Rejection of periodic disturbances of unknown and time-varying frequency

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SUMMARY

The paper reviews available methods for the rejection of periodic disturbances. Such disturbances are often encountered in active noise and vibration control, due to rotating machinery. The emphasis of the paper is on feedback control problems where reference sensors are not available. The case where the frequency of the disturbance is known is considered first. Two sets of algorithms are discussed: one based on the internal model principle of feedback control theory, and the second based on adaptive feedforward cancellation. An interesting observation is that algorithms originating from both approaches can be shown to be equivalent under certain conditions. When the frequency of the disturbance is unknown, an intuitive approach consists in combining a method for the rejection of disturbances of known frequency with a frequency estimator. Alternatively, one may seek to develop a stable adaptation mechanism so that the disturbance is cancelled asymptotically. While algorithms can be designed to adapt to plant and disturbance parameters, the most successful approaches use some limited plant information to adapt the magnitude, frequency, and phase parameters of the control signal. Applications are discussed throughout the paper. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: adaptive algorithms; active noise control; active vibration control; disturbance rejection; periodic signals; phase-locked loops

1. INTRODUCTION

The rejection of disturbances is a major objective of control system design. In active noise control, disturbance rejection is actually the sole purpose of the feedback system [1,2]. For active vibration suppression, tracking of a reference position is important, but the need to reject persistent disturbances is the source of most of the difficulty. Often, disturbances affecting control systems are periodic. For example, in a computer disk drive, the eccentricity of the track on the disk requires a periodic movement of the read/write head at the frequency of rotation of the disk [3]. Similarly, the rotating engine of a helicopter causes periodic noise and vibration [4]. In space, cryogenic coolers produce periodic vibrations that transmit through flexible structures,

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and must be isolated if precise pointing of instruments is to be achieved [5]. Altogether, periodic disturbances are encountered in a surprisingly large number of applications, spreading across the whole field of engineering. The paper begins with an overview of methods that are applicable when the frequency of the disturbance is known exactly, or with reasonable accuracy. Next, the more complicated problem with unknown frequency is considered.

2. DISTURBANCE REJECTION

2.1. Periodic disturbances

We consider the standard feedback system of Figure 1, where r is the reference input, u is the control input, d is the disturbance, n is the measurement noise, P is the system to be controlled or plant, and C is the control law. We assume that the reference input $r = 0$ (pure disturbance rejection problem), $n = 0$ (noise-free case), and the plant is linear time-invariant. Then, the system is described by

$$y(s) = P(s)(u(s) - d(s)) \qquad (1)$$

where $P(s)$ is the transfer function of the plant and $y(s)$, $u(s)$, and $d(s)$ are the Laplace transforms of the plant output, control input, and disturbance signal, respectively. The goal of the control system is to generate a control input $u(t)$ such that $y(t) \to 0$ as $t \to \infty$.

The disturbance is assumed to contain one or more sinusoidal components, i.e.

$$d(t) = \sum_{k=1}^{n} d_k \cos(\alpha_{d,k}(t)), \quad \frac{\mathrm{d}\alpha_{d,k}(t)}{\mathrm{d}t} = \omega_{d,k} \qquad (2)$$

where $\omega_{d,k}$ $(k = 1, \ldots, n)$ are the frequencies of the sinusoidal components of the disturbance. Two special cases arise when:

- $n = 1$, i.e. the disturbance is *sinusoidal*;
- $\omega_{d,k} = k \, \omega_d$, i.e. the disturbance is *periodic* with a fundamental period $T = 2\pi/\omega_d$.

The disturbance $d(t)$ is not known or measured, except through its effect at the output of the system.

In this paper, the plant will typically be assumed to be stable. Then, the objective can be achieved with $u(t) = d(t)$, regardless of any initial conditions in the plant. If the plant is unstable, a stabilizing controller can be designed and the techniques can be applied to the closed-loop system. Similarly, if the reference input is different from zero, an inner feedback loop can be
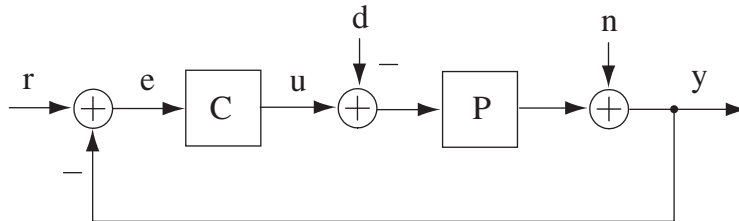

Figure 1. Standard feedback system.

constructed, and the techniques of the paper can be applied by subtracting $P_{CL}(s)r(s)$ from the plant output, where $P_{CL}(s)$ is the transfer function of the inner loop. The disturbance cancelling signal can be added to the reference input or to the plant input.

2.2. Input vs output disturbances

In active noise control problems, it is typical to represent the disturbance as an output disturbance, appearing at the location of the signal n in Figure 1. Given an input disturbance d, one can always define an equivalent output disturbance n, with $n(s) = -P(s)d(s)$. The reverse is not true in general, but an equivalent input disturbance *can* be defined if the disturbance is periodic and the system is single-input single-output. The frequency response of the plant must be non-zero at the frequencies of the components of the disturbance, but the problem cannot be solved unless the condition is satisfied.

In the multi-input multi-output case with a square frequency response matrix, an equivalent input disturbance can be defined as long as the matrix is full rank. If there are more outputs than inputs, perfect disturbance cancellation may not be possible, even if the matrix is full rank. Nevertheless, one can define an input disturbance that represents the part of the output disturbance that can be cancelled, and a residual output disturbance that lies outside the range of the plant operator [6]. In this paper, we assume that the disturbance can be modelled as an input disturbance. The same algorithms can be used in the general case, but analysis is more complicated.

2.3. Feedforward and feedback control

Adaptive algorithms for disturbance rejection can be found in the literature on active noise and vibration control. Typically, they fall in the class of *adaptive feedforward cancellation* (*AFC*) algorithms. Their essential structure is shown in Figure 2. In active noise control, $P(s)$ is the transfer function matrix from the speaker signals to the microphone signals (including A/D's, D/A's, filters, amplifiers, and sound propagation effects). The disturbance signal $d(t)$ is assumed to originate from a noise process with input $z(t)$ and transfer function $H_1(s)$. Further, a measurement $x(t)$ of the variable $z(t)$ is assumed to be available through a second process with transfer function $H_2(s)$. In the signal processing literature, the signal $x(t)$ is called a *reference*
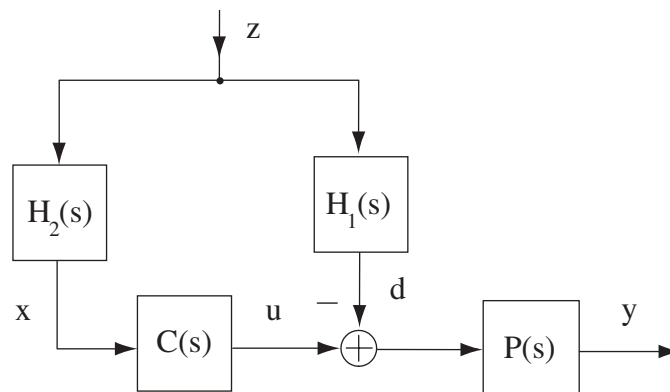

Figure 2. Feedforward cancellation of a disturbance.

signal, and should not be confused with the reference input $r(t)$ defined earlier. If the transfer function $H_1(s)H_2(s)^{-1}$ is causal and stable, a simple compensator $C(s) = H_1(s)H_2(s)^{-1}$ can be effective at cancelling the disturbance. Uncertainties in the transfer functions $H_1(s)$ and $H_2(s)$ make the adaptation of this compensator imperative. However, a linear parameterization of $C(s)$ as a discrete-time finite-impulse-response filter is often used, which makes it possible to use simple gradient techniques effectively.

