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A lifting technique for linear periodic systems with applications to sampled-data control *

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Abstract: A lifting technique is developed for periodic linear systems and applied to the $\mathscr{H}^\infty$ and $\mathscr{H}^2$ sampled-data control problems.

Keywords: Sampled-data system; lifted system; $\mathscr{H}^\infty$ optimal control; Riccati equation; operator norm.

1. Introduction

Given the success of $\mathscr{H}^\infty$-norm based optimization methods for analog control systems, there has recently been interest in applying such techniques to sampled-data systems [3,4,15,18]. The key point in utilizing such methods would be in their extension to certain periodic time-varying systems. An example of such a system is the sampled-data control system shown in Figure 1 below.

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The generalized plant G is a continuous-time, time-invariant system, K_d is discrete-time, time-invariant, S is the ideal periodic sampler with period h, and H the synchronized zero-order hold. Continuous-time signals are represented by continuous lines, discrete-time signals by dotted lines. The behavior of the system from the exogenous input w to the controlled output z is in general time-varying, in fact, periodic with period h.

To analyze the behavior of continuous-time periodic systems, we use a lifting technique similar to that used for discrete-time periodic systems in [16]. Once we develop the lifting technique, we apply it to describe a complete solution to the analysis problem of verifying that a given controller constrains the $\mathscr{L}^2$-induced norm of the sampled-data system to be less than some prespecified level. We will also show that the lifting technique is applicable in fact to all norm-based optimization problems, and in particular to sampled-data versions of the quadratic regulator and optimal filtering problems.

Given the success of $\mathscr{H}^\infty$-norm based optimization methods for analog control systems, there has recently been interest in applying such techniques to sampled-data systems [3,4,15,18,25].

The purpose of this note is to introduce the lifting technique itself and sketch how it can be applied to two optimal control problems. To our knowledge, such a lifting procedure was introduced into sampled-data systems by Toivonen [25], who also treats the $\mathscr{H}^\infty$ sampled-data problem. The details of the lifting in [25] are different from those given here ([25] represents certain finite-rank operators via SVD, which is avoided in our work). The mathematical basis of such lifting techniques may be found in [21]. Reference [1] gives a detailed account of the application of the lifting technique to the $\mathscr{H}^\infty$ sampled-data problem. Yamamoto [28] also uses lifting for sampled-data systems, but he lifts the state as well as the input and output. Consequently, his state space is infinite-dimensional, whereas ours is the original finite-dimensional one. Also, Yamamoto treats asymptotic tracking problems, while optimization problems are studied here.

While this paper was being reviewed, several others came into existence. For completeness we mention them here: a sampled-data $\mathscr{H}_2$ problem (different from that in [4]) in [2,17]; sample-data $\mathscr{L}_1$ (i.e., $\mathscr{L}_\infty$ induced norm) in [23,8]; and robust stability of sampled-data systems in [24].

In the operator norm design framework, this lifting technique was developed independently by the first two and the latter two authors. Reference [1] gives a detailed account of the application of this technique to the $\mathscr{H}^\infty$ sampled-data problem.

2. Lifting continuous-time signals

In this section we introduce a construction whereby one may 'lift' a continuous-time signal to a discrete-time one. This construction will also be used to associate a time-invariant discrete-time system to a continuous-time periodic one. The utility of this technique in feedback control is that all norms are preserved, as well as the feedback interconnection structure.

We will first work in a rather general framework before specializing to the case of interest. Let $\mathscr{X}$ denote a Banach space equipped with norm $\| \ \|_{\mathscr{X}}$. For every integer $p \geq 1$ we set

$$\mathscr{L}^p(\mathscr{X}) := \left\{ u : [0, \infty) \to \mathscr{X} : \int_0^\infty \| u(t) \|_{\mathscr{X}}^p \, \mathrm{d}t < \infty \right\}.$$

As is well-known, $\mathscr{L}^p(\mathscr{X})$ is a Banach space with norm

$$\| u \|_{p,\mathscr{X}} := \left(\int_0^\infty \| u(t) \|_{\mathscr{X}}^p \, \mathrm{d}t \right)^{1/p}.$$

For $p = 2$, $\mathscr{L}^2(\mathscr{X})$ may be given the structure of a Hilbert space in the usual way. For $p = \infty$, we have

$$\mathscr{L}^\infty(\mathscr{X}) := \{ u : [0, \infty) \to \mathscr{X} : \text{ess sup} \| u(t) \|_{\mathscr{X}} < \infty \}$$

(see, e.g., [22]). Finally, to each of the spaces $\mathscr{L}^p(\mathscr{X})$ we may associate the extended space $\mathscr{L}_e^p(\mathscr{X})$ in the standard way. For all the definitions see [6].

