

Sergio Bittanti
Patrizio Colaneri



Periodic Systems

Filtering and Control



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Periodic Systems

Filtering and Control

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*to Laura, Marta, Francesco
and to the memory of our mothers*

Authors' Note

Periodic phenomena are encountered in many different fields, from natural sciences to engineering, from physics to life sciences, not to mention economics and finance. Their study through mathematical modeling can be traced back to the 19th century, when differential equations with periodically varying coefficients began to be considered in physics and mathematics. However, the results upon which this book is mainly based were developed in the second part of the subsequent century, with the rising of modern control science.

Both authors studied at the Politecnico di Milano, and then joined the systems and control research team of that University (S.B. in the early 1970s, P.C. in the early 1980s). Among the various research lines there was periodic control, already cultivated in the sixties under the lead of Guido Guardabassi. The authors are grateful to him, as supervisor of both their Master of Science theses, and as inspiring guide in their early research steps.

Various events fostered the subsequent progress of the authors' interest in this field. The summer course on Periodic Optimization of 1972 is a milestone. Organized by CISM (Centre International des Sciences Mécaniques) in Udine (Italy), it had the merit of gathering distinguished researchers to discuss a number of topics.

After many years of joint research, the authors started writing this book in 1995. The writing was necessarily discontinuous, on account, in particular, of the need to investigate a number of open problems and to fill the blanks in the complex methodological mosaic reported in this volume. Many additional activities and human affairs delayed the completion of the book, in particular the organization of the IFAC Workshop on Periodic Control Systems (PSYCO) in Cernobbio-Como, Italy, 2001. This workshop was the first of a successful series continued in Yokohama, 2004, and St. Petersburg, 2007. Thirteen years were necessary to write the various chapters and crown the work with the finishing touch. The book is an expression of a life-long friendship and scientific collaboration.

During these intense years of research, the authors had the privilege of meeting and collaborating with many colleagues, in Italy and all around the world: warmest

thanks to all. A particular tribute goes to the memory of Giorgio Fronza (1945–1983), Diego Bricio Hernandez (1945–1993) and Claudio Maffezzoni (1946–2005).

Milan, September 2008

Sergio Bittanti
Patrizio Colaneri

Preface

The purpose of this volume is to provide a comprehensive view of the theory of periodic systems, by presenting the fundamental analytical tools for their analysis and the state of the art in the associated problems of filtering and control. This book is intended to be useful as a reference for all scientists, students, and professionals who have to investigate periodic phenomena or to study periodic control systems. For teaching purposes, the volume can be used as a textbook for a graduate course, especially in engineering, physics, economics and mathematical sciences. The treatment is self-contained; only a basic knowledge of input–output and state–space representations of dynamical systems is assumed.

Periodic Systems

Ordinary differential equations with periodic coefficients have a long history in physics and mathematics going back to the contributions of the 19th century by Faraday [134], Mathieu [230], Floquet [145], Rayleigh [250] and [251], Hill [184], and many others. As an intermediate class of systems bridging the time-invariant realm to the time-varying one, periodic systems are often included as a regular chapter in textbooks of differential equations or dynamical systems, such as [123, 175, 237, 256, 303]. In the second half of the 20th century, the development of systems and control theory has set the stage for a renewed interest in the study of periodic systems, both in continuous and in discrete-time, see e.g., the books [136, 155, 228, 252, 312] and the survey papers [29, 42]. This has been emphasized by specific application demands, in particular in industrial process control, see [1, 43, 76, 266, 267, 299], communication systems, [119, 144, 282], natural sciences, [225] and economics, [148, 161].

Periodic Control

The fact that a periodic operation may be advantageous is well-known to mankind since time immemorial. All farmers know that it is not advisable to always grow the same crop over the same field since the yield can be improved by adopting a crops rotation criterion. So, cycling is good.

In more recent times, similar concepts have been applied to industrial problems. Traditionally, almost every continuous industrial process was set and kept, in presence of disturbances, at a suitable steady state. It was the task of the designer to choose the optimal stationary regime. If the stationary requirement can be relaxed, the question arises whether a periodic action can lead to a better performance than the optimal stationary one. This observation germinated in the field of chemical engineering where it was seen that the performance of a number of catalytic reactors improved by cycling, see the pioneer contributions [17, 153, 189]. Suitable frequency domain tests have been developed to this purpose in the early 1970s, [25, 62, 162, 163, 172, 173, 275]. Unfortunately, as pointed out in [15], periodic control was still considered “too advanced” in the scenario of industrial control, in that “the steady-state operation is the norm and unsteady process behavior is taboo”. Its interest was therefore confined to advanced applications, such as those treated in [274] and [257]. However, in our days, the new possibilities offered by the control technology, together with the theoretical developments of the field, opened the way for a wide use of periodic operations. For example, periodic control is useful in a variety of problems concerning under-actuated systems, namely systems with a limited number of control inputs with respect to the degrees of freedom. In this field, control is often performed by imposing a stable limit cycle, namely an isolated and attractive periodic orbit, [97, 150, 179]. Another example comes from non-holonomic mechanical systems, where in some cases stabilization cannot be achieved by means of a time-invariant differentiable feedback control law, but it is achievable with a periodic control law, [14, 86].

In contemporary literature, the term periodic control takes a wider significance, and includes problems where either the controller or the system under control is a proper periodic system.

Periodic Models in Time-Series and Signal-Processing

Periodic models are useful in signal-processing and time-series analysis as well. In digital signal processing, periodicity arises whenever filter banks are used to impose different sampling rates for data compression, see [282] for a reference book. In time-series analysis, there are important phenomena that exhibit a periodic-like behavior. A typical example is constituted by the Wolf series of the sun spot numbers, a series which presents a sort of periodicity of 11.4 years. A distinctive feature of some seasonal series is that its lack of stationarity is not limited to the non-stationarity of the mean value: higher order statistics do not match those of a sta-

tionary process. In such cases, a model with periodic coefficients is more accurate than time-invariant modeling, especially for prediction purposes. The theory of such models is the subject of many publications, both theoretical and applied. The corresponding stochastic processes are named *cyclostationary processes* or *periodic correlated signals*, [79, 146, 155, 160, 190–192, 199, 222–225].

Motivating Applications

Among all possible fields where periodic control has a major impact, we focus here on a couple of motivating applications.

In aerospace, a considerable effort has been devoted to the development of active control systems for the attenuation of vibrations in helicopters. Besides improving the comfort of the crew and passengers, such attenuations would be profitable to reduce fatigue in the rotor structure of the aircraft and to protect on-board equipment from damage. Various approaches to this problem have been proposed in the literature, [280]. Among them, the focus is on the Individual Blade Control (IBC) strategies, [176, 196, 280] and references quoted therein. In an IBC framework, the control input is the pitch angle of each blade of the main rotor, while the output is the acceleration of the vibratory load transmitted from the blades to the rotor hub. This IBC vibration control problem arises in various flight conditions. While in hovering (motionless flight over a reference point), the dynamics of the rotor blade can be described by a time-invariant model; in forward flight the dynamics turns out to be time-periodic, with period $2\pi/\Omega$, where Ω is the rotor revolution frequency, see [198]. More precisely, the matrices of the model are periodic and take different values depending on the forward velocity of the aircraft. As for the vibration effect, if only the vertical component of the vibratory load is considered, it can be conveniently modeled by a periodic additive disturbance with frequency $N\Omega$, where N is the number of blades. The reader interested in more details is referred to [11, 71, 73, 75].

Another aerospace application can be found in satellite attitude control. The recent interest in small earth artificial satellites has spurred research activity in the attitude stabilization and control. The typical actuator is the magneto-torque, a device based on the interaction between the geomagnetic field and the magnetic field generated on-board by means of a set of three magnetic coils, typically aligned with the spacecraft principal axis. The current circulating in the coils is produced by means of solar paddles. A time history of the geomagnetic field encountered by the satellite along a (quasi) polar orbit shows a very strong periodic behavior, [302]. Consequently, the effect of the interaction between the geomagnetic field and the on-board magnetic field is periodically modulated with a period equal to the period of rotation of the satellite around the earth. It is therefore not surprising that the attitude model obtained by linearization of the satellite dynamics around the orbit is essentially periodic, [12, 135].

Turning to the field of economics and finance, a longstanding tradition in data analysis is to adopt additive models where trend, seasonality, and irregularity are treated as independent components none of which can be separately observed, [84, 177, 188, 232, 283]. In economic data, the seasonal term is determined by the natural rhythm of the series, typically 12 months or four quarters. The decomposition can be performed with various techniques. The US Bureau of Census seasonal adjustment toolkit known as $X - 11$ is one of the most used, [217]. The basic rationale behind “seasonal adjustment”, is to pre-filter data so as to obtain a new series that can be described by an ARMA (AutoRegressive Moving Average) model. Solid identification and forecasting methods are then available. Reality, however, is often complex, and periodicity may be hidden in a more subtle form. This is the case of the STANDARD & POOR’s 500 (S&P 500) stock index, or the UK non-durable consumer’s expenditures, [147, 148, 161, 240]. The higher-order periodicity is better captured by Periodic ARMA or Periodic GARCH (Generalized AutoRegressive Conditional Heteroscedastic) models, so obtaining a significant improvement in the accuracy of forecasts, [241].

Organization of the Book

As already said, this book aims to provide a theoretical corpus of results relative to the analysis, modeling, filtering, and control of periodic systems. The primary objective is to bring together in a coordinated view the innumerable variety of contributions appeared over many decades, by focusing on the theoretical aspects.

We concentrate on discrete-time signals and systems. However, in Chap. 1 we provide an overview of the entire field, including the continuous-time case. In particular, we will present the celebrated Π -test aiming at establishing whether a periodic operation of a time-invariant plant may lead to better performances than the optimal steady-state operation, see [25, 62, 162, 163, 172, 274–276], in continuous-time and [64, 65] in discrete-time.

The book proceeds with Chap. 2, devoted to the basic models and tools for the analysis. Among the key ingredients of this chapter, we mention the periodic transfer operator, the PARMA representation and the periodic state–space representation.

In Chap. 3, the discrete-time version of the Floquet theory is presented and the characterization of stability is provided also in terms of Lyapunov difference inequalities. Robust stability criteria are provided as periodic linear matrix inequalities. Finally, the companion forms for periodic systems are introduced.

Chapter 4 is devoted to structural properties. Here we provide a thorough analysis of reachability, controllability, observability, reconstructability, stabilizability, and detectability. From this study, the canonical decomposition of a periodic system follows. The structural properties allow the derivation of the so-called inertia results for the Lyapunov difference equations.

The study of periodic systems as input–output periodic operators is the subject of Chap. 5. Here, a single-input single-output (SISO) periodic system is seen as the (left or right) ratio of periodic polynomials. Instrumental to this study is the algebra of such polynomials. The chapter ends with the definition and characterization of input–output stability. A main tool for the analysis of periodic systems in discrete-time is provided by the reformulation technique. This amounts to the rewriting of a periodic system as a time-invariant one via a suitable transformation of variables. This is indeed a powerful technique, often encountered in the literature. However, in any design steps one has to explicitly take into account the constraints which a time-invariant system must meet in order to be transformed back into a periodic one.

These considerations are at the basis of Chap. 6, where three time-invariant reformulations are treated, namely, the time-lifted, the cyclic and the frequency-lifted reformulations.

The state–space realization problem, in its various facets, is studied in Chap. 7. After the definition of minimality for periodic systems, we first consider the determination of a state–space model starting from the time-lifted transfer function. Then, the realization of SISO systems from polynomial fractions is analyzed. The chapter ends with the study of balanced realization.

The notions of poles and zeros are dealt with in Chap. 8. These notions are seen from different angles, depending on the adopted type of model. Related to this study is the analysis of the delay structure, leading to the notion of spectral interactor matrix.

Chapter 9 is devoted to the norms of periodic systems. In particular, the notions of L_2 and L_∞ norms are widely discussed, and their characterizations are given both in the time and frequency domain. The definition of entropy is also introduced. Finally the role of the periodic Lyapunov difference equation in the computation of the l_∞ - l_2 gain is pointed out.

Chapter 10 deals with the problem of factorization of a periodic system. The spectral factorization and the J -spectral factorization are characterized in terms of the periodic solutions of suitable periodic Riccati equations. Finally, the parametrization of all stabilizing controllers is treated both in the field of rational periodic operators and in the field of polynomial periodic operators.

Cyclostationary processes are concisely studied in Chap. 11, with special references to their spectral properties.

Chapter 12 includes the typical estimation problems. First, filtering, prediction and fixed-lag smoothing in L_2 are considered via state–space and factorization tools. In particular, the celebrated periodic Kalman filtering theory is recovered. Next, the same problems are tackled in the L_∞ framework.

In the last chapter, namely Chap. 13, we treat a large number of stabilization and control problems. We begin with the characterization of the class of periodic state-feedback controllers in terms of periodic linear matrix inequalities. The extension to uncertain systems is also dealt with. Next, the classical problem of pole assignment is considered. Both sampled and memoryless state-feedback control laws are

considered. The chapter proceeds with the description of the exact model matching problem and the H_2 and H_∞ state-feedback control problems. The static output-feedback stabilization problem is dealt with next. The last section regards the dynamic output-feedback problem, including robust stabilization via a polynomial description, the LQG and the H_∞ problems.

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Chapter 1

Introduction

The aim of this chapter is to overview the main methodological themes arising in this area. Initially, both continuous-time and discrete-time systems will be considered. Then, from Sect. 1.4 on, we will consider only the continuous-time case since discrete-time systems will be treated in the remaining chapters of the book. Although the main focus is on linear systems, some important topics on non-linear periodic control are also touched upon. The chapter has an introductory nature, so that many technical details are omitted and the interested reader is referred to the quoted literature for a deeper insight. Starting from the subsequent chapter, we will move to an axiomatic approach for the presentation of the theory of linear discrete-time periodic systems.

1.1 Periodic Models

We will consider dynamical systems with input $u(t)$ and output $y(t)$. The input is a vector of m signals $u_1(t), u_2(t), \dots, u_m(t)$, and the output is a vector of p signals $y_1(t), y_2(t), \dots, y_p(t)$. A mathematical representation of a linear periodic system is based on a direct time-domain relationship between the input $u(t)$ and the output $y(t)$. In continuous-time, $t \in \mathbb{R}$, this leads to a model of the form:

$$\begin{aligned} \frac{d^r}{dt^r} y(t) = & F_1(t) \frac{d^{r-1}}{dt^{r-1}} y(t) + F_2(t) \frac{d^{r-2}}{dt^{r-2}} y(t) + \dots + F_r(t) y(t) + \\ & + G_1(t) \frac{d^{r-1}}{dt^{r-1}} u(t) + G_2(t) \frac{d^{r-2}}{dt^{r-2}} u(t) + \dots + G_s(t) \frac{d^{r-s}}{dt^{r-s}} u(t), \end{aligned} \quad (1.1)$$

whereas in discrete-time, $t \in \mathbb{Z}$,

$$\begin{aligned} y(t+r) = & F_1(t) y(t+r-1) + F_2(t) y(t+r-2) + \dots + F_r(t) y(t) + \\ & + G_1(t) u(t+r-1) + G_2(t) u(t+r-2) + \dots + G_s(t) u(t+r-s), \end{aligned} \quad (1.2)$$

where $F_i(\cdot)$ and $G_j(\cdot)$ are periodic real matrices with obvious dimensions and such that, for each t :

$$F_i(t) = F_i(t + T), \quad G_i(t) = G_i(t + T).$$

The smallest T satisfying these equalities is the system period.

Generally speaking, the input–output models (1.1), (1.2) are useful not only to describe cause-effect systems, where the input is a control variable, but also to characterize stochastic signals with hidden periodicity. In the latter case, the input $u(\cdot)$ is a remote signal often seen as a stochastic process. Then, the models are known as “PARMA models”, the acronym PARMA meaning “Periodic Auto-Regressive Moving Average”.

In the input–output framework, another important characterization is provided by the impulse response concept. The impulse response matrix, say $h(t, \tau)$, is a $p \times m$ matrix, function of two time-indexes t and τ , relating the input and output variables in the following way:

$$\begin{cases} y(t) = \int_0^t h(t, \tau) u(\tau) d\tau & \text{in continuous-time} \\ y(t) = \sum_{\tau=0}^t h(t, \tau) u(\tau) & \text{in discrete-time.} \end{cases} \quad (1.3)$$

The kernel $h(t, \tau)$ has the property of being bi-periodic, namely $h(t + T, \tau + T) = h(t, \tau)$. The distinctive feature of the kernel is that its i -th column provides the forced response¹ of the periodic system at time t to an impulse given at the i th input variable at time $\tau \geq 0$, namely

$$u(t) = e_i \delta(t - \tau),$$

where e_i denotes the i -th column of the m -dimensional identity matrix and $\delta(\cdot)$ is the impulse signal given by the Dirac function in continuous-time and the Kronecker function in discrete-time².

Alternatively, a widely used mathematical description of dynamical systems is the state–space model, i.e., the differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1.4a)$$

$$y(t) = C(t)x(t) + D(t)u(t) \quad (1.4b)$$

¹ The forced response is the output of the system associated with zero initial condition

² In discrete-time, the Kronecker function is the function

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0. \end{cases}$$

In continuous-time, the Dirac function is indeed a distribution, which can be roughly introduced as the function such that

$$\int_{t_1}^{t_2} g(\tau) \delta(\tau - t_0) d\tau = g(t_0), \quad t_0 \in [t_1, t_2].$$

in continuous-time, or the set of difference equations

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (1.5a)$$

$$y(t) = C(t)x(t) + D(t)u(t) \quad (1.5b)$$

in discrete-time. Here, besides $u(t)$ and $y(t)$, another vector of variables appears; the relationship between the input and the output takes place through such an intermediate variable, the so-called *state vector* $x(t)$. Correspondingly, (1.4) or (1.5) are known as state-space models. (1.4a) and (1.5a) are named state equations whereas (1.4b) and (1.5b) are the so-called output equations. The number n of elements composing vector $x(t)$ is the *system dimension*. Finally, matrix $A(\cdot)$ is named *dynamic matrix*.

The system periodicity manifests itself in the periodicity of the real matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$:

$$A(t+T) = A(t), \quad B(t+T) = B(t), \quad C(t+T) = C(t), \quad D(t+T) = D(t).$$

Again, the smallest T for which these periodicity conditions are met is called the system period. Normally, the dimension n is constant, so that the system matrices have constant dimension too. However, for some specific problems, it is necessary to allow for periodic time variability in this integer as well. Then, matrices $A(t)$, $B(t)$ and $C(t)$ will exhibit time-varying dimensionality.

In the whole class of periodic systems, one can recognize important subclasses. Among them, we mention the class of *sinusoidal systems* where all matrices are sinusoidally varying in time with the same frequency. By the way, most examples or counterexamples in the theory of periodic systems, and frequently in the theory of time-varying systems with any type of variability, resort to sinusoidal systems.

In general, given a periodic system, one can perform a Fourier expansion of the periodic matrices and then obtain an approximate model by Fourier truncation. In this regard, it is apparent that the simplest truncation is at the level of average values of the periodic matrices. In this way one obtains a time-invariant system, the *mean value model*. A more refined description is obtained by performing the truncation at the first harmonic. In such a way, an approximate *sinusoidal model* is obtained.

The above models are basic descriptions for intrinsically periodic phenomena encountered in science and engineering applications, as illustrated in the Preface. Furthermore, they arise in the study of non-linear systems operating in specific conditions, as shown in the following simple examples.

1.1.1 Periodicity Induced by Linearization

Since the early days of the theory of dynamical systems, periodic motion plays a fundamental role. **To analyze the behavior of a given system around a periodic motion, a most celebrated technique relies on linearization.** For example, consider the continuous-time non-linear system with input $v(t)$, state $\xi(t)$ and output $\eta(t)$:

$$\begin{aligned}\dot{\xi}(t) &= f(\xi(t), v(t)) \\ \eta(t) &= h(\xi(t), v(t))\end{aligned}$$

and let $\tilde{v}(\cdot)$, $\tilde{\xi}(\cdot)$, $\tilde{\eta}(\cdot)$ be a periodic regime of period T . This means that $\tilde{\xi}(\cdot)$ is a periodic solution of period T associated with the periodic input $\tilde{v}(\cdot)$ and that $\tilde{\eta}(t)$ is the corresponding periodic output. Then, the linearized equations result in a linear continuous-time system with

$$u(t) = v(t) - \tilde{v}(t), \quad x(t) = \xi(t) - \tilde{\xi}(t), \quad y(t) = \eta(t) - \tilde{\eta}(t)$$

and matrices

$$\begin{aligned}A(t) &= \left. \frac{\partial f(\xi, v)}{\partial \xi} \right|_{\xi=\tilde{\xi}, v=\tilde{v}} & B(t) &= \left. \frac{\partial f(\xi, v)}{\partial v} \right|_{\xi=\tilde{\xi}, v=\tilde{v}} \\ C(t) &= \left. \frac{\partial h(\xi, v)}{\partial \xi} \right|_{\xi=\tilde{\xi}, v=\tilde{v}} & D(t) &= \left. \frac{\partial h(\xi, v)}{\partial v} \right|_{\xi=\tilde{\xi}, v=\tilde{v}}.\end{aligned}$$

These matrices are obviously periodic of period T . *Mutatis mutandis* the same reasoning applies in discrete-time as well. The linearization rationale is illustrated in Fig. 1.1.

1.1.2 Periodicity Induced by Multirate Sampling

Multirate schemes arise in digital control and digital signal processing whenever it is necessary to sample the outputs and/or update the inputs with different rates, see [140] for sampling in digital signal processing and control. There is huge literature in the area covering different fields, see [105, 211, 231, 263, 282] and [92] for representative contributions. To explain the multirate mechanism in simple terms and how periodicity arises, let us focus on a system with two inputs and two outputs described by the time-invariant differential equations

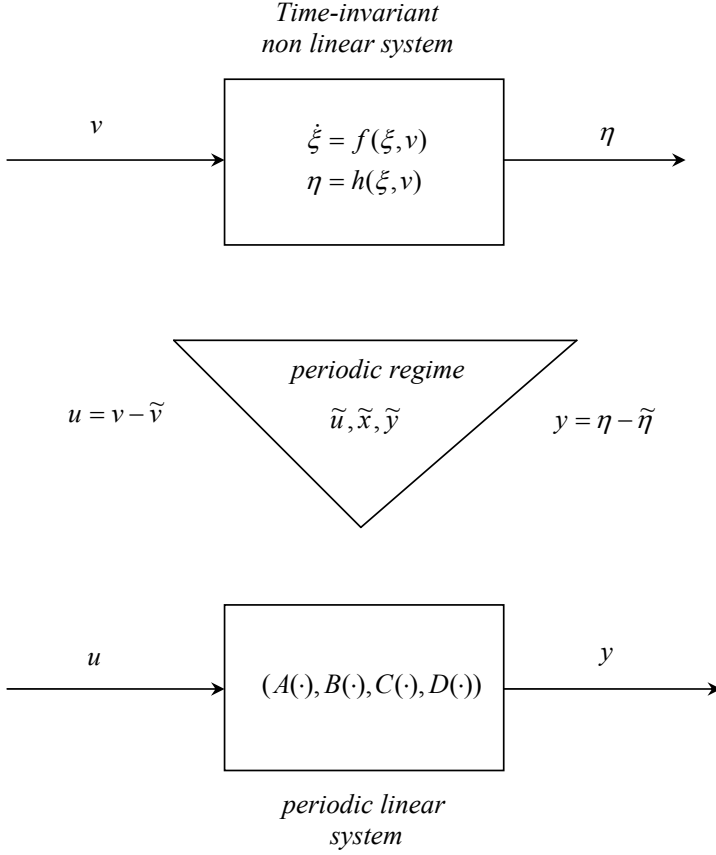


Fig. 1.1 Linearization around a periodic regime

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= Cx(t) + D \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.\end{aligned}$$

The two outputs $y_i(\cdot)$, $i = 1, 2$, are sampled with sampling intervals τ_{y_i} , $i = 1, 2$, whereas the two inputs $u_j(\cdot)$, $j = 1, 2$, are updated at the end of intervals of length τ_{u_j} , $j = 1, 2$, and kept constant in between. The sampling and updating instants are denoted by $t_{y_i}(k)$, $i = 1, 2$ and $t_{u_j}(k)$, $j = 1, 2$, k integer, respectively. Typically, these instants are taken as multiples of the basic clock period Δ . Moreover, for simplicity, assume that

$$t_{y_i}(k) = k\tau_{y_i}, \quad t_{u_j}(k) = k\tau_{u_j}.$$

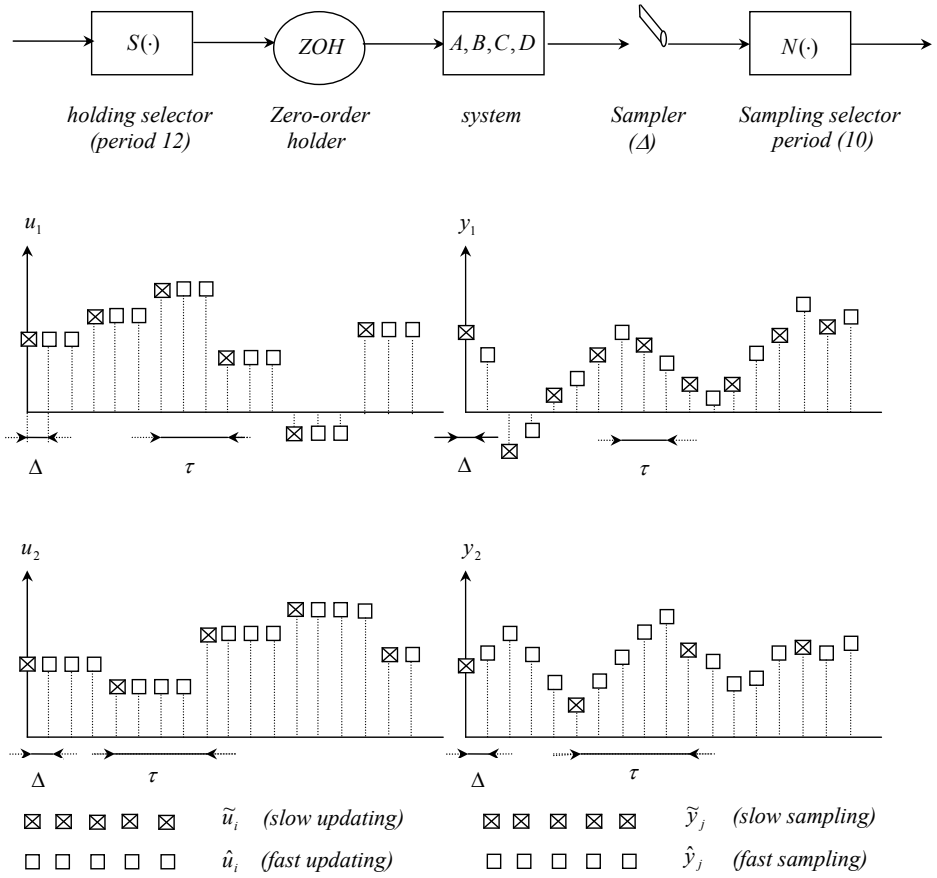


Fig. 1.2 Multirate system mechanism

The overall behavior of the obtained system is ruled by the discrete-time output variables $\tilde{y}_i(k)$, $i = 1, 2$ and the discrete-time input variables $\tilde{u}_j(k)$, $j = 1, 2$ defined as

$$\tilde{y}_i(k) = y(k\tau_{y_i}), \quad i = 1, 2$$

$$u(t) = \tilde{u}_j(k), \quad t \in [k\tau_{u_j}, k\tau_{u_j} + \tau_{u_j}), \quad j = 1, 2.$$

For the modelization, however, it is advisable to introduce also the “fast-rate” signals

$$\hat{y}(k) = y(k\Delta), \quad \hat{u}(k) = u(k\Delta).$$

In contrast, $\tilde{y}(k)$ and $\tilde{u}(k)$ can be considered as the “slow-rate” samples. The *sampling selector*, namely the device operating the passage from the fast sampled-output to the slow sampled-output, is described by the linear equation

$$\tilde{y}(k) = N(k)\hat{y}(k),$$

where

$$N(k) = \begin{bmatrix} n_1(k) & 0 \\ 0 & n_2(k) \end{bmatrix}$$

with

$$n_i(k) = \begin{cases} 1 & \text{if } k \text{ is a multiple of } \tau_{y_i}/\Delta \\ 0 & \text{otherwise.} \end{cases}$$

Notice that matrix $N(\cdot)$ is periodic with the period given by the integer T_y defined as the least common multiple of τ_{y_1}/Δ and τ_{y_2}/Δ .

As for the hold device, introduce the *holding selector* matrix

$$S(k) = \begin{bmatrix} s_1(k) & 0 \\ 0 & s_2(k) \end{bmatrix}$$

with

$$s_j(k) = \begin{cases} 0 & \text{if } k \text{ is a multiple of } \tau_{u_j}/\Delta \\ 1 & \text{otherwise.} \end{cases}$$

Then, the analog input signal $u(\cdot)$ is given by

$$u(t) = \hat{u}(k), \quad t \in [k\Delta, k\Delta + \Delta),$$

where the fast updated signal $\hat{u}(k)$ is obtained from the slow one $\tilde{u}(k)$ according to the holding selector mechanism

$$\begin{aligned} v(k+1) &= S(k)v(k) + (I - S(k))\tilde{u}(k) \\ \hat{u}(k) &= S(k)v(k) + (I - S(k))\tilde{u}(k). \end{aligned}$$

Matrix $S(k)$ is periodic of period T_u given by the least common multiple of τ_{u_1}/Δ and τ_{u_2}/Δ . This situation is schematically illustrated in Fig. 1.2, where $\tau_{u_1} = 3\Delta$, $\tau_{u_2} = 4\Delta$, $\tau_{y_1} = 2\Delta$, $\tau_{y_2} = 5\Delta$, so that $T_u = 12$ and $T_y = 10$.

The overall multirate sampled-data system is a discrete-time periodic system with state

$$\hat{x}(k) = \begin{bmatrix} x(k\Delta) \\ v(k) \end{bmatrix}$$

and equations

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(t) + B(k)\hat{u}(t) \\ \hat{y}(t) &= C(k)\hat{x}(k) + D(k)\hat{u}(k), \end{aligned}$$

where

$$\begin{aligned}\hat{A}(k) &= \begin{bmatrix} e^{A\Delta} \int_0^\tau e^{A\xi} d\xi B S(k) \\ 0 & S(k) \end{bmatrix} & \hat{B}(k) &= \begin{bmatrix} \int_0^\tau e^{A\xi} d\xi B(I - S(k)) \\ I - S(k) \end{bmatrix} \\ \hat{C}(k) &= N(k)C & \hat{D}(k) &= N(k)D.\end{aligned}$$

The resultant system is periodic with period T given by the least common multiple of T_u and T_y ($T = 60$ in the example of Fig. 1.2).

1.2 Floquet–Lyapunov Theory

The pioneer contributions relating to periodic systems deal with systems evolving autonomously, namely without any exogenous input. This amounts to considering the homogeneous equation

$$\begin{cases} \dot{x}(t) = A(t)x(t) & \text{in continuous-time} \\ x(t+1) = A(t)x(t) & \text{in discrete-time.} \end{cases}$$

Starting from an initial condition $x(\tau)$ at time τ , the solution is obtained as

$$x(t) = \Phi_A(t, \tau)x(\tau),$$

where the *transition matrix* $\Phi_A(t, \tau)$ is given by the solution of the differential matrix equation

$$\dot{\Phi}_A(t, \tau) = A(t)\Phi_A(t, \tau), \quad \Phi(\tau, \tau) = I$$

in continuous-time, and by

$$\Phi_A(t, \tau) = \begin{cases} I & t = \tau \\ A(t-1)A(t-2)\cdots A(\tau) & t > \tau \end{cases}$$

in discrete-time.

It is easily seen that the periodicity of the system entails the “bi-periodicity” of matrix $\Phi_A(t, \tau)$, namely:

$$\Phi_A(t+T, \tau+T) = \Phi_A(t, \tau).$$

The transition matrix over one period

$$\Psi_A(\tau) = \Phi_A(\tau+T, \tau)$$

deserves special attention and it is known as *monodromy matrix* at time τ .

In continuous-time, there is no analytic expression for the transition matrix. However, its determinant can be worked out from the so-called *Liouville–Jacobi* formula, i.e.,

$$\det[\Phi_A(t, \tau)] = e^{\int_{\tau}^t \text{trace}[A(\sigma)] d\sigma}. \quad (1.6)$$

Therefore, for any choice of t and τ , the transition matrix is invertible. This means that any continuous-time periodic system is *reversible*, in that the state $x(\tau)$ can be uniquely recovered from $x(t)$, $t > \tau$. This is no more true, in general, in discrete-time since the transition matrix is singular when $A(i)$ is singular for some i .

As for the time variability of the eigenvalues of the monodromy matrix, notice that, in continuous-time:

$$\Psi_A(\tau_1) = \Phi_A(\tau_1, \tau_2) \Psi_A(\tau_2) \Phi_A(\tau_2, \tau_1).$$

Since $\Phi_A(\tau_2, \tau_1) = \Phi_A(\tau_1, \tau_2)^{-1}$, this is a similarity transformation³. Hence the eigenvalues of the monodromy matrix do not depend on the particular time tag. The same statement is true in discrete-time, as will be proven in Chap. 2, although the proof is more complex due to the possible singularity of the transition matrix. The eigenvalues of the monodromy matrix are called *characteristic multipliers*.

1.2.1 Floquet–Lyapunov Transformation

One of the long-standing issues in periodic systems is whether it is possible to find a state-coordinate transformation leading to a periodic system with a *constant* dynamic matrix. In this way the eigenvalues of such a dynamic matrix would determine the modes of the system (up to the state periodic transformation). With reference to linear differential equations, this issue was considered by various mathematicians of the 19th century. Among them, a prominent role was played by Gaston Floquet (1847–1920) and Aleksandr Mikhailovich Lyapunov (1857–1918) who worked out a procedure to solve linear homogeneous periodic systems in continuous-time, now named after them [145, 227].

This theory can be outlined in a simple form as follows. If $S(\cdot)$ is a T -periodic invertible state–space transformation,

$$\hat{x}(t) = S(t)x(t)$$

then, in the new coordinates, the dynamic matrix $\hat{A}(t)$ is given by

$$\hat{A}(t) = \begin{cases} S(t)A(t)S(t)^{-1} + \dot{S}(t)S(t)^{-1} & \text{in continuous-time} \\ S(t+1)A(t)S(t)^{-1} & \text{in discrete-time.} \end{cases}$$

³ Two square matrices M and N are similar if there exists an invertible square matrix S such that $M = SNS^{-1}$. Similar matrices share the same characteristic polynomial.

In continuous-time, it is assumed that $S(\cdot)$ and $S(\cdot)^{-1}$ be continuously differentiable. The Floquet problem is then to find $S(t)$ (if any) in order to obtain a constant dynamic matrix $\hat{A}(t) = \hat{A}$.

Focus now on the continuous-time case. From the above expression of $\hat{A}(t)$ it follows that, if $\hat{A}(t)$ is constant, i.e., $\hat{A}(t) = \hat{A}$, then the transformation $S(t)$ must satisfy

$$\dot{S}(t) = \hat{A}S(t) - S(t)A(t). \quad (1.7)$$

This is a matrix differential equation. Considering τ as initial time point and $S(\tau)$ as initial condition, the solution is given by

$$S(t) = e^{\hat{A}(t-\tau)}S(\tau)\Phi_A(\tau, t). \quad (1.8)$$

Indeed, recalling that

$$\frac{de^{\hat{A}(t-\tau)}}{dt} = \hat{A}e^{\hat{A}(t-\tau)}$$

and that for any differentiable non-singular matrix $W(t)$ with differentiable inverse, the time derivative of $W(t)^{-1}$ is

$$\frac{dW(t)^{-1}}{dt} = -W(t)^{-1} \frac{dW(t)}{dt} W(t)^{-1},$$

we have

$$\begin{aligned} \dot{S}(t) &= \frac{de^{\hat{A}(t-\tau)}}{dt} S(\tau)\Phi_A(\tau, t) + e^{\hat{A}(t-\tau)} S(\tau) \frac{d(\Phi_A(t, \tau))^{-1}}{dt} \\ &= \hat{A}e^{\hat{A}(t-\tau)} S(\tau)\Phi_A(\tau, t) - e^{\hat{A}(t-\tau)} S(\tau)\Phi_A(t, \tau)^{-1} \frac{d\Phi_A(t, \tau)}{dt} \Phi_A(t, \tau)^{-1} \\ &= \hat{A}e^{\hat{A}(t-\tau)} S(\tau)\Phi_A(\tau, t) - e^{\hat{A}(t-\tau)} S(\tau)\Phi_A(\tau, t)A(t)\Phi_A(t, \tau)\Phi_A(t, \tau)^{-1} \\ &= \hat{A}S(t) - S(t)A(t). \end{aligned}$$

Take now $t = \tau + T$ and impose the periodicity condition $S(\tau + T) = S(\tau)$ to the solution $S(\cdot)$ given by (1.7):

$$S(\tau) = e^{\hat{A}T} S(\tau)\Phi_A(\tau, \tau + T).$$

Thus, the Floquet problem amounts to finding a pair of matrices \hat{A} and \bar{S} solving the algebraic equation

$$\bar{S} = e^{\hat{A}T} \bar{S} \Psi_A(\tau)^{-1}.$$

This system of algebraic equations admits infinitely many solutions (\bar{S}, \hat{A}) . If one takes $S = I$, then \hat{A} can be obtained from the condition

$$e^{\hat{A}T} = \Psi_A(\tau).$$

Correspondingly, the transformation $S(\cdot)$ is given by

$$S(t) = e^{\hat{A}(t-\tau)} \Phi_A(\tau, t)$$

which takes the identity value at $t = \tau + kT$, where k is any integer, so that $\hat{x}(\tau + kT) = x(\tau)$.

However, one can consider any other invertible matrix \bar{S} and compute \hat{A} according to the condition

$$e^{\hat{A}T} = \bar{S} \Psi_A(\tau) \bar{S}^{-1}. \quad (1.9)$$

Then

$$S(t) = e^{\hat{A}(t-\tau)} \bar{S} \Phi_A(\tau, t)$$

and $\hat{x}(\tau + kT) = \bar{S}x(\tau)$.

Obviously, any \hat{A} satisfying (1.9) depends not only upon the chosen matrix \bar{S} but also on the considered time instant τ .

The discrete-time case is rather involved and has been fully clarified only in recent times. We will discuss in more detail the Floquet problem in discrete-time in Chap. 3. For the moment we limit ourselves to the observation that there exist certain discrete-time systems which do not admit a Floquet representation. This is easily seen for the non-reversible system with $T = 2$, $A(0) = 0$, $A(1) = 1$. Indeed, the equation $S(t+1)A(t)S(t)^{-1} = \text{constant}$ does not admit any solution in such a case. Focus then on the discrete-time reversible case (i.e., $A(t)$ non-singular for each t). An easy computation shows that

$$S(t) = \hat{A}^{t-\tau} S(\tau) \Phi_A(\tau, t) \quad (1.10)$$

so that the Floquet problem becomes that of finding a pair (\bar{S}, \hat{A}) such that

$$\bar{S} = \hat{A}^T \bar{S} \Psi_A(\tau)^{-1}.$$

For $\bar{S} = I$, this leads to

$$\hat{A}^T = \Psi_A(\tau).$$

If instead one consider any non-singular \bar{S} , the condition is

$$\hat{A}^T = \bar{S} \Psi_A(\tau) \bar{S}^{-1}. \quad (1.11)$$

We now elaborate upon Expressions (1.8), (1.10), from which we obtain

$$\Phi_A(t, \tau) = \begin{cases} S(t)^{-1} e^{\hat{A}(t-\tau)} S(\tau) & \text{in continuous-time} \\ S(t)^{-1} \hat{A}^{t-\tau} S(\tau) & \text{in discrete-time,} \end{cases} \quad (1.12)$$

and, for the choice $S(\tau) = I$,

$$\Phi_A(t, \tau) = \begin{cases} S(t)^{-1} e^{\hat{A}(t-\tau)} & \text{in continuous-time} \\ S(t)^{-1} \hat{A}^{t-\tau} & \text{in discrete-time.} \end{cases} \quad (1.13)$$

This latter expression of the transition matrix for periodic systems is reminiscent of the corresponding one for time-invariant systems: the classical exponential function ($e^{\hat{A}(t-\tau)}$ in continuous-time, $\hat{A}^{t-\tau}$ in discrete-time) is modulated by the periodic matrix $S(t)^{-1}$.

Any solution \hat{A} of the Floquet problem, in continuous or discrete-time, is called the *Floquet factor*. Notice that the entries of \hat{A} may be complex numbers.

1.2.2 Characteristic Exponents

We have seen that, given a periodic system, the corresponding transition matrix can be given an exponential-like expression by means of a Floquet factor \hat{A} . The eigenvalues of \hat{A} are named *characteristic exponents*. From Expression (1.12) it is apparent that

$$\Psi_A(\tau) = \begin{cases} S(\tau)^{-1} e^{\hat{A}T} S(\tau) & \text{in continuous-time} \\ S(\tau)^{-1} \hat{A}^T S(\tau) & \text{in discrete-time.} \end{cases}$$

For the choice $S(\tau) = I$, Expression (1.13) entails

$$\Psi_A(\tau) = \begin{cases} e^{\hat{A}T} & \text{in continuous-time} \\ \hat{A}^T & \text{in discrete-time.} \end{cases}$$

Of course, given a monodromy matrix $\Psi_A(\tau)$, this equation leads to a multiplicity of (equivalent) Floquet factors \hat{A} . According to the above relationship, if ρ is a characteristic exponents, the associated characteristic multiplier λ is given by

$$\lambda = \begin{cases} e^{\rho T} & \text{in continuous-time} \\ \rho^T & \text{in discrete-time.} \end{cases}$$

Since the characteristic multipliers depend only on matrix $A(\cdot)$, from such a relationship one can conclude that the same holds for the characteristic exponents, which are determined by matrix $A(\cdot)$ only. This is also apparent from Eqs. (1.9) and (1.11), which are just similarity transformations.

Notice that the characteristic exponents of a periodic matrix $A(\cdot)$ can be given a *geometric interpretation* as follows: the complex number ρ is a characteristic exponent if and only if there exists a T -periodic vector $\bar{x}(t)$, not identically zero, such that

$$\begin{cases} \bar{x}(t) + \rho \bar{x}(t) = A(t) \bar{x}(t) & \text{in continuous-time} \\ \rho \bar{x}(t+1) = A(t) \bar{x}(t) & \text{in discrete-time.} \end{cases} \quad (1.14)$$

Indeed, by referring to continuous-time, assume that ρ satisfies (1.14) for some non-identically zero periodic vector $\bar{x}(t)$. Take a time-point τ for which $\bar{x}(\tau) \neq 0$

and integrate (1.14) over one period, so obtaining $\bar{x}(\tau + T) = \Psi_A(\tau)e^{-\rho T}\bar{x}(\tau)$. In view of periodicity, we have

$$\Psi_A(\tau)\bar{x}(\tau) = e^{\rho T}\bar{x}(\tau).$$

so that ρ is a characteristic exponent. Vice versa, assume that ρ is a characteristic exponent and consider an eigenvector ξ of the monodromy matrix, namely $\Psi_A(\tau)\xi = e^{\rho T}\xi$. Then

$$\bar{x}(t) = \Phi_A(t, \tau)e^{-\rho(t-\tau)}\xi$$

is a non-identically zero periodic vector function satisfying (1.14). The analogous proof for discrete-time is omitted.

The above-given geometric interpretation of a characteristic exponent lends itself to be interpreted in algebraic terms in the realm of operator polynomials with periodic coefficients. Here we first make reference to continuous-time systems.

Introduce a differential operator σ which acts on a differentiable function $f(t)$ as

$$\sigma \cdot f(t) = \dot{f}(t).$$

Note that

$$\sigma \cdot (f(t)g(t)) = \dot{f}(t)g(t) + f(t)\dot{g}(t) = (\dot{f}(t) + f(t)\sigma) \cdot g(t).$$

From such an identity one can deduce a pseudo-commutative property

$$\sigma f(t) = f(t)\sigma + \dot{f}(t).$$

Equation (1.14) can then be written as

$$[(\sigma + \rho)I - A(t)] \cdot \bar{x}(t) = 0. \quad (1.15)$$

This can be seen as a set of first-order linear differential equations in the components of vector $\bar{x}(t)$. By a sequence of elementary differential operations, one can reduce the system to a unique differential equation of higher order in $\sigma + \rho$, of polynomial form, say

$$p(\sigma + \rho, t) \cdot v(t) = 0,$$

where $v(\cdot)$ is a suitable non-identically zero periodic function. Here, setting $\rho = 0$, one obtains an operator polynomial $p(\sigma, t)$ with periodic coefficients that can be seen as the *characteristic operator polynomial* of the periodic matrix $A(\cdot)$.

If $A(t)$ is scalar, the monodromy matrix is

$$\Psi_A(0) = e^{\int_0^T A(\tau) d\tau}$$

which can be written as

$$\Psi_A(0) = e^{\rho T}, \quad \rho = \frac{1}{T} \int_0^T A(\tau) d\tau.$$

The characteristic operator polynomial is obviously given by $p(\sigma, t) = \sigma - A(t)$. Therefore the characteristic exponent is the average value of the periodic function $A(\cdot)$.

In addition, if $A(t)$ is a $n \times n$ matrix in *companion form*, i.e.,

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix},$$

then the characteristic operator polynomial is given by

$$p(\sigma, t) = \sigma^n + a_1(t)\sigma^{n-1} + \cdots + a_n(t).$$

However, for a periodic matrix with a generic structure, the characteristic operator polynomial may take complex expressions. For example, consider the matrix

$$A(t) = \begin{bmatrix} \alpha(t) & 1 \\ 1 & \beta(t) \end{bmatrix},$$

where $\alpha(t)$ and $\beta(t)$ are suitable periodic functions. Then, in view of (1.15) it follows

$$(\sigma + \rho - \alpha(t)) \cdot \bar{x}_1(t) = \bar{x}_2(t), \quad (\sigma + \rho - \beta(t)) \cdot \bar{x}_2(t) = \bar{x}_1(t),$$

where $\bar{x}_i(t)$, $i = 1, 2$ are the components of $\bar{x}(t)$. Therefore

$$(\sigma + \rho - \beta(t))(\sigma + \rho - \alpha(t)) \cdot x_1(t) = x_1(t).$$

Taking into account the pseudo-commutative property $\sigma\alpha(t) = \alpha(t)\sigma + \dot{\alpha}(t)$, it follows that the *characteristic operator polynomial* associated with $A(\cdot)$ is

$$p(\sigma, t) = \sigma^2 - (\alpha(t) + \beta(t))\sigma - \dot{\alpha}(t) + \alpha(t)\beta(t) - 1.$$

This kind of reasoning can be extended to cope with systems of any order n .

A typical factorized expression of the characteristic polynomial is

$$p(\sigma, t) = (\sigma - p_1(t))(\sigma - p_2(t)) \cdots (\sigma - p_n(t)).$$

The average values of $p_i(t)$, $i = 1, 2, \dots, n$ are then the characteristic exponents of the system.

As for the discrete-time case, we only limit ourselves to a basic observation concerning the first-order system $x(t+1) = A(t)x(t)$. The monodromy matrix can be written as

$$\Psi_A(0) = \prod_{i=0}^{T-1} A(i) = \rho^T, \quad \rho = \prod_{i=0}^{T-1} A(i)^{1/T}.$$

Consequently, the logarithm of the characteristic exponent is the average of the logarithms of the coefficients over one period. The notion of characteristic polynomial in discrete-time will be extensively dealt with in Chap. 5.

1.2.3 Stability

The monodromy matrix $\Psi_A(\tau)$ relates the value of the state in free motion at a given time-point τ to the value after one period $\tau + T$:

$$x(\tau + T) = \Psi_A(\tau)x(\tau).$$

Therefore, the sampled state $x_\tau(k) = x(\tau + kT)$ is governed in the free motion by the time-invariant discrete-time equation

$$x_\tau(k+1) = \Psi_A(\tau)^k x_\tau(k).$$

This is why the eigenvalues of $\Psi_A(\tau)$ play a major role in the modal analysis of periodic systems. The monodromy matrix is the basic tool in the stability analysis of periodic systems. Indeed, the free motion goes to zero asymptotically if and only if all characteristic multipliers have modulus lower than one. Hence,

a periodic system (in continuous or discrete-time) is stable⁴ if and only if its characteristic multipliers belong to the open unit disk in the complex plane.

Obviously, the previous condition can be stated in terms of characteristic exponents. In continuous-time a periodic system is stable if and only if the real part of all characteristic exponents is negative.

Notice that there is no direct relation between the eigenvalues of $A(t)$ and the system stability. In particular, it may well happen that all eigenvalues of $A(t)$ belong to the stability region (i.e., the left half plane in continuous-time and the unit disk in discrete-time) and nevertheless the system be unstable, as pointed out in the following

Example 1.1 *Vinograd [254] Consider the matrix*

$$A(t) = \begin{bmatrix} -1 - 9\cos^2(6t) + 12\sin(6t)\cos(6t) & 12\cos^2(6t) + 9\sin(6t)\cos(6t) \\ -12\sin^2(6t) + 9\sin(6t)\cos(6t) & -1 - 9\sin^2(6t) - 12\sin(6t)\cos(6t) \end{bmatrix}.$$

⁴ For simplicity, in the above statement we used the word *stable* in place of the word *asymptotically stable*, adopted in various textbooks. The same terminology will be used in the rest of the book.

This matrix, periodic with period $T = \pi/3$, has constant eigenvalues equal to -1 and -10 . However, the monodromy matrix is given by

$$\Psi_A(0) = 0.2 \begin{bmatrix} e^{2T} + 4e^{-13T} & 2e^{2T} - 2e^{-13T} \\ 2e^{2T} - 2e^{-13T} & 4e^{2T} + e^{-13T} \end{bmatrix}$$

whose eigenvalues are e^{2T} and e^{-13T} . Since $e^{2T} > 1$, the system is unstable. ■

Notable exceptions in continuous-time are slowly varying matrices or high-frequency perturbed matrices, see [127,273] and [24], respectively. In this connection, we mention the so-called *averaging technique*, which has been used in various applications. In brief, consider a T -periodic system of the form

$$\frac{1}{\varepsilon} \dot{x}(t) = A(t)x(t),$$

where ε is a positive parameter. Associated to it, we can define the averaged matrix

$$A_{av} = \frac{1}{T} \int_0^T A(t) dt.$$

It can be shown that if this matrix has all its eigenvalues in the open left-half plane, then there exists ε^* such that the periodic system is uniformly exponentially stable for all $\varepsilon < \varepsilon^*$. This result can be given a high-frequency interpretation by setting $\tau = \varepsilon t$, $\omega = \varepsilon^{-1}$, and rewriting the system as

$$\frac{d\tilde{x}(\tau)}{d\tau} = A(\omega\tau)\tilde{x}(\tau).$$

Obviously, the previous condition $\varepsilon \rightarrow 0$ corresponds to $\omega \rightarrow \infty$. For a proof of the averaging technique (even in a more general non-linear setting), the interested reader is referred to [175], see also [186].

A general and celebrated stability condition can be formulated in terms of the so-called *Lyapunov equation*, which takes the form

$$\dot{P}(t) = P(t)A(t)' + A(t)P(t) + W(t)$$

in continuous-time and

$$P(t+1) = A(t)P(t)A(t)' + W(t)$$

in discrete-time, where $W(\cdot)$ is any periodic ($W(t+T) = W(t), \forall t$) and positive definite ($x'W(t)x > 0, \forall t, \forall x \neq 0$) matrix⁵. It turns out that

the periodic system is stable if and only if the Lyapunov equation admits a (unique) periodic positive definite solution $P(\cdot)$.

⁵ Note that the discrete-time Lyapunov equation is also known as the Stein equation.

This statement can be expressed in a more general form by referring to positive semi-definite matrices $W(\cdot)$ provided that further technical assumptions on the pair $(A(\cdot), W(\cdot))$ are met with, see [81] for more details on the theoretical aspects and [285] and [165] for the numerical issues.

The above condition, known as *Lyapunov stability condition*, can also be stated in a variety of different forms. In particular, it is worth mentioning that one can resort to a *Lyapunov inequality* in place of a Lyapunov equation, i.e.,

$$\dot{P}(t) > A(t)P(t) + P(t)A(t)'$$

in continuous-time and

$$P(t+1) > A(t)P(t)A(t)'$$

in discrete-time⁶. Then, an equivalent stability condition is that the system is stable if and only if the Lyapunov inequality admits a periodic positive definite solution, see e.g., [46, 129]. The advantage of expressing the condition in this form is that no auxiliary matrix $W(\cdot)$ is required.

A main reference for the Floquet–Lyapunov theory in continuous-time is [312], see also [123], and [175].

1.3 Lagrange Formula and External Stability

Turning to input-driven periodic systems, namely systems (1.5) and (1.4), the solution of the state equation with a generic initial state $x(\tau)$ and input function $u(\cdot)$ is

$$x(t) = \Phi_A(t, \tau)x(\tau) + \int_{\tau}^t \Phi_A(t, \sigma)B(\sigma)u(\sigma)$$

in continuous-time and

$$x(t) = \Phi_A(t, \tau)x(\tau) + \sum_{j=\tau+1}^t \Phi_A(t, j)B(j-1)u(j-1)$$

in discrete-time. Correspondingly, the output $y(t)$ is given by

$$y(t) = C(t)\Phi_A(t, \tau)x(\tau) + \int_{\tau}^t C(t)\Phi_A(t, \sigma)B(\sigma)u(\sigma) + D(t)u(t)$$

in continuous-time and

$$y(t) = C(t)\Phi_A(t, \tau)x(\tau) + \sum_{j=\tau+1}^t C(t)\Phi_A(t, j)B(j-1)u(j-1) + D(t)u(t)$$

⁶ Given two square real symmetric matrices M and N of the same dimension, the symbol $M > N$ means that $M - N$ is positive definite. Similarly, $M \geq N$ means that $M - N$ is positive semi-definite.

in discrete-time. These expressions are known as *Lagrange formulae*.

Comparing these formulae with those of the input–output representation, one can conclude that the relation between the kernel $h(t, \tau)$ in (1.3) and the state–space matrices is as follows:

$$h(t, \tau) = \begin{cases} C(t)\Phi_A(t, \tau)B(\tau) + D(\tau)\delta(t - \tau) & \text{in continuous-time} \\ \begin{cases} C(t)\Phi_A(t, \tau)B(\tau) & t > \tau \\ D(t) & t = \tau \end{cases} & \text{in discrete-time.} \end{cases}$$

An input–output characterization of stability is based on the requirement that the forced output is bounded for bounded inputs (BIBO stability). More precisely, we say that the system is BIBO stable if for any bounded input, namely for any signal $u(t)$ for which there exists a constant U_M such that

$$\sup_{t > 0} \|u(t)\| < U_M,$$

the corresponding forced output⁷ $y(t)$ is bounded as well, i.e., there exists Y_M such that

$$\sup_{t > 0} \|y(t)\| < Y_M.$$

One can prove that a linear periodic system (in discrete or continuous-time) is BIBO stable if and only if, for some τ ,

$$\lim_{t \rightarrow \infty} \|h(t, \tau)\| = 0. \quad (1.16)$$

Indeed, for our class of impulse responses, condition (1.16) is equivalent to requiring that the impulse response $h(t, \tau)$ tends to zero exponentially fast. For continuous-time systems, this means that there exists a bi-periodic non-negative function $\alpha(t, \tau)$ and a negative constant β such that

$$\|h(t, \tau)\| \leq \alpha(t, \tau)e^{\beta(t-\tau)}.$$

Then, from (1.3) it is apparent that, for each $t \geq 0$,

$$\|y(t)\| \leq U_M \int_0^t \|h(t, \tau)\| d\tau = \frac{U_M \alpha(1 - e^{\beta t})}{\beta} \leq \frac{U_M \alpha}{\beta} = Y_M.$$

The converse statement is easily obtained by contradiction. The derivation for discrete-time systems is provided in a subsequent chapter.

⁷ By forced output we mean the output signal corresponding to the zero initial state.

1.4 Relative Degrees and Zeros

In the reminder of this first chapter, we will consider only continuous-time periodic systems. The corresponding concepts in discrete-time will be thoroughly discussed in the subsequent chapters.

1.4.1 Relative Degrees and Normal Form

For SISO systems, the relative degree is defined by considering subsequent derivatives of the output. The relative degree is the order of the derivative of minimal order where the input signal appears explicitly. This notion can be translated to multivariable systems by considering the output channels each at a time. Of course this leads to a set of p relative degrees, which can be organized in a vector.

Consider the periodic system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t),\end{aligned}$$

where $A(t)$, $B(t)$ and $C(t)$ are T -periodic matrices, with smooth entries. Let $c_i(t)$ be the i -row of matrix $C(t)$ and introduce the operator:

$$(L_A h(t)) = h(t)A(t) + \dot{h}(t) \quad (1.17)$$

for any differentiable $h(t)$ of compatible dimension. Then

$$(L_A^2 h(t)) = h(t)A(t)^2 + \dot{h}(t)A(t) + \dot{h}(t)A(t) + h(t)\dot{A}(t) + \ddot{h}(t),$$

and so on. We also define $(L_A^0 h(t)) = h(t)$.

It is apparent that the first derivative $y_i^{(1)}(t)$ of the i -th output signal is given by

$$\begin{aligned}y_i^{(1)}(t) &= c_i(t)\dot{x}(t) + \dot{c}_i(t)x(t) = \\ &= (c_i(t)A(t) + \dot{c}_i(t))x(t) + c_i(t)B(t)u(t) = \\ &= (L_A c_i(t))x(t) + c_i(t)B(t)u(t).\end{aligned}$$

If $c_i(t)B(t)$ is identically zero, then the second derivative of $y_i(t)$ is

$$y_i^{(2)}(t) = (L_A^2 c_i(t))x(t) + (L_A c_i(t))B(t)u(t).$$

In general, if $(L_A^j c_i(t))B(t)$ is identically zero for $j = 0, 1, \dots, k-1$, then the k -th derivative is

$$y_i^{(k)}(t) = (L_A^k c_i(t))x(t) + (L_A^{k-1} c_i(t))B(t)u(t).$$

We are now in a position to introduce the notion of relative degree, [126].

Suppose that at time t there exists an integer k such that the following two conditions hold:

1. $c_i(\tau)B(\tau) = (L_A c_i(\tau))B(\tau) = \dots = (L_A^{k-2} c_i(\tau))B(\tau) = 0$ for all τ in a neighborhood of t .
2. in every neighborhood of t there exists τ such that $(L_A^{k-1} c_i(\tau))B(\tau) \neq 0$.

Then, $r_i(t) = k$ is named *relative degree* of the i -th output signal at time t .

If such a $r_i(t)$ exists for each $i = 1, 2, \dots, p$, the vector $r(t) = [r_1(t) \ r_2(t) \ \dots \ r_p(t)]$ is named *relative degree vector* of the system at time t .

If $r(t)$ exists for each time instant, then the function $r(\cdot)$ is called *relative degree* of the system.

Consider now a periodic system with relative degree $r(\cdot)$ and define the $p \times p$ matrix

$$\Gamma(t) = \begin{bmatrix} (L_A^{r_1(t)-1} c_1(t))B(t) \\ (L_A^{r_2(t)-1} c_2(t))B(t) \\ \vdots \\ (L_A^{r_p(t)-1} c_p(t))B(t) \end{bmatrix}. \quad (1.18)$$

This matrix summarizes how input $u(\cdot)$ acts on the first derivatives of the various outputs on which $u(\cdot)$ has an explicit influence. It is therefore named *instantaneous interactor matrix* at time t .

For the sake of simplicity, we focus on square systems, i.e., $p = m$, from now on. Under this assumption matrix $\Gamma(t)$ is square.

The periodic system is said to have *uniform relative degree* if $r(t)$ is constant and matrix $\Gamma(t)$ has a bounded inverse for each t .

In such a case, the relative degree vector is denoted by $r = [r_1 \ r_2 \ \dots \ r_p]$. Each r_i is the relative degree of the i -th output signal while $r_0 = \sum_i r_i$ is named *global relative degree* of the system.

Suppose now that a periodic system (of order n) has uniform relative degree. Then, it is possible to prove that $r_0 \leq n$ and that it is possible to work out a suitable state-space transformation $\hat{x}(t) = S(t)x(t)$ such that the original system is rewritten in the so-called *normal form*. To be precise, the vector $\hat{x}(t)$ can be partitioned in a number of sub-vectors as

$$\hat{x}(t) = \begin{bmatrix} \hat{x}_{(1)}(t) \\ \hat{x}_{(2)}(t) \\ \vdots \\ \hat{x}_{(p)}(t) \\ \hat{x}_{(0)}(t) \end{bmatrix}. \quad (1.19)$$

Each sub-vector $\hat{x}_{(i)}(t)$, $i = 1, 2, \dots, p$ has dimension r_i and is constituted by $y_i(t)$ and its derivatives up to the $r_i - 1$ -th order.

In other words,

$$\hat{x}_{(i)}(t) = \begin{bmatrix} y_i(t) \\ y_i^{(1)}(t) \\ \vdots \\ y_i^{(r_i-1)}(t) \end{bmatrix}, \quad i = 1, 2, \dots, p.$$

Vector $x_{(0)}(t)$ is a complementary sub-vector of dimension $n - r_0$. The state-space transformation $S(\cdot)$, differentiable with differentiable inverse, leading to such a new representation, can be given a partitioned expression as follows: the transformed state can be written as

$$\hat{x}(t) = S(t)x(t) = \begin{bmatrix} S_{(1)}(t) \\ S_{(2)}(t) \\ \vdots \\ S_{(p)}(t) \\ S_{(0)}(t) \end{bmatrix} x(t),$$

where

$$S_{(i)}(t) = \begin{bmatrix} c_i(t) \\ L_A c_i(t) \\ \vdots \\ L_A^{r_i-1} c_i(t) \end{bmatrix}, \quad i = 1, 2, \dots, p,$$

and $S_{(0)}(t)$ is a $(n - r_0) \times n$ matrix such that $S_{(0)}(t)B(t) = 0$, for all t . The existence of such a transformation is ensured by the so-called Dolezal's Theorem, [130].

This leads to the *normal form*:

$$\dot{\hat{x}}_{(i)}(t) = \hat{A}_{(i)}\hat{x}_{(i)}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ L_A^{r_i} c_i(t) \end{bmatrix} S(t)^{-1} \hat{x}(t) + \hat{B}_{(i)}(t)u(t), \quad i = 1, 2, \dots, p \quad (1.20a)$$

$$\dot{\hat{x}}_{(0)}(t) = Z(t)\hat{x}_{(0)}(t) + Y(t)y(t), \quad (1.20b)$$

where

$$\hat{A}_{(i)}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{B}_{(i)}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (L_A^{r_i-1} c_i(t))B(t) \end{bmatrix},$$

and $Z(\cdot)$, $Y(\cdot)$ are suitable periodic matrices. From the structure of matrices $\hat{A}_{(i)}(t)$ and $\hat{B}_{(i)}(t)$, it is apparent that the first $r_i - 1$ elements of sub-vector $\hat{x}_{(i)}(t)$ form a

chain of integrators whereas the derivative of the last element of $\hat{x}_{(i)}(t)$ depends on the input $u(t)$, besides all state variables but $\hat{x}_{(0)}(t)$. As for this last sub-vector, it obeys a different logic in that its evolution is determined by the output $y(t)$ only.

We will now provide the transformation $S(\cdot)$ in explicit form in the case of SISO systems. In such a case, there is a unique relative degree, assumed to be time independent, and denoted by r . The transformed state $\hat{x}(t)$ is therefore partitioned in two sub-vectors $\hat{x}_{(1)}(t)$ and $\hat{x}_{(0)}(t)$, of dimension r and $n - r$, respectively. Correspondingly, matrix $S(t)$ is constituted by two blocks

$$S(t) = \begin{bmatrix} S_{(1)}(t) \\ S_{(0)}(t) \end{bmatrix},$$

where $S_{(1)}(t)$ and $S_{(0)}(t)$ have dimensions $r \times n$ and $n - r \times n$, respectively. The first block is provided by expression

$$S_{(1)}(t) = \begin{bmatrix} C(t) \\ L_A C(t) \\ \vdots \\ L_A^{r-1} C(t) \end{bmatrix}, \quad i = 1, 2, \dots, p.$$

As for $S_{(0)}(t)$, we introduce the dual operator of (1.17), namely

$$(\tilde{L}_A B(t)) = -A(t)B(t) + \dot{B}(t)$$

with which we construct the $n \times r$ matrix

$$\tilde{S}_{(1)}(t) = [B(t) \ (\tilde{L}_A B(t)) \ \dots \ (\tilde{L}_A^{r-1} B(t))] .$$

Finally, we introduce the $n \times n - r$ matrix $V(t)$ as any matrix whose columns span the null space of $S_{(1)}(t)$, in other words $S_{(1)}(t)V(t) = 0$, for all t . Then, the block $S_{(0)}(t)$ is given by

$$S_{(0)}(t) = (V(t)'V(t))^{-1}V(t)'(I - \tilde{S}_{(1)}(t)(S_{(1)}(t)\tilde{S}_{(1)}(t))^{-1}S_{(1)}(t)) .$$

Notice that

$$S(t)^{-1} = [\tilde{S}_{(1)}(t)(S_{(1)}(t)\tilde{S}_{(1)}(t))^{-1} \quad V(t)] .$$

By means of this formula, together with the expression of the normal form (1.20b), it is possible to compute

$$\begin{aligned} Z(t) &= (S_0(t)A(t) + \dot{S}_0(t))V(t) \\ &= (V(t)'V(t))^{-1}V(t)'((A(t) - B(t)\Gamma(t)^{-1}L_A^r C(t))V(t) - \dot{V}(t)) . \end{aligned}$$

Here $\Gamma(t) = L_A^{r-1}C(t)B(t)$ is the (scalar) instantaneous interactor matrix.

For further details on the normal form of periodic systems, we refer to [248] and [194].

1.4.2 Zeros

We are now in a position to introduce and discuss the notion of zero for a periodic system. Again, we refer to square systems, with the input–output transformation $D(t) = 0, \forall t$. We first introduce the concept of zero⁸ by referring to the so-called *blocking property*. Then, we interpret this notion by making reference to the normal form.

We say that a complex number $z = e^{\rho T}$ is a *zero* of the system if there exist two T -periodic vector functions $\bar{x}(\cdot)$ and $\bar{u}(\cdot)$ (not both identically zero) such that

$$\begin{bmatrix} \rho I - A(t) & -B(t) \\ C(t) & 0 \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{u}(t) \end{bmatrix} + \begin{bmatrix} \bar{x}(t) \\ 0 \end{bmatrix} = 0. \quad (1.21)$$

This definition is equivalent to saying that, choosing $u(t) = \bar{u}(t)e^{\rho t}$ as input signal and $\bar{x}(0)$ as initial state, the output $y(t)$ is identically equal to zero (blocking property). Notice that, with such an input and such an initial condition, the state vector is given by $x(t) = \bar{x}(t)e^{\rho t}$.

As expected, the zeros are invariant with respect to a state–space coordinate change. Indeed, by performing the change of basis $\hat{x}(t) = S(t)x(t)$, where $S(\cdot)$ is a periodic Lyapunov transformation, the transformed system is described by matrices

$$\hat{A}(t) = S(t)A(t)S(t)^{-1} + \dot{S}(t)S(t)^{-1}, \quad \hat{B}(t) = S(t)B(t), \quad \hat{C}(t) = C(t)S(t)^{-1}.$$

It is easy to see that, in the new coordinates, condition (1.21) becomes

$$\begin{bmatrix} \rho I - \hat{A}(t) & -\hat{B}(t) \\ \hat{C}(t) & 0 \end{bmatrix} \begin{bmatrix} S(t)\bar{x}(t) \\ \bar{u}(t) \end{bmatrix} + \begin{bmatrix} \frac{d}{dt}(S(t)\bar{x}(t)) \\ 0 \end{bmatrix} = 0. \quad (1.22)$$

Therefore, the zeros are not affected by a change of basis.

We can now relate the notion of zero to the normal form. First, we prove that any zero is a characteristic exponent of matrix $Z(\cdot)$ appearing in (1.20b). For, consider the identity (1.22) with $S(\cdot)$ given by the transformation leading to the normal form (1.20). With the choice $u(t) = \bar{u}(t)e^{\rho t}$ as input signal and $S(0)\bar{x}(0)$ as initial state, it turns out that the output $y(t)$ is identically zero and the state $\hat{x}(t)$ is given by $\hat{x}(t) = S(t)\bar{x}(t)e^{\rho t}$. Consequently, from Expression (1.19) it follows that $\hat{x}_{(0)}(t) = S_{(0)}(t)\bar{x}(t)e^{\rho t}$. On the other hand, $\hat{x}_{(0)}(t)$ must satisfy Eq. (1.20b) of the normal form. Then, by differentiating $\hat{x}_{(0)}(t)$, it turns out that

$$(Z(t) - \rho I)\bar{x}_{(0)}(t) = \bar{x}_{(0)}(t),$$

⁸ In the literature one can find a variety of notions of zeros. Here we make reference to the notion of invariant zeros, that hinges on the state–space description of the underlying dynamical system.

where $\bar{x}_{(0)}(t) = S_{(0)}(t)\bar{x}(t)$. Hence, ρ is a characteristic exponent of matrix $Z(\cdot)$ appearing in the normal form. This means that the zero $z = e^{\rho T}$ is a characteristic multiplier of $Z(\cdot)$.

Secondly, we prove that any characteristic exponent of $Z(\cdot)$ is a zero. In fact, take a periodic solution of $(Z(t) - \rho I)\bar{x}_{(0)}(t) = \bar{\dot{x}}_{(0)}(t)$ and form the pair

$$\bar{\hat{x}}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{x}_{(0)}(t) \end{bmatrix}, \quad \bar{u}(t) = -\Gamma(t)^{-1}Q(t)S(t)^{-1}\bar{\hat{x}}(t),$$

where

$$Q(t) = \begin{bmatrix} L_A^{r_1} c_1(t) \\ \vdots \\ L_A^{r_p-1} c_p(t) \end{bmatrix}. \quad (1.23)$$

In view of the structure of the normal form, it follows that

$$\begin{bmatrix} \rho I - \hat{A}(t) & -\hat{B}(t) \\ \hat{C}(t) & 0 \end{bmatrix} \begin{bmatrix} \bar{\hat{x}}(t) \\ \bar{u}(t) \end{bmatrix} + \begin{bmatrix} \frac{d}{dt}\bar{\hat{x}}(t) \\ 0 \end{bmatrix} = 0.$$

Hence, $e^{\rho T}$ is a zero. In conclusion,

ρ satisfies the blocking property if and only if ρ is a characteristic exponent of $Z(\cdot)$.

