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A new shift-invariant representation for periodic linear systems

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Abstract: This paper presents a new family of shift-invariant representations for periodic linear discrete-time systems. This representation has the advantages over the standard representation [5] of preserving the time step of the original system, and of preserving causality under arbitrary feedback. The disadvantage is that the realization is not minimal. An application in attitude determination is presented.

Keywords: Periodic systems; time-invariant representations; linear systems; periodic steady-state; Kalman filtering.

1. Introduction

In this paper, we demonstrate a new shift-invariant representation for periodic linear discrete-time systems, which we use here to formulate a problem with which we can find the periodic steady-state Kalman filter gains for such systems by solving the algebraic Riccati equation association with the shift-invariant system. A preliminary version of this paper appeared in [3].

There has been much recent interest in such representations, largely in the context of two problems: In Khargonekar et al. [6] the authors demonstrate that for linear time-invariant systems it is possible to improve performance over that attainable by linear time-invariant feedback by using using linear periodic feedback. This configuration leads to a periodic closed-loop system, for which analysis and synthesis issues continue to be explored extensively (see, for example, Francis and Georgiou [4]).

The second problem of wide interest leading to periodic linear systems is that of multirate control systems. This problem has a long history going back at least to Kranc [7] for the frequency domain formulation and Kalman and Bertram [5] for the state space formulation, and the model used for analysis of these systems is essentially unchanged since these references.

Both the models presented in this paper and the 'standard' model used in the references cited above are time-invariant representations of a periodic linear system. The essential difference is that the shift-invariant model presented below retains the time step of the original system, whereas the standard model uses the period of the system for the time step of the time-invariant system. A consequence of this is that minimal state-space realizations of the standard model have for their state the state of the original system once per period, whereas the state of minimal realizations of the proposed model at each step includes the state of the original system at that time step. In the present paper, this feature is exploited to solve a Kalman filtering problem, which also was a motivating example for the model. Other applications involving stability analyses of periodic systems are also under investigation by the author.

The advantages of the new model over the standard model are that causality of the original system is built into this model, and that the time step of this model is the same time step as the original system. The first advantage is significant in the Kalman filtering problem considered below. The second advantage has import in stability analyses, and will be not be treated here. The idea of including in a time-invariant representation of a periodic linear system state information for all the states within a period is also carried out, although in a different manner, in Araki and Yamamoto [1]. However, that formulation is essentially an embellishment of the standard approach, and has neither of the advantages listed above.

2. A motivating example

The following example may help the reader to visualize the developments below: A satellite orbits the earth, maintaining a local vertical orientation by means of a control system which uses an estimate of attitude based upon measurements of three gyroscopes ('gyros') and a star sensor. For concreteness, we use the attitude determination system described in Thompson and Quasius [8]. We make the following assumptions: the vehicle angular rate has the constant value

$$[\omega_x \quad \omega_y \quad \omega_z]^{\mathrm{T}} = [0 \quad \omega_0 \quad 0]^{\mathrm{T}},$$

and stars are sighted every Δ seconds at the same angle within the sensor field of view. In between star observations the attitude estimate is propagated by the 'strapdown' integration equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Theta = -\Omega\Theta,$$

where

$$\Omega = \begin{bmatrix} 0 & -\hat{\omega}_z & \hat{\omega}_y \\ \hat{\omega}_z & 0 & -\hat{\omega}_x \\ -\hat{\omega}_y & \hat{\omega}_x & 0 \end{bmatrix}$$

results from compensated gyro outputs, and Θ is the 3×3 'direction cosine' matrix. We assume that the gyro measurements are corrupted by a random walk (bias) and a white noise. Every star sighting is incorporated into the attitude estimate using an extended Kalman filter which has as its state components three small-angle rotations which correct the attitude and three gyro biases. If t_i is the time of the i-th star sighting, it can be shown that, in appropriate coordinates, the state of this error estimator is given by

$$x(t_i) = \Phi(\Delta_i)x(t_{i-1}) + n_x(\Delta_i),$$

i.e., the state transition matrix and the statistics of the equivalent noise n_x depend only upon the time elapsed since the last observation (and update)

$$\Delta_i = t_i - t_{i-1}.$$

For each individual star sighted, this particular star sensor actually forms three observations separated in time

$$y_{mj} = \left((N_j \times S_m)^{\mathrm{T}} \ 0 \ 0 \ 0\right)x(m\Delta + \tau(j-1)) + (n_s)_{mj}, \quad j = 1, 2, 3.$$