The same types of definitions are of course valid in the discrete-time case for sequences. A sequence will be written as a column vector, for example,

$$\psi = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \end{bmatrix}.$$

Again for any Banach space $\mathscr{X}$, define

$$l^p(\mathscr{X}) = \left\{ \psi : \psi_i \in \mathscr{X}, \sum_{i=0}^\infty \| \psi_i \|_{\mathscr{X}}^p < \infty \right\}, \quad 1 \leq p < \infty,$$

$$l^\infty(\mathscr{X}) = \left\{ \psi : \sup_i \| \psi_i \|_{\mathscr{X}} < \infty \right\}.$$

The norms are given by

$$\| \psi \|_{l^p(\mathscr{X})} = \left(\sum_{i=0}^\infty \| \psi_i \|_{\mathscr{X}}^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\| \psi \|_{l^\infty(\mathscr{X})} = \sup_i \| \psi_i \|_{\mathscr{X}}.$$

Equipped with this norm $l^p(\mathscr{X})$ is a Banach space for all $1 \leq p \leq \infty$. Once again for $p = 2$, $l^2(\mathscr{X})$ may be given a Hilbert structure in the usual way [22], and the associated extended space $l_e^p(\mathscr{X})$ may be defined: it is just the linear space of all sequences in $\mathscr{X}$.

We are now ready to describe the lifting procedure. For fixed $h > 0$ let

$$\mathscr{K}^p := \{ u \in \mathscr{L}^p(\mathscr{X}) \text{ with support in } [0, h) \}.$$

Once again $\mathscr{K}^p$ is a Banach space in the natural way with norm induced by $\| \ \|_{p,\mathscr{X}}$. Suppose u is an element of $\mathscr{L}_e^p(\mathscr{X})$. Chop u up into its components as follows:

$$u_0(t) = u(t), \quad 0 \leq t < h,$$
$$u_1(t) = u(t+h), \quad 0 \leq t < h,$$
$$u_2(t) = u(t+2h), \quad 0 \leq t < h,$$

etc.

Each piece, u_i, belongs to $\mathscr{K}^p$. Now form the sequence

$$\psi = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \end{bmatrix}.$$

Define the *lifting operator* W_p to be the map $u \mapsto \psi$. It maps $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$. We sometimes write just W when p is irrelevant.

It is important to note that W is a linear bijection from $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$ whose inverse is given by

$$u = W^{-1}\psi \quad \Leftrightarrow$$
$$u(t) = \psi_i(t - hi), \quad hi \le t < h(i+1).$$

It is easy to show that the restriction of W to the Banach space $\mathscr{L}^p(\mathscr{X}) \subset \mathscr{L}_e^p(\mathscr{X})$ is an isometry, $\mathscr{L}^p(\mathscr{X}) \to l^p(\mathscr{K}^p)$.

To recap, W is a bijective linear mapping from $\mathscr{L}_e^p(\mathscr{X})$ to $l_e^p(\mathscr{K}^p)$, and a bijective linear *isometry* from $\mathscr{L}^p(\mathscr{X})$ to $l^p(\mathscr{K}^p)$.

Of course, one may also lift systems. Let $G: \mathscr{L}_e^p(\mathscr{X}) \to \mathscr{L}_e^q(\mathscr{X})$ be a linear operator. Then the lifted system is defined to be $\tilde{G} = W_q G W_p^{-1}$, mapping $l_e^p(\mathscr{K}^p)$ to $l_e^q(\mathscr{K}^q)$. By the linearity of each of the defining operators, $\tilde{G}$ is linear. Moreover, if G is also bounded $\mathscr{L}^p(\mathscr{X}) \to \mathscr{L}^q(\mathscr{X})$, then $\tilde{G}$ is bounded too. Since W_p and W_q are isometries, one sees that $\|G\| = \|\tilde{G}\|$, that is, the system (operator) norm is preserved by the lifting. Furthermore, since the lifting procedure is isometric and preserves all the standard algebraic and feedback interconnection operations, feedback stability is also preserved under lifting.

Now if the system to be lifted is h-periodic, then the lifted system will be time-invariant. To see this, introduce the *delay operator* D_h, defined by $(D_h f)(t) = f(t-h)$. Gives a (causal) system $G: \mathscr{L}_e^p(\mathscr{X}) \to \mathscr{L}_e^q(\mathscr{X})$, we say that G is *h-periodic* if it commutes with D_h, that is, $D_h G = G D_h$. (G is *time-invariant* if it is h-periodic for every $h > 0$.) Let U be the unilateral shift operator on sequences:

$$U \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \psi_0 \\ \psi_1 \\ \vdots \end{bmatrix}, \quad \psi_i \in \mathscr{X}.$$

It is easy to compute that $W_p D_h W_p^{-1} = U$ on $l_e^p(\mathscr{K}^p)$ for any $p \ge 1$. Consequently, for $G: \mathscr{L}_e^p(\mathscr{X}) \to \mathscr{L}_e^q(\mathscr{X})$ h-periodic,

$$\begin{aligned} U\tilde{G} &= W_q D_h W_q^{-1} W_q G W_p^{-1} \\ &= W_q D_h G W_p^{-1} \\ &= W_q G D_h W_p^{-1} \\ &= W_q G W_p^{-1} W_p D_h W_p^{-1} \\ &= \tilde{G} U, \end{aligned}$$

so $\tilde{G}$ is time-invariant. Consequently, $\tilde{G}$ has a convolution representation.