Interestingly, it is possible to supply a closed-loop characterization of the zeros. Indeed, recalling definition (1.18) of the instantaneous interactor matrix $\Gamma(\cdot)$ and Expression (1.23) for $Q(\cdot)$, under the action

$$u(t) = -\Gamma(t)^{-1}Q(t)x(t) \quad (1.24)$$

the closed-loop system becomes

$$\dot{x}(t) = (A(t) - B(t)\Gamma(t)^{-1}Q(t))x(t) \quad (1.25a)$$

$$y(t) = C(t)x(t). \quad (1.25b)$$

System (1.25) is known as *zero dynamics*, a terminology taken from the theory of non-linear systems, [195]. Under the action of the transformation $S(\cdot)$ it turns out that

$$\dot{\hat{x}}_{(i)}(t) = \hat{A}_{(i)}\hat{x}_{(i)}(t), \quad i = 1, 2, \dots, p \quad (1.26a)$$

$$\dot{\hat{x}}_{(0)}(t) = Z(t)\hat{x}_{(0)}(t) + Y(t)y(t) \quad (1.26b)$$

$$y_i(t) = [1 \ 0 \ \dots \ 0]\hat{x}_{(i)}(t), \quad i = 1, 2, \dots, p. \quad (1.26c)$$

From the equations in (1.26) it is apparent that the output is identically zero if one imposes the state-feedback law (1.24) with an initial state of the form

$$x(0) = S(0)^{-1}\hat{x}(0) = S(0)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \hat{x}_{(0)}(0) \end{bmatrix}.$$

Finally, letting

$$R(t) = A(t) - B(t)\Gamma(t)^{-1}Q(t),$$

observe from (1.26) that

$$\hat{A}(t) = (S(t)R(t) + \dot{S}(t))S(t)^{-1} = \begin{bmatrix} \hat{A}_{(1)} & 0 & \cdots & 0 & 0 \\ 0 & \hat{A}_{(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \hat{A}_{(p)} & 0 \\ \bar{Y}_1(t) & \bar{Y}_2(t) & \cdots & \bar{Y}_p(t) & Z(t) \end{bmatrix},$$

where

$$\bar{Y}_i(t) = [Y_i(t) \ 0 \ \cdots \ 0], \quad i = 1, 2, \dots, p,$$

and $Y_i(t)$ are the columns of matrix $Y(t)$ appearing in the normal form (1.20b).

Summing up, the closed-loop matrix $R(\cdot)$ has $r_0 = r_1 + r_2 + \cdots + r_p$ characteristic exponents at zero (which coincide with the eigenvalues of $\hat{A}_{(i)}$, $i = 1, 2, \dots, p$) and the remaining $n - r_0$ coincident with the characteristic exponents of $Z(\cdot)$. Hence, $n - r_0$ characteristic multipliers of $R(\cdot)$ coincide with the system's zeros, whereas the remaining r_0 are at one.

We end this section by considering a SISO periodic system subject to a memoryless and constant output-feedback $u(t) = ky(t)$. The corresponding closed-loop system has n characteristic exponents depending on k . As a function of k they describe a locus which, in analogy to the time-invariant case, can be named *root locus*. By means of a simple example, we can again verify that the closed-loop behavior of a periodic system can be remarkably different from that of the time-invariant approximation obtained by taking the mean values of the various matrices.

Example 1.2 Consider the system described by matrices

$$\begin{aligned} A(t) &= \begin{bmatrix} -1 + \sin(t) & 0 \\ 1 - \cos(t) & -3 \end{bmatrix} & B(t) &= \begin{bmatrix} -1 - \cos(t) \\ 2 - \sin(t) \end{bmatrix} \\ C(t) &= [0 \ 1] & D(t) &= 0. \end{aligned}$$

This is a periodic system of period 2π whose mean-value approximation is

$$A_0 = \begin{bmatrix} -1 & 0 \\ 1 & -3 \end{bmatrix} \quad B_0 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ C_0 = [0 \ 1] \quad D_0 = 0.$$

Since $A(t)$ is lower triangular, it is easy to see that the characteristic exponents of the periodic system are given by -1 and -3 . They coincide with the eigenvalues of A_0 .

Consider now the closed-loop matrix

$$R(t) = A(t) + kB(t)C(t) = \begin{bmatrix} -1 + \sin(t) & -k - k\cos(t) \\ 1 - \cos(t) & -3 + 2k - k\sin(t) \end{bmatrix},$$

and its mean-value approximation

$$R_0 = \begin{bmatrix} -1 & -k \\ 1 & -3 + 2k \end{bmatrix}.$$

The constant matrix R_0 has all eigenvalues with negative real parts for $k < 2$; however, the time-periodic $R(\cdot)$ has all characteristic multipliers inside the open unit disk only for $k < 1.6$. Hence, the closed-loop stability analysis cannot rely on the mean-value approximation. As the two open-loop systems have the same “poles”, the discrepancy in the closed-loop behavior can only be interpreted by looking at the zeros. Obviously, the mean-value system has a relative degree equal to 1. The same holds for the periodic system, as the product $C(t)B(t)$ is generically nonzero, as required by the definition of relative degree previously introduced. It is easy to see that the zero of the mean-value system is given by $z_0 = -0.5$. To compute the zero of the periodic system, one can first determine matrix $Z(\cdot)$ appearing in (1.20b); in this example $Z(t)$ is scalar and given by

$$Z(t) = \frac{-2 + 3\sin(t)}{2 - \sin(t)}.$$

The mean value of this function is the characteristic exponent ρ of $Z(\cdot)$. It turns out that $\rho = -0.69$. Clearly, the discrepancy between z_0 and ρ is responsible for the different behavior of the closed-loop system.

1.5 Periodic Systems in Frequency Domain

The frequency domain representation is a fundamental tool in the analysis and control of time-invariant linear systems. It is related to the well-known property that, for this class of systems, sinusoidal inputs result into sinusoidal outputs at the same frequency and different amplitude and phase.

→ Bode Plots

A similar tool can be worked out for periodic systems by making reference to their response to the so-called *exponentially modulated periodic* (EMP) signals. Given any complex number s , a (complex) signal $u(t)$, $t \in \mathbb{R}$, is said to be EMP of period T and modulation s if

$$u(t) = \sum_{k \in \mathbb{Z}} u_k e^{s_k t},$$

where

$$s_k = s + jk\Omega.$$

$$u(t) = \sum_{k \in \mathbb{Z}} u_k e^{st + jk\Omega t}$$

The quantity $\Omega/2\pi$ is named period of the EMP signal. The class of EMP signals is a generalization of the class of T -periodic signals: as a matter of fact, an EMP signal with $s = 0$ is just an ordinary time-periodic signal.

In much the same way as a time-invariant system subject to a (complex) exponential input admits an exponential regime, a periodic system of period T

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

subject to an EMP input of the same period admits an EMP regime. In such a regime, all signals of interest can be expanded as EMP signals as follows:

$$x(t) = \sum_{k \in \mathbb{Z}} x_k e^{s_k t}$$

$$\dot{x}(t) = \sum_{k \in \mathbb{Z}} s_k x_k e^{s_k t}$$

$$y(t) = \sum_{k \in \mathbb{Z}} y_k e^{s_k t}.$$

Consider now the Fourier series for the periodic matrix coefficients, i.e.,

$$A(t) = \sum_{k \in \mathbb{Z}} A_k e^{jk\Omega t},$$

and similarly for $B(t)$, $C(t)$ and $D(t)$, and plug the expansions of the signals $x(t)$, $u(t)$, $\dot{x}(t)$ and the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$ into the system equations. By equating all terms at the same frequency, one obtains an infinite-dimensional matrix equation of the following kind:

$$\begin{aligned} s\mathcal{X} &= (\mathcal{A} - \mathcal{N})\mathcal{X} + \mathcal{B}\mathcal{U} \\ \mathcal{Y} &= \mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{U}, \end{aligned}$$

where \mathcal{X} , \mathcal{U} and \mathcal{Y} are doubly infinite vectors formed with the harmonics of x , u and y respectively, organized in the following fashion:

$$\mathcal{X}^T = [\cdots, x_{-2}^T, x_{-1}^T, x_0^T, x_1^T, x_2^T, \cdots],$$

and similarly for \mathcal{U} and \mathcal{V} . \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are doubly infinite Toeplitz matrices formed with the harmonics of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ respectively as

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_0 & A_{-1} & A_{-2} & A_{-3} & A_{-4} & \cdots \\ \cdots & A_1 & A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ \cdots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ \cdots & A_3 & A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \cdots & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and similarly for \mathcal{B} , \mathcal{C} and \mathcal{D} . As for matrix \mathcal{N} , it is the block diagonal matrix

$$\mathcal{N} = \text{blkdiag} \{jk\Omega I, k \in \mathbb{Z}\}.$$

Then, one can define the *harmonic transfer function* as the operator:

$$\hat{\mathcal{G}}(s) = \mathcal{C}[s\mathcal{I} - (\mathcal{A} - \mathcal{N})]^{-1}\mathcal{B} + \mathcal{D}.$$

Such an operator provides a most useful connection between the input harmonics and the output harmonics (organized in the infinite vectors \mathcal{U} and \mathcal{V} , respectively). In particular, if one takes $s = 0$ (so considering the truly periodic regimes), the appropriate input/output operator is

$$\hat{\mathcal{G}}(0) = \mathcal{C}[\mathcal{N} - \mathcal{A}]^{-1}\mathcal{B} + \mathcal{D}.$$

If $u(\cdot)$ is a sinusoid, this expression enables one to compute the amplitudes and phases of the harmonics constituting the output signal $y(\cdot)$ in a periodic regime.

In general, the input/output operator representation of a periodic system may be somewhat impractical, given that it is infinite-dimensional. Of course, one can consider replacing this model with a finite-dimensional approximation obtained by truncation of the Fourier series of the system matrices, which in turn implies that matrices \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} and \mathcal{N} are also truncated and have therefore finite dimensions.

Analyzing the frequency domain behavior of a continuous-time periodic system in terms of the Fourier expansion of its coefficients is a long-standing idea in the field. See [123] for a classical reference, the paper [301] and the doctoral thesis [319].

An alternative procedure in order to work out the harmonic transfer function is based on the **impulse response** representation rather than the state-space one. Precisely, consider again the output response of a periodic system

$$y(t) = \int_0^t h(t, \tau) u(\tau) d\tau,$$

where $h(t, \tau)$ is the impulse response. Since $h(t, \tau) = h(t + T, \tau + T)$, the function $h(t, t - r)$ is, for any fixed r , periodic in the variable t , so that it can be expanded in

the Fourier series with coefficients $h_k(r)$. Therefore, letting $\Omega = \frac{2\pi}{T}$, one can write

$$h(t, \tau) = \sum_{k=-\infty}^{\infty} h_k(t - \tau) e^{jk\Omega t}.$$

Substituting this expression into the output-response formula, we obtain

$$y(t) = \sum_{k=-\infty}^{\infty} \int_0^t h_k(t - \tau) e^{jk\Omega(t-\tau)} u(\tau) e^{jk\Omega\tau} d\tau.$$

This is a series of convolution integrals. Therefore, the Laplace transform is given by

$$Y(s) = \sum_{k=-\infty}^{\infty} H_k(s - jk\Omega) U(s - jk\Omega), \quad (1.27)$$

where $U(s)$, $H_k(s)$ and $Y(s)$ are the Laplace transforms of $u(\cdot)$, $h_k(\cdot)$, and $y(\cdot)$. The relation between $\text{column}\{y(s + jk\Omega), k \in Z\}$ and $\text{column}\{u(s + jk\Omega), k \in Z\}$, is expressed by the infinite matrix

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & H_0(s - j2\Omega) & H_{-1}(s - j\Omega) & H_{-2}(s) & H_{-3}(s + j\Omega) & H_{-4}(s + j2\Omega) & \cdots \\ \cdots & H_1(s - j2\Omega) & H_0(s - j\Omega) & H_{-1}(s) & H_{-2}(s + j\Omega) & H_{-3}(s + j2\Omega) & \cdots \\ \cdots & H_2(s - j2\Omega) & H_1(s - j\Omega) & H_0(s) & H_{-1}(s + j\Omega) & H_{-2}(s + j2\Omega) & \cdots \\ \cdots & H_3(s - j2\Omega) & H_2(s - j\Omega) & H_1(s) & H_0(s + j\Omega) & H_{-1}(s + j2\Omega) & \cdots \\ \cdots & H_4(s - j2\Omega) & H_3(s - j\Omega) & H_2(s) & H_1(s + j\Omega) & H_0(s + j2\Omega) & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This matrix, known as (*frequency-lifted transfer operator*), coincides with the harmonic transfer function $\hat{\mathcal{G}}(s)$.

As a final remark, notice that the harmonic transfer function lends itself to an input-output definition of zeros of a periodic system, based on the so-called *input-output blocking property*. This somehow complements the definition of zeros previously given, based on the *input-state-output blocking property*. Indeed, it is not difficult to show that Eq. (1.21) implies that the complex number z is a zero of the system if and only if there exist an exponentially modulated periodic signal and an initial state $x(0)$ such that the corresponding output $y(t)$ is identically zero. Alternatively, it is also possible to write the dependence between the EMP input

$$u(t) = \sum_{k \in Z} u_k e^{jk\Omega t} e^{st},$$

and the EMP regime output

$$y(t) = \sum_{k \in Z} y_k e^{jk\Omega t} e^{st},$$

as follows

$$y_k = \sum_{r=-\infty}^{\infty} H_{r-k}(s + j\Omega r) u_r.$$

This means that if $s = z$ is a zero of the periodic system, then all components of the output harmonics should be zero, i.e., $\forall k$,

$$\sum_{r=-\infty}^{\infty} H_{r-k}(z + j\Omega r) u_r = 0.$$

1.6 Time-invariant Representations

Periodicity is often the result of *ad hoc* operations over time-invariant systems. Here, we concisely deal with the “backward” problem, namely the problem of “transforming” the periodic system into a time-invariant one. In such a way, one can resort to the results already available in the time-invariant realm.

1.6.1 Sample and Hold

The simplest way to achieve stationarity is to resort to a sample and hold procedure. Indeed, with reference to a continuous or a discrete-time periodic system, suppose that the input is kept constant over a period, starting from an initial time point τ , i.e.,

$$u(t) = \tilde{u}(k), \quad t \in [kT + \tau, kT + T + \tau).$$

In this way, the evolution of the system state sampled at $\tau + kT$, i.e., $x_\tau(k) = x(kT + \tau)$, is governed by a *time-invariant* equation in discrete-time. Precisely,

$$x_\tau(k+1) = \Phi_A(T + \tau, \tau) x_\tau(k) + \Gamma(\tau) \tilde{u}(k),$$

where

$$\Gamma(\tau) = \int_{\tau}^{T+\tau} \Phi_A(T + \tau, s) B(s) ds.$$

It is possible to slightly generalize the sample and hold representation by equipping the holding mechanism with a time-varying modulating function $H(\cdot)$ acting over the period as follows

$$u(t) = H(t) \tilde{u}(k), \quad t \in [kT + \tau, kT + T + \tau).$$

In this way, the evolution of the sampled state $\tilde{x}(k)$ is still governed by the previous equations provided that $B(t)$ is replaced by $H(t)B(t)$. Such a *generalized sample and hold representation* allows for a further degree of freedom in the design of periodic controllers. Indeed, function $H(\cdot)$ is a free parameter to be chosen by the designer.

1.6.2 Lifting

The idea underlying the lifting technique is very simple. To be precise, given an analog signal $v(\cdot)$, the corresponding lifted signal is obtained by splitting the time-axis into intervals of length T , and defining the segments v_k , k integer, as follows

$$v_k = \{u(t), \quad t \in [kT, kT + T)\}.$$

In this way, the *lifted* signal \tilde{v}_k is defined in discrete-time. This procedure can be applied to the input $u(\cdot)$ and output $y(\cdot)$ of a continuous-time T -periodic system thus yielding the discrete-time signals u_k and y_k , respectively. The dynamic system relating u_k to y_k is the *lifted system*.

In an input–output framework, consider the regime of a periodic system with input $u(\cdot)$, i.e.,

$$y(t) = \int_{-\infty}^t h(t, \tau) u(\tau) d\tau,$$

where $h(t, \tau)$ is the impulse response function, already defined. An easy computation shows that

$$y_n = \sum_{i=0}^{\infty} H_i u_{n-i} = H_0 u_n + H_1 u_{n-1} + H_2 u_{n-2} + \cdots,$$

where the *lifted operators* H_i are defined as follows

$$H_i v[s] = \int_0^T h(s + iT, \tau) v(\tau) d\tau, \quad s \in [0, T).$$

The linear operator regression resembles the celebrated infinite order *moving average* model of linear time-invariant systems. By now taking the bilateral z -transform of a lifted signal v_k , i.e.,

$$\hat{V}(z) = \sum_{k=-\infty}^{\infty} v_k z^{-k}$$

it follows

$$\begin{aligned} \hat{Y}(z) &= \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} H_l u_{k-l} z^{-k} \\ &= \left(\sum_{l=0}^{\infty} H_l z^{-l} \right) \left(\sum_{i=-\infty}^{\infty} u_i z^{-i} \right) = \hat{H}(z) \hat{U}(z). \end{aligned}$$

In this way, the input–output behavior of the continuous-time periodic system is characterized by the discrete-time operator $\hat{H}(z)$, the z -transform of the sequence of lifted operators H_i . This operator is referred to as the *time-lifted transfer function operator*.

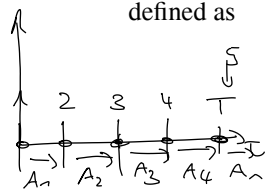
The time-lifted transfer function operator $\hat{H}(z)$ can be also derived by following a state-space path. To this purpose, one has to consider the following ingredients: (i) lifted input u_k , (ii) sampled state $x(kT)$ and (iii) lifted output y_k . By using the Lagrange formula in the interval $[kT, kT + T)$ one obtains

$$\begin{aligned} x_2 &= A_1 x_1 + B_1 u_1 \\ x_3 &= A_2 A_1 x_1 + A_2 B_1 u_1 + B_2 u_2 \\ &\vdots \\ x_{k+1} &= \Psi_A(T, 0) x_1 + \sum_{i=1}^k \Phi_A(T, T_i) B(T_i) u_i \end{aligned}$$

$$x(kT + T) = Fx(kT) + Gu_k$$

$$y_k = H_k x(kT) + Eu_k,$$

where $F = \Psi_A(T, 0)$ is the monodromy matrix and the operators G, H , and E are defined as



$$\begin{aligned} Gu_k &= \int_0^T \Phi_A(T, \tau) B(\tau) u(kT + \tau) d\tau \\ H &= C(t) \Phi_A(t, 0) \\ Eu_k &= \int_0^T (C(t) \Phi_A(t, \tau) + D(t) \delta(t - \tau)) u(kT + \tau) d\tau. \end{aligned}$$

Obviously, the time-lifted transfer function operator defined above coincides with the transfer function of this discrete-time system, namely

$$\hat{H}(z) = H(zI - F)^{-1}G + E.$$

As a final observation, note that the time-lifted transfer function operator $\hat{H}(z)$ and the frequency-lifted (harmonic) transfer function are related as $\hat{H}(e^{sT}) = \hat{\mathcal{G}}(s)$.

1.7 Periodic Control of Time-invariant Systems

The early developments of periodic control were concentrated on the problem of forcing a periodic regime in order to improve the performances of an industrial plant (*periodic optimization*). Nowadays, the term periodic control has taken a wider sense so as to include the design of control systems where the controller and/or the plant are described by periodic models.

In this section we outline some main results on periodic control of time-invariant systems, whereas periodic control of periodic systems is the subject of a subsequent section. We start with the problem of output stabilization by means of sampled and hold methodology. This technique has the advantage of reducing the problem to an equivalent one in the time-invariant realm. However, it suffers from basic drawbacks concerning the effect of disturbances in the inter-sample periods. Then we move to output-feedback without sampling. To probe the achievable performances, we start from a simple benchmark example, concerning the control of a second-order time-invariant system. Some general considerations on periodic output stabilization come next. The section ends with the issue of periodic optimization.

1.7.1 Output Stabilization of Linear Time-invariant Systems

The application of periodic controllers to time-invariant processes has been treated extensively in literature. The basic concern is to solve problems otherwise unsolvable with time-invariant controllers or to improve the achievable control performances.

A typical line of reasoning adopted in this context can be explained by making reference to the classical *output stabilization problem*, namely the problem of finding an algebraic feedback control law *based on the measurements of the output signal* in order to stabilize the overall control system. If the system is

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

with the control law

$$u(t) = Fy(t),$$

the problem is to find a matrix F (if any) such that the closed-loop system

$$\dot{x}(t) = (A + BFC)x(t)$$

is stable. Although a number of necessary and sufficient conditions concerning the existence of a stabilizing matrix F have been provided in the literature, it is not an easy matter to work out reliable algorithms for its determination, as discussed in the survey paper [278].

Periodic sampled control seems to offer a practical alternative to tackle this problem. Indeed, consider the time-varying control law based on the sampled measurements of $y(\cdot)$

$$u(t) = F(t)y(kT), \quad t \in [kT, kT + T).$$

The modulating function $F(\cdot)$ and the sampling period T have to be selected in order to stabilize the closed-loop system, now governed by the equation

$$x(kT + T) = A_c x(kT),$$

where

$$A_c = \left[e^{AT} + \int_0^T e^{A(T-s)} BF(s) C ds \right].$$

The crucial point is the selection of matrix $F(\cdot)$ for a given period T . A possibility, originally proposed in [201], is to consider an $F(\cdot)$ given by the following expression

$$F(t) = B' e^{A'(T-t)} \left[\int_0^T e^{A(T-s)} BB' e^{A'(T-s)} ds \right]^{-1} Z$$

with matrix Z still to be specified. Note that the formula above is valid provided that the matrix inversion can be performed (this is indeed the case under the so-called

reachability condition). In this way, the closed-loop matrix A_c takes the form

$$A_c = e^{AT} + ZC.$$

Then, provided that some weak condition on the pair (A, C) is met, period T and matrix Z can be selected so as to stabilize A_c ⁹.

The *generalized sample and hold* philosophy above outlined in the simple problem of stabilization has been pursued in many other contexts, ranging from the problem of simultaneous stabilization of a finite number of plants [201] to that of fixed poles removal in decentralized control [8], from the issue of pole and/or zero-assignment [5, 93, 94, 149, 202, 304], to that of gain margin or robustness improvement [208, 238], from adaptive control [239] to model matching [201], and so on.

When using generalized sample-data control, however, the intersample behavior can present some critical aspects, as pointed out in several papers, such as [233] and [141]. Indeed the action of the generalized sample and hold function is a sort of amplitude modulation which, in the frequency domain, may lead to additional high-frequency components centered on multiples of the sampling frequency. Consequently, there are non-negligible high-frequency components both in the output and control signals. To smooth out these ripples, remedies have been studied, see [10, 142]. An obvious possibility is to monitor *at each time instant* the output signal and to adopt the feedback control strategy

$$u(t) = F(t)y(t)$$

with a periodic gain $F(t)$, in place of the sampled strategy before seen. This leads to the issue of *memoryless output-feedback* control of time-invariant systems. This issue was studied in [3], where a pole assignment problem in discrete-time is considered, and in [234] where the stabilization problem for SISO continuous-time systems is studied. This issue will be elaborated upon by means of a simple example in the subsequent point.

Obviously, stabilization is a preliminary step for any control design problem. A further step forward leads to the *regulation problem*, in which the designer has to work out a controller aiming at zeroing the regulation error due to exogenous disturbances or reference signals. When the exosystem (the model generating the disturbances and/or the reference signals) is periodic, one is naturally led to design a periodic controller even if the process under control is time-invariant, see [318].

A final observation on periodic control of time-invariant systems concerns the possibility of designing an output-feedback controller by resorting to optimal control ideas, e.g., the so-called Linear Quadratic Gaussian (LQG) theory, [7]. One can wonder whether, by enlarging the family of LQG controllers from the time-invariant

⁹ The required condition for the existence of Z such that A_c is stable is that pair (e^{AT}, C) is detectable. This is equivalent to: (A, C) detectable and T such that $e^{\lambda_i T} \neq e^{\lambda_j T}$, for each eigenvalues $\lambda_i \neq \lambda_j$ of A .

class to the class of periodic controllers, the achievable performance can be improved. To this question, the reply may be negative, even in the presence of bounded disturbances, as argued in [95].

1.7.2 The Chocolate Problem

Consider the second-order time-invariant system defined by the transfer function

$$G(s) = \frac{s^2 - 1}{s^2 - 2\delta s + 1}, \quad (1.28)$$

where δ is a real parameter in the range $[0, 1)$. This system has two zeros, one of which is in the right half plane, and two unstable poles.

In [77], a kilogram of Belgian chocolate was offered for the solution of the following problem:

find the values of parameter δ for which there exists a linear, time-invariant, stable, and minimum-phase controller stabilizing the plant.

Finding all such δ s is a difficult task. In particular, it is apparent that for $\delta \approx 1$ there is a “near” unstable zero-pole cancellation; as is well-known, such a situation is extremely demanding for the stabilization problem, see e.g., the Klein’s bicycle design problem mentioned in [13]. In [78] it is shown that there exists a value $\delta^* < 1$ such that stabilization is possible for all $\delta < \delta^*$, and not possible for $\delta \geq \delta^*$. The exact value of δ^* has not been found yet, however in [245], an interval to which δ^* belongs has been provided. For $\delta = 0.9$ and $\delta \approx 0.924$, it was shown in [90] that a third-order feasible controller can be worked out. Moreover, a fourth-order controller was obtained for $\delta \approx 0.951$.

We now relax the assumption that the stable and minimum-phase controller be time-invariant and show that a simple stabilizing periodic controller can be designed for any given value of $\delta < 1$.

First of all, consider a state–space realization for the System (1.28) with transfer function $G(s)$:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + u, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2\delta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [-2 \quad 2\delta].$$

As for the control law, we will consider two possible strategies. In the first, the controller is memoryless, whereas in the second one, a first-order controller is considered.

Memoryless Control

Consider the control law

$$u(t) = F(t)y(t),$$

where $F(\cdot)$ is a periodic function of period T , to be designed. Notice that $F(t)$ must be different from 1 (the high-frequency gain) in order to have a well-behaved dynamic closed-loop system avoiding algebraic loops. Letting

$$K(t) = \frac{F(t)}{1 - F(t)},$$

the closed-loop system dynamic matrix becomes

$$\hat{A}(t) = A + BK(t)C = \begin{bmatrix} 0 & 1 \\ -1 - 2K(t) & 2\delta + 2\delta K(t) \end{bmatrix}.$$

From the Liouville–Jacobi formula (1.6) a necessary condition for asymptotic stability is that

$$T + \int_0^T K(t)dt < 0. \quad (1.29)$$

This condition can be easily satisfied. However, it is only necessary.

We restrict the class of periodic gains $K(t)$ to the set of piecewise constant laws of the form

$$K(t) = \begin{cases} K_1, & t \in [0, 0.5T) \\ K_2, & t \in [0.5T, T). \end{cases}$$

Correspondingly,

$$F(t) = \frac{K(t)}{1 + K(t)} = \begin{cases} F_1, & t \in [0, 0.5T) \\ F_2, & t \in [0.5T, T) \end{cases},$$

where

$$F_1 = \frac{K_2}{1 + K_2}, \quad F_2 = \frac{K_1}{1 + K_1}.$$

For simplicity we take $T = 2$, so that the monodromy matrix is given by

$$\Phi(2, 0) = \exp(\hat{A}_2) \exp(\hat{A}_1),$$

where

$$\hat{A}_i = \begin{bmatrix} 0 & 1 \\ -1 - 2K_i & 2\delta + 2\delta K_i \end{bmatrix}, \quad i = 1, 2.$$

Moreover, the necessary condition (1.29) simplifies to

$$K_1 + K_2 < -2.$$

A simple way to solve the stabilization problem is to impose

$$K_1 + K_2 = -2 - \varepsilon,$$

where ε is a given strictly positive real number, so that the necessary condition is satisfied. For our experiments, we have chosen $\varepsilon = 0.1$ and $\delta = 0.9$. For such choices, in Fig. 1.3 the maximum modulus of the characteristic multipliers is plotted as a function of K_1 . By inspection, it is apparent that there is a range of values of K_1 for which all characteristic multipliers belong to the open unit disk. In particular, a stabilizing switching gain is obtained for

$$K_1 = -4.2200, \quad K_2 = -K_1 - 2 - \varepsilon = 2.1200.$$

The output of the so-obtained closed-loop system with initial state $(0.1, 0.1)$ is represented in Fig. 1.4. This is a typical transient for non-smooth control laws.

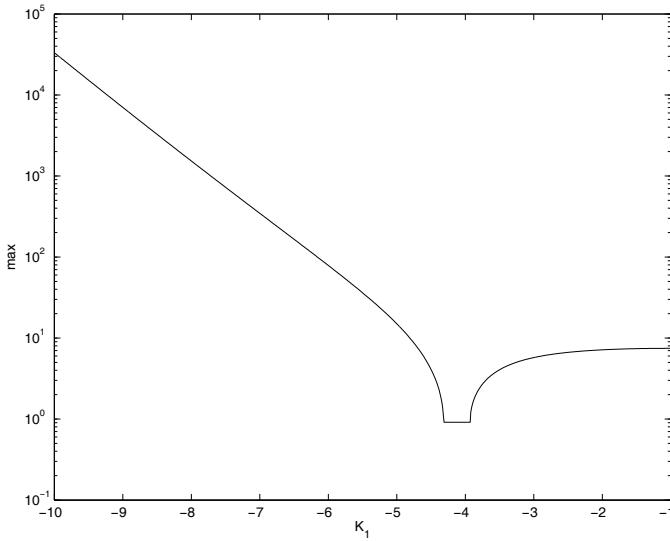


Fig. 1.3 Maximum modulus of the characteristic multipliers as a function of K_1 for $\delta = 0.9$ and $\varepsilon = 0.1$

The above analysis can be replicated for different values of $\delta < 1$, see [107] for further details.

An interesting observation concerns the possibility of working out a smooth controller gain $K(t)$ from the above switching law. In principle, this could be achieved by means of a sigmoidal shaped function connecting K_1 to K_2 over the period interval. Notice however that, in so doing, there would exist a time instant \bar{t} for which $K(\bar{t}) = -1$ and $F(\bar{t}) = \infty$, hence producing an impulsive control action.

We close this section by mentioning that an alternative approach to the design of memoryless periodic controllers for time-invariant systems is studied in [234] using

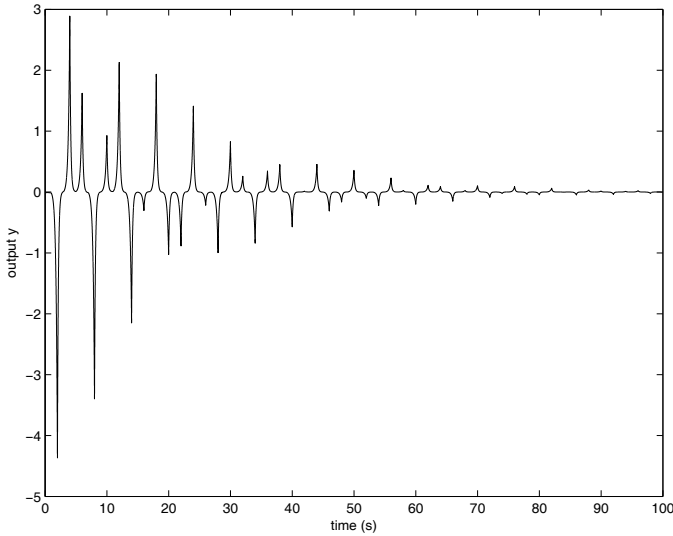


Fig. 1.4 Output response for $\delta = 0.9$ and a switching control law with $K_1 = -4.2200$ and $K_2 = 2.1200$

a high-frequency vibrational approach, which calls for the so-called *averaging technique*. However, this technique cannot be straightforwardly applied in this chocolate problem, since it concerns only systems with odd relative degree.

First-order Controller

The transfer function $G(s)$ of the chocolate system possesses a zero in $s = -1$. We now consider a first-order controller with a periodic gain $\alpha(t)$ and a pole located in $s = -1$ so as to cancel the mentioned zero. The overall control system takes on the structure depicted in Fig. 1.5 and the corresponding equations of the controller are:

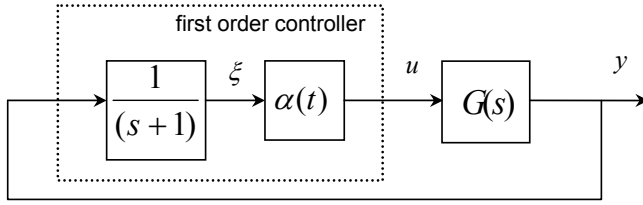
$$\begin{aligned}\dot{\xi} &= -\xi + y \\ u &= \alpha(t)\xi.\end{aligned}$$

In this way the original problem can be reformulated as that of designing a periodic memoryless gain $\alpha(\cdot)$ for the strictly proper system with transfer function

$$H(s) = \frac{s-1}{s^2 - 2\delta s + 1}.$$

Passing to a state-space description, the closed-loop matrix is:

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 - \alpha(t) & \alpha(t) + 2\delta \end{bmatrix}.$$

**Fig. 1.5** First order controller

By resorting to the differential operator σ , we can write the corresponding characteristic operator polynomial

$$p(\sigma, t) = \sigma^2 - (\alpha(t) + 2\delta)\sigma + 1 + \alpha(t). \quad (1.30)$$

We aim at finding a periodic function $\alpha(t)$ such that the characteristic exponents of $A(\cdot)$ are in the open left half-plane. As already observed in Sect. 1.2, once $p(\sigma, t)$ is written in the form

$$p(\sigma, t) = (\sigma - p_1(t))(\sigma - p_2(t)),$$

these characteristic exponents coincide with the average value of the periodic functions $p_1(t)$ and $p_2(t)$. On the other hand, recalling the commutative rule $\sigma p_2(t) = p_2(t)\sigma + \dot{p}_2(t)$, the above expression becomes

$$p(\sigma, t) = \sigma^2 - (p_1(t) + p_2(t))\sigma + p_2(t)p_1(t) - \dot{p}_2(t). \quad (1.31)$$

By comparing (1.30) with (1.31):

$$\alpha(t) + 2\delta = p_1(t) + p_2(t) \quad (1.32a)$$

$$\alpha(t) + 1 = p_2(t)p_1(t) - \dot{p}_2(t). \quad (1.32b)$$

Therefore, the problem is to find $\alpha(t)$ in such a way System (1.32) has a periodic solution $p_1(\cdot), p_2(\cdot)$ with

$$\text{av}(p_1) < 0, \quad \text{av}(p_2) < 0$$

where $\text{av}(\cdot)$ denotes the average value. In Eqs. (1.32), two unknown functions $p_1(t)$ and $p_2(t)$ appear. By expressing one of them as a function of the other one, the stability condition becomes

$$0 > \text{av}(p_2) \quad (1.33a)$$

$$0 > \text{av} \left(\frac{-1 + 2\delta - p_2 - \dot{p}_2}{1 - p_2} \right). \quad (1.33b)$$

and the set of stabilizing periodic functions $\alpha(t)$ s given by

$$\alpha(t) = \frac{-p_2(t)^2 + 2\delta p_2(t) - \dot{p}_2(t)}{1 - p_2(t)}. \quad (1.34)$$

One way to solve System (1.33) is to take as $p_2(t)$ a simple sinusoidal function of period $T = 1$ with average value satisfying (1.33a) and then to find the amplitude of such a sinusoid in order to satisfy (1.33b). In this way, a feasible parameter $p_2(t)$ is given by

$$p_2(t) = -0.3 - \varepsilon(\delta) - 1.3 \cos(2\pi t),$$

where $\varepsilon(\delta)$ is a small number depending on δ . In so doing, the closed-loop characteristic exponents are

$$\text{av}(p_2) = -0.3 - \varepsilon(\delta), \quad \text{av}(p_1) \approx -1.$$

For δ , we have considered the interval ranging from $\delta = 0.8$ to $\delta = 0.999$. It can be seen by simulation that $\varepsilon(\delta) = 0.00001$ can be chosen independent of δ in the entire given range. In the particular case $\delta = 0.999$, the function $\alpha(t)$ obtained from (1.34) is depicted in Fig. 1.6.

Notice that, being $H(s)$ with odd relative degree, the averaging technique proposed in [234] can be alternatively adopted to design a high-frequency gain $\alpha(t)$.

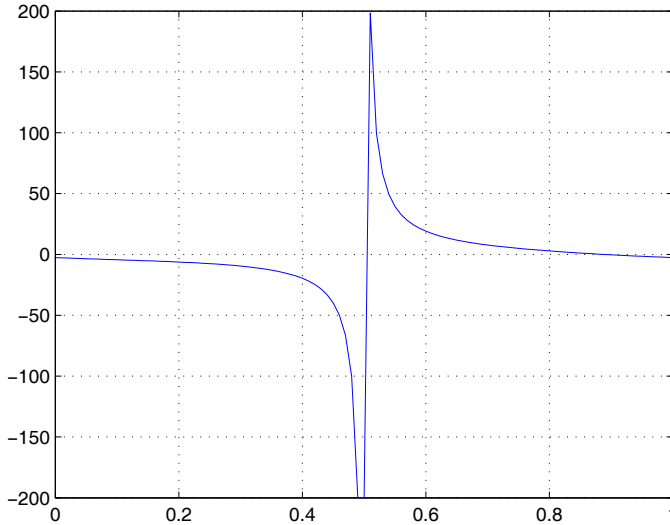


Fig. 1.6 The gain $\alpha(t)$ in a period interval ($0 \leq t \leq 1$) for $\delta = 0.999$

1.7.3 Periodic Optimization

One of the early applications of periodic control goes back to the 1970s and it is related to chemical plants regulation, [16]. Indeed, as stated in [15], “there is evidence that periodic operation [of catalytic reactors] can produce more reaction products or more valuable distribution of products, [and that] the production of wastes can perhaps be suppressed by cycling”. Ever since, the same idea has been further elaborated in other application fields, such as aeronautics [162, 274, 275], solar energy control [131], semi-conductor devices [1], and social and economic sciences [138, 139, 174, 236].

In continuous-time, the basic periodic optimization problem can be stated as follows. Consider the system

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t))\end{aligned}$$

subject to the periodicity constraint

$$x(T) = x(0),$$

and to further constraints of integral or pathwise type. The performance index to be maximized is

$$J(u(\cdot), T) = \frac{1}{T} \int_0^T g(y(t), u(t)) dt.$$

We will assume that the function $h(\cdot)$ is differentiable and functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are twice differentiable. If one limits the desired operation to steady-state conditions, then an algebraic optimization problem arises which can be tackled by mathematical programming techniques. More in detail, consider the problem in steady-state, namely when all variables are constant: $u(t) = \bar{u}$, $x(t) = \bar{x}$, $y(t) = h(\bar{x}) = \bar{y}$. Then the periodicity constraint $x(T) = x(0)$ is trivially satisfied and the performance index becomes

$$\bar{J} = g(h(\bar{x}, \bar{u})) \quad (1.35)$$

to be maximized with the constraint

$$f(\bar{x}, \bar{u}) = 0. \quad (1.36)$$

We will denote by \bar{u}^0 and \bar{x}^0 the values of \bar{u} and \bar{x} solving such a problem. Hence (\bar{u}^0, \bar{x}^0) is the optimal steady-state condition and $\bar{y}^0 = h(\bar{x}^0)$ is the corresponding optimal output. The associate performance index is $\bar{J}^0 = g(h(\bar{x}^0, \bar{u}^0))$.

When passing from steady-state to periodic operations, an important preliminary question is whether the optimal steady-state regime can be improved by cycling or not. A problem for which there exists a (truly) periodic operation with a better performance is said to be *proper*. To be precise, the optimization problem is proper if there exists a period \hat{T} and a control signal $\hat{u}(\cdot)$ such that the periodic solution of

period \hat{T} of the state equation, denoted by $\hat{x}(\cdot)$, is such that

$$J(\hat{u}(\cdot), \hat{T}) > J^0.$$

The issue of proper periodicity, originally dealt with in [62, 172], can be tackled by calculus of variation methods as follows: consider the perturbed signal

$$u(t) = \bar{u}^0 + \delta u(t), \quad (1.37)$$

where the periodic variation $\delta u(t)$ is expressed through its Fourier expansion:

$$\delta u(t) = \sum_{k=-\infty}^{\infty} U_k e^{jk\Omega t}, \quad \Omega = \frac{2\pi}{T}. \quad (1.38)$$

Define then the Hamiltonian function

$$H(x(t), u(t), \lambda(t)) = g(h(x(t), u(t))) + \lambda(t)' f(x(t), u(t)),$$

where $\lambda(t)$ is a n -dimensional vector. At steady-state, the non-linear programming problem of maximizing (1.35) with constraint (1.36) can be rephrased as the problem of solving a set of algebraic equations constituted by (1.36) and

$$H_x(\bar{x}, \bar{u}, \bar{\lambda}) = 0 \quad (1.39a)$$

$$H_u(\bar{x}, \bar{u}, \bar{\lambda}) = 0, \quad (1.39b)$$

for some $\bar{\lambda}$. These are the well-known *Lagrange conditions of optimality*. As a matter of fact, at the steady-state the Hamiltonian function is nothing but the Lagrange function and the elements of λ are the Lagrange multipliers. Overall, System (1.36), (1.39) is a set of $2n + m$ equations in $2n + m$ unknowns. Once a solution $\bar{x}^0, \bar{u}^0, \bar{\lambda}^0$ of this system is obtained, one computes

$$\begin{aligned} A &= f_x(\bar{x}^0, \bar{u}^0), \quad B = f_u(\bar{x}^0, \bar{u}^0) \\ P &= H_{xx}(\bar{x}^0, \bar{u}^0, \bar{\lambda}^0), \quad Q = H_{xu}(\bar{x}^0, \bar{u}^0, \bar{\lambda}^0), \quad R = H_{uu}(\bar{x}^0, \bar{u}^0, \bar{\lambda}^0). \end{aligned}$$

Then, the following fundamental function can be introduced:

$$\Pi(\omega) = G_x(-j\omega)' P G_x(j\omega) + Q' G_x(j\omega) + G_x(-j\omega)' Q + R,$$

where

$$G_x(s) = (sI - A)^{-1} B.$$

$\Pi(\omega)$ is a $n \times n$ matrix, obviously Hermitian since its conjugate transpose coincides with the matrix itself, i.e., $\Pi^*(\omega) = \Pi(\omega)$. It will be referred to as the “ Π matrix function”.

A simple computation, based on calculus of variation ideas, shows that, if $u(\cdot)$ is given by (1.37), (1.38), the second variation of the performance index is

$$\delta^2 J = \sum_{k=-\infty}^{\infty} U_k^* \Pi(k\Omega) U_k,$$

where U_k^* is the conjugate transpose of U_k . This expression is at the basis of the design. Suppose indeed that for some value $\hat{\Omega}$ of ω the matrix $\Pi(\omega)$ be positive definite, in other words

$$X^* \Pi(\hat{\Omega}) X > 0,$$

for some vector $X \neq 0$. Then take

$$\delta u(t) = \varepsilon (X e^{j\hat{\Omega}t} + X^* e^{-j\hat{\Omega}t}).$$

For ε small enough, the first variation of J around the optimal steady-state can be neglected whereas the second variation is

$$\delta^2 J = \varepsilon^2 X^* \Pi(\hat{\Omega}) X.$$

Being $\delta^2 J > 0$, the chosen $\delta u(\cdot)$ leads to a periodic regime that outperforms the results achievable at the optimal steady-state.

These considerations lead to the definition of *partial positivity* of the Π -function: $\Pi(\omega)$ is said to be partially positive if there exists a value of ω such that $\Pi(\omega)$ is positive definite. The partial positivity of the Π -function is a sufficient condition for the existence of an improving periodic regime. This is the so-called Π -condition.

A final remark concerns a graphical interpretation of the Π -condition when the system is linear and SISO:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t).$$

Denote by

$$p = g_{yy}(\bar{y}^0, \bar{u}^0), \quad q = g_{yu}(\bar{y}^0, \bar{u}^0), \quad r = g_{uu}(\bar{y}^0, \bar{u}^0).$$

Then, the previously introduced second-order derivatives of the Hamiltonian can be expressed as

$$P = C' C p, \quad Q = C q, \quad R = r,$$

and the proper periodicity condition becomes as follows. Assuming $p \neq 0$, define the circumference in the complex plane with center in the point q/p of the real axis and radius $\rho = \sqrt{(\frac{q}{p})^2 - \frac{r}{p}}$. Denote by \mathcal{L} the interior of such circle if $p < 0$ and the exterior if $p > 0$. \mathcal{L} includes the border of the region (as defined by the circumference). Then the proper periodicity condition can be stated as a *circle criterion* as follows: if the polar plot $\{G(j\omega) | \omega \geq 0\}$ of the transfer function $G(s)$ of the system enters \mathcal{L} then there is a periodic regime with a performance index better than the steady-state optimal one.

As pointed out in [172], this criterion is illustrated by the typical drawing of Fig. 1.7.3.

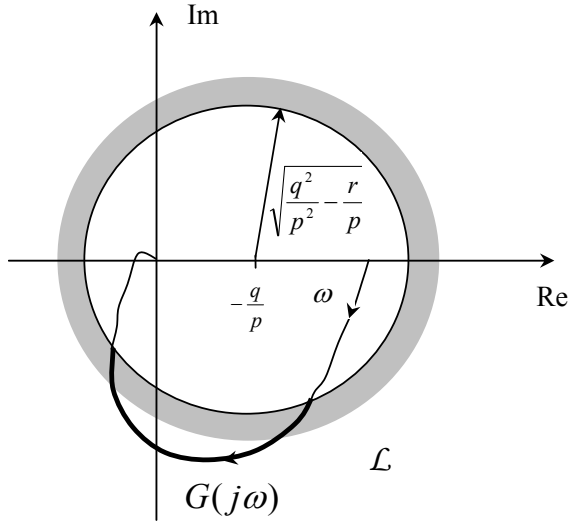


Fig. 1.7 The circle criterion

If $p = 0$, then the circle degenerates into the half plane at the right of the line parallel to the imaginary axis with real part $d = -\frac{r}{2q}$ if $q > 0$ or in the half plane at the left of such line if $q < 0$.

Example 1.3 *This example, originated in [18], and subsequently elaborated in [62], refers to the second-order linear system*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \alpha x_2 + u \\ y &= x_1\end{aligned}$$

with the transfer function

$$G(s) = \frac{1}{s^2 + \alpha s + 1}.$$

The performance index is

$$\begin{aligned}J &= \frac{1}{T} \int_0^T g(y(t), u(t)) dt \\ g(y, u) &= \phi(u) - \frac{1}{2} \phi(y) \\ \phi(u) &= 2u^2 - u^4 - \frac{1}{4}(u-1)^2.\end{aligned}$$

At the steady-state $\bar{x}_1 = \bar{u}$, $\bar{x}_2 = 0$, $\bar{y} = \bar{u}$.

The non-linear programming problem is that of maximizing

$$g(\bar{y}, \bar{u}) = \phi(\bar{u}) - \frac{1}{2}\phi(\bar{y})$$

with the constraint $\bar{y} = \bar{u}$. This leads to the curve of Fig. 1.8, with maximum at $\bar{u}^0 = 1$ and $J^0 = g(\bar{u}^0, \bar{u}^0) = 0.5$. This is the best performance achievable in steady-state conditions. As for the value of the second derivatives p , q , and r , from the expression of $g(y, u)$ it follows that $q = 0$, $p = -0.5$ and $r = 17/4$. Being $p > 0$, region \mathcal{L} is the exterior of the critical circumference having center in $-q/p = 0$ and radius $\rho = \sqrt{(\frac{q}{p})^2 - \frac{r}{p}} = \sqrt{2}$. In Fig. 1.9 the polar plot of $G(s)$ is represented for various values of parameter α . As can be verified, the polar plot enters the region \mathcal{L} for $\alpha < \sqrt{0.5}$. In Fig. 1.9, the bold curve represents the polar plot for $\alpha = 0.5$. This curve intersects the circumference for $\omega = 0.44$ and $\omega = 1.26$. ■

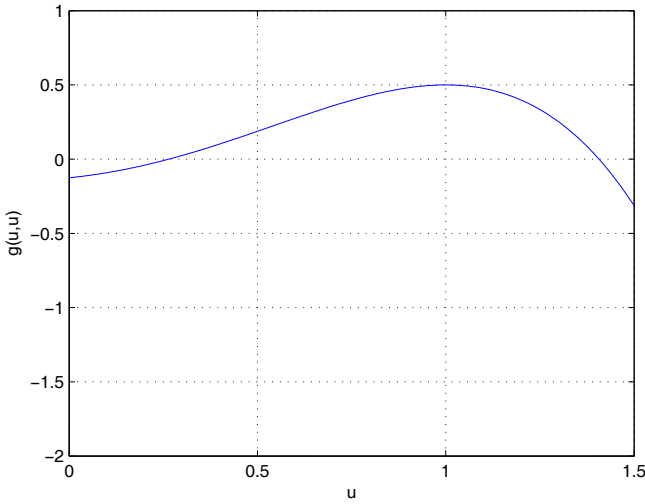


Fig. 1.8 The function $g(\bar{u}, \bar{u})$

Variational tools have been used in periodic control by many authors, see [72, 163, 276, 297, 298]. Moreover, along a similar line, it is worth mentioning the area of *vibrational control*, see [218], also dealing with the problem of forcing a time-invariant system to undertake a periodic movement in order to achieve a better stabilization property or a better performance specification.

In general, if one leaves the area of weak-variations, periodic optimization problems do not admit closed-form solutions. There is, however, a notable exception, as

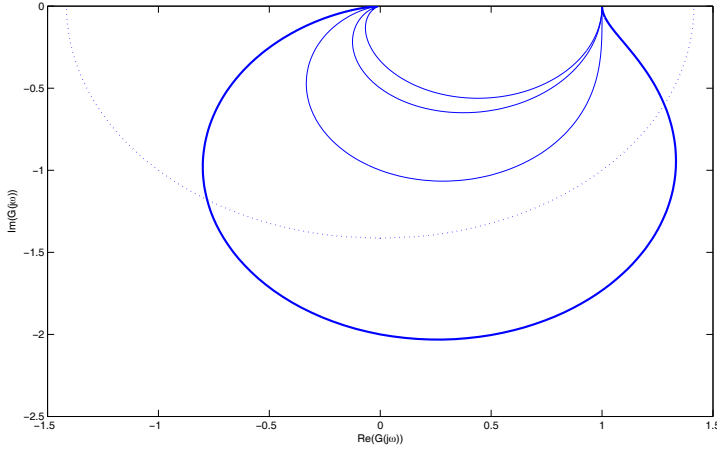


Fig. 1.9 The circle criterion applied in Example 1.3

pointed out in [63, 66, 67]. With reference to a linear system, if the problem is that of maximizing the output power

$$J = \frac{1}{T} \int_0^T y(t)' y(t) dt,$$

under the periodicity state constraint

$$x(0) = x(T),$$

and the input power constraint

$$\frac{1}{T} \int_0^T u(t)' u(t) dt \leq 1,$$

then, the optimal input function can be given an explicit expression as a sinusoidal signal of suitable frequency. In the single-input single-output case, denoting by $G(s)$ the system transfer function, the optimal frequency $\hat{\Omega}$ is that associated with the peak value of the Bode diagram, i.e.,

$$|G(j\hat{\Omega})| \geq |G(j\omega)|, \quad \forall \omega.$$

(In modern jargon, $\hat{\Omega}$ is the value of the frequency associated with the H_∞ norm of the system). In particular, the problem is proper if $\hat{\Omega} > 0$. Otherwise, the optimal steady-state operation cannot be improved by cycling.

1.7.4 Periodic Control of Non-linear Time-invariant Systems

Since the celebrated paper [86], it is well-known that there is a class of non-linear time-invariant systems that cannot be stabilized by a continuous-time-invariant state-feedback control law. The simplest example of such class is the following third-order system (known as Brockett integrator)

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1,\end{aligned}$$

where x_i are the state variables and u_j the exogenous inputs. Notice that this system can also be written in the form

$$\begin{aligned}\dot{x} &= u \\ \dot{y} &= xv \\ \dot{z} &= v,\end{aligned}$$

with the position

$$x = x_2, \quad y = 0.5(x_1 x_2 - x_3), \quad z = x_1.$$

While it is not possible to stabilize the system with a time-invariant continuous control law, it has also been proven that stabilization can be achieved either by means of a non-smooth control law or by means of a time-varying control law. As a matter of fact, the time-varying stabilizing controllers proposed in the literature are indeed periodic, [14, 116–118, 235, 264]. The relationship between discontinuous controllers and periodic controllers is studied in [118]. In particular, the origin of the state-space of the Brockett integrator is globally stabilized by means of the control law

$$u = y \sin(t) - (x + y \cos(t)), \quad v = -x(x + y \cos(t)) \cos(t) - (xy + x).$$

In general, it is possible to characterize a class of non-linear time-invariant systems (of order $n \geq 4$) which are stabilizable by means of periodic feedbacks, [116, 117].

1.8 Periodic Control of Periodic Systems

A typical way to control a plant described by a linear periodic model is to impose

$$u(t) = K(t)x(t) + S(t)v(t),$$

where $K(\cdot)$ is a periodic feedback gain ($K(t+T) = K(t)$, for each t), $R(\cdot)$ is a periodic feed-forward gain ($R(t+T) = R(t)$, for each t) and $v(t)$ is a new exogenous

signal. The associated closed-loop system is then

$$\dot{x}(t) = (A(t) + B(t)K(t))x(t) + B(t)S(t)v(t).$$

In particular, the closed-loop dynamic matrix is the periodic matrix $A(t) + B(t)K(t)$.

The main problems considered in the literature are:

1. *Stabilization.* Find a periodic feedback gain in such a way that the closed-loop system is stable (any $K(\cdot)$ meeting such a requirement is named stabilizing gain).
2. *Pole assignment:* Find a periodic feedback gain so as to position the closed-loop characteristic multipliers in given locations in the complex plane.
3. *Optimal control.* Set $v(\cdot) = 0$ and find a periodic feedback gain so as to minimize the quadratic performance index

$$J = \int_0^\infty z(t)'z(t)dt,$$

where $z(\cdot)$ is the “performance evaluation variable”. A typical choice is to assume a linear dependence of $z(\cdot)$ upon $x(\cdot)$ and $u(\cdot)$:

$$z(\tau) = E(\tau)x(\tau) + N(\tau)u(\tau).$$

4. *Invariantization.* Find a feedback control law so that the closed-loop system is time-invariant up to a periodic state–space coordinate change.
5. *Exact model matching.* Let $y(t) = C(t)x(t)$ be a system output variable. Find a feedback control law in a way that the closed-loop input–output relation (from $v(\cdot)$ to $y(\cdot)$) matches the input–output behavior of a given periodic system.
6. *Tracking and regulation.* Find a periodic controller in order to guarantee closed-loop stability and robust zeroing of the tracking errors for a given class of reference signals.

The above list of problems does not have any claim of completeness. In particular, it does not include some recent developments based on the mathematical machinery pertaining to the theory of non-linear control systems, see e.g. [269]. We now briefly elaborate on the above problems by surveying the main results available in the literature.

1.8.1 Stabilization via State-feedback

As a paradigm problem in control, the *stabilization* issue is the starting point of further performance requirement problems. A general parametrization of all periodic stabilizing gains can be worked out by means of a suitable matrix inequality. To be precise, by making reference to continuous-time systems, the *Lyapunov inequality condition* seen in Sect. 1.2 enables one to conclude that the closed-loop system associated with a periodic gain $K(\cdot)$ is stable if and only if there exists a positive definite

periodic matrix $P(\cdot)$ satisfying $\forall t$ the inequality:

$$\dot{P}(t) > (A(t) + B(t)K(t))P(t) + P(t)(A(t) + B(t)K(t))'.$$

Then, it is possible to show that a periodic gain is stabilizing if and only if it can be written in the form:

$$K(t) = W(t)'P(t)^{-1},$$

where $W(\cdot)$ and $P(\cdot)$ are periodic matrices (of dimensions $m \times n$ and $n \times n$, respectively) solving the matrix inequality:

$$\dot{P}(t) > A(t)P(t) + P(t)A(t)' + B(t)W(t)' + W(t)B(t)'.$$

The *pole assignment problem* (by state-feedback) is somehow strictly related to the *invariantization problem*. Both problems have been considered in the early paper [89], where continuous-time systems are treated, and subsequently in [101, 182, 210]. The basic idea is to render the system algebraically equivalent to a time-invariant one by means of a first periodic state-feedback (invariantization), and then to resort to the pole assignment theory for time-invariant systems in order to locate the characteristic multipliers. Thus the control scheme comprises two feedback loops, the inner for invariantization, and the outer for pole-placement. Analogous considerations can be applied in the discrete-time case, with some care for the possible non-reversibility of the system.

The *model matching* and the *tracking* problems are dealt with in [108] and [170], respectively.

1.8.2 LQ Optimal Control

As for the vast area of optimal control, the attention focuses herein on two main design methodologies, namely (1) linear quadratic control and (2) receding horizon control. Both will be presented by making reference to continuous-time.

The classical *finite horizon* optimal control problem is that of minimizing the quadratic performance index over the time interval (t, t_f) :

$$J(t, t_f, x_t) = x(t_f)'Q_{t_f}x(t_f) + \int_t^{t_f} z(\tau)'z(\tau)d\tau,$$

where x_t is the system initial state at time t , $Q_{t_f} \geq 0$ is the matrix weighting the final state $x(t_f)$. Considering that the second term of $J(t, t_f, x_t)$ is the “energy” of $z(\cdot)$, the definition of such a variable reflects a main design specification. As for $z(t)$ we assume

$$z(t) = E(t)x(t) + N(t)u(t),$$

where $E(t)$ and $N(t)$ are to be tuned by the designer. In this way, the performance index can also be written in the (perhaps more popular) form

$$J(t, t_f, x_t) = x(t_f)' Q_{t_f} x(t_f) + \int_t^{t_f} \{x(\tau)' E(\tau)' E(\tau) x(\tau) + 2u(\tau)' N(\tau)' E(\tau) x(\tau) + u(\tau)' N(\tau)' N(\tau) u(\tau)\} d\tau.$$

We will assume for simplicity that the problem is non-singular, i.e., $N(\tau)' N(\tau) > 0$, $\forall \tau$.

This problem is known as the *linear quadratic* (LQ) optimal control problem. To solve it, the auxiliary matrix equation

$$-\dot{Q}(t) = \tilde{A}(t)' Q(t) + Q(t) \tilde{A}(t) - Q(t) B(t) (N(t)' N(t))^{-1} B(t)' Q(t) + \tilde{E}(t)' \tilde{E}(t),$$

where

$$\tilde{A}(t) = A(t) - B(t) R(t)^{-1} S(t), \quad \tilde{E}(t)' \tilde{E}(t) = Q(t) - S(t)' R(t)^{-1} S(t)$$

is introduced. Notice that the matrix $Q(t) - S(t)' R(t)^{-1} S(t)$ is positive semi-definite for all t so that the definition of $\tilde{E}(t)$ makes sense. The above equation is the celebrated *differential Riccati equation*, in one of its many equivalent forms. To be precise, since the coefficients are periodic, the equation is referred to as the *periodic differential Riccati equation*.

Let $\bar{Q}(\cdot, t_f)$ be the backward solution of this equation with terminal condition $\bar{Q}(t_f, t_f) = Q_{t_f}$. Then, assuming that the state $x(\cdot)$ can be measured, the solution to the minimization problem can be easily written in terms of $\bar{Q}(\cdot, t_f)$ as follows

$$u(\tau) = \Lambda_{opt}(\tau, t_f) x(\tau),$$

where

$$\Lambda_{opt}(\tau, t_f) = -(N(\tau) N(\tau))^{-1} (B(\tau)' \bar{Q}(\tau, t_f) + N(\tau)' E(\tau))$$

Moreover, the value of the performance index associated with the optimal solution is

$$J^o(t, t_f, x_t) = x_t' \bar{Q}(t, t_f) x_t.$$

The passage from the finite horizon to the *infinite horizon* problem ($t_f \rightarrow \infty$) can be performed provided that $\bar{Q}(t, t_f)$ keeps bounded for each $t_f > t$ and converges as $t_f \rightarrow \infty$, i.e., there exists $Q(t)$ such that, $\forall t$

$$\lim_{t_f \rightarrow \infty} \bar{Q}(t, t_f) = Q(t).$$

Under suitable assumptions concerning the matrices $(A(\cdot), B(\cdot), Q(\cdot), S(\cdot))$, the limit matrix $P(\cdot)$ exists and is the unique positive semi-definite and T -periodic solution of the periodic differential Riccati equation.

The optimal control action is given by

$$u(\tau) = K_{opt}(\tau)x(\tau),$$

where $K_{opt}(\tau)$ is the periodic matrix obtained from $\Lambda_{opt}(\tau, t_f)$ by letting $t_f \rightarrow \infty$. Finally, the optimal infinite horizon performance index takes on the value

$$\lim_{t_f \rightarrow \infty} J^o(t, t_f, x_t) = x_t' Q(t) x_t.$$

If the state is not accessible, one has to rely instead on the measurable output

$$y(t) = C(t)x(t) + D(t)u(t)$$

A first task of the controller is then to determine the actual value of the state $x(t)$ from the past observation of $y(\cdot)$ and $u(\cdot)$ up to time t . This leads to the problem of finding an estimate $\hat{x}(t)$ of $x(t)$ as the output of a linear system (*filter*) fed by the available measurements. The design of such a filter can be carried out in a variety of ways, among which it is advisable to mention the celebrated *Kalman filter*, the implementation of which requires the solution of another matrix Riccati equation with periodic coefficients. Once $\hat{x}(t)$ is available, the control action is typically obtained as

$$u(\tau) = K_{opt}(\tau)\hat{x}(\tau).$$

Thus, the controller takes the form of a cascade of two blocks, as can be seen in Fig. 1.10. Periodic optimal filtering and control problems for periodic systems have been intensively investigated, see [23, 54, 55, 70, 91, 114, 128, 181, 207, 270]. For the numerical issue see e.g., [180].

1.8.3 Receding Horizon Periodic Control

The infinite horizon optimal control law can be implemented provided that the periodic solution of the matrix Riccati equation is available. Finding such a solution may be computationally demanding so that the development of simpler control design tools has been considered. Among them, an interesting approach is provided by the so-called *receding horizon* control strategy, which has its roots in optimal control theory and has remarkable connections with the field of adaptive and predictive control, see [27, 99]. Among the many research streams considered in such a context, the periodic stabilization of time-invariant systems was dealt with in [216] under the heading of “intervalwise receding horizon control”, see also [27], and then the approach extended to periodic systems [124]. The problem can be stated as follows. Consider the optimal control problem with $N(\cdot)'E(\cdot) = 0$ and $N(\cdot)'N(\cdot) = I$, and write the performance index over the interval $(t_f - T, t_f)$ as

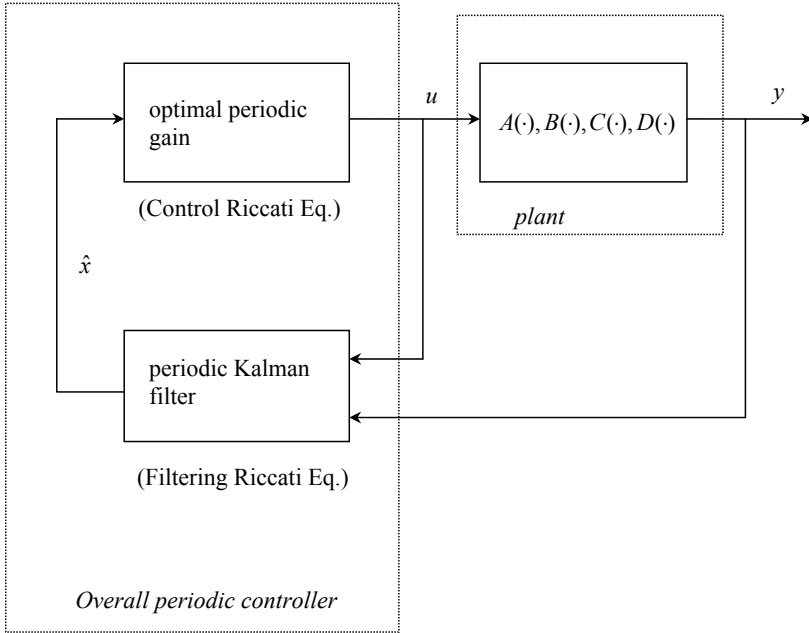


Fig. 1.10 The optimal controller structure

$$J(t, t_f, x_t) = x(t_f)' Q_{t_f} x(t_f) + \int_{t_f-T}^{t_f} \{x(\tau)' E(\tau)' E(\tau) x(\tau) + u(\tau)' u(\tau)\} d\tau.$$

Assume that the solution $\bar{Q}(\cdot, t_f)$ of the Riccati equation with terminal condition $\bar{Q}(t_f, t_f) = Q_{t_f}$ is such that

$$Q_{t_f} - \bar{Q}(t_f - T, t_f) \geq 0.$$

This condition is usually referred to as cyclomonotonicity condition. Now, consider the periodic extension $Q_e(\cdot)$ of $\bar{Q}(\cdot, t_f)$, i.e.,

$$Q_e(t + kT) = \bar{Q}(t, t_f), \quad t \in (t_f - T, t_f], \quad \forall \text{ integer } k.$$

Then, under mild assumptions on $(A(\cdot), B(\cdot), Q(\cdot))$ it turns out that the receding horizon control law

$$u(\tau) = -B(\tau)' Q_e(\tau) x(\tau)$$

is stabilizing. Although such a control law is suboptimal, it has the advantage of requiring the integration of the Riccati equation over a finite interval (precisely over an interval of length T , which has to be selected by the designer in the case of time-invariant plants, and coincides with the system period – or a multiple – in the periodic case). However, for feedback stability it is fundamental to check if the

cyclomonotonicity condition is met. If not, one is led to consider a different selection of matrix Q_{t_f} . The simplest way is to choose Q_{t_f} “indefinitely large”, ideally such that $Q_{t_f}^{-1} = 0$. Indeed, as is well-known in optimal control theory, such a condition guarantees that the solution of the differential Riccati equation enjoys the required monotonicity property.

Optimal control ideas have been used in a variety of contexts, and have been adequately shaped for the needs of the specific problem dealt with. In particular *ad hoc* control design techniques have been developed for the rejection or attenuation of periodic disturbances, a problem of major importance in the emerging field of active control of vibrations and noise, see [11, 73].

1.8.4 H_∞ Periodic Control

In the classical linear quadratic periodic control problem introduced above, the objective was the minimization of a mean-square performance criterion, based on the “performance evaluation variable” $z(t)$. An alternative measure of performance, which takes into account the requirement of robustness of the closed-loop system, is the so-called H_∞ criterion. To introduce such a criterion, the system dynamic equation is conveniently modified by introducing a disturbance $w(\cdot)$, with unspecified characteristics, as follows

$$\dot{x}(t) = A(t)x(t) + B_w(t)w(t) + B(t)u(t).$$

Moreover, for an initial time point, denoted again by t for simplicity, let the initial state $x(t)$ be zero, i.e., $x(t) = 0$. The rationale underlying H_∞ control is to achieve a satisfactory bound for the ratio between the energy of the performance variable $z(\cdot)$ and the energy of the disturbance $w(\cdot)$. In this direction, a possibility is to consider the following differential game cost index

$$J_\infty(t, t_f) = x(t_f)' P_{t_f} x(t_f) + \int_t^{t_f} (z(\tau)' z(\tau) - \gamma^2 w(\tau)' w(\tau)) d\tau,$$

where, as in the position of the optimal control problem, $Q_{t_f} \geq 0$. Moreover, γ is a positive scalar.

In optimal control, the design corresponds to a minimization problem. In the H_∞ framework, the objective is rather that of rendering $J_\infty(t, t_f) < 0$ for all possible disturbances $w(\cdot)$. Apart from the term penalizing the final condition $x(t_f)$, this objective corresponds to the wish of attenuating the effect of disturbance $w(\cdot)$ on the performance evaluation variable $z(\cdot)$. In this regard, parameter γ plays a major role in order to achieve a suitable tradeoff between the two energies, the one associated with $w(\cdot)$ and the one associated with $z(\cdot)$. Actually, the disturbance $w(\cdot)$ is unspecified, so that the real problem is to ensure $J_\infty(t, t_f) < 0$ in the *worst case*, namely for the disturbance maximizing the cost. This is a typical situation encountered in

differential game theory, where two players have conflicting objectives, leading to the so-called *min-max* techniques.

The determination of a solution of this H_∞ problem calls for the matrices $\tilde{A}(t)$, $\tilde{E}(t)' \tilde{E}(t)$ and $N(t)' N(t)$ previously defined. Furthermore, it requires the consideration of the modified periodic Riccati differential equation

$$-\dot{\bar{Q}}(t) = \tilde{A}(t)' \bar{Q}(t) + \bar{Q}(t) \tilde{A}(t) - \bar{Q}(t) B(t) (N(t) N(t))^{-1} B(t)' \bar{Q}(t) + \gamma^{-2} \bar{Q}(t) B_w(t) B_w(t)' \bar{Q}(t) + \tilde{E}(t)' \tilde{E}(t).$$

Let $\bar{Q}(\cdot, t_f)$ be the backward solution of this equation with final condition $\bar{Q}(t_f, t_f) = Q_{t_f}$. A solution of the problem is given by

$$u(\tau) = \Lambda_\infty(\tau, t_f) x(\tau), \quad \tau \in [t, t_f],$$

where

$$\Lambda_\infty(\tau, t_f) = -(N(\tau) N(\tau))^{-1} (B(\tau)' \bar{Q}(\tau, t_f) + N(\tau)' E(\tau)), \quad \tau \in [t, t_f].$$

Correspondingly,

$$w^*(\tau) = \frac{B_w(\tau) \bar{Q}(\tau, t_f)}{\gamma^2} x(\tau), \quad \tau \in [t, t_f]$$

is the *worst-case* disturbance.