Here S_m is the unit vector pointing to the star, τ is the time interval between each of a trio of observations, and N_j is one of three unit vectors (normals to slits in a reticle positioned in the sensor's focal plane). $(n_s)_{mj}$ is a sequence of independent identically distributed Gaussian random variables representing sensor noise. Under the assumption that the stars are always sighted at the same angle within the field of view, S_m in fact depends only upon the slit in which it is seen, the observations take the form

$$y(m\Delta) = \left(H_1^{\mathrm{T}} \ 0 \ 0 \ 0\right)x(m\Delta) + n_s(m\Delta),$$

$$y(m\Delta + \tau) = \left(H_2^{\mathrm{T}} \ 0 \ 0 \ 0\right)x(m\Delta + \tau) + n_s(m\Delta + \tau),$$

$$y(m\Delta + 2\tau) = \left(H_3^{\mathrm{T}} \ 0 \ 0 \ 0\right)x(m\Delta + 2\tau) + n_s(m\Delta + 2\tau).$$

Thus this system can be seen to be periodic with period Δ. We want to find the periodic steady state Kalman filter gains and the periodic steady state error covariance.

Our approach is to think of three identical satellites orbiting next to each other, one succeeding the other in each of the observation positions at each discrete time step. The composite of these three systems is shift-invariant, and the random behavior of each satellite is statistically independent of the others. Thus the the extended Kalman filter gains for the composite system will simply be the composite of the gains for the separate systems. Once the composite system is in steady state, following any discrete time step one can see a full period of the steady state gains by examining each satellite in turn.

3. Problem formulation

We consider causal finite-dimensional linear systems evolving in discrete time. Let us assume the system is given by a map

$$T: \mathbb{R}^n(z) \to \mathbb{R}^m(z).$$

Such a system which is periodic with period N has the shift-invariant property

$$z^N \cdot T(u(z)) = T(z^N \cdot u(z)),$$

and has a state space representation (assumed to be minimal with state dimension p) given by

$$x[k+1] = A_k x[k] + B_k u[k], \tag{1a}$$

$$y[k] = C_k x[k] + D_k u[k], \tag{1b}$$

where

$$B_{k+N} = B_k, \qquad C_{k+N} = C_k,$$
$$D_{k+N} = D_k, \qquad A_{k+N} = A_k,$$

for all integers $k \geqslant 0$.

4. A time-invariant equivalent

From this system we shall construct a time-invariant system

$$\hat{T}: \mathbb{R}^{n \cdot N}(z) \to \mathbb{R}^{m \cdot N}(z)$$

such that the following diagram commutes, and such that T is bounded-input/bounded-output stable if and only if $\hat{T}$ is:

$$\begin{array}{ccc} \mathbb{R}^{n \cdot N}(z) & \xrightarrow{\hat{T}} & \mathbb{R}^{m \cdot N}(z) \\ {\scriptstyle \mathscr{I}}\uparrow & & \downarrow{\scriptstyle \Pi} \\ \mathbb{R}^{n}(z) & \xrightarrow{T} & \mathbb{R}^{m}(z) \end{array}$$

where $\mathscr{I}$ is a (non-unique) injection and Π a surjection to be defined below.

The easiest way to describe the system $\hat{T}$ is that it consists of N copies of the original time-varying system, operating independently in parallel, and staggered in time by successive time steps. The construction of $\hat{T}$ is easiest to write using a realization. Take $\hat{T}$ to be the input/output map induced by the following difference equation:

$$\bar{x}[i+1] = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_N \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & A_{N-1} & 0 \end{bmatrix} \bar{x}[i] + \begin{bmatrix} 0 & 0 & \cdots & 0 & B_N \\ B_1 & 0 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & B_{N-1} & 0 \end{bmatrix} \bar{u}[i], \tag{2a}$$

$$\bar{y}[i] = \begin{bmatrix} C_1 & 0 & \cdots & 0 & 0 \\ 0 & C_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & C_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & C_N \end{bmatrix} \bar{x}[i] + \begin{bmatrix} D_1 & 0 & \cdots & 0 & 0 \\ 0 & D_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & D_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & D_N \end{bmatrix} \bar{u}[i], \tag{2b}$$

where

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_N \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_N \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_N \end{bmatrix},$$

$\bar{x}_i$, $\bar{u}_i$, and $\bar{y}_i$ have dimensions p, n and m, respectively, and $\bar{x}$, $\bar{u}$ and $\bar{y}$ have dimensions pN, nN and mN, respectively. It is easy to see that this time-invariant system amounts to N copies of the original system running in parallel, offset in time successively by one time step. Note that: (a) the systems can run independently in the sense that they are uncoupled and the inputs for each system can be specified independently, and (b) this realization is minimal if and only if (1) is.