Although the assumption of a 'preview' of the disturbance is unrealistic in typical control problems, it is often acceptable in active noise and vibration control applications. Either a sensor can be placed upstream in the transmission path of the noise or vibration, or a position/velocity sensor can be mounted on the machine that causes the disturbance. Nevertheless, there are applications where measurements are not feasible, or where they are undesirable for reliability or economic considerations. The paper's focus is on these applications.

3. PERIODIC DISTURBANCES OF KNOWN FREQUENCY

3.1. AFC

We first consider the problem of disturbance rejection for a sinusoidal disturbance $d(t)$ of known frequency. The problem can be solved using AFC algorithms because an equivalent signal $z(t)$ can be reconstructed, even if it cannot be directly measured. Nevertheless, the AFC algorithm is truly and purely a feedback control law.

For convenience, the disturbance is expressed in terms of its cos and sin components as

$$d(t) = \theta_c^* \cos(\alpha_d(t)) + \theta_s^* \sin(\alpha_d(t)), \quad \frac{\mathrm{d}\alpha_d(t)}{\mathrm{d}t} = \omega_d \quad (3)$$

The control input can be chosen to cancel the disturbance by setting

$$u(t) = \theta_c(t) \cos(\alpha_d(t)) + \theta_s(t) \sin(\alpha_d(t)) \quad (4)$$

where $\theta_c(t)$ and $\theta_s(t)$ are adaptive parameters. Exact cancellation occurs for $\theta_c(t) = \theta_c^*, \theta_s(t) = \theta_s^*$. The approach is shown schematically in Figure 3. If the disturbance is not sinusoidal, but nevertheless periodic, the control input may be replaced by

$$u(t) = \sum_{k=1}^{n} \theta_{c,k}(t) \cos(k\alpha_d(t)) + \theta_{s,k}(t) \sin(k\alpha_d(t)), \quad \frac{\mathrm{d}\alpha_d(t)}{\mathrm{d}t} = \omega_d \quad (5)$$

For simplicity, the presentation will focus on a sinusoidal disturbance, although extension to multiple frequencies is possible.
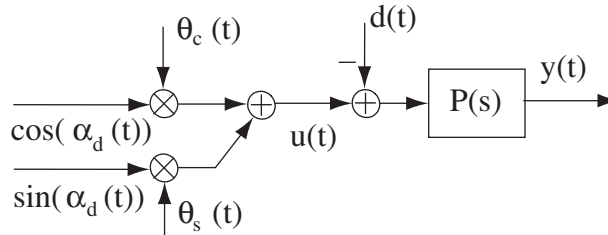

Figure 3. Feedforward cancellation of a sinusoidal disturbance of known frequency.

The problem of adapting the parameters can be formulated in vector notation by defining

$$\boldsymbol{\theta}(t) = \begin{pmatrix} \theta_c(t) \\ \theta_s(t) \end{pmatrix}, \quad \boldsymbol{\theta^*} = \begin{pmatrix} \theta_c^* \\ \theta_s^* \end{pmatrix}, \quad \mathbf{w}(t) = \begin{pmatrix} \cos(\alpha_d(t)) \\ \sin(\alpha_d(t)) \end{pmatrix} \tag{6}$$

so that

$$u(t) = \boldsymbol{\theta}^{\mathrm{T}}(t)\mathbf{w}(t), \quad d(t) = \boldsymbol{\theta}^{*\mathrm{T}}\mathbf{w}(t) \tag{7}$$

where the vector $\mathbf{w}(t)$ is called the *regressor vector*. Then

$$y(t) = P(s)[(\boldsymbol{\theta}(t) - \boldsymbol{\theta^*})^{\mathrm{T}}\mathbf{w}(t)] \tag{8}$$

The plant $P(s)$ is assumed to be a stable system. Note that we used a special notation that is convenient for adaptive control problems. Specifically, $P(s)[x(t)]$ is the time-domain output of the linear time-invariant system $P(s)$ with input $x(t)$.

Equation (8) provides the basis for the application of standard adaptive control theory [7–9]. The *pseudo-gradient algorithm* is the simplest algorithm, and is given by the differential equation

$$\frac{\mathrm{d}\boldsymbol{\theta}(t)}{\mathrm{d}t} = -gy(t)\mathbf{w}(t) \tag{9}$$

where $g > 0$ is a parameter called the *adaptation gain*. The regressor vector $\mathbf{w}(t)$ is persistently exciting, so that the algorithm provides exponential convergence of the adaptive parameters to their nominal values if the transfer function of the plant is *strictly positive real*.

In contrast, the *gradient algorithm* is given by

$$\frac{\mathrm{d}\boldsymbol{\theta}(t)}{\mathrm{d}t} = -gy(t)\mathbf{w}_F(t), \quad \mathbf{w}_F(t) = \hat{P}(s)[\mathbf{w}(t)] \tag{10}$$

where $\hat{P}(s)$ is an estimate of the plant transfer function. Because of the sinusoidal nature of $\mathbf{w}(t)$, $\hat{P}(s)[\mathbf{w}]$ may be implemented using measured frequency response data, rather than a transfer function estimate. Considering the steady-state response only, one has that

$$P(s)[\mathbf{w}(t)] = \begin{pmatrix} \hat{P}_{\mathrm{R}}(\omega_d) & -\hat{P}_{\mathrm{I}}(\omega_d) \\ \hat{P}_{\mathrm{I}}(\omega_d) & \hat{P}_{\mathrm{R}}(\omega_d) \end{pmatrix} \mathbf{w}(t) \tag{11}$$

where $\hat{P}_{\mathrm{R}}(\omega_d)$ and $\hat{P}_{\mathrm{I}}(\omega_d)$ are the real part and the imaginary part of $\hat{P}(\mathrm{j}\omega_d)$, the estimate of the frequency response of the plant evaluated at the frequency ω_d.

The gradient algorithm has not been much appreciated by the adaptive control community, mainly because of its instability for large adaptation gain. However, the adaptive signal processing community has used it extensively under the name of *filtered-X LMS* algorithm. If one accepts the restriction of sufficiently small adaptive gain, the algorithm provides exponential stability *without* a strict positive real condition on the plant (see Reference [10] for a comparison of the stability properties of various algorithms using averaging theory).