Note that, different from the optimal control problem, $\bar{Q}_\infty(\tau, t_f)$ may diverge in finite time. In that case, the problem does not admit any solution.

Finally, we discuss the infinite horizon problem ($t_f \rightarrow \infty$). For, assume that $\bar{Q}(\tau, t_f)$ exists for all $\tau \leq t_f$ and that, for each t , there exists

$$Q(t) = \lim_{t_f \rightarrow +\infty} \bar{Q}(t, t_f).$$

Then, $Q(t)$ is a T -periodic solution of the periodic modified Riccati (game differential) equation. Under mild assumptions this solution is stabilizing and

$$u(\tau) = \Lambda_\infty(\tau) x(\tau), \quad \Lambda_\infty(\tau) = \lim_{t_f \rightarrow +\infty} \Lambda_\infty(\tau, t_f)$$

provides a solution, named the *central solution*. The interpretation of this result is that, if $w(\cdot)$ is constrained to be a square integrable signal, the associated induced norm of the operator from $w(\cdot)$ to $z(\cdot)$ is less than γ . Alternatively, by resorting to a suitable frequency-domain definition of the H_∞ norm of a periodic system, one can say that the closed-loop periodic system has an H_∞ norm lower than γ . For further details, see [103, 104].

The rationale underlying the design of robust controllers via H_∞ optimization has been shown to be useful to cope with a variety of control problems, for instance, in

the Linear Parameter Varying (LPV) design methodology for the control of linear systems characterized by measurable time-varying parameters.

A number of further issues concerning linear periodic systems in continuous-time, such as the Bode sensitivity formula and the entropy measure, are addressed in [102].

1.8.5 Periodic Control of Non-linear Periodic Systems

For the control of non-linear periodic systems, various models have been considered. Among them, we focus the attention on a model constituted of a linear periodic system with a non-linear (possibly time-varying) feedback. To be precise, the linear system is described by the usual equations

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t),\end{aligned}$$

and the non-linear feedback is given by a non-linear memoryless characteristic

$$u(t) = \phi(t, y(t)),$$

or an hysteretic mechanism

$$u(t) = \phi(t, \{y(\tau), \tau \in [0, t]\}).$$

The general theory is treated in [309–311]. Here we limit ourselves to the simplest case where the system has scalar input and scalar output and the feedback is a memoryless characteristic $\phi(t, y)$ belonging to a sector, namely

$$\mu_1(t)y \leq \phi(t, y) \leq \mu_2(t)y.$$

The coefficients $\mu_i(t)$, $i = 1, 2$, are T -periodic and define a sector in which the non-linear memoryless characteristic $\phi(t, y)$ must lie for each t . Note that $\phi(t, y)$ is not required to be periodic in t .

Since $\phi(t, 0) = 0$, the origin of the state-space is an equilibrium point of the feedback system. The question is to assess the global asymptotic stability of the equilibrium point, whatever the characteristic $\phi(t, y)$ in the given sector may be. This type of stability is known as *absolute stability*. The solution of such a problem calls for the definition of the Hamiltonian matrix:

$$H(t) = \begin{bmatrix} A(t) + \frac{1}{2}(\mu_1(t) + \mu_2(t))B(t)C(t)' & B(t)B(t)' \\ -\frac{1}{4}(\mu_1(t) - \mu_2(t))^2 C(t)C(t)' & -A(t)' - \frac{1}{2}(\mu_1(t) + \mu_2(t))C(t)B(t)' \end{bmatrix}.$$

This is a periodic matrix of dimension $2n \times 2n$, which enjoys the Hamiltonian property

$$H(t)J + JH(t)' = 0, \quad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Denoting by Ψ_H the monodromy matrix of $H(\cdot)$ at 0, namely $\Psi_H = \Phi_H(T, 0)$, and recalling that in continuous-time the transition matrix is invertible, from this formula it follows that

$$\Psi_H = J^{-1} (\Psi_H')^{-1} J.$$

Therefore, the monodromy matrix and its inverse are similar matrices. Consequently, the characteristic multipliers of $H(\cdot)$ are reciprocal, that is, if λ is a characteristic multiplier, then λ^{-1} is a characteristic multiplier as well. Therefore, if no characteristic multipliers lie on the unit circle, the set of characteristic multipliers can be partitioned in two subsets, by splitting the *stable multipliers* ($|\lambda| < 1$) from the *unstable multipliers* ($|\lambda| > 1$). Define then a n -dimensional subspace \mathcal{V} formed by all (generalized) eigenvectors associated with the stable characteristic multipliers of $H(\cdot)$. This subspace can be written as

$$\mathcal{V} = sp \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix},$$

where \bar{X} and \bar{Y} are $n \times n$ matrices. The various column vectors of matrix

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}$$

constitute a basis for \mathcal{V} . Let $\tilde{\Psi}_H$ be the restriction of Ψ_H with respect to this basis, i.e.,

$$\Psi_H \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} = \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} \tilde{\Psi}_H.$$

By definition, the $n \times n$ matrix $\tilde{\Psi}_H$ has all eigenvalues inside the open unit disk. Now, define a T -periodic matrix $\tilde{H}(t)$ whose monodromy matrix coincides with $\tilde{\Psi}_H$ and let

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \Phi_H(t, 0) \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix} \Phi_{\tilde{H}}(0, t).$$

It turns out that this matrix is T -periodic.¹⁰

¹⁰ This matrix satisfies the differential equation

$$H(t) \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \tilde{H}(t) + \begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix},$$

which defines the restriction $\tilde{H}(\cdot)$ of $H(\cdot)$ with respect to the parametrization chosen for the T -periodic subspace generated by the matrix itself. Indeed, a T -periodic subspace $\mathcal{R}(t)$ is said to be invariant with respect to the linear T -periodic transformation $H(\cdot)$ if $H(t)\mathcal{R}(t) \subset \mathcal{R}(t) + \dot{\mathcal{R}}(t)$.

We are now in the position to state the result proven in [311]. The non-linear periodic feedback system is absolutely stable if

- (i) matrix $H(\cdot)$ does not have characteristic multipliers on the unit circle
- (ii) $X(t)$ is non-singular for each t .

1.9 Stochastic Periodic Signals

If the input $u(\cdot)$ in the state-space or input-output models is a white-noise, then $y(\cdot)$ is a stochastic process. Under suitable assumptions, $y(\cdot)$ converges to a *cyclostationary process*. A cyclostationary process is a stochastic process whose mean $m(t) = \mathcal{E}[y(t)]$ is periodic, $m(t) = m(t+T)$, and whose covariance function $\Gamma(t, \tau) = E[(y(t) - m(t))(y(\tau) - m(\tau))']$ satisfies the bi-periodicity condition $\Gamma(t+T, \tau+T) = \Gamma(t, \tau)$. In particular, its variance $\gamma(t, t)$ is T -periodic. The study of these processes in discrete-time is carried out in Chap. 11. Here we briefly discuss the continuous-time counterpart.

The periodic Lyapunov equation serves as a fundamental tool in the analysis of these processes. Indeed, consider System (1.4a) and assume that the initial state $x(\tau)$ is a random variable with zero mean and covariance matrix \bar{P}_τ , i.e., $\mathcal{E}[x(\tau)] = 0$ and $\mathcal{E}[x(\tau)x(\tau)'] = \bar{P}_\tau$. Moreover, we assume that $u(\cdot)$ is a white-noise process, independent of $x(0)$, with zero mean and unitary intensity, i.e., $\mathcal{E}[u(t)] = 0$ and $\mathcal{E}[u(t)u(\tau)'] = \delta(t - \tau)I$, where $\delta(\cdot)$ is the impulse (Dirac) function. Then, from the Lagrange equation

$$\dot{x}(t) = \Phi_A(t, \tau)x(\tau) + \int_\tau^t \Phi_A(t, \sigma)B(\sigma)u(\sigma)d\sigma,$$

it is easy to verify that the stochastic process $x(\cdot)$ has zero mean. Moreover, its covariance matrix is given by

$$\Gamma(t, \tau) = \mathcal{E}[x(t)x(\tau)'] = \Phi_A(t, \tau)\Gamma(\tau, \tau) + \int_\tau^t \Phi_A(t, \sigma)B(\sigma)\mathcal{E}[u(\sigma)x(\tau)']d\sigma,$$

so that

$$\Gamma(t, \tau) = \Phi_A(t, \tau)\Gamma(\tau, \tau).$$

This expression provides the process covariance at time points (τ, t) , i.e., $\Gamma(\tau, t)$, as a function of the variance matrix $\Gamma(\tau, \tau)$. Notice that, should the variance $\Gamma(\tau, \tau)$ be periodic in τ , $\Gamma(t, \tau)$ would be bi-periodic in (t, τ) , i.e., $\Gamma(t+T, \tau+T) = \Gamma(t, \tau)$.

As for the variance, again from the Lagrange formula

$$\begin{aligned}\Gamma(t, t) &= \Phi_A(t, \tau) \Gamma(\tau, \tau) \Phi_A(t, \tau)' \\ &+ \int_{\tau}^t \Phi_A(t, \sigma) B(\sigma) \mathcal{E}[u(\sigma) x(\tau)'] \Phi_A(t, \tau)' d\sigma \\ &+ \int_{\tau}^t \Phi_A(t, \tau) \mathcal{E}[x(\tau) u(\sigma)'] B(\sigma)' \Phi_A(t, \sigma)' d\sigma \\ &+ \int_{\tau}^t \int_{\tau}^t \Phi_A(t, \sigma_1) B(\sigma_1) \mathcal{E}[u(\sigma_1) u(\sigma_2)'] B(\sigma_2)' \Phi_A(t, \sigma_2)' d\sigma_1 d\sigma_2.\end{aligned}$$

Assuming that $u(\cdot)$ is independent upon the initial state $x(\tau)$, then $\mathcal{E}[u(s)x(\tau)'] = 0$, $s \geq \tau$. Moreover, $\mathcal{E}[u(s_1)u(s_2)'] = \delta(s_1 - s_2)I$, so that

$$\Gamma(t, t) = \Phi_A(t, \tau) \Gamma(\tau, \tau) \Phi_A(t, \tau)' + \int_{\tau}^t \Phi_A(t, s) B(s) B(s)' \Phi_A(t, s)' ds. \quad (1.40)$$

Now, letting

$$P(t) = \Gamma(t, t),$$

the derivative of the two members of Eq. (1.40) brings it to

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)' + B(t)B(t)'. \quad (1.41)$$

This is a differential Lyapunov equation, to be integrated with the initial condition

$$P(\tau) = \bar{P}_{\tau}.$$

We now assume that the periodic system is stable. Then, as $\tau \rightarrow -\infty$ the first term in the right-hand side of Eq. (1.40) tends to zero, whereas the second term converges to a periodic matrix $\tilde{P}(t)$. Precisely,

$$\tilde{P}(t) = \int_{-\infty}^t \Phi_A(t, s) B(s)' B(s) \Phi_A(t, s)' ds.$$

In conclusion, the state of the periodic system asymptotically converges to a zero mean cyclostationary process with variance matrix $P(t)$, which can be computed by finding the periodic solution of the periodic differential Lyapunov equation.

$$\lim_{t \rightarrow \infty} \{\Gamma(t, t) - \tilde{P}(t)\} = 0.$$

In practice, such a solution can be determined by computing first the so-called *periodic generator*. Precisely, by letting in (1.40) $\tau = 0$ and $t = T$, and imposing the periodicity constraint $\Gamma(T, T) = P(T) = \Gamma(0, 0) = P(0)$, it turns out that

$$P(0) = \Psi_A(0)P(0)\Omega(0)' + \int_0^T \Phi_A(T, s) B(s)' B(s) \Phi_A(T, s)' ds.$$

This is a discrete-time Lyapunov equation. Thanks to the stability condition, this equation admits a unique solution, which is positive semi-definite. This solution coincides with $\tilde{P}(0)$.

Chapter 2

Linear Periodic Systems

In this chapter, we set the basis for the study of linear periodic systems in discrete-time. We start with the impulse response description, thanks to which the notion of the periodic transfer operator is introduced. Then we move to the periodic input–output difference equation representation (PARMA model). This representation plays a major role in data analysis and periodic systems identification. The characterization in the frequency domain is carried out by means of the concept of exponentially modulated periodic signals. Finally, the state–space representation is discussed and related to the input–output descriptions. Historically, in the literature of the field, the state–space description has received more attention. However, herein we initially adopt an input–output viewpoint for reasons of generality, and postpone to later sections the state–space realm.

2.1 The Periodic Transfer Operator

In the description of linear dynamical systems, a most used model makes reference to the basic causal relationships supplying the output as a linear combination of past values of the input up to the current time instant. In discrete-time, by denoting with t the (integer) time variable, $u(t) \in \mathbb{R}^m$ the input and $y(t) \in \mathbb{R}^p$ the output, this amounts to writing:

$$\begin{aligned} y(t) &= M_0(t)u(t) + M_1(t)u(t-1) + M_2(t)u(t-2) + M_3(t)u(t-3) + \cdots \\ &= \sum_{j=0}^{\infty} M_j(t)u(t-j). \end{aligned} \tag{2.1}$$

The matrix coefficients $M_i(t)$, $i = 0, 1, \dots$, are known as *Markov coefficients* and completely capture the input–output behavior of the system. These Markov parameters are related to the impulsive response of the system.

Indeed, denote by $\delta(t)$ the impulsive function, i.e.,

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

When the input is a scalar variable ($m = 1$), from (2.1) it turns out that the response $y_{imp}^{(i)}(t)$ produced by $u(t) = \delta(t - i)$ is given by

$$y_{imp}^{(i)}(t) = M_{t-i}(t).$$

To be precise, this is the output at time t when the input is an impulse applied at time $t = i$. In the general multi-input case, the j -th column of the Markov coefficients $M_{t-i}(t)$ represents the output response of the system to an input with all elements set to zero except the j -th entry $u_j(t)$ which is an impulse applied at time $t = i$.

Herein, we deal with *periodic systems*, namely systems for which there exists a positive integer T such that

$$M_i(t + T) = M_i(t)$$

for each t and for each $i \geq 0$. Obviously, this class of periodic systems includes the class of time-invariant ones, characterized by constant Markov coefficients.

A compact notation is obtained by introducing the concept of *periodic transfer operator*. To this purpose, introduce first the *unit delay operator* σ^{-1} . With reference to a generic discrete-time signal $v(t)$, this operator acts as follows:

$$\sigma^{-1} \cdot v(t) = v(t - 1).$$

Then, Expression (2.1) can be given the form

$$y(t) = G(\sigma, t) \cdot u(t),$$

where the periodic transfer operator $G(\sigma, t)$ is defined as

$$G(\sigma, t) = M_0(t) + M_1(t)\sigma^{-1} + M_2(t)\sigma^{-2} + M_3(t)\sigma^{-3} + \dots \quad (2.3)$$

Notice that for a given t in the interval $[0, T - 1]$, this formula is just a power series in the variable σ^{-1} . Hence, convergence is ensured provided that the parameters $M_i(t)$ are bounded with respect to i .

The periodicity of the Markov parameters reflects into a specific structure of this operator. The first obvious consideration is that the operator $G(\sigma, t)$ is periodic:

$$G(\sigma, t + T) = G(\sigma, t).$$

As for the dependence upon σ , it is easy to verify that

$$\sigma^{-k} G(\sigma, t) = G(\sigma, t - k) \sigma^{-k}.$$

This property will be referred to as *pseudo-commutative property* with respect to the delays. By setting the integer k as a multiple of the period T , say $k = iT$, this expression together with the previous one implies that the operators σ^{-iT} and $G(\sigma, t)$ do commute. Notice in passing that, by means of the pseudo-commutative property, one can write

$$G(\sigma, t) = M_0(t) + \sigma^{-1}M_1(t+1) + \sigma^{-2}M_2(t+2) + \sigma^{-3}M_3(t+3) + \dots$$

in place of (2.3).

The periodicity of the Markov coefficients entails that the output response of the system at a generic time instant $t = kT + s$, with $s \in [0, T-1]$, can be written as a finite sum of the output responses of T time-invariant systems indexed in the integer s . As a matter of fact, consider again Eq. (2.1) and evaluate $y(t)$ in $t = kT + s$. It follows that

$$y(kT + s) = \sum_{i=0}^{T-1} \hat{y}_{i,s}(k),$$

where

$$\hat{y}_{i,s}(k) = \sum_{j=0}^{\infty} M_{jT+i}(s) \hat{u}_{i,s}(k-j),$$

and

$$\hat{u}_{i,s}(k) = u(kT + s - i).$$

From these expressions, it is apparent that $\hat{y}_{i,s}(k)$ is the output of a time-invariant system having $M_i(s), M_{i+T}(s), M_{i+2T}(s), \dots$, as Markov parameters. Note the role of the different indexes appearing in these expressions: s is the chosen tag time index for the output variable, $s - i$ is the tag time index for the input variable, and k is the sampled time current variable. For each $i \in [0, T-1]$ and $t \in [0, T-1]$, one can define:

$$H_i(z, s) = \sum_{j=0}^{\infty} M_{jT+i}(s) z^{-j}. \quad (2.4)$$

This is the transfer function from $u_{i,s}(k)$ to $\hat{y}_{i,s}(k)$ (both signals seen as function of k). By using z as the 1-step-ahead shift operator in time k (namely, z is the T -steps-ahead shift operator in time t), and resorting to a mixed z/k notation, one can write

$$\begin{aligned} y(kT + s) &= H_0(z, s)u(kT + s) + H_1(z, s)u(kT + s - 1) + \dots \\ &\quad + H_{T-2}(z, s)u(kT + s - T + 2) + H_{T-1}(z, s)u(kT + s - T + 1) \end{aligned} \quad (2.5)$$

or, equivalently

$$\begin{aligned} y(kT + s) &= H_0(z, s)u(kT + s) + H_{T-1}(z, s)z^{-1}u(kT + s + 1) + \dots \\ &\quad + H_2(z, s)z^{-1}u(kT + s + T - 2) + H_1(z, s)z^{-1}u(kT + s + T - 1). \end{aligned} \quad (2.6)$$

The function $H_i(z, s)$ will be referred to as the *sampled transfer function* at tag time s with input–output delay i .

The above “uncombing” procedure in the time-domain has a natural counterpart in the domain of transfer functions. Indeed, the transfer operator $G(\sigma, t)$ can be written as

$$G(\sigma, t) = \sum_{k=0}^{\infty} M_k(t) \sigma^{-k} = \sum_{i=0}^{T-1} \left[\sum_{j=0}^{\infty} M_{jT+i}(t) \sigma^{-jT} \right] \sigma^{-i}.$$

By taking into account the Expression (2.4), it finally follows

$$G(\sigma, t) = \sum_{i=0}^{T-1} H_i(\sigma^T, t) \sigma^{-i}. \quad (2.7)$$

This formula enables one to compute the periodic transfer operator from the sampled transfer functions, as depicted in Fig. 2.1. Expression (2.7) can be, so to say, “inverted”. To this purpose, we introduce for the first time a symbol ϕ which will be often used in this book:

$$\phi = \exp\left(\frac{2\pi j}{T}\right) = \cos\left(\frac{2\pi}{T}\right) + j \sin\left(\frac{2\pi}{T}\right). \quad (2.8)$$

Hence $1, \phi, \phi^2, \dots, \phi^{T-1}$ are the T -roots of the unit. Note that

$$\frac{1}{T} \sum_{k=0}^{T-1} \phi^{sk} = \begin{cases} 1 & s = 0, \pm T, \pm 2T, \dots \\ 0 & \text{otherwise.} \end{cases} \quad (2.9)$$

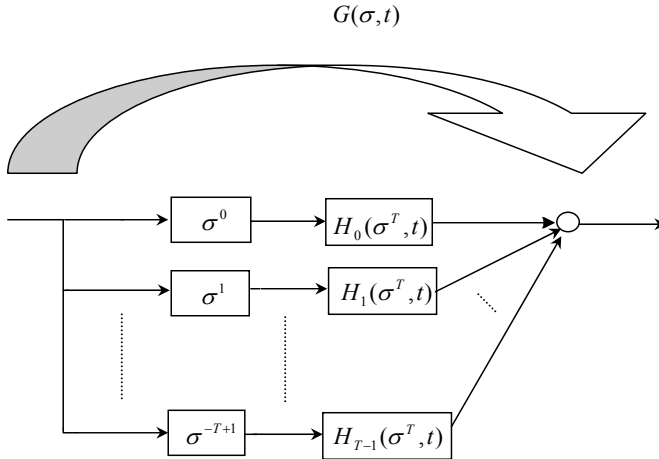


Fig. 2.1 The periodic transfer operator as a composition of (time-invariant) sampled transfer functions

Now, consider (2.7) evaluated in $\sigma\phi^k$. By multiplying both sides by ϕ^{kr} and summing w.r.t k , for k ranging from 0 to $T-1$, it easily follows from (2.9) that

$$H_r(\sigma^T, t) = \frac{\sigma^r}{T} \sum_{k=0}^{T-1} G(\sigma\phi^k, t-r)\phi^{kr}. \quad (2.10)$$

The input–output periodic model is said to be *rational* if all transfer functions (2.4) are indeed rational, i.e., they are transfer functions of finite–dimensional (time-invariant) systems. In view of (2.7), the periodic transfer operator $G(\sigma, t)$ is rational in σ for each t . In this case also the periodic system can be given a state–space finite–dimensional realization, as will be discussed in Chap. 7.

If the original periodic system is indeed time-invariant with transfer function $G(\sigma)$, the transfer function operator $G(\sigma, t)$ does not depend on t and $G(\sigma, t)$ actually reduces to $G(\sigma)$, i.e., $G(\sigma, t) = G(\sigma), \forall t$.

2.2 PARMA Model

A most useful input–output representation of a rational periodic system is given by the so-called *Periodic Auto-Regressive and Moving Average* (PARMA) model. This corresponds to writing:

$$y(t) = a_1(t)y(t-1) + a_2(t)y(t-2) + \cdots + a_r(t)y(t-r) \quad [AR \text{ part}] \\ + b_0(t)u(t) + b_1(t)u(t-1) + \cdots + b_s(t)u(t-s) \quad [MA \text{ part}], \quad (2.11)$$

where the coefficients $a_i(t)$ and $b_j(t)$ are T -periodic matrices of suitable dimensions and r and s are two positive integers. A generalization of this description is obtained by letting r and s be periodically time varying as well. However, as it is obvious, one can always rewrite the model with constant r and s , and for the sake of simplicity we will focus on the above description. Denoting by I the identity operator and letting

$$d_c(\sigma^{-1}, t) = I - a_1(t)\sigma^{-1} - a_2(t)\sigma^{-2} - \cdots - a_r(t)\sigma^{-r} \\ n_c(\sigma^{-1}, t) = b_0(t) + b_1(t)\sigma^{-1} + \cdots + b_s(t)\sigma^{-s}$$

the Eq. (2.11) can be given the compact (operator) form

$$d_c(\sigma^{-1}, t) \cdot y(t) = n_c(\sigma^{-1}, t) \cdot u(t).$$

Define now the rational matrix $d_c^{-1}(\sigma^{-1}, t)$ such that

$$d_c^{-1}(\sigma^{-1}, t)d_c(\sigma^{-1}, t) = d_c(\sigma^{-1}, t)d_c^{-1}(\sigma^{-1}, t) = I.$$

Then the periodic transfer operator of the PARMA system can be written as:

$$G(\sigma, t) = d_c^{-1}(\sigma^{-1}, t) n_c(\sigma^{-1}, t).$$

The fractional representation of a PARMA model can be equivalently written in terms of polynomial matrices in the forward operator σ as well. Indeed, letting

$$d(\sigma, t) = \sigma^{\max\{r,s\}} d_c(\sigma^{-1}, t), \quad n(\sigma, t) = \sigma^{\max\{r,s\}} n_c(\sigma^{-1}, t),$$

it follows that

$$G(\sigma, t) = d^{-1}(\sigma, t) n(\sigma, t). \quad (2.12)$$

Notice that the ring of polynomials with periodic coefficients is not commutative, so that multiplication must be handled with care. Particularly important is the case in which the input and output variables are scalar, i.e., the system is SISO (Single Input Single Output). Then, the two polynomial matrices are scalar polynomials.

In the input–output representations seen so far, the polynomial have periodic coefficients. It is important to observe that, at the price of increasing the polynomials order, it is possible to work out an equivalent input–output representation with the “numerator” or the “denominator” having time-invariant coefficients. This fact is illustrated in the following simple example. The general underlying theory will be duly developed in Chap. 5.

Example 2.1 *Consider the univariate PARMA model*

$$y(t) = a(t)y(t-2) + b_0(t)u(t) + b_1(t)u(t-1) + b_2(t)u(t-2),$$

where all parameters are periodic of period $T = 2$. The corresponding periodic transfer operator is

$$G(\sigma, t) = (\sigma^2 - a(t))^{-1} (b_0(t)\sigma^2 + b_1(t)\sigma + b_2(t)).$$

The sampled transfer functions can easily be computed from (2.10) as

$$H_0(\sigma^2, t) = \frac{b_0(t)\sigma^2 + b_2(t)}{\sigma^2 - a(t)}, \quad H_1(\sigma^2, t) = \frac{b_1(t)\sigma}{\sigma^2 - a(t+1)}.$$

Interestingly enough, the original periodic transfer operator can be equivalently rewritten with a denominator given by a polynomial in σ^2 with constant parameters. Indeed, write the PARMA model in the operator form (2.12) with

$$d(\sigma, t) = \sigma^2 - a(t), \quad n(\sigma, t) = b_0(t)\sigma^2 + b_1(t)\sigma + b_2(t).$$

Pre-multiplying both polynomials by

$$d_*(\sigma, t) = \sigma^2 - a(t+1),$$

the transfer function $G(\sigma, t)$ becomes

$$G(\sigma, t) = (d_*(\sigma, t)d(\sigma, t))^{-1}d_*(\sigma, t)n(\sigma, t),$$

and the new denominator is

$$d_*(\sigma, t)d(\sigma, t) = \sigma^4 - \bar{\alpha}_1\sigma^2 - \bar{\alpha}_2$$

with

$$\bar{\alpha}_1 = (a(t) + a(t+1)), \quad \bar{\alpha}_2 = -a(t)a(t+1).$$

This is a time-invariant polynomial in σ^2 . Notice that the numerator is given by

$$d_*(\sigma, t)n(\sigma, t) = \beta_0(t)\sigma^4 + \beta_1(t)\sigma^3 + \cdots + \beta_4(t),$$

where

$$\begin{aligned} \beta_0(t) &= b_0(t), & \beta_1(t) &= b_1(t), & \beta_2(t) &= b_2(t) - a(t+1)b_0(t) \\ \beta_3(t) &= -a(t+1)b_1(t), & \beta_4(t) &= -a(t+1)b_2(t). \end{aligned}$$

Thus, the numerator still has periodic coefficients. With this new transfer function, the corresponding PARMA representation becomes

$$y(t) = \bar{\alpha}_1 y(t-2) + \bar{\alpha}_2 y(t-4) + \beta_0(t)u(t) + \beta_1(t)u(t-1) + \cdots + \beta_4(t)u(t-4).$$

The distinctive feature of this new PARMA representation is that the autoregressive part of the model is a time-invariant polynomial in σ^2 . One can therefore conjecture that the underlying dynamics of the system is dominated by the roots of the polynomial

$$z^2 - \bar{\alpha}_1 z - \bar{\alpha}_2.$$

As a matter of fact, the roots of this polynomial determine the input–output stability of the system, as will be explained in a forthcoming chapter.

Conversely, it is also possible to give $G(\sigma, t)$ an expression in which the numerator is a polynomial with constant coefficients. To this end, it suffices to pre-multiply $n(\sigma, t)$ and $d(\sigma, t)$ by

$$n_*(\sigma, t) = b_0(t+1)\sigma^2 - b_1(t)\sigma + b_2(t+1),$$

so yielding

$$n_*(\sigma, t)n(\sigma, t) = \bar{\beta}_0\sigma^4 + \bar{\beta}_1\sigma^2 + \bar{\beta}_2,$$

where

$$\begin{aligned} \bar{\beta}_0 &= b_0(t+1)b_0(t) \\ \bar{\beta}_1 &= (b_0(t+1) - b_1(t+1))b_1(t) + b_0(t)b_2(t+1) \\ \bar{\beta}_2 &= b_2(t)b_2(t+1). \end{aligned}$$

In this second form, letting

$$\begin{aligned}\alpha_0(t) &= b_0(t+1), & \alpha_1(t) &= b_1(t), & \alpha_2(t) &= -b_2(t+1) + b_0(t+1)a(t) \\ \alpha_3(t) &= -b_1(t)a(t+1), & \alpha_4(t) &= b_2(t+1)a(t),\end{aligned}$$

the denominator is

$$n_*(\sigma, t)d(\sigma, t) = \alpha_0(t)\sigma^4 - \alpha_1(t)\sigma^3 - \cdots - \alpha_4(t).$$

This is a polynomial with periodic coefficients. The associated PARMA representation is

$$\alpha_0(t)y(t) = \alpha_1(t)y(t-1) + \cdots + \alpha_4(t)y(t-4) + \bar{\beta}_0 u(t) + \bar{\beta}_1 u(t-2) + \bar{\beta}_2 u(t-4).$$

Now, it is the moving-average part of the model to be time-invariant in the power of σ^2 . One can therefore conjecture that the underlying inverse dynamics is dominated by the roots of the polynomial

$$\bar{\beta}_0 z^2 + \bar{\beta}_1 z + \bar{\beta}_2.$$

Later on we will see that the roots of this polynomial are the transmission zeros of the periodic system. ■

Among other things, in the example it is shown that the AR part or the MA part of a PARMA model can be given an “invariantized” form. This holds in general. Precisely, consider any T -periodic polynomial $p(\sigma, t)$ whose coefficients are matrices of dimension $n \times m$. If $n \geq m$ [$n \leq m$], there exists an associated matrix periodic polynomial $p_*(\sigma, t)$ with coefficients of dimension $n \times n$ [$m \times m$] such that the product $p_*(\sigma, t)p(\sigma, t)$ [$p(\sigma, t)p_*(\sigma, t)$] is indeed a polynomial in σ^T with constant coefficients. This can be shown by preliminarily introducing an auxiliary matrix $P(\sigma^T, t)$, whose entries are polynomials in σ^T with periodic coefficients. Precisely, if $p(\sigma, t)$ is $n \times m$, the associated $P(\sigma^T, t)$ is the $nT \times mT$ matrix defined by the expression

$$\begin{bmatrix} I_n \\ \sigma I_n \\ \vdots \\ \sigma^{T-1} I_n \end{bmatrix} p(\sigma, t) = P(\sigma^T, t) \begin{bmatrix} I_m \\ \sigma I_m \\ \vdots \\ \sigma^{T-1} I_m \end{bmatrix}. \quad (2.13)$$

It is worthy to point out that the periodic coefficients of matrix $P(\sigma, t)$ obey the following recursive formula

$$P(\sigma^T, t+1) = \Delta'_n(\sigma^{-T})P(\sigma^T, t)\Delta_m(\sigma^T), \quad (2.14)$$

where the symbol $\Delta_k(z)$ denotes the matrix of dimension $kT \times kT$ given by

$$\Delta_k(z) = \begin{bmatrix} 0 & z^{-1}I_k \\ I_{k(T-1)} & 0 \end{bmatrix}.$$

This recursion is easily verified starting from Eq. (2.13). Matrix $P(\sigma, t)$ is called *lifted polynomial* at time t associated with the given periodic polynomial $p(\sigma, t)$. In the square case ($n = m$), the lifted polynomial has the distinctive feature that its determinant has constant coefficients

$$\det[P(\sigma^T, t)] = \bar{p}(\sigma^T),$$

namely $\bar{p}(\sigma^T)$ is a polynomial in σ^T with constant coefficients. This fact derives from (2.14) by noticing that $\det[\Delta_n(\sigma^T)] = \sigma^{-nT}$.

The representation of a periodic system in terms of a couple of lifted polynomials will be extensively studied in Chap. 5. Here we limit ourselves to exploiting the above representation in order to understand the rationale underlying the polynomial manipulations in Example 2.1. Precisely, the problem there is to pass from a PARMA scalar model to a new PARMA model in which the AR or MA part is time-invariant. In general, the procedure of “invariantization” of a given polynomial $p(\sigma, t)$ amounts to finding a polynomial $p_*(\sigma, t)$ such that $p_*(\sigma, t)p(\sigma, t)$ is a time-invariant polynomial in σ^T . This problem can be solved by pre-multiplying (2.13) by the row vector $[\det[P(\sigma^T, t)] \ 0 \ \cdots \ 0] (P(\sigma^T, t))^{-1}$ which is indeed polynomial in σ^T . In this way, one obtains

$$[\det[P(\sigma^T, t)] \ 0 \ \cdots \ 0] (P(\sigma^T, t))^{-1} \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{T-1} \end{bmatrix} p(\sigma, t) = \det[P(\sigma^T, t)].$$

As already observed $\det[P(\sigma^T, t)]$ has constant coefficients. Therefore the polynomial $p_*(\sigma, t)$ is simply given by

$$p_*(\sigma, t) = [\det[P(\sigma^T, t)] \ 0 \ \cdots \ 0] (P(\sigma^T, t))^{-1} \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{T-1} \end{bmatrix},$$

and it turns out that

$$p_*(\sigma, t)p(\sigma, t) = \det[P(\sigma^T, t)] = \bar{p}(\sigma^T).$$

2.3 The Transfer Operator in Frequency Domain

Periodic systems can be studied in the frequency domain by making reference to the concept of *exponentially modulated periodic* (EMP) signal.

A signal $v(\cdot)$ is said to be EMP if there exists a (complex) number $\lambda \neq 0$ such that

$$v(t + kT) = v(t)\lambda^{kT}, \quad t \in [\tau, \tau + T - 1].$$

Obviously, if one imposes $\lambda = 1$ (or equal to any T -th root of the unit), a T -periodic signal is eventually recovered.

Notice that if $v(\cdot)$ is EMP relative to a (non-null) complex number $\lambda \neq 0$, then $\bar{v}(t) = v(t)\lambda^{-t}$ is T -periodic. Conversely, if $\bar{v}(\cdot)$ is T -periodic, then, for each non-null λ , $v(t) = \bar{v}(t)\lambda^t$ is EMP. Therefore any EMP signal can be written as

$$v(t) = \sum_{k=0}^{T-1} \bar{v}_k \phi^{kt} \lambda^t, \quad (2.15)$$

where v_k are the coefficients of the Fourier expansion of the periodic signal $v(t)\lambda^{-t}$. Expression (2.15) will be referred to as the *EMP Fourier expansion*.

Suppose now we feed the T -periodic System (2.1) with a λ -EMP input $u(t)$, namely $u(t + kT) = u(t)\lambda^{kT}$, $t \in [0, T - 1]$ and assume that all the sampled transfer functions $H_i(z, t)$ are well defined in $z = \lambda^T$ for each t . Notice that $H_i(z, t)$ represents a time-invariant system operating over the sampled time-variable k . If such a system is subject to an exponential signal μ^k , then in an exponential regime, the output is simply obtained by evaluating the transfer function in $z = \mu$. In particular, this applies when $\mu = \lambda^T$. Then, considering that $z = \sigma^T$ is the one period ahead shift operator (one shift ahead in k), it follows that

$$H_i(z, t + r)u(kT + t + r) = H_i(\lambda^T, t)u(t + r)\lambda^{kT}, \quad \forall r \in [0, T - 1].$$

Consequently, from (2.6), for each $t = 0, 1, \dots, T - 1$:

$$\begin{aligned} y(t + kT) &= \sum_{i=0}^{T-1} H_i(z, t)u(t + kT - i) \\ &= H_0(z, t)u(t + kT) + \left[\sum_{i=1}^{T-1} H_i(z, t)z^{-1}u(t + kT + T - i) \right] \\ &= H_0(\lambda^T, t)\lambda^{kT}u(t) + \left[\sum_{i=1}^{T-1} H_i(\lambda^T, t)\lambda^{kT-T}u(t + T - i) \right] \\ &= \left[H_0(\lambda^T, t) + \left[\sum_{i=1}^{T-1} H_i(\lambda^T, t)\lambda^{-T}\sigma^{T-i} \right] u(t) \right] \lambda^{kT}. \end{aligned}$$

In an EMP regime, it suffices to set $y(t + kT) = y(t)\lambda^{kT}$. Hence in the EMP regime, the values of the input and output over one period are related as follows

$$y(t) = G_\lambda(\sigma, t) \cdot u(t), \quad t = 0, 1, \dots, T - 1 \quad (2.16)$$

with

$$G_\lambda(\sigma, t) = H_0(\lambda^T, t) + \sum_{i=1}^{T-1} H_i(\lambda^T, t) \lambda^{-T} \sigma^{T-i}. \quad (2.17)$$

It is worthy to point out the relationship between the periodic transfer function $G(\sigma, t)$ as characterized in (2.7) and the EMP transfer operator $G_\lambda(\sigma, t)$ in (2.17). A direct inspection leads to

$$G(\sigma, t) = G_\lambda(\sigma, t)|_{\lambda=\sigma}. \quad (2.18)$$

In conclusion, it is possible to pass from the periodic transfer operator to the EMP transfer operator and vice versa by means of (2.10), (2.17) and (2.18), respectively.

If the system is actually time-invariant, the EMP external signals take the form $u(t) = \bar{u}\lambda^t$ and $y(t) = \bar{y}\lambda^t$. Therefore, one obtains $G_\lambda(\sigma, t) = G(\lambda)$, which is indeed the gain between the input and output “amplitudes” \bar{u} - \bar{y} .

Remark 2.1 Notice that $G_\lambda(\sigma, t)$ is polynomial in σ . It is an anti-causal operator, in that it maps $u(t)$, $u(t+1)$, \dots , $u(t+T-1)$ to $y(t)$, $t \in [0, T-1]$. Instead, one can consider a causal operator $H_\lambda(\sigma, t)$ defined as

$$H_\lambda(\sigma, t) = G_\lambda(\sigma, t) \lambda^T \sigma^{-T},$$

which maps $u(t-T)$, $u(t-T+1)$, \dots , $u(t-1)$ to $y(t)$. If one makes reference to operator $H_\lambda(\sigma, t)$, then the relation

$$G(\sigma, t) = H_\lambda(\sigma, t)|_{\lambda=\sigma}.$$

follows. ■

The EMP transfer operator plays in the periodic realm the role that the transfer function plays in time-invariant systems. Indeed, consider, for each $t \in [\tau, \tau+T-1]$, the formal discrete-time series $\{u(t+kT)\}$ and $\{y(t+kT)\}$, obtained by uniformly sampling (with tag t) the input and output signals, and define the bilateral z -transform

$$U(z, t) = \sum_{k=-\infty}^{\infty} u(t+kT) z^{-k}$$

$$Y(z, t) = \sum_{k=-\infty}^{\infty} y(t+kT) z^{-k}.$$

Notice that $U(z^T, t)$ and $Y(z^T, t)$ are EMP signals in t , since $Y(z^T, t+kT) = Y(z^T, t) z^{kT}$, $t \in [\tau, \tau+T-1]$ and analogously for $U(z^T, t)$. Hence, Eq. (2.16) can be used yielding

$$Y(z^T, t) = G_z(\sigma, t) \cdot U(z^T, t).$$

The above equation shows that $G_z(\sigma, t)$ can be interpreted as “transfer function at t ” since it maps the input z -transform $U(z^T, t)$ into the z -transform of the forced output $Y(z^T, t)$. Of course, when the system is time-invariant ($T = 1$) one can set

$z = \sigma$. Therefore, if $G(z)$ is the transfer function of the time-invariant system, then $G_z(\sigma, t) = G(z)$.

Example 2.2 Consider a periodic system of period $T = 2$, characterized by the PARMA model

$$y(t) = a(t)y(t-1) + b(t)u(t-1).$$

The associated periodic transfer operator is

$$G(\sigma, t) = (1 - a(t)\sigma^{-1})^{-1}b(t)\sigma^{-1} = (\sigma - a(t+1))^{-1}b(t+1).$$

The sampled transfer functions can be derived by making reference to (2.10), thus obtaining:

$$H_0(\sigma^2, t) = 0.5 \left((\sigma - a(t+1))^{-1}b(t+1) - (\sigma + a(t+1))^{-1}b(t+1) \right).$$

The denominators of the two terms in this sum neither coincide nor commute. To work out a common denominator, we preliminarily pre-multiply the first term by $(\sigma + a(t))(\sigma + a(t))^{-1}$ and the second term by $(\sigma - a(t))(\sigma - a(t))^{-1}$. In this way, one obtains:

$$H_0(\sigma^2, t) = \frac{a(t)b(t+1)}{\sigma^2 - a(t)a(t+1)}, \quad t = 0, 1.$$

Analogously, it is possible to determine $H_1(\sigma^2, t)$ as

$$H_1(\sigma^2, t) = \frac{b(t)\sigma^2}{\sigma^2 - a(t)a(t+1)}, \quad t = 0, 1.$$

Notice that, as expected, all the four sampled transfer functions are constant coefficient rational functions of σ^2 , a fact which was not explicit in the previous expression. These expressions for $H_i(\sigma^2, t)$ can alternatively be obtained by considering the equivalent form of the periodic transfer operator

$$G(\sigma, t) = \frac{a(t)b(t+1) + b(t)\sigma}{\sigma^2 - a(t)a(t+1)}.$$

which can be derived from the previously given $G(\sigma, t)$ by pre-multiplying it by $(\sigma + a(t))(\sigma + a(t))^{-1}$.

Finally, the EMP transfer operator can be obtained from Eq. (2.17) in view of the previous expression of $H_i(\sigma, t)$:

$$G_\lambda(\sigma, t) = \frac{a(t)b(t+1) + b(t)\sigma}{\lambda^2 - a(t)a(t+1)}.$$

The causal equivalent form of this operator is

$$H_\lambda(\sigma, 0) = \frac{a(t)b(t+1)\sigma^{-2} + b(t)\sigma^{-1}}{1 - a(t)a(t+1)\lambda^{-2}}.$$

2.4 State–Space Description

A celebrated alternative to the input–output representation is provided by the state–space description. In the finite–dimensional case, this amounts to consider the model:

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (2.19a)$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad (2.19b)$$

where $u(t) \in R^m$, $x(t) \in R^n$, $y(t) \in R^p$ are the input, state and output vectors, respectively. Matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are real matrices, of appropriate dimensions, which depend periodically on t :

$$A(t+T) = A(t), \quad B(t+T) = B(t), \quad C(t+T) = C(t), \quad D(t+T) = D(t).$$

A most useful concept is that of state-transition matrix; this matrix denoted by $\Phi_A(t, \tau)$ with $t \geq \tau$ is defined as:

$$\Phi_A(t, \tau) = \begin{cases} A(t-1)A(t-2) \cdots A(\tau), & t > \tau \\ I, & t = \tau. \end{cases}$$

The solution of System (2.19a) is given by

$$x(t) = \Phi_A(t, \tau)x(\tau) + \sum_{j=\tau}^{t-1} \Phi_A(t, j+1)B(j)u(j). \quad (2.20)$$

In this expression we distinguish the free state motion $x_L(t)$ and the forced state motion $x_F(t)$

$$x_L(t) = \Phi_A(t, \tau)x(\tau), \quad x_F(t) = \sum_{j=\tau}^{t-1} \Phi_A(t, j+1)B(j)u(j).$$

In this way, one can see the state evolution $x(\cdot)$ as the superposition of the free and forced motions $x(t) = x_L(t) + x_F(t)$.

Remark 2.2 *The free motion expression can be derived in an operator way. For, write the state–space equation for the free motion originating in $x(\tau-1) = 0$ as*

$$x(t+1) = A(t)x(t) + x(\tau)\delta(t+1-\tau),$$

where $\delta(\cdot)$ is the Kronecker function (2.2). By resorting to the σ operator, one obtains

$$x(t) = (\sigma I - A(t))^{-1} \sigma x(\tau) \delta(t-\tau)$$

This operator expression can be elaborated as follows:

$$\begin{aligned}
 x(t) &= (\sigma I - A(t))^{-1} \sigma x(\tau) \delta(t - \tau) \\
 &= (I - \sigma^{-1} A(t))^{-1} x(\tau) \delta(t - \tau) \\
 &= \sum_{k=0}^{\infty} \Phi_A(t, t-k) \sigma^{-k} x(\tau) \delta(t - \tau) \\
 &= \sum_{k=0}^{\infty} \Phi_A(t, t-k) x(\tau) \delta(t - \tau - k) \\
 &= \Phi_A(t, \tau) x(\tau)
 \end{aligned}$$

so that the classical free motion formula is recovered. ■

As for the output, it is straightforward to see that

$$y(t) = C(t) \Phi_A(t, \tau) x(\tau) + \sum_{j=\tau}^{t-1} C(t) \Phi_A(t, j+1) B(j) u(j) + D(t) u(t). \quad (2.21)$$

Again one can write $y(t) = y_L(t) + y_F(t)$ where

$$y_L(t) = C(t) \Phi_A(t, \tau) x(\tau), \quad y_F(t) = \sum_{j=\tau}^{t-1} C(t) \Phi_A(t, j+1) B(j) u(j) + D(t) u(t)$$

are the free and forced output motion, respectively. Letting

$$\begin{aligned}
 M_0(t) &= D(t), \quad M_1(t) = C(t) B(t-1), \quad M_2(t) = C(t) A(t-1) B(t-2), \\
 M_3(t) &= C(t) A(t-1) A(t-2) B(t-3), \dots,
 \end{aligned}$$

the forced output motion can be written as

$$y_F(t) = \sum_{j=0}^{t-\tau} M_j(t) u(t-j).$$

In this formula, if one let $\tau \rightarrow -\infty$, then Expression (2.1) is eventually recovered. As already said, the parameters $M_i(t)$ are known as *Markov coefficients*.

A particularly significant role is played by the so-called *monodromy matrix*, which is defined as the transition matrix over a period $[\tau, \tau + T - 1]$:

$$\Psi_A(\tau) = \Phi_A(\tau + T, \tau).$$

The eigenvalues of this matrix are named *characteristic multipliers*. The monodromy matrix and the characteristic multipliers are fundamental tools in the analysis of periodic systems, noticeably for the stability analysis, and will be extensively studied in the next chapter.

Thanks to periodicity, the Markov parameters can be written as

$$M_0(t) = D(t), \quad M_{jT+i}(t) = C(t)\Psi_A(t)^j\Phi_A(t, t-i+1)B(t-i), \quad (2.22)$$

where $j = 0, 1, 2, \dots$ and $i \in [1, T]$. Moreover, a simple computation shows that the sampled transfer functions $H_i(z, t)$ defined in (2.4) are given by:

$$H_0(z, t) = D(t) + C(t)(zI - \Psi_A(t))^{-1}\Phi_A(t, t-T+1)B(t) \quad (2.23a)$$

$$H_i(z, t) = zC(t)(zI - \Psi_A(t))^{-1}\Phi_A(t, t-i+1)B(t-i), \quad i \in [1, T-1]. \quad (2.23b)$$

Hence, all $H_i(z, t)$ are rational functions of z , and, according to the definition in Sect. 2.1, the system is rational. From Expressions (2.7), (2.23), the periodic transfer operator $G(\sigma, t)$ can be given the compact form

$$G(\sigma, t) = D(t) + C(t)(\sigma^T I - \Psi_A(t))^{-1}\mathcal{B}(\sigma, t) \quad (2.24a)$$

$$\mathcal{B}(\sigma, t) = \sum_{j=0}^{T-1} \Phi_A(t, t+j-T+1)B(t+j)\sigma^j. \quad (2.24b)$$

This expression is reminiscent of the classical formula of a transfer function of a time-invariant system, the only difference being that the input matrix is replaced by a polynomial matrix of σ -powers up to $T-1$.

Finally, consider the periodic System (2.19) fed by the EMP input function

$$u(t+kT) = u(t)\lambda^{kT}, \quad t \in [\tau, \tau+T-1],$$

and assume that λ^T does not coincide with any characteristic multiplier of $A(\cdot)$. Then the initial state

$$x_\lambda(\tau) = (\lambda^T I - \Psi_A(\tau))^{-1} \sum_{i=\tau}^{\tau+T-1} \Phi_A(\tau+T, i+1)B(i)u(i) \quad (2.25)$$

is such that both the state and the corresponding output are still EMP signals, i.e.,

$$x(t+kT) = x(t)\lambda^{kT}, \quad t \in [\tau, \tau+T-1]$$

$$y(t+kT) = y(t)\lambda^{kT}, \quad t \in [\tau, \tau+T-1].$$

Note that, for any $t \in [\tau, \tau+T-1]$, the EMP output $y(t)$ can be written as

$$y(t) = C(t)\Phi_A(t, \tau)x_\lambda(\tau) + C(t) \sum_{i=\tau}^{t-1} \Phi_A(t, i+1)B(i)u(i) + D(t)u(t).$$

Since $x_\lambda(\tau)$ is given by (2.25) and the following property holds true

$$\Phi_A(t, \tau)(\lambda^T I - \Psi_A(\tau))^{-1} = (\lambda^T I - \Psi_A(t))^{-1}\Phi_A(t, \tau),$$

a cumbersome computation shows that $y(\cdot)$ can be equivalently rewritten as

$$y(t) = C(t)(\lambda^T I - \Psi_A(t))^{-1} \sum_{i=t}^{t+T-1} \Phi_A(t+T, i+1) B(i) u(i) + D(t) u(t).$$

This expression can be given a compact operator form as

$$y(t) = G_\lambda(\sigma, t) u(t),$$

where

$$G_\lambda(\sigma, t) = D(t) + C(t)(\lambda^T I - \Psi_A(t))^{-1} \mathcal{B}(\sigma, t), \quad (2.26)$$

and $\mathcal{B}(\sigma, t)$ was defined in (2.24b). This is the state-space version of the EMP transfer operator introduced in the previous section.

Remark 2.3 *As seen in Remark 2.1, the EMP transfer operator can be given a causal expression $H_\lambda(\sigma, t)$. In the state-space context, the causal expression is*

$$H_\lambda(\sigma, t) = D(t) + C(t)(I - \Psi_A(t)\lambda^{-T})^{-1} \sigma^{-T} \mathcal{B}(\sigma, t). \quad (2.27)$$

Example 2.3 *Consider System (2.19) with $T = 2$, $D(0) = D(1) = 0$ and*

$$A(t) = \begin{cases} 2 & t = 0 \\ -5 & t = 1 \end{cases}, \quad B(t) = \begin{cases} 1 & t = 0 \\ -2 & t = 1 \end{cases}, \quad C(t) = \begin{cases} 0.5 & t = 0 \\ 3 & t = 1 \end{cases}.$$

From (2.23), it follows that

$$\begin{aligned} H_0(z, 0) &= \frac{-2.5}{z+10}, & H_1(z, 0) &= \frac{-z}{z+10} \\ H_0(z, 1) &= \frac{-12}{z+10}, & H_1(z, 1) &= \frac{3z}{z+10}. \end{aligned}$$

As for the polynomial $\mathcal{B}(\sigma, t)$, it can be computed from (2.24b):

$$\mathcal{B}(\sigma, t) = \begin{cases} -5 - 2\sigma & t = 0 \\ -4 + \sigma & t = 1, \end{cases}$$

so that the periodic transfer operator as in (2.24a) turns out to be given by:

$$G(\sigma, 0) = \frac{-2.5 - \sigma}{\sigma^2 + 10}, \quad G(\sigma, 1) = \frac{-12 + 3\sigma}{\sigma^2 + 10}.$$

Finally, from (2.26) and (2.27), the following relations are obtained:

$$\begin{aligned} G_\lambda(\sigma, 0) &= \frac{-2.5 - \sigma}{\lambda^2 + 10}, & G_\lambda(\sigma, 1) &= \frac{-12 + 3\sigma}{\lambda^2 + 10} \\ H_\lambda(\sigma, 0) &= \frac{-2.5\sigma^{-2} - \sigma^{-1}}{1 + 10\lambda^{-2}}, & H_\lambda(\sigma, 1) &= \frac{-12\sigma^{-2} + 3\sigma^{-1}}{1 + 10\lambda^{-2}}. \end{aligned}$$

■

A different way to work out the transfer operator from the state–space description of the system is that of considering directly the action of the 1-step-ahead shift operator σ on the state, input and output signals. Indeed, consider the System (2.19) and notice that it is possible to regard the action of the initial state $x_0 = x(0)$ in the following way

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + x_0\delta(t+1) \\ y(t) &= C(t)x(t) + D(t)u(t) \\ x(-1) &= 0, \end{aligned} \quad (2.28)$$

where $\delta(t)$ is the (discrete) impulse function defined in (2.2). Obviously in (2.28) is has tacitly assumed that $u(\tau) = 0$ for $\tau < 0$. By resorting to the operator σ it follows that

$$x(t) = (\sigma I - A(t))^{-1} \sigma x_0 \cdot \delta(t) + (\sigma I - A(t))^{-1} B(t) \cdot u(t),$$

and

$$y(t) = C(t)(\sigma I - A(t))^{-1} \sigma x_0 \cdot \delta(t) + [C(t)(\sigma I - A(t))^{-1} B(t) + D(t)] \cdot u(t).$$

Of course the transfer operator can be written as

$$G(\sigma, t) = C(t)(\sigma I - A(t))^{-1} B(t) + D(t). \quad (2.29)$$

Equation (2.29) is formally identical to the usual formula of the transfer function for time-invariant systems. Obviously, when computing the inverse of $\sigma I - A(t)$, the skew commutative role of the operator σ must be taken into account, as illustrated in the subsequent example.

Example 2.4 Consider the matrix

$$A(t) = \begin{bmatrix} 0 & 1 \\ 1 & \alpha(t) \end{bmatrix}$$

where $\alpha(t)$ is a 2-periodic scalar function. The inverse of the operator matrix $\sigma I - A(t)$ is given by:

$$\begin{aligned} (\sigma I - A(t))^{-1} &= \begin{bmatrix} (\sigma - \alpha(t)) & 1 \\ 1 & \sigma \end{bmatrix} \begin{bmatrix} (\sigma^2 - \alpha(t+1)\sigma - 1)^{-1} & 0 \\ 0 & (\sigma^2 - \alpha(t)\sigma - 1)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} (\sigma^2 - \alpha(t)\sigma - 1)^{-1} & 0 \\ 0 & (\sigma^2 - \alpha(t+1)\sigma - 1)^{-1} \end{bmatrix} \begin{bmatrix} (\sigma - \alpha(t)) & 1 \\ 1 & \sigma \end{bmatrix} \end{aligned}$$

The easy check of the above expression is left to the reader. ■

Compare the Expression (2.29) with the EMP transfer function $G_\lambda(\sigma, t)$ as given by Eq. (2.26). Since Relation (2.18) holds true, such comparison suggests that

$$(\sigma^T I - \Psi_A(t))^{-1} \mathcal{B}(\sigma, t) = (\sigma I - A(t))^{-1} B(t).$$

The validity of such identity can be proven as follows:

$$\begin{aligned}
 (\sigma I - A(t))^{-1}B(t) &= (I - \sigma^{-1}A(t))^{-1}\sigma^{-1}B(t) \\
 &= \sum_{k=0}^{\infty} \Phi_A(t, t-k)\sigma^{-k-1}B(t) \\
 &= \sum_{p=0}^{\infty} \sum_{i=0}^{T-1} \Phi_A(t, t-i-pT)B(t-i-1)\sigma^{-i-1-pT} \\
 &= \sum_{p=0}^{\infty} \Psi_A(t)\sigma^{-pT} \sum_{i=0}^{T-1} \Phi_A(t, t-i)B(t-i-1)\sigma^{-i-1} \\
 &= (I - \sigma^{-T}\Psi_A(t))^{-1} \sum_{i=0}^{T-1} \Phi_A(t, t-i)B(t-i-1)\sigma^{-i-1} \\
 &= (\sigma^T I - \Psi_A(t))^{-1} \sum_{j=0}^{T-1} \Phi_A(t, t+j-T+1)B(t+j)\sigma^j \\
 &= (\sigma^T I - \Psi_A(t))^{-1}\mathcal{B}(\sigma, t).
 \end{aligned}$$

Example 2.5 Consider again the 2-periodic 1-dimensional system defined in Example 2.3. The periodic transfer operator, as defined in (2.29), is

$$G(\sigma, t) = C(t)(\sigma - A(t))^{-1}B(t).$$

By noticing that

$$C(t)(\sigma - A(t))^{-1} = (\sigma - A(t)C(t+1)C(t)^{-1})^{-1}C(t+1),$$

it follows

$$G(\sigma, t) = (\sigma - A(t)C(t+1)C(t)^{-1})^{-1}C(t+1)B(t).$$

Multiplying by

$$(\sigma + C(t)C(t+1)^{-1}A(t+1))(\sigma + C(t)C(t+1)^{-1}A(t+1))^{-1},$$

we have

$$\begin{aligned}
 G(\sigma, t) &= (\sigma^2 - A(t)A(t+1))^{-1}(B(t+1)C(t)\sigma + C(t)A(t+1)B(t)) \\
 &= \frac{B(t+1)C(t)\sigma + C(t)A(t+1)B(t)}{\sigma^2 - A(t)A(t+1)},
 \end{aligned}$$

which is easily proven to coincide with the one worked out in Example 2.3. ■

We end this section by pointing out the effect of a change of basis in the state-space.

Consider the new state coordinates

$$\tilde{x}(t) = S(t)x(t)$$

where $S(\cdot)$ is a $n \times n$ T -periodic matrix, invertible for each t . The new representation of the given system is

$$\begin{aligned}\tilde{x}(t+1) &= \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t) \\ y(t) &= \tilde{C}(t)\tilde{x}(t) + \tilde{D}(t)u(t),\end{aligned}$$

with

$$\tilde{A}(t) = S(t+1)A(t)S(t)^{-1}, \quad \tilde{B}(t) = S(t+1)B(t) \quad (2.30a)$$

$$\tilde{C}(t) = C(t)S(t)^{-1}, \quad \tilde{D}(t) = D(t). \quad (2.30b)$$

In system theory, two systems defined by the quadruplets $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ and $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot))$ related as in (2.30) are said to be *algebraically equivalent*. It is easy to see that the transition matrices and the monodromy matrices associated with $A(\cdot)$ and $\tilde{A}(\cdot)$ are related to each other as follows

$$\Phi_{\tilde{A}}(t, i) = S(t)\Phi_A(t, i)S(i)^{-1} \quad (2.31a)$$

$$\Psi_{\tilde{A}}(t) = S(t)\Psi_A(t)S(t)^{-1}. \quad (2.31b)$$

Formula (2.31b) is a similarity transformation which, as is well-known, preserves eigenvalues. This means that the characteristic multipliers of a periodic system are independent of the state representation.

In the same vein, one can observe that the Markov parameters are invariant with respect to the change of basis, as an easy computation starting from Expressions (2.22) and (2.31a) reveals. As a consequence, the periodic transfer operator, the sampled transfer functions, and the EMP transfer operators are invariant as well. This conclusion is expected since all these quantities have to deal with the input–output behavior of the system, and, as such, must be insensitive to the choice of basis in state–space.

2.5 Adjoint Operator and Its State–Space Description

For a (discrete-time) time-invariant system with transfer function $W(z)$, the adjoint system is defined as the system with transfer function $W^\sim(z) = W(z^{-1})'$. Notice that the adjoint transfer function $W^\sim(z)$ may correspond to a system which is not causal. This is indeed a major difference between discrete-time and continuous-time systems. If (A, B, C, D) is a state–space realization of $W(z)$, then

$$W(z) = D + C(zI - A)^{-1}B, \quad W^\sim(z) = D' + B'(z^{-1}I - A')^{-1}C'.$$

Now, consider the periodic System (2.19) and the transfer operator

$$\begin{aligned} G(\sigma, t) &= M_0(t) + M_1(t)\sigma^{-1} + M_2(t)\sigma^{-2} + M_3(t)\sigma^{-3} + \dots \\ &= D(t) + C(t)(\sigma I - A(t))^{-1}B(t). \end{aligned}$$

The adjoint operator is defined as

$$\begin{aligned} G^\sim(\sigma, t) &= M_0(t)' + \sigma M_1(t)' + \sigma^2 M_2(t)' + \sigma^3 M_3(t)' + \dots \\ &= M_0(t)' + M_1(t+1)'\sigma + M_2(t+2)'\sigma^2 + M_3(t+3)'\sigma^3 + \dots \\ &= D(t)' + B(t)'(\sigma^{-1}I - A(t)')^{-1}C(t)'. \end{aligned}$$

It is easy to check that the state-space system corresponding to the adjoint operator is written in the form

$$A(t)'\lambda(t+1) = \lambda(t) - C(t)'\nu(t) \quad (2.32a)$$

$$\zeta(t) = B(t)'\lambda(t+1) + D(t)'\nu(t). \quad (2.32b)$$

This form is referred to in the literature as the *descriptor form*. In general, a periodic system in descriptor form is characterized by the modified state equation $E(t)x(t+1) = A(t)x(t) + B(t)u(t)$, where $E(\cdot)$ is T -periodic.

2.6 Bibliographical Notes

Most of the fundamental concepts presented in this chapter for periodic systems are rooted in the core of basic system theory; classical textbooks are [85, 203, 204, 255, 316]. For time-varying systems, the notion of transfer function operator goes back to the early 1950s, [315], and has been subsequently studied and elaborated further by a number of authors, see e.g., [205, 206]. For the fractional representation of PARMA models we refer to [132] and [47].

In the developments of the theory, further families of periodic models have been introduced in the literature. Among them, we mention the so-called *descriptor* periodic systems which, as said above, are characterized by the modified state equation $E(t)x(t+1) = A(t)x(t) + B(t)u(t)$, where $E(t)$ is T -periodic too. Another interesting approach is supplied by the behavioral modelization. We will not cover these aspects, and the interested reader can refer to the literature, e.g., [215, 277].

Chapter 3

Floquet Theory and Stability

In this chapter, the free-input periodic system

$$x(t+1) = A(t)x(t) \quad (3.1)$$

is considered, with $A(t)$ of constant dimension $n \times n$. First, the properties of the *monodromy matrix* are pointed out. This opens the way to the celebrated Floquet theory, which deals with the problem of **finding a periodic state-space transformation**, so that, in the new basis, the dynamic matrix is constant. The issue of periodic system stability follows, together with the notion of periodic Lyapunov function and periodic Lyapunov inequality. Finally, the concept of quadratic stability is introduced as a tool for the stability analysis of periodic systems subject to parameter uncertainty.

3.1 Monodromy Matrix

In Sect. 2.4, the notion of monodromy matrix of $A(\cdot)$ was already introduced as the transition matrix over one period:

$$\text{Transition matrix } \Phi_A(t, \tau) = \begin{cases} I & t = \tau \\ A(t)A(t-1) \cdots A(\tau) & t > \tau \end{cases} \quad (3.2)$$

$$\text{Monodromy matrix } \Psi_A(t) = \Phi_A(t+T, t).$$

As already stated, the eigenvalues of the monodromy matrix at time t are named characteristic multipliers at t .

The transition matrix governs the behavior of the solution of (3.1)

$$x(t) = \Phi_A(t, \tau)x(\tau), \quad t \geq \tau,$$

and is “biperiodic”, namely

$$\Phi_A(t+T, \tau+T) = \Phi_A(t, \tau), \quad \forall t \geq \tau.$$

The monodromy matrix $\Psi_A(t)$ is periodic

$$\Psi_A(t+T) = \Psi_A(t),$$

and supplies the T -sampled solution of (3.1), in that

$$x(t+kT) = \Psi_A(t)^k x(t), \quad t \in [0, T-1], \quad k \text{ positive integer.} \quad (3.3)$$

Since this matrix plays a major role in the analysis, we will now elaborate on its main properties.

3.1.1 The Characteristic Polynomial of a Monodromy Matrix

The characteristic polynomial of $\Psi_A(t)$

$$p_c(\lambda) = \det[\lambda I - \Psi_A(t)]$$

is independent of t .

Indeed, consider the monodromy matrices $\Psi_A(t)$ and $\Psi_A(\tau)$, with $\tau, t \in [0, T-1]$. These matrices can be written as:

$$\Psi_A(\tau) = FG, \quad \Psi_A(t) = GF \quad (3.4)$$

with

$$F = \begin{cases} \Phi(\tau, t) & \text{if } \tau \geq t \\ \Phi(\tau+T, t) & \text{if } \tau < t \end{cases}, \quad G = \begin{cases} \Phi(t+T, \tau) & \text{if } \tau \geq t \\ \Phi(t, \tau) & \text{if } \tau < t. \end{cases} \quad (3.5)$$

Now, take the singular value decomposition of G , i.e.,

$$G = U\Sigma V,$$

where

$$\Sigma = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \det[G_1] \neq 0, \quad (3.6)$$

and U, V are suitable orthogonal matrices. It follows that

$$VFGV' = (VFU)\Sigma, \quad U'GFU = \Sigma(VFU).$$

One can partition the matrix VFU as

$$VFU = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}, \quad (3.7)$$

Denoting by n_1 the dimension of the sub-matrix G_1 it is easy to show that:

$$\begin{aligned}\det[\lambda I_n - FG] &= \det[\lambda I_n - VFGV'] = \lambda^{n-n_1} \det[\lambda I_{n_1} - F_1 G_1] \\ \det[\lambda I_n - GF] &= \det[\lambda I_n - U'GFU] = \lambda^{n-n_1} \det[\lambda I_{n_1} - G_1 F_1].\end{aligned}$$

Since G_1 is invertible, matrices $F_1 G_1$ and $G_1 F_1$ are similar ($G_1^{-1}[G_1 F_1]G_1 = F_1 G_1$), so that $\det[\lambda I_{n_1} - F_1 G_1] = \det[\lambda I_{n_1} - G_1 F_1]$. Therefore, the characteristic polynomials of $\Psi_A(t)$ and $\Psi_A(\tau)$ do coincide.

3.1.2 Invariance of the Characteristic Multipliers

An important consequence of the previous observation is that the characteristic multipliers at time t , namely the eigenvalues of $\Psi_A(t)$, are in fact independent of t . Notice that not only the eigenvalues are constant, but also their multiplicities are time-invariant. This is why one can make reference to such eigenvalues as characteristic multipliers without any specification of the time point. In conclusion, by denoting by r the number of distinct non-zero eigenvalues λ_i of $\Psi_A(t)$, the characteristic polynomial can be given the form

$$p_c(\lambda) = \lambda^{h_c} \prod_{i=1}^r (\lambda - \lambda_i)^{k_{ci}},$$

where λ_i , h_c and k_{ci} do not depend on time. The characteristic multipliers are the λ_i 's and $\lambda = 0$ if $h_c \neq 0$.

Remark 3.1 *In some cases, one can encounter a periodic system where the dimension of the state-space periodically changes with time. Then matrix $A(t)$ is no longer a square matrix and its dimension is $n(t+1) \times n(t)$. Anyhow, it is still possible to define the monodromy matrix $\Psi_A(t)$ of $A(\cdot)$ in the usual way, i.e.,*

$$\Psi_A(t) = A(t+T-1)A(t+T-2) \cdots A(t).$$

Of course $\Psi_A(t)$ is an $n(t) \times n(t)$ square matrix. The characteristic multipliers at t of $A(\cdot)$ are still defined as the eigenvalues of $\Psi_A(t)$. The non-zero multipliers are still independent of time along with their multiplicity.

Conversely, the null multiplier may have a time-varying multiplicity. Indeed, by mimicking the proof above for the constant-dimensional case, one can write

$$\begin{aligned}\det[\lambda I_{n(t)} - FG] &= \det[\lambda I_{n(t)} - VFGV'] = \lambda^{n(t)-n_1} \det[\lambda I_{n_1} - F_1 G_1] \\ \det[\lambda I_{n(\tau)} - GF] &= \det[\lambda I_{n(\tau)} - U'GFU] = \lambda^{n(\tau)-n_1} \det[\lambda I_{n_1} - G_1 F_1],\end{aligned}$$

so that

$$\det[\lambda I - \Psi_A(t)] = \lambda^{n(t)-n(\tau)} \det[\lambda I - \Psi_A(\tau)].$$

In particular, take τ as the time point for which $n(\cdot)$ is minimum, i.e., $n(\tau) = n_m = \min\{n(t)\}, t = 0, 1, \dots, T-1$ and let $p_{cmin}(\lambda)$ be the corresponding characteristic polynomial. Then, it is apparent that $p_{cmin}(\lambda)$ divides the characteristic polynomial of the monodromy matrix at any other time point t . The quotient polynomial is $\lambda^{n(t)-n_m}$. In conclusion, n_m characteristic multipliers of $A(\cdot)$ are time-independent and constitute the so-called core spectrum of $\Psi_A(\cdot)$. Among them, there are in general both non-zero and zero multipliers. At a generic time point t , additional $n(t) - n_m$ zero multipliers appear.

3.1.3 The Minimal Polynomial of a Monodromy Matrix

Another important property is connected with the null space of $\lambda I_n - \Psi_A(t)$, where λ is any complex number different from zero. Precisely, one can show that, for any integer $k > 0$, if $\lambda \neq 0$, the dimension of the null space of $(\lambda I_n - \Psi_A(t))^k$ is independent of t , i.e.,

$$\text{Ker}[(\lambda I_n - \Psi_A(t))^k] = \text{constant}.$$

Indeed, consider again partition (3.4)–(3.7), from which by an easy computation, it follows that

$$(\lambda I_n - FG)^k = \begin{bmatrix} \lambda I_{n_1} - F_1 G_1 & 0 \\ * & \lambda^k I_{n-n_1} \end{bmatrix}, \quad (\lambda I_n - GF)^k = \begin{bmatrix} \lambda I_{n_1} - G_1 F_1 & * \\ 0 & \lambda^k I_{n-n_1} \end{bmatrix},$$

where $*$ is a term whose value does not matter. Due to the block-triangular structure of these matrices, if $\lambda \neq 0$ one can conclude that

$$\begin{aligned} \dim[\text{Ker}[(\lambda I_n - FG)^k]] &= \dim[\text{Ker}[(\lambda I_{n_1} - F_1 G_1)^k]] \\ \dim[\text{Ker}[(\lambda I_n - GF)^k]] &= \dim[\text{Ker}[(\lambda I_{n_1} - G_1 F_1)^k]], \end{aligned}$$

G_1 being non-singular, matrices $\lambda I_{n_1} - G_1 F_1$ and $\lambda I_{n_1} - F_1 G_1$ are similar, so that, if $\lambda \neq 0$, $\dim[\text{Ker}[(\lambda I_n - FG)^k]] = \dim[\text{Ker}[(\lambda I_n - GF)^k]]$ implies

$$\dim[\text{Ker}[(\lambda I_n - \Psi_A(t))^k]] = \text{constant}. \quad (3.8)$$

In particular, Eq. (3.8) can be applied to a non-zero characteristic multiplier $\lambda_i \neq 0$. Thus, by letting $\lambda = \lambda_i$, it is easy to see that the dimensions of the Jordan blocks associated with the non-zero characteristic multipliers of $A(\cdot)$ are independent of t . Obviously, this implies that the multiplicity of a non-zero characteristic multiplier λ_i in the *minimal polynomial* of $\Psi_A(t)$ does not change with time. Therefore, the minimal polynomial can be given the form

$$p_m(\lambda, t) = \lambda^{h_m(t)} \prod_{i=1}^r (\lambda - \lambda_i)^{k_{m_i}}, \quad (3.9)$$

where the multiplicities k_{m_i} are constant. As for the null characteristic multipliers (if any), the multiplicity $h_m(t)$ can vary with time, as shown by the following simple example.

