values nT, $nT+t_1$, $\ldots$, $nT+t_{n-1}$. Obviously, $M\leq N$. Note that the time origin has been selected so that $t=nT$ separates intervals of time invariability.

To illustrate the meaning of N and M more fully, consider several examples. For a conventional sampled-data system with sampling at $t=nT$, $N=M=1$. For a cyclic-rate sampled-data system with five sampling instants in each fundamental period, $N=M=5$. A finite pulsed system with one pulse per period gives $N=2$ and $M=0$. For the finite pulse-clamping operation given by equation 1, $N=2$, $M=1$. As will be seen, the mixed system of Fig. 2 can have several N, M combinations depending on the formulation chosen. It is sometimes useful to think of time invariant systems as a special case where $N=1$, $M=0$.

The $(i-1)-i$ switching operation, shown schematically in Fig. 7 by a bar switch, is defined by:

$$
\begin{aligned}
c&=0,\ 0<t-nT\leq t_{i-1}\\
&=r,\ t_{i-1}<t-nT\leq t_i\\
&=0,\ t_i<t-nT\leq T
\end{aligned}
\tag{5}
$$

Although N switching operators exist, they may not all be used. An example of this is shown in Figs. 8(A) and (B) where the pulsed error:

$$
\begin{aligned}
e_p(t)&=e(t),\ 0<t-nT\leq t_1\\
&=0,\ t_1<t-nT\leq T
\end{aligned}
$$

is obtained in two different ways. Depending on the application, one formulation may be preferable to another.

The switching operations describe any piecewise constant parameter variation. The only other necessary operation is that of sampling. The sampling operator considered here differs from the usual impulse train multiplication. The new method of representation is illustrated in Fig. 9(A) for a 3-sample system where sampling occurs at $t=nT$, $nT+t_1$, and $nT+t_2$. The sampling operation, which always occurs in conjunction with integration is represented by arrow switches, as opposed to the bar switches, feeding the lower side of an integrator. The integrator equations in Fig. 9(A) are:

$$
\begin{aligned}
&\dot{y}_1=r_1\\
&y_1(nT+)=r\,(nT)\\
&y_1(nT+t_1+)=y_1(nT+t_1)\\
&y_1(nT+t_2+)=y_1(nT+t_2)
\end{aligned}
\tag{6}
$$

The first equation is the integrator differential equation; the last three equations are the reset or transition equations that specify the integrator initial condition at $t=nT+$, $nT+t_1+$, and $nT+t_2+$. For an M sample system each integrator would have M sample switches associated with it. To simplify the block diagrams, the sample switches will not be shown if they connect an integrator output to its lower side. This is reasonable since such operations give continuity in the integrator variable. Fig. 9(B) shows this simplification for the system of Fig. 9(A). Other examples are illustrated in Figs. 3, 4, 5, and 6 for invariant systems where the integrator outputs are obviously continuous.

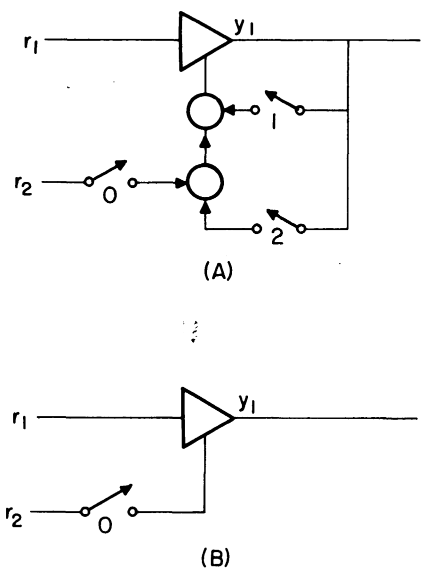

Fig. 9. Symbolic sampling elements

A—Complete form
B—Simplified form

To illustrate the use of the sampling operator consider several examples. Fig. 10(A) shows the usual way of representing linear interpolation between data points. Here:

$$
r^*(t)=\sum_{n=-\infty}^{\infty} r(nT)\delta(t-nT)
$$

$$
c(t)=r[(n-1)T]+\frac{1}{T}\{r(nT)-r[(n-1)T]\}\,(t-nT),\ 0<t-nT\leq T]
$$

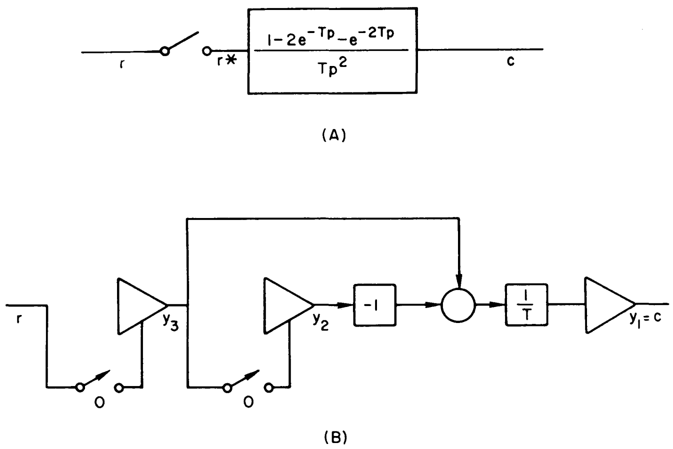

Fig. 10. Linear interpolation device

A—Conventional sampled-data system representation
B—New representation

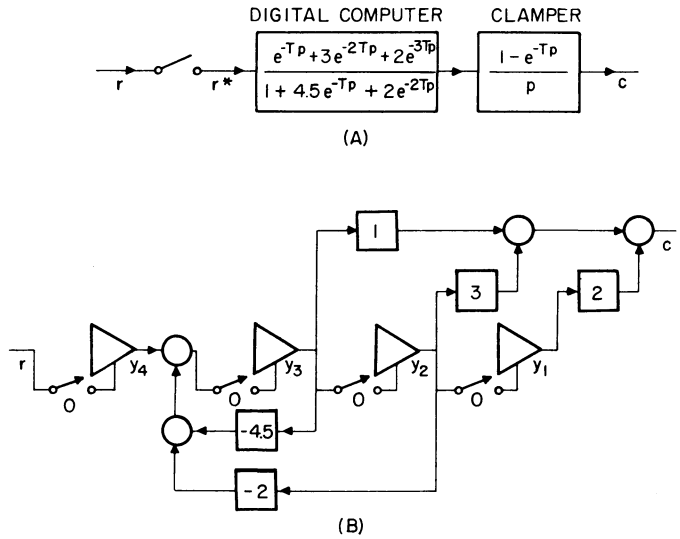

Fig. 11. Digital computer and clamper

A—Conventional sampled-data system representation
B—New representation

The new description, using the basic symbolism, is shown in Fig. 10(B). The variable y_3 is the sampled and clamped $r(t)$; y_2 is the sampled and clamped signal delayed by the period T; y_1 is $1/T\,\{r(nt)-r[(n-1)T]\}$ for $0<t-nT<T$; and therefore y_1 equals the desired $c(t)$ if $y_1(nT)=r[(n-1)T]$, which is the case if the system is initially at rest (say at $t=0$). The state vector formed by y_1, y_2, and y_3 completely describes the device. The variables are governed by the differential equations:

$$\dot{y}_1=-\frac{1}{T}y_2+\frac{1}{T}y_3$$

$$\dot{y}_2=0$$

$$\dot{y}_3=0$$

and the reset or transition equations:

$$y_1(nT+)=y_1(nT)$$

$$y_2(nT+)=y_3(nT)$$

$$y_3(nT+)=r(nT)$$

A somewhat more general example of a sampled-data system is the digital computer and clamper shown in the usual notation in Fig. 11(A) and in the new notation in Fig. 11(B). The sampled and clamped r is given by the y_4 integrator while the digital computer program is accomplished in the remainder of the block diagram. Note that the digital computer transfer function is the same as equation 4 with p^{-1} replaced by ϵ^{-Tp}. For this reason, the diagram is analogous to the direct decomposition of Fig. 4. The only difference is the replacement of integrator inputs by reset operations. Other forms of decomposition are of course possible. Fig. 12 illustates the cascade decomposition analogous to Fig. 6. Other decompositions are also possible; for example, the sampling and clamping accomplished in y_4 could take place following the computer representation instead of preceding it. The equations for the state variables of Fig. 11(B) or Fig. 12 are easily written. For the Fig. 12 representation, the differential equations are:

$$\dot{y}_1=0$$

$$\dot{y}_2=0$$

$$\dot{y}_3=0$$

$$\dot{y}_4=0$$

and the reset or transition equations are:

$$y_1(nT+)=(1-0.5)y_2(nT)+(2-4)y_3(nT)+y_4(nT)$$

$$y_2(nT+)=-0.5y_2(nT)+(2-4)y_3(nT)+y_4(nT)$$

$$y_3(nT+)=-4y_3(nT)+y_4(nT)$$

$$y_4(nT+)=r(nT)$$

These transition equations have the same coefficients as the differential equations for the system of Fig. 6.