Finally, we remark that all the standard results about the discrete Fourier transform go over to the space $l^2(\mathscr{X})$. We refer the reader to [21] for the details. This may be summarized by the following result.

Proposition 1. (i) *The discrete Fourier transform is an isometric isomorphism from the time-domain space $l^2(\mathscr{X})$ to the frequency-domain space $\mathscr{H}^2(\mathscr{X})$ (the space of square integrable $\mathscr{X}$-valued analytic functions defined on the unit disk).*

(ii) *If G is a bounded analytic $\mathscr{X}$-valued function on the unit disk, it defines a bounded operator on $\mathscr{H}^2(\mathscr{X})$ by multiplication, and its induced norm equals exactly $\|G\|_\infty$.*

By the equivalence between an h-periodic system and its lifting, this theorem provides a 'frequency-domain' characterization of the $\mathscr{L}^2$-induced norm of an h-periodic system.

3. Lifting: some examples

Now we look at what lifting means for state-space models. In what follows, G is a continuous-time finite-dimensional time-invariant linear system. Its input, state, and output evolve in finite-dimensional Euclidean spaces. Because the dimensions of these spaces will be irrelevant, they will all be denoted by $\mathscr{E}$. Thus G is considered as a linear operator on $\mathscr{L}_e^2(\mathscr{E})$. Suppose it has the realization A, B, C, D.

3.1. Lifting G

We begin by lifting G itself. The lifted system, WGW^{-1}, acts on $l_e^2(\mathscr{K}^2)$ and consequently has a matrix representation of the form

$$\begin{bmatrix} G_{11} & 0 & 0 & \cdots \\ G_{21} & G_{22} & 0 & \cdots \\ G_{31} & G_{32} & G_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Because G is time-invariant, so is WGW^{-1}, so this matrix is Toeplitz:

$$\begin{bmatrix} G_0 & 0 & 0 & \cdots \\ G_1 & G_0 & 0 & \cdots \\ G_2 & G_1 & G_0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

The computation below will show that this matrix in fact has the form

$$\begin{bmatrix} \tilde{D} & 0 & 0 & \cdots \\ \tilde{C}\tilde{B} & \tilde{D} & 0 & \cdots \\ \tilde{C}\tilde{A}\tilde{B} & \tilde{C}\tilde{B} & \tilde{D} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \tag{1}$$

so WGW^{-1} could be modeled by the discrete-time equations

$$\xi(k+1) = \tilde{A}\xi(k) + \tilde{B}u_k, \quad \xi(0) = 0,$$
$$y_k = \tilde{C}\xi(k) + \tilde{D}u_k.$$

Here u_k and y_k are the k-th components of the lifted input and output of G. Denote matrix (1) by

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right].$$

To determine these four operators $\tilde{A}, \ldots, \tilde{D}$, apply an input to G having support in $[0, h)$:

$$u(t) = \begin{cases} u_0(t), & 0 \le t < h, \\ 0, & t \ge h. \end{cases}$$

The output is thus

$$y(t) = \begin{cases} Du_0(t) + \int_0^t C\,\mathrm{e}^{(t-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau, \\ \qquad\qquad\qquad\qquad 0 \le t < h, \\ \int_0^h C\,\mathrm{e}^{(t-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau, \quad t \ge h. \end{cases}$$

The corresponding input and output of WGW^{-1} are

$$\tilde{u} = Wu = \begin{bmatrix} u_0 \\ 0 \\ \vdots \end{bmatrix}, \quad \tilde{y} = Wy = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix},$$

where

$$y_0(t) = Du_0(t) + \int_0^t C\,\mathrm{e}^{(t-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau,$$
$$y_1(t) = y(t+h)$$
$$= C\,\mathrm{e}^{tA}\int_0^h \mathrm{e}^{(h-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau,$$
$$y_2(t) = y(t+2h)$$
$$= C\,\mathrm{e}^{tA}\,\mathrm{e}^{hA}\int_0^h \mathrm{e}^{(h-\tau)A}Bu_0(\tau)\,\mathrm{d}\tau,$$

etc.