Example 3.1 Consider the 2×2 matrix $A(\cdot)$ of period $T = 2$

$$A(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$\Psi_A(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Psi_A(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so that the two polynomials

$$p_m(\lambda, 0) = \lambda, \quad p_m(\lambda, 1) = p_c(\lambda) = \lambda^2$$

completely characterize the minimal polynomial. ■

Although $h_m(t)$ can vary with time, its variations are limited. Precisely, as it will be shown below, it turns out that

$$|h_m(t) - h_m(\tau)| \leq 1, \quad \forall t, \tau \in [0, T - 1]. \quad (3.10)$$

Obviously by denoting by

$$v_m(t) = h_m(t) + \sum_{i=0}^r k_{m_i}$$

the degree of the minimal polynomial $p_m(\lambda, t)$, Eq. (3.10) implies that

$$|v_m(t) - v_m(\tau)| \leq 1, \quad \forall t, \tau \in [0, T - 1].$$

In order to prove (3.10), recall that the minimal polynomial is the annihilating polynomial of minimal degree and monic; hence, considering again expression $\Psi_A(t) = GF$, Eq. (3.9) entails

$$(GF)^{h_m(t)} \prod_{i=1}^r (GF - \lambda_i I_n)^{k_{m_i}} = 0.$$

Pre-multiplying by F and post-multiplying by G , one obtains

$$(FG)^{h_m(t)} F \prod_{i=1}^r (GF - \lambda_i I_n)^{k_{m_i}} G = 0.$$

Since $(GF - \lambda_i I_n)G = G(FG - \lambda_i I_n)$, the above is equivalent to

$$(FG)^{h_m(t)+1} \prod_{i=1}^r (FG - \lambda_i I_n)^{k_{m_i}} = 0.$$

Recalling that $FG = \Psi_A(\tau)$, this expression implies that $h_m(\tau) \leq h_m(t) + 1$. By reversing the roles of t and τ , (3.10) follows.

We summarize the results obtained above on the characteristic and minimal polynomials in the following

Proposition 3.1 *The characteristic polynomial of the monodromy matrix $\Psi_A(t)$ does not depend on t . As for the minimal polynomial, the only possible reason for the dependence upon t is limited to the multiplicity $h_m(t)$ of the null singularities. Such multiplicity can change under the constraint*

$$|h_m(t) - h_m(\tau)| \leq 1, \quad \forall t, \tau \in [0, T - 1].$$

■

When matrix $A(t)$ is non-singular for each t , all characteristic multipliers are all different from zero. In such a case, the system is *reversible*, in that the state $x(\tau)$ can be uniquely recovered from $x(t)$, $t > \tau$.

3.2 Floquet Theory

One of the long-standing issues in periodic systems is whether it is possible to reduce the study of the original *periodic* system to that of a *time-invariant* one. When focusing on the free System (3.1), this leads to the so-called Floquet problem precisely consisting in finding a T -periodic invertible state-space transformation $\hat{x}(t) = S(t)x(t)$ such that, in the new coordinates, the system is time-invariant:

$$\hat{x}(t+1) = \hat{A}\hat{x}(t), \quad (3.11)$$

where matrix \hat{A} , known as *Floquet factor* is constant. Should this be possible, $A(\cdot)$ would be *algebraically equivalent* to a *constant* matrix.

In this respect, a preliminary observation is that, if the periodic system could be written in the time-invariant form (3.11), then the monodromy matrix $\Psi_A(t)$ would be related to \hat{A}^T in a simple way by means of a similarity transformation. Indeed, from

$$\hat{x}(t) = S(t)x(t), \quad \hat{x}(t+T) = \hat{A}^T \hat{x}(t), \quad x(t+T) = \Psi_A(t)x(t)$$

it follows that

$$\Psi_A(t) = S(t)^{-1} \hat{A}^T S(t). \quad (3.12)$$

Hence, the characteristic multipliers are given by the T -th power of the eigenvalues of the Floquet factor \hat{A} .

We are now in a position to tackle the Floquet problem. To this purpose, it is advisable to distinguish the case when the system is reversible from the case when it is not reversible.

3.2.1 Discrete-time Floquet Theory for the Reversible Case

If $\Psi_A(t)$ is non-singular, Eq. (3.12) admits a solution \hat{A} for any invertible matrix $S(t)$. Indeed, from the stated reversibility assumption, \hat{A}^T can be determined from (3.12). Hence, a T -th root can be computed.

Moreover, if the system is reversible, there is a simple procedure to determine the change of basis enabling the “invariantization”. To be precise, by considering the reference time point $t = 0$, select any invertible $S(0)$ and find \hat{A} by solving (3.12). Then, compute $S(i)$, $i = 1, 2, \dots, T - 1$ according to the following chain of conditions

$$\begin{aligned} S(1) &= \hat{A}S(0)A(0)^{-1} \\ S(2) &= \hat{A}S(1)A(1)^{-1} \\ &\vdots \\ S(T-1) &= \hat{A}S(T-2)A(T-2)^{-1}. \end{aligned}$$

Now, taking the periodic extension $S(kT + i) = S(i)$, $k \geq 0$, Eq. (3.12) can be exploited to verify that

$$\hat{A} = S(t+1)A(t)S(t)^{-1} \quad (3.13)$$

holds true for each t . In conclusion, in the reversible case the Floquet problem is solved by

$$\hat{A} = [S(0)\Psi_A(0)S(0)^{-1}]^{1/T}, \quad S(i) = \hat{A}^i S(0)\Phi_A(i, 0)^{-1},$$

$S(0)$ being any invertible matrix.

3.2.2 Discrete-time Floquet Theory for the Non-reversible Case

If the system is not reversible, the theory is much more involved. First of all it is worthy pointing out that in this case the “invariantization” is not always achievable.

Example 3.2 *The 2-periodic system*

$$A(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

can be transformed into a time-invariant one. Indeed, it is easy to see that an appropriate choice for the state-space transformation is

$$S(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which yields

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example 3.3 *The periodic system defined in Example 3.1 cannot be transformed into an invariant one. This fact can be easily understood by noticing that $\hat{A}^2 S(0) = S(0)A(1)A(0) = 0$ whereas $\hat{A}^2 S(1) = S(1)A(0)A(1) \neq 0$. ■*

One can easily derive a necessary condition for “time-invariantization”, concerning the rank of the transition matrix. Indeed, if a non-singular $S(\cdot)$ satisfying (3.13) exists, then

$$S(\tau+k)\Phi_A(\tau+k, \tau)S(\tau)^{-1} = \hat{A}^k.$$

This implies that the rank of $\Phi_A(\tau+k, \tau)$ coincides with the rank of \hat{A}^k and as such, does not depend on τ . In general, however, $\text{rank}[\Phi_A(\tau+k, \tau)]$ depends upon k , and is therefore denoted by r_k . Only for $k \geq n$ such rank becomes constant; indeed \hat{A} is a $n \times n$ matrix so that $\text{rank}[\hat{A}^k] = \text{rank}[\hat{A}^n]$, whenever $k \geq n$. In conclusion, a necessary condition for “time-invariantization” is

$$\text{rank}[\Phi_A(\tau+k, \tau)] = r_k \text{ independent of } \tau, \forall \tau \in [0, T-1], \forall k \in [1, n]. \quad (3.14)$$

Condition (3.14) is also sufficient. The proof of this claim is much more complex and can be outlined as follows. For each t , let $X(t)$ be a T -periodic matrix such that $\text{Im}[X(t)] = \text{Ker}[A(t)]$, and then define a state-space transformation $V(\cdot)$ with

$$V(t)^{-1} = \begin{bmatrix} X(t) & Y(t) \end{bmatrix},$$

where the T -periodic matrix $Y(t)$ can be freely chosen so as to render $V(t)^{-1}$ invertible. It follows that $A(t)X(t) = 0$ so that, in the new basis, matrix $A(t)$ can be represented as

$$\tilde{A}(t) = V(t+1)A(t)V(t)^{-1} = \begin{bmatrix} 0 & A_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix}.$$

We now distinguish between two cases. First, we assume that $A_{22}(t)$ is invertible for each t , and then we will consider the singular case.

Suppose that matrix $A_{22}(t)$ is invertible, $\forall t$. In this case, it is easy to work out an invertible transformation $W(t)$ such that $W(t+1)V(t+1)A(t)V(t)^{-1}W(t)^{-1} = \hat{A}$ is constant. Indeed, the monodromy matrix of $\tilde{A}(\cdot)$ is

$$\Psi_{\tilde{A}}(0) = \begin{bmatrix} 0 & A_{12}(T-1)\Phi_{A_{22}}(T-1, 0) \\ 0 & \Psi_{A_{22}}(0) \end{bmatrix}.$$

Now, in view of the structure of $\tilde{A}(\cdot)$, one can partition $W(t)$ and \hat{A} as follows

$$W(t) = \begin{bmatrix} W_{11}(t) & W_{12}(t) \\ 0 & W_{22}(t) \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}.$$

It turns out that

$$\hat{A}^T = \begin{bmatrix} 0 & \hat{A}_{12}\hat{A}_{22}^{T-1} \\ 0 & \hat{A}_{22}^T \end{bmatrix}.$$

Equating \hat{A}^T with $\Psi_{\hat{A}}(0)$ it follows that \hat{A}_{22} can be found from the condition $\hat{A}_{22}^T = \Psi_{\hat{A}_{22}}(0)$. This equation admits an invertible solution \hat{A}_{22} since $\Psi_{\hat{A}_{22}}(0)$ is invertible. As for \hat{A}_{12} , it can be uniquely computed as

$$\hat{A}_{12} = A_{12}(T-1)\Phi_{A_{22}}(T-1,0)\hat{A}_{22}^{1-T}.$$

Finally, the block elements of matrix $W(t)$ must solve the system of equations

$$\begin{aligned} \hat{A}_{12}W_{22}(t) &= W_{11}(t+1)A_{12}(t) + W_{12}(t+1)A_{22}(t) \\ \hat{A}_{22}W_{22}(t) &= W_{22}(t+1)A_{22}(t). \end{aligned}$$

Since $A_{22}(t)$ is non-singular $\forall t$, the last equation admits the solution

$$W_{22}(t) = \hat{A}_{22}^t W_{22}(0) \Phi_{A_{22}}(t, 0)^{-1}$$

with $W_{22}(0)$ invertible and satisfying the linear equation

$$W_{22}(0) = \hat{A}_{22}^T W_{22}(0) \Phi_{A_{22}}(T, 0)^{-1}.$$

Once $W_{22}(\cdot)$ has been computed, from the first equation, one can evaluate

$$W_{12}(t+1) = (\hat{A}_{12}W_{22}(t) - W_{11}(t+1)A_{12}(t))A_{22}(t)^{-1},$$

where $W_{11}(\cdot)$ is any invertible T -periodic matrix.

Turn now to the case when matrix $A_{22}(t)$ is singular for some t . Then, one can exploit the fact that the rank of $[A_{12}(t)' A_{22}(t)']'$ is maximum to prove that $A_{22}(t)$ inherits all rank properties of $A(t)$. Namely, the rank of $\Phi_{A_{22}}(\tau+k, \tau)$ is independent of τ for $k > 0$. Hence, all the above arguments may be well applied to $A_{22}(t)$ and so on so forth. It can be shown that this procedure always converges if condition (3.14) is met with.

To summarize, the following result holds true.

Proposition 3.2 *Given a T -periodic matrix $A(\cdot)$ there exists an invertible T -periodic transformation $S(\cdot)$ and a constant real matrix \hat{A} such that $S(t+1)A(t)S(t)^{-1} = \hat{A}$ if and only if $\text{rank}[\Phi_A(\tau+k, \tau)]$ is independent of τ for all $k \in [1, n]$.*

Example 3.4 *Consider again Example 3.2. As for point (i), notice that*

$$A(0)A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(1)A(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so that the rank condition (3.14) is satisfied. Vice versa, in Example 3.3

$$A(0)A(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A(1)A(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and (3.14) is not satisfied. ■

In conclusion, in discrete-time, not all periodic matrices can be “time-invariantized” by a change of coordinates. The time-invariantization can be performed if and only if the rank condition (3.14) is met with, a condition which is obviously satisfied for reversible systems.

3.3 Stability

The stability analysis of any time-varying linear system reduces to the study of the behavior of the free motion as a function of the initial state. A system is stable if, for any initial condition, the free-motion vanishes. In the time-invariant case, the property of stability is determined by the position in the complex plane of the eigenvalues and their multiplicity as roots of the minimal polynomial of the dynamical matrix. A similar condition holds for periodic systems by making reference to the monodromy matrix, as we will see in Sect. 3.3.1. On the contrary, the eigenvalues of the dynamic matrix $A(\cdot)$ do not play, in general, any role for stability (Sect. 3.3.2). The Lyapunov paradigm, a major tool for stability analysis, is the subject of Sect. 3.3.3. Some issues regarding robust stability are finally discussed.

3.3.1 Stability and Characteristic Multipliers

As already mentioned, in the study of stability of periodic systems, one can focus on the free-state Eq. (3.1) from the very beginning. If one considers t as a given time point in the period, Eq. (3.3) describes the sampled behavior of the system. On the other hand, in the time variable k , (3.3) is a discrete-time and *time-invariant* system, the dynamic matrix of which is the monodromy matrix $\Psi_A(t)$. Now, we know that the monodromy matrix has distinctive features, including the fact that its eigenvalues and the multiplicity of the non-null roots of the minimal polynomial are time-independent (see Sect. 3.1). This leads to the following celebrated result.

Proposition 3.3 (i) *The System (3.1) is stable if and only if the characteristic multipliers of $A(\cdot)$ have modulus lower than 1.* (ii) *The System (3.1) is stable if and only if the characteristic multipliers of $A(\cdot)$ have modulus lower than or equal to 1 and those characteristic multipliers with unit-modulus are simple roots of the minimal polynomial of $\Psi_A(t)$.* ■

For linear time-invariant stable systems, the rate of decay of the free motion is exponential. In view of (3.3), this property is *ipso facto* transferred to periodic systems. In other words, the periodic System (3.1) is stable if and only if it is exponentially stable.

Often one says that $A(\cdot)$ is stable to say that the periodic system is stable, namely its characteristic multipliers belong to the open unit disk.

Remark 3.2 *Following Remark 3.1, one can also consider systems with periodically state-space dimensions $n(\cdot)$. In this case, matrix $A(t)$ has dimensions $n(t+1) \times n(t)$ and its monodromy matrix $\Psi_A(t)$ at t is a square $n(t)$ -dimensional matrix. One can say that $A(\cdot)$ is stable if the core spectrum of $\Psi_A(t)$ is in the open unit disc.*

3.3.2 Time-freezing and Stability

Periodic systems are frequently used as examples or counterexamples for the study of stability of time-varying systems. However, the existing literature mainly insists on continuous-time systems.

First we address the issue whether the stability of a discrete-time periodic system can be related to the eigenvalues of the periodic matrix $A(\cdot)$. We will perform the analysis by means of a few elementary examples.

Example 3.5 *For $T = 2$, $n = 2$, let*

$$A(t) = \begin{bmatrix} 0 & a(t) \\ 2 - a(t) & 0 \end{bmatrix}, \quad a(t) = \begin{cases} 0 & t = 0 \\ 2 & t = 1. \end{cases}$$

The monodromy matrix is given by

$$\Psi_A(T, 0) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix},$$

so that the system is unstable. Nevertheless, the eigenvalues of $A(t)$ are located at the origin for each t .

Example 3.6 *For $T = 3$, $n = 1$, let*

$$A(t) = \begin{cases} 0.5, & t = 0 \\ 0.5, & t = 1 \\ 2, & t = 2. \end{cases}$$

The monodromy matrix is $\Psi_A(T, 0) = 0.5$ so that the system is stable. However, $|A(t)|$ exceeds 1 at $t = 3$.

Example 3.7 For $T = 2$, $n = 2$, let

$$A(0) = \begin{bmatrix} 2 & 0 \\ -3.5 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

The monodromy matrix is given by

$$\Psi_A(T, 0) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix},$$

and the system is stable despite the fact that the one eigenvalue of $A(t)$ is equal to 2 for each t . ■

Example 3.5 shows that the system may be unstable even if the “frozen matrices” $A(t)$ are stable for each $t \in [0, T - 1]$. The second example illustrates that one can encounter an stable periodic system with matrix $A(t)$ having an eigenvalue of modulus larger than 1 at some time point. Even more, Example 3.7 shows that an eigenvalue of $A(t)$ may always be of modulus larger than 1 and nevertheless, the system is stable.

One of the few links that can be established between the stability of the frozen matrices $(A(t), t \in [0, T - 1])$, and that of the periodic system is that, if all eigenvalues of $A(t)$ have modulus larger than 1 at each time point in the period, then the periodic system is unstable. Indeed, being $\Psi_A(T, 0) = A(T - 1)A(T - 2) \cdots A(0)$ it is apparent that $\det[\Psi_A(T, 0)]$ is larger than one if all eigenvalues of $A(0)$, $A(1)$, \dots and $A(T - 1)$ are larger than 1.

Perhaps not surprisingly, it is possible to prove that a periodic system is stable if $A(\cdot)$ evolves around a stable constant matrix A with sufficiently small variations. In order to precisely formalize this claim, the notion of matrix norm is preliminarily needed.

Throughout this text, given any real matrix $M = [m_{ij}]$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, h$, by $\|M\|$ we mean

$$\|M\| = \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|},$$

where the norm of a vector x is $\|x\| = (x'x)^{1/2}$. As easily verified, this is indeed a norm, known as *spectral norm*. Among other things, $\|M\|$ coincides with the maximum singular value of M , i.e., the square of the largest eigenvalue of $M'M$ (or MM' equivalently).

We recall that this norm has the additional property that $\|MN\| \leq \|M\|\|N\|$, an inequality which will be used in the proof below.

Proposition 3.4 Consider a stable matrix \bar{A} , namely a matrix with all eigenvalues with modulus lower than 1. Then there exist $\lambda \in (0, 1)$ and $K \geq 1$ such that

$$\|\bar{A}^t\| \leq K\lambda^t, \quad \forall t \geq 0. \quad (3.15)$$

Then, any T -periodic matrix $A(\cdot)$ such that

$$\sum_{t=0}^{T-1} \|A(t) - \bar{A}\| < \frac{\lambda}{K} \ln\left(\frac{1}{K\lambda^T}\right), \quad (3.16)$$

is stable (its characteristic multipliers belong to the open unit disk).

Proof The periodic system

$$x(t+1) = A(t)x(t)$$

can be seen as a perturbation of a time-invariant system with dynamic matrix \bar{A} as follows

$$\begin{aligned} x(t+1) &= \bar{A}x(t) + \varepsilon(t) \\ \varepsilon(t) &= (A(t) - \bar{A})x(t). \end{aligned}$$

A trivial application of the Lagrange formula leads to

$$\Phi_A(t, 0)x(0) = \bar{A}^t x(0) + \sum_{i=0}^{t-1} \bar{A}^{t-1-i} (A(i) - \bar{A}) \Phi_A(i, 0)x(0), \quad \forall x(0), \forall t \geq 1,$$

whence

$$\Phi_A(t, 0) = \bar{A}^t + \sum_{i=0}^{t-1} \bar{A}^{t-1-i} (A(i) - \bar{A}) \Phi_A(i, 0), \quad \forall t \geq 1.$$

In view of (3.15),

$$\|\Phi_A(t, 0)\| \leq K\lambda^t + \sum_{i=0}^{t-1} K\lambda^{t-1-i} \|A(i) - \bar{A}\| \|\Phi_A(i, 0)\|, \quad \forall t \geq 1.$$

Now, letting

$$v(t) = \|\Phi_A(t, 0)\| \lambda^{-t}, \quad (3.17)$$

it follows that

$$v(t) \leq K + \sum_{i=0}^{t-1} \frac{K}{\lambda} \|A(i) - \bar{A}\| v(i), \quad \forall t \geq 1. \quad (3.18)$$

Being $v(0) = 1$, Inequality (3.18) leads to

$$v(t) \leq \prod_{i=1}^{t-1} \left(1 + \frac{K}{\lambda} \|A(i) - \bar{A}\|\right) \left(K + \frac{K}{\lambda} \|A(0) - \bar{A}\|\right).$$

Now, recall that $K \geq 1$ so that $K \leq K^2$. Hence

$$v(t) \leq K \prod_{i=0}^{t-1} \left(1 + \frac{K}{\lambda} \|A(i) - \bar{A}\|\right).$$

Now using (3.17) for $t = T$ one obtains

$$\|\Psi_A(0)\| = \|\Phi_A(T, 0)\| \leq K\lambda^T \prod_{i=0}^{T-1} (1 + \frac{K}{\lambda} \|A(i) - \bar{A}\|),$$

i.e.,

$$\ln(\|\frac{\Psi_A(0)}{\lambda^T}\|) \leq \sum_{i=0}^{T-1} \ln(1 + \frac{K}{\lambda} \|A(i) - \bar{A}\|) \leq \frac{K}{\lambda} \sum_{i=0}^{T-1} \|A(i) - \bar{A}\|.$$

Thanks to (3.16), it finally follows that

$$\ln(\frac{\|\Psi_A(0)\|}{K\lambda^T}) < \ln(\frac{1}{K\lambda^T}),$$

i.e., $\|\Psi_A(0)\| < 1$. This last inequality entails system stability. ■

Note that condition (3.16) is rather conservative, as illustrated in the following example.

Example 3.8 Let $\bar{A} = 0.5$ so that one can take $\lambda = 0.5$ and $K = 1$. With this choice, condition (3.15) is obviously satisfied. The class of scalar periodic systems of period $T = 2$ implicitly defined by condition (3.16) can be easily computed. This class is defined by a pair $\{A(0), A(1)\}$ satisfying

$$|A(0) - 0.5| + |A(1) - 0.5| < \ln(2).$$

The corresponding set of pairs $\{A(0), A(1)\}$ is depicted in Fig. 3.1. As can be seen, this set is much smaller than the set of all pairs $\{A(0), A(1)\}$ for which the periodic system is stable.

3.3.3 The Lyapunov Paradigm

One of the most fundamental tools for the stability analysis of linear systems is constituted by the Lyapunov conditions. These conditions usually require the existence of a positive semi-definite¹ solution of a matrix equation, or matrix inequality.

Lyapunov inequalities.

Herein, we will deal with the following *difference Lyapunov inequalities* (DLI)

$$A(t)P(t)A(t)' - P(t+1) < 0 \quad (3.19)$$

$$A(t)'Q(t+1)A(t) - Q(t) < 0. \quad (3.20)$$

¹ As usual, we say that a square real matrix M is positive definite [positive semi-definite] if it is symmetric and $x'Mx > 0$ for each $x \neq 0$ [$x'Mx \geq 0$ for each $x \neq 0$]. If M and N are square matrices of the same dimension, $M > N$ [$M \geq N$] means that $M - N$ is positive definite [positive semi-definite]. We also use the symbol $M < N$ [$M \leq N$] to say that $M - N$ is negative definite [negative semi-definite].

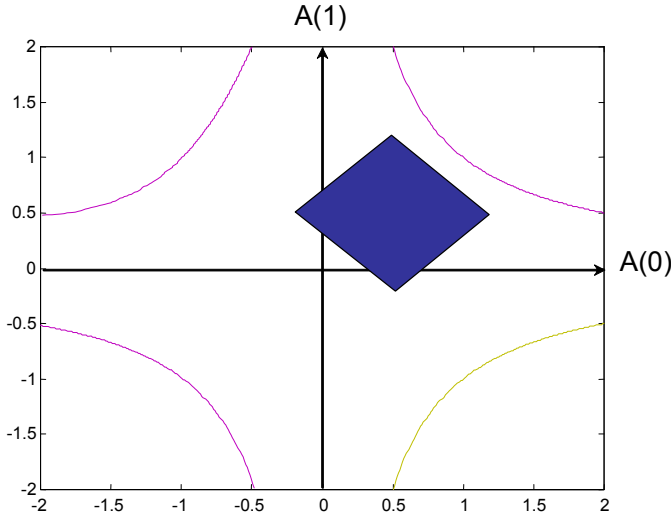


Fig. 3.1 The set of all values of $\{A(0), A(1)\}$ satisfying the sufficient condition (3.16) with $\bar{A} = 0.5$ (filled region), versus the set of $\{A(0), A(1)\}$ for which a scalar periodic system of period $T = 2$ is stable (delimited by the dotted curve)

In the system and control realm, it is well-known that Inequality (3.19) can be associated with a filtering problem, whereas (3.20) corresponds to a control problem. Therefore these inequalities are often referred to as the *filtering DLI* and *control DLI*, respectively. In the sequel, we will focus on Inequality (3.20) for the derivation of the various results. For conciseness, we omit the analogous derivations relative to Inequality (3.19).

We first introduce a celebrated matrix equality known as *Schur complement Lemma*. By means of this lemma, it will be possible to write the DLI's in equivalent higher dimension inequalities, most useful for analysis purposes.

Lemma 3.1 (Schur complement Lemma)

Consider the partitioned matrix:

$$R = \begin{bmatrix} E & F \\ F' & H \end{bmatrix}$$

with $E = E'$ and $H = H'$. Then,

(a) Matrix R is negative definite if and only if

$$H < 0, \quad E - FH^{-1}F' < 0.$$

(b) Matrix R is negative semi-definite if and only if²

$$H \leq 0, \quad E - FH^\dagger F' \leq 0, \quad F(I - HH^\dagger) = 0. \quad \blacksquare$$

Thanks to the Schur complement lemma, one can re-write the DLI's in equivalent forms.

Lemma 3.2 *The following statements are equivalent.*

(i) *There exists a T -periodic positive definite solution $Q(\cdot)$ of the Lyapunov inequality*

$$A(t)'Q(t+1)A(t) - Q(t) < 0.$$

(ii) *There exists a T -periodic solution $Q(\cdot)$ of the inequality*

$$\begin{bmatrix} -Q(t) & A(t)' \\ A(t) & -Q(t+1)^{-1} \end{bmatrix} < 0. \quad (3.21)$$

(iii) *There exist two T -periodic matrices $Q(\cdot)$ and $Z(\cdot)$ solving the following inequality*

$$\begin{bmatrix} -Q(t) & A(t)'Z(t) \\ Z(t)'A(t) & Q(t+1) - Z(t) - Z(t)' \end{bmatrix} < 0. \quad (3.22)$$

Proof Statements (i) and (ii) are indeed equivalent thanks to the Schur complement lemma. As for the equivalence between (ii) and (iii), assume first that (ii) holds. Then, taking $Z(t) = Q(t+1)$ and pre-multiplying and post-multiplying (3.22) by

$$\begin{bmatrix} I & 0 \\ 0 & Q(t+1)^{-1} \end{bmatrix},$$

one obtains (3.21). Conversely, assume that condition (iii) is met with. Then, $Z(t) + Z(t)' > Q(t+1) > 0$ so that $Z(t)$ is invertible (for each t). Consider now $M(t) = (Z(t)' - Q(t+1))Q(t+1)^{-1}(Z(t) - Q(t+1))$. Obviously, $M(t) \geq 0$. This entails that $Q(t+1) - Z(t) - Z(t)' \geq -Z(t)'Q(t+1)^{-1}Z(t)$. Therefore

$$\begin{aligned} 0 &> \begin{bmatrix} -Q(t) & A(t)'Z(t) \\ Z(t)'A(t) & Q(t+1) - Z(t) - Z(t)' \end{bmatrix} \\ &\geq \begin{bmatrix} -Q(t) & A(t)'Z(t) \\ Z(t)'A(t) & -Z(t)'Q(t+1)^{-1}Z(t) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & Z(t)' \end{bmatrix} \begin{bmatrix} -Q(t) & A(t)' \\ A(t) & -Q(t+1)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Z(t) \end{bmatrix}. \end{aligned}$$

Since $Z(t)$ is invertible, condition (ii) follows. \blacksquare

² The symbol \dagger denotes the Moore-Penrose pseudo-inverse, namely, for a given L , matrix L^\dagger satisfies $L^\dagger L L^\dagger = L^\dagger$ and $LL^\dagger L = L$. For instance, if L is full-row rank, then $L^\dagger = L'(LL')^{-1}$, and, if L is full-column rank, then $L^\dagger = (L'L)^{-1}L'$.

Notice that the original Lyapunov Inequality (3.20) is linear in the unknown $Q(\cdot)$ but non-linear in the system matrix $A(\cdot)$. On the contrary, the equivalent block formulation (3.21) is non-linear in the unknown $Q(\cdot)$ but affine in $A(\cdot)$. This fact can be properly exploited to deal with the robust filtering and stabilization problems. As for Inequality (3.22), note that by letting $Z(t) = Q(t+1)$, it follows

$$\begin{bmatrix} -Q(t) & A(t)'Q(t+1) \\ Q(t+1)A(t) & -Q(t+1) \end{bmatrix} < 0.$$

Premultiplying and postmultiplying by

$$\begin{bmatrix} I & 0 \\ 0 & Q(t+1)^{-1} \end{bmatrix} < 0,$$

this inequality boils down to Inequality (3.21). Therefore it can be seen as a generalization of the previous DLI. The new unknown $Z(\cdot)$ is useful when trying to reduce the intrinsic conservatism of the certain robust stability conditions, as we will seen in the sequel.

The analogous statement for Inequality (3.19) is given below.

Lemma 3.3 *The following statements are equivalent.*

(i) *There exists a T -periodic positive definite solution $P(\cdot)$ of the Lyapunov inequality*

$$A(t)P(t)A(t)' - P(t+1) < 0.$$

(ii) *There exists a T -periodic solution $P(\cdot)$ of the inequality*

$$\begin{bmatrix} -P(t+1) & A(t) \\ A(t)' & -P(t)^{-1} \end{bmatrix} < 0.$$

(iii) *There exist two T -periodic matrices $Q(\cdot)$ and $W(\cdot)$ solving the following inequality*

$$\begin{bmatrix} -P(t+1) & A(t)W(t) \\ W(t)'A(t)' - P(t) + W(t) + W(t)' \end{bmatrix} < 0.$$

Remark 3.3 *As seen above, the conditions expressed by Inequalities (3.19) and (3.20) are necessary and sufficient for stability. As such, they are equivalent to each other. This equivalence can also be proven directly. Precisely, suppose that $P(\cdot)$ is a positive definite solution of (3.19). Hence, there exists a periodic and positive definite $R(\cdot)$ such that*

$$P(t+1) = A(t)P(t)A(t)' + R(t).$$

Then, by resorting to the matrix inversion lemma³, the expression for the inverse matrix $P(t+1)^{-1}$ is obtained. Furthermore, pre-multiplying by $A(t)'$ and post-multiplying by $A(t)$, and letting $Q(t) = P(t)^{-1}$, one obtains

$$A(t)'Q(t+1)A(t) - Q(t) = -Q(t)(Q(t) + A(t)'R(t)^{-1}A(t))^{-1}Q(t) < 0$$

so that Inequality (3.20) is recovered. The converse implication can be proven by similar considerations.

3.3.4 Lyapunov Stability Condition

We are now in a position to derive the main stability result in terms of Lyapunov inequalities. With this objective in mind, observe that, by subsequent iterations on Eq. (3.20), one obtains

$$\begin{aligned} Q(t) &> A(t)'Q(t+1)A(t) \\ &> A(t)'[A(t+1)'Q(t+2)A(t+1)]A(t) \\ &> \dots > \\ &> \Psi_A(t)'Q(t+T)\Psi_A(t). \end{aligned}$$

If (3.20) admits a T -periodic solution, then one can set $Q(t+T) = Q(t)$, so that

$$Q(t) > \Psi_A(t)'Q(t)\Psi_A(t).$$

The positive semi-definiteness of $Q(\cdot)$ implies that $\Psi_A(t)$ must be contractive, i.e., $A(\cdot)$ is stable. Vice versa, suppose that $A(\cdot)$ is a T -periodic stable matrix, and let

$$Q(t) = \sum_{i=0}^{\infty} \Phi_A(t+i, t)' \Phi_A(t+i, t).$$

This matrix series is convergent. Indeed, for any positive integer r ,

$$\sum_{i=0}^{rT-1} \Phi_A(t+i, t)' \Phi_A(t+i, t) = \sum_{i=0}^{r-1} \Psi_A(t)'^k M \Psi_A(t)^k$$

with

$$M = \sum_{j=0}^{T-1} \Phi_A(t+j, t)' \Phi_A(t+j, t).$$

³ Given a square invertible matrix expressed as $F + GHK$, with F and H invertible too, its inverse is given by $(F + GHK)^{-1} = F^{-1} - F^{-1}G(H^{-1} + KF^{-1}G)^{-1}KF^{-1}$. This identity is known as the matrix inversion lemma. The proof can be found in most textbooks of matrix theory.

The stability assumption implies that $\|\Psi_A(t)^k\| \leq K\lambda^k$ for some $K \geq 1$ and $\lambda < 1$. Hence

$$\left\| \sum_{i=0}^{rT-1} \Phi_A(t+i, t)' \Phi_A(t+i, t) \right\| \leq K^2 \|M\| \sum_{k=0}^{r-1} \lambda^{2k}.$$

Therefore the quantity

$$\sum_{i=0}^{\tau} \Phi_A(t+i, t)' \Phi_A(t+i, t)$$

is bounded for any τ ; moreover it is obviously monotonically non-decreasing in τ . In conclusion,

$$Q(t) = \sum_{i=0}^{\infty} \Phi_A(t+i, t)' \Phi_A(t+i, t)$$

exists. Moreover, $Q(t)$ is T -periodic and positive semi-definite. Finally, a simple computation shows that

$$Q(t) = A(t)' Q(t+1) A(t) + I.$$

Hence (3.20) is satisfied. This completes the proof of the following main Lyapunov stability condition.

Proposition 3.5 (Periodic Lyapunov Lemma).

The T -periodic matrix $A(\cdot)$ is stable if and only if there exists a T -periodic positive semi-definite solution of Inequality (3.20) or Inequality (3.19). ■

As already stated, the two inequalities express the same condition in a different form; namely the existence of a T -periodic positive semi-definite solution $P(\cdot)$ to Eq. (3.19) is equivalent to the existence of a T -periodic positive semi-definite solution to Eq. (3.20). Notice also that if Inequality (3.20) admits a positive semi-definite solution $Q(\cdot)$, then such a solution must satisfy inequality $Q(t) > A(t)' Q(t+1) A(t)$, and is therefore positive definite. The same is true for Eq. (3.19).

Remark 3.4 *The periodic Lyapunov Lemma stated in Proposition 3.5 can be easily generalized to cope with the property of the exponential stability of $A(\cdot)$. Precisely, given a real T -periodic function $r(\cdot)$, with $0 < r(t) < 1$, $\forall t$, matrix $A(\cdot)$ is said to be r -exponentially stable if $r(\cdot)^{-1} A(\cdot)$ is stable. The necessary and sufficient condition for $A(\cdot)$ to be r -exponentially stable is that one of the following Lyapunov inequalities*

$$\begin{aligned} A(t)P(t)A(t)' - r(t)^2 P(t+1) &< 0 \\ A(t)' Q(t+1) A(t) - r(t)^2 Q(t) &< 0 \end{aligned}$$

admits a T -periodic positive semi-definite solution. Of course, the existence of a T -periodic positive semi-definite solution $P(\cdot)$ to the first equation is equivalent to the existence of a T -periodic positive semi-definite solution $Q(\cdot)$ to the second one. Notice that r -exponential stability implies that the characteristic multipliers of $A(\cdot)$

are inside the disk of radius $r(T-1)r(T-2)\cdots r(0)$. Such a radius becomes r^T in case $r(t)$ is constant, $r(t) = r$.

3.3.5 Lyapunov Function

The Lyapunov Inequality (3.20) has a tight link with another concept, that of Lyapunov function. Precisely, a Lyapunov function for system (3.1) is any function $V(x, t)$ such that

- (i) $V(x, t)$ is continuous in x , $x \neq 0$.
- (ii) $V(x, t) > 0$, $\forall t$.
- (iii) $V(0, t) = 0$, $\forall t$
- (iv) $V(x(t+1), t+1) - V(x(t), t) < 0$, $\forall x(\cdot)$ solving (3.1).

It is easy to see that the existence of a Lyapunov function is equivalent to the stability of System (3.1). Indeed, if one has a T -periodic positive definite solution of Inequality (3.20), then it is straightforward to verify that

$$V(x, t) = x'Q(t)x$$

is a Lyapunov function.

3.3.6 Robust Stability

Up to this point, we have considered the stability problem for a system described by an unique dynamical model. In order to cope with uncertainties, one is often led to consider a *family* of models describing the dynamics of the given system. This has spurred the research activity in the field of robust stability. Precisely, one may consider a periodic system

$$x(t+1) = A(t)x(t), \quad A(\cdot) \in \mathcal{A}, \quad (3.23)$$

where \mathcal{A} denotes a family of T -periodic matrices. Then, robust stability is typically characterized according to the following definitions.

Definition 3.1 System (3.23) is \mathcal{A} -robustly stable if $A(\cdot)$ is stable $\forall A(\cdot) \in \mathcal{A}$. ■

A useful tool to assess robust stability is provided by the notion of *quadratic stability*.

Definition 3.2 System (3.23) is said to be quadratically stable if there exists a function $V(x, t) = x'Q(t)x$, with $Q(t+T) = Q(t)$, $\forall t$ that can serve as a Lyapunov function for each $A(\cdot) \in \mathcal{A}$. ■

Obviously quadratic stability implies robust stability.

In the sequel, we will consider two possible characterizations for the set \mathcal{A} .

3.3.7 Quadratic Stability for Norm-bounded Uncertainty

The family of the so-called norm-bounded perturbations is defined as

$$\mathcal{A} = \{A(\cdot) = A_n(\cdot) + L(\cdot)\Delta(\cdot)N(\cdot)\}, \quad (3.24)$$

where $A_n(\cdot)$ is a given T -periodic matrix (nominal dynamical matrix) and

- (i) $L(\cdot)$ and $N(\cdot)$ are T -periodic.
- (ii) $\|\Delta(t)\| \leq 1, \forall t$.

Matrices $L(\cdot)$ and $N(\cdot)$ characterize the structure of uncertainty. For instance, if

$$L(t) = N(t)' = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

then only the element $(1, 1)$ of matrix $A(\cdot)$ is uncertain and given by

$$a_{11}(t) = a_{n11}(t) + \delta_{11}(t), \quad |\delta_{11}(t)| \leq 1,$$

whereas all remaining elements coincide with the corresponding elements of $A_n(\cdot)$ and are therefore known.

In the case of norm-bounded perturbations, it is possible to derive a sufficient condition for quadratic stability based on a periodic linear matrix inequality (PLMI). Indeed, consider the following inequality

$$\begin{bmatrix} -P(t+1) + A_n(t)P(t)A_n(t)' + L(t)L(t)' & A_n(t)P(t)N(t)' \\ N(t)P(t)A_n(t)' & N(t)P(t)N(t)' - I \end{bmatrix} < 0. \quad (3.25)$$

In view of the Schur complement Lemma 3.1, by letting

$$S(t) = I - N(t)P(t)N(t)',$$

it is easy to see that this periodic matrix inequality is equivalent to this pair of inequalities:

$$P(t+1) > A_n(t) (P(t) + P(t)N(t)'S(t)^{-1}N(t)P(t)) A_n(t)' + L(t)L(t)' \quad (3.26a)$$

$$S(t) > 0. \quad (3.26b)$$

These inequalities have the advantage of being of lower dimension in comparison with (3.25). However, they are no longer affine in $P(\cdot)$.

Recall now that $\|\Delta(t)\| \leq 1, \forall t$. Therefore from (3.26):

$$\begin{aligned}
P(t+1) &> A_n(t) (P(t) + P(t)N(t)'S(t)^{-1}N(t)P(t))A_n(t)' + L(t)L(t)' \\
&\quad + A(t)P(t)A(t)' - A(t)P(t)A(t)' \\
&= A(t)P(t)A(t)' - L(t)\Delta(t)N(t)P(t)A_n(t)' \\
&\quad - A_n(t)P(t)N(t)' \Delta(t)'L(t)' - L(t)\Delta(t)N(t)P(t)N(t)' \Delta(t)'L(t)' \\
&\quad + A_n(t)P(t)N(t)'S(t)^{-1}N(t)P(t)A_n(t)' + L(t)L(t)' \\
&= (A_n(t)P(t)N(t)' - L(t)\Delta(t)S(t))S(t)^{-1}(A_n(t)P(t)N(t)' - L(t)\Delta(t)S(t))' \\
&\quad + A(t)P(t)A(t)' + L(t)L(t)' - L(t)\Delta(t)N(t)P(t)N(t)' \Delta(t)'L(t)' \\
&\quad - L(t)\Delta(t)S(t)\Delta(t)'L(t)' \\
&= (A_n(t)P(t)N(t)' - L(t)\Delta(t)S(t))S(t)^{-1}(A_n(t)P(t)N(t)' - L(t)\Delta(t)S(t))' \\
&\quad + A(t)P(t)A(t)' + L(t)(I - \Delta(t)\Delta(t)')L(t)' \\
&\geq A(t)P(t)A(t)'.
\end{aligned}$$

Hence $P(\cdot)$ is a positive semi-definite solution of the Lyapunov Inequality (3.19). Since $P(\cdot)$ is independent of Δ (see (3.26)), quadratic stability is ensured by considering $V(x, t) = x'Q(t)x$, where $Q(t) = P(t)^{-1}$, as a Lyapunov function, see Remark 3.3.

Note that, contrary to the time-invariant case, the above result cannot be stated as a necessary condition, even if the uncertain $\Delta(\cdot)$ is allowed to belong to the set of complex matrices. This will be shown in a subsequent chapter, precisely in Sect. 9.2.2, by means of a counterexample. In conclusion, for the quadratic stability of periodic systems, the basic result can be stated as follows.

Proposition 3.6 (Quadratic stability for norm-bounded uncertainty)

The T -periodic linear System (3.23) with dynamic matrix $A(t)$ subject to a norm-bounded perturbation (3.24) with $\|\Delta(t)\| \leq 1, \forall t$, is quadratically stable if there exists a T -periodic positive semi-definite matrix $P(\cdot)$ to the PLMI (3.25). ■

The previous result can be given an useful dual version by referring to the dual PLMI

$$\begin{bmatrix} -Q(t) + A_n(t)'Q(t+1)A_n(t) + N(t)'N(t) & A_n(t)'Q(t+1)L(t) \\ L(t)'Q(t+1)A_n(t) & L(t)'Q(t+1)L(t) - I \end{bmatrix} < 0. \quad (3.27)$$

The proof of the following result is omitted since it follows the same rationale as that of the previous one.

Proposition 3.7 (Quadratic stability for norm-bounded uncertainty)

The T -periodic linear System (3.23) with dynamic matrix $A(t)$ subject to a norm-bounded perturbation (3.24) with $\|\Delta(t)\| \leq 1, \forall t$, is quadratically stable if there exists a T -periodic positive semi-definite matrix $P(\cdot)$ to the PLMI (3.27).

3.3.8 Robust and Quadratic Stability for Polytopic Uncertainty

Another widely used description of uncertainty refers to the so-called *polytopic* family

$$\mathcal{A} = \{A(\cdot) = \sum_{i=1}^N \alpha_i A_i(t), \quad \alpha_i > 0, \quad \sum_{i=1}^N \alpha_i = 1\}. \quad (3.28)$$

This set \mathcal{A} is obtained by the convex linear combination of matrices $A_i(\cdot)$ which are known as *vertices* of the polytope.

For polytopic uncertainty, consider the Lyapunov inequalities

$$A_i(t)'Q(t+1)A_i(t) - Q(t) < 0, \quad i = 1, 2, \dots, N,$$

and assume that for each i there exists the same T -periodic positive definite solution $Q(\cdot)$. Now, recalling (3.21), it is apparent that $Q(\cdot)$ satisfies

$$\begin{bmatrix} -Q(t) & A_i(t)' \\ A_i(t) & -Q(t+1)^{-1} \end{bmatrix} < 0, \quad i = 1, 2, \dots, N.$$

Multiplying each of these matrices by α_i and summing up, one obtains

$$\begin{bmatrix} -Q(t) & A(t)' \\ A(t) & -Q(t+1)^{-1} \end{bmatrix} < 0, \quad i = 1, 2, \dots, N,$$

which is equivalent to $Q(\cdot) > 0$ and

$$A(t)'Q(t+1)A(t) - Q(t) < 0.$$

This means that stability is ensured for each $A(\cdot) \in \mathcal{A}$. Conversely, if the system is quadratically stable, then there exists a periodic positive definite matrix $Q(\cdot)$ satisfying the Lyapunov inequality for all matrices $A(\cdot) \in \mathcal{A}$. In particular, this matrix must satisfy the Lyapunov inequality for all vertices.

In conclusion, the following proposition can be stated.

Proposition 3.8 (Quadratic stability for polytopic uncertainty)

The T -periodic linear System (3.23), with dynamic matrix $A(t)$ belonging to the polytopic family (3.28), is quadratically stable if and only if the following Lyapunov inequality

$$A_i(t)'Q(t+1)A_i(t) - Q(t) < 0, \quad i = 1, 2, \dots, N.$$

admits a T -periodic positive definite solution $Q(\cdot)$. ■

Notice that the quadratic Lyapunov function is $V(x, t) = x'Q(t)x$.

The property of quadratic stability has the disadvantage of requiring one Lyapunov function for all members of the uncertainty set, and this may be too conservative since there might exist sets of matrices that are stable in a robust sense without being

quadratically stable. In the case of polytopic uncertainty, it is possible to provide a sufficient condition of robust stability not necessarily implying quadratic stability.

Proposition 3.9 (Robust stability for polytopic uncertainty)

The T -periodic linear System (3.23), with dynamic matrix $A(t)$ belonging to the polytopic family (3.28), is robustly stable if there exist a T -periodic matrix $Z(\cdot)$ and T -periodic matrices $Q_i(\cdot)$, $i = 1, 2, \dots, N$, such that

$$\begin{bmatrix} -Q_i(t) & A_i(t)'Z(t) \\ Z(t)'A_i(t) & Q_i(t+1) - Z(t) - Z(t)' \end{bmatrix} < 0. \quad (3.29)$$

Proof Multiply (3.29) by α_i and sum up with respect to i , $i = 1, 2, \dots, N$. Then, recalling that $\sum_{i=1}^N \alpha_i = 1$ it follows

$$\begin{bmatrix} -\bar{Q}(t) & A(t)'Z(t) \\ Z(t)'A(t) & \bar{Q}(t+1) - Z(t) - Z(t)' \end{bmatrix} < 0,$$

where $\bar{Q}(\cdot) = \sum_{i=1}^N \alpha_i Q_i(\cdot)$. The stability of any $A(\cdot) \in \mathcal{A}$ is a result of the periodic Lyapunov Lemma and Lemma 3.2. ■

Notice that the Lyapunov function for robust stability of the polytopic family is $V(x, t) = x' \sum_{i=1}^N \alpha_i Q_i(t) x$. Such a function depends on the particular matrix $A(\cdot)$ within the set \mathcal{A} , so that quadratic stability is not guaranteed.

We end this section by generalizing about the polytopic structure of the uncertainty, in the sense of allowing time-variability in the coefficients α_i , namely

$$\mathcal{A} = \{A(\cdot) = \sum_{i=1}^N \alpha_i(t) A_i(t), \quad \alpha_i(t) > 0, \quad \sum_{i=1}^N \alpha_i(t) = 1\}. \quad (3.30)$$

The coefficients $\alpha_i(t)$ are assumed to be periodic of period T . As can be easily seen, the necessary and sufficient condition for quadratic stability given in Proposition 3.8, still holds true. On the contrary, Proposition 3.9 is no longer valid for periodically varying coefficients $\alpha_i(t)$. However, it is possible to provide the following extended version of such a proposition, valid for time-varying coefficients as well.

Proposition 3.10 (Robust stability for polytopic and periodic uncertainty)

The T -periodic linear System (3.23), with dynamic matrix $A(t)$ belonging to the polytopic family (3.30) is robustly stable if there exist T -periodic matrices $Z_j(\cdot)$, $j = 1, 2, \dots, N$ and T -periodic matrices $Q_i(\cdot)$, $i = 1, 2, \dots, N$ such that

$$M_{ij}(t) = \begin{bmatrix} -Q_i(t) & A_i(t)'Z_j(t) \\ Z_j(t)'A_i(t) & Q_j(t+1) - Z_j(t) - Z_j(t)' \end{bmatrix} < 0, \quad \forall i, j. \quad (3.31)$$

Proof First of all, multiply matrix (3.31) by $\alpha_j(t+1)$ and take the sum for $j = 1, 2, \dots, N$. Then, multiplying again by $\alpha_i(t)$ and taking the sum for $i = 1, 2, \dots, N$,

it follows,

$$\sum_{j=1}^N \alpha_j(t+1) \sum_{i=1}^N \alpha_i(t) M_{ij}(t) = \begin{bmatrix} -\bar{Q}(t) & A(t)' \bar{Z}(t) \\ \bar{Z}(t)' A(t) & \bar{Q}(t+1) - \bar{Z}(t) - \bar{Z}(t)' \end{bmatrix} < 0,$$

where $\bar{Q}(t) = \sum_{i=1}^N \alpha_i(t) Q_i(t)$ and $\bar{Z}(t) = \sum_{i=1}^N \alpha_i(t+1) Z_i(t)$. Hence, the system is robustly stable with Lyapunov function $V(x, t) = x' \bar{Q}(t) x$.

Remark 3.5 A further generalization can be achieved by considering the polytopic class (3.30) with generically time-varying coefficients. Then, it is apparent that Proposition 3.10 provides a sufficient condition of robust exponential stability. Indeed, the positive matrix function $\bar{Q}(t)$ satisfies $A(t)' \bar{Q}(t) A(t) - \bar{Q}(t) < 0$, for each t , so that $V(x, t) = x' \bar{Q}(t) x$ is a Lyapunov function for the time-varying (non-periodic) system $x(t+1) = A(t)x(t)$.

3.4 Companion Forms

A basic system-theoretic issue is whether it is possible to work out a change of basis in order to represent in a simplified way a dynamical matrix $A(\cdot)$. This leads to the definition of the so-called *canonical forms*. Among them, we will focus on the following two forms.

h-companion form
The $n \times n$ matrix

$$A_{hc}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n(t) & -\alpha_{n-1}(t) & -\alpha_{n-2}(t) & \cdots & -\alpha_1(t) \end{bmatrix},$$

where $\alpha_i(t)$, $i = 1, 2, \dots, n$, are T -periodic coefficients, is said to be in *h-companion form*.

v-companion form
The $n \times n$ matrix

$$A_{vc}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_n(t) \\ 1 & 0 & \cdots & 0 & -\beta_{n-1}(t) \\ 0 & 1 & \cdots & 0 & -\beta_{n-2}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\beta_1(t) \end{bmatrix},$$

where $\beta_i(t)$, $i = 1, 2, \dots, n$, are T -periodic coefficients, is said to be in v -companion form.

We now characterize the conditions under which $A(\cdot)$ is algebraically equivalent to a matrix in h -companion or v -companion form. To this end, the following definition of T -cyclic matrix is in order.

Definition 3.3 (Periodic cyclic matrix)

An $n \times n$ T -periodic matrix $A(\cdot)$ is said to be T -cyclic if there exists a T -periodic vector $x(\cdot)$, $x(t) \neq 0$, $\forall t$, such that the $n \times n$ matrix

$$R(t) = \begin{bmatrix} x(t) & \Phi_A(t, t-1)x(t-1) & \cdots & \Phi_A(t, t-n+1)x(t-n+1) \end{bmatrix} \quad (3.32)$$

is invertible for all t . If such a vector $x(\cdot)$ exists, it is said to be a periodic cyclic generator. ■

If matrix $R(t)$ is invertible for each t any vector of \mathbb{R}^n , and, in particular vector $\Phi_A(t, t-n)x(t-n)$, can be seen as a linear combination of the columns of $R(t)$. This means that for a T -cyclic matrix $A(\cdot)$, it is possible to find n coefficients $\alpha_i(\cdot)$, $i = 1, 2, \dots, n$, such that

$$\begin{aligned} & \alpha_n(t-1)x(t) + \alpha_{n-1}(t-2)\Phi_A(t, t-1)x(t-1) + \cdots \\ & \alpha_1(t-n)\Phi_A(t, t-n+1)x(t-n+1) + \Phi_A(t, t-n)x(t-n) = 0. \end{aligned} \quad (3.33)$$

Moreover, such coefficients are T -periodic as a consequence of the periodicity of $R(t)$ and $\Phi_A(t, t-n)x(t-n)$.

A T -periodic matrix in h -companion form or in v -companion form is a T -cyclic matrix. Precisely, for such matrices, the cyclic generators are

$$x_{hc} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}', \quad x_{vc} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}', \quad (3.34)$$

respectively.

We are now in the position to characterize the class of T -cyclic matrices in terms of companion forms.

Proposition 3.11 *With reference to an $n \times n$ T -periodic matrix $A(\cdot)$, the following statements are equivalent.*

- (i) $A(\cdot)$ is T -cyclic.
- (ii) $A(\cdot)$ is algebraically equivalent to a T -periodic matrix in v -companion form.
- (iii) $A(\cdot)$ is algebraically equivalent to a T -periodic matrix in h -companion form.

Proof (i) \leftrightarrow (ii)

Assume that (i) holds. Then there exists a T -periodic vector $x(\cdot)$ such that $R(t)$ defined in (3.32) is invertible for each t . Then it is immediate to see that for any $i = 1, 2, \dots, n-1$, the i -th column of $A(t)R(t)$ coincides with the $i+1$ -th column of $R(t+1)$. As for the last column of $A(t)R(t)$, consider Eq. (3.33) with t replaced by $t+1$. It follows that the last column of $A(t)R(t)$ is a linear combination of the

preceding columns of $R(t+1)$, the i -th being weighted by coefficient $\alpha_{n-i+1}(t-i+1)$. Hence,

$$A(t)R(t) = R(t+1)A_{vc}(t)$$

with $\beta_{n-i}(t) = \alpha_{n-i}(t-i)$, $i = 0, 1, \dots, n-1$. This proves point (ii). Vice versa, if point (ii) holds, then there exists a T -periodic invertible matrix $S(\cdot)$ such that $S(t+1)A(t)S(t)^{-1} = A_{vc}(t)$. Hence, recalling (3.34), a cyclic generator for $A(\cdot)$ is $x(t) = S(t)^{-1}x_{vc}$, as it is easy to verify.

(i) \leftrightarrow (iii)

Assume again that (i) holds and let $x(\cdot)$ be a vector such that $R(t)$ defined in (3.32) is invertible for each t . By means of such $x(\cdot)$ and the coefficients $\alpha_i(\cdot)$ defined in (3.33), define the n vectors

$$v_0(t) = x(t) \tag{3.35a}$$

$$v_1(t) = A(t-1)v_0(t-1) + \alpha_1(t-1)x(t) \tag{3.35b}$$

$$v_2(t) = A(t-1)v_1(t-1) + \alpha_2(t-1)x(t) \tag{3.35c}$$

$$\vdots$$

$$v_{n-1}(t) = A(t-1)v_{n-2}(t) + \alpha_{n-1}(t-1)x(t). \tag{3.35d}$$

These vectors $v_i(t)$, $i = 1, 2, \dots, n-1$ constitute a triangular combination of the columns of $R(t)$. As such, they span the entire space \mathbb{R}^n for each t . It is then possible to define the invertible state-space transformation

$$S(t) = [v_{n-1}(t) \ v_{n-2}(t) \ \cdots \ v_0(t)]^{-1}. \tag{3.36}$$

From the definition of $v_i(t)$, it is straightforwardly seen that the last $n-1$ columns of $A(t)S(t)^{-1}$ are given by

$$A(t)v_0(t) = v_1(t+1) - \alpha_1(t)x(t+1)$$

$$A(t)v_1(t) = v_2(t+1) - \alpha_2(t)x(t+1)$$

$$A(t)v_2(t) = v_3(t+1) - \alpha_3(t)x(t+1)$$

$$\vdots$$

$$A(t)v_{n-2}(t) = v_{n-1}(t+1) - \alpha_{n-1}(t)x(t+1).$$

The first column of $A(t)S(t)^{-1}$ can be computed by means of a back propagation of the sequence (3.35) and by taking into account (3.33). Hence,

$$\begin{aligned} A(t)v_{n-1}(t) &= A(t)A(t-1)v_{n-2}(t) + \alpha_{n-1}(t-1)A(t)v_0(t) \\ &= \cdots = \Phi_A(t+1, t-n+1)v_0(t-n+1) \\ &\quad + \sum_{i=2}^n \alpha_{i-1}(t-n+i-1)\Phi_A(t+1, t-n+i)v_0(t-n+i) \\ &= -\alpha_n(t)v_0(t+1). \end{aligned}$$

Therefore, $A(t)S(t)^{-1} = S(t+1)^{-1}A_{hc}(t)$ thus yielding point (ii). Conversely, if point (ii) holds, then $S(t+1)A(t)S(t)^{-1} = A_{hc}(t)$ for some invertible T -periodic state-space transformation $S(\cdot)$. Then, recalling that x_{hc} defined in (3.34) is a cyclic generator for $A_{hc}(t)$, it turns out that the vector $x(t) = S(t)^{-1}x_{hc}(t)$ is a periodic cyclic generator for $A(\cdot)$.

3.5 Bibliographical Notes

The idea underlying the “time-invariantization” of the dynamical matrix of a periodic system can be traced back to the work of Gustav Floquet regarding linear differential equations [145], 1883. Notice that in continuous-time the Floquet problem was solved long time ago. This is in contrast with discrete-time where the Floquet problem has been solved at the end of the 20th century only, see [284]. Along this road, important instrumental results of matrix theory are needed, such as the existence of the T -th root of a matrix in the real field, see [154].

Turning to stability, a discussion on the stability of time-varying systems can be found in many references, such as [96, 123, 175, 209, 315, 316], to name only a few.

The use of Lyapunov equations for stability analysis can be traced back to the beginning of the 20th century [226]. It is worth to point out that Lyapunov himself studied linear differential equations with periodic coefficients [227]. However, only in the last decades of the past century the periodic Lyapunov equation has been extensively used as a main tool of analysis for periodic systems. The notion of quadratic stability is even a more recent subject of study, see e.g., [46, 129]. The characterization for polytopic stability with time-varying coefficients can be found in [120]. As for frozen stability, notice that Proposition 3.4 can be considered as a discretized version of Theorem 1.11 in [175].

For the study of companion forms, see [47], where the concept of cyclicity of a periodic system and its relation with reachability has been introduced.

Chapter 4

Structural Properties

Along the route to analyzing state–space systems, one encounters the main issue of the structural properties. The study of such properties has a long history, going back to the early days of system theory, as explained in the sequel. We will first discuss about reachability and controllability, and the associated canonical decomposition. Then, we move to observability and reconstructability. The four-part decomposition of periodic systems can then be worked out. The extended structural properties, namely stabilizability and detectability, are finally studied.

4.1 Reachability and Controllability

4.1.1 Basic Definitions

The reachability concept deals with the problem of reaching a point in the state–space of system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad (4.1)$$

starting from the origin by applying a proper input sequence; controllability is concerned with the problem of driving a state–point to the origin by feeding the system with a suitable input sequence. For the mathematical developments, it is advisable to articulate the necessary definitions in a formal way as follows.

Definition 4.1

(i) The state $x \in \mathbb{R}^n$ is reachable over (τ, t) , $\tau < t$, if there exists an input function for System (4.1) which brings the event $(\tau, 0)$ into (t, x) . The state $x \in \mathbb{R}^n$ is controllable over (t, τ) , $\tau > t$, if there exists an input function for System (4.1) which carries the event (t, x) into $(\tau, 0)$.

(ii) The state $x \in \mathbb{R}^n$ is reachable at t in k steps [controllable at t in k steps], $k > 0$, if x is reachable over $(t-k, t)$ [controllable over $(t, t+k)$].

(iii) The state $x \in \mathbb{R}^n$ is reachable at t [controllable at t] if there exists a $k, k > 0$, such that x is reachable [controllable] at t in k steps.

(iv) System (4.1) (or, equivalently, the pair $(A(\cdot), B(\cdot))$) is reachable over (τ, t) [controllable over (t, τ)] if any state is reachable over (τ, t) [controllable over (t, τ)].

(v) System (4.1) (or, equivalently, the pair $(A(\cdot), B(\cdot))$) is reachable [controllable] at t in k steps, $k > 0$, if the system is reachable over $(t - k, t)$ [controllable over $(t, t - k)$].

(vi) System (4.1) (or, equivalently, the pair $(A(\cdot), B(\cdot))$) is reachable [controllable] at time t if there exists a $k, k > 0$, such that the system is reachable [controllable] at t in k steps.

(vii) System (4.1) (or, equivalently, the pair $(A(\cdot), B(\cdot))$) is reachable [controllable] if it is reachable [controllable] at any time point t . ■

We will also introduce the symbols

$$\begin{aligned}\mathcal{X}_r(\tau, t) & \text{ set of reachable states over } (\tau, t) \\ \mathcal{X}_c(t, \tau) & \text{ set of controllable states over } (\tau, t) \\ X_r(t) & \text{ set of reachable states at } t \\ X_c(t) & \text{ set of controllable states at } t.\end{aligned}$$

Such sets are in fact subspaces of \mathbb{R}^n , as it is easy to prove.

Obviously, $\mathcal{X}_r(\tau + T, t + T) = \mathcal{X}_r(\tau, t)$, $X_r(t + T) = X_r(t)$ and analogously for the controllability subspaces.

4.1.2 Reachability

The characterization of reachability and controllability of periodic systems can be carried out by means of both system theoretic considerations and algebraic tools.

Quiescence property. Suppose that the reachability subspace over k periods coincides with the reachability subspace over $k + 1$ periods, i.e., $\mathcal{X}_r(t - kT, t) = \mathcal{X}_r(t - kT - T, t)$; then the reachability subspace over any interval including k periods keeps unchanged, i.e., for each $j \geq 0$

$$\mathcal{X}_r(t - kT - T, t) = \mathcal{X}_r(t - kT, t) \longrightarrow \mathcal{X}_r(t - kT - j, t) = \mathcal{X}_r(t - kT, t). \quad (4.2)$$

This is known as the quiescence property.

The above conclusion is easily drawn by inspecting Fig. 4.1. Precisely, take a state x reachable at t in $k + 2$ periods. The reachability transition is pictorially illustrated in Fig. 4.1. Consider the state \bar{x} along such a transition at time $t - T$. Obviously,

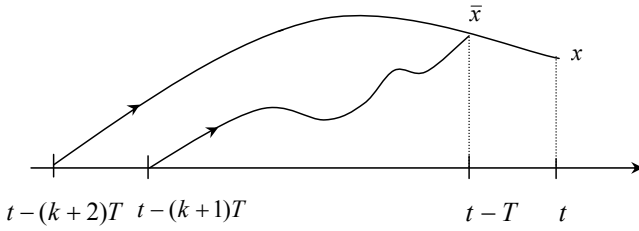


Fig. 4.1 The “quiescence” property of the reachability subspace

\bar{x} is reachable at $t - T$ starting from $t - kT - 2T$. In view of periodicity, the assumption $\mathcal{X}_r(t - kT, t) = \mathcal{X}_r(t - kT - T, t)$ is equivalent to $\mathcal{X}_r(t - kT - T, t - T) = \mathcal{X}_r(t - kT - 2T, t - T)$. Hence, \bar{x} is reachable at $t - T$ starting from $t - kT - T$. By concatenating the input function bringing the state from the origin at $t - kT - T$ to \bar{x} at $t - T$ with the input transferring the state \bar{x} at $t - T$ to x at t , it follows that x is reachable at t in $k + 1$ periods. In conclusion $\mathcal{X}_r(t - kT - 2T, t) \subseteq \mathcal{X}_r(t - kT - T, t)$. On the other hand, it is apparent that any system (not necessarily periodic) satisfies the property

$$\mathcal{X}_r(i, t) \subseteq \mathcal{X}_r(j, t), \quad \forall j < i < t. \quad (4.3)$$

Therefore $\mathcal{X}_r(t - kT - 2T, t) = \mathcal{X}_r(t - kT - T, t)$ and by induction, $\mathcal{X}_r(t - kT - iT, t) = \mathcal{X}_r(t - kT, t)$, $\forall i \geq 0$. Finally, (4.3) leads to the quiescence property (4.2).

Analytic characterization of the reachability set. In order to characterize analytically the reachability subspace, recall the Lagrange formula (2.20). By letting $\tau = t - j$, $x(\tau) = 0$ and $x(t) = x$ one obtains

$$x = R_j(t) \begin{bmatrix} u(t-1) \\ u(t-2) \\ \vdots \\ u(t-j) \end{bmatrix},$$

where

$$R_j(t) = [B(t-1) \Phi_A(t, t-1)B(t-2) \cdots \Phi_A(t, t-j+1)B(t-j)]. \quad (4.4)$$

Therefore a state x is reachable at t in j steps if and only if x belongs to the range¹ of matrix $R_j(t)$, denoted as $\text{Im}[R_j(t)]$. In other words

$$\mathcal{X}_r(t-j, t) = \text{Im}[R_j(t)].$$

¹ The range, or image space, of matrix is the set of vectors spanned by its columns, namely the subspace generated by all linear combinations of the columns.

As already noted, this is a subspace of \mathbb{R}^n periodic in t , and, in view of (4.3), its dimension is non-decreasing with respect to j . It is important to point out the particular structure of $R_j(t)$, which is more easily introduced by considering the case $j = kT$:

$$R_{kT}(t) = [R_T(t) \Psi_A(t) R_T(t) \cdots \Psi_A(t)^{k-1} R_T(t)] . \quad (4.5)$$

Letting

$$F_t := \Psi_A(t), \quad G_t := R_T(t), \quad (4.6)$$

it is apparent that the structure of (4.5) is that of a reachability matrix for the time-invariant pair (F_t, G_t) .

Length of the reachability transition interval. A straightforward corollary of the quiescence property is that any reachability transition can be performed in n periods at the most, i.e.,

$$X_r(t) = \mathcal{X}_r(t - nT, t). \quad (4.7)$$

One can strengthen this result as follows. Define the *reachability index*² $\mu_r(t)$ at t of the periodic pair $(A(\cdot), B(\cdot))$ as the reachability index of the invariant pair (4.6), namely

$$\text{rank}[R_{jT}(t)] = \text{rank}[R_{\mu_r(t)T}(t)], \quad \forall j \geq \mu_r(t).$$

From the very definition of $\mu_r(t)$, one can easily conclude that

$$X_r(t) = \mathcal{X}_r(t - \mu_r(t)T, t). \quad (4.8)$$

Thus, not only one can state that all reachability transitions require n periods at the most, but that $\mu_r(t)$ periods suffice to reach any reachable state.

Note that, in general, $\mu_r(t)$ may vary in time. However, the variability is limited as will be now pointed out. Indeed, let

$$\bar{\mu}_r = \min_{t \in [0, T-1]} \mu_r(t),$$

and introduce the time point $\bar{t} \in [t - T + 1, t]$ such that $\mu_r(\bar{t}) = \bar{\mu}_r$. If x is reachable at t , x is reachable starting from $t - \mu_r(t)T$, see Fig. 4.2. Let \bar{x} be the state of such a transition at time \bar{t} . It turns out that \bar{x} is reachable in $\bar{\mu}_r$ periods. By a concatenation argument, it follows that x is reachable in fewer than $\bar{\mu}_r + 1$ periods. Therefore

$$|\mu_r(t) - \mu_r(\tau)| \leq 1, \quad \forall t, \tau \in [0, T - 1].$$

Remark 4.1 *The reachability index of a time-invariant system cannot exceed the degree of the minimal polynomial of the dynamic matrix. By applying this statement to the pair (F_t, G_t) , the conclusion is drawn that $\mu_r(t)$ cannot exceed the degree of the minimal polynomial of $\Psi_A(t)$, namely $\mu_r(t) \leq v_m(t)$. Hence a corollary of (4.8)*

² The reachability index μ_r of (F, G) is defined as the minimum integer required to perform the reachability transition for all reachable states. It can be characterized as the minimum integer such that the columns of $F^{\mu_r}G$ are linear combinations of the columns of $\{F^iG, i = 0, 1, \dots, \mu_r - 1\}$. Obviously $\mu_r \leq v_m$, where v_m is the degree of the minimal polynomial of F .

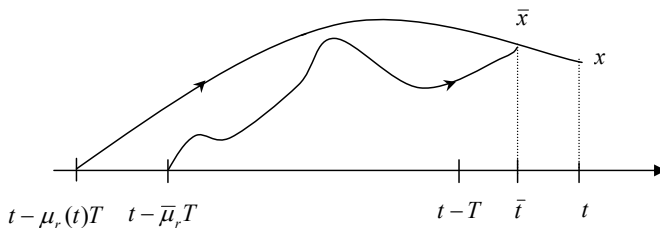


Fig. 4.2 Reachability interval length

is that

$$X_r(t) = \mathcal{X}_r(t - v_m(t)T, t).$$

Therefore, all reachable states at t can be reached in $v_m(t)$ periods at the most. ■

It is worth pointing out that the dimension of the reachability subspace $X_r(t)$ may undergo time-variations as illustrated by the following example.