To represent the finite pulse-clamping operation defined in equation 1, both the switching and sampling operations are necessary. This is obvious from Fig. 13 where the 0–1 and the 1–2 interval equations for e_c are:

$$e_c(t)=e(t),\quad 0<t-nT\leq t_1$$

$$=y_1,\quad t_1<t-nT\leq T$$

with $\dot{y}_1=0$ and

$$y_1(nT+)=y_1(nT)$$

$$y_1(nT+t_1+)=e(nT+t_1)$$

Time delay is a dynamic characteristic that can be represented when it occurs in a path containing sampling. As an example, suppose that a sampler and a clamper are followed by a time delay of $2T+t_1$, where $0<t_1<T$. The symbolic representation is shown in Fig. 14. Here y_4 is the sampled and clamped r where it is assumed sampling occurs at $t=nT$. The y_2 and y_3 integrators provide a delay of $2T$ while the remaining delay t_1 occurs in the y_1 integrator.

This completes the symbolic description of components necessary to represent more complex systems such as the one in Fig. 2. As described earlier, this system

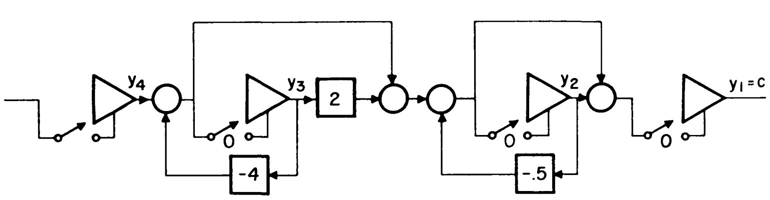

Fig. 12. Alternative representation or digital computer and clamper

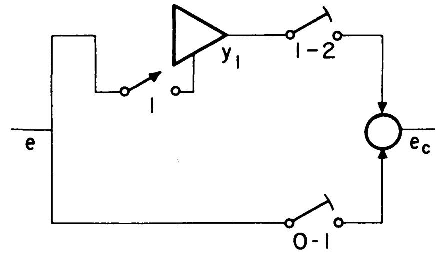

Fig. 13. Symbolic representation of finite pulse clamping

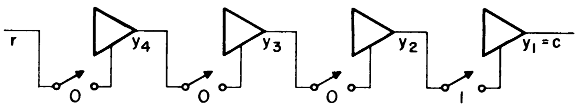

Fig. 14. Representation of a time delay

is a 3-interval 3-sample problem. The three intervals of the fundamental period are 0 to t_1, t_1 to t_2, and t_2 to T. The sampling occurs in the error pulse clamper at $nT+t_1$, and in the digital computer at $nT+t_2$ and nT. However, the sampling at $nT+t_2$ could equally well occur at $(n+1)T$ since the digital computer has constant input for $nT+t_1<t\leq(n+1)T$. This reduces the problem to a 2-interval 2-sample problem. One representation of the simplified system is shown in Fig. 15. Since there are seven state variables, two intervals, and two samples times, fourteen constant coefficient differential equations and fourteen transition equations must be written. The differential equations in interval 0–1 are:

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -a_4 y_2 + K_1[a_3 y_3 + (a_2 - b_2 a_0) y_4 + (a_1 - b_1 a_0) y_5 + a_0 y_6] \quad (7)$$

$$\dot{y}_3 = \dot{y}_4 = \dot{y}_5 = \dot{y}_6 = \dot{y}_7 = 0$$

and in interval 1–2 are:

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -a_4 y_2 + K_2[a_3 y_3 + (a_2 - b_2 a_0) y_4 + (a_1 - b_1 a_0) y_5 + a_0 y_6] \quad (8)$$

$$\dot{y}_3 = \dot{y}_4 = \dot{y}_5 = \dot{y}_6 = \dot{y}_7 = 0$$

The transition equations at $t=nT$ and $t=nT+t_1$ are:

$$y_1(nT+) = y_1(nT),\ y_1(nT+t_1+) = y_1(nT+t_1)$$

$$y_2(nT+) = y_2(nT),\ y_2(nT+t_1+) = y_2(nT+t_1)$$

$$y_3(nT+) = y_4(nT),\quad y_3(nT+t_1+) = y_3(nT+t_1)$$

$$y_4(nT+) = y_5(nT),\ y_4(nT+t_1+) = y_4(nT+t_1)$$

$$y_5(nT+) = y_6(nT) - b_1 y_5(nT) - b_2 y_4(nT),\ y_5(nT+t_1+) = y_5(nT+t_1)$$

$$y_6(nT+) = y_7(nT),\ y_6(nT+t_1+) = y_6(nT+t_1)$$

$$y_7(nT+) = y_7(nT),\ y_7(nT+t_1+) = x_1(nT+t_1) - y_1(nT+t_1) \quad (9)$$

It is worthy to note that the 0–1 switch in the error pulse clamper is superfluous in this system since the sampler following operates at $t=nT$ only.

Input Description

Up to now, input variables have been unspecified in functional form. For analysis, it will be useful to think of input quantities as additional state variables described by differential equations and, possibly, transition equations. For a deterministic input such as a step function, ramp function, general polynomial in t, exponential, sinusoid, or certain periodic functions, the formulation of the input state vector offers no particular difficulty. Fortunately, such inputs occur in many important response and synthesis problems. Methods for obtaining the input state variables will now be discussed.

Polynomial inputs are easily generated by the series of integrators shown in Fig. 16(A). Here:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = 0$$

With the general set of initial conditions $x_1(0)$ $x_2(0)$, $\ldots$, $x_{n-1}(0)$, $x_n(0)$ any function of the type

$$x_1 = r = r_0 + r_1 t + r_2 t^2 + \ldots + r_{n-1} t^{n-1},\ t>0$$

can be generated. Special cases are the step and ramp functions. Exponential functions are generated by the system represented in Fig. 16(B). Here:

$$\dot{x}_1 = a x_1 \quad (10)$$

and with the initial condition $x_1(0)=1$,

$$x_1 = r = \epsilon^{at},\ t>0 \quad (11)$$

Response to a cosine wave is obtained by setting $a=j\omega$ and taking the real part of the response. If desired, a sine and/or cosine function can be obtained from the system in Fig. 16(C). Here:

$$\dot{x}_1 = \omega x_2$$

$$\dot{x}_2 = -\omega x_1$$

and with the initial conditions $x_1(0)$ and $x_2(0)$:

$$r = x_1(0)\cos\omega t + x_2(0)\sin\omega t$$

Other functions that are similar or combinations of the above are readily obtained.

More general inputs are obtained using sampling and/or switching. As one example consider the symbolic diagram in Fig. 17 defining a 1-interval 1-sample system. The differential equations are:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 0$$

$$\dot{x}_3 = 0$$

the transition equations are:

$$x_1(nT+) = x_3(nT)$$

$$x_2(nT+) = x_2(nT)$$

$$x_3(nT+) = x_3(nT)$$

and the initial conditions at $t=0$ are:

$$x_1(0) = 0$$

$$x_2(0) = \frac{1}{T}$$

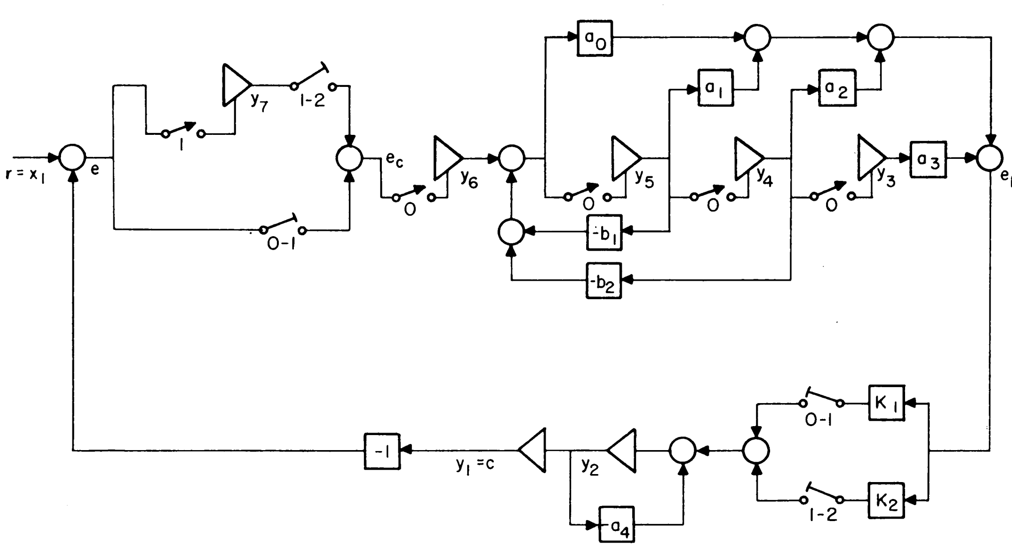

Fig. 15. The symbolic representation of feedback system example

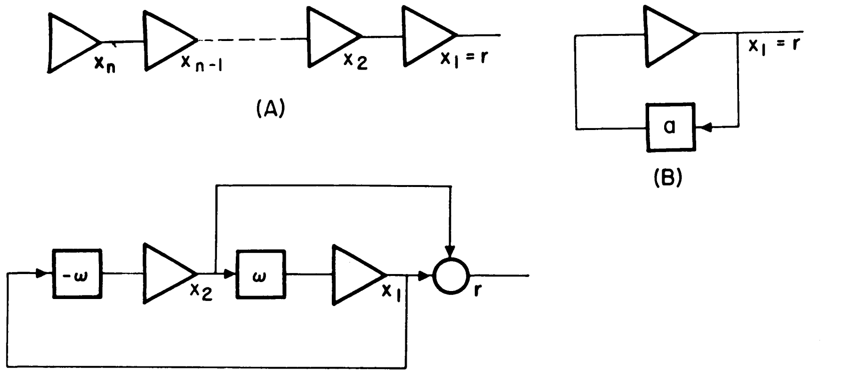

Fig. 16. Input representation

A—Polynomial B—Exponent C—Sinusoid

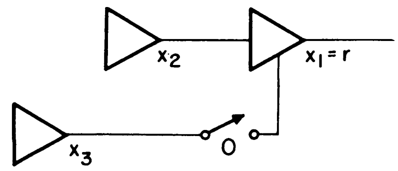

Fig. 17. Sawtooth wave representation

$x_3(0)=0$

For this system:

$$x_1=r=\frac{t-nT}{T},\ 0<t-nT\leq T,\ t>0$$

This is the equation for a sawtooth wave of unit amplitude and period T. Other periodic inputs, such as a square wave or parabolic wave, are obtainable through similar methods. These more general periodic inputs are limited in that their period or an integral multiple of it must equal the system fundamental period T. It is also possible to generate aperiodic functions of a more general class than defined.

System Equations in Vector Form

By defining the input state vector in the described manner, it is now possible to state concisely equations governing system response. Assume that the system vector $\mathbf{y}$ has k components $y_1, y_2, \ldots, y_k$ and that the input vector $\mathbf{x}$ has m components $x_1, x_2, \ldots, x_m$. Furthermore, assume that the input and system has N invariant intervals and M sampling instants in the fundamental period T. If $\mathbf{w}$ is defined as the vector with the $m+k$ components $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_k$, the system is described by the N vector differential equations:

$$\begin{aligned}\dot{\mathbf{w}}&=A_1\mathbf{w},\ 0<t-nT\leq t_1\\ \dot{\mathbf{w}}&=A_2\mathbf{w},\ t_1<t-nT\leq t_2\\ &\ \vdots\\ \dot{\mathbf{w}}&=A_N\mathbf{w},\ t_{N-1},<t-nT\leq T\end{aligned}\tag{12}$$

and the N vector transition equations:

$$\begin{aligned}\mathbf{w}(nT+)&=\bar{A}_0\mathbf{w}(nT)\\ \mathbf{w}(nT+t_1+)&=\bar{A}_1\mathbf{w}(nT+t_1)\\ &\ \vdots\\ \mathbf{w}(nT+t_{N-1}+)&=\bar{A}_{N-1}\mathbf{w}(nT+t_{N-1})\end{aligned}\tag{13}$$

where A_i and $\bar{A}_i$ are $(m+k)\mathrm{x}(m+k)$ matrices. N–M of the matrices $\bar{A}_i$ are equal to the identity matrix I while the remaining M matrices $\bar{A}_i$ are not. Equations 12 and 13 and an initial condition on $\mathbf{w}$ at some time completely specifies the system response.

The matrices A_i and A_i are easily obtained from the input and system state variable differential and transition equations. Consider the system in Fig. 15 forced by the exponential of Fig. 16(B). The differential and transition equations are given by 7, 8, 9, 10, and 11. From these the four necessary 8×8 matrices are:

$$A_1=\begin{array}{c}\begin{array}{cccccccc} x_1 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7\end{array}\\ \begin{Vmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -a_4 & K_1a_3 & K_1(a_2-b_2a_0) & K_1(a_1-b_1a_0) & K_1a_0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{Vmatrix}\end{array}\begin{array}{c}\\ \dot{x}_1\\ \dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\\ \dot{y}_4\\ \dot{y}_5\\ \dot{y}_6\\ \dot{y}_7\end{array}$$

$$A_2=\begin{Vmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -a_4 & K_2a_3 & K_2(a_2-b_2a_0) & K_2(a_1-b_1a_0) & K_2a_0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{Vmatrix}\tag{14}$$

$$\bar{A}_0=\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -b_2 & -b_1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{Vmatrix}$$

$$\bar{A}_1=\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\end{Vmatrix}$$

Above and to the right side of A_1 are the variables corresponding to the respective columns and rows of the matrix.

From this example, it is apparent that the matrices can be written by inspection of the system and input symbolic diagrams. A particular response problem is specified by an initial condition on $\mathbf{w}$. As an example, suppose that the system is initially at rest and the input $x_1=\epsilon^{at}$ is applied at $t=0$. Then

$$\mathbf{w}(0)=\begin{vmatrix} x_1(0)\\ y_1(0)\\ y_2(0)\\ y_3(0)\\ y_4(0)\\ y_5(0)\\ y_6(0)\\ y_7(0)\end{vmatrix}=\begin{vmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{vmatrix}$$

In this example many of the matrix components are zero. Frequently, but not always, this is the case. However, the i, j components of the A and $\bar{A}$ matrices for $i=1$ to m and $j=m+1$ to $m+k$ are always zero, since the output components y_i cannot influence the input components x_i. This is seen by partitioning equations 12 and 13:

$$\begin{vmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{y}}\end{vmatrix}=\begin{Vmatrix}F_i & 0\\ G_i & S_i\end{Vmatrix}\begin{vmatrix}\mathbf{x}\\ \mathbf{y}\end{vmatrix},\ i=1\text{ to }N\tag{15}$$

$$\begin{vmatrix}\mathbf{x}(nT+t_i+)\\ \mathbf{y}(nT+t_i+)\end{vmatrix}=\begin{Vmatrix}\bar{F}_i & 0\\ \bar{G}_i & \bar{S}_i\end{Vmatrix}\begin{vmatrix}\mathbf{x}(nT+t_i)\\ \mathbf{y}(nT+t_i)\end{vmatrix}$$

$$i=0\text{ to }N-1,\ t_0=0$$

The $m\mathrm{x}m$ matrices F_i and $\bar{F}_i$ determine the input; the $k\mathrm{x}k$ matrices S_i and $\bar{S}_i$ define the system dynamics; the $k\mathrm{x}m$ matrices G_i and $\bar{G}_i$ are gain factors indicating the effect of the input on the system; and 0 is an mxk null matrix.

When the input cannot be expressed in components of an input state vector, separate system equations are desirable. From equation 15.

$$\dot{\mathbf{y}} = S_i\mathbf{y} + G_i\mathbf{x},\ i = 1 \text{ to } N$$

$$\mathbf{y}(nT+t_i+) = \bar{S}_i\mathbf{y}(nT+t_i) + \bar{G}_i\mathbf{x}(nT+t_i),\ i = 0 \text{ to } N-1 \tag{16}$$

where $\mathbf{x}$ is now a scalar input.

