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On the Theory of Linear Dynamic System with Periodic Parameters*

IMSONG LEE

Information Studies Section, General Electric Research Laboratory, Schenectady, New York

This paper presents an extension of the classical theory of Floquet to provide a unified treatment on the state transformation properties and stability of linear dynamic systems with continuous and piecewise continuous periodic parameters. The linear dynamic system with periodic parameters is represented mathematically by a linear differential equation with periodic coefficients.

It is shown that the linear differential equation with periodic coefficients can always be reduced to linear difference equations with *constant* coefficient matrices. The coefficient matrices of difference equations are all related to each other by similarity transformations. Consequently, it is sufficient to study any one of the difference equations to determine stability of the original differential equation with periodic coefficients. The entire discussion is developed from the point of view of the state transformation which has been proven to be of great use in the study of dynamic systems.

LIST OF SYMBOLS

$\mathbf{x}(t)$	vector representing the state of the system
$\dot{\mathbf{x}}(t)$	a time derivative of $\mathbf{x}(t)$
$A(t)$	coefficient matrix of the vector differential equation
$\Phi(t, t_0)$	fundamental matrix of the differential equation
$\mathbf{b}(t)$	vector forcing function
I	identity matrix
C	constant coefficient matrix of the vector difference equation
$\exp Bt$	exponential of the square matrix Bt
β_i	eigenvalues of B
γ_i	eigenvalues of C

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Re β_i	real part of β_i
τ_i^+	right limit to τ_i
τ_i^-	left limit to τ_i
$\|\mathbf{x}(t)\|$	The norm of $\mathbf{x}(t)$ defined to be $\sum_i \mid x_i(t) \mid$

I. INTRODUCTION

The linear feedback systems with periodically time-varying parameters represent an important class of linear dynamic systems. This class of systems is described mathematically by a linear differential equation with periodic coefficients.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + \mathbf{b}(t), \qquad t_0 \leqq t < \infty \\ A(t+1) &= A(t), \qquad -\infty < t < \infty \end{aligned} \tag{1}$$

where $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ column vectors and $A(t)$ is an $n \times n$ matrix whose elements are continuous functions of time $a_{ij}(t)$ of normalized period 1. The period has been normalized to 1 as a matter of convenience.

This equation, of course, represents not only the class of linear feedback systems mentioned above, but also many important physical phenomena (Malkin, 1952). The basic properties of Eq. (1), i.e., the normal modes and characteristic frequencies etc. are readily obtained from the classical theory of Floquet (Cesari, 1959; Lefschetz, 1957; Coddington and Levinson, 1955; Floquet, 1883).

There are many examples of modern engineering systems in which one or more system parameters may vary periodically but piecewise continuously. Recently the linear feedback systems containing product type pulse-modulators have been studied extensively by various specialized techniques (Farmanfarma, 1958; Kalman and Bertram, 1959; Gilbert, 1959).

This paper represents an attempt to treat the class of linear dynamic systems containing periodically varying but *piecewise continuous* elements from a single, unified point of view. The periodically pulsed feedback systems, carrier-frequency servo systems and many carrier-type electronic dc feedback amplifiers all belong to this class.

II. STATEMENT OF THE PROBLEM

The linear dynamic system with periodically varying but piecewise continuous parameters is represented by a linear differential equation with *piecewise continuous*, periodic coefficients. The form of equation is the same as (1), but some of the elements $a_{ij}(t)$ of matrix $A(t)$ have a finite number of discontinuities within the period, $0 \leqq t \leqq 1$. We wish to study the state transformation characteristics and stability properties of this system.

It is desired in particular to *properly* extend the classical theory of Floquet so as to provide a satisfactory mathematical framework for study of the linear dynamic systems with periodic parameters from a unified point of view.