3.2. Internal model principle (IMP)

The rejection of disturbances of known frequency is a classical problem of control theory. The IMP [11] prescribes that the controller should include a model of the disturbance and, therefore, have poles on the jω-axis at locations corresponding to the disturbance frequency ω_d. Figure 4. shows the structure of the feedback loop. The compensator $C_1(s)$ should be designed to ensure a stable closed-loop system.
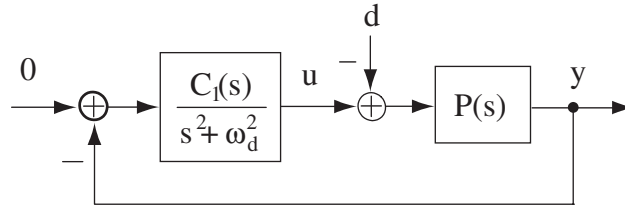
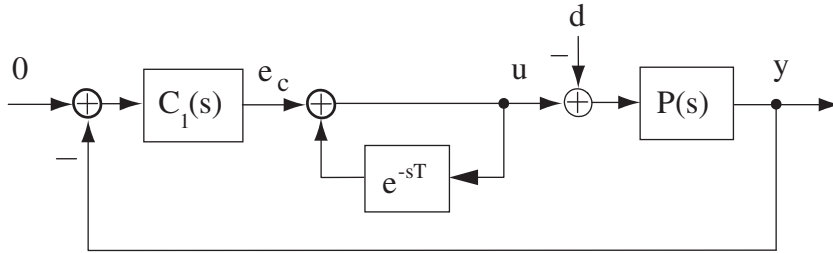Figure 4. Controller based on the internal model principle.

Figure 5. Repetitive controller.

For a periodic disturbance with multiple harmonics, additional poles at $2\omega_d$, $3\omega_d, \ldots$ can be added. A special case is the compensator based on the concept of *repetitive control* [12], as shown in Figure 5. Note that

$$u(t) = u(t - T) + e_c(t) \tag{12}$$

so that the control signal repeats itself every T seconds, except for a correction term $e_c(t)$ based on the error signal. For a disturbance with frequency ω_d, the period T is adjusted to be equal to the period of the signal, $T = 2\pi/\omega_d$. The overall compensator has transfer function

$$C(s) = \frac{C_1(s)}{1 - e^{-sT}} \tag{13}$$

and has poles at $s = \pm j\omega_d$, $\pm j2\omega_d, \ldots$ Therefore, repetitive control is closely related to IMP control. The fundamental component of the disturbance and all its harmonics are rejected asymptotically, so that a periodic disturbance can be cancelled exactly.

3.3. Equivalence between IMP and AFC controllers

Surprisingly, the pseudo-gradient and gradient algorithms discussed earlier are *exactly equivalent* to linear time-invariant (LTI) operators. The LTI system equivalence applies for a single sinusoid, as well as for multiple frequencies. In particular, it can be shown that the AFC system with pseudo-gradient algorithm is equivalent to the IMP controller with

$$C_1(s) = gs \tag{14}$$

The equivalence is also shown in Figure 6. The result can be proved using well-known properties of the Laplace transform [13]. The comparable result for discrete-time algorithms can also be found in the signal processing literature [14] (with some preliminary results in Reference [15]).
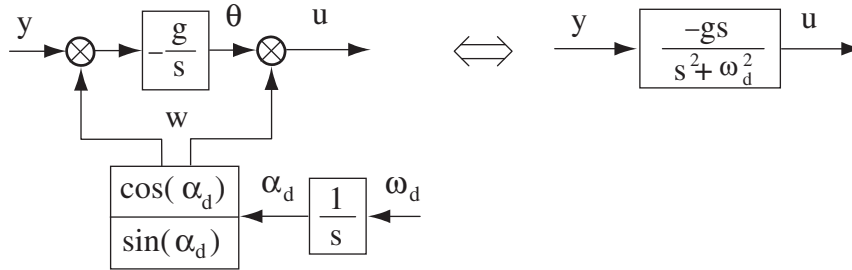
Figure 6. Equivalence between AFC and IMP controllers.

The equivalence can be used for a number of purposes. Stability properties can be determined for plants that are not strictly positive real. In Reference [13], the equivalence was used to explain results obtained when an algorithm designed for a single sinusoidal component was applied to a system with multiple harmonic components. The reduction or amplification of the harmonics could be predicted exactly using the LTI equivalence.

In the same manner, the gradient algorithm can be shown to be equivalent to an IMP controller with

$$C_1(s) = g(\hat{P}_R(\omega_d)s + \hat{P}_I(\omega_d)\omega_d) \tag{15}$$

The compensator again embodies the internal model principle, although with a different zero in the transfer function. Interestingly, this zero is non-minimum phase for typical systems, including for a first-order system. This feature and related design principles were discussed in Reference [16]. An equivalent LTI system was derived for an adaptive algorithm using an augmented error in Reference [17], where a modified algorithm called the *filtered augmented error* (*FAE*) was also proposed. The purpose of the FAE algorithm was to enhance the convergence and robustness of the algorithm for plants with resonant modes. In Reference [18] general conditions were obtained under which the LTI system equivalence applies.

3.4. Disturbances of known but varying frequency

In some applications, the frequency of the disturbance is known, but varies significantly over time. For example, researchers at the University of Strasbourg [19] have considered the problem of regulating the tension in a paper rolling machine. A picture of their testbed is shown in Figure 7. Note that this testbed exhibits problems encountered in other, similar applications such as rolling machines for steel and magnetic tape drives.

Experiments on the paper machine testbed indicate that, when a simple controller is used, the spectrum of the tension has large components at the frequencies of rotation of the winding and unwinding rolls. The periodic disturbances are due to the eccentricity and to the non-circularity of the rolls. Figure 8, which was derived from the spectrograms of Reference [19], shows the frequencies of the sinusoidal components, as functions of time. The frequency f_1 is the fundamental component associated with the unwinding roll. It increases with time due to the decreasing radius. Conversely, the frequency f_2 is associated with the winding roll and decreases over time. Second harmonics are also present. Note that the frequencies can be obtained through measurements of the angular positions of the rolls (using encoders), but there is a significant variation with time.
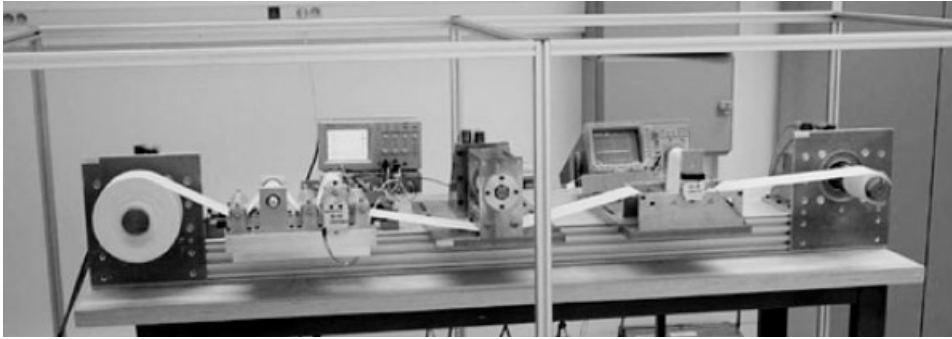
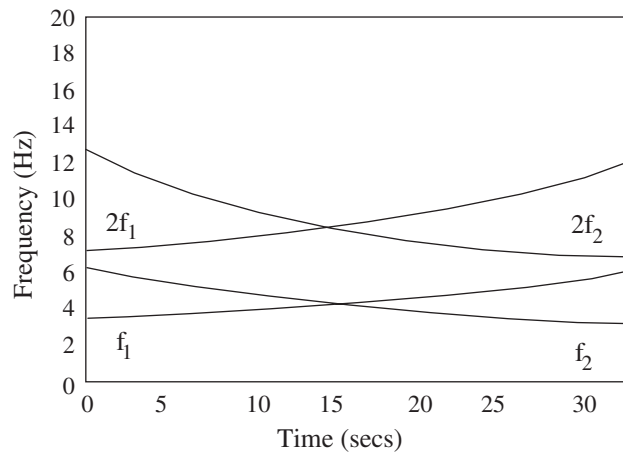Figure 7. Paper machine testbed (University of Strasbourg).