System Response

Basic Techniques

In the previous section an extensive class of periodic system response problems have been concisely formulated in the $2N$ differential and transition equations 12 and 13. The solution of these equations will now be obtained.

The basic procedure is to work from interval to interval piecing the solution together. Start at $t = nT$ where the solution, yet unknown, is $\mathbf{w}(nT)$. Application of the first transition equation gives:

$$\mathbf{w}(nT+) = \bar{A}_0\mathbf{w}(nT) \tag{17}$$

This serves as the initial condition on the first differential equation of 12 that defines the solution for $0 < t - nT \leq t_1$. Thus the vector equation $d\mathbf{w}/dt = A_1\mathbf{w}$ must be solved with the initial condition $\mathbf{w}(nT+)$. This is conveniently accomplished by the Laplace transform. First define the new time scale $\tau = t - nT$ translating $t = nT$ into $\tau = 0$. The problem is now stated:

$$\frac{d\mathbf{w}}{d\tau} = A_1\mathbf{w},\ \mathbf{w}(\tau)\big|_{\tau=0} = \mathbf{w}(nT+) \tag{18}$$

Using the notion $\bar{\mathbf{w}}(s) = L[\mathbf{w}(\tau)]$ to indicate the Laplace transform of $\mathbf{w}(\tau)$ with respect to τ, the transform of equation 18 is:

$$s\bar{\mathbf{w}}(s) - \mathbf{w}(nT+) = A_1\bar{\mathbf{w}}(s) \tag{19}$$

Solving for $\bar{\mathbf{w}}$ and using the inverse Laplace transform gives:

$$\mathbf{w}(\tau) = L^{-1}[(sI - A_1)^{-1}]\mathbf{w}(nT+) \tag{20}$$

The matrix function $L^{-1}[(sI-A_1)^{-1}]$ deserves special notation: define the matrix time function:

$$\epsilon^{A_1\tau} = L^{-1}[(sI - A_1)^{-1}] \tag{21}$$

This function and methods for simplifying its computation are discussed in the Appendix. The solution for $0 < t - nT \leq t_1$, is now completed by substituting for τ in equation 21, using equations 20 and 17. The result is:

$$\mathbf{w}(t) = \epsilon^{A_1(t-nT)}\bar{A}_0\mathbf{w}(nT),\ 0 < t - nT \leq t_1 \tag{22}$$

A similar procedure is used in the second interval.

First:

$$\mathbf{w}(nT+t_1+) = \bar{A}_1\mathbf{w}(nT+t_1) \tag{23}$$

where $\mathbf{w}(nT+t_1)$ is obtained by evaluating equation 22 at $t = nT + t_1$. Defining another new time scale $\tau = t - (nT + t_1)$, the second equation of 12 becomes:

$$\frac{d\mathbf{w}}{d\tau} = A_2\mathbf{w},\ \mathbf{w}(\tau)\big|_{\tau=0} = \mathbf{w}(nT+t_1+) \tag{24}$$

Use of the Laplace transform, substitution for τ, and use of equation 23 yields:

$$\mathbf{w}(t) = \epsilon^{A_2(t-nT-t_1)}\bar{A}_1\mathbf{w}(nT+t_1),\quad t_1 < t - nT \leq t_2 \tag{25}$$

where

$$\epsilon^{A_2\tau} = L^{-1}[(sI - A_2)^{-1}] \tag{26}$$

Substituting in equation 25 for $w(nT+t_1)$ from equation 22 gives:

$$\mathbf{w}(t) = \epsilon^{A_2(t-nT-t_1)}\bar{A}_1\epsilon^{A_1t_1}\bar{A}_0\mathbf{w}(nT),\quad t_1 < t - nT \leq t_2 \tag{27}$$

The solution for the remaining N-2 intervals of $0 < t - nT \leq T$ is obtained in a similar manner.

The solution derived is expressed more concisely by defining the matrix time function:

$$\begin{aligned} W(\tau) &= \epsilon^{A_1\tau}\bar{A}_0,\ 0 < \tau \leq t_1 \\ &\ \vdots \\ &= \epsilon^{A_i(\tau - t_{i-1})}\bar{A}_{i-1}\epsilon^{A_{i-1}(t_{i-1}-t_{i-2})}\ldots \bar{A}_0;\ t_{i-1} < \tau \leq t_i \\ &\ \vdots \\ &= \epsilon^{A_N(\tau - t_{N-1})}\bar{A}_{N-1}\ldots\bar{A}_0,\quad t_{N-1} < \tau \leq t_N = T \end{aligned} \tag{28}$$

where of course

$$\epsilon^{A_i\tau} = L^{-1}[(sI - A_i)^{-1}] \tag{29}$$

If $M < N$, $W(\tau)$ is simpler in form since $N - M$ of the $\bar{A}_i$ equal the identity matrix. The solution $\mathbf{w}(t)$ is now easily written as:

$$\mathbf{w}(t) = W(t - nT)\mathbf{w}(nT),\ 0 < t - nT \leq T \tag{30}$$

This is the equation for the solution interior to any fundamental interval.

The first step in obtaining the solution at multiples of T is to evaluate equation 30 for $t = (n+1)T$ giving the vector difference equation:

$$\mathbf{w}[(n+1)T] = W(T)\mathbf{w}(nT) \tag{31}$$

Since this equation is valid for all n, start with $n = 0$ and the known initial condition $\mathbf{w}(0)$. After n applications of equation 31, it is found that:

$$\mathbf{w}(nT) = W^n\mathbf{w}(0) \tag{32}$$

where the notation $W = W(T)$ has been used. The problem of solving for $\mathbf{w}(nT)$ is now reduced to obtaining the nth power of W. This is accomplished by obtaining $\mathbf{w}^*(z)$, the Z-transform of $\mathbf{w}(t)$. Using equation 32, the Z-transform can of written:

$$\mathbf{w}^*(z) = \sum_{n=0}^{\infty}\mathbf{w}(nT)z^{-n} = \left[\sum_{n=0}^{\infty}W^nz^{-n}\right]\mathbf{w}(0) \tag{33}$$

where it is understood that $W^0 = I$. But the bracketed infinite matrix series is recognized as the matrix function $[I - z^{-1}W]^{-1}$.[18] Thus:

$$\mathbf{w}^*(z) = [I - z^{-1}W]^{-1}\mathbf{w}(0) \tag{34}$$

is the closed form of the Z-transform. Application of the inverse Z-transform to equation 34 gives the desired values $\mathbf{w}(nT)$. Alternatively, the components of equation 34, ratios of polynomials in z^{-1}, can be divided out and the coefficient of z^{-n} equated to the components of $\mathbf{w}(nT)$.

As an illustration of the method, consider the response of a first-order system to a suddenly applied sawtooth wave. The transfer function and input are defined by

$$\frac{c}{r} = \frac{1}{\dfrac{1}{a}p + 1}$$

and

$$\begin{aligned} r &= 0,\ t < 0 \\ &= \frac{1}{T}(t - nT),\ 0 < t - nT \leq T \end{aligned}$$

Initially the system is at rest so $c(0) = 0$. Fig. 18 shows the symbolic representation of the input, identical to that of Fig. 17, and the system. By inspection of the diagram, the two required matrices:

$$A_1 = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & -a \end{Vmatrix} \qquad \bar{A}_0 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \tag{35}$$

are obtained. Since $N = M = 1$, postmultiplying e^{A_1t}, which is evaluated in the Appendix, by $\bar{A}_0$ gives the desired:

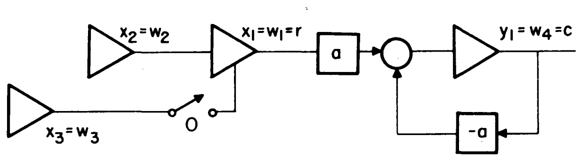

Fig. 18. Example of input and the system

$$W(\tau)=\begin{Vmatrix}0 & \tau & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \tau-\frac{1}{a}(1-\epsilon^{-a\tau}) & (1-\epsilon^{-a\tau}) & \epsilon^{-a\tau}\end{Vmatrix} \tag{36}$$

Evaluating equation 36 for $\tau=T$, $\mathbf{w}^*(z)$ is readily obtained from equation 34 using the initial condition:

$$\mathbf{w}(0)=\begin{vmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\\ y_1(0)\end{vmatrix}=\begin{vmatrix}0\\ \frac{1}{T}\\ 0\\ 0\end{vmatrix} \tag{37}$$