Fortunately, it is possible to extend the conclusions of the classical Floquet theory to the above class of systems without elaborate modifications. We shall first review briefly the important basic properties of the linear system with *continuous* periodic parameters and afterwards proceed to a discussion of the linear system with *piecewise continuous* periodic parameters.

III. THE LINEAR SYSTEM WITH CONTINUOUS PERIODIC PARAMETERS: A REVIEW OF THE FLOQUET THEORY

The solution of Eq. (1) is formally given by

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\mathbf{b}(\tau)\, d\tau \tag{2}$$

where $\Phi(t, \tau)$ is the fundamental matrix. It may be regarded as a generalized impulse response matrix and can be always factored into the form

$$\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau) \tag{3}$$

and

$$\Phi(t) \equiv \Phi(t, 0), \qquad \Phi(t, t) = I.$$

Although general properties of the fundamental matrix associated with the linear differential system with continuous periodic coefficients are well-known, they are usually not presented in a form most suitable to study of the dynamics of physical systems. The time history of a dynamic system is completely described by the laws governing the transformation of the state from the initial time to the final time. Consequently, we derive important basic properties of $\Phi(t, \tau)$ from the concept of linear transformation of the state. Since the structure and properties of $\Phi(t, \tau)$ are independent of the driving force, we set $\mathbf{b}(t) = \mathbf{0}$ in (1).

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t), & t_0 \leqq t < \infty \\
A(t+1) &= A(t) \quad \text{or} & \\
A(t+k) &= A(t), \qquad k = 0, 1, 2, \cdots &
\end{aligned} \tag{4}
$$

(a) The solution of (4) is

$$\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0), \qquad t \geqq t_0 \tag{5}$$

and $\Phi(t, t_0)$ represents the transformation of the initial state $\mathbf{x}(t_0)$ to the new state $\mathbf{x}(t)$. However, due to the periodicity of $A(t)$, the transformation must be also of periodic character. This property may be illustrated with the aid of a graphical representation in Fig. 1.

The periodicity of the fundamental matrix $\Phi(t)$ is readily derived from Eqs. (4) and (5). It is analytically expressed by

$$\Phi(k + \tau, k) = \Phi(\tau, 0) = \Phi(\tau), \qquad 0 \leqq \tau \leqq 1, k = 0, 1, 2, \cdots \tag{6}$$

or

$$\mathbf{x}(k + \tau) = \Phi(k + \tau, k)\mathbf{x}(k) = \Phi(\tau)\mathbf{x}(k).$$

The significance of this equation is that the fundamental matrix $\Phi(t)$ is known for all $t \geqq 0$, if it is known over a period $0 \leqq t \leqq 1$.

(b) If we set $\tau = 1$ in (6), we obtain

$$\mathbf{x}(k + 1) = \Phi(1)\mathbf{x}(k) \equiv C\mathbf{x}(k) \tag{7}$$

where $\Phi(k + 1, k) = \Phi(1) \equiv C$: a constant, nonsingular matrix. We have thus reduced the differential equation with periodic coefficients (4) to a difference equation with *constant* coefficients. This is an extremely important fact.

(c) Next we derive the matrix difference equation satisfied by $\Phi(t)$. Any positive real number t may be represented by $t = k + \tau$ where $0 \leqq \tau \leqq 1$, $k = 0, 1, 2, \cdots$. Consequently, we have

$$
\begin{aligned}
\mathbf{x}(t + 1) &= \mathbf{x}(k + \tau + 1) = \Phi(k + \tau)\mathbf{x}(1) = \Phi(t)C\mathbf{x}(0) \\
&= \Phi(t + 1)\mathbf{x}(0)
\end{aligned} \tag{8}
$$

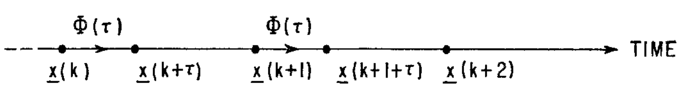

FIG. 1. A graphical representation of the periodic transformation of the state. Given an initial state $\mathbf{x}(k)$, a subsequent state is $\mathbf{x}(k + \tau) = \Phi(\tau)\mathbf{x}(k)$ for $k = 0, 1, 2, \cdots$.