Figure 8. Frequencies of the components of the web tension.

Other problems with eccentricity compensation were considered in Reference [20], while Reference [21] discussed a problem with similar characteristics in stepper motors. Specifically, an adaptive algorithm was applied to reduce torque pulsations having a fundamental component related to the frequency of rotation of the motor. In all these applications, one may assume that an angular measurement of the position of the motor provides direct knowledge of the frequency of the disturbance, which may vary rapidly. The AFC algorithms are attractive, because their theory as well as their implementation does not depend on the frequency being constant. In fact, when the angle α_d is available, it can be directly incorporated in the algorithms (see Figure 6) without reconstructing the frequency ω_d [21].

The IMP algorithms can also be implemented for time-varying frequency, provided that a state-space realization of Figure 4 is chosen. For example, one may pick

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_d^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y, \quad u = \begin{pmatrix} 0 & -g \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{16}$$

This implementation works well if the frequency of the disturbance is slowly varying, but errors occur if the variation is rapid compared to ω_d. Indeed, consider simulations performed with the AFC and IMP algorithms for a plant $P(s) = 100/(s + 100)$. The disturbance has magnitude equal to 1. The two figures on the top of Figure 9 show the plant output for a fixed disturbance frequency at 30 rad/s. The AFC and IMP responses are identical (as predicted by the LTI equivalence of Reference [13]). The bottom figures show the results when the frequency varies from 30 to 60 rad/s. Note that the error goes to zero for the AFC, but not for the IMP controller.

An explanation for this phenomenon can be found in the work of Wulich *et al.* [22], which addresses a similar problem in the case of time-varying notch filters. Translated into continuous-time, their observation is that the differential equation for $x(t) = A\cos(\alpha_d(t))$ has an additional term when the frequency varies

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega_d^2 x + \frac{\mathrm{d}\omega_d}{\mathrm{d}t}\frac{1}{\omega_d}\frac{\mathrm{d}x}{\mathrm{d}t} \tag{17}$$

Therefore, the state-space system (16) needs to be corrected so that the (2,2) element of the matrix is replaced by $(\mathrm{d}\omega_d/\mathrm{d}t)/\omega_d$ when the rate of variation of the frequency is high. Interestingly, this correction is not needed if the state-space representation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega_d \\ -\omega_d & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}y, \quad u = (0 \quad -g)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{18}$$
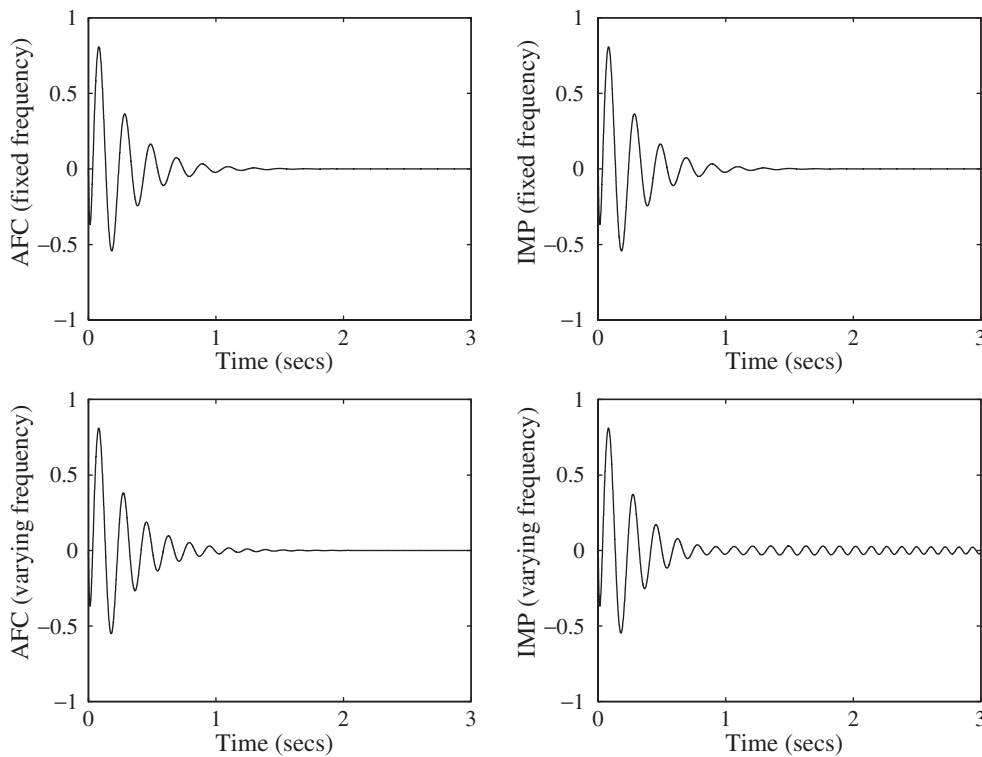

Figure 9. Comparison of AFC and IMP with fixed and linearly varying frequency.

is used instead [23]. The paper [23] also shows how the LTI equivalence result can be extended to time-varying systems. Design of the feedback controller can be performed based on linear parameter-varying (LPV) control theory [19], where ω_d is the parameter appearing linearly in the system description.

4. REJECTION OF DISTURBANCES OF UNKNOWN FREQUENCY

4.1. Adaptive IMP

A natural approach to the rejection of periodic disturbances of unknown frequency is the adaptive IMP scheme shown in Figure 10. To make sense of the diagram, a state-space realization of the compensator (such as (16) or (18)) must be chosen for the adaptation of the parameters. In general, both the frequency ω_d and the compensator $C_1(s)$ need to be adjusted, although a fixed compensator $C_1(s)$ can sometimes be used.

Researchers have considered the application of adaptive control theory for the ambitious goal of adapting compensators to unknown disturbance parameters as well as to unknown plants [8,24]. Typically, the algorithms implement some form of adaptive pole placement where two poles of the compensator are forced to lie on the jω-axis (or on the unit circle in discrete-time). For example, Figure 11 shows the structure of a model reference adaptive control algorithm found in Reference [8]. The plant is a first-order system and the disturbance is sinusoidal. The parameters k, c_2, d_1, d_2, and θ are updated according to a gradient algorithm, while the constraint $c_1 = a_1$ is applied so that the feedback loop has a pair of poles on the jω-axis (for fixed parameters, the inner loop with $c_1 = a_1$ has poles determined by $s^2 + a_2 - c_2 = 0$).