Example 4.1 For the system defined by

$$A(t) = B(t) = C(t) = \begin{cases} 0 & t \text{ even} \\ 1 & t \text{ odd}, \end{cases}$$

$X_r(t) = \mathbb{R}$ for t even and $X_r(t) = \{0\}$ for t odd. ■

An exception is the reversible case, for which $\dim[X_r(t)]$ is constant. Indeed, consider the reachable states at time τ ; the free motion originating in such states produces the set $\Phi_A(t, \tau)X_r(\tau)$ at time t . Obviously these states are reachable at t , and hence $\Phi_A(t, \tau)X_r(\tau) = X_r(t)$. Being $\Phi_A(t, \tau)$ invertible, $\dim[\Phi_A(t, \tau)X_r(\tau)] = \dim[X_r(\tau)]$, so that $\dim[X_r(\tau)] \leq \dim[X_r(t)]$. By periodicity arguments, the stated conclusion about the invariance of the reachability subspace dimension in the reversible case follows.

Invariance of the reachability subspace. An important property is that the reachability subspace $X_r(t)$ is $A(\cdot)$ -invariant³. This can be easily proven as follows: if $x \in X_r(t)$ then the state obtained at $t + 1$ by free motion starting from x at time t is obviously reachable at $t + 1$, so that $A(t)x \in X_r(t + 1)$. Now, one can introduce a basis matrix $Q_r(\cdot)$ for $X_r(\cdot)$, i.e., a matrix so that $\text{Im}[Q_r(t)] = X_r(t)$, $\forall t$. Obviously, thanks to the invariance property, $A(t)Q_r(t)$ is a matrix whose columns belong to $X_r(t + 1) = \text{Im}[Q_r(t + 1)]$. Hence there exists a matrix $A_r(\cdot)$ so that

$$A(t)Q_r(t) = Q_r(t + 1)A_r(t).$$

³ A subspace $V(\cdot)$ is said to be $A(\cdot)$ -invariant if $A(t)V(t) \subseteq V(t + 1)$.

Notice that in general, matrix $Q_r(\cdot)$ has a periodically time-varying number of columns, and $A_r(\cdot)$ has periodically time-varying dimensions. Moreover, it is easy to see that $\text{Im}[B(t-1)] = \mathcal{X}_r(t-1, t) \subseteq X_r(t)$, so that there exists a matrix (with periodically time-varying dimensions) $B_r(\cdot)$ in a way that

$$B(t) = Q_r(t+1)B_r(t).$$

As a matter of fact, the reachability subspace $X_r(t)$ is the smallest $A(\cdot)$ -invariant subspace containing $\text{Im}[B(t-1)]$. To prove this statement, assume that $V(t)$ is another $A(\cdot)$ -invariant subspace containing $\text{Im}[B(t-1)]$. The result follows if one shows that $X_r(t) \subseteq V(t)$. Indeed, $\forall j \in [0, T-1]$ it follows that

$$F_t^i \Phi_A(t+T, t+j+1) \text{Im}[B(t+j)] \subseteq F_t^i \Phi_A(t+T, t+j+1) V(t+1), \forall i > 0.$$

Moreover, the system periodicity together with the invariance property entails that

$$F_t^i \Phi_A(t+T, t+j+1) V(t+1) \subseteq V(t), \forall i > 0, \forall j \in [0, T-1].$$

The two inclusions above lead to

$$F_t^i \Phi_A(t+T, t+j+1) \text{Im}[B(t+j)] \subseteq V(t), \quad i = 0, 1, \dots, \quad j \in [0, T-1]$$

The matrices $F_t^i \Phi_A(t+T, t+j+1)B(t+j)$ evaluated for $i = 0, \dots, n-1$ and $j = 0, \dots, T-1$ form the reachability matrix $R_{nT}(t)$ so that $X_r(t) = \text{Im}[R_{nT}] \subseteq V(t)$.

Remark 4.2 Under a change of basis $\tilde{x}(t) = Q(t)x(t)$ in the state-space, the system matrices change accordingly, i.e., $\tilde{A}(t) = Q(t+1)A(t)Q(t)^{-1}$ and $\tilde{B}(t) = Q(t+1)B(t)$. Correspondingly, the reachability subspace modifies as

$$\tilde{\mathcal{X}}_r(t-j, t) = Q(t)\mathcal{X}_r(t-j, t).$$

Since $Q(t)$ is invertible, this means that the dimension of the reachability subspace and the reachability index are not affected by the change of basis. In particular, if the system is reachable, any algebraically equivalent representation is reachable too and the length of the reachability interval keeps unchanged.

4.1.3 Reachability Canonical Form

In many applications, it is useful to consider specific parametrizations for the pair of matrices $A(\cdot)$, $B(\cdot)$ representing a given system. This task is particularly important for parameter estimation from real data (system identification problem) or state-space construction from input-output representation (realization problem). This leads to the issue of *canonical forms*.

For simplicity, we will restrict our analysis to single-input systems only ($m = 1$) and begin with the definition of *reachable canonical form*. A single-input T -periodic

system is said to be in reachable canonical form when

$$A(t) = A_{rch}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{bmatrix}$$

$$B(t) = B_{rch}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

where the coefficients $a_i(t)$ are T -periodic. Notice that matrix $A_{rch}(t)$ is in h -companion form and as such is a T -cyclic matrix, see Sect. 3.4.

A well-known result of time-invariant system theory claims that a single-input system can be put in the reachable canonical form (via state-space change of basis) if and only if it is reachable. This statement is not valid as such for a periodic system, since there are reachable systems which are not algebraically equivalent to the reachable canonical form. To illustrate this fact, consider the 1-dimensional 2-periodic system

$$x(t+1) = x(t) + b(t)u(t)$$

with $b(0) = 0$ and $b(1) = 1$. It is apparent that this system is reachable for each t . However, it cannot be transformed into the reachable canonical form since, for any invertible $q(t)$ if follows $\tilde{b}(t) = q(t+1)b(t) \neq 1$. Notice that the reachability interval is of length 1 for $t = 0$ and length 2 for $t = 1$. The point is that the class of n -dimensional periodic systems which can be put into reachable canonical form is not that of reachable periodic systems but that of periodic systems reachable in n steps. This fact is precisely stated in the following result.

Proposition 4.1 *Consider a single input T -periodic system of dimension n . The system is reachable in n steps for each time point t if and only if it is algebraically equivalent to a system in reachable canonical form.*

Proof If the system is in the reachable canonical form, then it is reachable in n steps for each t . Indeed, the peculiar structure of $A_{rch}(t)$ and $B_{rch}(t)$ is such that the matrix

$$\left[\Phi_{A_{rch}}(t, t-n+1) B_{rch}(t-n) \cdots A_{rch}(t-1) B_{rch}(t-2) B_{rch}(t-1) \right]$$

is triangular and invertible for each t , so that the system is reachable in n -steps. In order to prove the converse statement, notice that, since the system is reachable in n steps, the matrix

$$\left[\Phi_A(t, t-n+1) B(t-n) \cdots A(t-1) B(t-2) B(t-1) \right]$$

is, by assumption, invertible for each t . As such, the vector $x(t) = B(t-1)$ is a cyclic generator, see Sect. 3.4 for the definition of such a notion. Now, recall the derivation carried out in Proposition 3.11. In particular, the change of basis which brings matrix $A(\cdot)$ in the h -companion form $A_{rch}(t)$ is given by $S(\cdot)$ defined in (3.36), where the vectors $v_i(t)$ are given by the sequence (3.35) and the coefficients $\alpha_i(t)$ are obtained as in (3.33). The change of basis $S(\cdot)$ is also such that $S(t+1)B(t) = B_{rch}(t)$. Hence,

$$A(t)S(t)^{-1} = S(t+1)^{-1}A_{rch}(t), \quad S(t+1)B(t) = B_{rch}(t),$$

which proves the result.

4.1.4 Controllability

Turning to controllability, one can establish many properties similar to those seen for reachability. However, some subtle differences may arise and therefore it is appropriate to revisit the properties one by one.

Quiescence property. The *quiescence property* holds true for controllability too. Indeed, assume that for an integer $k > 0$, $\mathcal{X}_c(t, t+kT) = \mathcal{X}_c(t, t+kT+T)$. The invariance of the controllability subspaces over $(t, t+iT)$, $\forall i > k$ is then proven by induction. As a matter of fact, assume that $x \in \mathcal{X}_c(t, t+kT+2T)$ and consider the controllability transition, namely the transition transferring a state x at time t into the origin at time $t+kT+2T$ by means of an input function $u(\cdot)$. Consider the state, say \bar{x} , associated with such a transition at time $t+T$. Obviously, such a state is controllable over $(t+T, t+kT+2T)$, and in view of periodicity, over $(t, t+kT+T)$. Therefore, our opening assumption implies that \bar{x} is controllable over $(t, t+kT)$, or equivalently, over $(t+T, t+kT+T)$. Thus, by a concatenation argument, x is controllable over $(t, t+kT+T)$ so that $\mathcal{X}_c(t, t+kT+2T) \subseteq \mathcal{X}_c(t, t+kT+T)$. Since for any linear system, $\mathcal{X}_c(t, j) \subseteq \mathcal{X}_c(t, i)$, $\forall i > j > t$, the conclusion follows that, $\forall j \geq 0$

$$\mathcal{X}_c(t, t+kT+T) = \mathcal{X}_c(t, t+kT+T) \longrightarrow \mathcal{X}_c(t, t+kT+j) = \mathcal{X}_c(t, t+kT).$$

This is the quiescence property for controllability. An obvious corollary of this property is that

$$X_c(t) = \mathcal{X}_c(t, t+nT). \quad (4.9)$$

Analytic characterization of the controllability subspace. In the Lagrange formula (2.20), let $x(t) = x$, $\tau = t+j$ and $x(t+j) = 0$. Then

$$R_j(t+j) \begin{bmatrix} u(t+j-1) \\ u(t+j-2) \\ \vdots \\ u(t) \end{bmatrix} = -\Phi_A(t+j, t)x,$$

where $R_j(\cdot)$ was defined in (4.4). Hence,

$$x \text{ controllable over } (t, t+j) \iff \Phi_A(t+j, t)x \in \text{Im}[R_j(t+j)]. \quad (4.10)$$

In other words, the controllability subspace is related to the reachability subspace as follows⁴

$$\mathcal{X}_c(t, t+j) = \Phi_A(t+j, t)^- \mathcal{X}_r(t, t+j). \quad (4.11)$$

If one takes $j = kT$, the relationship

$$\mathcal{X}_c(t, t+kT) = [\Psi_A(t)^k]^- \mathcal{X}_r(t, t+kT) \quad (4.12)$$

is obtained. This expression, together with (4.4), implies that the controllability subspace over $(t, t+kT)$ of the periodic pair $(A(t), B(t))$ coincides with the controllability subspace of the invariant pair (F_t, G_t) over $(0, k)$.

Another important corollary of (4.11) is that

$$\dim[\mathcal{X}_c(t, t+kT)] \geq \dim[\mathcal{X}_r(t, t+kT)]. \quad (4.13)$$

Finally, if we set $k = n$ in (4.12), Eqs. (4.7) and (4.9) imply that

$$X_c(t) = [F_t^n]^- X_r(t), \quad (4.14)$$

or, equivalently⁵,

$$X_c(t) = X_r(t) + \text{Ker}[F_t^n]. \quad (4.15)$$

Therefore

$$\dim[X_c(t)] \geq \dim[X_r(t)], \quad (4.16)$$

as already expected from (4.13). As a matter of fact, the dimensionality Inequality (4.16) can be strengthened by the subspace inclusion

$$X_r(t) \subseteq X_c(t) \quad (4.17)$$

This is easily derived from (4.14) by taking into account that

$$X_r(t) = \text{Im} [G_t \ F_t G_t \ \cdots \ F_t^{n-1} G_t].$$

However, note that in general, inclusion $\mathcal{X}_r(t, t+kT) \subseteq \mathcal{X}_c(t, t+kT)$ is false.

Interestingly enough, inclusion (4.17) turns into an equality in the case of reversible systems. Indeed, consider a state x controllable at time t , i.e., $x \in X_c(t)$. Thanks to the reversibility assumption, there exists a unique vector \tilde{x} such that the free motion starting from \tilde{x} at time $t - nT$ ends in x at time t , i.e., $x = F_t^n \tilde{x}$. Obviously, such a state \tilde{x} is controllable, since it can be carried to zero by a suitable concatenation of

⁴ If X is a subspace of \mathbb{R}^n , by F^-X we mean the subspace $Y = \{y : x = Fy, x \in X\}$. Now letting $Y = F^-X$, an important dimensional inequality is that $\dim[X] \leq \dim[Y]$.

⁵ The symbol $\text{Ker}[M]$ denotes the kernel, or null space, of a matrix M . This is a subspace formed by all vectors x such that $Mx = 0$.

input functions. On the other hand, in view of the properties of periodic systems, there exists an input function $u(\cdot)$ which brings \bar{x} to zero in nT steps. Then, the superposition principle requires that if one feeds the system with the input function $-u(\cdot)$ starting from the zero state at $t - nT$, then after n periods one ends in x . Therefore x is also reachable at $t - nT$ which is equivalent to say that $x \in X_r(t)$. In conclusion, $X_c(t) \subseteq X_r(t)$, so that $X_r(t) = X_c(t)$.

Length of the controllability transition interval. The property expressed in Eq. (4.9) implies that any state which is controllable can be transferred to the origin in no more than n periods. In analogy with the discussion on reachability, one can strengthen the result concerning the controllability interval length given in (4.9) by means of the *controllability index* concept⁶. Precisely, we define a controllability index $\mu_c(t)$ at time t for the periodic pair $(A(\cdot), B(\cdot))$ as the controllability index of the invariant pair (F_t, G_t) , where F_t and G_t were defined in (4.6). From the previous consideration, it is easy to conclude that

$$X_c(t) = \mathcal{X}_c(t, t + \mu_c(t)T). \quad (4.18)$$

Combining conclusion (4.18) with the necessary and sufficient controllability condition (4.10), one can infer that

$$(A(\cdot), B(\cdot)) \text{ controllable at } t \iff \text{Im}[F_t^{\mu_c(t)}] \subseteq \text{Im}[R_{\mu_c(t)T}(t)]. \quad (4.19)$$

Although the index $\mu_c(t)$ may vary, one can show that

$$|\mu_c(t) - \mu_c(\tau)| \leq 1, \quad \forall t, \tau \in [0, T - 1].$$

Indeed, denote by

$$\bar{\mu}_c := \min_{t \in [0, T-1]} \mu_c(t),$$

⁶ Given a discrete-time system characterized by the pair (F, G) , the controllability index μ_c is the minimum integer necessary to perform the controllability transition for all controllable states of the system. Such index can be related to the reachability index μ_r by considering the Kalman canonical decomposition

$$F \longrightarrow \begin{bmatrix} F_R & * & * \\ 0 & F_{\bar{R}C} & * \\ 0 & 0 & F_{\bar{C}} \end{bmatrix}, \quad G \longrightarrow \begin{bmatrix} G_R \\ 0 \\ 0 \end{bmatrix}$$

where (F_R, G_R) is the reachable part whereas

$$\left(\begin{bmatrix} F_R & * \\ 0 & F_{\bar{R}C} \end{bmatrix}, \begin{bmatrix} G_R \\ 0 \end{bmatrix} \right)$$

is the controllable part. Matrix $F_{\bar{R}C}$ corresponds to the part of the system which is controllable and not reachable. It can be shown that all eigenvalues of $F_{\bar{R}C}$ are equal to zero. Therefore,

$$\mu_c = \max\{\mu_r, \mu_z\}$$

where μ_z is the dimension of the largest Jordan block associated with $F_{\bar{R}C}$. Obviously if F is non-singular, then $\mu_c = \mu_r$. Letting again v_m be the degree of the minimal polynomial of F , recalling that $\mu_r \leq v_m$ and noting that $\mu_z \leq v_m$, the conclusion $\mu_c \leq v_m$ is drawn.

and let $\bar{t} \in [t, t + T - 1]$ be such that $\mu_c(\bar{t}) = \bar{\mu}_c$. If x is controllable at t , it can be driven to the origin at $t + \mu_c(t)T$ at the most. Let \bar{x} be the state of such a transition at \bar{t} . Since \bar{x} is controllable over $(\bar{t}, \bar{t} + \bar{\mu}_c)$, x is controllable over $(t, \bar{t} + \bar{\mu}_c T)$. Being $\bar{t} \in [t, t + T - 1]$, x is controllable in $\bar{\mu}_c + 1$ periods at the most.

A notable difference between the reachability and the controllability subspaces is that the dimension of the latter cannot be subject to time variations:

$$\dim[X_c(t)] = \text{constant}. \quad (4.20)$$

Indeed, consider the set of states at time $t - j$, $j > 0$ which evolve by free motion into a controllable state at time t : $\Phi_A(t - j, t)^- X_c(t)$. Obviously, such states are controllable at $t - j$:

$$\Phi_A(t - j, t)^- X_c(t) \subseteq X_c(t - j).$$

Therefore

$$\dim[X_c(t)] \leq \dim[\Phi_A(t - j, t)^- X_c(t)] \leq \dim[X_c(t - j)].$$

By periodicity arguments, (4.20) follows. Consequently, condition (4.19) can also be stated as

$$(A(\cdot), B(\cdot)) \text{ controllable} \iff \text{Im}[F_t^{\mu_c(t)}] \subseteq \text{Im}[R_{\mu_c(t)T}(t)], \text{ for some } t.$$

Remark 4.3 *The controllability index of a time-invariant system cannot exceed the degree of the minimal polynomial of the dynamic matrix (recall the previous note). This statement can be applied to the pair (F_t, G_t) , thus leading to the conclusion that $\mu_r(t) \leq \nu_m(t)$, $\forall t$. Hence,*

$$X_c(t) = \mathcal{X}_c(t, t + \nu_m(t)T),$$

which is a weaker statement of (4.18). ■

The controllability subspace is $A(\cdot)$ -invariant. Indeed, if $x \in X_c(t)$, then in view of (4.14) and (4.17), $F_t^n x \in X_c(t)$. Since $F_t^n x$ is the state at time $t + nT$ reached by the free motion starting at time t , the intermediate states along the trajectory are controllable as well. In particular, $A(t)x \in X_c(t)$.

Finally, in analogy with reachability, it can be shown that $X_c(t)$ is the smallest $A(\cdot)$ -invariant subspace containing $\text{Im}[B(t - 1)] + \text{Ker}[F_t^n]$.

4.1.5 Reachability and Controllability Grammians

The $n \times n$ positive semi-definite matrix

$$W_r(\tau, t) = \sum_{j=\tau+1}^t \Phi_A(t, j) B(j-1) B(j-1)' \Phi_A(t, j)' \quad (4.21)$$

is known as reachability Grammian over (τ, t) . Comparing this formula with (4.4) it is apparent that

$$W_r(t - j, t) = R_j(t)R_j(t)', \quad (4.22)$$

so that⁷

$$\text{Im}[W_r(t - j, t)] = \text{Im}[R_j(t)] = \mathcal{R}_r(t - j, t).$$

This is why reachability can be analyzed by means of the Grammian matrix. For a T -periodic system it is obvious that

$$W_r(\tau + T, t + T) = W_r(\tau, t).$$

Moreover, letting

$$\bar{W}_i(t) = W_r(t - iT, t),$$

it is easy to see that the following recursive expression holds true

$$\bar{W}_{i+1}(t) = \bar{W}_i(t) + \Psi_A(t)^i \bar{W}_1(t) \Psi_A(t)^{i'}. \quad (4.23)$$

From (4.4), (4.8), and (4.22) we can conclude the following

Proposition 4.2 *The periodic System (4.1) is reachable at t if and only if the following condition*

$$W_r(t - \mu_r(t)T, t) > 0$$

holds. ■

The Grammian reachability matrix is also useful to work out a controllability test.

Proposition 4.3 *The periodic System (4.1) is controllable at t if and only if there exists no $\eta \neq 0$ such that*

$$\eta' W_r(t, t + \mu_c(t)T) \eta = 0$$

with

$$\Phi_A(t + \mu_c(t)T, t)' \eta \neq 0. \quad (4.24)$$

Proof The proof moves from the necessary and sufficient controllability condition (4.19) which can be equivalently written as

$$(A(\cdot), B(\cdot)) \text{ controllability} \iff \text{Im}[\Psi_A(\tau)^{\mu_c(\tau)}] \subseteq \text{Im}[W_r(\tau - \mu_c(\tau)T, \tau)].$$

Since⁸ $\text{Im}[P] = \text{Ker}[P']^\perp$ and the Grammian is symmetric,

$$\text{Ker}[W_r(\tau - \mu_c(\tau)T, \tau)] \subseteq \text{Ker}[\Psi_A(\tau)^{\mu_c(\tau)}].$$

$W_r(\cdot, \cdot)$ being positive semi-definite, this last inclusion obviously leads to the Gramian controllability criterion stated above, provided to set $t = \tau - \mu_c(\tau)$. ■

⁷ Recall that for any matrix P , it is $\text{Im}[PP'] = \text{Im}[P]$.

⁸ The symbol \perp denotes orthogonal complement.

Obviously, if the system is reversible, (4.24) is trivially satisfied for all $\eta \neq 0$, so that the controllability condition of Proposition 4.3 is equivalent to $W_r(\tau, \tau + \mu_c(\tau)T) > 0$. Thus a reversible system is reachable at t if and only if it is controllable at t . Note that in the reversible case, a controllability Grammian can be defined as

$$W_c(\tau, t) = \sum_{j=\tau+1}^t \Phi_A(\tau, j) B(j-1) B(j-1)' \Phi_A(\tau, j)',$$

and controllability at t can be characterized by requiring that $W_c(t - \mu_c(t)T, t)$ is positive definite.

4.1.6 Modal Characterization of Reachability and Controllability

It is possible to perform a test of reachability (controllability) by making reference to the eigenvectors of the monodromy matrix. In such a way, the characteristic multipliers can be split into two groups: the reachable (controllable) and the unreachable (uncontrollable) multipliers. To derive such characterization we begin by considering the quadratic form $\eta^* \bar{W}_i(t) \eta$ where η is a complex vector. In view of (4.23), this quadratic form can be written as

$$\begin{aligned} \eta^* \bar{W}_i(t) \eta &= \eta^* \Psi_A(t)^{i-1} \bar{W}_1(t) \Psi_A(t)^{i-1} \eta + \\ &+ \eta^* \Psi_A(t)^{i-2} \bar{W}_1(t) \Psi_A(t)^{i-2} \eta + \\ &+ \dots + \eta^* \bar{W}_1(t) \eta. \end{aligned}$$

Suppose now that η is an eigenvector of $\Psi_A(t)'$ associated with the characteristic multiplier λ , i.e.,

$$\Psi_A(t)' \eta = \lambda \eta.$$

Then,

$$\eta^* \bar{W}_i(t) \eta = (|\lambda|^{2(i-1)} + |\lambda|^{2(i-2)} + \dots + |\lambda|^2 + 1) \eta^* \bar{W}_1(t) \eta.$$

Therefore, $\eta^* \bar{W}_1(t) \eta = 0, \forall i > 0$ if and only if $\eta^* \bar{W}_i(t) \eta = 0$. Recalling the expression of the reachability Grammian, one can then easily come to the following conclusion:

Proposition 4.4 *System (4.1) is reachable at time t if and only if, for each characteristic multiplier λ*

$$\Psi_A(t)' \eta = \lambda \eta \tag{4.25a}$$

$$B(j-1)' \Phi_A(t, j)' \eta = 0, \quad \forall j \in [t-T+1, t], \tag{4.25b}$$

imply $\eta = 0$. ■

In the time-invariant case, this condition is known as the Popov–Belevitch–Hautus modal test of reachability.

As already anticipated, the modal notion leads to the following

Definition 4.2 A characteristic multiplier λ is said to be reachable at time t if

$$\begin{aligned}\Psi_A(t)' \eta &= \lambda \eta \\ B(j-1)' \Phi_A(t, j)' \eta &= 0, \quad \forall j \in [0, T]\end{aligned}$$

imply $\eta = 0$. ■

Thus we can state the following conclusion.

Proposition 4.5 A periodic system is reachable at time t if all characteristic multipliers are reachable at t . ■

Moreover, Definition 4.2 enables one to state the controllability criterion in a compact and intuitive form.

Proposition 4.6 A periodic system is controllable if all non-zero characteristic multipliers are reachable at some t .

Proof Assume that all non-zero characteristic multipliers are reachable but nevertheless, the system is not controllable. Then, in view of the Grammian controllability criterion, there exists $\eta \neq 0$ such that

$$\eta' W_{\mu_c(t)}(t) \eta = 0, \quad \Psi_A(t)'^{\mu_c(t)} \eta \neq 0.$$

Without any loss of generality, one can take η as an eigenvector of $\Phi_A(t)'$ associated with a non-zero eigenvalue, i.e., $\Phi_A(t)' \eta = \lambda \eta$, with $\lambda \neq 0$. Moreover, from (4.23),

$$\eta^* \bar{W}_{i+1}(t) \eta = \eta' \bar{W}_i(t) \eta + |\lambda|^i \eta' \bar{W}_1(t) \eta.$$

Hence, $\eta^* W_{\mu_c(t)}(t) \eta = 0$ implies

$$\eta^* \bar{W}_1(t) \eta = 0,$$

which, together with $\Phi_A(t)' \eta = \lambda \eta$ is equivalent to (4.25). This violates the assumption that all non-zero characteristic multipliers are reachable.

Vice versa, suppose that the system is controllable at t and nevertheless, that there exists a non-zero characteristic multiplier λ which is not reachable. This means that, for some $\eta \neq 0$

$$B(j-1)' \Phi_A(t, j)' \eta = 0, \quad j \in [0, T], \quad \Psi_A(t)' \eta = \lambda \eta,$$

i.e.,

$$\eta^* \bar{W}_i(t) \eta = 0, \quad \Psi_A(t)' \eta \neq 0,$$

thus violating the controllability of the system in view of Proposition 4.3.

4.1.7 Canonical Controllability Decomposition

When a periodic system is not controllable, it is possible to decompose it into a controllable and uncontrollable part by means of a suitable time-varying state-space transformation. Precisely, for a non-controllable system, consider the subspace $X_c(t)$ and a complementary subspace $\tilde{X}_c(t)$, i.e., $X_c(t) \oplus \tilde{X}_c(t)$ covers the entire space⁹. Notice that $X_c(t)$ is evolving in time according to a periodic pattern. The dimension of this time-varying subspace is however constant, say n_c . The complementary subspace $\tilde{X}_c(t)$ will also be evolving in time with dimension $n - n_c$ constant.

Let $S_c(t)$ $[\tilde{S}_c(t)]$ be a matrix of dimension $n \times n_c$ $[n \times n - n_c]$ whose columns constitute a basis for $X_c(t)$ $[\tilde{X}_c(t)]$, and let

$$S(t) = [S_c(t) \ \tilde{S}_c(t)] .$$

Obviously, $S(t)$ is periodic and non-singular for each t . Then by defining

$$\tilde{x}(t) = S(t)^{-1}x(t) ,$$

and recalling the properties of the subspace $X_c(t)$ discussed in Sect. 4.1.4, one obtains a new system

$$\tilde{x}(t+1) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)u(t)$$

with

$$\tilde{A}(t) = S(t+1)^{-1}A(t)S(t) = \begin{bmatrix} A_c(t) & -^* \\ 0 & \tilde{A}_c(t) \end{bmatrix} \quad (4.26a)$$

$$\tilde{B}(t) = S(t+1)^{-1}B(t) = \begin{bmatrix} B_c(t) \\ 0 \end{bmatrix} . \quad (4.26b)$$

Therefore the subsystem $(A_c(t), B_c(t))$ is a periodic n_c -dimensional system which is controllable for each t , and is therefore named *controllable part* of the original system. The remaining dynamics, characterized by matrix $\tilde{A}_c(t)$, corresponds to the so-called uncontrollable part. As a matter of fact, the input signal does not have any effect on the associated group of state variables.

One can also introduce a reachability decomposition provided that the requirement of constant dimensionality of the matrices is removed. Precisely, as indicated at the end of Sect. 4.1.2, thanks to the fact that the reachable subspace $X_r(t)$ is $A(\cdot)$ -invariant and contains $\text{Im}[B(t-1)]$, it is possible to find a matrix $Q_r(\cdot)$ such that $\text{Im}[Q_r(t)] = X_r(t)$ and thus to compute matrices $A_r(\cdot)$ and $B_r(\cdot)$ satisfying

$$A(t)Q_r(t) = Q_r(t+1)A_r(t), \quad B(t) = Q_r(t+1)B_r(t). \quad (4.27)$$

⁹ Given two disjoint subspaces X and Y , the direct sum $X \oplus Y$ denotes the set (subspace) of vectors which can be written as $x+y$, where $x \in X$ and $y \in Y$.

The pair $(A_r(\cdot), B_r(\cdot))$ is indeed the reachable part of the system, since it is completely reachable for each t . Matrix $Q_r(t)$ has dimensions $n_r(t+1) \times n_r(t)$, where $n_r(t) = \dim[X_r(t)]$. Correspondingly, $A_r(t)$ has the dimension $n_r(t+1) \times n_r(t)$ as well. Define then the state motion of the reachable system with time-varying dimensions

$$x_r(t+1) = A_r(t)x_r(t) + B_r(t)u(t). \quad (4.28)$$

Multiplying both members by $Q_r(t+1)$,

$$Q_r(t+1)x_r(t+1) = Q_r(t+1)A_r(t)x_r(t) + Q_r(t+1)B_r(t)u(t),$$

so that, in view of (4.27)

$$Q_r(t+1)x_r(t+1) = A(t)Q_r(t)x_r(t) + B(t)u(t).$$

Hence,

$$x(t) = Q_r(t)x_r(t)$$

is a solution of the original system. System (4.28) is an unconventional dynamical system in that $x_r(t)$ belongs to a space with periodically time-varying dimensions. This taken for granted, this system is completely reachable. Therefore, the pair $(A_r(\cdot), B_r(\cdot))$ can be named *bona fide* as the reachable part of the original system.

It is also possible to define an unreachable part of the system by taking a matrix basis $\bar{Q}_r(t)$ for any subspace complementary to $X_r(t)$. In this way, letting $Q(t) = [Q_r(t) \ \bar{Q}_r(t)]$, and recalling the properties of the subspace $X_r(t)$ discussed in Sect. 4.1.2, the system can be decomposed into the reachable/unreachable parts

$$\tilde{A}(t) = Q(t+1)A(t)Q(t)^{-1} = \begin{bmatrix} A_r(t) & \bar{A}_r^*(t) \\ 0 & A_r(t) \end{bmatrix} \quad (4.29a)$$

$$\tilde{B}(t) = Q(t+1)B(t) = \begin{bmatrix} B_r(t) \\ 0 \end{bmatrix}. \quad (4.29b)$$

In view of the previous considerations, it is not surprising that the controllable part can be further fragmented so as to separate the controllable and reachable part from the controllable and unreachable one. Indeed, thanks to (4.15) it is not difficult to conclude that it is always possible to select the matrix basis $S_c(t)$ for the controllability subspace $X_c(t)$ as follows:

$$S_c(t) = [Q_r(t) \ \bar{Q}_{rc}(t)],$$

where $\bar{Q}_{rc}(t)$ allows $S_c(t)$ to have full column rank. Therefore, by means of (4.27), the controllable part $(A_c(t), B_c(t))$ can be further decomposed in the following way

$$A_c(t) = \begin{bmatrix} A_r(t) & \bar{A}_{rc}^*(t) \\ 0 & A_{rc}(t) \end{bmatrix}, \quad B_c(t) = \begin{bmatrix} B_r(t) \\ 0 \end{bmatrix}. \quad (4.30)$$

It is apparent that the input signal $u(\cdot)$ cannot have any influence on the states associated with the dynamic matrix $A_{rc}(\cdot)$. Since the pair $(A_c(\cdot), B_c(\cdot))$ is controllable, any state which is not subject to the action of the exogenous signal must necessarily go to zero autonomously. This means that the monodromy matrix associated with $A_{rc}(\cdot)$ is nilpotent, i.e., its eigenvalues are all located in the origin. The same conclusion can be drawn from an algebraic viewpoint by making reference to (4.15).

A final comment concerns the unreachable part of the system. From the structure of $\bar{A}_c(\cdot)$ specified in (4.30) and in view of (4.26a) and (4.29a), it can be concluded that $\bar{A}_{rc}(\cdot)$ can be partitioned as follows

$$\bar{A}_r(t) = \begin{bmatrix} \bar{A}_{rc}(t) & * \\ 0 & \bar{A}_c(t) \end{bmatrix}. \quad (4.31)$$

In other words, the unreachable parts can be decomposed according to the controllability property.

Remark 4.4 *Partitions (4.26a), (4.29a) induce analogous partitions of the monodromy matrix. For instance, the monodromy matrix associated with the reachability partition (4.29a) takes the form*

$$\Psi_{\bar{A}}(t) = \begin{bmatrix} \Psi_{A_r}(t) & * \\ 0 & \Psi_{\bar{A}_r}(t) \end{bmatrix}.$$

Matrices $\Psi_{A_r}(t)$ and $\Psi_{\bar{A}_r}(t)$ are square matrices with dimensions $n_r(t)$ and $n - n_r(t)$, respectively. The characteristic multipliers of $A(\cdot)$ can therefore be grouped into $n_r(t)$ reachable multipliers and into $n - n_r(t)$ unreachable multipliers. Obviously these two groups may have common elements. As for controllability, analogous considerations hold, the only difference being the constant dimensionality of the controllability subspace. Moreover, from (4.30) it appears that the n_c controllable characteristic multipliers of $A(\cdot)$ can be subdivided into two groups: indeed, among them $n_r(t)$ coincide with the characteristic multipliers that are reachable at t ; the remaining $n_c - n_r(t)$ multipliers are all located in the origin.

Example 4.2 *Consider a periodic system with period $T = 2$ defined by the matrices*

$$\begin{aligned} A(0) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & A(1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ B(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} & B(1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

The system is completely controllable, as can be easily checked. However, the reachability subspace is the entire state-space for t even, whereas it reduces to $\text{Im}[[1 \ 0]']$ for t odd. Therefore

$$Q_r(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad Q_r(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and

$$\begin{aligned} A_r(0) &= \begin{bmatrix} 0 & 1 \end{bmatrix} & A_r(1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ B_r(0) &= 1 & B_r(1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

4.2 Observability and Reconstructability

4.2.1 Basic Definitions

The observability concept is associated with the possibility of distinguishing two *initial* states from the observation of the corresponding output signals when the system is subject to the same input function. The concept of reconstructability is analogous, the only difference being that one has to distinguish two *final* states from past measurements of the output variable. As is well-known for linear systems, this is equivalent to considering the possibility of distinguishing a given state from zero by observing the free motion of the output in the past (reconstructability) or in the future (observability). Correspondingly, we will consider the system

$$x(t+1) = A(t)x(t) \quad (4.32a)$$

$$y(t) = C(t)x(t). \quad (4.32b)$$

Definition 4.3

- (i) The state $x \in \mathbb{R}^n$ is *unobservable* over (t, τ) , $t < \tau$ if to the free motion starting in $x(t) = x$ there corresponds the output $y(i) = 0$, $\forall i \in [t, \tau - 1]$. The state $x \in \mathbb{R}^n$ is *unreconstructable* over (τ, t) , $t > \tau$ if there exists a free motion ending in $x(t) = x$ which results in the output $y(i) = 0$, $\forall i \in [\tau, t - 1]$.
- (ii) The state $x \in \mathbb{R}^n$ is *unobservable* at t in k steps [unreconstructable at t in k steps] if x is *unobservable* over $(t, t+k)$ [unreconstructable over $(t-k, t)$].
- (iii) The state $x \in \mathbb{R}^n$ is *unobservable* at t [unreconstructable at t] if for all k , x is *unobservable* at t in k steps [unreconstructable at t in k steps].
- (iv) System (4.32) (or, equivalently, the pair $(A(\cdot), C(\cdot))$) is *observable* over (t, τ) [reconstructable over (τ, t)] if any non-zero state is not *unobservable* over (t, τ) [not unreconstructable over (τ, t)].
- (v) System (4.32) (or, equivalently, the pair $(A(\cdot), C(\cdot))$) is *observable* [reconstructable] in k steps if the system is *observable* over $(t, t+k)$ [reconstructable over $(t-k, t)$].
- (vi) System (4.32) (or, equivalently, the pair $(A(\cdot), C(\cdot))$) is *observable* [reconstructable] at time t if there exists a k , $k > 0$, such that the system is *observable* [reconstructable] at t in k -steps.

(vii) System (4.32), (or, equivalently, the pair $(A(\cdot), C(\cdot))$) is observable [reconstructable] if it is observable [reconstructable] at any time point.

We will also introduce the symbols

$$\begin{aligned}\bar{\mathcal{X}}_\omega(t, \tau) & \text{ set of unobservable states over } (t, \tau) \\ \bar{\mathcal{X}}_\rho(\tau, t) & \text{ set of unreconstructible states over } (\tau, t) \\ \bar{X}_\omega(t) & \text{ set of unobservable states at } t \\ \bar{X}_\rho(t) & \text{ set of unreconstructible states at } t.\end{aligned}$$

Such sets are in fact subspaces of \mathbb{R}^n , as it is easy to prove. Correspondingly, we introduce the observable and reconstructable subspaces as

$$\begin{aligned}\mathcal{X}_\omega(t, \tau) &= \bar{\mathcal{X}}_\omega(t, \tau)^\perp \\ \mathcal{X}_\rho(\tau, t) &= \bar{\mathcal{X}}_\rho(\tau, t)^\perp \\ X_\omega(t) &= \bar{X}_\omega(t)^\perp \\ X_\rho(t) &= \bar{X}_\rho(t)^\perp.\end{aligned}$$

The system periodicity obviously entails that $\mathcal{X}_\omega(t+T, \tau+T) = \mathcal{X}_\omega(t, \tau)$ and $X_\omega(t+T) = X_\omega(t)$, and similarly for $\mathcal{X}_\rho(\tau, t)$ and $X_\rho(t)$.

In the analysis of observability and reconstructability, a main role is played by the observability matrix defined as

$$\bar{O}_j(t) = \begin{bmatrix} C(t) \\ C(t+1)\Psi_A(t+1, t) \\ \vdots \\ C(t+j-1)\Psi_A(t+j-1, t) \end{bmatrix}. \quad (4.33)$$

As apparent from the definition, a state x is unobservable over $(t, t+j)$ if and only if

$$\bar{O}_j(t)x = 0.$$

Therefore the unobservable subspace over $(t, t+j)$ coincides with the null space of $\bar{O}_j(t)$, i.e.,

$$\bar{\mathcal{X}}_\omega(t, t+j) = \text{Ker}[\bar{O}_j(t)], \quad \mathcal{X}_\omega(t, t+j) = \text{Im}[\bar{O}_j(t)'].$$

Therefore the null space of $\bar{O}_j(t)$ supplies the unobservability subspace over $(t, t+j)$ and the range of $\bar{O}_j(t)$ is the observable subspace over $(t, t+j)$.

4.2.2 Duality

The properties of a periodic system relative to observability and reconstructability can be worked out in a strict analogy with reachability and controllability, analyzed in the previous sections. This is a consequence of the well-known system-theoretic concept of *duality*, which establishes the following correspondences among the structural properties.

$$\begin{aligned} \text{reachability} &\Longleftrightarrow \text{observability} \\ \text{controllability} &\Longleftrightarrow \text{reconstructibility}. \end{aligned}$$

Precisely, given a T -periodic system

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (4.34a)$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad (4.34b)$$

the associated dual system is defined as

$$x^d(t+1) = A^d(t)x^d(t) + B^d(t)u^d(t) \quad (4.35a)$$

$$y^d(t) = C^d(t)x^d(t) + D^d(t)u^d(t), \quad (4.35b)$$

where

$$A^d(t) = A(-t-1)' \quad B^d(t) = C(-t-1)' \quad (4.36a)$$

$$C^d(t) = B(-t-1)' \quad D^d(t) = D(-t-1)'. \quad (4.36b)$$

It is easy to see that the transition matrix $\Phi_{A^d}(t, \tau)$ of $A^d(\cdot)$ is related to $\Phi_A(t, \tau)$ as follows

$$\Phi_{A^d}(t, \tau) = \Phi_A(-\tau, -t)'. \quad (4.36c)$$

In particular, the relationship between the monodromy matrices is $\Psi_{A^d}(t) = \Psi_A(-t)'$, so that the system and its dual share the same characteristic multipliers. Moreover, recalling the definitions of the reachability and unobservability matrices as given in (4.4), (4.33), and denoting by $R_j^d(t)$ the reachability matrix for the dual system, it turns out that

$$\bar{O}_j(t)' = R_j^d(-t),$$

so that

$$\mathcal{X}_\omega(t, \tau) = X_r^d(-\tau, -t), \quad (4.37)$$

where the symbol $X_r^d(\cdot, \cdot)$ obviously denotes the reachability subspace of the dual system. As for reconstructability, one can exploit formula (4.11) in order to conclude

$$\mathcal{X}_\rho(\tau, t) = X_c^d(-t, -\tau), \quad (4.38)$$

where $X_c^d(\cdot, \cdot)$ denotes the controllability subspace of the dual system.

In particular, the system is observable [reconstructable] if and only if its dual is reachable [controllable]. Formulas (4.37), (4.38) are important in that they allow one to freely recover any observability and reconstructability properties from the reachability and controllability properties of the dual system. For this reason, the results in the subsequent sections are stated without proof.

Remark 4.5 *Previously, the duality concept has been introduced with reference to systems given in state–space form, see (4.35), (4.36). This concept translates to input–output maps as well. To be precise, if $G(\sigma, t)$ is the transfer operator associated with the periodic System (4.34), then by recalling (4.36), it can be seen that the dual operator $G^d(\sigma, t)$ is given by*

$$G^d(\sigma, t) = G(\sigma, -t - 1)'$$

4.2.3 Observability

The observability subspace $\mathcal{X}_\omega(t, \tau)$ is “monotonically non-decreasing” in the sense that

$$\mathcal{X}_\omega(t, j) \subseteq \mathcal{X}_\omega(t, i), \quad t < j < i.$$

It enjoys the quiescence property, i.e if $\mathcal{X}_\omega(t, t + kT) = \mathcal{X}_\omega(t, t + kT + T)$, then $\mathcal{X}_\omega(t, t + iT) = \mathcal{X}_\omega(t, t + kT)$, $\forall i \geq k$. Consequently an unobservable state can be detected in an interval of time not larger than nT , i.e.,

$$X_\omega(t) = \mathcal{X}_\omega(t, t + nT).$$

If one considers an interval of length multiple of the period T , the unobservability matrix can be seen as the unobservability matrix of a time-invariant system. Indeed,

$$\bar{O}_{kT}(t) = \begin{bmatrix} \bar{O}_T(t) \\ \bar{O}_T(t)\Psi_A(t) \\ \vdots \\ \bar{O}_T(t)\Psi_A(t)^{k-1} \end{bmatrix}.$$

Therefore, letting

$$F_t := \Psi_A(t), \quad H_t := \bar{O}_T(t),$$

the unobservability matrix above coincides with that associated with the time-invariant pair (F_t, H_t) . In this way, one can introduce the *observability index* $\mu_o(t)$ as the minimum index for which the time-invariant system characterized by the pair (F_t, H_t) can be observed, i.e.,

$$X_\omega(t) = \mathcal{X}_\omega(t, t + \mu_o(t)T).$$

The index $\mu_o(t)$ is periodic and its variability is limited by the condition

$$|\mu_o(t) - \mu_o(\tau)| \leq 1, \forall t, \tau \in [0, T-1].$$

In general, the observability subspace has a time-varying dimension. For reversible systems, the dimension is constant.

It turns out that the unobservability subspace $\bar{X}_\omega(t)$ is $A(\cdot)$ -invariant. Again, this can be seen by duality. However, we prefer to work out this conclusion by direct analysis of the unobservability matrix. Consider an unobservable state $x \in \bar{X}_\omega(t)$, namely $x \in \text{Ker}[O_{nT}(t)] = \text{Ker}[O_{nT+1}(t)]$. By considering the row-block structure of the unobservability matrix, vector x must satisfy the following set of equations

$$\begin{aligned} C(t)\Psi_A(t)^i x &= 0, \quad i = 0, 1, \dots, n \\ C(t+1)A(t)\Psi_A(t)^i x &= 0, \quad i = 0, 1, \dots, n-1 \\ &\dots \\ C(t+T-1)A(t+T-2)A(t+T-3) \cdots A(t)\Psi_A(t)^i x &= 0, \quad i = 0, 1, \dots, n-1. \end{aligned}$$

On the other hand, it is easy to verify that

$$A(t)\Psi_A(t)^i = \Psi_A(t+1)^i A(t),$$

so that, for $i = 0, 1, \dots, n-1$

$$\begin{aligned} C(t+1)\Psi_A(t+1)^i A(t)x &= 0, \\ C(t+2)A(t+1)\Psi_A(t+1)^i A(t)x &= 0, \\ &\dots \\ C(t+T)A(t+T-1)A(t+T-2) \cdots A(t+1)\Psi_A(t+1)^i A(t)x &= 0. \end{aligned}$$

This means that $A(t)x \in \bar{X}_\omega(t+1)$. Therefore, $\bar{X}_\omega(t)$ is $A(\cdot)$ -invariant.

One can introduce a matrix basis $\bar{Q}_o(t)$ for the unobservability subspace, i.e., $\text{Im}[\bar{Q}_o(t)] = \bar{X}_\omega(t)$. The invariance property implies that there exists a periodic matrix $A_o(t)$ such that

$$A(t)\bar{Q}_o(t) = \bar{Q}_o(t+1)\bar{A}_o(t).$$

Moreover, being $\bar{X}_\omega(t) \subseteq \text{Ker}[C(t)]$, it follows that

$$C(t)\bar{Q}_o(t) = 0.$$

In this way, one can prove that the unobservability subspace is not only $A(\cdot)$ -invariant but also that it is the largest $A(\cdot)$ -invariant subspace contained in $\text{Ker}[C(t)]$.

4.2.4 Observability Canonical Form

A periodic system can be described with a limited number of free parameters by means of the so-called observability canonical form. We will develop this subject by making reference to single output periodic systems. Precisely, a single-output T -periodic system is said to be in observable canonical form if

$$A(t) = A_{obs}(t) = \begin{bmatrix} 0 & \cdots & 0 & 0 & -\beta_n(t) \\ 1 & \cdots & 0 & 0 & -\beta_{n-2}(t) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & -\beta_2(t) \\ 0 & \cdots & 0 & 0 & -\beta_1(t) \end{bmatrix}$$

$$C(t) = C_{obs}(t) = [0 \ 0 \ 0 \ \cdots \ 1] .$$

Notice that matrix $A_{obs}(t)$ is in v -companion form, and as such, it is a T -cyclic matrix (Sect. 3.4). The following result shows that a T -periodic system of dimension n can be put in the observable canonical form if and only if it is n -step observable for each t . This is reminiscent of the dual result in Sect. 4.1.3.

Proposition 4.7 *Consider a single output T -periodic system of dimension n . The system is observable in n steps for each time point t if and only if it is algebraically equivalent to a system in observable canonical form.*

Proof If the system is in the observable canonical form, then the matrix

$$\begin{bmatrix} C_{obs}(t+n-1)\Phi_{A_{obs}}(t+n-1,t) \\ C_{obs}(t+n-2)\Phi_{A_{obs}}(t+n-2,t) \\ \vdots \\ C_{obs}(t+1)A_{obs}(t) \\ C_{obs}(t) \end{bmatrix}$$

is triangular and invertible for each t . This means that the pair $(A_{obs}(t), C_{obs}(t))$ is observable in n steps for each t . The converse statement can be proven by duality as follows. First notice that by assumption, the matrix

$$\bar{O}_n(t) = \begin{bmatrix} C(t) \\ C(t+1)A(t) \\ \vdots \\ C(t+n-2)\Phi_A(t+n-2,t) \\ C(t+n-1)\Phi_A(t+n-1,t) \end{bmatrix}$$

is invertible for each t . Now define the dual pair $(\hat{A}(\cdot), \hat{B}(\cdot))$ as

$$\hat{A}(t) = A(-t-1)', \quad \hat{B}(t) = C(-t-1)' .$$

Then

$$\bar{O}_n(-t)' = [\hat{B}(t-1) \hat{A}(t-1) \hat{B}(t-2) \cdots \Phi_{\hat{A}}(t, t-n+1) \hat{B}(t-n)]$$

is invertible for each t as well. In passing from matrix $\bar{O}_n(t)$ to $\bar{O}_n(-t)'$, we took advantage of the relation between the transition matrix of $A(\cdot)$ and that of $\hat{A}(\cdot)$, namely

$$\Phi_{\hat{A}}(t, t-n+1) = \Phi_A(-t+n-1, -t)'$$

By mimicking the proof of Proposition 4.1, it follows that there exists a T -periodic change of basis $S(\cdot)$ so that the pair

$$A_{reach}(t) = \hat{S}(t+1) \hat{A}(t) \hat{S}(t)^{-1}, \quad B_{reach}(t) = \hat{S}(t+1) \hat{B}(t)$$

is in the reachable canonical form. In conclusion, this means that the transformation $S(t) = (\hat{S}(-t)^{-1})'$ yields

$$A_{obs}(t) = S(t+1) A(t) S(t)^{-1}, \quad C_{obs}(t) = C(t) S(t)^{-1}$$

in observable canonical form.

4.2.5 Reconstructability

In this section, we overview the properties of reconstructability subspace of periodic systems. The first observation is that the quiescence property still holds true, as is apparent from the dual property concerning controllability. This means that

$$X_\rho(t) = \mathcal{X}_\rho(t - nT, t).$$

In an analogy with the controllability notion, the dimension of the reconstructability subspace is constant.

The unreconstructability subspace can be related to the unobservability one; indeed, by virtue of the very definition, any unreconstructable state is the result of a free motion starting from an unobservable state, so that:

$$\bar{\mathcal{X}}_\rho(\tau, t) = \Phi_A(t, \tau) \bar{\mathcal{X}}_\omega(\tau, t).$$

In turn, by letting $\tau = t - nT$, this inclusion becomes

$$\bar{X}_\rho(t) = F_t^n \bar{X}_\omega(t), \tag{4.39}$$

so that

$$X_\omega(t) \subseteq X_\rho(t).$$

Of course, in the case of reversible systems, $X_\omega(t) = X_\rho(t)$.

One can also define the *reconstructability index* $\mu_\rho(t)$ at time t of the periodic pair $(A(\cdot), C(\cdot))$ as the reconstructability index of the time-invariant pair¹⁰ (F_t, H_t) . Again, this index satisfies

$$|\mu_\rho(t) - \mu_\rho(\tau)| \leq 1, \quad \forall t, \tau \in [0, T-1].$$

It can be shown that

$$\dim[X_\rho(t)] = \text{constant},$$

and that a necessary and sufficient condition for the system reconstructability is the following one:

$$(A(\cdot), C(\cdot)) \text{ reconstructable} \iff \text{Ker}[\bar{O}_{\mu_\rho(t)} T(t)] \subseteq \text{Ker}[F_t^{\mu_\rho(t)}], \text{ for some } t.$$

4.2.6 Observability and Reconstructability Grammians

The observability and reconstructability properties can be studied by analyzing the corresponding Grammian matrices. The observability Grammian over the time-interval (τ, t) is the $n \times n$ positive semi-definite matrix

$$W_\omega(\tau, t) = \sum_{j=\tau}^{t-1} \Phi_A(j, \tau)' C(j)' C(j) \Phi_A(j, \tau),$$

which is obviously bi-periodic $W_\omega(\tau + T, t + T) = W_\omega(\tau, t)$. The unobservability matrix $O_j(t)$ is a Cholewsky factor of such a matrix, i.e.,

$$W_\omega(t, t + j) = \bar{O}_j(t)' \bar{O}_j(t),$$

so that¹¹

$$\bar{\mathcal{X}}_\omega(t, t + j) = \text{Ker}[W_\omega(t, t + j)].$$

The Grammian matrix computed over intervals of time of length multiple of the period can be given a recursive expression as

$$\bar{W}_{i+1}(t) = \bar{W}_i(t) + \Psi_A(t)' \bar{W}_i(t) \Psi_A(t)^i,$$

where $\bar{W}_i(t) = W_\omega(t, t + iT)$. Therefore, the system is observable at t if and only if

$$W_\omega(t, t + \mu_o(t)T) > 0.$$

¹⁰ For a time-invariant system, the reconstructability index is defined as the minimum integer for which the unreconstructability subspace ceases to decrease. Analogous to the reachability index, it can be related to the observability index by resorting to the Kalman canonical decomposition. Precisely $\mu_\rho(t) = \max\{\mu_o(t), \mu_w(t)\}$, where $\mu_w(t)$ is the dimension of the largest Jordan block of the unobservable and reconstructable part of the time-invariant pair (F_t, H_t) .

¹¹ For any matrix P , $\text{Ker}[P'P] = \text{Ker}[P]$.

As for reconstructability, it can be shown that the periodic system is reconstructable if and only if no vector η exists such that

$$\begin{aligned}\eta' W_\omega(t, t + \mu_\rho(t)T) \eta &= 0 \\ \Psi_A(t)^{\mu_\rho(t)T} \eta &\neq 0.\end{aligned}$$

For a reversible system, one can introduce a reconstructability Grammian as

$$W_\rho(\tau, t) = \sum_{j=\tau+1}^{t-1} \Phi_A(j, t)' C(j)' C(j) \Phi_A(j, t).$$

A necessary and sufficient condition for a reversible system to be reconstructable is $W_\rho(t, t + \mu_\rho(t)T) > 0$.

4.2.7 Modal Observability and Reconstructability

The recursive scheme used in Sect. 4.1.6 can be replicated in order to derive a modal notion for observability and reconstructability. The simple derivation leads to the following conclusions.

Proposition 4.8 *System (4.32) is observable at time t if and only if, for each characteristic multiplier λ , the conditions*

$$\begin{aligned}\Psi_A(t) \eta &= \lambda \eta \\ C(j)' \Phi_A(j, t) \eta &= 0, \quad \forall j \in [t, t + T - 1]\end{aligned}$$

imply $\eta = 0$. ■

This leads to the following definition of an unobservable characteristic multiplier.

Definition 4.4 *A characteristic multiplier λ is said to be unobservable at time t if the conditions*

$$\begin{aligned}\Psi_A(t) \eta &= \lambda \eta \\ C(j) \Phi_A(j, t) \eta &= 0, \quad \forall j \in [0, T].\end{aligned}$$

are satisfied for some vector $\eta \neq 0$. ■

This definition enables one to state the reconstructability condition below, the proof of which can be obtained by dualizing the proof of Proposition 4.6.

Proposition 4.9 *System (4.32) is reconstructable if all non-zero characteristic multipliers are observable at some t .*

4.2.8 Canonical Reconstructability Decomposition

As seen above, although the reconstructability subspace $X_\rho(t)$ is periodically time-varying, its dimension is constant, say n_ρ . This motivates the use of this subspace to derive a state-space transformation which decomposes the periodic system into the reconstructable and unreconstructable parts. To this purpose, define $\bar{X}_\rho(t)$ as a complementary subspace of the unreconstructability subspace $\bar{X}_\rho(t)$, and let $\bar{S}_\rho(t)$ and $S_\rho(t)$ be two matrices of dimensions $n \times n_\rho$ and $n \times n - n_\rho$, whose columns constitute, respectively, a basis for $\bar{X}_\rho(t)$ and $X_\rho(t)$. Then one can introduce the $n \times n$ invertible matrix

$$S(t) = \begin{bmatrix} \bar{S}_\rho(t) & S_\rho(t) \end{bmatrix}.$$

Obviously, $S(t)$ is periodic and non-singular for each t . Then, by defining

$$\tilde{x}(t) = S(t)^{-1}x(t),$$

one obtains in the new coordinates

$$\begin{aligned} \tilde{A}(t) &= S(t+1)^{-1}A(t)S(t) = \begin{bmatrix} A_\rho(t) & 0 \\ \star & \bar{A}_\rho(t) \end{bmatrix} \\ \tilde{C}(t) &= C(t)S(t) = \begin{bmatrix} 0 & C_\rho(t) \end{bmatrix}, \end{aligned}$$

where \star denotes a no matter matrix. Therefore, the pair $(A_\rho(t), B_\rho(t))$ is a periodic n_ρ -dimensional pair that is reconstructable for each t , and is named the *reconstructable part* of the original system. The remaining set of states, characterized by the dynamic matrix $\bar{A}_\rho(t)$, are completely undistinguishable from the free motion and correspond to the so-called unreconstructable part of the system.

It is also possible to define the *unobservable part* of the system by considering matrices with time-varying dimensions. Indeed, recall that the unobservability subspace $\bar{X}_\omega(t)$ is $A(\cdot)$ -invariant and is contained in the null space of $C(\cdot)$. Then, letting $\bar{Q}_\omega(t)$ be a matrix-basis for $\bar{X}_\omega(t)$, i.e., $\text{Im}[\bar{Q}_\omega(t)] = \bar{X}_\omega(t)$, it follows that

$$\begin{aligned} A(t)\bar{Q}_\omega(t) &= \bar{Q}_\omega(t+1)\bar{A}_\omega(t) \\ C(t)\bar{Q}_\omega(t) &= 0. \end{aligned} \tag{4.40}$$

Notice that $\bar{A}_\omega(t)$ has dimension $\bar{n}_\omega(t+1) \times \bar{n}_\omega(t)$, where $\bar{n}_\omega(t) = \dim[\bar{X}_\omega(t)]$. Now consider the free-motion of the system with time-varying dimensions

$$\bar{x}_\omega(t+1) = \bar{A}_\omega(t)\bar{x}_\omega(t). \tag{4.41}$$

If one multiplies both members by $\bar{Q}_\omega(t+1)$, then

$$\bar{Q}_\omega(t+1)\bar{x}_\omega(t+1) = A(t)\bar{Q}_\omega(t)\bar{x}_\omega(t).$$

Hence,

$$x(t) = \bar{Q}_\omega(t)\bar{x}_\omega(t)$$

is a free motion for the original system. Associated with such a motion, the output is identically zero since

$$y(t) = C(t)x(t) = C(t)\bar{Q}_\omega(t)\bar{x}_\omega(t) = 0.$$

Therefore, System (4.41) is completely unobservable, and the matrix $\bar{A}_\omega(\cdot)$ can be named the unobservable part of the original system.

One can now wonder how the unreconstructable part $\bar{A}_\rho(t)$ is related to the unobservable part $\bar{A}_\omega(t)$. To figure out this relation, notice that in view of (4.39), the matrix basis $\bar{Q}_\omega(t)$ for the unobservability subspace $\bar{X}_\omega(t)$ can be partitioned in the following way

$$\bar{Q}_\omega(t) = [* (t) \bar{S}_\rho(t)],$$

where $\bar{S}_\rho(t)$ is any matrix which renders \bar{Q}_ω a full column rank matrix. Thanks to (4.40), it turns out that the finer structure of $A_\omega(t)$ is

$$\bar{A}_\omega(t) = \begin{bmatrix} \bar{A}_{\omega\rho}(t) & 0 \\ \star & \bar{A}_\rho(t) \end{bmatrix}, \quad (4.42)$$

where again \star denotes a no matter matrix. The characteristic multipliers associated with $\bar{A}_{\omega\rho}$ are all located in the origin of the complex plane. Their number can change in time due to the time-variability of the matrix dimension.

4.3 Canonical Decomposition

It has been already shown that the state variables of a periodic system can be grouped in two sets, namely the controllable/uncontrollable ones for the input-to-state relation and the reconstructable/unreconstructable sets for the state-to-output map. Therefore, it is not surprising that it is possible to perform a refined decomposition into four sets of states obtained by the combination of the previous properties. Indeed, consider the controllable subspace $X_c(\cdot)$ and the unreconstructable subspace $\bar{X}_\rho(\cdot)$. As already said, such subspaces are $A(\cdot)$ -invariant. This implies that $X_c(t)$ and $X_\rho(t)$ are F_t -invariant, where $F_t = \Psi_A(t)$. Therefore their intersection

$$X_1(t) = X_c(t) \cap \bar{X}_\rho(t)$$

is also F_t -invariant. Now define $X_2(t)$ and $X_3(t)$ as any subspaces complementary to $X_1(t)$ within $X_c(t)$ and $\bar{X}_\rho(t)$, respectively, i.e.,

$$\begin{aligned} X_c(t) &= X_1(t) \oplus X_2(t) \\ \bar{X}_\rho(t) &= X_1(t) \oplus X_3(t). \end{aligned}$$

Finally, introduce the subspace $X_4(t)$ as the subspace such that

$$R^n = (X_c(t) + \bar{X}_\rho(t)) \oplus X_4(t).$$

Notice that $X_1(t)$, $X_2(t)$ and $X_3(t)$ do not have non-zero common vectors. This entails that

$$X_1(t) \oplus X_2(t) \oplus X_3(t) = X_c(t) + \bar{X}_\rho(t),$$

whence

$$R^n = X_1(t) \oplus X_2(t) \oplus X_3(t) \oplus X_4(t).$$

The four subspaces $X_i(t)$, $i = 1, \dots, 4$, are periodic with constant dimensions, say n_i , $i = 1, \dots, 4$. Defining $S_i(t)$, $i = 1, \dots, 4$ any $n \times n_i$ matrices such that $Im[S_i(t)] = X_i(t)$, one can introduce the $n \times n$ T-periodic state-space transformation $\tilde{x}(t) = S(t)x(t)$, where

$$S(t) = [S_1(t) \ S_2(t) \ S_3(t) \ S_4(t)].$$

In the new coordinates, the system matrices

$$\tilde{A}(t) = S(t+1)A(t)S(t)^{-1}, \quad \tilde{B}(t) = S(t+1)B(t), \quad \tilde{C}(t) = C(t)S(t)^{-1}$$

take the form

$$\tilde{A}(t) = \begin{bmatrix} A_{11}(t) & \underline{A}_{12}(t) & A_{13}(t) & A_{14}(t) \\ 0 & A_{22}(t) & 0 & A_{24}(t) \\ 0 & 0 & A_{33}(t) & A_{34}(t) \\ 0 & 0 & 0 & A_{44}(t) \end{bmatrix} \quad \tilde{B}(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C}(t) = [0 \ C_2(t) \ 0 \ C_4(t)].$$

The system state $\tilde{x}(t)$ is then partitioned into four parts, namely controllable and unreconstructable ($\tilde{x}_1(t)$), controllable and reconstructable ($\tilde{x}_2(t)$), uncontrollable and unreconstructable ($\tilde{x}_3(t)$), uncontrollable and reconstructable ($\tilde{x}_4(t)$).

Obviously, the controllable part ($A_c(t), B_c(t)$) defined in Sect. 4.1.7 can be extracted from $(\tilde{A}(t), \tilde{B}(t))$ as follows

$$A_c(t) = \begin{bmatrix} A_{11}(t) & \underline{A}_{12}(t) \\ 0 & A_{22}(t) \end{bmatrix}, \quad B_c(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}.$$

Analogously, the reconstructable part ($A_\rho(t), C_\rho(t)$) defined in Sect. 4.2.8 is given by

$$A_\rho(t) = \begin{bmatrix} A_{22}(t) & \underline{A}_{24}(t) \\ 0 & A_{44}(t) \end{bmatrix}, \quad C_\rho(t) = [C_2(t) \ C_4(t)].$$

Example 4.3 Consider a periodic system with period $T = 2$ defined by the matrices

$$\begin{aligned} A(0) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & A(1) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ B(0) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & B(1) &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ C(0) &= [0 \ 0 \ 1] & C(1) &= [0 \ 1 \ 0]. \end{aligned}$$

The monodromy matrix is given by

$$\Phi_A(0) = F_0 = A(1)A(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \Phi_A(1) = F_1 = A(0)A(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

so that

$$F_0^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad F_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

We first study the system reachability. To this end, consider the reachability matrix (4.4). Letting $j = nT = 6$, it turns out that

$$\begin{aligned} R_6(0) &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ R_6(1) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Consequently, in this case the reachability subspace is time-invariant and given by the x_1 line. The controllability subspace, which can be deduced from the reachability one on the basis of (4.14) or (4.15), is the plane (x_1, x_2) for each time point.

Turning now to observability, consider the unobservability matrix (4.33)

$$\bar{O}_6(0) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{O}_6(1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

Therefore, the unobservability subspace is periodically time-varying. Precisely, $X_\omega(0)$ coincides with the plane (x_1, x_2) , whereas $X_\omega(1)$ reduces to the line (x_1) .

In view of the link between the unobservability subspace and the unreconstructibility subspace given by (4.39), it easily follows that $X_p(0) = \bar{X}_p(1) = \{0\}$ so that the system is reconstructable.

In view of these structural properties of the system, it is apparent that the controllable/reconstructable canonical decomposition introduced above reduces to a decomposition into the controllable and uncontrollable parts. Indeed, the original system is already decomposed accordingly:

$$\begin{aligned} A_c(0) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & B_c(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A_c(1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & B_c(1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

If one is interested in extracting the reachable and unobservable part, the first step is to compute the reachable and unobservable subspace $X_r(t) \cap \bar{X}_\omega(t)$. This subspace, being the intersection of $(A(\cdot))$ -invariant subspaces, is $A(\cdot)$ -invariant as well. In our case, this subspace turns out to have constant dimensions, since it coincides, for each time point, with the (x_1) line. The reachable and unobservable part is then described by

$$\xi(t+1) = \bar{A}_{\omega r}(t)\xi(t) + \bar{B}_{\omega r}u(t),$$

where

$$\bar{A}_{\omega r}(0) = 0, \quad \bar{A}_{\omega r}(1) = 1, \quad \bar{B}_{\omega r}(0) = \bar{B}_{\omega r}(1) = 1.$$

If $\xi(t)$ is a motion of the system above, then

$$x(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xi(t)$$

is a motion of the original system, constituted of reachable states and resulting in an identically null output.

4.4 Extended Structural Properties

4.4.1 Stabilizability

The problem of stabilizing a given system is a key problem in feedback control theory. If one focuses on *state-feedback*, the problem can be stated as follows: Find, if possible, a T -periodic matrix $K(\cdot)$ so that the feedback control law $u(t) = K(t)x(t)$ applied to System (4.1) results in a stable closed-loop system. Note that the closed-loop system is characterized by the dynamic matrix $\hat{A}(\cdot) = A(\cdot) + B(\cdot)K(\cdot)$.

Definition 4.5 *The periodic system is said to be stabilizable if there exists a T -periodic matrix $K(\cdot)$ such that $A(\cdot) + B(\cdot)K(\cdot)$ is stable. ■*

A useful characterization of stabilizability can be obtained by making reference to the canonical decomposition. Indeed, consider the state-transformation that brings the system into the controllable/uncontrollable canonical decomposition $(\tilde{A}(\cdot), \tilde{B}(\cdot))$, see (4.26). From the structure of these matrices it is apparent that the characteristic multipliers of $\tilde{A}_c(\cdot)$ are part of the characteristic multipliers of $A(\cdot) + B(\cdot)K(\cdot)$, for each T -periodic matrix $K(\cdot)$. This leads to the following characterization of stabilizability.

Proposition 4.10 *The system is stabilizable if and only if its uncontrollable part is stable.*

Proof If the system is stabilizable, then the previous considerations about the canonical decomposition show that the uncontrollable part $\tilde{A}_c(\cdot)$ is stable. Conversely, suppose that $\tilde{A}_c(\cdot)$ is stable. Since one can stabilize the controllable part (see e.g., Sect. 13.3 on pole assignment), the whole system is stabilizable. ■

Another useful characterization is the modal one, which can be stated by slightly generalizing the modal notions introduced in Sect. 4.1.6.

Proposition 4.11 *The system is stabilizable if and only if, for each characteristic multiplier λ of $A(\cdot)$, with $|\lambda| \geq 1$, the conditions*

$$\Psi_A(t)' \eta = \lambda \eta \quad (4.43a)$$

$$B(j-1)' \Phi_A(t, j)' \eta = 0, \quad \forall j \in [t-T+1, t] \quad (4.43b)$$

imply $\eta = 0$.

Proof Without any loss in generality, assume that $A(\cdot)$ and $B(\cdot)$ are decomposed into the controllable/uncontrollable canonical form, see (4.26). Correspondingly, the transition matrix takes a triangular form

$$\Phi_A(t, j) = \begin{bmatrix} \Phi_{A_c}(t, j) & \Delta(t, j) \\ 0 & \Phi_{\tilde{A}_c}(t, j) \end{bmatrix}.$$

Now assume that the system is not stabilizable, so that in view of Proposition 4.10, $\tilde{A}_c(\cdot)$ has a characteristic multiplier λ with a modulus greater than or equal to one. Correspondingly, take an associated eigenvector $\bar{\eta}_c$. It is immediately seen that the vector

$$\eta = \begin{bmatrix} 0 \\ \bar{\eta}_c \end{bmatrix}$$

violates the statement of the theorem. Vice versa, assume that there exists a non-null vector

$$\eta = \begin{bmatrix} \eta_c \\ \bar{\eta}_c \end{bmatrix},$$

which violates the statement of the theorem for some λ with $|\lambda| \geq 1$, i.e.,

$$\begin{aligned}\Psi_{A_c}(t)' \eta_c &= \lambda \eta_c \\ \Delta(t+T, t)' \eta_c + \Psi_{A_c}^-(t)' \bar{\eta}_c &= \lambda \bar{\eta}_c \\ B_c(j-1)' \Phi_{A_c}(t, j)' \eta_c &= 0, \quad \forall j \in [t-T+1, t].\end{aligned}$$

In view of Propositions 4.4, 4.6, the controllability of $(A_c(\cdot), B_c(\cdot))$ implies $\eta_c = 0$ so that the second equation becomes $\Phi_{A_c}^-(t)' \bar{\eta}_c = \lambda \bar{\eta}_c$. Being $|\lambda| \geq 1$, the uncontrollable part turns out to be unstable. In view of Proposition 4.10, this is equivalent to say that the system is not stabilizable. ■

Notice the role of time t in Proposition 4.11 is immaterial; for the test (4.43) any time point may be adopted.