Defining

$$\tilde{A}: \mathscr{E} \to \mathscr{E}, \quad \tilde{A}x = \mathrm{e}^{hA}x,$$
$$\tilde{B}: \mathscr{K}^2 \to \mathscr{E}, \quad \tilde{B}u = \int_0^h \mathrm{e}^{(h-\tau)A}Bu(\tau)\,\mathrm{d}\tau,$$
$$\tilde{C}: \mathscr{E} \to \mathscr{K}^2, \quad (\tilde{C}x)(t) = C\,\mathrm{e}^{tA}x,$$
$$\tilde{D}: \mathscr{K}^2 \to \mathscr{K}^2,$$
$$(\tilde{D}u)(t) = Du(t) + \int_0^t C\,\mathrm{e}^{(t-\tau)A}Bu(\tau)\,\mathrm{d}\tau,$$

we have

$$y_0 = \tilde{D}u_0, \qquad y_1 = \tilde{C}\tilde{B}u_0,$$
$$y_2 = \tilde{C}\tilde{A}\tilde{B}u_0, \qquad \text{etc.}$$

as required for matrix (1).

The important point to observe is how finite-dimensionality of G is manifest in G, namely, $\tilde{A}$ acts on the same state-space as does the original A: its matrix is e^{hA}, which would appear in a discretization of G using sample and hold. Similarly, operators $\tilde{B}$ and $\tilde{C}$ have finite rank. Operator $\tilde{D}$ is the compression of G to $\mathscr{K}^2$.

Now suppose A has all its eigenvalues in the open right half-plane, so G is bounded on $\mathscr{L}^2(\mathscr{E})$; then $\tilde{A}$ has all its eigenvalues in the open unit disk, and consequently WGW^{-1} is bounded on $l^2(\mathscr{K}^2)$. These operators have equal norm, and this equals the $\mathscr{H}^\infty$ norm of the transfer matrix of G. It is worthwhile to see how to compute $\| WGW^{-1} \|$, because exactly the same procedure could be used to compute the $\mathscr{L}^2(\mathscr{E})$-induced norm of the sampled-data system in Figure 1. This will be done in Section 4. Iglesias and Glover [14] give a state-space test for a discrete-time system to have $\mathscr{H}^\infty$ norm less than 1. In their system the input and output evolve on the finite-dimensional space $\mathscr{E}$, whereas the input and output of WGW^{-1} evolve on the infinite-dimensional space $\mathscr{K}^2$. Nevertheless, since the state-space of WGW^{-1} is finite-dimensional, the result of [14] carries over. The development goes as follows.

Starting with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ as above, define the operators

$$Q = I - \tilde{D}\tilde{D}^*, \quad R = I - \tilde{D}^*\tilde{D}$$

mapping $\mathscr{K}^2$ to $\mathscr{K}^2$, and define the pencil

$$S = \lambda \begin{bmatrix} I & -\tilde{B}R^{-1}\tilde{B}^* \\ 0 & \tilde{A}^* + \tilde{C}^*\tilde{D}R^{-1}\tilde{B}^* \end{bmatrix} - \begin{bmatrix} \tilde{A} + \tilde{B}R^{-1}\tilde{D}^*\tilde{C} & 0 \\ -\tilde{C}^*Q^{-1}\tilde{C} & I \end{bmatrix}.$$

Observe, for example, that $\tilde{B}R^{-1}\tilde{B}^*$ maps $\mathscr{E}$ to $\mathscr{E}$, i.e., it is a finite matrix. So S is a finite matrix pencil. Suppose S has no eigenvalues on the unit circle; then it must have n inside the unit disc. Let $\mathscr{X}_-$ denote the corresponding spectral subspace. It can be represented as

$$\mathscr{X}_- = \operatorname{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

where X_1 and X_2 are both $n \times n$. Assuming X_1 is invertible, we can define $X := X_2 X_1^{-1}$. This defines the Riccati operator Ric: $S \mapsto X$ and its domain. Lemma 2.3 of [14] provides the following.

Lemma 1. $\| WGW^{-1} \| < 1$ *iff the following three conditions hold*:

(a) $\| \tilde{D} \| < 1$;
(b) *S belongs to the domain of* Ric;
(c) $R - \tilde{B}^*XB > 0$, *where* $X = \mathrm{Ric}(S)$.

To compute $\| WGW^{-1} \|$ in this way, we would have to

– compute the matrices $\tilde{B}R^{-1}\tilde{B}^*$, $\tilde{C}^*Q^{-1}\tilde{C}$, $\tilde{B}R^{-1}\tilde{D}^*\tilde{C}$ in the definition of S,
– compute $\| \tilde{D} \|$, and
– check if $R - \tilde{B}^*XB > 0$ for a given matrix X.