Now to define $\mathscr{I}$ we require

$$\bar{u}_i[k] = u[k]$$

for all integers $k \geqslant 0$ and the integer i with $1 \leqslant i \leqslant N$ such that $i = (k-1) \pmod N + 1$, and for Π we require

$$\Pi(\bar{y})[k] = \bar{y}_i[k]$$

for $i = (k-1) \pmod N + 1$ for all integers $k \geqslant 1$. It is easy to check that with these definitions the diagram commutes, and in that sense we have embedded (1) in a time-invariant system.

This completely defines Π, but there is substantial remaining freedom in picking the remaining $\bar{u}_j[k]$ to define $\mathscr{I}$. This time-invariant system is equivalent to the original system (in the sense of the above diagram commuting) if we pick any set of inputs and initial states for the remaining $N-1$ copies of the original system and any initial state which satisfies $\bar{x}_1[0] = x[0]$. So long as these conditions are satisfied, we can check that $x[k] = \bar{x}_i[k]$ and $y[k] = \bar{y}_i[k]$ for $i = (k-1) \pmod N + 1$.

The choice we make to completely define $\mathscr{I}$ depends upon the application. In the present paper we shall focus on solving a steady state Kalman filtering problem, and therefore we take the input sequence

$$u[i] = \begin{pmatrix} \nu[i] \\ \xi[i] \end{pmatrix}$$

to be a sequence of independent identically distributed zero-mean Gaussian random variables with covariance function

$$K_u(i, j) = \begin{bmatrix} Q_i & 0 \\ 0 & R_i \end{bmatrix} \delta_{ij}, \tag{3}$$

and we shall assume that

$$B_i = \begin{pmatrix} \tilde{B}_i & 0 \end{pmatrix}, \qquad D_i = \begin{pmatrix} 0 & \tilde{D}_i \end{pmatrix} \tag{4}$$

(with appropriate dimensions).

For the *shift invariant* system we defined above we take the noise inputs and observation noise to the parallel systems to be identically distributed, independent, and having statistics cyclically delayed in time. In equations, we can write (2) for this problem as

$$\bar{x}[i+1] = \begin{bmatrix} 0 & 0 & \cdots & 0 & A_N \\ A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & A_{N-1} & 0 \end{bmatrix} \bar{x}[i] + \begin{bmatrix} 0 & 0 & \cdots & 0 & \tilde{B}_N \\ \tilde{B}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{B}_2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \tilde{B}_{N-1} & 0 \end{bmatrix} \bar{\nu}[i], \tag{5a}$$

$$\bar{y}[i] = \begin{bmatrix} C_1 & 0 & \cdots & 0 & 0 \\ 0 & C_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & C_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & C_N \end{bmatrix} \bar{x}[i] + \begin{bmatrix} \tilde{D}_1 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{D}_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \tilde{D}_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \tilde{D}_N \end{bmatrix} \bar{\xi}[i], \tag{5b}$$

with

$$K_{\bar{\nu}}(i, j) = \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & Q_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & Q_N \end{bmatrix} \delta_{ij}, \tag{6a}$$

$$K_{\bar{\xi}}(i, j) = \begin{bmatrix} R_1 & 0 & \cdots & 0 & 0 \\ 0 & R_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & R_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & R_N \end{bmatrix} \delta_{ij}. \tag{6b}$$

Similarly, we take the initial conditions for the sub-states to be independent and satisfying

$$\mathscr{E}\left(\bar{x}_i(0)\bar{x}_j^{\mathrm{T}}(0)\right) = P[i-1]\delta_{ij},$$

where $P[l] = \mathscr{E}(x(l)x^{\mathrm{T}}(l))$.