Because of the zero components in $\mathbf{w}(0)$, only the second columns of $(I-z^{-1}W)^{-1}$ need be computed. Multiplication of this second column by $1/T$ then gives

$$\mathbf{w}^*(z)=\begin{vmatrix}\dfrac{z^{-1}}{1-z^{-1}}\\ \dfrac{\frac{1}{T}}{1-z^{-1}}\\ 0\\ \dfrac{\left[1-\frac{1}{Ta}(1-\epsilon^{-aT})\right]z^{-1}}{(1-z^{-1})(1-\epsilon^{-aT}z^{-1})}\end{vmatrix} \tag{38}$$

Evaluating the inverse Z-transform of each component at $t=nT$ yields the desired solution:

$$\mathbf{w}(nT)=\begin{vmatrix}u(nT-T)\\ \frac{1}{T}\,u(nT)\\ 0\\ u(nT)\left(\dfrac{1}{1-\epsilon^{-aT}}-\dfrac{1}{aT}\right)(1-\epsilon^{-anT})\end{vmatrix} \tag{39}$$

where the step function $u(t)$ is zero for $t<0$ and unity for $t\geq 0$. The first three components, $x_1(nT)$, $x_2(nT)$, and $x_3(nT)$, obviously agree with specified input state vector. System response is given by the fourth component, $w_4(nT)=y_1(nT)=c(nT)$. From equations 30 and 39 the response $w_4(t)=c(t)$ for $t>0$ is:

$$c(t)=\frac{t-nT}{T}-\frac{1}{aT}+\left[\frac{1}{aT}+\left(\frac{1}{1-\epsilon^{-aT}}-\frac{1}{aT}\right)(1-\epsilon^{-anT})\right]\epsilon^{-a(t-nT)},\quad 0<t-nT\leq T \tag{40}$$

Note that the input and output values $x_2(nT)$ and $y_1(nT)$ were needed to obtain equation 40.

Detailed Solution for N = M = 2

The general method just presented has the advantage of notational simplicity. A more detailed form of the solution with some computational advantage can be obtained using the partitioned equations 15. Unfortunately, equations are excessively long and burdensome to derive if N is large. For this reason only the $N=M=2$ case will be considered here. The solution for higher N and M would proceed in a similar manner. Actually many problems fall into $N=M=2$ class.

To obtain the desired solution, it is necessary to express $\epsilon^{A_i\tau}$ and $W(\tau)$ in partitioned form. From equations 29 and 15:

$$\epsilon^{A_i\tau}=L^{-1}\left[\begin{Vmatrix}(sI-F_i) & 0\\ -G_i & (sI-S_i)\end{Vmatrix}^{-1}\right]=L^{-1}\left[\begin{Vmatrix}(sI-F_i)^{-1} & 0\\ (sI-S_i)^{-1}G_i(sI-F_i)^{-1} & (sI-S_i)^{-1}\end{Vmatrix}\right] \tag{41}$$

This is expressed more simply as:

$$\epsilon^{A_i\tau}=\begin{Vmatrix}\epsilon^{F_i\tau} & 0\\ G_i(\tau) & \epsilon^{S_i\tau}\end{Vmatrix} \tag{42}$$

by defining

$$\epsilon^{F_i\tau}=L^{-1}[(sI-F_i)^{-1}]$$
$$\epsilon^{S_i\tau}=L^{-1}[(sI-S_i)^{-1}]$$
$$G_i(\tau)=L^{-1}[(sI-S_i)^{-1}G_i(sI-F_i)^{-1}] \tag{43}$$

where $i=1$ and 2. Substituting equation 42 and the partitioned transition matrices of 15, equation 28 then gives:

$$W(\tau)=\begin{Vmatrix}X(\tau) & 0\\ G(\tau) & Y(\tau)\end{Vmatrix} \tag{44}$$

where

$$X(\tau)=\epsilon^{F_1\tau}\bar{F}_0,\ 0<\tau\leq t_1$$
$$=\epsilon^{F_2(\tau-t_1)}\bar{F}_1\epsilon^{F_1t_1}\bar{F}_0,\ t_1<\tau\leq T$$
$$Y(\tau)=\epsilon^{S_1\tau}\bar{S}_0,\ 0<\tau\leq t_1$$
$$=\epsilon^{S_2(\tau-t_1)}\bar{S}_1\epsilon^{S_1t_1}\bar{S}_0,\ t_1<\tau\leq T$$
$$G(\tau)=G_1(\tau)\bar{F}_0+\epsilon^{S_1\tau}\bar{G}_0,\ 0<\tau\leq t_1$$
$$=[G_2(\tau-t_1)\bar{F}_1+\epsilon^{S_2(\tau-t_1)}\bar{G}_1]\epsilon^{F_1t_1}\bar{F}_0+\epsilon^{S_2(\tau-t_1)}\bar{S}_1[G_1(t_1)\bar{F}_0+\epsilon^{S_1t_1}\bar{G}_0],\ t_1<\tau\leq T \tag{45}$$

The solution for $t=nT$ is obtained by the Z-transform as before. By defining $X=X(T)$, $Y=Y(T)$, and $G=G(T)$, equation 34 becomes

$$\mathbf{w}^*(z)=\begin{vmatrix}\mathbf{x}^*(z)\\ \mathbf{y}^*(z)\end{vmatrix}=\begin{Vmatrix}(I-z^{-1}X) & 0\\ -z^{-1}G & (I-z^{-1}Y)\end{Vmatrix}^{-1}\begin{vmatrix}\mathbf{x}(0)\\ \mathbf{y}(0)\end{vmatrix}$$
$$=\begin{Vmatrix}(I-z^{-1}X)^{-1} & 0\\ z^{-1}(I-z^{-1}Y)^{-1}G(I-z^{-1}X)^{-1} & (I-z^{-1}Y)^{-1}\end{Vmatrix}\begin{vmatrix}\mathbf{x}(0)\\ \mathbf{y}(0)\end{vmatrix} \tag{46}$$

From this

$$\mathbf{y}^*(z)=z^{-1}(I-z^{-1}Y)^{-1}G\mathbf{x}^*(z)+(I-z^{-1}Y)^{-1}\mathbf{y}(0) \tag{47}$$

where

$$\mathbf{x}^*(z)=(I-z^{-1}X)^{-1}\mathbf{x}(0) \tag{48}$$

The forced part of the solution 47 is the first term; the part of the solution due to an initial $\mathbf{y}(0)$ is the second term, obviously zero for a system initially at rest. This equation for $\mathbf{y}^*(z)$ is correct in form for any N and M. Higher N and M only result in increased complexity of the equations analogous to equation 45.

After $\mathbf{y}(nT)$ is obtained from equation 47 by the inverse Z-transform or component division, equations 30 and 44 can be used to write:

$$\mathbf{y}(t)=Y(t-nT)\mathbf{y}(nT)+G(t-nT)\mathbf{x}(nT),\quad 0<t-nT\leq T \tag{49}$$

Note that the input vector at $t=nT$ is required.

The complete solution described by equations 47, 48, and 49 is simple in form. For complicated systems, determination of $X(\tau)$, $Y(\tau)$, $G(\tau)$, and the Z-transforms may be lengthy, but the work is straightforward and systematic. To illustrate the method and allow comparison with other methods, when other methods apply, several examples will now be discussed.