While such an adaptive algorithm solves a far more general problem than has been considered in this paper, the control laws are difficult to implement in practice and are easily destabilized in the presence of unmodelled dynamics [25,26]. Indeed, Figure 12, shows the plant output resulting from a simulation of the example of Figure 11. The conditions are the same as those of Reference [8], and Figure 12 is essentially a replica of a figure found on p. 302 of Reference [8]. The dashed line shows the output of the reference model, which the plant output is supposed to follow. Note that the plant is unstable, so that both stabilization of the unknown plant and tracking of the reference input are achieved. However, rejection of the disturbance is not obtained, with a large oscillation remaining after 6 s. While the theoretical results guarantee the
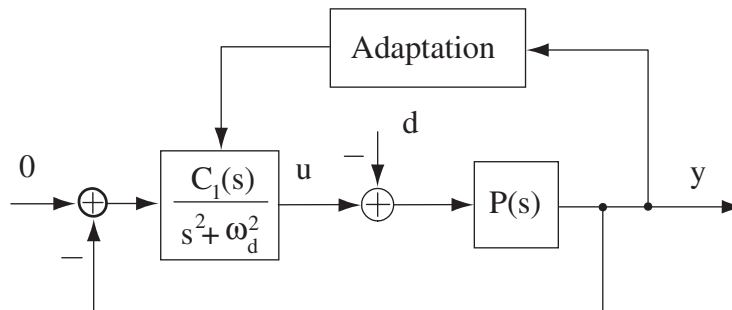

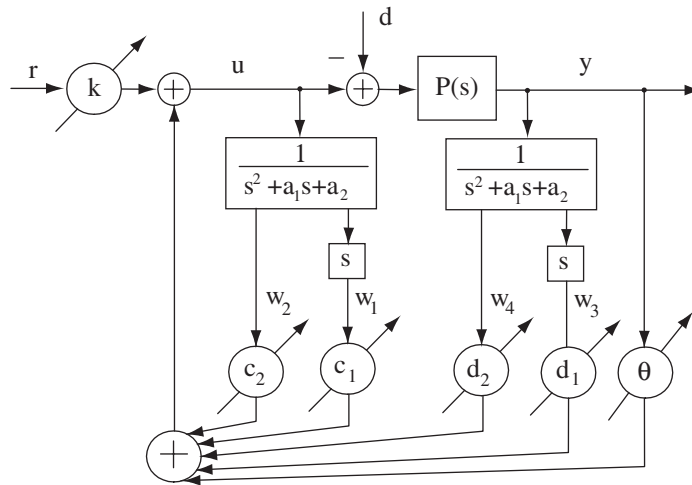Figure 10. Adaptive internal model principle.

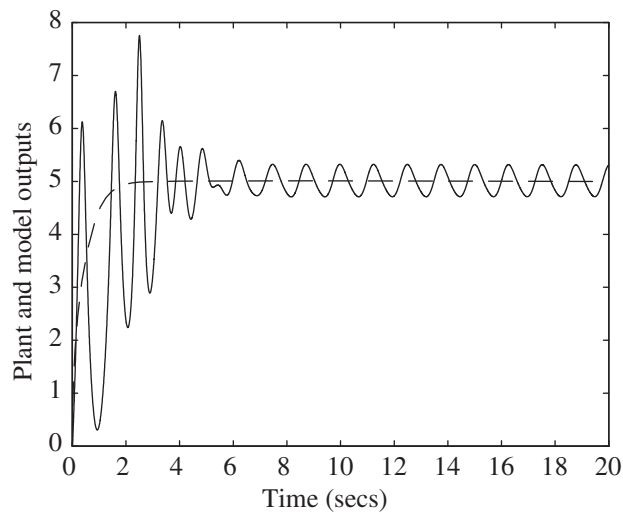Figure 11. Narendra and Annaswamy's example.

Figure 12. Plant output for Narendra and Annaswamy's example.

stability of the adaptive system and the convergence of the error to zero, the actual convergence is extremely slow. It was found that convergence of the amplitude of the error was approximately of the order of $1/\sqrt{t}$, and remained very slow even when the adaptive gain was multiplied by 10.

The lack of good convergence properties for this scheme can be expected to lead to poor robustness properties. Indeed, Figure 13 shows the result of a simulation with the plant replaced by

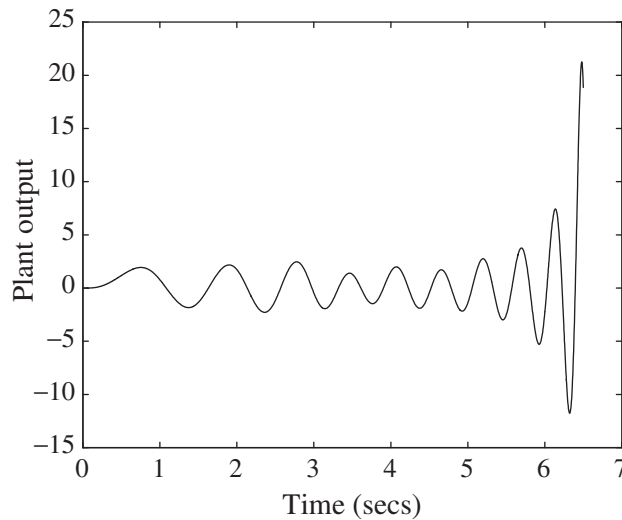$$P(s) = P_{\text{nom}}(s) \, \frac{229}{s^2 + 30s + 229} \tag{19}$$

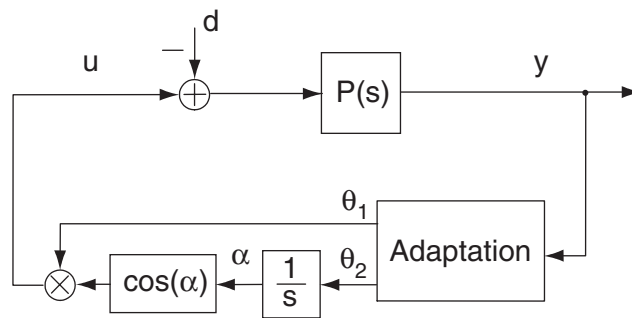Figure 13. Plant output for Narendra and Annaswamy's example with unmodelled dynamics.

Figure 14. Direct approach to sinusoidal disturbance cancellation.

where $P_{nom}(s) = 2/(s-1)$ is the nominal transfer function used in the previous simulation. The unmodelled dynamics in (19) are those of the well-known 'Rohrs examples' [9,27] and constitute relatively mild modifications to the original plant. Yet, the output of the plant, shown in Figure 13, rapidly diverges. As in the Rohrs examples, the weak stability properties of the system in ideal conditions mean that the system can be easily destabilized.

The adaptive internal model principle approach is not hopeless, but it is likely that successful approaches will rely on information about the plant. For example, recent algorithms [28,29] use a direct adaptive control approach where the plant parameters are assumed to be known.

4.2. Direct approach to disturbance cancellation

A class of algorithms for the cancellation of periodic disturbances is based on the concept shown in Figure 14. The idea, for a single harmonic component, is to generate a sinusoidal control signal and to adapt its parameters so that the disturbance is cancelled. The parameters of the

signal are its magnitude, frequency, and phase, with the third parameter determined as the integral of the second. This concept was proposed in Reference [30], under the name of *direct approach*.