These subproblems are similar. We will mention two methods for the second subproblem. First of all, let $\Pi: \mathscr{L}^2(\mathscr{E}) \to \mathscr{K}^2$ denote orthogonal projection. Observe that $\tilde{D}$ is the compression of the unlifted system G to $\mathscr{K}^2$, i.e., $\tilde{D} = \Pi G \mid \mathscr{K}^2$. Note that the Laplace transform is an isomorphism of $\mathscr{K}^2$ onto $\mathscr{H}^2 \ominus e^{-hs}\mathscr{H}^2$. Thus computing $\| \tilde{D} \|$ amounts to computing the norm of the operator 'multiplication by the transfer matrix for G' compressed to $\mathscr{H}^2 \ominus e^{-hs}\mathscr{H}^2$. In [9] this computation is reduced to a linear two-point boundary value problem. See also [1]. In [10] a second, frequency-domain ('skew Toeplitz') approach is given for the computation of $\| \tilde{D} \|$.

In summary, the computation of $\| WGW^{-1} \|$ involves the standard iteractive search of scaling, and then using Lemma 1 to check if the norm of the scaled system is less than one.

3.2. Lifting SG

The ideal sampling operator with period h is defined by

$$\psi = Su \;\Rightarrow\; \psi(k) = u(kh).$$

We shall lift SG, where G is as before except with $D = 0$. Operator SG maps $\mathscr{L}_e^2(\mathscr{E})$ to $l_e^2(\mathscr{E})$; G is assumed strictly causal so that SG is bounded on these spaces. The output from SG is already discrete-time, so we need lift only the input. The lifted system, SGW^{-1}, acts from $l_e^2(\mathscr{K}^2)$ to $l_e^2(\mathscr{E})$. Its matrix is easily derived to be

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]$$

where

$$\tilde{A}: \mathscr{E} \to \mathscr{E}, \quad \tilde{A}x = e^{hA}x,$$

$$\tilde{B}: \mathscr{K}^2 \to \mathscr{E}, \quad \tilde{B}u = \int_0^h e^{(h-\tau)A}Bu(\tau)\,d\tau,$$

$$\tilde{C}: \mathscr{E} \to \mathscr{E}, \quad \tilde{C}x = Cx.$$

3.3. Lifting GH

Finally, we shall lift GH, where H is the ideal hold operator with period h, defined by

$$y = H\psi \;\Leftrightarrow\; y(t) = \psi(k), \; kh \le t < (k+1)h.$$

This is an operator from $l_e^2(\mathscr{E})$ to $\mathscr{L}_e^2(\mathscr{E})$. The input to GH is already discrete-time, so we need lift only the output. The lifted system, WGH, acts from $l_e^2(\mathscr{E})$ to $l_e^2(\mathscr{K}^2)$ and its matrix is

$$\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

where

$$\tilde{A}: \mathscr{E} \to \mathscr{E}, \quad \tilde{A}x = e^{hA}x,$$

$$\tilde{B}: \mathscr{E} \to \mathscr{E}, \quad \tilde{B}v = \int_0^h e^{\tau A}\,d\tau Bv,$$

$$\tilde{C}: \mathscr{E} \to \mathscr{K}^2, \quad (\tilde{C}x)(t) = C\,e^{tA}x,$$

$$\tilde{D}: \mathscr{E} \to \mathscr{K}^2, \quad (\tilde{D}v)(t) = \left[D + \int_0^t C\,e^{\tau A}\,d\tau B\right]v.$$

4. Application to $\mathscr{H}^\infty$ optimization of sampled-data systems

In this section we outline an application to optimizing the $\mathscr{L}^2(\mathscr{E})$-induced norm from w to z in Figure 1. Let T denote the linear system mapping w to z. If K_d is internally stabilizing (suitably defined) and under mild assumptions on G, T is a bounded operator on $\mathscr{L}^2(\mathscr{E})$. It is time-varying. Our approach is to lift T up to WTW^{-1}, which will be a time-invariant operator on $l^2(\mathscr{K}^2)$. The optimization of $\|T\|$ is thus reduced to a discrete-time, time-invariant $\mathscr{H}^\infty$ optimization problem, a problem whose solution is formally the same as the standard discrete-time matrix-valued $\mathscr{H}^\infty$ problem for which there exist solutions [13,14,19,20]. (An alternative but equivalent approach is taken in [1] where the operator valued $\mathscr{H}^\infty$ problem is solved through an intermediate step of reducing it to an equivalent matrix-valued discrete-time $\mathscr{H}^\infty$ problem.)

The details of our approach are as follows. Partition G as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

and let a corresponding minimal realization be

$$\left[\begin{array}{c|c} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} \end{array}\right].$$

In Figure 1, bring S and H around and adsorb them into G to get the setup shown in Figure 2 below. Matrix D_{21} is taken to be zero so that w is low-pass filtered (through G_{21}) before being sampled; the system could not in general be internally stabilized without this assumption.

