Communications and Control Engineering Series

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Analysis of Periodically Time-Varying Systems

With 73 Figures

Springer-Verlag Berlin Heidelberg New York 1983

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ISBN-13:978-3-642-81875-2 e-ISBN-13:978-3-642-81873-8
DOI: 10.1007/978-3-642-81873-8

Library of Congress Cataloging in Publication Data
Richards, John Alan, 1945–.
Analysis of periodically time-varying systems.
(Communications and control engineering series)
Bibliography: p. Includes index. 1. System analysis.
I. Title. II. Series.
QA402.R47 1983 003 82-5978 AACR2

Softcover reprint of the hardcover 1st edition 1983

Typesetting: Asco Trade Typesetting Ltd., Hong Kong

2061/3020-543210

To Dick Huey

Preface

Many of the practical techniques developed for treating systems described by periodic differential equations have arisen in different fields of application; consequently some procedures have not always been known to workers in areas that might benefit substantially from them. Furthermore, recent analytical methods are computationally based so that it now seems an opportune time for an applications-oriented book to be made available that, in a sense, bridges the fields in which equations with periodic coefficients arise and which draws together analytical methods that are implemented readily. This book seeks to fill that role, from a user's and not a theoretician's view.

The complexities of periodic systems often demand a computational approach. Matrix treatments therefore are emphasized here although algebraic methods have been included where they are useful in their own right or where they establish properties that can be exploited by the matrix approach. The matrix development given calls upon the nomenclature and treatment of H. D'Angelo, Linear Time-Varying Systems: Analysis and Synthesis (Boston: Allyn and Bacon 1970) which deals with time-varying systems in general. It is recommended for its modernity and comprehensive approach to systems analysis by matrix methods. Since the present work is applications-oriented no attempt has been made to be complete theoretically by way of presenting all proofs, existence theorems and so on. These can be found in D'Angelo and classic and well-developed treatises such as

McLachlan, N. W.: Theory and application of Mathieu functions. Clarendon: Oxford U.P. 1947. Reprinted by Dover, New York 1964.
Arscott, F. M.: Periodic differential equations. Oxford: Pergamon 1964.
Magnus, W.; Winkler, S.: Hill's equation. New York: Wiley 1966.

Instead, this book relates theory to applications via analytical methods that are, in the main, computationally-based.

The book is presented in two parts. The first deals with theory, and techniques for applying that theory in the analysis of systems. A highlight of this (chapter five) is the development of modelling procedures that allow intractable periodic differential equations to be handled. This is regarded as significant since the great majority of differential equations with periodic coefficients cannot be treated by closed form methods of analysis. The second part presents an overview of the applications of periodic equations. The particular applications chosen have been done so to illustrate the variety of ways periodic differential equation descriptions arise and to demonstrate that the modelling procedures of Part I can be useful in determining system properties.

The developments in Part I and the applications of Part II are all related to a standard form for the equations, which in the case of second order systems is the canonical Hill equation used by McLachlan, viz.

$$\ddot{x} + (a - 2q\psi(t))x = 0, \psi(t) = \psi(t + \pi)$$

where $\psi(t)$ is a general periodic coefficient. Adopting such a canonical form is of value if the results of the theory and techniques chapters are to be used directly. Original system equations for particular applications presented in Part II are thus transformed into the appropriate canonical form before drawing upon the material of Part I.

The treatment is not intended to provide a text for the study of periodic differential equations but could be used for a single semester senior undergraduate or graduate level subject in systems with periodic parameters, particularly if applications are to be emphasised.

The encouragement and assistance of others is of course essential in producing a book. It is the author's pleasure to acknowledge the inspiration of his mentor and friend Professor Dick Huey to whom this book is dedicated; over the years of their association he has done much to encourage in a quiet, yet effective, manner the completion of this work. The manuscript was typed by Mrs. Gellisinda Galang to whom the author is grateful for the competent and patient manner in which she undertook the task.

Kensington, Australia, March 1982 J. A. Richards

Contents

List of Symbols

a	constant coefficient in a Hill or Mathieu equation
$a_0, a_1 \cdots$	stability boundaries for a Hill or Mathieu equation
A_{ir}	residues of the Hill determinant
$b_1, b_2 \cdots$	stability boundaries for a Hill or Mathieu equation
B_1, B_2	boundary modes for a Hill equation
c	capacitance per unit length
C	capacitance; discrete transition matrix; solution type of a Hill equation corresponding to complex eigenvalues of the discrete transition matrix
C_r	expansion coefficients in a Floquet solution
ce_m	Mathieu function of the first kind of order m
Ce_m	modified Mathieu function of the first kind of order m
ce_ν	Mathieu function of the first kind of fractional order ν
ceu_ν	unstable Mathieu function of the first kind of fractional order ν
$d_i(t)$	weighting coefficients
$\boldsymbol{D}$	electric displacement vector
e	eccentricity; charge on ion
E	Young's modulus
$\boldsymbol{E}$	electric field vector
$\boldsymbol{E}(t)$	time varying electric field
E_g	generator voltage
f	semi interfocal distance of an ellipse
$f(t), \boldsymbol{f}(t)$	forcing function, and vector form, in a periodic system; general function of time
F	force
$g_i(t)$	periodic coefficient in a general order Hill equation
$G(t)$	periodic coefficient matrix
G_n	complex Fourier coefficient of $g(t)$.
h	eigenvalue of a waveguide mode
h_k, h_r	step heights in a staircase periodic coefficient
$h(t)$	impulse response of a linear, time-invariant system
$\hbar$	$h/2\pi$, h is Planck's constant
$\boldsymbol{H}$	magnetic field vector
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors in a cartesian coordinate system
$i(t)$	current
I	current phasor; second moment of area
J_ν	Bessel function of the first kind of order ν
k	wave number
l	inductance per unit length
L	inductance
m	mass of an ion; distributed mass per unit length; waveguide mode number
M_+, M_-	state transition matrices
n	waveguide mode number
N	solution type of a Hill equation corresponding to a negative eigenvalue of the discrete transition matrix