The direct algorithm proposed in Reference [30] is remarkably simple, and consists of the following equations:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 2 \begin{pmatrix} \hat{P}_R(\omega_d) & -\hat{P}_I(\omega_d) \\ \hat{P}_I(\omega_d) & \hat{P}_R(\omega_d) \end{pmatrix}^{-1} \begin{pmatrix} y(t) & \cos(\alpha(t)) \\ -y(t) & \sin(\alpha(t)) \end{pmatrix}$$

$$\begin{pmatrix} \theta_1(s) \\ \theta_2(s) \end{pmatrix} = -\frac{1}{s} \begin{pmatrix} g_1 & 0 \\ 0 & g_2(s+a)/(s+b) \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}$$

$$\alpha(s) = \frac{1}{s} \theta_2(s) \qquad (20)$$

where $\hat{P}_R(\omega_d)$ and $\hat{P}_I(\omega_d)$ are the estimated real and imaginary parts of the frequency response of the plant at the disturbance frequency. The frequency response values do not have to be very precise and, if ω_d varies over a significant range, the frequency response may be evaluated at the estimate of the frequency θ_2. Implementation of the control law requires estimates of the frequency response and of the disturbance magnitude d_1 for tuning of the parameters (an upper bound on d_1 is sufficient).

Despite the simplicity of the adaptive algorithm, it is quite nonlinear. In fact, the closed-loop system can be shown to be similar to a phase-locked loop (PLL). Yet, a main difference with a PLL is that the feedback system locks both the phase *and* the magnitude of the signal to those of the exogeneous signal.

A most interesting aspect of the algorithm is that an accurate approximation of the system has been found, which is useful for analysis and design. Specifically, the approximation is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \simeq \begin{pmatrix} \theta_1(t) - d_1 \cos(\alpha(t) - \alpha_d(t)) \\ d_1 \sin(\alpha(t) - \alpha_d(t)) \end{pmatrix} \simeq \begin{pmatrix} \theta_1(t) - d_1 \\ d_1(\alpha(t) - \alpha_d(t)) \end{pmatrix} \qquad (21)$$

where d_1 and α_d are the magnitude and angle corresponding to the sinusoidal disturbance, and it was assumed that $\hat{P} = P$. The first approximation in (21) retains the essential non-linearity of the system, while the second approximation is a linearization of the first. The linear approximation, together with the update for θ_1, θ_2 in (20) and the plant description (1), provide a linear time-invariant approximation of the whole adaptive system.

A representative simulation result is shown in Figure 15. The solid line gives the response of the adaptive parameter estimating the magnitude of the disturbance, θ_1. The two dashed lines are the results of the non-linear and linear approximations. The non-linear approximation is very accurate, and follows the 'capture' portion of the response during which the frequency of disturbance is acquired. The linear approximation is poorer, but is indicative of the asymptotic convergence of the algorithm. It is very useful for design, in order to set the parameter gains. Note that the direct algorithm can be designed to achieve specific targets for asymptotic convergence rates, as well as for noise rejection properties [31].

The direct approach of Bodson and Douglas [30] was extended to multi-input multi-output systems and to periodic disturbances [32]. Active noise control experiments were performed in a small room, which is a more difficult problem than either a duct (because of the

three-dimensional sound distribution) or open space (because of reflections on the walls of the room). Figure 16 shows the power spectra of two microphone signals in an experiment with two microphones, two speakers, and a disturbance with a fundamental and second harmonic. The solid lines show the power spectra without compensation, and the dashed lines show the results obtained with the adaptive algorithm. Due to the sweeping of the fundamental frequency from 130 to 150 Hz, one finds a large component of the uncompensated spectrum in that range as well
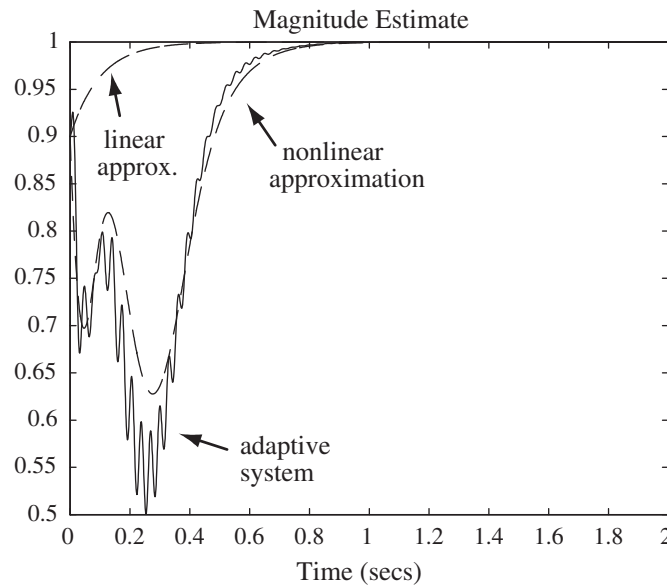

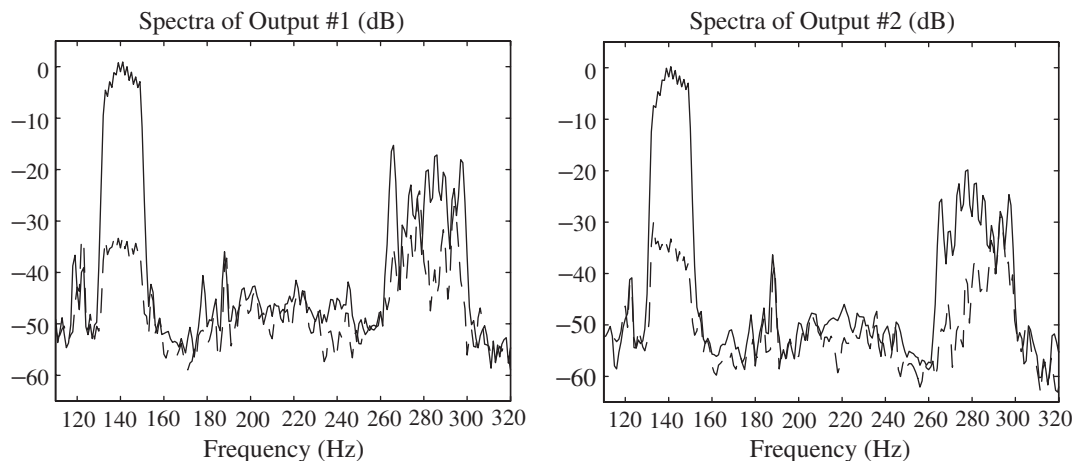Figure 15. Magnitude parameter responses in the direct approach.

Figure 16. Spectra of the two output signals without cancellation (solid) and with cancellation (dashed) using the direct approach.

as at the double of the range. Overall, the adaptive controller was successful at significantly reducing the effects of the disturbance, despite the lack of reference sensor, or knowledge of the frequency of the disturbance.

The concept proposed in Reference [30] was also applied successfully for the rejection of periodic disturbances in a 6 DOF Stewart platform at the University of Wyoming [33]. A variation of the algorithm was applied to the paper rolling machine of Figure 7 in Reference [19]. Although the frequency of the disturbance was known, a modification of algorithm (20) was found to be the most successful approach, among a few that were tried. The algorithm of Algrain *et al.* [34] is also similar in concept to the one proposed in Reference [30]. It was applied to the vibration control of smart structures, but no stability analysis was provided that would predict the dynamic behaviour of the adaptive system.

4.3. Indirect approach to disturbance cancellation

An alternate approach to the cancellation of sinusoidal disturbances of unknown frequency is shown in Figure 17. The concept was proposed in Reference [30] under the name of *indirect approach*. This terminology was chosen in analogy to adaptive control, where an indirect scheme is such that plant estimates are used in an inner control loop as if they were the exact plant parameters. Here, the idea is to use an algorithm for the cancellation of periodic disturbances of known frequency, and to replace the frequency parameter in the algorithm by an estimate (denoted θ_f in the figure). The difference between the direct and indirect approaches can be fuzzy, since a direct approach typically also includes a frequency estimate. The distinction made here is basically between schemes that are designed in an integrated fashion and schemes that are designed in a two-step procedure.