Examples

Example I

Fig. 19 shows the symbolic representation for a sampled-data feedback system where the sampled error is clamped and fed into the transfer function $G(p)=K/p(p+a)$. This is an $N=M=1$ system and the results for the $N=M=2$ system apply by taking $t_1=T$ and $\bar{S}_1=I$. The system matrices are:

$$S_1=\begin{Vmatrix}0 & 1 & 0\\ 0 & -a & K\\ 0 & 0 & 0\end{Vmatrix} \tag{50}$$

$$\bar{S}_0=\begin{Vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 0 & 0\end{Vmatrix}$$

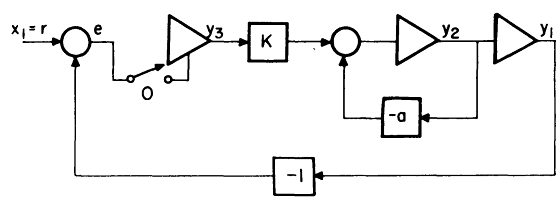

Fig. 19. Sampled-data system example

Now

$$\epsilon^{S_1\tau}=\begin{Vmatrix}1 & \frac{1}{a}(1-\epsilon^{-a\tau}) & \frac{K}{a}\left[\tau-\frac{1}{a}(1-e^{-a\tau})\right]\\ 0 & \epsilon^{-a\tau} & \frac{K}{a}(1-\epsilon^{-a\tau})\\ 0 & 0 & 1\end{Vmatrix} \tag{51}$$

and thus

$$Y=Y(T)=\epsilon^{S_1T}\bar{S}_0=$$
$$\begin{Vmatrix}\left\{1-\frac{K}{a}\left[T-\frac{1}{a}(1-\epsilon^{-aT})\right]\right\} & \frac{1}{a}(1-\epsilon^{-aT}) & 0\\ -\frac{K}{a}(1-\epsilon^{-aT}) & \epsilon^{-a\tau} & 0\\ -1 & 0 & 0\end{Vmatrix} \tag{52}$$

Since $t_1=T$, equation 45 shows that $G=G(T)=G_1(T)+\epsilon^{S_1T}\bar{G}_0$ for $\bar{F}_0=I$. But $G_1=0$ and

$$\bar{G}_0=\begin{vmatrix}0\\0\\1\end{vmatrix} \tag{53}$$

Thus:

$$G=\begin{vmatrix}\frac{K}{a}\left[T-\frac{1}{a}(1-\epsilon^{-aT})\right]\\ \frac{K}{a}(1-\epsilon^{-aT})\\ 1\end{vmatrix} \tag{54}$$

In this case G is the same for any input, so for all $r^*(z)=x_1^*(z)$, $\mathbf{y}^*(z)=z^{-1}(I-z^{-1}Y)^{-1}Gx_1^*(z)$ where it is assumed that system is initially at rest. Computation of the component $y_1^*=c^*(z)$ gives:

$$c^*(z)=\frac{K}{a^2}\frac{\{(aT-1+\epsilon^{-aT})z^{-1}+[1-(aT+1)\epsilon^{-aT}]z^{-2}\}r^*(z)}{1-\left[1-\frac{K}{a^2}(aT-1)+\left(1-\frac{K}{a^2}\right)\epsilon^{-aT}\right]z^{-1}+\left[\frac{K}{a^2}+\left(1-\frac{K}{a^2}-\frac{KT}{a}\right)\epsilon^{-aT}\right]z^{-2}} \tag{55}$$

a result easily checked with the conventional Z-transform theory of sampled-data systems. In the matrix method, manipulation is somewhat more complicated but not excessively so. The solution between samples, conventionally handled by the modified Z-transform, is given by equation 49.

Example 2

Next consider a finite pulsed system analyzed by Farmanfarma.[8] In this system the plant input is $e=r-c$ for $0<t-n<0.6$ and zero for $0.6<t-n<1$. The plant transfer function is $G(p)=6/p(p+5)$. Fig. 20 shows the symbolic diagram for the system and input when $r=x_1=\epsilon^{at}$. Since in this case $N=2$ and $M=0$, $\bar{S}_0=\bar{S}_1=\bar{F}_0=\bar{F}_1=I$ and $\bar{G}_0=\bar{G}_1=0$. By inspection:

$$S_1=\begin{Vmatrix}0 & 1\\ -6 & -5\end{Vmatrix},\quad G_1=\begin{vmatrix}0\\6\end{vmatrix}$$

$$S_2=\begin{Vmatrix}0 & 1\\ 0 & -5\end{Vmatrix},\quad G_2=\begin{vmatrix}0\\0\end{vmatrix} \tag{56}$$

From equations 10 and 11, desired input is obtained when $F_1=F_2=a$. After obtaining $\epsilon^{A_1\tau}$ and $\epsilon^{A_2\tau}$ and computing $Y=\epsilon^{A_10.4}\epsilon^{A_20.6}$, it is possible to find.

$$(I-z^{-1}Y)^{-1}=\begin{Vmatrix}\frac{1+0.0144z^{-1}}{1-0.417z^{-1}+0.0067z^{-2}} & \frac{0.1175z^{-1}}{1-0.417z^{-1}+0.0067z^{-2}}\\ (\quad) & (\quad)\end{Vmatrix} \tag{57}$$

Only $y_1^*(z)=c^*(z)$ is of interest so the second row is not computed.

If stability is the only question, only the denominator of each component, equal to the $\det(I-z^{-1}Y)$, need be computed. In this case the system is stable since both the z roots are within the unit circle. Bertram and Kalman in a discussion of Farmanfarma's paper[8] discussed this method of stability determination.

To continue with the response problem, G is obtained from equation 45, which in this case is the simpler form:

$$G(\tau)=\epsilon^{S_2(\tau-t_1)}G_1(t_1)$$

where

$$G_1(\tau)=L^{-1}\left[(sI-S_1)^{-1}G_1\frac{1}{s-a}\right]$$

Carrying out the computation for $G=G(1)$,

$$G=\frac{1}{\left(\frac{a}{2}+1\right)\left(\frac{a}{3}+1\right)}\begin{vmatrix}-0.1175a-0.432+(1+0.173a)\epsilon^{0.6a}\\ 0.0144a-0.1104+0.1353a\epsilon^{0.6r}\end{vmatrix} \tag{58}$$

In this case, G depends on the input indicating the greater complexity of finite pulsed systems as compared with sampled-data systems.

Assuming the system is initially at rest,

$$y_1^*(z)=c^*(z)=$$
$$\frac{[-0.1175a-0.432+(1+0.173a)\epsilon^{0.6a}]z^{-1}+[0.0067+(0.0144+0.0184a)\epsilon^{0.6a}]z^{-2}}{\left(\frac{a}{2}+1\right)\left(\frac{a}{3}+1\right)(1-0.417z^{-1}+0.0067z^{-2})(1-\epsilon^{a}z^{-1})} \tag{59}$$

is obtained from equation 47. Setting $a=0$ and dividing out the polynomial gives $c(n)$ values that check with Farmanfarma's Fig. 10.[8]

Example 3

As a final example consider the finite pulse-clamped system shown in Fig. 21. Take $T=1$ so that the plant receives the actual error for $0<t-n\leq t_1$, and the t_1 clamped error for $t_1<t-n<1$, where the plant has the transfer function $G(p)=2/p$. The initial conditions $x_1(0)=1$, and $y_1(0)=y_2(0)=0$ specify a step-response problem.

By inspection:

$$S_1=\begin{Vmatrix}-2 & 0\\ 0 & 0\end{Vmatrix},\quad \bar{S}_0=\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}$$

$$S_2=\begin{Vmatrix}0 & 2\\ 0 & 0\end{Vmatrix},\quad \bar{S}_1=\begin{Vmatrix}1 & 0\\ -1 & 0\end{Vmatrix} \tag{60}$$

$$G_1=\begin{vmatrix}2\\0\end{vmatrix},\quad \bar{G}_0=\begin{vmatrix}0\\0\end{vmatrix}$$

$$G_2=\begin{vmatrix}0\\0\end{vmatrix},\quad \bar{G}_1=\begin{vmatrix}0\\1\end{vmatrix}$$

$$F_1=F_2=0,\quad \bar{F}_0=\bar{F}_1=1$$

After $\epsilon^{A_1\tau}$ and $\epsilon^{A_2\tau}$ are obtained,

$$Y=\epsilon^{A_2(1-t_1)}\bar{A}_1\epsilon^{A_1t_1}A_0=\begin{Vmatrix}\epsilon^{-2t_1}(2t_1-1) & 0\\ -\epsilon^{-2t_1} & 0\end{Vmatrix} \tag{61}$$

is readily found. Then:

$$(I-z^{-1}Y)^{-1}=\begin{Vmatrix}\frac{1}{1-\epsilon^{-2t_1}(2t_1-1)z^{-1}} & 0\\ \frac{-\epsilon^{-2t_1}z^{-1}}{1-\epsilon^{-2t_1}(2t_1-1)z^{-1}} & 1\end{Vmatrix} \tag{62}$$

Stability is determined by the component denominators. In this case the system is stable for all t_1 except $t_1=0$, where neutral stability exists.