The system in the upper block is

$$\begin{bmatrix} G_{11} & G_{12}H \\ SG_{21} & SG_{22}H \end{bmatrix}.$$

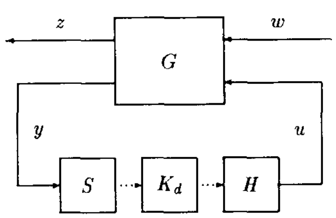

Fig. 1. Sampled-data control system.

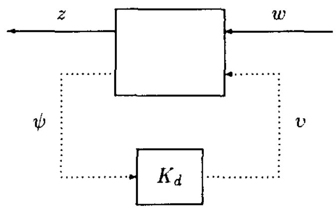

Fig. 2.

Now lift w and z in the previous figure to arrive at the setup in Figure 3 below.

System P is obviously given by

$$P = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G_{11} & G_{12}H \\ SG_{21} & SG_{22} \end{bmatrix} \begin{bmatrix} W^{-1} & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} WG_{11}W^{-1} & WG_{12}H \\ SG_{21}W^{-1} & SG_{22}H \end{bmatrix}.$$

Realizations of the three liftings $WG_{11}W^{-1}$, $WG_{12}H$, $SG_{21}W^{-1}$ were obtained in Section 3. Furthermore, $SG_{22}H$ is just G_{22} discretized: a realization is well-known to consist of the four matrices

$$e^{hA}, \quad \int_0^h e^{\tau A}\, d\tau B_2, \quad C_2, \quad D_{22}.$$

In this way we get the realization of P,

$$\left[\begin{array}{c|c} \tilde{A} & \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & \tilde{D}_{22} \end{bmatrix} \end{array}\right]$$

where

$$\tilde{A} : \mathscr{E} \to \mathscr{E}, \quad \tilde{A}x = e^{hA}x,$$

$$\tilde{B}_1 : \mathscr{K}^2 \to \mathscr{E}, \quad \tilde{B}_1 w = \int_0^h e^{(h-\tau)A} B_1 w(\tau)\, d\tau,$$

$$\tilde{B}_2 : \mathscr{E} \to \mathscr{E}, \quad \tilde{B}_2 v = \int_0^h e^{\tau A}\, d\tau B_2 v,$$

$$\tilde{C}_1 : \mathscr{E} \to \mathscr{K}^2, \quad (\tilde{C}_1 x)(t) = C_1\, e^{tA}x,$$

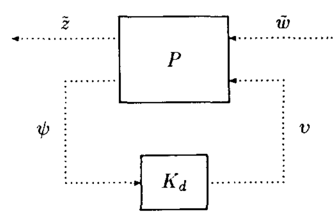

Fig. 3.

$$\tilde{C}_2 : \mathscr{E} \to \mathscr{E}, \quad \tilde{C}_2 x = C_2 x,$$

$$\tilde{D}_{11} : \mathscr{K}^2 \to \mathscr{K}^2,$$

$$(\tilde{D}_{11} w)(t) = D_{11} w(t) + C_1 \int_0^t e^{(t-\tau)A} B_1 w(\tau)\, d\tau,$$

$$\tilde{D}_{12} : \mathscr{E} \to \mathscr{K}^2,$$

$$(\tilde{D}_{12} v)(t) = D_{12} v + C_1 \int_0^t e^{(t-\tau)A}\, d\tau B_2 v,$$

$$\tilde{D}_{22} : \mathscr{E} \to \mathscr{E}, \quad \tilde{D}_{22} v = D_{22} v.$$

Figure 3 is a discrete-time setup. Iglesias and Glover's solution [14] to the discrete-time $\mathscr{H}^\infty$ problem is in the style of the continuous-time solution of Doyle et al. [7]. We will illustrate how the solution of [14] can be applied to the setup at hand by looking at the analysis problem, which is easier than the synthesis problem. Namely, for a fixed stabilizing K_d we will show how to compute the $\mathscr{L}^2(\mathscr{E})$-induced norm.