$p(t)$	periodic staircase modulating function
P	solution type of a Hill equation corresponding to a positive eigenvalue of the discrete transition matrix
$P(t)$	periodic matrix
P_m	complex Fourier coefficient of $p(t)$
q	half amplitude of the periodic coefficient in a Hill or Mathieu equation
$q_{mn}^{c \text{ or } s}$	nth value of q that gives a zero of the mth order modified Mathieu function of the first or second kind
q	charge
$q(t)$	periodic staircase modulating function
Q_m	complex Fourier coefficient of $q(t)$
r	radial polar coordinate
r_0	field radius in a quadrupole mass filter
R	resistance
R_g	generator impedance
R_L	load impedance
R_s	series resistance of varactor diode
s	complex frequency variable; Laplace transform variable
s_i	root of system characteristic equation
se_m	Mathieu function of the second kind of order m
Se_m	modified Mathieu function of the second kind of order m
t	real time
T	period; tension
$T_r(x)$	Chebyshev polynomial
U	dc potential
v	phase velocity
$v(t)$	voltage
V	Voltage phasor; magnitude of periodic potential
$W(t)$	Wronskian matrix; unit periodic sampling function; impulse response of an N path network
W_0	Wronskian (determinant)
$\boldsymbol{x}(t)$	state vector for a periodically time-varying system
$\bar{X}(s)$	single sided Laplace transform of the state vector $\boldsymbol{x}(t)$
Y_ν	Bessel function of the second kind of order ν
α	real part of the characteristic exponent
α_k	constant coefficient in a general periodic differential equation
β	imaginary part of the characteristic exponent; phase constant inside a waveguide
β_0	phase constant in an unbounded medium
β_r	phase constant for the rth space harmonic in a periodic structure
γ	propagation constant inside a waveguide
γ_k	amplitude parameter in a general periodic differential equation
γ_0	propagation constant in an unbounded medium
Γ	exponent matrix associated with the discrete transition matrix; Gamma function
δ	duty cycle parameter for a rectangular or trapezoidal waveform
$\delta(t)$	Delta (impulse) function
$\Delta(\ )$	Hill infinite determinant
ε	permittivity; rise or fall time in a trapezoidal waveform
ζ	canonical displacement variable; damping constant
η	normalised rise and fall time in a trapezoidal waveform; elliptic coordinate
θ	period of the periodic coefficients in a general order Hill equation
λ_c	cut-off wavelength in a waveguide
λ_i	eigenvalue of the discrete transition matrix
Λ	matrix of eigenvalues of the discrete transition matrix
μ	permeability
μ, μ_i	characteristic exponent in the solution to a periodic differential equation

ξ	canonical time variable; elliptic coordinate
$\Xi(\ )$	periodic function in the solution to a periodic differential equation
π	pi; period of the periodic coefficient in a Hill or Mathieu equation
ρ_r	pole of the Hill determinant
σ	conductivity
τ	positive porch length in a rectangular or trapezoidal waveform
τ_r, τ_f	rise and fall times in a trapezoidal waveform
ϕ	scalar electric potential; polar coordinate
$\phi(t, 0)$	state transition matrix over the interval $(0, t)$
$\phi(\theta, 0)$	discrete state transition matrix for a general periodic differential equation
$\phi(\pi, 0)$	discrete state transition matrix for a Hill or Mathieu equation
χ	canonical time interval less than π
χ_{mn}^{e}	nth zero of the mth order Bessel function of the first kind
χ_{mn}^{h}	nth zero of the first derivative of the mth order Bessel function of the first kind
ψ	matrix of eigenvectors of the discrete transition matrix
$\psi(t)$	periodic function; voltage applied to the electrodes of a quadrupole mass filter
ψ_i	eigenvector of the discrete transition matrix
Ψ_n	complex Fourier coefficient of $\psi(t)$
ω	frequency
ω_c	cut-off frequency of a waveguide
ω_i	ith natural frequency of a linear, time-invariant system; system input (forcing) frequency
ω_l	idler frequency
ω_0	resonant frequency of a second order system
ω_p	pump frequency
∇^2	Laplacian in three dimensions
∇_t^2	transverse (two dimensional) Laplacian