and

$$\Phi(t + 1) = \Phi(t)C; \qquad \Phi(0) = I, t \geqq 0.$$

(d) The equation (7) represents the transformation of the state for a particular sequence of "sampling time," $t = 0, 1, 2, 3, \cdots$. For a general choice of "sampling time," the equation becomes

$$\begin{aligned} \mathbf{x}(k + \tau + 1) &= \Phi(k + \tau + 1, k + \tau)\mathbf{x}(k + \tau) \\ &= C(\tau)\mathbf{x}(k + \tau); \quad 0 \leqq \tau \leqq 1 \end{aligned} \tag{9}$$

and this may be illustrated graphically as shown in Fig. 2.

It is clear from Fig. 2 and Eqs. (8) and (9) that C and $C(\tau)$ are related by the similarity transformation

$$C(\tau) = \Phi(\tau)C\Phi^{-1}(\tau). \tag{10}$$

Consequently, a difference equation derived from the differential equation with periodic coefficients (4) is *not* unique, but C and $C(\tau)$ have the same eigenvalues because of Eq. (10). Therefore, it is sufficient to examine Eq. (7) for stability of the system.

(e) It was first shown by Floquet that the analytic form of the fundamental matrix is

$$\Phi(t) = P(t) \exp Bt, \qquad P(t + 1) = P(t) \text{ and continuous.} \tag{11}$$

It follows from this,

$$C = \exp B \text{ (or: } \exp BT \text{ if } A(t + T) = A(t)) \tag{12}$$

where B is an $n \times n$ constant matrix.

If we denote the eigenvalues of B and C by β and γ respectively, then they are related by

$$\gamma_i = e^{\beta_i + j2k\pi} \qquad k = 0, 1, 2, \cdots, i \leqq n; j^2 = -1 \tag{13}$$

or,

$$\gamma_i = e^{\beta_i T + j2\pi k\Omega} \qquad \text{if } A(t + T) = A(t) \text{ and } \Omega = 2\pi/T.$$

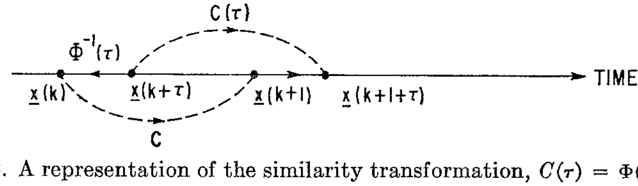

FIG. 2. A representation of the similarity transformation, $C(\tau) = \Phi(\tau)C\Phi^{-1}(\tau)$

It is clear from Eqs. (11), (12), and (13) that the differential equation (4) is asymptotically stable if, and only if, the eigenvalues of C all lie within the unit circle, i.e., $|\gamma_i| < 1$ which is equivalent to the statement, Re $\beta_i < 0$. This is also the necessary and sufficient condition for $\mathbf{x}(t)$ to be bounded for all $t \geqq t_0$, given that $\mathbf{b}(t)$ is bounded in Eq. (2).

(f) Calculation of the characteristic roots γ_i has been a difficult problem for a long time. Today, however, it is possible to compute the matrix C and its eigenvalues to any desired degree of accuracy by machine computation. The matrix $C = \Phi(1)$ is obtained by integrating the matrix differential equation associated with (4) over a period.

$$\begin{aligned} \dot{\Phi}(t) &= A(t)\Phi(t), \qquad \Phi(0) = I, 0 \leqq t \leqq 1 \\ A(t+1) &= A(t). \end{aligned} \tag{14}$$

IV. THE LINEAR SYSTEM WITH PIECEWISE CONTINUOUS PERIODIC PARAMETERS: AN EXTENSION OF THE FLOQUET THEORY (Lee, 1962, pp. 25)