In Figure 3 the equations for P are

$$\xi_P(k+1) = \tilde{A}\xi_P(k) + \tilde{B}_1 w_k + \tilde{B}_2 v(k),$$

$$z_k = \tilde{C}_1 \xi_P(k) + \tilde{D}_{11} w_k + \tilde{D}_{12} v(k),$$

$$\psi(k) = \tilde{C}_2 \xi_P(k) + \tilde{D}_{22} v(k).$$

Suppose K_d is strictly causal for simplicity, and its equations are

$$\xi_K(k+1) = A_K \xi_K(k) + B_K \psi(k),$$

$$v(k) = C_K \xi_K(k).$$

Then the matrix of the closed-loop system is

$$\left[\begin{array}{c|c} A_{CL} & B_{CL} \\ \hline C_{CL} & D_{CL} \end{array}\right]$$

where A_{CL} is the map from $\mathscr{E}$ to $\mathscr{E}$ given by

$$A_{CL}\xi = \begin{bmatrix} \tilde{A} & \tilde{B}_2 C_K \\ B_K \tilde{C}_2 & A_K \end{bmatrix} \xi = \begin{bmatrix} e^{hA} & \int_0^h e^{\tau A}\, d\tau B_2 C_K \\ B_K C_2 & A_K \end{bmatrix} \xi,$$

B_{CL} is the map from $\mathscr{K}^2$ to $\mathscr{E}$ given by

$$B_{CL} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},$$

C_{CL} is the map from $\mathscr{E}$ to $\mathscr{K}^2$ given by

$$C_{CL} = \begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} C_K \end{bmatrix},$$

and $D_{CL} = \tilde{D}_{11}$, mapping $\mathscr{K}^2$ to $\mathscr{K}^2$. Internal stability means that all eigenvalues of A_{CL} are inside the unit disk. Computing now proceeds as in Subsection 3.1.

5. Application to $\mathscr{H}^2$ optimization of sampled-data systems

In this section, we would like to make some remarks about the lifting technique applied to other types of norms. Since the lifting is an isometry in any given norm, we can apply it to other $\mathscr{L}^p$ spaces. First we would like to make some remarks about the induced operator norm on l^p.

Consider $\mathscr{E}$ equipped with the l^r-norm,

$$\| v \|_r = \left[\sum_{j=1}^{n} | v_j |^r \right]^{1/r}, \quad 1 \le r < \infty,$$

$$\| v \|_\infty = \max_{1 \le j \le n} | v_j |,$$

where the v_j denote the components of the vector $v \in \mathscr{E}$. With $\mathscr{E}$ equipped with the r-norm we will set $\mathscr{L}_r^p(\mathscr{E}) := \mathscr{L}^p(\mathscr{E})$ and denote the norm by $\| \ \|_{p,r}$. Also, $\mathscr{K}_r^p$ will denote the subspace of $\mathscr{L}_r^p(\mathscr{E})$ of functions with support in $[0, h)$. By slight abuse of notation, $\| \ \|_{p,r}$ will also denote the norm on $\mathscr{K}_r^p$.

By the lifting construction, we see that there exists an isometry $W_{p,r} : \mathscr{L}_r^p(\mathscr{E}) \to l^p(\mathscr{K}_r^p)$ for each $1 \le p,\ r \le \infty$. Recall that the induced norm of a bounded linear operator T from one Banach space $\mathscr{X}_1$ to another Banach space $\mathscr{X}_2$ is

$$\| T \| := \sup_{v \ne 0} \frac{\| Tv \|_{\mathscr{X}_2}}{\| v \|_{\mathscr{X}_1}}.$$

We consider the problem, then, of computing the induced norm of a discrete-time causal convolution operator $F : l^p(\mathscr{K}_r^p) \to l^q(\mathscr{K}_s^q)$. When $p = q = r = s = 2$ we have seen that the induced norm is in fact the $\mathscr{H}^\infty$-norm of the discrete Fourier transform of the pulse response of F. But this of course is not the only possibility, and one can ask for choices of p, q, r, s which will induce a 2-norm which would correspond to a quadratic type sampled-data optimization problem. We should note that in [4] the authors consider an optimization problem with the Hilbert–Schmidt norm, which is not an operator-induced norm.

Before stating the result, we will need some additional notation. First if $F: l^p(\mathcal{K}_r^p) \to l^q(\mathcal{K}_s^q)$ is a (causal) convolution operator, the equality $y = F(u)$ means that

$$y_k = \sum_{i=0}^{k} F_{k-i}(u_i), \quad \forall k \geq 0,$$

where the $F_k : \mathcal{K}_r^p \to \mathcal{K}_s^q$ are linear operators. In our case, the F_k will come from the lifted closed-loop operator T_{zw}, and so will have the form

$$C_{\mathrm{CL}} A_{\mathrm{CL}}^{k-1} B_{\mathrm{CL}}, \quad k \geq 1, \tag{2}$$

in the notation of Section 4. Note that the closed-loop impulse response F_k for $k = 0$ is the operator $\tilde{D}_{11}$. This fact will not affect our discussion below, since the controller only enters into the closed loop operator F_k for $k \geq 1$, and so from the way in which the norm is computed, the controller that minimizes the cost (norm) with F_0 included is the same as the controller that minimizes the cost without including F_0, hence the answer given below is valid.