In principle, one could also consider the possibility of defining stabilizability in terms of stability of the unreachable part $A_r(t)$, see (4.31). Notice however that, in general, such a part has time-varying dimensions: precisely, denoting by $n_r(t)$ the dimension of the reachability subspace at t , $A_r(t)$ has the dimension $(n - n_r(t+1)) \times (n - n_r(t))$. The monodromy matrix $\Psi_{A_r}^-(t)$ of $A_r(t)$ is a square matrix with dimension $(n - n_r(t)) \times (n - n_r(t))$. In view of (4.31), its characteristic multipliers are those of $A_c(\cdot)$ along with the characteristic multipliers of $A_{rc}(t)$ that are in fact all zero. In other words, only the number of zero-characteristic multipliers of $A_r(\cdot)$ may change with time. Moreover, the uncontrollable part $A_c(\cdot)$ differs from the unreachable part $A_r(\cdot)$ only for the number of null characteristic multipliers. In conclusion, Proposition 4.10 holds true even if, in the statement, the notion of uncontrollability is replaced with that of unreachability, provided that the stability for a time-varying dimensional matrix is defined properly (recall Remark 3.1).

4.4.2 Detectability

The roots of the detectability concept lie in the problem of finding a stable filter that, fed by the output $y(\cdot)$ of a given system, is capable of generating an estimate $\hat{x}(t)$ of the system state converging to the true state $x(t)$. This problem is known as the *observer problem* and the filter is named *observer*.

With reference to System (4.32), a classical way to tackle the problem is to resort to an observer of the form

$$\begin{aligned}\hat{x}(t+1) &= A(t)\hat{x}(t) + B(t)u(t) + L(t)(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C(t)\hat{x}(t) + D(t)u(t).\end{aligned}$$

Matrix $L(\cdot)$ is the so-called *observer gain*. The system being periodic, matrix $L(\cdot)$ is assumed to be T -periodic as well. The estimation error $v(t) = x(t) - \hat{x}(t)$ satisfies the equation

$$v(t+1) = (A(t) + L(t)C(t))v(t).$$

Therefore, it is obvious that the observer is convergent ($v(t) \rightarrow 0$) if and only if $A(\cdot) + L(\cdot)C(\cdot)$ is stable. This leads to the following definition.

Definition 4.6 *The periodic system is said to be detectable if there exists a T -periodic matrix $L(\cdot)$ such that $A(\cdot) + L(\cdot)C(\cdot)$ is stable.* ■

By comparing this definition with Definition 4.5, it is apparent that detectability and stabilizability are dual concepts. Therefore, all results presented in the previous section can be transformed into analogous ones by duality and their associated proofs are therefore omitted. In this way, one can work out the link between detectability and canonical decomposition.

Proposition 4.12 *The system is detectable if and only if its unreconstructable part is stable.* ■

The modal characterization of detectability coincides with the modal observability criterion of Proposition 4.8 by restricting it to the non-stable characteristic multipliers of the system.

Proposition 4.13 *The system is detectable if and only if, for each characteristic multiplier λ of $A(\cdot)$, with $|\lambda| \geq 1$, the conditions*

$$\begin{aligned}\Psi_A(t)\eta &= \lambda\eta \\ C(j)'\Phi_A(j,t)\eta &= 0, \quad \forall j \in [t, t+T-1]\end{aligned}$$

imply $\eta = 0$. ■

Notice that the dependence of this criterion on t is immaterial, as can be inferred from the fact that the dimension of the unreconstructable subspace is in fact time-independent.

Again, one can consider a characterization of detectability in terms of the stability of the unobservable part of the system $A_\omega(\cdot)$, see (4.42). Despite the time-varying nature of the dimensions of $A_\omega(\cdot)$, this characterization is equivalent to that given in Proposition 4.12 since the only difference between $A_\omega(\cdot)$ and $A_\rho(\cdot)$ is given by the set of characteristic multipliers (the number of which is possibly time-varying).

4.5 The Periodic Lyapunov Equation

In the study of the stability of periodic systems (Sect. 3.3), we encountered the Lyapunov paradigm stated there in terms of (difference) Lyapunov inequalities. We now proceed to the study of the celebrated (difference) Lyapunov equation, in the light of the analysis of the structural properties developed in the previous sections of this chapter. In this way, it will be possible to precisely characterize a number of important properties of the solutions, such as existence, uniqueness, and inertia.

The periodic Lyapunov equations considered herein are:

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' \quad (4.44)$$

$$Q(t) = A(t)'Q(t+1)A(t) + C(t)'C(t), \quad (4.45)$$

where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are T -periodic matrices. Equations (4.44) and (4.45), which are reminiscent of Inequalities (3.20) and (3.19), are known as *filter Lyapunov equation* and *control Lyapunov equation*, respectively. For simplicity, we will focus on the filter Lyapunov equation in the sequel and we will refer to it as the *Periodic Lyapunov Equation* (PLE). All results can be appropriately “dualized” to the control Lyapunov equation, and are therefore omitted.

By solving recursively Eq. (4.44) starting from an initial condition $P(\tau) = P_\tau$, one obtains the solution for $t \geq \tau$

$$P(t) = \Phi_A(t, \tau)P_\tau\Phi_A(t, \tau)' + W_\tau(\tau, t), \quad (4.46)$$

where $W_\tau(\tau, t)$ is the reachability Grammian defined in (4.21). Since we focus on periodic solutions, we set $t = \tau + T$ and impose $P(\tau + T) = P_\tau$. In this way, a discrete-time *Algebraic Lyapunov Equation* (ALE) in the unknown P_τ is obtained. Precisely, recalling the definition of the reachability factor G_τ as given in (4.6), the ALE is:

$$P_\tau = F_\tau P_\tau F_\tau' + G_\tau G_\tau', \quad (4.47)$$

where we have set, as in (4.6), $F_\tau = \Psi_A(\tau)$. Obviously there is a one-to-one correspondence between the T -periodic solutions of the PLE and the solutions of this ALE. Precisely, the following result is an immediate consequence of Eq. (4.46).

Proposition 4.14

- (i) The PLE (4.44) admits a T -periodic solution $P(\cdot)$ if and only if the ALE (4.47) admits a solution P_τ .
- (ii) The PLE (4.44) admits an unique T -periodic solution $P(\cdot)$ if and only if the ALE (4.47) admits an unique solution P_τ .
- (iii) The PLE (4.44) admits a T -periodic solution $P(\cdot)$ that is positive semi-definite for each t if and only if the ALE (4.47) admits a positive semi-definite solution P_τ .
- (iv) The PLE (4.44) admits a T -periodic solution $P(\cdot)$ that is positive definite at τ if and only if the ALE (4.47) admits a positive definite solution P_τ . ■

To precisely state a number of significant properties, we are well advised to recall the canonical controllability decomposition (4.26):

$$\begin{aligned} \tilde{A}(t) &= S(t+1)^{-1}A(t)S(t) = \begin{bmatrix} A_c(t) & \hat{A}(t) \\ 0 & A_c(t) \end{bmatrix} \\ \tilde{B}(t) &= S(t+1)^{-1}B(t) = \begin{bmatrix} B_c(t) \\ 0 \end{bmatrix}. \end{aligned}$$

According to such a partition, matrices F_t and G_t transform into

$$\tilde{F}_t = S(t)^{-1} F_t S(t) = \begin{bmatrix} F_{ct} & \hat{F}_{ct} \\ 0 & \bar{F}_{ct} \end{bmatrix} \quad (4.48a)$$

$$\tilde{G}_t = S(t)^{-1} G_t = \begin{bmatrix} G_{ct} \\ 0 \end{bmatrix}. \quad (4.48b)$$

Obviously, F_{ct} and \bar{F}_{ct} are the monodromy matrices of $A_c(\cdot)$ and $\bar{A}_c(\cdot)$, respectively, at time t . Now, the given canonical decomposition reflects into a corresponding decomposition for the ALE (4.47). Indeed, it is easy to see that P_τ is a solution of the ALE (4.47) if and only if

$$\tilde{P}_\tau = S(\tau)^{-1} P_\tau (S(\tau))^{-1}$$

is a solution of

$$\tilde{P}_\tau = \tilde{F}_\tau \tilde{P}_\tau \tilde{F}_\tau' + \tilde{G}_\tau \tilde{G}_\tau'. \quad (4.49)$$

Sign definite solutions: existence and uniqueness.

We are now in a position to work out the main existence and uniqueness theorem for the PLE. According to Proposition 4.14 and the previous reasoning, the proof completely relies on the properties of the ALE (4.49).

Proposition 4.15 *Consider the PLE (4.44) and the controllability partition of the monodromy matrix (4.48). Then*

(a) *The PLE admits a unique T -periodic solution if and only if $A(\cdot)$ does not have reciprocal characteristic multipliers.*

(b) *The PLE admits a T -periodic solution if and only if, for each $\mu \in C$, $y \in C^n$ and $z \in C^n$ such that $F_\tau' y = \mu y$ and $F_\tau' z = \mu^{-1} z$, it results $y^* G_\tau G_\tau' z = 0$.*

(c) *The PLE admits a T -periodic solution which is positive definite for each t if and only if the following conditions hold:*

(c.1) $A_c(\cdot)$ is stable

(c.2) all characteristic multipliers of $\bar{A}_c(\cdot)$ are located on the unit circle

(c.3) \bar{F}_{ct} is diagonalizable

(c.4) the reachability subspace of the pair $(A(\cdot), B(\cdot))$ and the controllability subspace of the pair $(A(\cdot), B(\cdot))$ do coincide for each t .

(d) *The PLE admits a T -periodic solution which is positive semi-definite for each t if and only if $A_c(\cdot)$ is stable.*

(e) *The PLE admits a unique T -periodic positive definite solution if and only if the following conditions hold:*

(e.1) $A(\cdot)$ is stable

(e.2) $(A(\cdot), B(\cdot))$ is reachable for any time point.

(f) *The PLE admits a unique T -periodic positive semi-definite solution if and only if the following conditions hold:*

(f.1) $A_c(\cdot)$ is stable

(f.2) The characteristic multipliers of $\bar{A}_c(\cdot)$ are not located in the unit circle.

Proof Thanks to Proposition 4.14, the proof of all statements will be carried out only for the ALE (4.47).

Points (a), (b). Thanks to the Kronecker calculus, the ALE can be rewritten as

$$Hp = g,$$

where $p = \text{vec}(P_\tau)$, $g = \text{vec}(G_\tau G'_\tau)$, and $H = I_{n^2} - F_\tau \otimes F_\tau$. The eigenvalues of H are given by $1 - \lambda_i(F_\tau)\lambda_j(F_\tau)$ for all pairs i, j . Hence, the equation $Hp = g$ admits a unique solution if and only if $\lambda_i(F_\tau)\lambda_j(F_\tau) \neq 1$, for each (i, j) , i.e., H is nonsingular. This completes the proof of part (a). As for point (b), notice first that equation $Hp = g$ admits a solution if and only if $g \in \text{Im}[H] = \ker[H']^\perp$. This is equivalent to say that for all $v \in \mathbb{R}^{2n}$, $v'H = 0$ implies $v'g = 0$. From linear algebra concepts, it is known that any vector v satisfying $v'H = 0$ can be expressed as $v = z \otimes y$, where $y \in C^n$ and $z \in C^n$ are such that $F'_\tau y = \lambda y$ and $F'_\tau z = \lambda^{-1}z$, for some $\lambda \in C$. By using standard concepts of Kronecker calculus, it follows that

$$\begin{aligned} v'g &= (z \otimes y)^* \text{Vec}(G_\tau G'_\tau) \\ &= (z^* \otimes y^*) (G_\tau \otimes G_\tau) \text{Vec}(I_n) \\ &= (z^* G) \otimes (y^* G) \text{Vec}(I_n) \\ &= \text{Vec}(y^* G_\tau G'_\tau z) = y^* G_\tau G'_\tau z. \end{aligned}$$

Points (c), (d). To prove these points, we make reference to partition of the ALE (4.49), induced by Decomposition (4.48). Letting

$$\tilde{P}_\tau = \begin{bmatrix} P_1 & P_2 \\ P'_2 & P_3 \end{bmatrix},$$

it follows that

$$P_1 = F_{c\tau} P_1 F'_{c\tau} + \hat{F}_{c\tau} P'_2 F_{c\tau} + F_{c\tau} P_2 F'_{c\tau} + \hat{F}_{c\tau} P_3 \hat{F}'_{c\tau} + G_{c\tau} G'_{c\tau} \quad (4.50a)$$

$$P_2 = F_{c\tau} P_2 \bar{F}_{c\tau} + \hat{F}_{c\tau} P_3 \bar{F}'_{c\tau} \quad (4.50b)$$

$$P_3 = \bar{F}_{c\tau} P_3 \bar{F}'_{c\tau}. \quad (4.50c)$$

Necessity of point (c). Let P_τ be a solution of the original ALE (4.47). Consider an eigenvalue μ of F_τ and let z be an eigenvector of F'_τ , i.e., $F'_\tau z = \mu z$. By pre-multiplying both sides of Eq. (4.47) by z^* and post-multiplying them by z , we obtain

$$(|\mu|^2 - 1)z^* P_\tau z + z^* G_\tau G'_\tau z = 0.$$

The condition $z^* P_\tau z > 0$ implies that $|\mu| \leq 1$. Hence, no eigenvalue of F_τ lies outside the unit disk. As a consequence of this conclusion, without any loss of generality¹²

¹² The meaning of this statement is that F_τ can be driven to the given block-partitioned form by means of a change of coordinates.

one may assume that the dynamic matrix F_τ is partitioned as follows

$$F_\tau = \begin{bmatrix} F_a & 0 \\ 0 & F_b \end{bmatrix},$$

where F_a has all eigenvalues with modulus strictly less than one and F_b has all eigenvalues with modulus equal to one. Correspondingly, the solution P_τ and matrix G_τ transform into

$$P_\tau = \begin{bmatrix} P_a & P_c \\ P'_c & P_b \end{bmatrix}, \quad G_\tau = \begin{bmatrix} G_a \\ G_b \end{bmatrix}.$$

Equation ALE (4.47) can be split into:

$$P_a = F_a P_a F'_a + G_a G'_a \quad (4.51a)$$

$$P_c = F_a P_c F'_b + G_a G'_b \quad (4.51b)$$

$$P_b = \bar{F}_b P_b \bar{F}'_b + G_b G'_b. \quad (4.51c)$$

We now show that F_b is similar to a diagonal matrix. Let y be a vector so that $F'_b y = \mu y$. Recalling that $|\mu| = 1$, from (4.51c) one obtains

$$(|\mu|^2 - 1)y^* P_b y + y^* G_b G'_b y = y^* G_b G'_b y = 0,$$

so that $G_b y = 0$. Suppose by contradiction that F_b is not diagonalizable. This means that there exists, for at least one eigenvalue μ of F_b , a generalized eigenvector z so that $F'_b y = \mu y$ and $F'_b z = \mu z + y$. Again from (4.51c) it follows

$$\begin{aligned} 0 &= z^* F_b P_b F'_b y - z^* P_b y + z^* G_b G'_b y \\ &= (\mu^* z^* + y^*) P_b \mu y - z^* P_b y \\ &= (|\mu|^2 - 1)z^* P_b y + \mu y^* P_b y \\ &= \mu y^* P_b y, \end{aligned}$$

which contradicts the positive definiteness of P_b . Hence the n_b -dimensional matrix F_b is diagonalizable and its eigenvectors span the whole subspace of dimension n_b . The diagonalizability of F_b , together with the fact that $G'_b y_i = 0$ for all eigenvectors y_i , implies that $G_b = 0$. In turn, this conclusion and the fact that F_b has non-zero eigenvalues lead to $n_b \leq \dim[X_{nc}(F, G)]$, where $X_{nc}(F, G)$ is the uncontrollable subspace associated with the pair (F, G) .

Consider the ALE (4.49) and its partition (4.50). From (4.50c), it follows

$$(|\mu|^2 - 1)z^* P_3 z = 0,$$

for any eigenvector z of F_3 . Hence, since P_3 is a positive definite solution of (4.50c), the conclusion is drawn that the eigenvalues of the uncontrollable part lie on the unit circle. Therefore, $n_b \geq \dim[X_{nc}(F, G)]$. This conclusion, together with the previous opposite inequality, obviously implies that $n_b = \dim[X_{nc}(F, G)]$. This means that the pair (F, G) is algebraically equivalent to the pair $(\tilde{F}_\tau, \tilde{G}_\tau)$ so that F_a is similar

to $\tilde{F}_{c\tau}$ and F_b is similar to $\bar{F}_{c\tau}$. This proves points (c.1)-(c.3). As for (c.4), assume by contradiction that the controllable pair (F_a, G_a) is not reachable. This may be possible only for the presence of null eigenvalues that are unreachable, i.e., $F'_a x = 0$, $G'_a x = 0$ for some $x \neq 0$. In view of (4.51a), this would imply that $x^* P_a x = 0$, which contradicts the positive definiteness of P_a . Therefore point (c.4) is proven as well.

Sufficiency of point (c). From the assumptions, matrices $\bar{F}_{c\tau}$ and $\bar{F}_{c\tau}$ do not have common eigenvalues. Therefore, there exists a matrix S satisfying

$$\hat{F}_{c\tau} + \bar{S}\bar{F}_{c\tau} = F_{c\tau}\bar{S}.$$

Hence, taking

$$S = \begin{bmatrix} I & \bar{S} \\ 0 & I \end{bmatrix},$$

it follows

$$F = SF_{c\tau}S^{-1} = \begin{bmatrix} I & \bar{S} \\ 0 & I \end{bmatrix}, \quad G = \begin{bmatrix} G_{c\tau} \\ 0 \end{bmatrix}.$$

The ALE (4.47) can be reformulated in view of this new state-space basis, yielding

$$P_1 = F_{c\tau}P_1F'_{c\tau} + G_{c\tau}G'_{c\tau} \quad (4.52a)$$

$$P_2 = F_{c\tau}\hat{F}_{c\tau}P_3\bar{F}'_{c\tau} \quad (4.52b)$$

$$P_3 = \bar{F}_{c\tau}P_3\bar{F}'_{c\tau}. \quad (4.52c)$$

In view of the assumptions, the pair $(F_{c\tau}, G_{c\tau})$ is reachable and $F_{c\tau}$ is stable. Hence Eq. (4.52a) admits a solution P_1 which is positive definite. Moreover, matrix $\bar{F}_{c\tau}$ is similar to an orthogonal matrix, so that there exists \hat{S} such that $\hat{S}^{-1}\bar{F}_{c\tau}\hat{S}$ is orthogonal. Hence, a solution P can be constructed by taking $P_2 = 0$, $P_1 > 0$ satisfying (4.52a) and $P_3 = \alpha\hat{S}\hat{S}^*$, for some $\alpha > 0$.

Necessity of point (d). Consider Eq. (4.49) and assume that \tilde{F}_τ has an unstable eigenvalue, i.e.,

$$\tilde{F}_\tau x = \lambda x, \quad |\mu| \geq 1.$$

From (4.49) it follows that

$$x^* \tilde{P}_\tau x (1 - |\mu|^2) + x^* \tilde{G}_\tau \tilde{G}'_\tau x = 0.$$

Being $\tilde{P}_\tau \geq 0$ and $|\mu| \geq 1$ the conclusion holds that $\tilde{G}'_\tau x = 0$. Moreover, recalling Decomposition (4.48) it is:

$$\mu x = \tilde{F}'_\tau x = \begin{bmatrix} F'_{c\tau} & 0 \\ \hat{F}'_{c\tau} & \bar{F}'_{c\tau} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{G}'_\tau x = \begin{bmatrix} G'_{c\tau} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Thanks to the controllability property, $|\mu| \geq 1$, $F'_{c\tau}x_1 = \mu x_1$ and $G'_{c\tau}x_1 = 0$ imply $x_1 = 0$, so that $\bar{F}_{c\tau}x_2 = \mu x_2$. This means that any unstable eigenvalue of \tilde{F}_τ is not an eigenvalue of $F_{c\tau}$, which is in turn a stable matrix.

Sufficiency of point (d). Consider again the ALE (4.49) decomposed according to (4.50). It is apparent that $P_2 = 0$ and $P_3 = 0$ solve (4.50b)-(4.50c). Hence, (4.50a) reduces to

$$P_1 = F_{c\tau} P_1 F_{c\tau}' + G_{c\tau} G_{c\tau}'$$

with $F_{c\tau}$ stable. Hence,

$$P_1 = \sum_{k=0}^{\infty} F_{c\tau}^k G_{c\tau} G_{c\tau}' (F_{c\tau}^k)'$$

solves (4.50a) with P_1 positive semi-definite. Therefore, \tilde{P}_τ with the entries $P_1 \geq 0$, $P_2 = 0$ and $P_3 = 0$ is a positive semi-definite solution of (4.49).

Necessity of point (e). If the PLE admits a (unique) T -periodic positive definite solution, then the conclusions of point (c) hold. This means that $F_{c\tau}$ is stable, the pair $(\underline{F}_{c\tau}, G_{c\tau})$ is reachable, $\underline{F}_{c\tau}$ has all its eigenvalues with modulus equal to one and $\underline{F}_{c\tau}$ is diagonalizable. In the proof of the sufficiency part of point (c), it was shown that in these conditions, all positive definite solutions P can be constructed by taking $P_2 = 0$, $P_1 > 0$ satisfying (4.52a) and $P_3 = \alpha \hat{S} \hat{S}^*$, for any $\alpha > 0$. Since by assumption, the T -periodic positive definite solution is unique, then $\underline{F}_{c\tau}$ must vanish, thus concluding the proof.

Sufficiency of point (e). If F_τ is stable, the unique solution of (4.47) is

$$P_\tau = \sum_{i=0}^{\infty} F_\tau^i G_\tau G_\tau' (F_\tau^i)'$$

The fact that P_τ is positive definite for each τ directly follows from the reachability of the pair (F_τ, G_τ) for each τ .

Necessity of point (f). If the PLE admits a (unique) T -periodic positive semi-definite solution, then the conclusions of point (d) hold, so that point (f.1) is proven. To prove point (f.2), without any loss of generality, matrices F_τ and G_τ can be written as follows:

$$F_\tau = \begin{bmatrix} F_c & 0 & 0 \\ 0 & F_d & 0 \\ 0 & 0 & F_e \end{bmatrix}, \quad G_\tau = \begin{bmatrix} G_c \\ 0 \\ 0 \end{bmatrix},$$

where F_c is stable, F_d has eigenvalues on the unit circle and F_e is anti-stable. The reason why G_τ has the given zero structure lies in the fact that the controllable part of the pair (F_τ, G_τ) is stable (point (f.1)). Obviously, the pair (F_c, G_c) is controllable. The solution P_τ can be partitioned accordingly:

$$P_\tau = \begin{bmatrix} P_c & P_f & P_g \\ P_f' & P_d & P_h \\ P_g' & P_h' & P_e \end{bmatrix},$$

so that

$$\begin{aligned} P_c &= F_c P_c F_c' + G_c G_c' \\ P_f &= F_c P_f F_d' \\ P_d &= F_d P_d F_d' \\ P_e &= F_e P_e F_e'. \end{aligned}$$

The antistability of F_e implies that the only positive semi-definite solution of the last equation is $P_e = 0$. Consequently, since P_τ must be the positive semi-definite, also P_h and P_g must be zero. Again, since F_c and F_d do not have reciprocal eigenvalues, the conclusion follows that the only solution of the second equation is $P_f = 0$. Thus, any positive semi-definite solution P_τ has the form:

$$P_\tau = \begin{bmatrix} P_c & 0 & 0 \\ 0 & P_d & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $P_c \geq 0$ and $P_d \geq 0$. Now consider the third equation above, namely $P_d = F_d P_d F_d'$. The eigenvalues of F_d lie on the unit circle. Therefore, this equation admits infinitely many positive semi-definite solutions. The uniqueness assumptions can be satisfied only if such part is empty, that is, only if matrix F_τ does not possess eigenvalues on the unit circle (point (f.2)).

Sufficiency of point (f). If point (f.1) holds, then, from point (d), it is known that the PLE admits a T -periodic positive semi-definite solution. Hence, the ALE admits a positive semi-definite solution P_τ . Point (f.2) enables one to partition matrices F_τ and G_τ , without any loss of generality, as follows:

$$F_\tau = \begin{bmatrix} F_c & 0 \\ 0 & F_e \end{bmatrix}, \quad G_\tau = \begin{bmatrix} G_c \\ 0 \end{bmatrix},$$

where F_c is stable and F_e is anti-stable. Again, G_τ has the given zero structure because the controllable part of the pair (F_τ, G_τ) is stable (point (f.1)). Obviously, the pair (F_c, G_c) is controllable. The solution P_τ can be partitioned accordingly:

$$P_\tau = \begin{bmatrix} P_c & P_g \\ P_g' & P_e \end{bmatrix},$$

so that

$$\begin{aligned} P_c &= F_c P_c F_c' + G_c G_c' \\ P_g &= F_c P_g F_e' \\ P_e &= F_e P_e F_e'. \end{aligned}$$

The unique solution of the last equation is $P_e = 0$. Then the fact that $P_\tau \geq 0$ implies $P_g = 0$. Finally, the first equation admits a unique positive semi-definite solution thanks to the stability of F_c . ■

From this proposition, a straightforward corollary is as follows.

Proposition 4.16

- (i) Suppose that the pair $(A(\cdot), B(\cdot))$ is reachable at any time point. Then $A(\cdot)$ is stable if and only if the PLE admits a unique T -periodic positive definite solution.
- (ii) Suppose that the pair $(A(\cdot), B(\cdot))$ is stabilizable. Then $A(\cdot)$ is stable if and only if the PLE admits a unique T -periodic positive semi-definite solution. ■

In general, the results stated so far have to do with the positive definiteness (or positive semi-definiteness) of the periodic solutions. More in general, if one is interested in any periodic solution (sign indefinite) of the Lyapunov equation, a very interesting characterization is supplied by the so-called *inertia theory*.

Inertia Theory

In order to state the main result, we first need to recall the basic definition of *inertia* of a matrix. Given a square matrix M , its inertia is defined as the triplet of integers:

- $\nu(M)$: number of stable eigenvalues of M
- $\pi(M)$: number of unstable eigenvalues of M
- $\delta(M)$: number of eigenvalues of M in the stability-instability border line.

In the sequel, we have to deal with both the continuous-time and discrete-time stability regions. Correspondingly, we use the subscript c and d to distinguish the two cases. For instance, $\nu_c(M)$ denotes the number of eigenvalues of M with strictly negative real parts whereas $\nu_d(M)$ denotes the number of eigenvalues of M with modulus strictly smaller than one. Of course, should M be a symmetric matrix, then $\nu_c(M)$ denotes the number of negative eigenvalues. Finally, if M is symmetric, then the inertia of M coincides with the inertia of SMS' , where S is any invertible matrix.

Proposition 4.17 *Let the PLE admit a T -periodic solution $P(\cdot)$*

- (a) *If $(A(\cdot), B(\cdot))$ is reachable at time t , then*

$$\begin{aligned}\pi_c(P(t)) &= \nu_d(F_t) \\ \nu_c(P(t)) &= \pi_d(F_t) \\ \delta_c(P(t)) &= \delta_d(F_t) = 0.\end{aligned}$$

- (b) *If $(A(\cdot), B(\cdot))$ is controllable, then for each t*

$$\begin{aligned}\pi_c(P(t)) &= \nu_d(F_t) - \delta_c(P(t)) \\ \nu_c(P(t)) &= \pi_d(F_t) \\ \delta_d(F_t) &= 0,\end{aligned}$$

and $\delta_c(P(t))$ coincides with the dimension of the unreachability subspace of $(A(\cdot), B(\cdot))$ at time t .

(c) If $(A(\cdot), B(\cdot))$ is stabilizable, then for each t

$$\begin{aligned}\pi_c(P(t)) &= v_d(F_t) - \delta_c(P(t)) \\ v_c(P(t)) &= \pi_d(F_t) \\ \delta_d(F_t) &= 0.\end{aligned}$$

(d) If $(A(\cdot), B(\cdot))$ is stabilizable and $P(\cdot)$ is the unique T -periodic solution, then for each t the same conclusions of point (b) hold.

Proof Again, the proofs of all statements rely on ALE (4.47).

Point (a). Assume by contradiction that $\delta_d(F_t) \neq 0$, i.e., $F_t'x = \lambda x$, $x \neq 0$ with $|\lambda| = 1$. Then, pre-multiply by x^* and post-multiply by x the ALE:

$$x^* P_t x = |\lambda|^2 x^* P_t x + x^* G_t G_t' x = x^* P_t x + x^* G_t G_t' x,$$

so that $G_t x = 0$, together with $F_t'x = 0$, $x \neq 0$ and $|\lambda| = 1$, contradicts the reachability of the pair (F_t, G_t) . Hence, the first conclusion $\delta_d(F_t) = 0$ follows. Thanks to this, and without any loss of generality¹³, one can consider matrix F_t in the form

$$F_t = \begin{bmatrix} F_s & 0 \\ 0 & F_a \end{bmatrix}$$

where F_s is stable and F_a is anti-stable. Accordingly, matrix G_t and (any) symmetric solution P_t of the ALE can be given the form:

$$G_t = \begin{bmatrix} G_s \\ G_a \end{bmatrix}, \quad P_t = \begin{bmatrix} P_s & P_{sa} \\ P_{sa}' & P_a \end{bmatrix}$$

Obviously, the reachability of the pair (F_t, G_t) implies the reachability of both pairs (F_s, G_s) and (F_a, G_a) . The ALE splits as follows:

$$\begin{aligned}P_s &= F_s P_s F_s' + G_s G_s' \\ P_{sa} &= F_s P_{sa} F_a' + G_s G_a' \\ P_a &= F_a P_a F_a' + G_a G_a' .\end{aligned}$$

Thanks to the stability of F_s and the reachability of (F_s, G_s) , by Proposition 4.15-point (e), the first sub-equation may only admit a unique solution P_s which is positive definite. Notice now that the third equation can be rewritten as

$$\bar{P}_a = \bar{F}_a \bar{P}_a \bar{F}_a' + \bar{G}_a \bar{G}_a'$$

¹³ If a change of coordinates $\bar{x} = Sx$ is performed on the underlying system, then F_t becomes $\bar{F}_t = S F_t S^{-1}$, G_t becomes $\bar{G}_t = S G_t$ and a solution P_t is transformed in $\bar{P}_t = S P_t S'$. It is not difficult to see that the structural properties do not change and the inertia of \bar{F}_t and F_t coincide. Finally, the inertia of P_t coincides with the inertia of \bar{P}_t . This means that without any loss of generality, it is possible to consider matrix F_t in a suitable canonical form.

with

$$\bar{F}_a = F_a^{-1}, \quad \bar{P}_a = -P_a, \quad \bar{G}_a = A_a^{-1} G_a.$$

As can be easily seen, anti-stability of \bar{F}_a and reachability of (F_a, G_a) are equivalent to stability of F_a and reachability of (F_a, \bar{G}_a) . As such, this last equation may only admit a unique positive definite solution \bar{P}_a , which in turn, means $P_a < 0$. Therefore, any solution P_t is such that $P_s > 0$ and $P_a < 0$. Finally, notice that

$$\tilde{P}_t = \begin{bmatrix} I & -P_{as}P_a^{-1} \\ 0 & I \end{bmatrix} P_t \begin{bmatrix} I & 0 \\ -P'_{as}P_a^{-1} & I \end{bmatrix} = \begin{bmatrix} P_s - P_{as}P_a^{-1}P'_{as} & 0 \\ 0 & P_a \end{bmatrix}$$

The proof follows by observing that $P_s - P_{as}P_a^{-1}P'_{as} > 0$, $P_a < 0$ and recalling that \tilde{P}_t and P_t share the same inertia.

Point (b). First, we can assume that the controllable pair (F_t, G_t) is decomposed in the reachable and unreachable parts, i.e.,

$$F_t = \begin{bmatrix} F_r & \tilde{F}_r \\ 0 & F_{nr} \end{bmatrix}, \quad G_t = \begin{bmatrix} G_r \\ 0 \end{bmatrix},$$

where (F_r, G_r) is reachable and the matrix F_{nr} associated with the non-reachable part is nilpotent (has all eigenvalues in the origin). Correspondingly, the solution of the ALE reads as:

$$P_t = \begin{bmatrix} P_r & \tilde{P}_r \\ \tilde{P}'_r & P_{nr} \end{bmatrix}.$$

Correspondingly, the ALE splits as:

$$\begin{aligned} P_r &= (F_r P_r + \tilde{F}_r \tilde{P}'_r) F'_r + (F_r \tilde{P}_r + \tilde{F}_r P_{nr}) \tilde{F}'_r + G_r G'_r \\ \tilde{P}_r &= (F_r \tilde{P}_r + \tilde{F}_r P_{nr}) F'_{nr} \\ P_{nr} &= F_{nr} P_{nr} F'_{nr}. \end{aligned}$$

Now consider the third sub-equation. Since F_{nr} is nilpotent, there exists a power $k \geq 0$ such that $F_{nr}^k = 0$. Consequently,

$$P_{nr} = F_{nr} P_{nr} F'_{nr} = F_{nr}^2 P_{nr} (F'_{nr})^2 = F_{nr}^3 P_{nr} (F'_{nr})^3 = \dots = 0.$$

Hence, $P_{nr} = 0$. In turn, the second sub-equation becomes

$$\tilde{P}_r = F_r \tilde{P}_r F'_{nr}.$$

Again,

$$\tilde{P}_r = F_r \tilde{P}_r F'_{nr} = F_r^2 \tilde{P}_r (F'_{nr})^2 = \dots = 0,$$

so that $\tilde{P}_r = 0$. Finally, the first sub-equation becomes:

$$P_r = F_r P_r F'_r + G_r G'_r$$

with (F_r, G_r) reachable. For this equation, the conclusion of point (a) is drawn. From the structure of

$$P_t = \begin{bmatrix} P_r & 0 \\ 0 & 0 \end{bmatrix}$$

the proof follows.

Point (c). The proof proceeds following the same lines as in the proof of point (a). In particular, it follows that $\delta_d(F_t) \neq 0$. Again, as in point (a) consider F_t , G_t and P_t decomposed as:

$$F_t = \begin{bmatrix} F_s & 0 \\ 0 & F_a \end{bmatrix}, \quad G_t = \begin{bmatrix} G_s \\ G_a \end{bmatrix}, \quad P_t = \begin{bmatrix} P_s & P_{sa} \\ P'_{sa} & P_a \end{bmatrix},$$

where F_s is stable and F_a is anti-stable. The ALE splits as follows:

$$\begin{aligned} P_s &= F_s P_c F'_s + G_s G'_s \\ P_{sa} &= F_s P_{as} F'_a + G_s G'_a \\ P_a &= F_a P_a F'_a + G_a G'_a. \end{aligned}$$

The stabilizability of the pair (F_t, G_t) implies the stabilizability of both pairs (F_s, G_s) and (F_a, G_a) . Since F_a is anti-stable, this means that (F_a, G_a) is indeed reachable. Hence, as in point (a) it follows that $P_a < 0$. Moreover, the stability of F_s implies, from the first sub-equation, that $P_s \geq 0$. The inertia of P_t coincides with the inertia of

$$\tilde{P}_t = \begin{bmatrix} P_s - P_{as} P_a^{-1} P'_{as} & 0 \\ 0 & P_a \end{bmatrix},$$

where $P_s - P_{as} P_a^{-1} P'_{as} \geq 0$. This concludes the proof of point (c).

Point (d). Assume that the pair (F_t, G_t) is decomposed in the reachable and unreachable parts, i.e.,

$$F_t = \begin{bmatrix} F_r & \tilde{F}_r \\ 0 & F_{nr} \end{bmatrix}, \quad G_t = \begin{bmatrix} G_r \\ 0 \end{bmatrix},$$

where (F_r, G_r) is reachable and the matrix F_{nr} associated with the non-reachable part is stable. The solution of the ALE is:

$$P_t = \begin{bmatrix} P_r & \tilde{P}_r \\ \tilde{P}'_r & P_{nr} \end{bmatrix},$$

and the ALE splits as:

$$\begin{aligned} P_r &= (F_r P_r + \tilde{F}_r \tilde{P}'_r) F'_r + (F_r \tilde{P}_r + \tilde{F}_r P_{nr}) \tilde{F}'_r + G_r G'_r \\ \tilde{P}_r &= (F_r \tilde{P}_r + \tilde{F}_r P_{nr}) F'_{nr} \\ P_{nr} &= F_{nr} P_{nr} F'_{nr}. \end{aligned}$$

Now consider the third sub-equation. Since F_{nr} is stable, the only solution is $P_{nr} = 0$.

In turn, the second sub-equation becomes

$$\tilde{P}_r = F_r \tilde{P}_r F_r'.$$

Apparently, a solution is $\tilde{P}_r = 0$. Finally, the first sub-equation becomes:

$$P_r = F_r P_r F_r' + G_r G_r'$$

with (F_r, G_r) reachable. Moreover, thanks to the uniqueness of the solution and in view of Proposition 4.15 point (a), matrix F_r , and in turn, matrix F_r , do not possess reciprocal eigenvalues. Therefore, this sub-equation admits only one solution P_r and the conclusions of point (a) can be drawn, thanks to the reachability of the pair (F_r, G_r) . This means that the structure of the unique solution of the ALE is such that: i.e.,

$$\begin{aligned} P_{nr} &= 0 \\ \tilde{P}_r &= 0 \\ \pi_c(P_r) &= \nu_d(F_r) \\ \nu_c(P_r) &= \pi_d(F_r) \\ \delta_c(P_r) &= \delta_d(F_r) = 0. \end{aligned}$$

In view of these conclusions, and recalling that the dimension of P_{nr} coincides with the dimension of the unreachability subspace of (F_r, G_r) , the proof follows.

4.6 Bibliographical Notes

Since the early days of system theory [204], structural properties have been the subject of a wide literature, see [187, 271, 272] for time-varying systems. For periodic continuous-time systems, a pioneer study is reported in [88] where it is shown that the reachability transition in a reachable periodic system can be performed in n periods at the most. The stronger statement that the reachability transition can be performed in one period was given in Theorem 2.26 of [204] and a proof was provided in [183] (Theorem 2.15). Unfortunately, this proof was erroneous, and a counterexample to the statement itself was presented in [68]. Further significant references in continuous-time are [207] and [31, 36, 39, 53, 121, 300]. The Master Thesis [28] was also devoted to such issue. The discrete-time case was elaborated in later years, the most relevant papers being [29] and [34, 35, 49, 168]. The canonical decomposition of discrete-time periodic systems is treated in [32, 33, 37, 167]. For the numerical aspects, see [291]. Finally, for the inertia theorems related to the T -periodic Lyapunov equation, see e.g., [38, 81] and [80].

Chapter 5

Periodic Transfer Function and BIBO Stability

In this chapter, we elaborate on the concept of periodic transfer function introduced in Chap. 2. For simplicity, we will focus on single-input single-output (SISO) systems. We will start with the study of the family of polynomials with periodic coefficients. This opens the way to conceive a periodic system as the ratio of two periodic polynomials. In this way, one can work out the notion of the periodic transfer function, thanks to which many typical system-theoretic issues can be addressed, in particular the notion of zeros, poles, and input–output stability.

5.1 Elementary Algebra of Periodic Polynomials

Denote by $p_i(t)$, $i = 0, 1, \dots, n$, a set of T -periodic real coefficients and let

$$p(\sigma, t) = p_0(t)\sigma^n + p_1(t)\sigma^{n-1} + \dots p_n(t)$$

be the associated T -periodic polynomial in the symbol σ . Such a symbol denotes the 1-step-ahead operator. The degree of the polynomial is defined as the T -periodic function $\rho(t)$ given by the power associated with the maximum-power coefficient which is non-zero at time t . If $p_0(t) \neq 0$, $\forall t$, then $\rho(t) = n$, $\forall t$, and the polynomial is called *regular*. A regular polynomial with $p_0(t) = 1$, $\forall t$, is said to be *monic*.

The sum of two T -periodic polynomials, say $p(\sigma, t)$ and

$$q(\sigma, t) = q_0(t)\sigma^n + q_1(t)\sigma^{n-1} + \dots q_n(t)$$

is still a T -periodic polynomial, given by

$$p(\sigma, t) + q(\sigma, t) = (p_0(t) + q_0(t))\sigma^n + (p_1(t) + q_1(t))\sigma^{n-1} + \dots (p_n(t) + q_n(t)).$$

As for the product, a preliminary observation is in order. Since σ is the 1-step-ahead operator, it can be postponed to a coefficient $p_i(t)$ provided that a corresponding time-shift is performed on the coefficient.

In other words, for any non-negative k ,

$$\sigma^k p(\sigma, t) = p(\sigma, t + k) \sigma^k$$

holds. This is named *pseudo-commutative property*. Obviously, the commutative property is recovered if k is a multiple of the period.

We can now introduce the polynomial product, as

$$p(\sigma, t)q(\sigma, t) = r_0(t)\sigma^{2n} + r_1(t)\sigma^{2n-1} + \cdots + r_{2n-1}\sigma + r_{2n}(t),$$

where

$$\begin{aligned} r_0(t) &= p_0(t)q_0(t+n) \\ r_1(t) &= p_0(t)q_1(t+n) + p_1(t)q_0(t+n-1) \\ &\vdots \\ r_{2n-1} &= p_{n-1}q_n(t+1) + p_n(t)q_{n-1}(t) \\ r_{2n}(t) &= p_n(t)q_n(t). \end{aligned}$$

In this way, the set of T -periodic polynomials forms a non-commutative ring. However, the subset constituted by the polynomials in the symbol σ^T is a commutative ring.

Following standard algebraic terminology, a T -periodic polynomial $p(\sigma, t)$ is said to be *unimodular* if there exists a T -periodic polynomial $q(\sigma, t)$ so that for each t ,

$$p(\sigma, t)q(\sigma, t) = 1,$$

where 1 denotes the identity operator. Note that, if $q(\sigma, t)$ is a polynomial such that this condition holds, then

$$q(\sigma, t)p(\sigma, t) = 1$$

holds as well. Contrary to the case of constant polynomials, there might exist unimodular T -periodic polynomials with degree different from zero at each time point. For instance, the 2-periodic polynomial

$$p(\sigma, t) = p_0(t)\sigma + p_1(t)$$

with $p_0(0) = 0$, $p_0(1) = 1$, $p_1(0) = 1$, $p_1(1) = -1$, is unimodular in that

$$(p_0(t)\sigma + p_1(t))(p_0(t)\sigma - p_1(t+1)) = 1.$$

Notice that the unimodular T -periodic polynomials with non-identically zero degree are all non-regular polynomials.

Remark 5.1 *The above definitions and developments are all based on the representation of polynomials in terms of the 1-step-ahead operator σ . These considerations can be equivalently made by referring to the delay operator σ^{-1} . For example, one*

can define the polynomial

$$r(\sigma^{-1}, t) = r_0(t)\sigma^{-n} + r_1(t)\sigma^{-n+1} + \cdots r_n(t).$$

Then the pseudo-commutative rule becomes

$$\sigma^{-k}r(\sigma^{-1}, t) = r(\sigma^{-1}, t - k)\sigma^{-k}.$$

The passage from a polynomial in σ^{-1} to the corresponding polynomial in σ can be easily performed as follows:

$$r(\sigma^{-1}, t)\sigma^n = p(\sigma, t),$$

where

$$p(\sigma, t) = r_0(t) + r_1(t)\sigma + \cdots r_n(t)\sigma^n.$$

5.2 Lifted Periodic Polynomials and Zeroes

Given a T -periodic polynomial $p(\sigma, t)$, the associated lifted polynomial $P(\sigma^T, t)$ is a $T \times T$ polynomial matrix with T -periodic coefficients uniquely defined by the condition

$$P(\sigma^T, t) \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{T-1} \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{T-1} \end{bmatrix} p(\sigma, t). \quad (5.1)$$

For instance, as it is easy to check,

$$p(\sigma, t) = \sigma \implies P(\sigma^T, t) = \begin{bmatrix} 0 & I_{T-1} \\ \sigma^T & 0 \end{bmatrix}.$$

A number of basic properties of the lifted polynomial are in order:

(i) *Time-recursion of $P(\sigma^T, t)$*

For any given T -periodic polynomial $p(\sigma, t)$, the associated lifted polynomial matrix $P(\sigma, t)$ satisfies the following recursive formula:

$$P(\sigma^T, t+1) = \Delta'(\sigma^{-T})P(\sigma^T, t)\Delta(\sigma^T), \quad (5.2)$$

where

$$\Delta(z) = \begin{bmatrix} 0 & z^{-1} \\ I_{T-1} & 0 \end{bmatrix}. \quad (5.3)$$

To prove this recursive Expression (5.2), denote by $v(\sigma)$ the vector

$$v(\sigma) = \begin{bmatrix} 1 \\ \sigma \\ \vdots \\ \sigma^{T-1} \end{bmatrix}. \quad (5.4)$$

Notice that $\Delta(\sigma^{-T})'$ is the lifted polynomial matrix associated with the polynomial σ , i.e.,

$$\Delta(\sigma^{-T})'v(\sigma) = v(\sigma)\sigma.$$

Hence,

$$v(\sigma)p(\sigma, t+1)\sigma = P(\sigma^T, t+1)v(\sigma)\sigma = P(\sigma^T, t+1)\Delta(\sigma^{-T})'v(\sigma)$$

On the other hand,

$$\begin{aligned} v(\sigma)p(\sigma, t+1)\sigma &= v(\sigma)\sigma p(\sigma, t) \\ &= \Delta(\sigma^{-T})'v(\sigma)p(\sigma, t) \\ &= \Delta(\sigma^{-T})'P(\sigma^T, t)v(\sigma). \end{aligned}$$

In conclusion

$$(P(\sigma^T, t+1)\Delta(\sigma^{-T})' - \Delta'(\sigma^{-T})P(\sigma^T, t))v(\sigma) = 0,$$

so that, since the matrix within the brackets is a polynomial function of only σ^T , formula (5.2) follows.

(ii) *Upper-triangularity of $P(0, t)$*

Another significant property of a lifted polynomial matrix $P(\sigma^T, t)$ is that $P(0, t)$ is upper triangular for each t in the period. This fact can be assessed by studying the structure of each column of $P(0, t)$. Indeed, recall again the definition of a lifted polynomial matrix (5.1). By letting in this formula $\sigma = 0$, it is apparent that only the first element of the first column of $P(0, t)$ may be non-zero. Obviously, this fact is true for all $t \in [0, T-1]$, i.e., the first column of $P(0, t)$ is given by

$$[P(0, t)]^{(1)} = \begin{bmatrix} \star \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (5.5)$$

where \star is a generically non-zero element.

As for the second column, consider again the recursion (5.2), from which it follows

$$[P(\sigma^T, t+1)]^{(1)} = \begin{bmatrix} [P(\sigma^T, t)]_{2,2} \\ [P(\sigma^T, t)]_{3,2} \\ \vdots \\ [P(\sigma^T, t)]_{T,2} \\ \sigma^T [P(\sigma^T, t)]_{1,2} \end{bmatrix}, \quad (5.6)$$

where $[P(\sigma^T, t)]_{i,j}$ denotes the (i, j) entry of matrix $P(\sigma^T, t)$. Since the first column of $P(0, t+1)$ must match the structure (5.5), then (5.6) implies that the second column $[P(0, t)]^{(2)}$ of $P(0, t)$ must have the following structure:

$$[P(0, t)]^{(2)} = \begin{bmatrix} \star \\ \star \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

By iterating this argument, the conclusion is drawn that $P(0, t)$ must be upper triangular for each t , i.e.,

$$P(0, t) = \begin{bmatrix} \star & \star & \star & \cdots & \star \\ 0 & \star & \star & \cdots & \star \\ 0 & 0 & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \star \end{bmatrix}. \quad (5.7)$$

(iii) *One-to-one correspondence*

The set of T -periodic polynomial matrices in σ^T satisfying the recursion (5.2) and the upper triangularity condition (5.7) is in one-to-one correspondence with the set of T -periodic polynomials. Precisely, given a T periodic polynomial $p(\sigma, t)$, the lifted T -periodic polynomial matrix is uniquely defined through (5.1). Conversely, given a lifted polynomial matrix, i.e., a T -periodic polynomial matrix in σ^T satisfying both conditions (5.2) and (5.7), the T -periodic polynomial can be computed as

$$p(\sigma, t) = [P(\sigma^T, t)]_{(1)} v(\sigma),$$

where $[P(\sigma, t)]_{(1)}$ denotes the first row of $P(\sigma, t)$ and $v(\sigma)$ is defined in (5.4). This expression is an obvious consequence of (5.1).

(iv) *Algebraic closure*

The set of lifted polynomials is closed with respect to the sum and the product. This means that the lifted polynomial associated with the sum or the product of

two T -periodic polynomials $p_1(\sigma, t)$ and $p_2(\sigma, t)$ is given by the sum or, respectively, the product of the two lifted polynomials $P_1(\sigma^T, t)$ and $P_2(\sigma^T, t)$. This claim is easily verified and the details are therefore omitted. We limit ourselves to proving the product rule which straightforwardly follows from

$$v(\sigma)p_1(\sigma, t)p_2(\sigma, t) = P_1(\sigma^T, t)v(\sigma)p_2(\sigma, t) = P_1(\sigma^T, t)P_2(\sigma^T, t)v(\sigma).$$

(v) *Time-invariance of the lifted determinant*

The determinant of a lifted polynomial matrix is a polynomial in the powers of σ^T with constant coefficients, i.e.,

$$\det[P(\sigma^T, t)] = \bar{p}(\sigma^T), \quad \text{independent of } t.$$

This is a consequence of the recursive formula (5.2).

(vi) *Singularities of a periodic polynomial*

The previous property allows one to properly define the *singularities (zeros)* of a periodic polynomial $p(\sigma, t)$ as the roots of $\det[P(\sigma^T, t)]$. Therefore the singularities of a T -periodic polynomial are constant. Obviously, contrary to the case of constant coefficient polynomials, one can construct infinitely many T -periodic monic polynomials with the same set of singularities.

The definition above can be equivalently reformulated in terms of “blocking property” of a periodic signal as follows: λ^T is a singularity of a T -periodic polynomial $p(\sigma, \cdot)$ if and only if there exists a non-identically zero periodic function $x(\cdot)$ so that

$$p(\lambda\sigma, t) \cdot x(t) = 0. \quad (5.8)$$

Notice that the above equation reduces to the characteristic equation of the time-invariant case if the coefficients and the function $x(\cdot)$ are constant ($T = 1$). Thanks to this analogy, the Expression (5.8) will be referred to as *characteristic periodic difference equation* associated with $p(\sigma, t)$.

The proof of the above statement can be worked out as follows. Assume that (5.8) holds and pre-multiply it by $v(\sigma)$. It follows

$$0 = v(\sigma)p(\lambda\sigma, t)x(t) = P(\lambda^T\sigma^T, t) \begin{bmatrix} x(t) \\ x(t+1) \\ x(t+2) \\ \vdots \\ x(t+T-1) \end{bmatrix}.$$

Since $x(\cdot)$ is T -periodic this implies that

$$P(\lambda^T, t) \begin{bmatrix} x(t) \\ x(t+1) \\ x(t+2) \\ \vdots \\ x(t+T-1) \end{bmatrix} = 0,$$

so that λ^T is a root of the determinant of $P(\sigma^T, t)$, i.e., is a singularity of $p(\sigma, t)$. Vice versa, if λ^T is a singularity, then

$$P(\lambda^T, t) \begin{bmatrix} y_0(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_{T-1}(t) \end{bmatrix} = 0,$$

for some T -periodic functions $y_i(t)$, $i = 0, 1, \dots, T-1$. Recalling (5.2), the definition of $\Delta(\sigma^T)$ in (5.3) and the definition of $v(\sigma)$ in (5.4), it follows that it is possible to define a T -periodic function $x(\cdot)$ such that $x(t+i) = y_i(t)$, $i = 1, 2, \dots, T-1$, and

$$0 = P(\lambda^T, t) \begin{bmatrix} x(t) \\ x(t+1) \\ x(t+2) \\ \vdots \\ x(t+T-1) \end{bmatrix} = P(\lambda^T \sigma^T, t) v(\sigma) \cdot x(t) = v(\sigma, t) p(\lambda \sigma, t) \cdot x(t),$$

so that the conclusion $p(\lambda \sigma, t) \cdot x(t)$ follows.

(vii) *Unimodular and Z-polynomials*

It is important to notice that there are periodic polynomials of any degree without any zero. These polynomials exactly define the set of unimodular polynomials. Indeed it is possible to say that a periodic polynomial is unimodular if and only if it does not possess finite singularities. This follows from the fact that the lifted polynomial $P(\lambda^T, t)$ is unimodular, i.e., its determinant is a non-zero constant.

For the study of the structural properties of periodic systems and the study of periodic filters with finite impulse response, it is useful to introduce also the class of periodic *Z-polynomials* as the class of periodic polynomials whose finite singularities, if any, are at the origin. Of course, this class includes as a special case the class of unimodular polynomials (which do not have singularities at all). The class of periodic *Z-polynomials* $p(\sigma, t)$ can be formally defined as the class of periodic polynomials such that the associated lifted expression $P(\sigma, t)$ satisfies

$$\det[P(\sigma^T, t)] = \alpha \sigma^{kT}, \quad \text{for some } \alpha \neq 0, \quad \text{for some integer } k \geq 0.$$

Remark 5.2 The lifted theory can be worked out by referring to polynomials in the delay operator σ^{-1} as well. For example, the lifted polynomial matrix $R(\sigma^{-T}, t)$ associated with a periodic polynomial $r(\sigma^{-1}, t)$ is determined according to the following equation

$$R(\sigma^{-T}, t) \begin{bmatrix} 1 \\ \sigma^{-1} \\ \vdots \\ \sigma^{-T+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \sigma^{-1} \\ \vdots \\ \sigma^{-T+1} \end{bmatrix} r(\sigma^{-1}, t),$$

and the recursive Eq. (5.2) takes the form:

$$R(\sigma^{-T}, t-1) = \nabla(\sigma^{-T})R(\sigma^{-T}, t)\nabla(\sigma^{-T})', \quad \nabla(\sigma^{-T}) = \begin{bmatrix} 0 & I_{T-1} \\ \sigma^{-T} & 0 \end{bmatrix}.$$

5.3 Weak and Strong Coprimeness

We say that a T -periodic polynomial $q(\sigma, t)$ is a *left divisor* of a T -periodic polynomial $p(\sigma, t)$ if there exists a T -periodic polynomial $r(\sigma, t)$ such that

$$p(\sigma, t) = q(\sigma, t)r(\sigma, t)$$

Analogously, $q(\sigma, t)$ is said to be a *right divisor* of $p(\sigma, t)$ if there exists a T -periodic polynomial $s(\sigma, t)$ such that

$$p(\sigma, t) = s(\sigma, t)q(\sigma, t).$$

Along this route we define the notion of it common left (right) divisor. To this purpose, consider two T -periodic polynomials $p(\sigma, t)$ and $q(\sigma, t)$. If there exists a T -periodic polynomial $r(\sigma, t)$ such that

$$p(\sigma, t) = r(\sigma, t)s_1(\sigma, t), \quad q(\sigma, t) = r(\sigma, t)s_2(\sigma, t)$$

for some T -periodic polynomials $s_1(\sigma, t)$ and $s_2(\sigma, t)$, then $r(\sigma, t)$ is a *common left divisor* of $p(\sigma, t)$ and $q(\sigma, t)$.

Analogously if

$$p(\sigma, t) = s_1(\sigma, t)r(\sigma, t), \quad q(\sigma, t) = s_2(\sigma, t)r(\sigma, t), \quad (5.9)$$

then $r(\sigma, t)$ is a *common right divisor* of $p(\sigma, t)$ and $q(\sigma, t)$.

By means of the concept of left (right) divisor, it is possible to define the notions of weak left (right) coprimeness. Precisely, two T -periodic polynomials are said to be *weakly left coprime* if any left divisor is unimodular. Analogously, they are said to be *weakly right coprime* if any right divisor is unimodular.

A stronger concept of coprimeness can be directly formulated in terms of the celebrated *Bezout identity*. Precisely, we say that two T -periodic polynomials $p(\sigma, t)$ and $q(\sigma, t)$ are *strongly left coprime* if there exists two T -periodic polynomials $x(\sigma, t)$ and $y(\sigma, t)$ such that the identity

$$p(\sigma, t)x(\sigma, t) + q(\sigma, t)y(\sigma, t) = 1$$

holds for each t . Analogously, they are said to be *strongly right coprime* if the identity

$$x(\sigma, t)p(\sigma, t) + y(\sigma, t)q(\sigma, t) = 1 \quad (5.10)$$

holds for some T -periodic polynomials $x(\sigma, t)$ and $y(\sigma, t)$.

It is important to stress that, differently from the time-invariant case, the notions of weak and strong coprimeness are not equivalent. Strong coprimeness implies weak coprimeness, but the converse is not true in general. To see this, assume that (5.10) holds for some T -periodic polynomials $x(\sigma, t)$ and $y(\sigma, t)$. Now, let $r(\sigma, t)$ be any common right divisor according to (5.9). Therefore it is possible to write

$$1 = (x(\sigma, t)s_1(\sigma, t) + y(\sigma, t)s_2(\sigma, t))r(\sigma, t),$$

which shows that the generic right divisor $r(\sigma, t)$ is unimodular, so that strong right coprimeness implies weak right coprimeness. Analogously, it can be seen that strong left coprimeness implies weak left coprimeness.

In order to see that weak coprimeness does not imply strong coprimeness, it suffices to consider the simple 2-periodic counterexample:

$$p(\sigma, t) = \sigma - a(t), \quad q(\sigma, t) = \sigma - b(t), \quad a(0) = b(0) = 0, \quad a(1) = 1, \quad b(1) = 2.$$

It is apparent that the polynomials have only unimodular (zero degree) divisors, and as such they are weakly right and left coprime. However, the Bezout identity is not satisfied by any pair $x(\sigma, t)$, $y(\sigma, t)$, $a(t)$ and $b(t)$ being both zero at the same time instant $t = 0$.

Another difference with respect to the case of time-invariant polynomials is that it may happen that two polynomials with the same singularities are strongly coprime. An elementary example illustrating this fact can be built by considering the two 2-periodic polynomials

$$p(\sigma, t) = \sigma + a(t), \quad q(\sigma, t) = \sigma + a(t+1), \quad a(0) = 1, \quad a(1) = 5.$$

It is apparent that both polynomials share the same singularity, namely $\lambda^T = 5$. On the other hand, the Bezout equation

$$x(\sigma, t)p(\sigma, t) + y(\sigma, t)q(\sigma, t) = 1$$

with

$$x(\sigma, t) = k_1(t) - \sigma, \quad y(\sigma, t) = k_2(t) + \sigma,$$

where $k_1(0) = 4.75$, $k_1(1) = 1.25$, $k_2(0) = -0.75$, $k_2(1) = -5.25$, holds true.

An effective way to characterize strong coprimeness is to resort to the associated characteristic periodic difference equations (5.8). Precisely, two polynomials $p(\sigma, t)$ and $q(\sigma, t)$ are not strongly right coprime if and only if their characteristic periodic difference equations have a common solution. i.e., there exists a complex number λ and a non-identically zero T -periodic function $x(\cdot)$ such that

$$p(\lambda\sigma, t) \cdot x(t) = 0 \quad (5.11a)$$

$$q(\lambda\sigma, t) \cdot x(t) = 0. \quad (5.11b)$$

Interestingly, if (5.11) hold for some $\lambda \neq 0$, then the two polynomials have one common right divisor, i.e., the two polynomials are not weakly right coprime.

Example 5.1 Take again the 2 – periodic polynomials

$$p(\sigma, t) = \sigma - a(t), \quad q(\sigma, t) = \sigma - b(t), \quad a(0) = b(0) = 0, \quad a(1) = 1, \quad b(1) = 2.$$

These polynomials are not strongly right coprime, since (5.11) hold for $\lambda = 0$ and $x(1) = 0$. However, they are weakly right coprime, as already discussed.

Now, take any two periodic polynomials of degree one, i.e.,

$$p(\sigma, t) = \sigma - a(t), \quad q(\sigma, t) = \sigma - b(t),$$

and assume that (5.11) a hold for some $\lambda \neq 0$. Then

$$\lambda x(t+1) = a(t)x(t), \quad \lambda x(t+1) = b(t)x(t).$$

Therefore $a(t) = b(t)$ for each t , so that the two polynomials do coincide.

Remark 5.3 The definitions of divisors, strong coprimeness and weak coprimeness of two polynomials can be also stated for polynomials in the delay operator σ^{-1} .

5.4 Rational Periodic Operators

In this section, we introduce the notion of rational operator with periodic coefficients. To this aim, consider the formal power series in the symbol σ^{-1} :

$$f(\sigma, t) = \sum_{i=0}^{\infty} f_i(t) \sigma^{-i}, \quad (5.12)$$

where $f_i(t)$ are T -periodic coefficients.

By seeing σ^{-1} as the unit delay operator, this formal series can be seen as an operator processing a time-signal to supply another time-signal. Such an operator can be given a more compact expression as will be now discussed.

One can rewrite (5.12) as follows

$$f(\sigma, t) = \sum_{k=0}^{T-1} \left[\sum_{j=0}^{\infty} f_{k+jT}(t) \sigma^{-jT} \right] \sigma^{-k}.$$

We say that $f(\sigma, t)$ is *rational* if the functions

$$\hat{f}_k(\sigma^T, t) = \sum_{j=0}^{\infty} f_{k+jT}(t) \sigma^{-jT}$$

are rational in σ^T , for each t in the period. If this is the case, it is possible to write

$$\hat{f}_k(\sigma^T, t) = \frac{\hat{n}_k(\sigma^T, t)}{\hat{d}_k(\sigma^T, t)}.$$

Recall now that the symbol σ^T commutes with T -periodic functions. This means that it is possible to perform the product of the functions $\hat{d}_k(\sigma^T, t)$ and $\hat{n}_k(\sigma^T, t)$ in any possible order. Letting

$$\bar{d}(\sigma^T, t) = \prod_{k=0}^{T-1} \hat{d}_k(\sigma^T, t), \quad \bar{n}_i(\sigma, t) = \prod_{k \neq i, k=0}^{T-1} \hat{d}_k(\sigma^T, t) \hat{n}_i(\sigma^T, t),$$

it follows that $f(\sigma, t)$ can be written as

$$f(\sigma, t) = (\sigma^{T-1} \bar{d}(\sigma^T, t))^{-1} \left(\sum_{i=0}^{T-1} \bar{n}_i(\sigma^T, t + T - 1) \sigma^{T-1-i} \right),$$

or, equivalently, as

$$f(\sigma, t) = \left(\sum_{i=0}^{T-1} \bar{n}_i(\sigma^T, t) \sigma^{T-1-i} \right) (\bar{d}(\sigma, t) \sigma^{T-1})^{-1}.$$

In general, a causal rational operator $f(\sigma, t)$ can be written in a left or right factorized form as the ratio of two periodic polynomials:

$$\begin{aligned} f(\sigma, t) &= d_L(\sigma, t)^{-1} n_L(\sigma, t) \\ &= n_R(\sigma, t) d_R(\sigma, t)^{-1}, \end{aligned}$$

where the four polynomials satisfy

$$d_L(\sigma, t) n_R(\sigma, t) = n_L(\sigma, t) d_R(\sigma, t),$$

and, moreover, the degree of $d_L(\sigma, t)$ ($d_R(\sigma, t)$) is greater than that of $n_L(\sigma, t)$ ($n_R(\sigma, t)$), for each t .

Example 5.2 Consider the periodic polynomials of period 3:

$$a(\sigma, t) = a_0(t)\sigma^2 + \sigma - 2, \quad b(\sigma, t) = b_0(t)\sigma,$$

where $a_0(0) = a_0(1) = 0$, $a_0(2) = 1$, $b_0(0) = 1$, $b_0(1) = 0$, $b_0(2) = 2$ and define the rational left factorized operator

$$f(\sigma, t) = a(\sigma, t)^{-1}b(\sigma, t).$$

In order to find a right factorization

$$f(\sigma, t) = c(\sigma, t)d(\sigma, t)^{-1},$$

one has to solve

$$a(\sigma, t)c(\sigma, t) = b(\sigma, t)d(\sigma, t),$$

where, in principle,

$$d(\sigma, t) = d_0(t)\sigma^2 + d_1(t)\sigma + d_2(t), \quad c(\sigma, t) = c_0(t)\sigma + c_1(t).$$

Therefore,

$$\begin{aligned} a_0(t)c_0(t+2) &= b_0(t)d_0(t+1) \\ a_0(t)c_1(t+2) + c_0(t+1) &= b_0(t)d_1(t+1) \\ c_1(t+1) - 2c_0(t) &= b_0(t)d_2(t+1) \\ -2c_1(t) &= 0. \end{aligned}$$

One feasible solution of this system is

$$c(\sigma, t) = c_0(t)\sigma, \quad d(\sigma, t) = d_1(t)\sigma + d_2(t),$$

where $c_0(1) = c_0(2) = 0$, $c_0(0) = 1$, $d_2(0) = d_2(2) = 0$, $d_2(1) = -2$, $d_1(0) = 0.5$, $d_1(1) = d_1(2) = 0$. ■

It is also important to point out that if the polynomial at the denominator is not regular, the system may not be properly causal even if the degree of the denominator is not smaller than that of the numerator for each t . The following example clarifies this issue.

Example 5.3 Consider a 2-periodic system with transfer function operator

$$G(\sigma, t) = \frac{1}{p_0(t)\sigma + 1},$$

where $p_0(0) = 0$, $p_0(1) = 1$. The denominator degree fluctuates periodically between 0 and 1, whereas the numerator has constant degree equal to zero. Notice that the denominator is unimodular so that it does not possess finite singularities. This implies that the transfer operator does not correspond to a proper dynamical

system. To see this, it suffices to left-multiply both sides of

$$(p_0(t)\sigma + 1) \cdot y(t) = u(t)$$

by $p_0(t)\sigma - 1$, so that

$$y(t) = -p_0(t)u(t+1) + u(t).$$

This is not a dynamical system since the output at odd time points depends on the value of future inputs. ■

A T -periodic rational operator $f(\sigma, t)$ can be given a lifted reformulation, induced by the reformulation already defined for polynomials. Indeed, preliminarily observe that if $D(\sigma^T, t)$ is the lifted reformulation of a T -periodic polynomial $d(\sigma, t)$, then $D(\sigma^T, t)^{-1}$ can be properly considered as the lifted reformulation of the T -periodic rational operator $d(\sigma, t)^{-1}$, since, according to (5.1) and the definition of vector $v(\sigma)$ given in (5.4)

$$v(\sigma)d(\sigma, t)^{-1} = D(\sigma^T, t)^{-1}v(\sigma).$$

Assume now that $D(\sigma^T, t)$ and $N(\sigma^T, t)$ are the lifted reformulations of $d(\sigma, t)$ and $n(\sigma, t)$, and consider the rational operator $f(\sigma, t) = d(\sigma, t)^{-1}n(\sigma, t)$. Hence,

$$v(\sigma)f(\sigma, t) = D(\sigma^T, t)^{-1}N(\sigma^T, t)v(\sigma).$$

This means that

$$F(\sigma^T, t) = D(\sigma^T, t)^{-1}N(\sigma^T, t)$$

is the lifted reformulation of $f(\sigma, t)$. Analogously, if $f(\sigma) = n(\sigma, t)d(\sigma, t)^{-1}$, then the lifted reformulation of $f(\sigma, t)$ is

$$F(\sigma^T, t) = N(\sigma^T, t)D(\sigma^T, t)^{-1}.$$

In any case, $F(\sigma^T, t)$ is a matrix of rational functions in σ^T . However, both the “numerator matrix” $N(\sigma^T, t)$ and the “denominator matrix” $D(\sigma^T, t)$ have constant determinants.

From the lifted reformulation it becomes clear whether a T -periodic transfer operator $G(\sigma, t)$ corresponds to a dynamical system. Indeed, this happens when its lifted reformulation $F(\sigma^T, t)$ corresponds to a time-invariant proper dynamical system.

Example 5.4 Consider again the transfer operator defined in Example (5.3). Then,

$$D(\sigma^2, t) = \begin{bmatrix} 1 & p_0(t) \\ p_0(t+1)\sigma^2 & 1 \end{bmatrix}, \quad N(\sigma, t) = I_2,$$

and the lifted reformulation associated with $G(\sigma, t)$ is

$$F(\sigma^2, t) = \begin{bmatrix} 1 & -p_0(t) \\ -p_0(t+1)\sigma^2 & 1 \end{bmatrix}.$$

Notice that $F(\sigma^2, 0)$ is the transfer function from $u_L^0(k) = [u(2k) \ u(2k+1)]'$ to $y_L^0(k) = [y(2k) \ y(2k+1)]'$, whereas $F(\sigma^2, 1)$ is the transfer function from $u_L^1(k) = [u(2k+1) \ u(2k+2)]'$ to $y_L^1(k) = [y(2k+1) \ y(2k+2)]'$. As apparent, $F(\sigma^2, 0)$ is a polynomial matrix, so that $G(\sigma, t)$ does not correspond to any T -periodic dynamical system.

Remark 5.4 Consider the following peculiar transfer operator

$$G(\sigma, t) = \sigma^{-n} p(\sigma, t),$$

where $Pp(\sigma, t)$ is a polynomial. As already observed in Remark 5.1, $G(\sigma, t)$ is ipso facto a polynomial in the delay operator σ^{-1} . Stated in another way, this means that a polynomial in σ^{-1} is a dynamical periodic system with all poles in the origin. Again, any transfer operator $G(\sigma, t)$ can be written as the ratio of two polynomials in σ^{-1} , i.e., as the ratio of two systems with all poles in the origin. Indeed, consider a system in the left fractional representation:

$$G(\sigma, t) = d(\sigma, t)^{-1} n(\sigma, t).$$

Polynomials $d(\sigma, t)$ and $n(\sigma, t)$ can be written as:

$$\begin{aligned} d(\sigma, t) &= \sum_{i=0}^n d_i(t) \sigma^{n-i} = \sigma^n \sum_{i=0}^n d_i(t-n) \sigma^{-i} = \sigma^n \hat{d}(\sigma^{-1}, t) \\ n(\sigma, t) &= \sum_{i=0}^m n_i(t) \sigma^{-i} = \sigma^m \sum_{i=0}^m n_i(t-m) \sigma^{-i} = \sigma^m \hat{n}(\sigma^{-1}, t), \end{aligned}$$

so that

$$G(\sigma, t) = \hat{d}(\sigma^{-1}, t)^{-1} \bar{n}(\sigma^{-1}, t),$$

where $\bar{n}(\sigma^{-1}, t) = \sigma^{-n+m} \hat{n}(\sigma^{-1}, t)$.

5.5 BIBO Stability and Finite Impulse Response

The concept of BIBO (Bounded Input - Bounded Output) stability is well-known in system theory. A system is BIBO-stable if the forced output response is bounded for any bounded input signals. For a time-invariant and finite-dimensional linear system, the BIBO stability can be assessed from the impulse response. Precisely, the system is BIBO stable if and only if the impulse response tends asymptotically to zero as time goes to infinity. In discrete-time, the values taken by the impulse response are the so-called Markov parameters.