There is a subtlety in Figure 17, in that the signal used for frequency estimation is not the plant output, but a modified signal y_d obtained using an estimate of the plant transfer function. If $\hat{P}(s) = P(s)$, the signal y_d is equal to the plant output that would be observed if no control signal was applied (i.e. if only the disturbance was present). Simulations in Reference [30] showed that an algorithm using the output signal y for frequency estimation suffered from slow asymptotic convergence, due to the disappearance of the output signal when the disturbance
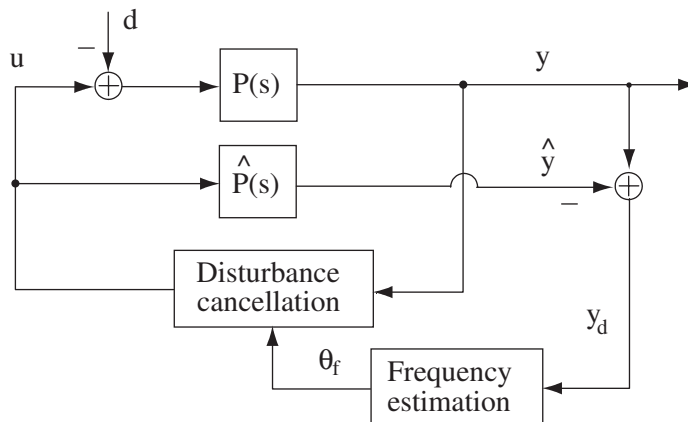

Figure 17. Indirect approach to sinusoidal disturbance cancellation.

was cancelled. In contrast, the alternate implementation of Figure 17 was found to work well even when the plant estimate $\hat{P}(s)$ was a crude estimate of the transfer function $P(s)$. It turns out that the concept was proposed in a patent [35], although not all papers taking an indirect approach adopt this useful modification.

A significant advantage of the indirect approach is the flexibility that it gives in choosing independently algorithms for disturbance cancellation and for frequency estimation. A few papers have appeared that make various choices and testify to the broad range of applications where periodic disturbances of unknown frequency must be rejected. For example:

- [36] develops a new frequency estimator and combines it with a repetitive controller. Application: periodic load disturbances in a continuous steel casting process.
- [37] uses an internal model controller for cancellation and an extended Kalman filter for frequency estimation. Application: control of a voltage source inverter to reject harmonic disturbances in a power network.
- [38] uses repetitive control and adaptive notch filtering. Application: compensation of the eccentricity in a compact disc mechanism.
- [39] uses a k-incremental controller with recursive least-squares estimation for cancellation. The frequency is estimated off-line using a maximum entropy method. Application: a shell-and-tube heat exchanger.
- [40] uses a filtered-X LMS algorithm for cancellation and an adaptive notch filter for frequency estimation. Application: active control of noise in a passenger ship and in a passenger vehicle.

Different types of cancellation and frequency estimation algorithms were compared in Reference [6]. In an active noise control experiment with two inputs and two outputs, the frequency estimates were obtained using an adaptive notch filter, a PLL, and the dual PLL described in the next section. In the dual PLL, both output signals were used in the frequency estimator. Figure 18 shows the spectra of the output signals with and without cancellation. The
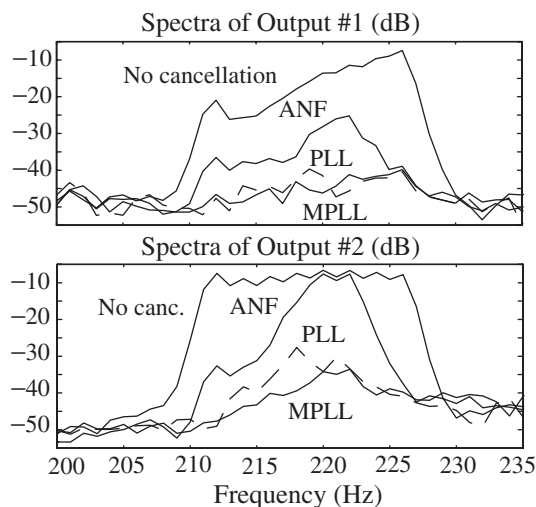

Figure 18. Spectra of two outputs with different algorithms using the indirect approach.

indirect schemes with the PLL and the dual PLL were the most effective overall. The scheme with the dual PLL had somewhat better performance because the fundamental became small at output #1 during the course of the experiment. The experiments demonstrated the ability of the dual PLL to use multiple signals in order to better estimate the fundamental frequency. As a result, better disturbance cancellation was achieved.

Generally, an advantage of the indirect approach is that it is very flexible. The modularity of the design allows one to select the cancellation and frequency estimation algorithms independently, and each component can be selected from a large number of possible choices. Testing can also be performed separately for both components, before the whole system is put together. On the other hand, the stability properties of the algorithms designed using a direct approach are better understood.

4.4. Frequency estimation

Motivated in part by applications requiring the rejection of periodic disturbances of unknown frequency, there has been a renewed interest in frequency estimation methods, especially those that enable the tracking of frequency in real-time. In Reference [41], the classical estimation of the power spectral density using the fast Fourier transform was applied in an indirect adaptive cancellation scheme. Other established methods in signal processing include adaptive notch filtering [42,43] and extended Kalman filtering [44]. Some recent work has also focused on deriving globally convergent algorithms for frequency estimation [45,46].

There is a close connection between disturbance rejection and frequency estimation. Indeed, an algorithm with the structure of Figure 14 can be used for the estimation of the frequency of a signal $d(t)$ by taking $P(s) = 1$. The resulting algorithm will not only be useful for frequency estimation, but also for *signal reconstruction*. Conversely, methods for signal reconstruction with a feedback structure can potentially be used for disturbance rejection as well. The issue is whether their stability can be guaranteed when a non-unity transfer function is inserted in the feedback path.

Based on these observations, a *magnitude/phase-locked loop* (MPLL) concept was proposed in Reference [47]. The algorithm enables the tracking of the magnitude, frequency, and phase of a sinusoidal signal. In addition, a *dual phase-locked loop* was developed where two separate components contributed to a single frequency estimate. The algorithm, shown in Figure 19, combines the fundamental components of two signals y_1 and y_2 for frequency estimation (the scheme assumes that both signals contain information about a single source). The figure shows a standard PLL in the dashed box. A dual path was added and integrated for the use of the other signal.