5. A Kalman filtering solution

Now we consider the design of Kalman filters for the two systems:

Proposition 1. *Suppose the system given by* (1), (3) *and* (4) *is completely controllable and completely reconstructible. The periodic steady state Kalman filter gains $\hat{G}_i$ for that system are given by the block* *lower diagonal and circulant optimal Kalman filter gains $\bar{G}$ for the problem* (5) *and the conditions* (6), *as indicated in the equation*

$$\bar{G} = \begin{bmatrix} 0 & 0 & \dots & 0 & \hat{G}_N \\ \hat{G}_1 & 0 & \dots & 0 & 0 \\ 0 & \hat{G}_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & 0 & \hat{G}_{N-1} & 0 \end{bmatrix}.$$

Proof. There are two ways to prove this claim. The first is system-theoretic: Since the parallel systems have independent noises, the observations of one sub-system contain no information about the state of any of the other sub-systems. Thus at any time step of the time-invariant system the best estimate of a given sub-state will depend only on the observations of the state of that sub-system at the previous time step, and that corresponds to the cyclically *previous* sub-vector of the state at the previous step, the sub-vector having index lower by 1 cyclicly. Therefore the optimal filter gains will have the same block circulant structure as $\bar{A}$, with only a single non-zero circulant block diagonal.

Because the diagram commutes, and because of the block circulant and block zero structure for the shift-invariant system (which amounts to the interpretation of the sub-vectors of the state as parallel and independent subsystems), at finite times the optimal gains for the periodic system rotate through the blocks of the non-zero circulant diagonal of the optimal gains for the shift-invariant system.

We use the following easy lemma:

Lemma 1. *If* (1) *is completely controllable and completely reconstructible, then* (2) *is stabilizable and detectable.*

As a consequence of this lemma, a unique positive semi-definite solution to the corresponding discrete-time algebraic Riccati equation exists, and the steady state Kalman filter gains for (5) are obtained from this solution. It follows that the optimal gains for the periodic system approach the periodic gains given by non-zero blocks of the circulant gain matrix $\bar{G}$.

The second proof merely consists of proving the block zero and circulant structure of the optimal steady-state gain for the time-invariant system by appealing to the block structure of the realization, and noting that the solution to the Riccati difference equation has diagonal block structure. This may be seen by explicitly writing down the Riccati difference equation. Therefore so must the steady-solution have diagonal block structure.

The argument connecting this solution to the periodic system is the same as before: In the steady state the effect of initial conditions has disappeared, and the statistical behaviors of the parallel systems are identical but for delays. The block-circulant Kalman filter gain for the parallel system has for its i-th non-zero block the gain for the j-th parallel system at the times $nN + j + i - 1$ for n large. Fixing $j = 1$, we see that the solution to this time-invariant Kalman filtering problem also provides the steady-state Kalman filter gains for the original periodic problem. □

6. Applications to the example

This approach has been used to examine the sensitivity of this attitude estimation problem to various parameters, and the results have been validated by Monte Carlo simulation with a substantially more realistic model. (Actually, a slightly more complex observation model was used for forming the shift-invariant system: Alternate stars were assumed to appear above and below the celestial sensor line of sight, leading to six different observations which repeat cyclically.)

For example, it is relatively easy to compute attitude estimation error variance as a function of true noise variance and nominal noise variance. This allows 'tuning' of the noise variance parameters within the filter to accomodate the expected range of true noise within known performance limits. This essentially amounts to an extension of the single-axis rotation and observation computations of Farrenkopf [2].

A similar use was to compute the dependence of estimation error variance on the angular width of the star sensor field of view, and on the angle at which the sensor is mounted on the satellite.

7. Conclusions and remarks

Our time-invariant system consists of N independent systems which have identical statistical behavior in the (periodic) steady state, synchronized to make transitions simultaneously and staggered in time by one time step. The state of each of the parallel systems is not a fixed sub-state of the composite state but rather a sub-state which shifts in index cyclically. In the steady state the statistics of the systems shift cyclically from one sub-state to the next. At each time step only one of the systems has statistics which none of the others had during the last time step.

The picture may be easiest to visualize in the orbital example above, because not only is the structure of the problem cyclic, but sub-states have geographical interpretation.

The standard technique for finding a time-invariant representation of periodic systems can also be used to solve the Kalman filtering problem above. At its simplest, this would involve solving an algebraic Riccati equation of order n (rather than order nN), along with some auxiliary calculations to determine the equivalent state and observation noise statistics. However, in this case, state and observation noise would be correlated. The approach above is much easier to implement, at the expense of more computation in the solution of the algebraic Riccati equation. The choice of preferred method would depend upon the application. It might well be possible to utilize the sparse circulant structure of the algebraic Riccati equation for the present realization to equalize the computational effort of the two approaches.

This approach is also applicable to stability analyses of periodic linear systems, such as multirate control systems. In this case another injection $\mathscr{I}$ is used, and this is the subject of current work.

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