This class of dynamic systems may be defined by the following set of equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t); \qquad t \geqq 0, t \neq k + \tau_i \\ A(t+1) &= A(t) \\ A(t) &= A_i(t), \qquad k + \tau_{i-1}^{+} \leqq t \leqq k + \tau_i^{-} \\ i &= 1, 2, \cdots m, \qquad k = 0, 1, 2, \cdots \end{aligned} \tag{15}$$

where

$$T_i = \tau_i^- - \tau_{i-1}^+, \qquad \sum_{i=1}^{m} T_i = 1, \qquad \tau_0 = 0 \text{ and } \tau_m = 1;$$

and the boundary conditions

$$\mathbf{x}(\tau_i^+ + k) = S_i \mathbf{x}(\tau_i^- + k) \tag{16}$$

where S_i are constant, nonsingular $n \times n$ matrices. If the solution $\mathbf{x}(t)$ is continuous despite the discontinuities of $A(t)$, then $S_i = I$, $i = 1, 2, \cdots m$.

It should be clearly understood that $\mathbf{x}(t)$ is *not* necessarily discontinuous (Ince, 1956) and it is always possible, at least in principle, to choose the state variables so as to be continuous everywhere in a given interval. For instance, if we choose the stored charges and flux linkages as the state variables in electrical systems, then the basic conservation laws require that they be continuous with respect to time in a given interval. Normally the boundary conditions are given or derived from the constraint and conservation laws applicable to the physical systems under investigation.

(a) The solution of (15) plus (16) may be written as

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t, k^+)\mathbf{x}(k^+), \qquad t \neq k + \tau_i, t > k \\ k &= 0, 1, 2, \cdots; \qquad i = 1, 2, \cdots m. \end{aligned} \tag{17}$$

Because of the periodicity of $A(t)$, we obtain the matrix difference equation

$$\Phi(t + 1, k^+) = \Phi(t, k^+)C \tag{18}$$

and the vector difference equation

$$\mathbf{x}(k + 1^+) = C\mathbf{x}(k^+). \tag{19}$$

(b) The effect of discontinuity on the transformation characteristics may be "visualized" again by means of a graphical sketch in Fig. 3. One can write down the matrix C from Fig. 3 by inspection.

$$C = S_m\Phi(\tau_m^-, \tau_{m-1}^+)S_{m-1}\Phi(\tau_{m-1}^-, \tau_{m-2}^+) \cdots S_1\Phi(\tau_1^-, 0^+) \tag{20}$$

(c) In order to compute C, we need to solve the set of matrix differential equations associated with (15) and (16).

$$\begin{aligned} \dot{\Phi}(t) &= A(t)\Phi(t), \qquad \Phi(0^+) = I, \; 0^+ \leqq t \leqq 1^+ \\ A(t) &= A_i(t) \qquad \tau_i < t < \tau_{i-1}; \; i = 1, 2, \cdots m \end{aligned} \tag{21}$$

and the boundary conditions may be obtained from (16) or Fig. 3.

$$\Phi(\tau_i^+) = S_i\Phi(\tau_i^-) \qquad i = 1, 2, \cdots m \tag{22}$$

It is clear from Eq. (18), $C = \Phi(1^+, 0^+) = \Phi(1^+)$.

(d) The analytical form of the fundamental matrix $\Phi(t)$ is the same as shown by (11) in the classical Floquet theory, but the periodic matrix $P(t)$ is no longer continuous in this case.