The algorithm can be modified so that the fundamental component and the Nth harmonic of a single signal can contribute to the frequency estimate. Figure 20 shows frequency tracking results of a basic PLL and of a dual PLL for a periodic signal having a fundamental and a second harmonic. The fundamental frequency increases linearly from 148 to 176 Hz in 3.5 s. The experiment was designed so that the fundamental component vanished from 1 to 1.5 s, resulting in a loss of tracking for the basic PLL. However, the dual PLL was able to extract the frequency information from both components and to maintain tracking throughout the experiment. Extensions of the algorithm to arbitrary combinations of signals and/or harmonics may be derived. An advantage of the modified PLL is that it is not necessary to know *a priori* which components exist or which component is the most suitable to base the frequency estimation on [47].
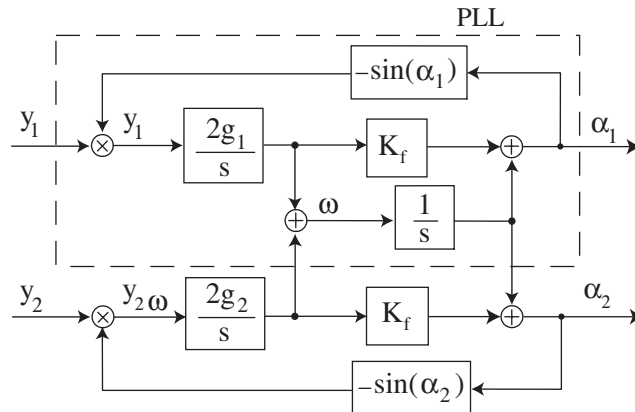
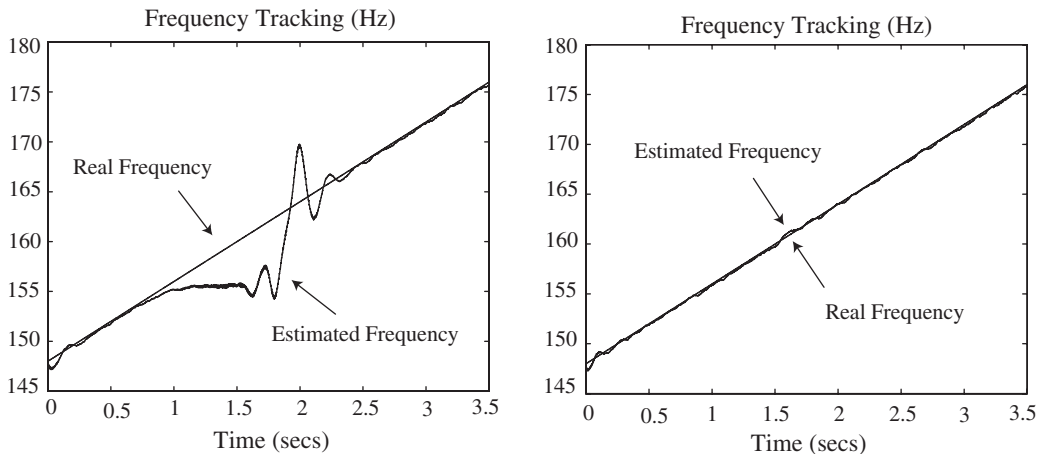Figure 19. Frequency estimation based on a dual PLL.

Figure 20. Frequency tracking of a PLL (left) and dual PLL (right).

Recently, researchers have also considered problems where the frequencies of two independent sinusoidal signals must be tracked, including cases where the two frequencies are very close. For example, Reference [48] describes a problem where a sensor must be developed to measure mass flow in an agricultural machine. The spectrum of the sensor data shows a peak at 13.2 Hz, corresponding to the mass flow, together with a parasitic signal at 11.6 Hz that corresponds to the resonance frequency of the sensor. In Reference [49], the problem of pitch tracking for automatic music transcription is considered. When multiple notes are played together (polyphonic case), the algorithm must track more than one sinusoidal component. The paper reports data for notes at 262 and 393 Hz, but closer spacing may be encountered.

An extension of the MPLL algorithm for the estimation of two sinusoids with closely spaced frequencies was proposed in Reference [50]. The structure of the algorithm is shown in Figure 21. A discrete-time formulation was used in Reference [50], in order to implement the frequency update and separation block shown in the figure.

Figure 21 shows two estimators placed in parallel. ω_1 and ω_2 are the two frequency estimates, while m_1 and m_2 are the magnitude estimates. Note that the two sinusoidal components of the signal are reconstructed as $\hat{d}_1$ and $\hat{d}_2$. The frequency separation block prevents the frequency estimates from converging to the same value (for details, see Reference [50]). Figure 22 shows results of an experiment with two closely spaced frequencies (equal to the 11.6 and 13.2 Hz of
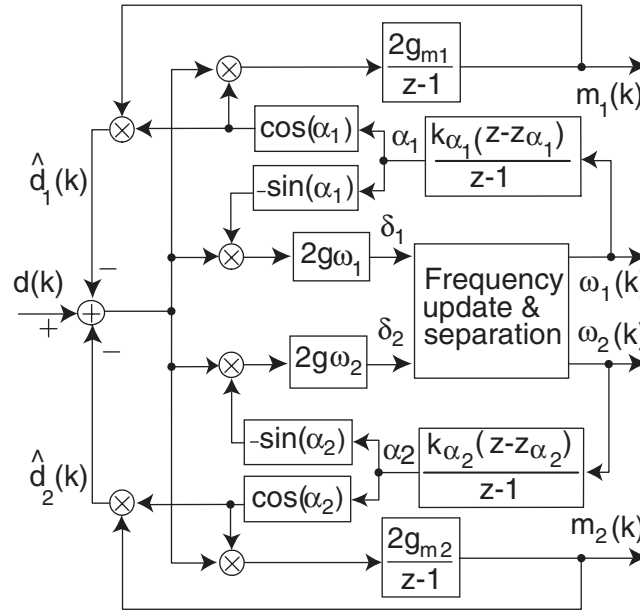

Figure 21. Parameter estimation for a signal with two independent sinusoidal components.
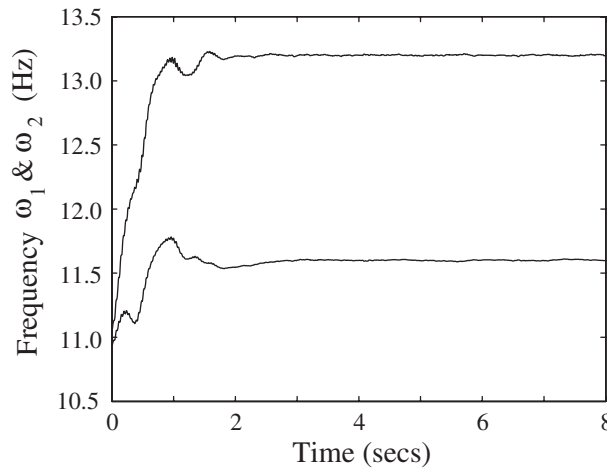

Figure 22. Frequency estimates with two components.

Reference [48]). The figure shows that the estimator is able to determine the two frequencies, even though the initial estimates of frequency are close, and smaller than both true frequencies.

5. CONCLUSIONS

To reject periodic disturbances of known frequency, control laws can be designed based on the internal model principle, or using the closely related repetitive control approach. In addition, cancellation of a disturbance can be performed using adaptive algorithms. The adaptive algorithms have advantages in some applications, although they can be proved to be equivalent to controllers implementing the internal model principle under certain conditions. When the frequency of the disturbance is unknown, the most intuitive approach consists in combining a frequency estimator together with an algorithm for the rejection of disturbances of known frequency. This approach gives a lot of flexibility in the design of the feedback system. The rejection of periodic disturbances of unknown frequency can also be achieved by direct adaptation of the frequency, magnitude, and phase of the components of the control input. The dynamic properties of such feedback systems are better understood, but their design and implementation is more difficult. Adaptive algorithms have also been proposed to reject unknown disturbances affecting unknown plants. However, existing methods have been found to lack the robustness required for practical implementation.

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