By equations 45 and 60,

$$G=G(1)=\epsilon^{S_2(1-t_1)}\bar{G}_1+\epsilon^{S_2(1-t_1)}\bar{S}_1G_1(t_1) \tag{63}$$

where by equation 43

$$G_1(\tau)=L^{-1}\left[\begin{Vmatrix}\frac{1}{s+2} & 0\\ 0 & \frac{1}{s}\end{Vmatrix}\begin{vmatrix}2\\0\end{vmatrix}\frac{1}{s}\right] \tag{64}$$

Carrying out the indicated operations:

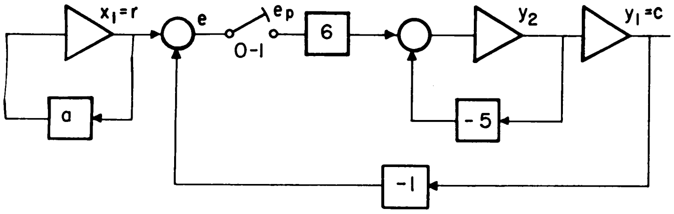

Fig. 20 (above). Finite pulsed system example

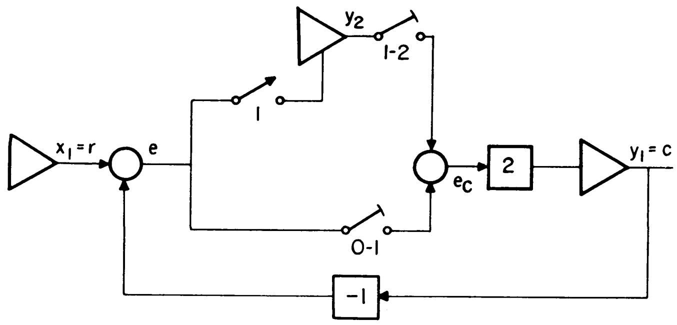

Fig. 21 (right). Finite-pulse clamped-system example

$$G=\begin{vmatrix}1-(2t_1-1)\epsilon^{-2t_1}\\ \epsilon^{-2t_1}\end{vmatrix} \tag{65}$$

Substituting in equation 47 from 62 and 65, the component $y_1^*(z)=c^*(z)$ is given by:

$$y_1^*(z)=c^*(z)$$
$$=\frac{[1-(2t_1-1)\epsilon^{-2t_1}]z^{-1}}{[1-(2t_1-1)\epsilon^{-2t_1}z^{-1}](1-z^{-1})} \tag{66}$$

This result reduces to the sampled-data case when $t_1=0$ and the continuous case when $t_1=1$. Applications of the final value theorem shows that $c(\infty)=1$ as expected. The system has dead-beat response for $t_1=1/2$. More complicated examples of finite pulse-clamped systems are analyzed in a similar way.

Conclusions

This paper presents methods of symbolic representation and analysis applicable to an extensive class of periodic linear systems. The symbolic representations are easily obtained and lead directly to a concise mathematical formulation in terms of a state vector representing both the input and system variables. System response at multiples of the fundamental period, obtained by Z-transform methods, is presented in a concise vector matrix notation. The usual Z-transform methods for stability prediction and final value determination apply. Response interior to the fundamental periods is readily obtained using time matrices and the solution at multiples of the fundamental period.

Appendix

The matrix time function

$$\epsilon^{A\tau}=L^{-1}[(sI-A)^{-1}] \tag{67}$$

results naturally from the vector differential equation and initial condition

$$\frac{d\mathbf{w}}{d\tau}=A\mathbf{w},\ \mathbf{w}(0) \tag{68}$$

This is obvious from the Laplace transform of equation 68 which gives:

$$\bar{\mathbf{w}}(s)=(sI-A)^{-1}\mathbf{w}(0) \tag{69}$$

The computation of $\epsilon^{A\tau}$ by equation 67 is perfectly straightforward but may become lengthy if the matrix order Q is high. One difficulty is the rapidly increasing complexity of matrix inversion with increasing Q; another problem is that the $(sI-A)^{-1}$ components, in general $Q-1$ order polynomials in s divided by Q order polynomials in s, rapidly increase in number and complexity with increasing Q. However, there are frequently many zero entries in the A matrix. As a result, matrix inversion is less complex than expected. Also, $(sI-A)^{-1}$ typically has many components that are zero or simple in form.

In most cases the computation of $\epsilon^{A\tau}$ is greatly simplified by relating the components of $\epsilon^{A\tau}$ to the symbolic diagram. To see this, write:

$$\mathbf{w}(\tau)=\epsilon^{A\tau}\mathbf{w}(0) \tag{70}$$

obtained by applying the inverse Laplace transform to equation 69 and using equation 67. Assume that the jth component of $\mathbf{w}(0)$ is one and that all other components of $\mathbf{w}(0)$ are zero. Then by equation 70, the ith component of $\mathbf{w}(\tau)$ is the i, j component of $\epsilon^{A\tau}$. In terms of the symbolic diagram, the i, j component of $\epsilon^{A\tau}$ is the ith integrator output for unit initial condition on the jth integrator and zero initial condition on all other integrators. This interpretation of $\epsilon^{A\tau}$ immediately gives all the components of $\epsilon^{A\tau}$ which are zero and frequently allows simple evaluation of the remaining nonzero components.

To illustrate, consider the system in Fig. 18. Since $\epsilon^{A\tau}$ describes the system between sample intervals, neglect the sample switch. Consider first-unit initial value on w_1. By inspection of Fig. 18 i, 1 components, forming the first column of $\epsilon^{A\tau}$, are 1, 0, 0, and $1-\epsilon^{A\tau}$. For unit initial value on w_2, the i, 2 components forming the second column of $\epsilon^{A\tau}$ are τ, 1, 0, and $\tau-1/a\ (1-\epsilon^{-a\tau})$. The other columns are obtained similarly. Thus:

$$\epsilon^{A\tau}=\begin{Vmatrix}1 & \tau & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 1-\epsilon^{-a\tau} & \tau-\frac{1}{a}(1-\epsilon^{-a\tau}) & 0 & \epsilon^{-a\tau}\end{Vmatrix} \tag{71}$$

In more complicated systems, any component of $(sI-A)^{-1}$ can be obtained from the symbolic block diagram using the same basic idea. Thus the i, j component to $(sI-A)^{-1}$ is the transfer function relating the ith integrator output to an artificial input summed in directly at the jth integrator output. This is made clear in Fig. 22 for the system of Fig. 4. Here the i, 2 components of $(sI-A)^{-1}$ are the transfer functions relating y_i to the artificial input u_2. For example, the 3, 2 component of $(sI-A)^{-1}$ is the transfer function:

$$\frac{-2s}{s^2+4.5s+2}$$

relating y_3 to u_2.

Another method, particularly useful in approximation, is to use the infinite series representation of $\epsilon^{A\tau}$. By substitution into equations 70 and 68, it is easily shown that the infinite matrix series:

$$\epsilon^{A\tau}=I+A\tau+A^2\frac{\tau^2}{2!}+A^3\frac{\tau^3}{3!}+\ldots \tag{72}$$

is valid. When τ is small, two or three terms of this series yield a good approximation to the desired function.

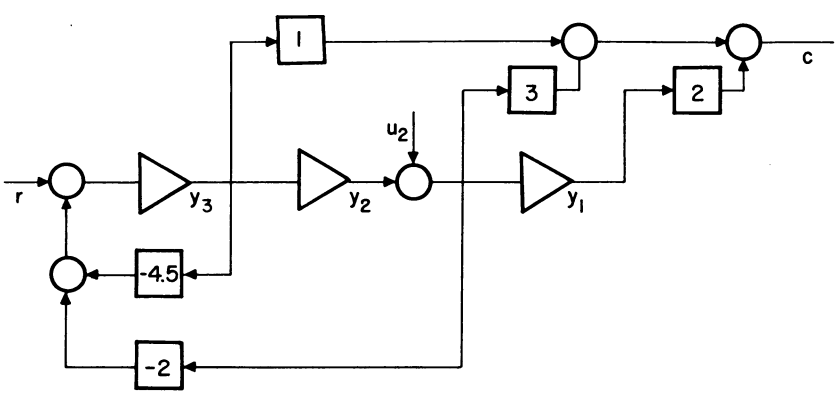

Fig. 22. Symbolic diagram method for determining $(sI-A)^{-1}$ components

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Discussion

E. I. Jury (University of California, Berkeley, Calif.): The author is to be commended for the general analysis of a large class of periodic feedback systems in a unified symbolic representation. This form of representation is very useful in interpreting the steps of the mathematical formulation and is, in some cases, useful in the simulation studies of these systems. Among the important classes this method describes is the case of periodically varying parameters. This includes the category of gain, time constants, pulse duration or the sampling period, and periodic variations. This category is among the topics of recent investigation in this field and this paper appears timely and illuminating.