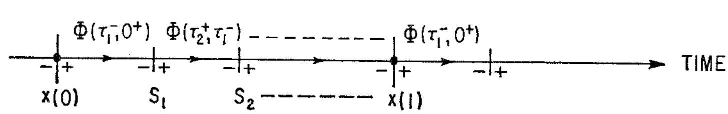

FIG. 3. A piecewise continuous periodic transformation of the state

$$\begin{aligned} \Phi(t) &= P(t) \exp Bt & t > 0 \\ P(t+1) &= P(t), & P(0^+) = I \end{aligned} \tag{23}$$

and

$$\begin{gathered} P(\tau_i^+ + k) = S_i P(\tau_i^- + k) \\ k = 0, 1, 2, \cdots; \qquad i = 1, 2, \cdots m. \end{gathered} \tag{24}$$

The discontinuities in $\Phi(t)$ are transferred to the periodic matrix $P(t)$ since $\exp Bt$ is a continuous matrix.

Fourier expansion of $P(t)$ may be expressed in a form, $P(t) = \sum_{n=-\infty}^{\infty} Q_n e^{jnt}$ where Q_n is an $n \times n$ matrix with complex elements. Consequently the Fourier spectrum of $\Phi(t)$ contains in general all the harmonics of the fundamental frequency of the periodic parameters. In some cases, however, $P(t)$ is expressed by a finite sum instead of an infinite sum.

(e) One can derive the same stability criterion for the linear system with piecewise continuous periodic coefficients as for the classical case. We consider Eq. (15) with a finite forcing term,

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t), \qquad 0 \leqq t < \infty \tag{25}$$

where

$$\|\mathbf{b}(t)\| \leqq b_0 \text{ for } t \geqq 0$$

with the same boundary conditions as given by Eq. (16). Now we state the following theorem as an extension of the classical Floquet theory.

THEOREM. *The solution* $\mathbf{x}(t)$ *of Eq.* (25) *is bounded for all* $t \geqq 0$ *if, and only if, all the eigenvalues of the discrete transformation matrix* C *as given by* (20) *lie within the unit circle.*

PROOF: It follows from (18) and (23) that $C = \exp B$. Consequently the eigenvalues of B all have the negative real parts, by virtue of Eq. (13). The solution of (25) is

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\, \mathbf{x}(0) + \int_0^t \Phi(t)\, \Phi^{-1}(\tau)\, \mathbf{b}(\tau)\, d\tau \\ &= \Big[P(t) \exp Bt\Big]\, \mathbf{x}(0) + \int_0^t P(t) \Big[\exp B(t-\tau)\Big] P^{-1}(\tau)\, \mathbf{b}(\tau)\, d\tau \end{aligned} \tag{26}$$

and it is clear that $\mathbf{x}(t)$ is bounded if all the eigenvalues of B have negative real parts, since $P(t)$ and $P^{-1}(t)$ are periodic and bounded.

(f) Finally we consider a linear system with piecewise *constant* periodic coefficients as a special example. We study a generalized Hill-Meissner equation (Meissner, 1918; Pipes, 1953) which is defined by the following:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t), & 0 < t < \infty,\ t \neq k,\ t \neq k + \tfrac{1}{2} \\ A(t) &= A_1 & k < t < k + \tfrac{1}{2} \\ A(t) &= A_2 & k + \tfrac{1}{2} < t < k + 1 \\ & & k = 0, 1, 2, \cdots \end{aligned} \tag{27}$$

$$\begin{aligned} \mathbf{x}(k + \tfrac{1}{2}^{+}) &= S_1\mathbf{x}(k + \tfrac{1}{2}^{-}) \\ \mathbf{x}(k^{+}) &= S_2\mathbf{x}(k^{-}). \end{aligned} \tag{28}$$

The solution $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \Phi(t, k^{+})\mathbf{x}(k^{+}) \qquad t \geqq k^{+} \tag{29}$$

and the difference equation is

$$\begin{aligned} \mathbf{x}(k + 1^{+}) &= \Phi(k + 1^{+}, k^{+})\mathbf{x}(k^{+}) = \Phi(1^{+})\mathbf{x}(k^{+}) \\ &= C\mathbf{x}(k^{+}) \end{aligned} \tag{30}$$

where the matrix C is obtained from (20), (27), and (28).