Turning to periodic systems, recall that the forced output (2.3) can be written as the combination of the forced outputs of T time-invariant systems fed by independent inputs, see (2.1), (2.4), (2.5) and (2.7). This means that the T -periodic system is BIBO stable if and only if all these time-invariant systems are BIBO stable. In turn,

this happens if and only if the Markov parameters $M_{jT+i}(t)$ tend to zero as j tends to infinity, for each $t \in [0, T-1]$ and $i \in [0, T-1]$. This is equivalent to say that

$$\lim_{k \rightarrow \infty} M_k(t) = 0, \quad \forall t \in [0, T-1].$$

Denote by $h_j^{(i)}(t)$ the j -th column of $M_{t-j}(t)$, namely the response at time t of the system forced by an impulse injected at the j -th input channel at time i . Then the system is BIBO stable if and only if such responses tend asymptotically to zero, i.e.,

$$\lim_{t \rightarrow \infty} h_j^{(i)}(t) = 0, \quad \forall i \in [0, T-1], \forall j = 1, 2, \dots, m.$$

For single-input systems ($m = 1$), one can say that the system is BIBO stable if and only if the impulse response tends asymptotically to zero for each time-point of application of the impulse in the period.

The relation between internal stability and BIBO stability is simple. Indeed, if e_j is the j -th column of the identity matrix, from (2.22) it follows that

$$h_j^{(i)}(kT+r) = M_{kT+r-i}(r) = C(r)\Psi_A(r)^k\Phi_A(r,i)B(i). \quad (5.13)$$

Asymptotic stability means that $\Psi_A(r)^k$ tends to zero as $k \rightarrow \infty$ (recall that the eigenvalues of $\Psi_A(r)$ do not depend on r). Therefore, asymptotic (internal) stability implies BIBO stability. The converse is not true in general, due to possible hidden unstable inner modes.

A particular class of BIBO stable systems is that of *Periodic Finite Impulse Response (PFIR) systems*. A generic system in this class is characterized by the fact that any transient ends up in a finite number of time-steps. In particular, the impulse response goes to zero in finite time. To build up a PFIR system from a state-space description, it suffices one to consider a dynamical matrix $A(\cdot)$ with all characteristic multipliers located in the origin. This is equivalent to say that $\Psi_A(r)^k$ is zero for a certain integer $k \geq k$, with $k \leq n$, $\forall r$. Then, from (5.13) it follows that

$$h_j^{(i)}(t) = 0, \quad t \geq \bar{k}T.$$

Alternatively, starting from an input-output context, a PFIR system is characterized by a fractional representation in which the polynomial $d(\sigma, t)$ at the denominator has all singularities at zero. This means that the determinant of its lifted reformulation $D(\sigma^T, t)$ is a power of σ^T , i.e.,

$$\det[D(\sigma, t)] = \alpha \sigma^{kT}, \quad \text{for some } \alpha \neq 0, \quad \text{for some integer } k \geq 0.$$

Example 5.5 Consider the 3-periodic system with transfer operator

$$G(\sigma, t) = (1 - \sigma)(\sigma^2 + \alpha(t)\sigma + \beta(t))^{-1},$$

where $\alpha(0) = 1$, $\alpha(1) = 0$, $\alpha(2) = -1$, $\beta(0) = 0$, $\beta(1) = 1$, $\beta(2) = 1$. The lifted reformulation of the polynomial

$$d(\sigma, t) = \sigma^2 + \alpha(t)\sigma + \beta(t)$$

is given by

$$D(\sigma, t) = \begin{bmatrix} \beta(t) & \alpha(t) & 1 \\ \sigma^3 & \beta(t+1) & \alpha(t+1) \\ \alpha(t+2)\sigma^3 & \sigma^3 & \beta(t+2) \end{bmatrix},$$

whose determinant is σ^6 . Hence, this system is a PFIR and the impulse response starting at time $t = 0$ ends in 2 periods at the most. In order to compute such a response, let us factorize the system. This can be done in a number of different ways. One possibility is to recognize that the polynomial

$$d^*(\sigma, t) = \sigma^4 - \alpha(t+1)\sigma^3 + (\alpha(t)\alpha(t+1) - \beta(t+1))\sigma^2 - \alpha(t)\beta(t+2)\sigma + \beta(t+1)\beta(t+2)$$

is such that $d(\sigma, t)d^*(\sigma, t) = \sigma^6$. Therefore,

$$\begin{aligned} G(\sigma, t) &= (1 - \sigma)d^*(\sigma, t)(d(\sigma, t)d^*(\sigma, t))^{-1} \\ &= \gamma_0(t)\sigma^{-1} + \gamma_1(t)\sigma^{-2} + \gamma_2(t)\sigma^{-3} + \gamma_3(t)\sigma^{-4} + \gamma_5(t)\sigma^{-5}, \end{aligned}$$

where

$$\begin{aligned} \gamma_0(t) &= 1, \forall t \\ \gamma_1(t) &= 1 + \alpha_1(t+2) \\ \gamma_2(t) &= -\alpha(t+1) - \alpha(t+1)\alpha(t+2) + \beta(t+2) \\ \gamma_3(t) &= \alpha(t+1)\beta(t) + \alpha(t)\alpha(t+1) - \beta(t+1) \\ \gamma_4(t) &= -\alpha(t)\beta(t+2) - \beta(t)\beta(t+2) \\ \gamma_5(t) &= \beta(t+1)\beta(t+2), \end{aligned}$$

hence the response to an impulse at time $t = 0$ is given by

$$\begin{aligned} y(0) &= 0, y(1) = \gamma_0(1) = -1, y(2) = \gamma_1(2) = 1, y(3) = \gamma_2(3) = 1, \\ y(4) &= \gamma_3(4) = -2, y(5) = \gamma_4(5) = 0, y(6) = \gamma_5(6) = 1, y(i) = 0, i \geq 7. \end{aligned}$$

5.6 Bibliographical Notes

The study of time-varying systems through operators is a long-standing issue in system theory, see [205, 206]. The algebra of periodic polynomials, dealt with in [132], has been used in the solution of a number of periodic control and filtering problems in [47, 48, 56, 110].

Chapter 6

Time-invariant Reformulations

The correspondence between periodic and time-invariant systems is a long-standing issue in mathematics, signals, and systems. The purpose of this chapter is to present a complete and clear picture of such time-invariant representations. To this purpose, we will preliminarily study the effect induced by sampling in the frequency domain. Such data processing is then applied to the input and output variables of a periodic system.

We articulate the time-invariant representations into three main classes.

(i) *Time-lifted reformulation.*

The operation of lifting consists in packing the values of a signal over one period into a new enlarged signal. The lifted reformulation relates the lifted output signal to the lifted input signal. It traces back to the 1950s and it has often been used ever since.

(ii) *Cyclic reformulation.*

For a signal, the operation of cycling amounts to picking up a once-per-period sample and transferring it into a given position of an augmented vector. As time proceeds, the sample of the signal shifts over the various cells of the augmented vector, periodically restarting from the top position. Again, by applying such a processing to the external variables of a periodic system, one obtains a time-invariant model, the cyclic reformulation.

(iii) *Frequency-lifted reformulation.*

In an Exponentially Modulated Periodic (EMP) regime (Chap. 2), the number of possible harmonics compatible with a periodic discrete-time system coincides with the system period. Collecting all harmonics of a signal in a vector is at the basis of the frequency-lifted reformulation. In the state–space context, when considering all signals (input, state, output) in an EMP regime, this reformulation corresponds to taking the Fourier expansion of the periodic system matrices. In this way, it is possible to work out a Fourier representation, expressing the “gain” between the input–output harmonics of the EMP regime.

?

In our presentation, we will make the constant effort of relate each reformulation to the periodic transfer function introduced in Chap. 2. In so doing, the inter-relations among the reformulations is better clarified. We begin with the study of the effect of sampling a signal in the frequency domain.

6.1 Sampling in the z -Domain

Consider a discrete-time signal $v(\cdot)$. Its z -transform is defined as

$$V(z) = \sum_{t=0}^{\infty} v(t)z^{-t}.$$

Hence, the z -transform of the periodically sampled signal

$$v^{(i)}(k) = v(kT + i - 1)$$

is given by

$$V^{(i)}(z) = \sum_{k=0}^{\infty} v(kT + i - 1)z^{-k}.$$

Now, evaluating this expression in z^T and denoting by $\delta(t)$ the Kronecker symbol, i.e., $\delta(0) = 1$, $\delta(t) = 0$, $\forall t \neq 0$, it is easy to see that

$$\begin{aligned} V^{(i)}(z^T) &= \sum_{k=0}^{\infty} v(kT + i - 1)z^{-kT} \\ &= \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} v(t)z^{-t}z^{i-1}\delta(t - kT - i + 1) \\ &= z^{i-1} \sum_{t=0}^{\infty} v(t)f_i(t)z^{-t}, \end{aligned} \tag{6.1}$$

where

$$f_i(t) = \sum_{k=0}^{\infty} \delta(t - kT - i + 1) \tag{6.2}$$

is a T -periodic sequence of discrete pulses. Now, recall definition (2.8) of the T -roots of the unit and the associated property (2.9). It turns out that

$$f_i(t) = \frac{1}{T} \sum_{k=0}^{T-1} \phi^{k(i-1-t)}.$$

Notice in passing that this expression is just the Fourier sum of the periodic function $f_i(t)$.

From (6.1) and (6.2) it follows that

$$\begin{aligned} V^{(i)}(z^T) &= \frac{z^{i-1}}{T} \sum_{t=0}^{\infty} v(t) z^{-t} \sum_{k=0}^{T-1} \phi^{k(i-1)} \phi^{-kt} \\ &= \frac{z^{i-1}}{T} \sum_{k=0}^{T-1} \phi^{k(i-1)} \sum_{t=0}^{\infty} v(t) (z\phi^k)^{-t}. \end{aligned}$$

In conclusion, the z -transform of the sampled signal can be given the form

$$V^{(i)}(z^T) = \frac{z^{i-1}}{T} \sum_{k=0}^{T-1} V(z\phi^k) \phi^{k(i-1)}. \quad (6.3)$$

Conversely, the z -transform of the original signal $v(t)$ can be recovered from the z -transform of the sampled signals $v(kT + i)$. Indeed,

$$\begin{aligned} V(z) &= \sum_{t=0}^{\infty} v(t) z^{-t} \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{T-1} v(iT + k) z^{-iT-k} \\ &= \sum_{k=0}^{T-1} z^{-k} \sum_{i=0}^{\infty} v(iT + k) z^{-iT}. \end{aligned}$$

Therefore, the converse relation is

$$V(z) = \sum_{k=1}^T V^{(k)}(z^T) z^{-k+1}. \quad (6.4)$$

6.2 Time-lifted Reformulation

In this section, we introduce the most classical time-invariant reformulation, namely the lifted reformulation. The simplest way to explain the idea of lifting is to make reference to a (discrete-time) signal, say $v(\cdot)$. Associated with $v(\cdot)$ one can introduce the augmented (*lifted*) signal

$$v_{\tau}(k) = [v(kT + \tau)' \ v(kT + \tau + 1)' \ \dots \ v(kT + \tau + T - 1)']',$$

where τ is a tag point, arbitrarily chosen. We will also use the symbol $v^{(i)}(k)$ for the components of vector $v_{\tau}(k)$, i.e., $v_{\tau}^{(i)}(k) = v(kT + \tau + i - 1)$, $i = 1, 2, \dots, T - 1$. As is apparent, $v_{\tau}(k)$ is constituted by the samples of the original signal over the interval $[\tau, \tau + T - 1]$, T being any positive integer. For a proper comprehension of the way in which signal lifting is reflected into the input/output representation of a

system, we start by considering the effect of sampling in the z -transform domain. Then, we will pass to analyze the action of lifting on time-invariant and T -periodic systems, both in state-space and input-output forms.

6.2.1 Lifting a Time-invariant System

Now consider a time-invariant system with transfer function $G(z)$:

$$Y(z) = G(z)U(z).$$

If one performs a uniform sampling of the input and output signals with sampling interval T , the relationship between the z -transform $U^{(j)}(z)$ of the sampled input and the z -transform $Y^{(i)}(z)$ of the sampled output can be worked out by referring to Expressions (6.3), (6.4). It follows that

$$\begin{aligned} Y^{(i)}(z^T) &= \frac{z^{i-1}}{T} \sum_{k=0}^{T-1} G(z\phi^k)U(z\phi^k)\phi^{k(i-1)} = \\ &= \frac{z^{i-1}}{T} \sum_{k=0}^{T-1} G(z\phi^k)\phi^{k(i-1)} \sum_{r=1}^T U^{(r)}(z^T)(z\phi^k)^{-r+1} = \\ &= \sum_{r=1}^T \left[\frac{z^{i-r}}{T} \sum_{k=0}^{T-1} G(z\phi^k)\phi^{k(i-r)} \right] U^{(r)}(z^T). \end{aligned} \quad (6.5)$$

A moment's reflection shows that the term in the square brackets in this expression is a rational function in the variable z^T and depends only on the difference $i - r$. In view of (6.5), the transfer function from the r -th input channel to the i -th output channel turns out to be $G^{(i-r)}(z)$, where

$$G^{(s)}(z^T) = \frac{z^s}{T} \sum_{k=0}^{T-1} G(z\phi^k)\phi^{ks}. \quad (6.6)$$

Notice that

$$G^{(s+T)}(z^T) = z^T G^{(s)}(z^T). \quad (6.7)$$

By letting i and r span the interval $[0, T - 1]$, one can determine the transfer function from the lifted input signal $u_0(k)$ to the lifted output signal $y_0(k)$. This will be called the *lifted transfer function* at time 0 and will be denoted by $W_0(z)$. In view of Expression (6.5), it is apparent that the (i, j) block of $W_0(z)$ only depends on the difference $j - i$.

Taking also into account Eq. (6.7), an easy computation leads to

$$W_0(z) = \begin{bmatrix} G^{(0)}(z) & z^{-1}G^{(T-1)}(z) & z^{-1}G^{(T-2)}(z) & \cdots & z^{-1}G^{(1)}(z) \\ G^{(1)}(z) & G^{(0)}(z) & z^{-1}G^{(T-1)}(z) & \cdots & z^{-1}G^{(2)}(z) \\ G^{(2)}(z) & G^{(1)}(z) & G^{(0)}(z) & \cdots & z^{-1}G^{(3)}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G^{(T-1)}(z) & G^{(T-2)}(z) & G^{(T-3)}(z) & \cdots & G^{(0)}(z) \end{bmatrix} \quad (6.8)$$

Interestingly enough, the relationship between $y_\tau(k)$ and $u_\tau(k)$ coincides with that between $y_0(k)$ and $u_0(k)$, for any integer τ . In other words, the transfer function $W_\tau(z)$ from $u_\tau(k)$ to $y_\tau(k)$ is such that

$$W_\tau(z) = W_0(z), \quad \forall \tau. \quad (6.9)$$

This can be easily verified by observing that, due to the time-invariance of the system, the transfer function from $u(kT + \tau + i)$ to $y(kT + \tau + j)$ depends only on the time lag $j - i$, and, as such, it is not affected by the specific value of τ .

Note that the enlarged transfer function (6.8) obtained by means of the lifting procedure has indeed a peculiar Toeplitz structure.

We close the section by pointing out that the original transfer function $G(z)$ can be recovered from the T -sampled transfer functions by means of the inverse formula

$$G(z) = \sum_{j=0}^{T-1} G^{(j)}(z^T) z^{-j}. \quad (6.10)$$

6.2.2 Lifting a Periodic System in the Input–Output Framework

The lifted reformulation of a periodic system is based on the obvious consideration that a periodic system has an invariant behavior with respect to time shifts which are multiple of the period. Therefore if one performs a uniform sampling of the input at time $i, i + T, i + 2T$, etc., and of the output signals at $j, j + T, j + 2T$, etc., the dynamics underlying these sampled signals is time-invariant. Of course, such a dynamics depends on the particular pair of time instants i, j . The lifted reformulation is just the collection of all these time-invariant dynamics for i, j ranging in the period. If one considers the generic tag instant τ , the *lifted transfer function* at τ has the form

$$W_\tau(z) = \begin{bmatrix} [W_\tau(z)]_{1,1} & [W_\tau(z)]_{1,2} & \cdots & [W_\tau(z)]_{1,T} \\ [W_\tau(z)]_{2,1} & [W_\tau(z)]_{2,2} & \cdots & [W_\tau(z)]_{2,T} \\ \cdots & \cdots & \ddots & \cdots \\ [W_\tau(z)]_{T,1} & [W_\tau(z)]_{T,2} & \cdots & [W_\tau(z)]_{T,T} \end{bmatrix},$$

where the generic block entry $[W_\tau(z)]_{i,j}$ is the transfer function from the sampled input $u(kT + \tau + i - 1)$ to the sampled output $y(kT + \tau + j - 1)$. Obviously, matrix $W_\tau(z)$ is of dimension $pT \times mT$.

It is important to point out that there is a simple recursive formula in order to pass from $W_\tau(z)$ to $W_{\tau+1}(z)$. This can be seen, for instance, by the very definition of the packed input $u_\tau(k)$ and packed output $y_\tau(k)$. Indeed, interpreting the symbol z as a 1-step-ahead shift operator in the variable k (1-period-ahead shift operator in the variable t),

$$\begin{bmatrix} u(kT + \tau) \\ u(kT + \tau + 1) \\ u(kT + \tau + 2) \\ \vdots \\ u(kT + \tau + T - 1) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 0 & z^{-1}I_m \\ I_m & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & I_m & 0 & 0 \\ 0 & \cdots & 0 & I_m & 0 \end{bmatrix} \begin{bmatrix} u(kT + \tau + 1) \\ u(kT + \tau + 2) \\ u(kT + \tau + 3) \\ \vdots \\ u(kT + \tau + T) \end{bmatrix}$$

$$\begin{bmatrix} y(kT + \tau + 1) \\ y(kT + \tau + 2) \\ y(kT + \tau + 3) \\ \vdots \\ y(kT + \tau + T) \end{bmatrix} = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 \\ 0 & 0 & I_p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I_p \\ zI_p & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} y(kT + \tau) \\ y(kT + \tau + 1) \\ y(kT + \tau + 2) \\ \vdots \\ y(kT + \tau + T - 1) \end{bmatrix}.$$

Thus, defining

$$\Delta_k(z) = \begin{bmatrix} 0 & z^{-1}I_k \\ I_{k(T-1)} & 0 \end{bmatrix}. \quad (6.11)$$

the above relationships can be rewritten as

$$u_\tau(k) = \Delta_m(z)u_{\tau+1}(k), \quad y_{\tau+1}(k) = \Delta_p(z^{-1})'y_\tau(k).$$

Therefore, considering that $y_\tau(k) = W_\tau(z)u_\tau(k)$ and $y_{\tau+1}(k) = W_{\tau+1}(z)u_{\tau+1}(k)$, it is possible to work out a recursion for the lifted transfer functions as

$$W_{\tau+1}(z) = \Delta_p'(z^{-1})W_\tau(z)\Delta_m(z). \quad (6.12)$$

Notice that the matrix $\Delta_k(z)$ is inner, i.e.,

$$\Delta_k'(z^{-1})\Delta_k(z) = I_{kT}.$$

Moreover, for $k = 1$, this matrix coincides with the one defined in Chap. 5, see Eq. (5.3).

We now express the lifted transfer function in terms of the sampled transfer functions $H_i(z, t)$.

As seen in Chap. 2, the output of the system can be given the following form

$$y(kT + s) = H_0(z, s)u(kT + s) + H_1(z, s)u(kT + s - 1) + \cdots \\ + H_{T-2}(z, s)u(kT + s - T + 2) + H_{T-1}(z, s)u(kT + s - T + 1),$$

or, equivalently

$$y(kT + s) = H_0(z, s)u(kT + s) + H_{T-1}(z, s)z^{-1}u(kT + s + 1) + \cdots \\ + H_2(z, s)z^{-1}u(kT + s + T - 2) + H_1(z, s)z^{-1}u(kT + s + T - 1).$$

Now, taking s ranging from τ to $\tau + T - 1$, it is apparent that, for any $i, j = 1, 2, \dots, T$:

$$[W_\tau(z)]_{i,j} = \begin{cases} H_{T-j+i}(z, i-1+\tau)z^{-1} & i < j \\ H_{i-j}(z, i-1+\tau) & i \geq j \end{cases}, \quad (6.13)$$

so that

$$W_\tau(z) = \begin{bmatrix} H_0(z, \tau) & H_{T-1}(z, \tau)z^{-1} & \cdots & H_1(z, \tau)z^{-1} \\ H_1(z, \tau+1) & H_0(z, \tau+1) & \cdots & H_2(z, \tau+1)z^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{T-1}(z, \tau+T-1) & H_{T-2}(z, \tau+T-1) & \cdots & H_0(z, \tau+T-1) \end{bmatrix}. \quad (6.14)$$

Notice that this is a Toeplitz-like matrix, in that, along the diagonals, one can find the same functions with different time-shifts.

Expression (6.13) can be given a converse formulation, enabling one to recover the sampled transfer functions from the lifted matrix. Precisely, for $i = 1, 2, \dots, T$:

$$H_k(z, i-1+\tau) = \begin{cases} [W_\tau(z)]_{i,i-k} & k = 0, 1, \dots, i-1 \\ [W_\tau(z)]_{i,i+T-k}z & k = i, i+1, \dots, T-1 \end{cases}. \quad (6.15)$$

It can be seen from (6.14) that an important property of the lifted transfer function $W_\tau(z)$ is that it has a lower block-triangular structure at infinity, i.e.,

$$\lim_{z \rightarrow \infty} [W_\tau(z)]_{i,j} = 0, \quad i < j. \quad (6.16)$$

This triangularity property originates from the causality property which implies the absence of direct feed-through between past output and future inputs.

Remark 6.1 As seen above, the lifted transfer function $W_\tau(z)$ is composed of T^2 blocks of dimensions $p \times m$. However, it is worthy pointing out that T transfer matrices of dimensions $p \times m$ suffice to work out the entire matrix $W_\tau(z)$.

Indeed, recall the definition of the periodic transfer function $G(\sigma, t)$ introduced in Chap. 2, and its link with the sampled transfer functions $H_i(z, t)$, namely

$$G(\sigma, t) = \sum_{i=0}^{T-1} H_i(\sigma^T, t) \sigma^{-i}.$$

In view of (6.15) one obtains

$$G(\sigma, \tau) = \sum_{j=0}^{T-1} [W_\tau(\sigma^T)]_{1, j+1} \sigma^j, \quad \tau = 0, 1, \dots, T-1 \quad (6.17)$$

or, equivalently

$$G(\sigma, \tau) = \sum_{j=0}^{T-1} [W_0(\sigma^T)]_{1+\tau, j+1} \sigma^{j-\tau}, \quad \tau = 0, 1, \dots, T-1. \quad (6.18)$$

Conversely, in view of (6.13), and taking into account that

$$H_r(\sigma^T, t) = \frac{\sigma^r}{T} \sum_{k=0}^{T-1} G(\sigma \phi^k, t-r) \phi^{kr},$$

it follows that

$$[W_\tau(z^T)]_{i,j} = \frac{z^{i-j}}{T} \sum_{k=0}^{T-1} G(z\phi^k, i-1+\tau) \phi^{k(i-j)} \quad i, j = 1, 2, \dots, T. \quad (6.19)$$

Expression (6.19) points out that all block entries of $W_\tau(z)$ can be built from the T generating functions $G(z, t)$, $t = 0, 1, \dots, T-1$, defined in (2.3).

6.2.3 Lifting a Periodic System in the State–Space Framework

The lifted reformulation can be worked out by starting from a state–space description of the periodic System (2.19). This can be obtained by making reference to the sampled state

$$x^{(\tau)}(k) = x(kT + \tau).$$

In this way, the 1-period-ahead state $x^{(\tau)}(k+1) = x((k+1)T + \tau)$ is determined by actual state $x^{(\tau)}(k) = x(kT + \tau)$ along with the values of the input in the period; these values are packed altogether in the lifted input $u_\tau(k)$. The lifted output $y_\tau(k)$ can be obtained from the sampled state $x^{(\tau)}(k)$ and the lifted input $u_\tau(k)$.

More precisely, define $F_\tau \in \mathbb{R}^{n \times n}$, $G_\tau \in \mathbb{R}^{n \times mT}$, $H_\tau \in \mathbb{R}^{pT \times n}$, $E_\tau \in \mathbb{R}^{pT \times mT}$ as

$$F_\tau = \Psi_A(\tau) \quad (6.20a)$$

$$G_\tau = [\Phi_A(\tau + T, \tau + 1)B(\tau) \ \Phi_A(\tau + T, \tau + 2)B(\tau + 1) \ \cdots \ B(\tau + T - 1)] \quad (6.20b)$$

$$H_\tau = [C(\tau)' \ \Phi_A(\tau + 1, \tau)'C(\tau + 1)' \ \cdots \ \Phi_A(\tau + T - 1, \tau)'C(\tau + T - 1)']' \quad (6.20c)$$

$$E_\tau = \{(E_\tau)_{ij}\}, \quad i, j = 1, 2, \dots, T \quad (6.20d)$$

$$(E_\tau)_{ij} = \begin{cases} 0 & i < j \\ D(\tau + i - 1) & i = j \\ C(\tau + i - 1)\Psi_A(\tau + i - 1, \tau + j)B(\tau + j - 1) & i > j \end{cases} \quad (6.20e)$$

The lifted reformulation in state-space takes the form

$$x^{(\tau)}(k + 1) = F_\tau x^{(\tau)}(k) + G_\tau u_\tau(k) \quad (6.21a)$$

$$y_\tau(k) = H_\tau x^{(\tau)}(k) + E_\tau u_\tau(k). \quad (6.21b)$$

Equations (6.21) define the so-called *lifted system*. This system is a state-sampled representation of the original System (2.19), fed by T input *channels* (the entries of the lifted input) and producing T output *channels* (the entries of the lifted output).

Obviously the transfer function of System (6.21) coincides with $W_\tau(z)$ so that one can write

$$W_\tau(z) = H_\tau(zI - F_\tau)^{-1}G_\tau + E_\tau. \quad (6.22)$$

Remark 6.2 *The dynamic matrix of the lifted System (6.21) coincides with the monodromy matrix of $A(\cdot)$. Therefore, the periodic system is stable if and only if the time-invariant lifted system is stable.*

This analogy extends to the structural properties. In particular, System (6.21) is controllable (reconstructable, detectable, stabilizable) if and only if System (2.19) is controllable (reconstructable, detectable, stabilizable, respectively). As for reachability and observability, some attention must be paid to the possible time-variability of the dimensions of the reachability and observability subspaces of a periodic system. However, for a given τ , it is still true that System (6.21) is reachable (observable) if and only if the time-invariant System (2.19) is reachable (observable) at time τ . These statements can be easily checked on the basis of the treatment of the structural properties in Chap. 4.

Example 6.1 *Making reference to the system considered in Example 2.3, the lifted transfer function (6.22) can be deduced from the transfer function operator $G(\sigma, t)$ by exploiting Eq. (6.19). Precisely,*

$$W_0(z) = \frac{1}{z + 10} \begin{bmatrix} -2.5 & -1 \\ 3z & -12 \end{bmatrix}, \quad W_1(z) = \frac{1}{z + 10} \begin{bmatrix} -12 & 3 \\ -z & -2.5 \end{bmatrix}.$$

The state–space lifted system can be worked out from (6.20) thus obtaining

$$F_0 = -10, \quad G_0 = \begin{bmatrix} -5 & -2 \end{bmatrix}, \quad H_0 = \begin{bmatrix} 0.5 \\ 6 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix},$$

and

$$F_1 = -10, \quad G_1 = \begin{bmatrix} -4 & 1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 3 \\ -2.5 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Remark 6.3 It is possible to work out the relation between the lifted transfer function and that of its dual (4.35). Actually, recalling (4.36) and (6.20), it can be seen that the transfer function at τ , say $W_\tau^d(z)$, of the dual lifted system is related to the transfer function $W_\tau(z)$ of the original system as follows

$$W_\tau^d(z) = \begin{bmatrix} 0 & \cdots & 0 & I \\ 0 & \cdots & I & 0 \\ \vdots & \ddots & \vdots & \vdots \\ I & \cdots & 0 & 0 \end{bmatrix} W_{-\tau}(z)' \begin{bmatrix} 0 & \cdots & 0 & I \\ 0 & \cdots & I & 0 \\ \vdots & \ddots & \vdots & \vdots \\ I & \cdots & 0 & 0 \end{bmatrix}.$$

Remark 6.4 Consider the time-invariant reformulation of the periodic system at τ , i.e., System (6.21). The transfer function of this system is $W_\tau(z) = E_\tau + H_\tau(zI - F_\tau)^{-1}G_\tau$. Hence, $W_\tau(z^{-1})' = E_\tau' + G_\tau'(z^{-1}I - F_\tau')^{-1}H_\tau'$ is the transfer function of the adjoint system

$$F_\tau' \lambda_\tau(k+1) = \lambda_\tau(k) - H_\tau' v_\tau(k) \quad (6.23a)$$

$$\zeta_\tau(k) = G_\tau' \lambda_\tau(k+1) + E_\tau' v_\tau(k). \quad (6.23b)$$

If one set

$$v_\tau(k) = \left[v(kT + \tau)' \ v(kT + \tau + 1)' \ \dots \ v(kT + \tau + T - 1)' \right]'$$

$$\zeta_\tau(k) = \left[\zeta(kT + \tau)' \ \zeta(kT + \tau + 1)' \ \dots \ \zeta(kT + \tau + T - 1)' \right]',$$

the adjoint System (6.23) is exactly the time-lifted reformulation of the adjoint periodic system at τ , see (2.32).

6.3 Cyclic Reformulation

To explain the basic idea underlying the cyclic reformulation, we start again by considering a discrete-time signal $v(\cdot)$, of arbitrary dimension, say r . Having chosen an arbitrary tag τ , select a uniformly distributed grid $\mathcal{T}_\tau(i)$ of time points:

$$\mathcal{T}_\tau(i) = \{\tau + i - 1 + kT, \quad | \ k = 0, 1, \dots\}$$

and define a new signal $\hat{v}_\tau^{(i)}(t)$ which is the restriction of the original one over the grid:

$$\hat{v}_\tau^{(i)}(t) = v(t), \quad t \in \mathcal{T}_\tau(i).$$

It is now possible to define the *cyclic signal* $\hat{v}_\tau(t)$ as

$$\hat{v}_\tau(t) = \left[\hat{v}_\tau^{(1)}(t)' \quad \hat{v}_\tau^{(2)}(t)' \quad \cdots \quad \hat{v}_\tau^{(T)}(t)' \right]'$$

Notice that this augmented vector, of dimension rT , is peculiar in that each block-component is defined over a different set of time points. However, what really matters is that there is a one-to-one correspondence between the augmented signal $\hat{v}_\tau(\cdot)$ and the original one $v(\cdot)$.

Remark 6.5 *To avoid the oddity of having a vector with undefined components at certain time points, one can alternatively define the cyclic vector as follows*

$$\hat{v}_\tau^{(i)}(t) = \begin{cases} v(t) & t = \tau + i - 1 + kT \\ 0 & \text{otherwise} \end{cases}. \quad (6.24)$$

In other words, at each time-point the vector $\hat{v}_\tau(t)$ has a unique non-zero subvector, which cyclically shifts along the column blocks:

$$\hat{v}_\tau(\tau) = \begin{bmatrix} v(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \hat{v}_\tau(\tau+1) = \begin{bmatrix} 0 \\ v(\tau+1) \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \hat{v}_\tau(\tau+T-1) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v(\tau+T-1) \end{bmatrix}.$$

The introduction of the slack null values is redundant, but has the advantage of leading to signals defined for any time points.

6.3.1 Cycled Signal in the z -Domain

It is possible to establish a link between the z -transform $\hat{V}_\tau(z)$ of the cycled signal $\hat{v}_\tau(t)$, the z -transform $V_\tau(z)$ of the lifted signal $v_\tau(k)$, and the z -transform $V(z)$ of the original signal $v(t)$. Indeed, notice that:

$$\hat{V}_\tau^{(i)}(z) = \sum_{t=0}^{\infty} z^{-t} \hat{v}_\tau^{(i)}(t) = \sum_{t \in \mathcal{T}_\tau(i)} z^{-t} v(t).$$

Hence

$$\hat{V}_\tau^{(i)}(z) = z^{-\tau-i+1} \sum_{k=0}^{\infty} z^{-kT} v(\tau+kT+i-1) = z^{-\tau-i+1} \sum_{k=0}^{\infty} v_\tau^{(i)}(k) z^{-kT},$$

so that

$$\hat{V}_\tau^{(i)}(z) = z^{-\tau-i+1} V_\tau^{(i)}(z^T).$$

This expression brings into light a simple relationship between the z -transform of the augmented lifted and cycled signals. Precisely, letting r be the dimension of the vector $v(\cdot)$, it follows

$$\hat{V}_\tau(z) = z^{-\tau} \nabla_r(z) V_\tau(z^T), \quad (6.25)$$

where $\nabla_r(z)$ is the block-diagonal matrix

$$\nabla_r(z) = \begin{bmatrix} I_r & 0 & \cdots & 0 \\ 0 & z^{-1} I_r & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & z^{-T+1} I_r \end{bmatrix}. \quad (6.26)$$

Taking into account Eq. (6.3), one can also express the z -transform of any block entry of the cycled signal in terms of the z -transform of the original signal:

$$\hat{V}_\tau^{(i)}(z) = \frac{z^{-\tau}}{T} \sum_{k=0}^{T-1} V(z\phi^k) \phi^{k(i-1)}.$$

Conversely, the z -transform of the original signal is easily obtained from (6.4):

$$V(z) = z^\tau \sum_{k=1}^T V_\tau^{(k)}(z).$$

6.3.2 Cycling a Time-invariant System

As in the lifting case, it is advisable to study at first the action of cycling a time-invariant system with transfer function $G(z)$. This is made easy by applying the Relation (6.25) between the lifted and the cycled signal to both the input and output of the system. Then, the *cyclic transfer function* $\hat{W}_\tau(z)$ from $\hat{u}_\tau(t)$ to $\hat{y}_\tau(t)$ can be computed from the lifted one $W_\tau(z)$ as follows

$$\hat{W}_\tau(z) = \nabla_p(z) W_\tau(z^T) \nabla_m(z)^{-1},$$

so that, in view of the formula of the lifted transfer function (6.8), and Eqs. (6.6) and (6.9), it follows that

$$\hat{W}_\tau(z) = \hat{W}_0(z) = \begin{bmatrix} \hat{G}^{(0)}(z) & \hat{G}^{(T-1)}(z) & \hat{G}^{(T-2)}(z) & \cdots & \hat{G}^{(1)}(z) \\ \hat{G}^{(1)}(z) & \hat{G}^{(0)}(z) & \hat{G}^{(T-1)}(z) & \cdots & \hat{G}^{(2)}(z) \\ \hat{G}^{(2)}(z) & \hat{G}^{(1)}(z) & \hat{G}^{(0)}(z) & \cdots & \hat{G}^{(3)}(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{G}^{(T-1)}(z) & \hat{G}^{(T-2)}(z) & \hat{G}^{(T-3)}(z) & \cdots & \hat{G}^{(0)}(z) \end{bmatrix},$$

where

$$\hat{G}^{(s)}(z) = \frac{1}{T} \sum_{k=0}^{T-1} G(z\phi^k) \phi^{ks}. \quad (6.27)$$

Noticing that $\hat{G}^{(s)}(z) = z^{-s} G^{(s)}(z^T)$, it is possible to exploit Eq. (6.10) in order to recover the transfer function $G(z)$ of the time-invariant system from the T -functions $\hat{G}^{(s)}(z)$, $s = 0, 1, \dots, T-1$:

$$G(z) = \sum_{j=0}^{T-1} \hat{G}^{(j)}(z). \quad (6.28)$$

Therefore, the transfer function of the original time-invariant system can be computed from the cyclic transfer function by summing the block entries of any row (or any column).

6.3.3 Cycling a Periodic System in the Input–Output Framework

The very definition of the time-grids $\mathcal{T}_\tau(k)$, $k = 1, 2, \dots, T$, obviously requires that the dynamics relating the input of a periodic system over $\mathcal{T}_\tau(j)$ to its output over $\mathcal{T}_\tau(i)$ is indeed time-invariant.

The associated transfer function will be denoted by $[\hat{W}_\tau(z)]_{i,j}$. Hence, it is possible to define the *cyclic transfer function* at τ of a periodic system as

$$\hat{W}_\tau(z) = \begin{bmatrix} [\hat{W}_\tau(z)]_{1,1} & [\hat{W}_\tau(z)]_{1,2} & \cdots & [\hat{W}_\tau(z)]_{1,T} \\ [\hat{W}_\tau(z)]_{2,1} & [\hat{W}_\tau(z)]_{2,2} & \cdots & [\hat{W}_\tau(z)]_{2,T} \\ \cdots & \cdots & \ddots & \cdots \\ [\hat{W}_\tau(z)]_{T,1} & [\hat{W}_\tau(z)]_{T,2} & \cdots & [\hat{W}_\tau(z)]_{T,T} \end{bmatrix}.$$

Obviously matrix $\hat{W}_\tau(z)$ is of dimension $pT \times mT$.

There is a simple recursive formula to pass from $\hat{W}_\tau(z)$ to $\hat{W}_{\tau+1}(z)$. Indeed, it is apparent that

$$\mathcal{T}_{\tau+1}(i) = \begin{cases} \mathcal{T}_\tau(i+1) & i = 1, \dots, T-1 \\ \mathcal{T}_\tau(1) & i = T \end{cases}.$$

This implies that the z -transform $\hat{U}_{\tau+1}(z)$ of $\hat{u}_{\tau+1}(t)$ is a permutation of the z -transform $\hat{U}_\tau(z)$ of $\hat{u}_\tau(t)$:

$$\hat{U}_{\tau+1}(z) = \hat{\Delta}_m \hat{U}_\tau(z),$$

where

$$\hat{\Delta}_k = \begin{bmatrix} 0 & I_k \\ I_{k(T-1)} & 0 \end{bmatrix}. \quad (6.29)$$

As can be seen, the matrix Δ_k is orthogonal i.e., $\hat{\Delta}_k' \hat{\Delta}_k = I_{kT}$. By applying the same considerations to the output signal, one obtains the following recursion for the cyclic

transfer function:

$$\hat{W}_{\tau+1}(z) = \hat{\Delta}'_p \hat{W}_\tau(z) \hat{\Delta}_m. \quad (6.30)$$

Thanks again to (6.25), (6.26), it is apparent that the relation

$$\hat{W}_\tau(z) = \nabla_m(z) W_\tau(z^T) \nabla_p(z)^{-1} \quad (6.31)$$

holds, so that the single entries are simply related as

$$[\hat{W}_\tau(z)]_{i,j} = z^{j-i} [W_\tau(z^T)]_{i,j}. \quad (6.32)$$

As for the relation between the cyclic transfer function and the sampled transfer functions $H_i(z, t)$ defined in Chap. 2, it can be easily verified from (6.13), (6.15), (6.31), in this way obtaining:

$$[\hat{W}_\tau(z)]_{i,j} = \begin{cases} H_{T-j+i}(z^T, i-1+\tau) z^{j-i-T} & i < j \\ H_{i-j}(z^T, i-1+\tau) z^{j-i} & i \geq j \end{cases}$$

$$H_k(z^T, i-1+\tau) = \begin{cases} [\hat{W}_\tau(z)]_{i,i-k} z^k & k = 0, 1, \dots, i-1 \\ [\hat{W}_\tau(z)]_{i,i+T-k} z^k & k = i, i+1, \dots, T-1 \end{cases}.$$

Consequently,

$$\hat{W}_\tau(z) = \begin{bmatrix} H_0(z^T, \tau) & H_{T-1}(z^T, \tau) z^{-T+1} & \dots & H_1(z^T, \tau) z^{-1} \\ H_1(z^T, \tau+1) z^{-1} & H_0(z^T, \tau+1) & \dots & H_2(z^T, \tau+1) z^{-2} \\ \dots & \dots & \ddots & \dots \\ H_{T-1}(z^T, \tau+T-1) z^{-T+1} & H_{T-2}(z^T, \tau+T-1) z^{-T+2} & \dots & H_0(z^T, \tau+T-1) \end{bmatrix}.$$

This expression ensures that the cyclic transfer function $\hat{W}_\tau(z)$ is block-diagonal at infinity, i.e.,

$$\lim_{z \rightarrow \infty} [\hat{W}_\tau(z)]_{i,j} = 0, \quad i \neq j$$

This fact can be explained by noticing that there is no direct feedthrough between an entry of the cycled input and an entry of the cycled output, whenever these entries are defined over the time sets $\mathcal{T}_\tau(i)$ and $\mathcal{T}_\tau(j)$, $i \neq j$, which have empty intersection.

Remark 6.6 *As seen in Remark 6.1, the lifted transfer function can be only constructed starting from T transfer matrices of dimensions $p \times m$. The same property holds for the cyclic transfer function. Indeed, from (6.17) and (6.32) one obtains*

$$G(z, \tau) = \sum_{j=0}^{T-1} [\hat{W}_\tau(z)]_{1,j+1}, \quad \tau = 0, 1, \dots, T-1. \quad (6.33)$$

The converse expression can be derived from (6.19) and (6.32):

$$[\hat{W}_\tau(z)]_{i,j} = \frac{1}{T} \sum_{k=0}^{T-1} G(z\phi^k, i-1+\tau)\phi^{k(i-j)} \quad i, j = 1, 2, \dots, T. \quad (6.34)$$

This last formula shows that the knowledge of $G(z, t)$, $t = 0, 1, \dots, T-1$, suffices to construct the entire cyclic transfer function.

6.3.4 Cycling a Periodic System in the State–Space Framework

The cyclic reformulation of a periodic system in the state–space form (2.19) is just the state–space relation among the cycled state, cycled input and cycled output signals $\hat{x}_\tau(t)$, $\hat{u}_\tau(t)$, and $\hat{y}_\tau(t)$. In this regard, notice that the block-component $\hat{x}_\tau^{(i)}(t+1)$ is defined for $t \in \mathcal{T}_\tau(i-1)$ and, as such, can be obtained from $\hat{x}_\tau^{(i-1)}(t)$ and $\hat{u}_\tau^{(i-1)}(t)$ (obviously, if $i = 1$, $\hat{x}_\tau^{(i)}(t+1)$ will depend on $\hat{x}_\tau^{(T)}(t)$ and $\hat{u}_\tau^{(T)}(t)$).

The obtained cyclic reformulation would be defined componentwise over different grids of time points. In order to come up with a dynamical system defined for all time instants, one should rather refer to Expression (6.24), where slack zero components are fictitiously introduced in the definition of cycled vectors (recall Remark 6.5).

We will adopt for simplicity this last viewpoint. Consider the first component $\hat{x}_\tau^{(1)}(t)$ of the state and the first component $\hat{y}_\tau^{(1)}(t)$ of the output. Computing them at time $t = 1$ one obtains

$$\begin{aligned} \hat{x}_\tau^{(1)}(1) &= A(\tau + T - 1)\hat{x}_\tau^{(T)}(0) + B(\tau + T - 1)\hat{u}_\tau^{(T)}(0) \\ \hat{y}_\tau^{(1)}(1) &= C(\tau)\hat{x}_\tau^{(1)}(1) + D(\tau)\hat{u}_\tau^{(1)}(0). \end{aligned}$$

Of course, these expressions can be generalized to any time-point. Analogously, one can derive the formulas for the remaining components. In this way, it is possible to work out the cyclic reformulation at τ in state–space:

$$\hat{x}_\tau(t+1) = \hat{F}_\tau \hat{x}_\tau(t) + \hat{G}_\tau \hat{u}_\tau(t) \quad (6.35a)$$

$$\hat{y}_\tau(t) = \hat{H}_\tau \hat{x}_\tau(t) + \hat{E}_\tau \hat{u}_\tau(t), \quad (6.35b)$$

where

$$\hat{F}_\tau = \begin{bmatrix} 0 & 0 & \cdots & 0 & A(\tau + T - 1) \\ A(\tau) & 0 & \cdots & 0 & 0 \\ 0 & A(\tau + 1) & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & A(\tau + T - 2) & 0 \end{bmatrix} \quad (6.36a)$$

$$\hat{G}_\tau = \begin{bmatrix} 0 & 0 & \cdots & 0 & B(\tau + T - 1) \\ B(\tau) & 0 & \cdots & 0 & 0 \\ 0 & B(\tau + 1) & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & B(\tau + T - 2) & 0 \end{bmatrix} \quad (6.36b)$$

$$\hat{H}_\tau = \begin{bmatrix} C(\tau) & 0 & \cdots & 0 \\ 0 & C(\tau + 1) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & C(\tau + T - 1) \end{bmatrix} \quad (6.36c)$$

$$\hat{E}_\tau = \begin{bmatrix} D(\tau) & 0 & \cdots & 0 \\ 0 & D(\tau + 1) & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & D(\tau + T - 1) \end{bmatrix}. \quad (6.36d)$$

The dimensions of the state, input and output spaces of the cyclic reformulation are those of the original periodic systems multiplied by T .

Obviously, the transfer function of the cyclic reformulation at τ is given by

$$\hat{W}_\tau(z) = \hat{H}_\tau(zI - \hat{F}_\tau)^{-1} \hat{G}_\tau + \hat{E}_\tau.$$

Remark 6.7 *The dynamical matrix (6.36a) has a peculiar structure. Its eigenvalues are given by the T -th roots of the characteristic multipliers of $A(\cdot)$. Hence, the periodic system is stable if and only if its cyclic reformulation is stable. The structural properties of the cyclic reformulation are also determined by the corresponding structural properties of the original system. However, there are some slight but remarkable differences with respect to the lifted case, due to the extended dimension of the state-space. For example, if System (2.19) is reachable at time τ , the cyclic System (6.35) is not necessarily reachable. The appropriate statement is that System (2.19) is reachable (observable) at each time if and only if System (6.35) is reachable (observable) (at any arbitrary time τ which parametrizes (6.35)). This reflects the fact that, if System (6.35) is reachable for a value of τ , it is reachable for any τ . As for the remaining structural properties, one can recognize that System (2.19) is controllable (reconstructable, detectable, stabilizable) if and only if System (6.35) is controllable (reconstructable, detectable, stabilizable, respectively).*

Example 6.2 *The cyclic reformulation can be determined from the lifted one by means of Relation (6.31). Therefore, by considering the system of Example (6.1), from its reformulation given in Example 2.3, one obtains*

$$\hat{W}_0(z) = \frac{1}{z^2 + 10} \begin{bmatrix} -2.5 & -z \\ 3z & -12 \end{bmatrix}, \quad \hat{W}_1(z) = \frac{1}{z^2 + 10} \begin{bmatrix} -12 & 3z \\ -z & -2.5 \end{bmatrix}.$$

As for the state-space description, it can be worked out by using Expressions (6.36). It turns out that

$$\begin{aligned} \hat{F}_0 &= \begin{bmatrix} 0 & -5 \\ 2 & 0 \end{bmatrix}, & \hat{G}_0 &= \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, & \hat{H}_0 &= \begin{bmatrix} 0.5 & 0 \\ 0 & 3 \end{bmatrix} \\ \hat{F}_1 &= \begin{bmatrix} 0 & 2 \\ -5 & 0 \end{bmatrix}, & \hat{G}_1 &= \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, & \hat{H}_1 &= \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}, \end{aligned}$$

whereas E_0 and E_1 are the null matrices.

Remark 6.8 *As for the lifted system (Remark 6.3), the relation between the cyclic transfer function and its dual can be derived. From (4.35), (4.36) and (6.36), it can be seen that the transfer function at τ , say $\hat{W}_\tau^d(z)$, of the dual cyclic system is related to the transfer function $\hat{W}_\tau(z)$ of the original system as follows*

$$\hat{W}_\tau^d(z) = \begin{bmatrix} 0 & \cdots & I & 0 \\ \vdots & \ddots & \vdots & \vdots \\ I & \cdots & 0 & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} W_{-\tau+1}(z)' \begin{bmatrix} 0 & \cdots & I & 0 \\ \vdots & \ddots & \vdots & \vdots \\ I & \cdots & 0 & 0 \\ 0 & \cdots & 0 & I \end{bmatrix}.$$

6.4 The Frequency-lifted Reformulation

We conclude this chapter by presenting a reformulation which is naturally set directly in the z -domain. For, with reference to a signal $v(t)$ with z -transfer $V(z)$, define first the frequency-lifted vector $\tilde{V}(z)$ as

$$\tilde{V}(z) = \begin{bmatrix} V(z) \\ V(z\phi) \\ V(z\phi^2) \\ \vdots \\ V(z\phi^{T-1}) \end{bmatrix}, \quad (6.37)$$

where again $\phi = e^{j\frac{2\pi}{T}}$. Notice that the i -th component of this vector is nothing but the z -transform of the complex signal obtained by multiplying signal $v(t)$ by the complex number $\phi^{(-i+1)t}$.

It is straightforward to link $\tilde{V}(z)$ to the z transform $V_0(z)$ of the lifted signal $v_0(k)$ introduced in Sect. 6.2. Indeed, consider the elements $v_0^{(i)}(k) = v(kT + i - 1)$, $i = 0, 1, \dots, T-1$, of $v_0(k)$ and their z -transform $V_0^{(i)}(z)$ which are the elements of $V_0(z)$. Evaluating (6.4) in $z\phi^i$, $i = 0, 1, \dots, T-1$ and noticing that $V^{(k)}(z^T) = V_0^{(k)}(z^T)$, it is easy to recognize that

$$\tilde{V}(z) = M(z)V_0(z^T), \quad (6.38)$$

where

$$M(z) = \begin{bmatrix} I & z^{-1}I & \dots & z^{-T+1}I \\ I & (z\phi)^{-1} & \dots & (z\phi)^{-T+1} \\ \dots & \dots & \dots & \dots \\ I & (z\phi^{T-1})^{-1} & \dots & (z\phi^{T-1})^{-T+1} \end{bmatrix}.$$

6.4.1 Frequency-lifting of a Periodic System

The relationship between the frequency-lifted input signal $\tilde{U}(z)$ and the frequency-lifted output signal $\tilde{Y}(z)$ of a periodic system can be worked out by exploiting the previous results regarding the lifted transfer function. Indeed, recalling the definition of $W_0(z)$, from Eq. (6.38), it follows that

$$\tilde{Y}(z) = M(z)W_0(z^T)M(z)^{-1}\tilde{U}(z).$$

This leads to the definition of the *frequency-lifted transfer function* of a periodic system as

$$\tilde{W}(z) = M(z)W_0(z^T)M(z)^{-1}. \quad (6.39)$$

This formula points out that, while $W_0(z)$ has real coefficients, $\tilde{W}(z)$ is a proper rational matrix with complex coefficients. The reason is that, as above observed, $\tilde{W}(z)$ relates input–output time-domain signals with *complex coefficients*.

As for the previous reformulations, it is possible to express $\tilde{W}(z)$ in terms of the transfer function $G(z, t)$. Indeed, from (6.4),

$$\begin{aligned} Y(z\phi^q) &= \sum_{k=0}^{T-1} z^{-k}\phi^{-qk}Y_k^{(1)}(z^T) \\ &= \sum_{k=0}^{T-1} z^{-k}\phi^{-qk} \sum_{i=0}^{T-1} [W_k(z^T)]_{1,i+1} U_k^{(i+1)}(z^T), \end{aligned}$$

where in view of (6.3),

$$U_k^{(i+1)}(z^T) = \frac{z^{k+i}}{T} \sum_{p=0}^{T-1} u(z\phi^p)\phi^{p(k+i)}.$$

Therefore,

$$Y(z\phi^q) = \frac{1}{T} \sum_{k=0}^{T-1} \sum_{i=0}^{T-1} [W_k(z^T)]_{1,i+1} (z\phi^p)^i \sum_{p=0}^{T-1} U(z\phi^p) \phi^{k(p-q)}.$$

Now, we can exploit Expression (6.17) to work out the generic element of the frequency-lifted transfer function $\tilde{W}(z)$ as a function of $G(z, t)$, i.e.,

$$[\tilde{W}(z)]_{q+1,p+1} = \frac{1}{T} \sum_{k=0}^{T-1} G(z\phi^p, k) \phi^{k(p-q)}. \quad (6.40)$$

Conversely, it is possible to obtain $G(z, t)$ from $\tilde{W}(z)$. Indeed, from (6.40) and recalling property (2.9) it follows that

$$\sum_{q=0}^{T-1} [\tilde{W}(z)]_{q+1,p+1} \phi^{tq} = G(z\phi^p, t) \phi^{pt}$$

so that

$$G(z, t) = \sum_{q=0}^{T-1} [\tilde{W}(z\phi^{-p})]_{q+1,p+1} \phi^{t(q-p)}.$$

Remark 6.9 For a time-invariant system with transfer function $G(z)$, the periodic transfer operator is in fact constant, i.e., $G(z, t) = G(z)$, $\forall t$, so that, from (6.40),

$$\begin{aligned} [\tilde{W}(z)]_{q+1,p+1} &= G(z\phi^p) \frac{1}{T} \sum_{k=0}^{T-1} \phi^{k(p-q)} \\ &= \begin{cases} G(z\phi^p) & p = q \\ 0 & p \neq q \end{cases} \\ &= \text{diag} \{G(z\phi^p), p = 0, 1, \dots, T-1\}. \end{aligned}$$

This expression points out that, in the time-invariant case, $\tilde{W}(z)$ is block-diagonal. This conclusion could also be seen as a straightforward consequence of the very definition of frequency-lifted signals.

6.4.2 Frequency-lifting in State-Space: the Fourier Reformulation

The Fourier reformulation is obtained by considering the periodic system in an EMP regime. Then, one can express all signals (input, state, output) in the EMP Fourier expansion and in the periodic system matrices in terms of their Fourier-sum coefficients.

Recalling the general EMP Fourier formula (2.15), this amounts to writing

$$\begin{aligned} u(t) &= \sum_{k=0}^{T-1} \bar{u}_k \phi^{kt} z^t \\ x(t) &= \sum_{k=0}^{T-1} \bar{x}_k \phi^{kt} z^t \\ y(t) &= \sum_{k=0}^{T-1} \bar{y}_k \phi^{kt} z^t. \end{aligned}$$

As for the periodic system matrices, the Fourier sum is

$$A(t) = \sum_{k=0}^{T-1} A_k \phi^{kt},$$

and similarly for $B(t)$, $C(t)$ and $D(t)$. Now, exploiting these expansions in the system equations and equating all terms at the same frequency, one obtains the following matrix equation

$$\begin{aligned} z\mathcal{N}\tilde{x} &= \mathcal{A}\tilde{x} + \mathcal{B}\tilde{u} \\ \tilde{y} &= \mathcal{C}\tilde{x} + \mathcal{D}\tilde{u} \end{aligned},$$

where \tilde{x} , \tilde{u} and \tilde{y} are vectors formed with the harmonics of x , u and y respectively, organized in the following fashion:

$$\tilde{x}' = [\vec{x}_0 \ \vec{x}_1 \ \cdots \ \vec{x}_{T-1}]',$$

and similarly for \tilde{u} and \tilde{y} . \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are block Toeplitz matrices formed with the harmonics of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ respectively, as

$$\mathcal{A} = \begin{bmatrix} A_0 & A_{T-1} & A_{T-2} & \cdots & A_1 \\ A_1 & A_0 & A_{T-1} & \cdots & A_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{T-2} & A_{T-3} & A_{T-4} & \cdots & A_{T-1} \\ A_{T-1} & A_{T-2} & A_{T-3} & \cdots & A_0 \end{bmatrix}.$$

and similarly for \mathcal{B} , \mathcal{C} and \mathcal{D} . As for matrix \mathcal{N} , it is the block diagonal matrix

$$\mathcal{N} = \text{blkdiag} \left\{ \phi^k I \right\}, \quad k = 0, 1, \dots, T-1.$$

Then, one can define the *harmonic transfer function* as:

$$\hat{\mathcal{G}}(z) = \mathcal{C}[z\mathcal{N}I - \mathcal{A}]^{-1}\mathcal{B} + \mathcal{D}.$$

Such a function provides a most useful connection between the input harmonics and the output harmonics (organized in the vectors \tilde{u} and \tilde{y} , respectively). In particular, if one takes $z = 1$ (thus considering the truly periodic regimes), this expression enables

one to compute the amplitudes and phases of the harmonics constituting the output signal $y(\cdot)$ in a periodic regime.

Interestingly, the transfer function $\hat{\mathcal{G}}(z)$ coincides with that of the frequency-lifted reformulation $\tilde{W}(z)$. Indeed, write the state equation in terms of the Fourier expansion of the $A(\cdot)$ and $B(\cdot)$ matrices, i.e.,:

$$x(t+1) = \sum_{k=0}^{\infty} A_k \phi^{kt} x(t) + \sum_{k=0}^{\infty} B_k \phi^{kt} u(t).$$

By letting $X(z)$ and $U(z)$ be the z -transforms of $x(\cdot)$ and $u(\cdot)$ respectively, and neglecting the role of the initial state, it then follows:

$$zX(z) = \sum_{k=0}^{\infty} A_k X(z\phi^{-k}) + \sum_{k=0}^{\infty} B_k U(z\phi^{-k}).$$

Proceeding in the same way for the z -transform $Y(z)$ of the output signal, one also obtains:

$$Y(z) = \sum_{k=0}^{\infty} C_k X(z\phi^{-k}) + \sum_{k=0}^{\infty} D_k U(z\phi^{-k}).$$

Therefore, by introducing the frequency-lifted vectors $\tilde{X}(z)$, $\tilde{U}(z)$ and $\tilde{Y}(z)$ (recall Eq. (6.37)), the two equations above can be rewritten as:

$$\begin{aligned} z\mathcal{N}\tilde{X}(z) &= \mathcal{A}\tilde{X}(z) + \mathcal{B}U(z) \\ \tilde{Y}(z) &= \mathcal{C}\tilde{X}(z) + \mathcal{D}\tilde{U}(z) \end{aligned}.$$

Therefore, the harmonic transfer function coincides with the transfer function of the frequency-lifted reformulation, i.e.,

$$\tilde{\mathcal{G}}(z) = \tilde{W}(z).$$

Example 6.3 *The frequency-lifted transfer function relative to our leading examples (Examples 2.3-2.4) can be derived in a number of equivalent ways. For instance, one can start from the lifted transfer function (Example 6.1) and then resort to Expression (6.39) for the computation of $\tilde{W}(z)$. Another possibility is to exploit Eq. (6.40) based on the transfer function operator $G(\sigma, t)$ (Example 2.3). In any case, one obtains*

$$\tilde{W}(z) = \frac{1}{z^2 + 10} \begin{bmatrix} -7.25 + z & 4.75 + 2z \\ 4.75 - 2z & -7.25 - z \end{bmatrix}.$$

If one is interested in the state-space description, then one can consider the Fourier expansion of the system matrices

$$A_0 = \frac{1}{2}(A(0) + A(1)) = -1.5, \quad A_1 = \frac{1}{2}(A(0) - A(1)) = 3.5,$$

and analogously

$$B_0 = -0.5, \quad B_1 = 1.5, \quad C_0 = 1.75, \quad C_1 = -1.25, \quad D_0 = D_1 = 0.$$

Hence \mathcal{D} is the null matrix and

$$\mathcal{A} = \begin{bmatrix} -1.5 & 3.5 \\ 3.5 & -1.5 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} -0.5 & 1.5 \\ 1.5 & -0.5 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1.75 & -1.25 \\ -1.25 & 1.75 \end{bmatrix}.$$

Taking into account that $\mathcal{N} = \text{diag}(1, -1)$ the Fourier expansion is fully defined. It is a matter of a simple computation to show that the associated harmonic transfer function $\tilde{G}(z) = \mathcal{C}(z\mathcal{N} - \mathcal{A})^{-1}\mathcal{D}$ coincides with the frequency transfer function $\tilde{W}(z)$.

6.5 Bibliographical Notes

The time-lifted reformulation goes back to [211], [231] and has been used to deal with many problems, such as the definitions of periodic zeros [83], [169], parametrization of stabilizing controllers [151], optimal control design [122], control performance improvement [208] and so on and so forth. The cyclic reformulation was introduced in [294], [143, 243] and it plays an important role in the model matching [108] and realization problems [214], where it is called twisted reformulation. The frequency-lifted representation has its roots in a long-standing idea in the field, see e.g., [123] as a classical reference, and has been recognized as being particularly useful in the performance analysis [317] and in the computation of system norms, [320]. A comprehensive survey on the time-invariant reformulation is [44].

Chapter 7

State–Space Realization

Generally speaking, a realization problem arises whenever it is necessary to build a state–space model from an input–output representation.

For time-invariant systems, the typical formulation of this problem is the following one: Find a state–space model, namely a quadruplet of matrices (A, B, C, D) , which matches a given (matrix) transfer function $G(z)$, i.e., a quadruplet so that $G(z) = C(zI - A)^{-1}B + D$. The state–space system defined by (A, B, C, D) is said to be a *realization* of $G(z)$. It's a matter of a moment's reflection to recognize that there are infinitely many realizations associated with a given transfer function. For simplicity reasons, the focus is on minimal complexity descriptions. The complexity is naturally characterized in terms of system order, namely the state–space dimension. This leads to the notion of *minimal realization*: A realization is minimal if there does not exist any other realization with a lower dimension. The order of a minimal realization is the so-called *McMillan degree* of the given transfer function. As to the question of whether the minimal realization is unique, the reply is negative. There are infinitely many minimal realizations, that differ from each other for a change of basis in the state–space. In this regard, it is worthy recalling that two time-invariant linear systems are said to be algebraically equivalent if, with a change of basis in the state–space, it is possible to pass from one realization to the other one. Therefore, we can restate the previous conclusion by saying that all minimal realizations are algebraically equivalent to each other. Finally, an important system theoretic characterization of minimality can be given in terms of structural properties: A realization is minimal if and only if it enjoys both the reachability and observability properties.

The minimal realization problem becomes much more involved, and somewhat intriguing, when leaving the time-invariant realm. The main source of trouble is that a time-varying system may be reachable (observable) at a time point and unreachable (unobservable) at another point, see Chap. 4. Correspondingly, in order to preserve the condition that a minimal realization is both reachable and observable, it is necessary to allow the time-variability of the dimension of the realization. This fact is illustrated in the following example, where a periodic system of period $T = 2$ is considered.

Example 7.1 Consider a SISO periodic system of period $T = 2$, with input–output model given by

$$y(t) = \begin{cases} 0 & t \text{ odd} \\ u(t-1) & t \text{ even} \end{cases}.$$

To find a state–space description, we explore here the simplest route. This amounts to elaborating first the given PARMA representation so as to derive the associated lifted transfer functions, one for time $t = 0$ and one for time $t = 1$. Then, once a conjecture is made concerning the order of a state–space realization, the periodic coefficients of such a model can be found by equating the lifted transfer functions derived from the state–space description to the corresponding transfer functions already obtained from the PARMA model.

From the definition of $y(t)$, it is straightforward to see that $y(2k) = u(2k-1)$ and $y(2k+1) = 0$ for any integer k . Hence, recalling that z^{-1} plays the role of one period delay,

$$\begin{aligned} \begin{bmatrix} y(2k) \\ y(2k+1) \end{bmatrix} &= \begin{bmatrix} 0 & z^{-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix} \\ \begin{bmatrix} y(2k+1) \\ y(2k+2) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(2k+1) \\ u(2k+2) \end{bmatrix} \end{aligned}$$

. This implies that the lifted transfer functions are given by

$$W_0(z) = \begin{bmatrix} 0 & z^{-1} \\ 0 & 0 \end{bmatrix}, \quad W_1(z) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider now a 2-periodic state–space model of constant dimension $n = 1$:

$$\begin{aligned} x(t+1) &= a(t)x(t) + b(t)u(t) \\ y(t) &= c(t)x(t) + d(t)u(t). \end{aligned}$$

According to the treatment of Sect. 6.2, one can then find $(F_\tau, G_\tau, H_\tau, E_\tau)$ for $\tau = 0, 1$ of the lifted system, and from this pass to the lifted transfer functions as

$$\begin{aligned} W_\tau(z) &= H_\tau(zI - F_\tau)^{-1}G_\tau + E_\tau \\ &= \frac{1}{z - a(\tau+1)a(\tau)} \begin{bmatrix} c(\tau)a(\tau+1)b(\tau) & c(\tau)b(\tau+1) \\ c(\tau+1)a(\tau)a(\tau+1)b(\tau) & c(\tau+1)a(\tau)b(\tau+1) \end{bmatrix} + \\ &\quad + \begin{bmatrix} d(\tau) & 0 \\ c(\tau+1)b(\tau) & d(\tau+1) \end{bmatrix}. \end{aligned}$$

Since $F_0 = F_1 = a(1)a(0) = 0$, it is apparent that $H_1G_1 = 0$. This means that any 1-dimensional periodic realization of the given input–output model is unreachable and/or unobservable at $t = 1$.

In the example above, the lifted transfer functions $W_0(z)$ and $W_1(z)$ have different McMillan degrees (order of the associated *time-invariant* minimal realizations) $n(0) = 1$ and $n(1) = 0$. If instead one imposes that the periodic realization has a constant dimension, $n(t) = 1$, $t = 0, 1$, then the obtained state–space model is not reachable and observable at any time point. In conclusion, a sensible way of approaching the minimal realization problem of periodic systems is to allow the dimension of the state–space to be periodically time-varying. This leads to an enlargement of the class of periodic systems considered so far. Precisely, one has to deal with systems in state–space whose dynamic matrices are rectangular rather than square. Indeed, if $n(t)$ is the system dimension, with $n(t + T) = n(t)$, then $A(t) \in \mathbb{R}^{n(t+1) \times n(t)}$. In the same way, the input and output matrices may have time-varying dimensions: $B(t) \in \mathbb{R}^{n(t+1) \times m}$ and $C(t) \in \mathbb{R}^{p \times n(t)}$. The fact that the matrices dimensions may vary in time does not have significant consequences on the characterization of the structural properties. For instance, the characterization of reachability at time τ of the pair $(A(\cdot), B(\cdot))$ can be given in terms of reachability of the associated lifted pair (F_τ, G_τ) . On the other hand, such a pair is just a pair of matrices defining an usual time-invariant system. Indeed, from (6.20a), it is apparent that F_τ is square with dimension $n(\tau)$. Correspondingly, in view of (6.20b), matrix G_τ has $n(\tau)$ rows and mT columns. Analogous considerations hold for observability.

7.1 Periodic Realization from Lifted Models

In this section, the periodic realization problem is tackled according to the following rationale. For a system with m inputs and p outputs, a transfer function $W(z)$ of *enlarged* dimension $pT \times mT$ is given. The core of the realization problem is whether there exists a T -periodic system – possibly with time-varying dimensions – whose lifted reformulation at a certain time point h has $W(z)$ as transfer function, see Sect. 6.2. Precisely, for a periodic system defined by the quadruplet $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$, denote as $\Sigma_h(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ the transfer function of its lifted reformulation at time h . The periodic realization problem consists in (i) establishing the existence of a suitable state–space model characterized by the T -periodic matrices $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ so that $\Sigma_h(A(\cdot), B(\cdot), C(\cdot), D(\cdot)) = W(z)$, and, in the affirmative, (ii) finding one such quadruplet.

7.1.1 Minimal, Quasi-minimal, and Uniform Periodic Realizations

As witnessed by Example 7.1, a crucial point for the solution of the realization problem is the conceptualization of the minimality notion. This motivates the following definitions.

Definition 7.1 (i) Given a transfer function $W(z) \in \mathbb{C}^{pT \times mT}$, the quadruplet $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is a T -periodic realization of $W(z)$ if, for an integer h ,

$$\Sigma_h((A(\cdot), B(\cdot), C(\cdot), D(\cdot))) = W(z).$$

(ii) A T -periodic realization of $W(z)$ with dimension $n(\cdot)$ is minimal if $n(t) \leq \tilde{n}(t)$ for all $t \in [0, T-1]$, where $\tilde{n}(\cdot)$ is the dimension of any other T -periodic realization of $W(z)$.

(iii) A T -periodic realization of $W(z)$ with dimension $n(\cdot)$ is quasi-minimal if, for at least one time instant $\bar{t} \in [0, T-1]$, $n(\bar{t}) \leq \tilde{n}(\bar{t})$, where $\tilde{n}(\cdot)$ is the dimension of any other T -periodic realization of $W(z)$.

(iv) A T -periodic realization of $W(z)$ is uniform if $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are matrices with constant dimensions. ■

A characterization of the minimality of any T -periodic realization is easily obtained in terms of structural properties, as outlined in the following lemma.

Lemma 7.1 (i) A T -periodic realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of $W(z)$ is quasi-minimal if and only if it is reachable and observable for at least one time point $t \in [0, T-1]$.

(ii) A T -periodic realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of $W(z)$ is minimal if and only if it is reachable and observable for all $t \in [0, T-1]$.

Proof We limit ourselves to the proof of point (i) since the derivation of (ii) follows the same lines. Assume that there exists a quasi-minimal T -periodic realization of order $n(\cdot)$, and, by contradiction, assume also that it is unreachable or unobservable at each time point. Then, in view of the discussion on the Kalman canonical decomposition introduced in Sect. 4.3, it is possible to work out a T -periodic realization with order $\tilde{n}(t) < n(t)$, $\forall t$. This contradicts the definition of quasi-minimality.

Conversely, assume that a T -periodic realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of order $n(\cdot)$ is reachable and observable at time point \bar{t} . We show that this realization is quasi-minimal by proving that $n(\bar{t}) \leq \tilde{n}(\bar{t})$, where $\tilde{n}(\cdot)$ is the dimension of any other T -periodic realization. Indeed, the time-lifted reformulation $(F_{\bar{t}}, G_{\bar{t}}, H_{\bar{t}}, E_{\bar{t}})$ at time \bar{t} associated with $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ inherits the reachability and observability property (see Remark 6.2) so that it is minimal with order $n(\bar{t})$. If, by contradiction, there would exist another T -periodic realization $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot))$ with order $\tilde{n}(\cdot)$ such that $n(\bar{t}) > \tilde{n}(\bar{t})$, then, its associated time-lifted reformulation at time \bar{t} would be of a lower order.