The analytical solution of the problem is generally based on two steps: 1. the formulation of the difference equations which describe the systems, and 2. the stability study and solutions of these equations for certain periodic or aperiodic inputs or disturbances. For the class of systems mentioned, the difference equations are generally linear with periodic coefficients and, for simpler cases, with constant coefficients.

To obtain the difference equations for these systems, the author has made use of the state vector[1] concept; however, at this point it might be well to mention that one can also arrive at this step by simple manipulation of the p-transform and Z-transform methods or by writing the difference equations in time domain if the transfer function is given (or can be easily obtained) in terms of differential equations.[2] For the solution of these difference equations, there also exist several methods, among them the Z-transform[3] method and the matrix form,[4] or a combination of both, as used by the author. The choice among the several methods of formulation and solution depends on the problem and on ease and convenience of arriving at the required results. However, the labor involved is more or less equal among the several methods.

I have recently proposed in a paper written with T. Nishimura the analysis of finite pulsed feedback systems with periodically varying sampling rate and pulse width. Fig. 23 illustrates the basic concepts involved in this problem. Although this case was not treated by the author, the method of the paper is applicable, for it falls under the general category (N, M) with $M=0$ for this case. Our method of approach, based on the p-transform and Z-transform methods known for the conventional cases, is quite different from the method of the paper, indicating an alternate approach to analysis of the general class of (N, M) systems. It might be indicated that for the simpler cases of linear sampled-data systems, the basic Z-transform or p-transform methods are easily applicable without having to use the general method of the paper. However, it is for larger values of (N, M) that this method becomes more effective and useful.

Furthermore, it is worthwhile to mention that the class of systems having piecewise constant-varying sampling rate[5] shown in Fig. 24, can also be solved by minor extension of this paper's method. Further work on investigating the various sampling schemes as well as the synthesis procedure that could be tackled with this method is indeed warranted.

In conclusion, this paper represents a useful contribution to the growing area of the field and the symbolic method introduced will undoubtedly enhance the methods and techniques available for the formulation and analysis of the general class of linear periodic feedback systems.

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Julius T. Tou (Purdue University, Lafayette, Ind.): Dr. Gilbert is to be commended for his valuable contribution to the literature of

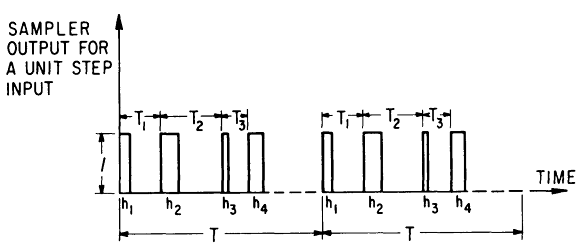

Fig. 23. Periodically time-varying sampling rate and pulse width

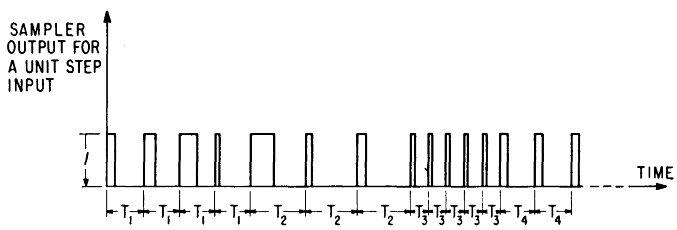

Fig. 24. Piecewise constant rate finite pulse system, settling into a fixed period

sampled-data control systems. The method presented by the author is systematic and straightforward and appears to be a better technique for analyzing finite pulsed feedback control systems by use of Z-transforms. However, it appears that the new symbolic representation by the author does not simplify the problem. On the other hand, the conventional symbolic representation is probably easier and simpler to use. The continuous-data part of the system and the continuous input signals can be represented by the conventional analog-computer simulation diagrams. The pulsed-data part of the system can be described by the conventional symbolic representation given in reference 3 of the paper, with each ideal sampler followed by a zero-order hold. When a sampled-data system is so represented, the state and the transition equations, like equations 8 and 9 of the paper, can readily be obtained.

It would be clearer if $t_0=0$ is added to equation 5 which defines the switching operation analytically. According to the definition of equation 5, Fig. 8(B) can represent a number of switching operations depending upon the value of t_2, and it is equivalent to Fig. 8(A) only if $t_2=T$.

In general, system analysis and synthesis may be carried out by two major approaches. One approach involves the determination of the transfer characteristics of the system components and the over-all transfer characteristic. This approach is usually effected by a block diagram representation and may be referred to as the block diagram approach. The other approach is based upon the characterization of a system by a number of simple first-order differential equations describing the state variables, with the initial conditions given by the transition equations. Each component of a system is decomposed into the basic mathematical elements describing it. This approach is usually effected by a state variable diagram and may be referred to as the state diagram approach.

The author has written a concise and lucid paper on the state diagram approach of the analysis of linear periodic feedback systems.

Edward O. Gilbert: The discussions by Professors Jury and Tou are indeed appreciated. Their discussions and recent papers,[1,2] published or submitted after submission of the author's paper, indicate high interest in more general time-variant operations.

Professor Jury is correct in pointing out the usefulness of the representation in system simulation. There is a 1-to-1 correspondence between the symbolic diagram and differential analyzer setup, including discrete components. Using this analogy, the author has successfully simulated digital computer components by approximating the sampling operation described in the paper. The method of the paper does not lead to difference equations with periodic coefficients as does the method of reference 2. The reduction of the periodic difference equations to constant coefficient difference equations[2,3] is inherent in the computation of the matrix $W(T)$. The p-transform and Z-transform methods for writing difference equations are certainly acceptable, but are more difficult to generalize to the extensive class considered in this paper. This, of course, does not invalidate the usefulness of such approaches, which for certain systems, primarily low (N, M) systems, may be more desirable. The same may be said for the various techniques of solving the resulting difference equations.

Professor Tou's statement that the symbolic representation does not simplify the problem and is not necessary to readily derive the state and transition equations is only true for the simplest of systems. For example, the finite pulse-clamping operation is not easily described by conventional representations. The correction in regard to Fig. 8(B) is appreciated.

References

1. See reference 1 of the Jury discussion.
2. See reference 2 of the Jury discussion.
3. See reference 3 of the Jury discussion.

Executive-Controlled Adaptive Systems

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THERE IS an immense body of literature on the general subject of feedback control theory. The greatest portion of this literature is devoted to the analysis and synthesis of single-loop or simple multiloop linear or piecewise-linear feedback systems. In most of this work, there is a common fundamental assumption, and that is that the designer has, or can obtain, reasonably complete knowledge of the properties of the device or process which is to be controlled. This knowledge generally consists of a fairly accurate characterization of the process or device under one set of environmental conditions, along with information indicating the manner and the limits within which the properties of the process vary with time and with different external conditions. It is the ability of closed-loop control to render the complete system relatively insensitive to inaccuracies in the characterization and to variations in the properties of the process to be controlled that has resulted in the phenonomenal growth of the field.

In this introductory era of supersonic flight, guided missiles, space travel, and complex industrial processes, the control engineer is frequently confronted with the problem of designing control systems for processes where little significant information is known about the process, where the properties of the process vary over an extraordinarily large range, and where the characteristics of the input signals change markedly with time. It is the purpose of this paper to present an approach to the design of feedback control systems which will enable the designer to cope with such situations when they arise.

Adaptive Systems

A block diagram of a single-loop feedback system is shown in Fig. 1. The compensator can be a passive or an active network, linear or nonlinear, a digital controller, or an analog computer. If the characteristics of the process and of the input signal are known, there are numerous procedures for designing the compensator so that some desired performance is obtained. Deviations from this desired performance due to changes in the process and/or the input signal often can be made acceptably small by increasing the loop gain. If, however, the parameter variations are extremely large, the gain required to achieve the specified system performance may become so high as to be unobtainable because of noise or saturation limitations. As an alternative to increasing the gain, the transfer char-

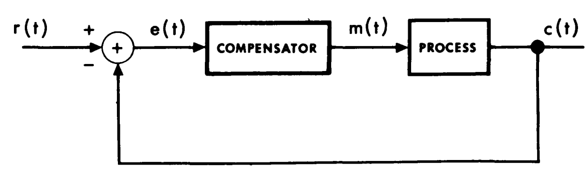

Fig. 1. General single-loop feedback system

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