Next notice that

$$B_{\mathrm{CL}} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix},$$

where

$$\tilde{B}_1 w = \int_0^h \mathrm{e}^{(h-\tau)A} B_1 w(\tau)\, \mathrm{d}\tau.$$

Thus $\tilde{B}_1$ acts as a convolution operator evaluated at h, and so we may express the action of the impulse response function (2) as an integral operator of the form

$$(F_k u)(t) = \int_0^h F_k(h-\tau, t) u(\tau)\, \mathrm{d}\tau, \quad k \geq 1.$$

Now for A a non-negative matrix, we let $\lambda_{\max}(A)$ denote the maximal eigenvalue, and $d_{\max}(A)$ the maximal diagonal entry.

The following result may be proven using a method similar to that in [26].

Proposition 2. *For all $k \geq 1$, set*

$$Q_1^k(\tau) := \int_0^h F_k(h-\tau, t)' F_k(h-\tau, t)\, \mathrm{d}t,$$

$$Q_2^k(\tau) := \int_0^h F_k(h-\tau, t) F_k(h-\tau, t)'\, \mathrm{d}t,$$

for $\tau \in [0, h)$, and set

$$R_1^k := \sup_{\tau \in [0,h)} \lambda_{\max}(Q_1^k(\tau)),$$

$$R_2^k := \sup_{\tau \in [0,h)} \lambda_{\max}(Q_2^k(\tau)),$$

$$S_1^k := \sup_{\tau \in [0,h)} d_{\max}(Q_1^k(\tau)),$$

$$S_2^k := \sup_{\tau \in [0,h)} d_{\max}(Q_2^k(\tau)).$$

Then (i) *the induced norm of $F: l^1(\mathcal{K}_2^1) \to l^2(\mathcal{K}_2^2)$ equals*

$$\Big(\sum_k R_1^k\Big)^{1/2};$$

(ii) *the induced norm of $F: l^2(\mathcal{K}_2^2) \to l^\infty(\mathcal{K}_2^\infty)$ equals*

$$\Big(\sum_k R_2^k\Big)^{1/2};$$

(iii) *the induced norm of $F: l^1(\mathcal{K}_1^1) \to l^2(\mathcal{K}_2^2)$ equals*

$$\Big(\sum_k S_1^k\Big)^{1/2};$$

(iv) *the induced norm of $F: l^2(\mathcal{K}_2^2) \to l^\infty(\mathcal{K}_\infty^\infty)$ equals*

$$\Big(\sum_k S_2^k\Big)^{1/2}.$$

Referring again to Figure 1, we can pose the problem of minimizing the operator norm of the transfer operator from w to z, where we allow the signals to be in the various spaces $\mathcal{L}_r^p(\mathcal{E})$. This problem may be lifted to get the equivalent discrete-time problem in the spaces $l^p(\mathcal{K}_r^p)$ and then one may apply the solution in [27]. For the full state information problem (this corresponds to the classical LQR problem) one can show that the classical LQR optimal controller is optimal in the case when the disturbances are in $\mathcal{L}_r^1$ for $r = 1, 2$, and the errors are in $\mathcal{L}_2^2$. For the optimal filtering problem, one can show that the optimal state estimator is again given by the classical formula with disturbances in $\mathcal{L}_2^2$ and errors in $\mathcal{L}_r^\infty$ for $r = 2, \infty$. The argument goes exactly as in the $\mathcal{H}^\infty$ case by considering the equivalent 'lifted' discrete time-invariant system and applying Proposition 2 and the results of [27]. Note that from our previous remarks the lifted operator $\hat{G}$ is finite dimensional.

To make this argument more concrete, we will consider the sampled-data version of the full state information (LQR) problem. Referring to Section 4, in this case the generalized plant G has the form

$$\left[\begin{array}{c|c} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ I \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \end{array}\right].$$

We are interested in minimizing the induced operator norm of T, the linear input/output operator from w to z taken over all the controllers K as in Figure 1. For our problem, we assume that $w \in \mathscr{L}_r^1$ ($r = 1$ or $r = 2$) and $z \in \mathscr{L}_2^2$.

Now in this case the lifted system will have the form

$$\left[\begin{array}{c|c} \tilde{A} & \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} \\ \hline \begin{bmatrix} \tilde{C}_1 \\ I \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ 0 & 0 \end{bmatrix} \end{array}\right]. \tag{3}$$

Note once again that all norms are preserved in the lifting procedure. Hence, arguing precisely as in Section 4 (and making the standard assumptions of stabilizability and detectability on (3)), and using the results of [27], the optimal feedback gain may be derived from the classical finite dimensional algebraic Stein (discrete Riccati equation) associated to the LQR problem with respect to the generalized time-invariant, discrete-time plant given in (3).

Unfortunately, at this point there is no separation principle available because of the incompatibility of the norms in the filtering and regulator problems.

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