$$C = S_2 \exp A_2(\tfrac{1}{2})S_1 \exp A_1(\tfrac{1}{2}) \tag{31}$$

For a different sequence of sample points, we obtain

$$\mathbf{x}(k + 1 + \tfrac{1}{4}) = C(\tfrac{1}{4})\mathbf{x}(k + \tfrac{1}{4}) \quad k = 0, 1, 2, \cdots \tag{32}$$

and

$$C(\tfrac{1}{4}) = \exp A_1(\tfrac{1}{4})C \exp A_1(-\tfrac{1}{4}). \tag{33}$$

These results are recognized as special cases of Eqs. (9) and (10) for the value of $\tau = \frac{1}{4}$.

The following numerical example shows an interesting case. Let the matrices A_1, A_2, S_1, and S_2 be given as

$$A_1 = \begin{bmatrix} -2 & 10 \\ 0 & -4 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & 0 \\ 10 & -4 \end{bmatrix}$$

$$S_1 = S_2 = I$$

The system is switching periodically between the two sets of parameters represented by A_1 and A_2. An analogue simulation of this example

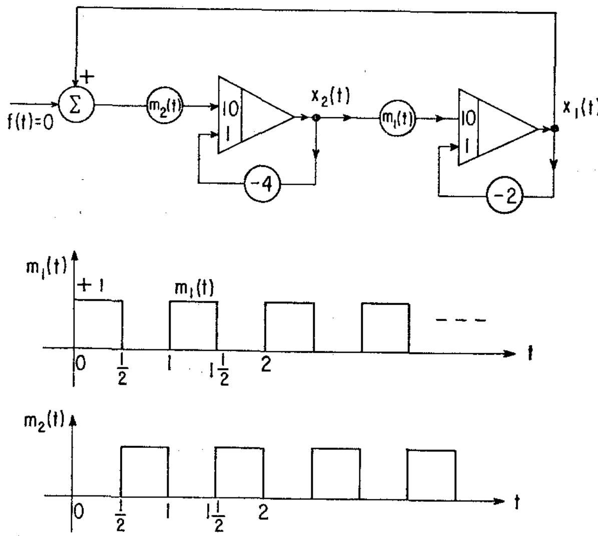

FIG. 4. Analogue simulation diagram

may take the form shown in Fig. 4. A_1 and A_2 have the same eigenvalues, namely, $\lambda_1 = -2$ and $\lambda_2 - =4$. Consequently, each dynamic system represented by $\dot{\mathbf{x}}(t) = A_1\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t) = A_2\mathbf{x}(t)$ is stable. But the resultant, periodically time-varying system is shown to be unstable.

$$C = \exp A_2(\tfrac{1}{2}) \exp A_1(\tfrac{1}{2}) = \begin{bmatrix} 0.368 & 0 \\ 1.16 & 0.136 \end{bmatrix} \begin{bmatrix} 0.368 & 1.16 \\ 0 & 0.136 \end{bmatrix}$$

$$= \begin{bmatrix} 1.36 & 0.427 \\ 0.427 & 1.364 \end{bmatrix}.$$

The characteristic roots of this matrix are $z_1 = 1.496$ and $z_2 = 0.004$. Since $|z_1| > 1$, the system is unstable, i.e., the solution becomes unbounded as time approaches infinity.

V. CONCLUSIONS

It has been shown that the extended Floquet theory provides a unified analytical method for study of linear dynamic systems with periodically time-varying parameters. For the special subclass of linear systems with piecewise constant periodic coefficients, it is not necessary to formally apply the full-powers of the extended Floquet theory to obtain the equations (30) and (31). However, it is more satisfying to treat the linear systems with continuous and piecewise continuous periodic parameters from a consistent, single point of view.

The application of this theory to stabilization of a certain class of linear feedback systems and adjustment of the design parameters will be published elsewhere.

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