7.1.2 A Necessary Condition

We now tackle the main issue of the derivation of a T -periodic realization from lifted models. Some preliminary considerations are in order. First, observe that any lifted transfer function has a lower block triangular structure at infinity, see (6.16). Hence, for the periodic realization problem to be solvable, it is necessary that the given

transfer function $W(z)$ satisfies this structural constraint. More precisely, define the class $\Xi(m, p, T)$ of $pT \times mT$ proper rational matrices $W(z)$ such that:

1. $W(z) = \begin{bmatrix} W_{11}(z) & W_{12}(z) & \cdots & W_{1T}(z) \\ W_{21}(z) & W_{22}(z) & \cdots & W_{2T}(z) \\ \vdots & \vdots & \ddots & \vdots \\ W_{T1}(z) & W_{T2}(z) & \cdots & W_{TT}(z) \end{bmatrix}, \quad W_{ij}(z) \in C^{p \times m}.$
2. $W_{ij}(\infty) = 0, i < j.$

Then, the necessary condition for a periodic realization to exist is that $W(z) \in \Xi(m, p, T)$. Interestingly, this is also a sufficient condition for the existence of a minimal and quasi-minimal realization. As for the uniform minimal realization, the existence condition is more complex. These conclusions will be drawn up in the following point, where a periodic realization is derived starting from the Markov parameters. An alternative procedure which requires only orthogonal transformations is provided in the subsequent section.

7.1.3 Periodic Realization from Markov Parameters

In this section, we tackle the realization problem via a set of Hankel matrices of Markov parameters. To this purpose, first define the rational functions $W_h(z)$, $h \in [0, T-1]$, recursively deduced from $W(z)$ as follows

$$W_{h+1}(z) = \Delta'_p(z^{-1})W_h(z)\Delta_m(z), \quad W_0(z) = W(z). \quad (7.1)$$

Recall that the inner function $\Delta_r(z)$ was already defined in (6.11). Comparing this expression with formula (6.12), it is apparent that if $W(z) \in \Xi(m, p, T)$, then also $W_h(z) \in \Xi(m, p, T), \forall h$. Each $W_h(z)$ is a transfer function of a causal time-invariant finite-dimensional system. We will denote by ρ_h the degree of the least common multiple of all denominators of $W_h(z)$.

Remark 7.1 *The integer ρ_h may vary with respect to h ; however, its variations are limited by the condition:*

$$|\rho_h - \rho_k| \leq 1, \quad \forall h, k. \quad (7.2)$$

Such an inequality can be easily verified by inspecting the denominators of the rational functions appearing in the formula below:

$$\begin{bmatrix} 0 & z^{-1}I_p \\ I_{p(T-1)} & 0 \end{bmatrix} W_{h+1}(z) = W_h(z) \begin{bmatrix} 0 & z^{-1}I_m \\ I_{m(T-1)} & 0 \end{bmatrix}.$$

This formula can be easily deduced from Eq. (7.1). ■

A well-known result of the theory of time-invariant linear systems ensures that ρ_h is an upper bound for the order of the minimal realization of the transfer function

$W_h(z)$. In order to move towards periodic systems, we need to introduce the expansion in power series of $W_h(z)$ as follows:

$$W_h(z) = W_h(\infty) + \sum_{i=1}^{\infty} J_h(i) z^{-i},$$

where the matrices $J_h(q)$, $q \in \mathbb{Z}^+$, are the associated Markov parameters. Finally, define the Hankel matrices $S_h(\gamma)$ of order γ associated with $W_h(z)$, i.e.,

$$S_h(\gamma) = \begin{bmatrix} J_h(1) & J_h(2) & \cdots & J_h(\gamma) \\ J_h(2) & J_h(3) & \cdots & J_h(\gamma+1) \\ \vdots & \vdots & \ddots & \vdots \\ J_h(\gamma) & J_h(\gamma+1) & \cdots & J_h(2\gamma-1) \end{bmatrix}. \quad (7.3)$$

As is well-known, the rank of $S_h(\infty)$ coincides with the order of the minimal realization associated with $W_h(z)$. Obviously, such an order is not greater than ρ_h . Thanks to Inequality (7.2) we can state that the order of any Hankel matrix $S_h(\gamma)$, $h = 0, 1, \dots, T-1$, cannot increase for $\gamma \geq \rho_0 + 1$. In other words,

$$\text{rank}[S_h(\gamma)] = \text{rank}[S_h(\rho_0 + 1)], \quad \forall \gamma \geq \rho_0 + 1. \quad (7.4)$$

Now, we are in the position to precisely state the main result of periodic realization.

Proposition 7.1 *Consider a multivariable rational matrix $W(z)$ of dimension $pT \times mT$.*

- (i) *There exists a periodic minimal realization of $W(z)$ if and only if $W(z) \in \Xi(m, p, T)$.*
- (ii) *There exists a periodic quasi-minimal uniform realization of $W(z)$ if and only if $W(z) \in \Xi(m, p, T)$.*
- (iii) *There exists a periodic uniform minimal realization of $W(z)$ if and only if $W(z) \in \Xi(m, p, T)$ and the Hankel matrix $S_h(\rho_0 + 1)$ has constant rank for $h = 0, 1, \dots, T-1$.*

Proof The necessity of the condition $W(z) \in \Xi(p, m, T)$ appearing at points (i)–(iii) has been already discussed. Hence, we have to prove the sufficiency conditions relative to points (i)–(iii) and the necessity of the rank condition for point (iii).

First, we introduce a partition of the Markov parameters $J_h(q) \in \mathbb{R}^{pT \times mT}$ associated with the various transfer functions $W_h(z)$ defined in (7.1). Precisely, we partition any $J_h(q)$ into blocks of dimensions $p \times m$, which will be indicated with the symbol $(J_h(q))_{i,j}$, $i, j = 0, 1, \dots, T-1$. In view of (7.1), these blocks are related in the following way:

$$(J_{h+1}(q))_{i,j} = \begin{cases} (J_h(q))_{i+1,j+1} & i < T, \quad j < T \\ (J_h(q-1))_{1,j+1} & i = T, \quad j < T \\ (J_h(q+1))_{i+1,1} & i < T, \quad j = T \\ (J_h(q))_{1,1} & i = T, \quad j = T. \end{cases} \quad (7.5)$$

Sufficiency of point (i). Associated with $W(z)$, introduce the double infinite block matrix $\Theta_W := [\theta_{ki}]_{k,i=-\infty}^{+\infty}$ by defining $\theta_{ki} \in \mathbb{R}^{p \times m}$ as

$$\theta_{h+rT+j, h+sT+\ell} := \begin{cases} (J_h(r-s))_{j+1, \ell+1} & r \geq s \\ 0 & r < s \end{cases} \quad (7.6)$$

for $h, j, \ell = 0, 1, \dots, T-1$, $r, s \in \mathbb{Z}$. It is easy to see that the various blocks have the property $\theta_{k+T, i+T} = \theta_{ki}$ with $k, i \in \mathbb{Z}$. Moreover, if $W(z) \in \Xi(p, m, T)$, matrix Θ_W is a block lower triangular, i.e., $\theta_{ki} = 0$, $i > k$, $i, k \in \mathbb{Z}$.

Now, denote by H_W^h the following infinite-dimensional matrices

$$H_W^h := [\theta_{h+i-1, h-j}]_{i,j=1}^{+\infty}, \quad h = 0, 1, \dots, T-1.$$

Obviously,

$$H_W^{k+T} = H_W^k, \quad k \in \mathbb{Z}.$$

Taking into account the structure of the Hankel matrices $S_h(\gamma)$ of $W_h(z)$, defined by (7.3), and the definition (7.6) of the matrices θ_{ki} , it is easy to see that

$$H_W^h = S_h(\gamma) \text{diag}\{\underbrace{M, \dots, M}_{\gamma \text{ blocks}}\}, \quad \gamma = \infty, \quad h = 0, 1, \dots, T-1,$$

where $M = \{M_{ij} \in \mathbb{R}^{m \times m}, i, j = 1, \dots, T\}$, with $M_{i, T+1-i} = I_m$, and $M_{ij} := 0$, $j \neq T+1-i$, $i = 1, \dots, T$. In view of (7.4), the above relation implies that

$$\text{rank } H_W^h = \text{rank } S_h(\rho_0 + 1\gamma) = \eta_h, \quad h = 0, 1, \dots, T-1. \quad (7.7)$$

In the sequel we pose $\eta_h = \text{rank } H_W^h$ and $\bar{\gamma} = \rho_0 + 1$. The construction of a minimal periodic realization is achieved by making use of the following matrices:

$$T_W^h(\bar{\gamma}T) := [\theta_{h+i-1, h-j}]_{i,j=1}^{\bar{\gamma}T}, \quad \hat{T}_W^h(\bar{\gamma}T) := [\theta_{h+i, h-j}]_{i,j=1}^{\bar{\gamma}T}, \quad h = 0, 1, \dots, T-1,$$

which can be expressed in terms of the Hankel matrices $S_h(\gamma)$ of $W_h(z)$ by the following identities, which hold for $h = 0, 1, \dots, T-1$:

$$T_W^h(\bar{\gamma}T) = S_h(\bar{\gamma}) \text{diag}\{\underbrace{M, \dots, M}_{\bar{\gamma} \text{ blocks}}\} \quad (7.8)$$

$$\hat{T}_W^h(\bar{\gamma}T) = \begin{bmatrix} 0_p & I_{pT\bar{\gamma}} & 0_{p(T-1)} \end{bmatrix} S_h(\bar{\gamma}+1) \text{diag}\{\underbrace{M, \dots, M}_{\bar{\gamma}+1 \text{ blocks}}\} \begin{bmatrix} I_{mT\bar{\gamma}} \\ 0_{mT} \end{bmatrix}, \quad (7.9)$$

where 0_n denotes a null matrix with n columns and a suitable number of rows.

In view of (7.7) and (7.8), for each $h = 0, 1, \dots, T-1$, the rank of $T_W^h(\bar{\gamma}T)$ coincides with the rank η_h of H_W^h . Then, there exist two matrices $U_h \in \mathbb{R}^{pT\bar{\gamma} \times pT\bar{\gamma}}$ and $V_h \in \mathbb{R}^{mT\bar{\gamma} \times mT\bar{\gamma}}$ such that

$$U_h T_W^h(\bar{\gamma}T) V_h = \begin{bmatrix} I_{\eta_h} & 0 \\ 0 & 0 \end{bmatrix}, \quad h = 0, 1, \dots, T-1. \quad (7.10)$$

Also let $U_T = U_0$, $T_W^T(\bar{\gamma}T) = T_W^0(\bar{\gamma}T)$ and $\eta_T = \eta_0$. Now, denote with L_a^b the $b \times a$ matrix defined as follows: $L_a^b := [I_b \ 0]$ if $b < a$, $L_a^b := I_b$ if $b = a$, $L_a^b := [I_a \ 0]^T$ if $b > a$. A T -periodic minimal realization of $W(z)$ is given by the following T -periodic matrices:

$$A(h) = L_{pT\bar{\gamma}}^{\eta_{h+1}} U_{h+1} \hat{T}_W^h(\bar{\gamma}T) V_h L_{\eta_h}^{mT\bar{\gamma}} \in \mathbb{R}^{\eta_{h+1} \times \eta_h}, \quad (7.11a)$$

$$B(h) = L_{pT\bar{\gamma}}^{\eta_{h+1}} U_{h+1} T_W^{h+1}(\bar{\gamma}T) L_m^{mT\bar{\gamma}} \in \mathbb{R}^{\eta_{h+1} \times m} \quad (7.11b)$$

$$C(h) = L_{pT\bar{\gamma}}^p \bar{T}_W^h(\bar{\gamma}T) V_h L_{\eta_h}^{mT\bar{\gamma}} \in \mathbb{R}^{p \times \eta_h}, \quad (7.11c)$$

$$D(h) = (J_h(0))_{11} \in \mathbb{R}^{p \times m}. \quad (7.11d)$$

Sufficiency of point (ii). Following part (i), take a T -periodic minimal realization $A(h) \in \mathbb{R}^{\eta_{h+1} \times \eta_h}$, $B(h) \in \mathbb{R}^{\eta_{h+1} \times m}$, $C(h) \in \mathbb{R}^{p \times \eta_h}$ and $D(h) \in \mathbb{R}^{p \times m}$, $h = 0, 1, \dots, T-1$, where $\eta_h = \text{rank } S_h(\bar{\gamma})$, $h = 0, 1, \dots, T-1$, $\eta_T := \eta_0$, and $\bar{\gamma} = \rho_0 + 1$. Denote with $\bar{\eta} := \max_{i \in [0, T-1]} \text{rank } S_h(\bar{\gamma})$, and define T -periodic matrices $A(h) \in \mathbb{R}^{\bar{\eta} \times \bar{\eta}}$, $B(h) \in \mathbb{R}^{\bar{\eta} \times m}$, $C(h) \in \mathbb{R}^{p \times \bar{\eta}}$, and $D(h) \in \mathbb{R}^{p \times m}$, in the following way:

$$\begin{aligned} A(h) &:= \text{diag}\{A(h), 0\}, & B(h) &:= [B(h)^T \ 0]^T, \quad \forall h \in Z \\ C(h) &:= [C(h) \ 0], & D(h) &:= D(h), \quad \forall h \in Z, \end{aligned}$$

where the zero blocks can vanish in some time instant. The T -periodic quadruplet $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ defines a T -periodic system that is completely reachable and observable in \bar{t} , where \bar{t} is such that $\text{rank } S_{\bar{t}}(\bar{\gamma}) = \bar{\eta}$. This realization is obviously quasi-minimal and uniform.

Point (iii). If $\text{rank } \eta_h$ of $S_h(\bar{\gamma})$ is constant for $h = 0, 1, \dots, T-1$, the derivation performed in part (i) can be applied, thus yielding a uniform realization. To complete the necessity of part (iii) suppose that there exists a uniform T -periodic minimal realization of $W(z)$, whose order is n . We can associate with this realization the reachable and observable lifted reformulations Σ_h , $h = 0, 1, \dots, T-1$, whose transfer matrices are $W_h(z)$, $h = 0, 1, \dots, T-1$. Each Σ_h is a minimal realization of order n , which equals the rank of the Hankel matrix $S_h(\bar{\gamma})$, for $h = 0, 1, \dots, T-1$. ■

From the above theorem, it is apparent that the rank of the Hankel matrix plays a main role. Indeed, we can say that the order $n(\cdot)$ of the minimal realization is

$$n(h) = \text{rank}[S_h(\rho_0 + 1)].$$

Hence, $n(h)$ coincides with the McMillan degree of the transfer function $W_h(z)$. Then, point (iii) of the theorem equivalently reads as follows: There exists a uniform minimal periodic realization if and only if $W(z) \in \Xi(m, p, T)$ and the McMillan degree of $W_h(z)$ is time-invariant. The possible non-existence of a uniform *minimal* periodic realization is connected with the fact that the dimension of the reach-

able/observable part of a periodic system may be time-varying, recall also Lemma 7.1.

It is important to point out that a uniform *quasi-minimal* periodic realization can always be obtained by complementing a minimal realization (which always exists provided that the basic condition $W(z) \in \Xi(m, p, T)$ is met) with suitable unreachable and/or unobservable parts. Then, the minimal order of a uniform quasi-minimal periodic realization is $\bar{n} = \max\{n(t)\}$, $t = 0, 1, \dots, T-1$. The complementing parts can be simply constructed by inserting extra poles in the origin. In this way, the minimal realization being reachable and observable at any time point, one obtains a periodic system which is controllable and reconstructable. As for reachability and observability, this still holds at a certain time-point at least. This discussion can be formalized as follows.

Proposition 7.2 *There exists a uniform quasi-minimal periodic realization that is controllable and reconstructable if and only if $W(z) \in \Xi(m, p, T)$.*

7.1.4 Algorithm for the Derivation of a Minimal Periodic Realization

The proof of the main theorem 7.1 is of a constructive type and it leads to a procedure which is now elaborated stepwise. Assume that the given transfer function $W(z)$ belongs to the feasible class $\Xi(m, p, T)$. Then a minimal periodic realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ can be obtained as follows:

Step 1

Compute $\rho_0 = \text{degree of the least common multiple of all denominators of the entries of } W(z)$. Let $\bar{\gamma} = \rho_0 + 1$ and $i = 0$.

Step 2

Compute matrices $T_W^h(\bar{\gamma}T)$, $\hat{T}_W^h(\bar{\gamma}T)$ according to (7.8), (7.9), (7.3) and relationships provided in (7.5) (if $i = T-1$ let $T_W^T(\bar{\gamma}T) = T_W^0(\bar{\gamma}T)$ and $\hat{\mathcal{T}}_W^T(\bar{\gamma}T) = \hat{T}_W^0(\bar{\gamma}T)$).

Step 3

Compute U_{i+1} and V_i according to (7.10) (if $i = T-1$, let $U_T = U_0$ and $\eta_T = \eta_0$).

Step 4

Compute $A(i) \in \mathbb{R}^{\eta_{i+1} \times \eta_i}$, $B(i) \in \mathbb{R}^{\eta_{i+1} \times m}$, $C(i) \in \mathbb{R}^{p \times \eta_i}$ and $D(i) \in \mathbb{R}^{p \times m}$, according to (7.11), where η_i is the rank of $\mathcal{S}_i(\bar{\gamma})$.

Step 5

Let $i = i + 1$. If $i = T$ then the procedure is completed, otherwise go to step 2.

Example 7.2 *The transfer function*

$$W(z) = \begin{bmatrix} z^{-1} & z^{-2} \\ 1 & z^{-1} \end{bmatrix}$$

has the property that it is lower triangular as z tends to infinity and therefore belongs to the feasible class $\Xi(1, 1, 2)$. Moreover, following (7.1), let $W_0(z) = W(z)$ and

$$W_1(z) = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} W_0(z) \begin{bmatrix} 0 & z^{-1} \\ 1 & 0 \end{bmatrix} = z^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Notice that $W_0(z)$ has McMillan degree equal to 2 whereas the McMillan degree of $W_1(z)$ is 1. Therefore, any minimal periodic realization has order $n(t)$ with $n(0) = 2$ and $n(1) = 1$ and, consequently, it is not uniform. The procedure above leads to the minimal realization as

$$\begin{aligned} A(0) &= \begin{bmatrix} 0 & 1 \end{bmatrix}, & A(1) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ B(0) &= 1, & B(1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & C(1) &= 1 \\ D(0) &= 0, & D(1) &= 0. \end{aligned}$$

This periodic system is reachable and observable for each t . From this system, one can pass to a uniform quasi-minimal realization by inserting slack zeros in the matrices:

$$\begin{aligned} A(0) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & A(1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ B(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & B(1) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & C(1) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\ D(0) &= 0, & D(1) &= 0. \end{aligned}$$

The obtained periodic system is controllable and reconstructable. However, this system is neither reachable nor observable at $t = 1$.

7.1.5 Periodic Realization via Orthogonal Transformations

The main idea behind this alternative procedure is to find the quadruplet $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ of a minimal periodic realization from a minimal (time-invariant) realization of the lifted transfer function at a given time point, say $\tau = 1$. Hence, consider the given lifted transfer function at time $\tau = 1$, i.e $W_1(z)$, and let a minimal realization of $W_1(z)$ be given by the quadruplet

$$F \in \mathbb{R}^{n_1 \times n_1}, \quad G \in \mathbb{R}^{n_1 \times mT}, \quad H \in \mathbb{R}^{pT \times n_1}, \quad E \in \mathbb{R}^{pT \times mT},$$

where n_1 be the McMillan degree of such a realization. Matrices G, H, E can be partitioned as follows

$$G = [G_1 \ G_2 \ \cdots \ G_T], \quad H = \begin{bmatrix} H_1 \\ H_2 \\ \vdots \\ H_T \end{bmatrix}, \quad E = \begin{bmatrix} E_{11} & 0 & \cdots & 0 \\ E_{2,1} & E_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{T,1} & E_{T,2} & \cdots & E_{T,T} \end{bmatrix}.$$

The upper block-triangularity of matrix E is inherited by the necessary condition for the existence of a periodic realization.

Recalling the structure of the lifted representation of a periodic system, (6.20), it is easy to conclude that

$$C(1) = H_1, \quad B(T) = G_T, \quad D(i) = E_{i,i}, \quad i = 1, 2, \dots, T. \quad (7.12)$$

In view of the above, the derivation of a minimal periodic realization is completed by specifying $A(i)$, $i = 1, 2, \dots, T$, $B(i)$, $i = 1, 2, \dots, T-1$ and $C(i)$, $i = 2, 3, \dots, T$. For the computation of these unknowns, consider, for $i = 1, 2, \dots, T-1$, the matrices of dimensions $(n_1 + (T-i)p) \times (n_1 + mi)$

$$K_i = \begin{bmatrix} F & G_1 & G_2 & \cdots & G_i \\ H_T & E_{T,1} & E_{T,2} & \cdots & E_{T,i} \\ H_{T-1} & E_{T-1,1} & E_{T-1,2} & \cdots & E_{T-1,i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{i+1} & E_{i+1,1} & E_{i+1,2} & \cdots & E_{i+1,i} \end{bmatrix}.$$

The rationale behind the realization algorithm can be explained by noticing that the above matrices admit the following factorizations:

$$K_i = K_i^+ K_i^-,$$

where

$$K_i^+ = \begin{bmatrix} \Phi_A(T+1, i+1) \\ C(T)\Phi_A(T, i+1) \\ C(T-1)\Phi_A(T-1, i+1) \\ \vdots \\ C(i+1) \end{bmatrix}$$

depends upon matrices $A(t)$ and $C(t)$ for $t = i+1, i+2, \dots, T$, whereas

$$K_i^- = [\Phi_A(i+1, 1) \ \Phi_A(i+1, 2)B(1) \ \Phi_A(i+1, 3)B(2) \ \cdots \ B(i)]$$

depends upon matrices $A(t)$ and $B(t)$ for $t = 1, 2, \dots, i$. Notice that, if matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ define a realization of dimension $n(\cdot)$, then K_i^+ has $n(i+1)$ columns and K_i^- has $n(i+1)$ rows. A minimal realization is such that these two matrices have minimum rank. This is the basic observation for working out the dimension of the

minimal realization. Secondly, observe that the last row block of K_i^+ supplies matrix $C(i+1)$ while the last column block of K_i^- is just $B(i)$. As for matrix $A(\cdot)$ note that $A(1)$ is given by the first column block of K_1^- . As for the remaining matrices $A(i)$, $i = 2, 3, \dots, T$, they can be worked out on the basis of the following recursion

$$\begin{aligned} K_{T-1}^+ &= \begin{bmatrix} A(T) \\ C(T) \end{bmatrix} \\ K_{T-2}^+ &= \begin{bmatrix} K_{T-1}^+ A(T-1) \\ C(T-1) \end{bmatrix} \\ K_{T-3}^+ &= \begin{bmatrix} K_{T-2}^+ A(T-2) \\ C(T-2) \end{bmatrix} \\ &\vdots \\ K_1^+ &= \begin{bmatrix} K_2^+ A(2) \\ C(2) \end{bmatrix}. \end{aligned}$$

At this point it is advisable to define matrix $K_i^{+(u)}$ as the sub-matrix of K_i^+ obtained by deleting the last p rows. In this way, it follows:

$$A(T) = K_{T-1}^{+(u)} \quad (7.13a)$$

$$K_{T-1}^+ A(T-1) = K_{T-2}^{+(u)} \quad (7.13b)$$

$$K_{T-2}^+ A(T-2) = K_{T-3}^{+(u)} \quad (7.13c)$$

$$\vdots \quad (7.13d)$$

$$K_2^+ A(2) = K_2^{+(u)}. \quad (7.13e)$$

The above formulas allow one to work out $A(2), A(3), \dots, A(T)$ from the known matrices K_i^+ , $i = 1, 2, \dots, T-1$ if obvious rank conditions hold.

Now that the rationale for the construction of a minimal realization has been explained, consider again the known matrices K_i and compute a full rank orthogonal factorization as:

$$K_i = K_i^+ K_i^-,$$

where K_i^+ has $n(i+1)$ orthogonal columns ($(K_i^+)' K_i^+ = I_{n(i+1)}$) and K_i^- has full row rank $n(i+1)$. Again, observe that $C(1) = H_1$, $B(T) = G_T$, $D(i) = E_{i,i}$ and $A(1)$ is given by the first column block of K_1^- . Moreover, the last row block of K_i^+ supplies matrix $C(i+1)$ while the last column block of K_i^- is just $B(i)$. Finally, letting $K_i^{+(u)}$ be the sub-matrix of K_i^+ obtained by deleting the last p rows, equations (7.13) are obtained. Provided that $\text{Im}[K_{i-1}^{+(u)}] \subseteq \text{Im}[K_i^+]$, it follows

$$A(i) = (K_i^+)' K_{i-1}^{+(u)}, \quad i = 2, 3, \dots, T.$$

To show that $Im[K_{i-1}^{+(u)}] \subseteq Im[K_i^+]$, notice that

$$K_i = [X_i \star], \quad K_{i-1} = \begin{bmatrix} X_i \\ \star \end{bmatrix}, \quad K_{i-1}^+ = \begin{bmatrix} K_{i-1}^{+(u)} \\ \star \end{bmatrix}.$$

Hence, $Im[K_{i-1}^{+(u)}] = Im[X_i] \subseteq Im[K_i] = Im[K_i^+]$. This concludes the construction of the realization. The proof that such a realization is indeed minimal (reachable and observable for each time point) is left to the reader.

7.2 Realization from a Fractional Representation

The starting point of the realization procedures seen above is a transfer function of lifted type. We now consider the case when the input–output model is given through the system periodic transfer function in a fractional representation. We limit our attention to the single-input single-output case. At first we deal with the right fractional representation, namely suppose that

$$y(t) = n(\sigma, t) d(\sigma, t)^{-1} \cdot u(t),$$

where $n(\sigma, t)$ and $d(\sigma, t)$ are the periodic polynomials

$$\begin{aligned} d(\sigma, t) &= \sigma^n + \alpha_1(t) \sigma^{n-1} + \alpha_2(t) \sigma^{n-2} + \cdots + \alpha_n(t) \\ n(\sigma, t) &= \gamma_0(t) \sigma^n + \gamma_1(t) \sigma^{n-1} + \gamma_2(t) \sigma^{n-2} + \cdots + \gamma_n(t). \end{aligned}$$

For simplicity reasons, we will initially set $\gamma_0(t) = 0, \forall t$. In such a case, the fractional representation can equivalently be written by defining the latent variable $z(\cdot)$ as follows

$$\begin{aligned} y(t) &= n(\sigma, t) \cdot z(t) \\ u(t) &= d(\sigma, t) \cdot z(t). \end{aligned}$$

This equation can be easily transformed into a state–space form by resorting to the so-called *reachable canonical form*

$$\begin{aligned} A(t) &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n(t) & -\alpha_{n-1}(t) & -\alpha_{n-2}(t) & \cdots & -\alpha_1(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}' \\ C(t) &= [\gamma_n(t) \ \gamma_{n-1}(t) \ \cdots \ \gamma_2(t) \ \gamma_1(t)], \quad D(t) = 0, \end{aligned}$$

where, as easily checkable, the variable $z(t)$ coincides with the first state variable $x_1(t)$.

The system in the canonical reachable form defined above is, by construction, a fully reachable periodic system (see Sect. 4.1). In addition, thanks to the structure of matrices $(A(\cdot), B(\cdot))$, the reachability interval is not longer than the system order n . Notice that this property does not hold in general since, as seen in Chap. 4, a reachable periodic system is reachable at most in nT steps.

Now, we want to check the observability properties of the system, which, as we will prove in the theorem below, depends on the coprimeness properties of the two polynomials $n(\sigma, t)$ and $d(\sigma, t)$. To this end, we take advantage of the coprimeness properties discussed in Chap. 5, and, in particular, of the concept of characteristic periodic difference equation introduced therein.

Proposition 7.3 *The system in the reachable canonical form is observable for each t if and only if the two polynomials $(d(\sigma, t), n(\sigma, t))$ are strongly right coprime. It is reconstructable if and only if the finite singularities of every common right divisors of the pair of periodic polynomials $(n(\sigma, t), d(\sigma, t))$ are all located at zero¹.*

Proof If the system is not observable, then

$$\begin{bmatrix} \lambda \sigma I - A(t) \\ C(t) \end{bmatrix} \cdot \bar{p}(t) = 0, \quad (7.14)$$

for some complex number λ and non-zero T -periodic function $\bar{p}(t)$. Hence, taking into account the structure of $A(\cdot)$ and $C(\cdot)$ it follows that

$$n(\lambda \sigma, t) \cdot \bar{p}_1(t) = 0, \quad d(\lambda \sigma, t) \cdot \bar{p}_1(t) = 0, \quad (7.15)$$

where $\bar{p}_1(\cdot)$ is the first entry of $\bar{p}(\cdot)$. Notice that $\bar{p}_1(\cdot)$ cannot be identically zero, otherwise the vector $\bar{p}(\cdot)$ would be zero for each t . Letting $p(t) = \lambda^t \bar{p}(t)$, it follows that

$$n(\sigma, t) \cdot \bar{p}_1(t) = 0, \quad d(\sigma, t) \cdot \bar{p}_1(t) = 0,$$

so that the Bezout identity cannot be satisfied by any $x(\sigma, t)$ and $y(\sigma, t)$, i.e., the two polynomials $n(\sigma, t)$, $d(\sigma, t)$ are not strongly right coprime.

Vice versa, if the polynomials are not strongly right coprime, there exists an exponentially modulated non-zero function $p_1(t)$ such that (7.15) is satisfied. By taking $p_{i+1}(t) = \lambda p_i(t+1)$, $i = 1, 2, \dots, n-1$ the conclusion (7.14) follows, so that the system is not observable for each t .

Assume now that the two polynomials are such that

$$\begin{aligned} n(\sigma, t) &= \tilde{n}(\sigma, t)k(\sigma, t) \\ d(\sigma, t) &= \tilde{d}(\sigma, t)k(\sigma, t), \end{aligned}$$

where the polynomial $k(\sigma, t)$ is not a Z -polynomial, i.e., not all its roots are located at the origin of the complex plane. For a root $\lambda \neq 0$, consider the T -periodic function

¹ This is equivalent to requiring that every right divisor is a Z -polynomial, see point (vii) of Sect. 5.2.

$\bar{p}_1(\cdot)$ such that

$$k(\lambda\sigma, t)\bar{p}_1(t) = 0.$$

Hence,

$$n(\lambda\sigma, t) \cdot \bar{p}_1(t) = 0$$

$$d(\lambda\sigma, t) \cdot \bar{p}_1(t) = 0.$$

Taking $\bar{p}_{i+1}(t) = \lambda p_i(t+1)$, condition (7.14) follows. Since $\lambda \neq 0$, the conclusion can be drawn that the system is not reconstructable.

Vice-versa, if the system is not reconstructable, Eq. (7.14) holds for some $\lambda \neq 0$, from which (7.15) follows. Hence, the two polynomials share the same factor $k(\sigma, t)$ with the same root $\lambda^T \neq 0$, i.e., it is possible to write

$$n(\sigma, t) = \tilde{n}(\sigma, t)k(\sigma, t)$$

$$d(\sigma, t) = \tilde{d}(\sigma, t)k(\sigma, t).$$

Consequently, the two polynomials admit the same right divisor $k(\sigma, t)$ and such a divisor is not a Z-polynomial.

Remark 7.2 *It is not difficult to show that, when $\gamma_0(t) \neq 0$, the reachable canonical realization requires only the modification of matrices $C(\cdot)$ and $D(\cdot)$. Precisely,*

$$D(t) = \gamma_0(t),$$

and

$$C(t) = [\tilde{\gamma}_n(t) \tilde{\gamma}_{n-1}(t) \tilde{\gamma}_{n-2}(t) \cdots \tilde{\gamma}_1(t)],$$

where

$$\tilde{\gamma}_i(t) = \gamma_i(t) - \alpha_i(t)\gamma_0(t), \quad i = 1, 2, \dots, n$$

■

Secondly, we consider the left fractional representation. Precisely, we assume that the input–output model is given via the left fractional representation

$$y(t) = d(\sigma, t)^{-1}n(\sigma, t) \cdot u(t),$$

where $n(\sigma, t)$ and $d(\sigma, t)$ are the periodic polynomials

$$d(\sigma, t) = \sigma^n + \sigma^{n-1}a_1(t) + \sigma^{n-2}a_2(t) + \cdots a_n(t)$$

$$n(\sigma, t) = b_0(t)\sigma^n + \sigma^{n-1}b_1(t) + \sigma^{n-2}b_2(t) + \cdots b_n(t).$$

As is easily seen, the time-domain model associated with the left fractional representation is the PARMA model (Sect. 2.2)

$$\begin{aligned} y(t) = & -a_1(t-1)y(t-1) - a_2(t-2)y(t-2) - \cdots - a_n(t-n)y(t-n) + \\ & + b_0(t)u(t) + b_1(t-1)u(t-1) + \cdots + b_n(t-n)u(t-n). \end{aligned} \quad (7.16)$$

In order to find a state–space realization, one can again resort to the procedure in Sect. 7.1 by first deriving the lifted version of the given periodic transfer function as explained in Sect. 6.2.2. Alternatively, one can work out the state–space system from the coefficients of the PARMA model directly, as done for the right fractional representation. Here, we will consider this direct passage.

We will initially set $b_0(t) = 0, \forall t$. In such a case, a straightforward way to construct a periodic realization is to resort to the following state–space representation

$$A(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n(t) \\ 1 & 0 & \cdots & 0 & -a_{n-1}(t) \\ 0 & 1 & \cdots & 0 & -a_{n-2}(t) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1(t) \end{bmatrix} \quad B(t) = \begin{bmatrix} b_n(t) \\ b_{n-1}(t) \\ b_{n-2}(t) \\ \vdots \\ b_1(t) \end{bmatrix} \quad (7.17)$$

$$C(t) = [0 \ 0 \ \cdots \ 0 \ 1] \quad D(t) = 0.$$

We leave to the reader the task of checking that from this state–space system Eq. (7.16) is recovered. The representation (7.17) is the celebrated *observable canonical form*.

A distinctive feature of this form (7.17) is that it is observable at each time-point. Moreover, observability can be performed at the maximum in n steps at any time point (see Sect. 4.2).

When $b_0(t) \neq 0$, the observer canonical realization requires the modification of only matrices $B(\cdot)$ and $D(\cdot)$. Precisely,

$$D(t) = b_0(t),$$

and

$$B(t) = \begin{bmatrix} \tilde{b}_n(t) \\ \tilde{b}_{n-1}(t) \\ \tilde{b}_{n-2}(t) \\ \vdots \\ \tilde{b}_1(t) \end{bmatrix}.$$

where

$$\tilde{b}_i(t) = b_i(t) - a_i(t)b_0(t), \quad i = 1, 2, \dots, n.$$

Example 7.3 Consider the following input–output representation

$$y(t+1) = a(t)y(t) + b(t)u(t),$$

where the two coefficients are periodic of period $T = 2$. If we define $x(t) = y(t)$, then the obtained system is obviously in the observable canonical form. With such a choice, $A(t) = a(t)$, $B(t) = b(t)$, $C(t) = 1$. The corresponding state–space system

is observable for each t since $H_t = [1 \ a(t)]^t$ has rank 1 for each t . As for reachability, one has to consider matrix $G_t = [a(t+1)b(t) \ b(t+1)]$. If $\text{rank } [G_t] = 1$, then the system is reachable at time t . For instance, the system with $a(1) = b(1) = 1$, and $a(0) = b(0) = 0$ is reachable for $t = 0$ and unreachable for $t = 1$. The system is reachable for each t if and only if $(a(t)b(t-1), b(t)) \neq (0, 0)$ for every t . One can also verify that the system is not controllable for each t if and only if both the following conditions hold: i) $b(\cdot)$ is identically zero and ii) $a(t) \neq 0$ for each t . ■

In the same way, as previously discussed for the observability property of the system in the reachable canonical form, the study of the reachability property of the observable canonical form (7.17) can be carried out by considering the coprimeness of the polynomial pair $(n(\sigma, t), d(\sigma, t))$. Taking advantage of the coprimeness properties discussed in Chap. 5, the following result can be established. The proof is here omitted since the result is the dual of the previous one established in Proposition 7.3.

Proposition 7.4 *System (7.17) is reachable for each t if and only if the pair of periodic polynomials $(n(\sigma, t), d(\sigma, t))$ is strongly left coprime. It is controllable if and only if the finite singularities of every common left divisors of the pair of periodic polynomials $(n(\sigma, t), d(\sigma, t))$ are all located at zero (every left divisor is a Z -polynomial).*

Example 7.4 *In order to illustrate the result in Proposition 7.4, consider again the system of Example (7.3). The corresponding Bezout identity is*

$$(\sigma - a(t))x(\sigma, t) + b(t)y(\sigma, t) = 1.$$

A solution of this Eq. (if any) has the form

$$x(\sigma, t) = x_0(t), \quad y(\sigma, t) = y_0(t) + y_1(t)\sigma.$$

Therefore a solution exists if and only if

$$x_0(t+1) + b(t)y_1(t) = 0, \quad -a(t)x_0(t) + b(t)y_0(t) = 1,$$

i.e.,

$$a(t)b(t-1)y_1(t-1) + b(t)y_0(t) = 1.$$

This happens if and only if $(a(t)b(t-1), b(t)) \neq (0, 0)$ for every t . This is again the condition for reachability at any time point.

Turning to controllability, one has to be sure that any left common divisor $x(\sigma, t)$ of $\sigma - a(t)$ and $b(t)$ has singularities in the origin. This means that controllability holds if and only if any $x(\sigma, t)$ satisfying the two polynomial equations

$$\sigma - a(t) = x(\sigma, t)\alpha(\sigma, t), \quad b(t) = x(\sigma, t)\beta(\sigma, t),$$

for some periodic polynomials $\alpha(\sigma, t)$ and $\beta(\sigma, t)$, is such that its lifted expression $X(\sigma^T, t)$ has its determinant of the type $c\sigma^{2k}$, where $c \neq 0$ and k non-negative integer. It is easy to check that this is the case if $b(t)$ is different from zero at least at one

time instant. Otherwise, if $b(\cdot)$ is identically zero, then one can set $x(\sigma, t) = \sigma - a(t)$, $\alpha(\sigma, t) = 1$ and $\beta(\sigma, t) = 0$. The necessary and sufficient condition for controllability in this case is that $\sigma - a(t)$ has all its singularities at the origin. This happens if and only if $a(t)$ is zero at least at one time-instant.

7.3 Balanced Realization

The balanced minimal realization is a classical issue in system theory. It originates in the attempt to work out a state-space description characterized by reachability and observability Grammians, see Sects. 4.1.5, 4.2.6, with “comparable weights”. Here we will consider only the case when the system is stable. The problem is tackled in two steps. First, it is assumed that the given representation is in minimal form, i.e., reachable and observable for each t . Then, this condition is relaxed and the general case is dealt with. Accordingly, consider the system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t), \end{aligned}$$

and assume that it is stable, reachable and observable for each t . For the sake of generality, we allow for time-varying dimensions of the state-space, i.e., $x(t) \in \mathbb{R}^{n(t)}$. This corresponds to the fact that a minimal realization may have time-varying dimensions, as seen in Sect. 7.1. The reachability and observability Grammian matrices $P(t)$ and $Q(t)$ are defined as follows:

$$\begin{aligned} P(t) &= \sum_{i=-\infty}^{t-1} \Phi_A(t, i+1) B(i) B(i)' \Phi_A(t, i+1)' \\ Q(t) &= \sum_{i=t}^{\infty} \Phi_A(i, t)' C(i)' C(i) \Phi_A(i, t). \end{aligned}$$

$P(t)$ and $Q(t)$ are square matrices with time-varying dimensions $n(t)$. As shown in Sect. 4.5, the stability of matrix $A(\cdot)$ implies that $P(t)$ and $Q(t)$ are the unique T -periodic solutions of the periodic Lyapunov equations

$$\begin{aligned} P(t+1) &= A(t)P(t)A(t)' + B(t)B(t)' \\ Q(t) &= A(t)'Q(t+1)A(t) + C(t)'C(t), \end{aligned}$$

respectively. Moreover, both $P(\cdot)$ and $Q(\cdot)$ are positive semi-definite for each t . Finally, since the system is reachable and observable for each t , these matrices are strictly positive definite for each t .

To work out a balanced realization, perform a Cholewski decomposition of the Grammians:

$$P(t) = R(t)'R(t), \quad Q(t) = S(t)'S(t).$$

Now, consider the product $S(t)R(t)'$ and take the singular value decomposition

$$S(t)R(t)' = U(t)\Sigma(t)V(t)'.$$

As is well-known, $U(t)$ and $V(t)$ are orthogonal matrices and $\Sigma(t)$ is diagonal with non-negative elements. In fact, the elements along the diagonal of $\Sigma(t)$ are strictly positive thanks to the invertibility of $R(t)$ and $S(t)$. Notice that the positive definiteness of $P(t)$ and $Q(t)$ implies the invertibility of matrix $S(t)R(t)'$ and in turn of $\Sigma(t)$. Hence, we can define the $n(t) \times n(t)$ transformation matrix $T(t)$ as

$$T(t) = \Sigma(t)^{-1/2}U(t)'S(t).$$

Notice that

$$T(t)^{-1} = R(t)'V(t)\Sigma(t)^{-1/2}.$$

With this position, let

$$\tilde{A}(t) = T(t+1)A(t)T(t)^{-1}, \quad \tilde{B}(t) = T(t+1)B(t), \quad \tilde{C}(t) = C(t)T(t)^{-1}.$$

The new Grammian matrices satisfy

$$\begin{aligned} \tilde{P}(t) &= \sum_{i=-\infty}^{t-1} \Phi_{\tilde{A}}(t, i+1)\tilde{B}(i)\tilde{B}(i)'\Phi_{\tilde{A}}(t, i+1)' = \Sigma(t) \\ \tilde{Q}(t) &= \sum_{i=t}^{\infty} \Phi_{\tilde{A}}(i, t)'\tilde{C}(i)'\tilde{C}(i)\Phi_{\tilde{A}}(i, t) = \Sigma(t), \end{aligned}$$

and it follows

$$\tilde{P}(t) = T(t)P(t)T(t)' = \Sigma(t), \quad \tilde{Q}(t) = [T(t)^{-1}]'Q(t)T(t)^{-1} = \Sigma(t).$$

This means that the transformed system, defined by the triplet $(\tilde{A}(t), \tilde{B}(t), \tilde{C}(t))$ has the distinctive feature of being fully balanced, namely $\tilde{P}(t) = \tilde{Q}(t) = \Sigma(t)$.

Finally, consider the case where the original system is not in minimal form, i.e., is not reachable and observable at each t . In this case, the stability assumption of $A(\cdot)$ ensures that the Grammian matrices $P(\cdot)$ and $Q(\cdot)$ are only positive semi-definite (see again Sect. 4.5). The balanced realization can still be worked out along the following lines. Write again the Cholewsky factors $S(\cdot)$ and $R(\cdot)$ and consider the product $S(t)R(t)'$. Notice however, that this matrix may be singular. Therefore, the associated singular value decomposition has the form

$$S(t)R(t)' = [U_1(t) \ U_2(t)] \begin{bmatrix} \Sigma_1(t) & 0 \\ 0 & 0 \end{bmatrix} [V_1(t) \ V_2(t)]'.$$

Matrix $\Sigma_1(t)$ is a square invertible matrix with dimension $r(t) \leq n(t)$, for each t , where $r(t) = \text{rank}(P(t)Q(t))$. In fact $r(t)$ is the dimension of the minimal realization of the system.

Indeed, let

$$T_1(t) = \Sigma_1^{-1/2} U_1(t)' S(t), \quad T_2(t) = R(t)' V_1(t) \Sigma_1(t)^{-1/2}.$$

Obviously, $T_1(t) \in \mathbb{R}^{r(t) \times n(t)}$ and $T_2(t) \in \mathbb{R}^{n(t) \times r(t)}$. A balanced minimal realization can be obtained as follows:

$$\tilde{A}(t) = T_1(t+1)A(t)T_2(t), \quad \tilde{B}(t) = T_1(t+1)B(t), \quad \tilde{C}(t) = C(t)T_2(t). \quad (7.18)$$

Notice that the dimensions of the matrices are: $\tilde{A}(t) \in \mathbb{R}^{r(t+1) \times r(t)}$, $\tilde{B}(t) \in \mathbb{R}^{r(t+1) \times m}$ and $\tilde{C}(t) \in \mathbb{R}^{p \times r(t)}$. It can be shown that this triplet defines a T -periodic system sharing the same input–output behavior (Markov parameters) and

$$\begin{aligned} \tilde{P}(t) &= T_1(t)P(t)T_1(t)' = \sum_{i=-\infty}^{t-1} \Phi_{\tilde{A}}(t, i+1) \tilde{B}(i) \tilde{B}(i)' \Phi_{\tilde{A}}(t, i+1)' = \Sigma_1(t) \\ \tilde{Q}(t) &= T_2(t)' Q(t) T_2(t) = \sum_{i=t}^{\infty} \Phi_{\tilde{A}}(i, t)' \tilde{C}(i)' \tilde{C}(i) \Phi_{\tilde{A}}(i, t) = \Sigma_1(t). \end{aligned}$$

Remark 7.3 *If the original system is poorly balanced, the above algorithm can lead to ill-conditioned matrices $T_1(t)$ and $T_2(t)$. To avoid this possibility, a square-root balanced-free approach can be used instead as follows:*

(i) *Compute the QR-decomposition*

$$S(t)' U_1(t) = Z(t) Y(t), \quad R(t)' V_1(t) = W(t) X(t),$$

where $X(t)$ and $Y(t)$ are non-singular and $W(t)$ and $Z(t)$ have orthogonal columns.

(ii) *Compute*

$$T_1(t) = (Z(t)' W(t))^{-1} Z(t)', \quad T_2(t) = W(t).$$

A minimal balanced realization can thus be obtained from matrices $T_1(t)$ and $T_2(t)$ as in (7.18).

7.4 Bibliographical Notes

Early references about the minimal realization problem for time-varying discrete-time systems are [41], [164]. The main reference for periodic system is [111]. There, the problem of passing from the lifted transfer function to the minimal state–space description is addressed. The term realization is also used to denote the construction of an ARMA model from the impulse response. This topic has been considered in [214]. In any case, the underlying ideas are rooted in the cornerstone paper [185]. The direct algorithm enabling the determination of a minimal realization from the transfer function, presented in Sect. 7.1.5, was proposed in [290].

The balanced realization is treated in [286], [287]. In particular, in [287] the case of unstable dynamic matrix (not considered in our chapter) is dealt with. The ideas underlying balanced realization are also useful for the problem of model reduction, [265, 289].

Chapter 8

Zeros, Poles, and the Delay Structure

The zeros and poles of a dynamical system play a major role in the characterization of its dynamical behavior. As is well-known, the poles are related to the modes and stability of the system and the zeros determine the fundamental limitations of the performance of a control system. As for the zeros, there are different definitions, depending upon the adopted model in terms of state–space (invariant zeros) or input–output (transmission zeros) representation. In the periodic realm, one can resort to the isomorphism with time-invariant systems, according to the time-invariant reformulations discussed in Chap. 6. This leads to the various definitions of cyclic or lifted zeros and to the associated characterization in the time-domain (blocking properties).

8.1 Cyclic Poles and Zeros

In this section, we focus upon the cyclic representation defined in Sect. 6.3. The definition of a cyclic zero is very natural. To be precise, we say that a complex number z is a cyclic invariant zero at time τ if it is an invariant zero of the associated cyclic System (6.35). This means that the cyclic invariant zeros of a periodic system are the roots of the invariant polynomials of the Smith form¹ associated with the polynomial matrix

$$\hat{\Sigma}_\tau(\lambda) = \begin{bmatrix} \lambda I - \hat{F}_\tau & -\hat{G}_\tau \\ \hat{H}_\tau & \hat{E}_\tau \end{bmatrix}.$$

This matrix is the system matrix of the cyclic reformulation at time τ . Hence, we can say that a cyclic invariant zero at τ is a complex number λ in such a way that this system matrix loses rank. In particular, if we consider a “tall system”, namely a system with a number of inputs not greater than the number of outputs, the complex

¹ The canonical Smith form is formally defined as follows. Let $N(z)$ be an $n \times m$ polynomial matrix and assume that $\text{rank}[N(z)] := r \leq \min[n, m]$. Then, there exist two polynomial and unimodular (i.e., with polynomial inverse) matrices $L(z)$ and $R(z)$ such that $N(z) = L(z)S(z)R(z)$ with

number λ is a cyclic invariant zero at τ if and only if there exists a vector $[\hat{x}'_\tau \hat{u}'_\tau]'$ so that

$$\begin{bmatrix} \lambda I - \hat{F}_\tau & -\hat{G}_\tau \\ \hat{H}_\tau & \hat{E}_\tau \end{bmatrix} \begin{bmatrix} \hat{x}_\tau \\ \hat{u}_\tau \end{bmatrix} = 0.$$

If the system has more inputs than outputs ("fat system"), the rank characterization can be expressed in the following way

$$[\hat{x}'_\tau \hat{y}'_\tau] \begin{bmatrix} \lambda I - \hat{F}_\tau & -\hat{G}_\tau \\ \hat{H}_\tau & \hat{E}_\tau \end{bmatrix} = 0,$$

where $[\hat{x}'_\tau \hat{y}'_\tau]$ is a suitable vector.

The cyclic invariant zeros do not depend upon the particular time point under consideration. Indeed, it can be easily verified that

$$\hat{\Sigma}_{\tau+1}(\lambda) = \begin{bmatrix} \hat{\Delta}'_n & 0 \\ 0 & \hat{\Delta}'_p \end{bmatrix} \hat{\Sigma}_\tau(\lambda) \begin{bmatrix} \hat{\Delta}_n & 0 \\ 0 & \hat{\Delta}_m \end{bmatrix},$$

where the constant matrix $\hat{\Delta}_k$ has been defined in (6.29). Therefore, the Smith forms of $\hat{\Sigma}_\tau(\lambda)$ and $\hat{\Sigma}_{\tau+1}(\lambda)$ do coincide.

The cyclic invariant zeros have an interpretation in terms of the *periodic system matrix operator* $\Sigma(\sigma, t)$ that is defined as

$$\Sigma(\sigma, t) = \begin{bmatrix} \sigma I - A(t) & -B(t) \\ C(t) & D(t) \end{bmatrix}. \quad (8.1)$$

To be precise, a complex number λ is a cyclic invariant zero (at any time-point) if

$$\Sigma(\lambda \sigma, t) \cdot \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = 0, \quad \forall t,$$

for some (non-identically zero) T -periodic vector $[x(\cdot)'] u(\cdot)']'$.

$$S(z) = \underbrace{\begin{bmatrix} \alpha_1(z) & 0 & \cdots & 0 & 0 \\ 0 & \alpha_2(z) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha_r(z) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{m-r \text{ columns}}, \quad \text{\textit{\textbf{}} } n-r \text{ rows}$$

where each polynomial $\alpha_i(z)$ is monic and divides the next one, i.e., $\alpha_i(z) | \alpha_{i+1}(z)$, $i = 1, 2, \dots, r-1$. Matrix $S(z)$ is said to be the Smith form of $N(z)$, and the polynomials $\alpha_i(z)$ are referred to as *invariant polynomials*.

Indeed, as can be easily verified, the above condition coincides with

$$\hat{\Sigma}_\tau(\lambda) \begin{bmatrix} \hat{x}_\tau \\ \hat{u}_\tau \end{bmatrix} = 0,$$

with $\hat{x}_\tau(i) = x(i)$ and $\hat{u}_\tau(i) = u(i)$, $i = 1, 2, \dots, T - 1$.

Passing now to the transmission zeros, consider the cyclic transfer function $\hat{W}_\tau(z)$ defined in Sect. 6.3.3. The cyclic transmission zeros at τ are defined as the transmission zeros of the cyclic transfer function $\hat{W}_\tau(z)$, i.e., the roots of the numerators of the associated Smith–McMillan form². The rank characterization of a cyclic transmission zero at τ is formulated by requiring that $\hat{W}_\tau(z)$ loses rank. To be precise, with reference to a system with a number of inputs not greater than the number of outputs, a complex number λ is a cyclic transmission zero if and only if there exists a non-identically zero polynomial vector $\hat{U}_\tau(z)$ such that

$$\lim_{z \rightarrow \lambda} \hat{W}_\tau(z) \hat{U}_\tau(z) = 0.$$

For a system with a number of outputs lower than the number of inputs one can consider $\hat{W}_\tau(z)'$ instead of $\hat{W}_\tau(z)$ in the rank characterization above.

As for the dependence upon τ , notice again that the recursive relation (see also Eq. (6.30))

$$\hat{W}_{\tau+1}(z) = \hat{\Delta}'_p \hat{W}_\tau(z) \hat{\Delta}_m$$

holds, so that the two transfer functions share the same Smith–McMillan form.

In the same way, the cyclic poles at τ are the roots of the denominators of the Smith–McMillan form of $\hat{W}_\tau(z)$. Thanks again to the recursive formula above, such a pole structure is not dependent upon τ . In conclusion, one can refer to the cyclic zeros and poles without any specification of the time tag.

² This form is defined as follows. Let $G(z)$ be a proper rational matrix with n rows, m columns and $\text{rank}[G(z)] = r \leq \min[n, m]$. Then there exist two polynomial and unimodular (i.e., with polynomial inverse) matrices $L(z)$ and $R(z)$ such that $G(z) = L(z)M(z)R(z)$, where

$$M(z) = \begin{bmatrix} \frac{\varepsilon_1(z)}{\psi_1(z)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\varepsilon_2(z)}{\psi_2(z)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\varepsilon_r(z)}{\psi_r(z)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \left. \begin{array}{l} n - r \text{ rows} \\ m - r \text{ columns} \end{array} \right\}$$

where $\varepsilon_i(z)$ and $\psi_i(z)$ are monic and coprime, $\varepsilon_i(z)$ divides $\varepsilon_{i+1}(z)$, and $\psi_{i+1}(z)$ divides $\psi_i(z)$. The transmission zeros are the roots of the polynomials $\varepsilon_i(z)$, whereas the poles are the roots of the polynomials $\psi_i(z)$.

Remark 8.1 *A change of basis in the state-space amounts to transforming the state vector $x(t)$ into a new vector $\tilde{x}(t) = S(t)x(t)$, where $S(t)$ is T -periodic and non-singular for each t . If one performs a change of basis on System (2.19), the new system is characterized by the quadruplet $(\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot))$, where*

$$\tilde{A}(t) = S(t+1)A(t)S(t)^{-1}$$

$$\tilde{B}(t) = S(t+1)B(t)$$

$$\tilde{C}(t) = C(t)S(t)^{-1}$$

$$\tilde{D}(t) = D(t).$$

As usual, we will say that the original system and the one associated with the new quadruplet are algebraically equivalent. As with stability, reachability and observability, the (invariant) zeros are not affected by a change of state coordinates.

To conclude this section, in an analogy with the time-invariant case, we define the notion of a minimum phase periodic system as a system with all cyclic transmission zeros and poles in the open unit disk. Of course, this notion is not dependent on the time-tag τ .

8.2 Lifted Poles and Zeros

We now turn to the definition of zeros and poles based on the lifted reformulation that was defined in Sect. 6.2. The complex number z is a lifted invariant zero at time τ of System (2.19) if it is an invariant zero of the associated lifted System (6.21) at time τ , i.e., a root of a polynomial of the Smith form of

$$\Sigma_\tau(\lambda) = \begin{bmatrix} \lambda I - F_\tau & -G_\tau \\ H_\tau & E_\tau \end{bmatrix}.$$

This matrix is the system matrix of the lifted reformulation at time τ . For a tall system, the complex number λ is a lifted invariant zero at τ if and only if there exists a vector $[x'_\tau \ u'_\tau]'$ such that

$$\begin{bmatrix} \lambda I - F_\tau & -G_\tau \\ H_\tau & E_\tau \end{bmatrix} \begin{bmatrix} x_\tau \\ u_\tau \end{bmatrix} = 0.$$

If the system is fat, the rank characterization can be expressed by considering the transposed lifted system.

As illustrated in the example below, the lifted case requires a more careful treatment than the cyclic one, due to the fact that the multiplicity of zeros located at the origin may change with time.

Example 8.1 Consider the periodic System (2.19) with period $T = 2$ and

$$\begin{aligned} A(0) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & A(1) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, & B(0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & B(1) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ C(0) &= \begin{bmatrix} 0 & 0 \end{bmatrix}, & C(1) &= \begin{bmatrix} 1 & 1 \end{bmatrix}, & D(0) &= D(1) = 1. \end{aligned}$$

Hence,

$$\Sigma_0(\lambda) = \begin{bmatrix} \lambda & -1 & -1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}, \quad \Sigma_1(\lambda) = \begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

whose correspondent Smith form are $\text{diag}\{1, 1, 1, z^2\}$ and $\text{diag}\{1, 1, z, z\}$, respectively. These forms demonstrate the time dependence of the multiplicity of the invariant lifted zeros at the origin. ■

The lifted invariant zeros can be related to the *periodic system matrix operator* $\Sigma(\sigma, t)$ defined in the previous section. Restricting the attention to tall systems, a complex number λ^T is a lifted invariant zero (at some particular time-point) if

$$\Sigma(\lambda\sigma, t) \cdot \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = 0, \quad \forall t.,$$

for some (non-identically zero) T -periodic vector $[x(\cdot)' u(\cdot)']'$. Indeed, notice that there exists a time-instant, say τ , such that $x(\tau)$ and $u(\tau)$ are not both zero. Hence, the above condition implies that

$$\Sigma_\tau(\lambda^T) \begin{bmatrix} x_\tau \\ u_\tau \end{bmatrix} = 0,$$

with

$$x_\tau = x(\tau), \quad u_\tau = \begin{bmatrix} u(\tau) \\ \lambda u(\tau+1) \\ \lambda^2 u(\tau+2) \\ \vdots \\ \lambda^{T-1} u(\tau+T-1) \end{bmatrix}.$$

The characterization of a lifted zero in terms of the so-called blocking property calls for the definition of an exponentially modulated periodic signal, introduced in Sect. 2.3. Indeed, assume that λ is a lifted zero at τ . Hence, there exist $\eta = [\eta'_0 \ \eta'_1 \ \cdots \ \eta'_{T-1}]'$ and $x_\tau(0)$, not contemporarily zero, such that

$$\Sigma_\tau(\lambda) \begin{bmatrix} x_\tau(0) \\ \eta \end{bmatrix} = 0,$$

i.e., $F_\tau x_\tau(0) + G_\tau \eta = \lambda x_\tau(0)$ and $H_\tau x_\tau(0) + E_\tau \eta = 0$. Furthermore, the well-known blocking property for the invariant zeros of time-invariant systems entails that System (6.21), with input signal given by $u_\tau(k) = \eta \lambda^k$, $k \geq 0$, and initial state $x_\tau(0)$, results in the null output: $y_\tau(k) = 0$, $k \geq 0$. This implies that there exists an *Exponentially Modulated Periodic (EMP)* signal $u(t + kT) = u(t) \lambda^k$, $t \in [\tau, \tau + T - 1]$, with $u(\tau + i) = \eta_i$, $i \in [0, T - 1]$, and initial state $x(\tau) = x_\tau(0)$ such that $y(t) = 0$, $t \geq \tau$.

It is easy to see that if $\lambda \neq 0$ is a lifted invariant zero at time τ , then it is also a lifted invariant zero at time $\tau + 1$. In fact, it suffices to select as input $u(t + kT) = u(t) \lambda^k$, $t \in [\tau + 1, \tau + T]$, with $u(\tau + i + 1) = \eta_i$, $i \in [0, T - 2]$, and $u(\tau + T) = \lambda \eta_0$, and initial state $x(\tau + 1) = A(\tau) x_\tau(0) + B(\tau) \eta_0$ to ensure that $y(t) = 0$, $t \geq \tau + 1$. Note that for consistency, one has to check that $\eta = [\eta'_1 \cdots \eta'_{T-1} \lambda \eta'_0]'$ and $x(\tau + 1) = A(\tau) x_\tau(0) + B(\tau) \eta_0$ are not contemporarily zero. If $\eta \neq 0$, this is obvious. If instead $\eta = 0$, then the system matrix interpretation leads to $F_\tau x_\tau(0) = \lambda x_\tau(0)$. Moreover, $\eta = 0$ and $x(\tau + 1) = A(\tau) x_\tau(0)$. To show that $x(\tau + 1) \neq 0$, notice that $A(\tau) x_\tau(0) = 0$ would imply that $A(\tau + T - 1) A(\tau + T - 2) \cdots A(\tau) x_\tau(0) = 0$, i.e., $F_\tau x_\tau(0) = 0$. Therefore, it would turn out that $\lambda = 0$, in contradiction to our initial assumption.

In the previous section we have seen that the cyclic invariant zero structure (Smith form of $\hat{\Sigma}_\tau(\lambda)$) does not depend on τ whereas the lifted invariant zeros structure (Smith form of $\Sigma_\tau(\lambda)$) may depend on τ , only as far as the multiplicity of the zeros at the origin is concerned. To render this fact more precise, it is possible to work out the exact relationship between the two system matrices. Indeed, it turns out that

$$\Sigma_\tau(z^T) = A(z) \hat{\Sigma}_\tau(z) B(z),$$

where

$$A(z) = \begin{bmatrix} A_1(z) & 0 \\ 0 & \nabla_p(z) \end{bmatrix}, \quad B(z) = \begin{bmatrix} B_1(z) & B_2(z) \\ 0 & \nabla_m(z^{-1}) \end{bmatrix},$$

with $\nabla_i(z)$ given by Eq. (6.26) and

$$A_1(z) = [I_n \ 0 \ \cdots 0], \quad B_1(z) = \begin{bmatrix} z^{T-1} I_n \\ z^{T-2} A(\tau) \\ z^{T-3} A(\tau+1) A(\tau) \\ \vdots \\ \Phi_A(\tau+T-1, \tau) \end{bmatrix}$$

$$B_2(z) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ B(\tau) z^{T-2} & 0 & \cdots & 0 & 0 \\ A(\tau+1) B(\tau) z^{T-3} & B(\tau+1) z^{T-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Phi_A(\tau+T-1, \tau+1) B(\tau) & \Phi_A(\tau+T-1, \tau+2) B(\tau+1) & \cdots & B(\tau+T-2) & 0 \end{bmatrix}.$$

The above formulas show that the non-null cyclic invariant zeros structure can be computed from the non-null lifted invariant zeros structure by replacing z with z^T .

The lifted transmission zeros and poles at time τ are the zeros and poles of the associated lifted transfer function $W_\tau(z)$. In this regard, notice that in Sect. 6.2.2 we have worked out the recursive expression

$$W_{\tau+1}(z) = \Delta'_p(z^{-1})W_\tau(z)\Delta_m(z),$$

where

$$\Delta_k(z) = \begin{bmatrix} 0 & z^{-1}I_k \\ I_{k(T-1)} & 0 \end{bmatrix}.$$

The above relationship implies that the structure of the zeros and poles different from zero does not depend upon the chosen time-tag τ . This can be also seen by considering the blocking property for tall systems. Indeed, if λ is a transmission zero for a periodic system at time τ , then there exists a polynomial vector $\eta(z) \neq 0$ such that $\lim_{z \rightarrow \lambda} W_\tau(z)\eta(z) = 0$. Since $\Delta_k(\lambda)$ is invertible if $\lambda \neq 0$, it is apparent that $\lim_{z \rightarrow \lambda} W_{\tau+1}(z)\tilde{\eta}(z) = 0$ with the polynomial vector $\tilde{\eta}(z) = \Delta_m(z)^{-1}\eta(z) \neq 0$. Hence the non-zero transmission zeros of the periodic system do not change with τ . On the contrary, as illustrated by the examples below, the multiplicity of the lifted transmission zeros and lifted poles at the origin may change in time.

Example 8.2 Consider the scalar periodic System (2.19) with period $T = 2$ and $A(0) = 2$, $A(1) = 0$, $B(0) = 1$, $B(1) = 0$, $C(0) = C(1) = 1$, $D(0) = D(1) = 1$. The transfer function of the lifted reformulation at $\tau = 0$ is given by

$$W_0(z) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

This matrix is constant and invertible; therefore there are neither lifted transmission poles nor zeros at $\tau = 0$, both finite and infinite. On the contrary, at $\tau = 1$ the lifted reformulation turns out to be given by

$$W_1(z) = \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix}.$$

This is no more a constant matrix. Since the corresponding Smith–McMillan form is given by

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix},$$

we have a transmission lifted zero and a lifted pole at the origin of the complex plane. The reason for the discrepancy between the finite zero structure of $W_0(z)$ and $W_1(z)$ lies in the different property of the characteristic multipliers of the periodic system at the origin as far as reachability and/or observability is concerned. Indeed, it is easy to show that the characteristic multiplier $\lambda = 0$ is observable at any time point, whereas it is reachable only at time $\tau = 1$.

Example 8.3 Consider again the previous example with $B(0)$ and $B(1)$ interchanged with each other, i.e., System (2.19) with period $T = 2$ and $A(0) = 2$, $A(1) = 0$, $B(0) = 0$, $B(1) = 1$, $C(0) = C(1) = 1$, $D(0) = D(1) = 1$. It is easy to

verify that this system is reachable and observable at each time point. The transfer function of the lifted reformulation at $\tau = 0$ and $\tau = 1$ are given by

$$W_0(z) = \begin{bmatrix} 1 & z^{-1} \\ 0 & z^{-1}(z+2) \end{bmatrix}, \quad W_1(z) = \begin{bmatrix} z^{-1}(z+2) & 0 \\ 1 & 1 \end{bmatrix}.$$

These two transfer functions share the same pole-zero structure, given by

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & (z+2) \end{bmatrix}.$$

Therefore the periodic system has a lifted pole in the origin (with constant multiplicity) at any time point. ■

To end this section, let us briefly discuss the relationship between the transmission lifted and cyclic zeros (poles). The relationship between the transfer functions is (recall Eq. (6.31)),

$$\hat{W}_\tau(z) = \nabla_m(z) W_\tau(z^T) \nabla_p(z)^{-1},$$

where $\nabla_r(z) = \text{blkdiag}\{I_r, z^{-1}I_r, \dots, z^{-T+1}I_r\}$. Hence, the structure of the non-zero poles and zeros is inherited by this relationship, in the sense that it is enough to replace the variable z of the lifted case for the variable z^T for the cyclic case.

Example 8.4 *It is interesting to compare the previous results concerning the lifted singularities with those relative to the cyclic reformulation. Consider first Example 8.2. The transfer function of the cyclic reformulation at $\tau = 0$ and $\tau = 1$ are respectively:*

$$\hat{W}_0(z) = \begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix}, \quad \hat{W}_1(z) = \begin{bmatrix} 1 & z^{-1} \\ 0 & 1 \end{bmatrix}$$

Apparently, these two transfer functions have the same pole-zero structure given by

$$\begin{bmatrix} z^{-1} & 0 \\ 0 & z \end{bmatrix}$$

Therefore, for each time point, there is a cyclic zero and a cyclic pole at the origin. Notice that this form does not coincide with the Smith–McMillan form associated with $W_0(z)$ or $W_1(z)$, with z replaced by z^2 .

Finally, consider Example 8.3. The transfer functions of the cyclic reformulation at $\tau = 0$ and $\tau = 1$ are

$$\hat{W}_0(z) = \begin{bmatrix} 1 & z^{-1} \\ 0 & z^{-2}(z^2+2) \end{bmatrix}, \quad \hat{W}_1(z) = \begin{bmatrix} z^{-2}(z^2+2) & 0 \\ z^{-1} & 1 \end{bmatrix},$$

which have coincident pole zero structure $\text{diag}\{z^{-2}, z^2+2\}$. Despite the presence of zero singularities, notice that this structure coincides with the Smith–McMillan form associated with $W_0(z)$ and $W_1(z)$ with z replaced with z^2 .

As for the poles, it is apparent that they constitute a subset of the characteristic multipliers of system. Similarly to the zeros, the non-zero poles, together with their multiplicities, are in fact independent of τ . The proof of this statement, omitted here for the sake of brevity, can be worked out in terms of an impulse response characterization of the system.

Notice that an analogous definition of a minimum phase system, given at the end of the previous section, makes reference to the lifted transmission zeros and poles as well. As a matter of fact, these lifted zeros and poles do have modulus that are less than one if and only if the cyclic transmission zeros and poles do.

8.3 Zeros via a Geometric Approach

In the previous sections, we studied the zeros by making reference to the algebraic properties of the underlying dynamical system. A parallel analysis can be carried over by pursuing a different path that is centered in the study of the geometric structure of the system. This method will provide a new insight into the characterization of the invariant zeros.

In what follows, we assume that the periodic system is strictly proper, i.e., $D(t) = 0$, $\forall t$. The characterization of the so-called structural zeros of a periodic system calls for two concepts, namely the concept of $(A(\cdot), B(\cdot))$ -invariant subspace³, and the concept of a $(A(\cdot), B(\cdot))$ -reachability subspace⁴. The concept of $A(\cdot), B(\cdot)$ -invariant subspace has an interesting characterization in terms of state-feedback. Indeed, a subspace $\mathbf{V}(t)$ is $A(\cdot), B(\cdot)$ -invariant if and only if there exists a periodic matrix $K(\cdot)$ such that $\mathbf{V}(t)$ is $A(\cdot) + B(\cdot)K(\cdot)$ -invariant (see note in Sect. 4.1.2). The proof of this statement is very simple. If $\mathbf{V}(t)$ is $A(\cdot) + B(\cdot)K(\cdot)$ -invariant, for some $K(\cdot)$, then $A(t)x = -B(t)u + y$, for some $x \in \mathbf{V}(t)$ and $y \in \mathbf{V}(t+1)$ with $u = K(t)x$. This means that $A(t)\mathbf{V}(t) \subseteq \mathbf{V}(t+1) + \text{Im}[B(t)]$. Vice versa, if $\mathbf{V}(t)$ is an $(A(\cdot), B(\cdot))$ -invariant subspace, then for each $x(t) \in \mathbf{V}(t)$ there exist $u(t) \in \text{Im}[B(t)]$ and $y(t) \in \mathbf{V}(t+1)$ such that $A(t)x(t) = B(t)u(t) + y(t)$. Of course, it is always possible to find $K(\cdot)$ such that $u(t) = -K(t)x(t)$, and therefore the thesis follows.

Notice that the class of $(A(\cdot), B(\cdot))$ -invariant subspaces is closed with respect to the sum. Moreover, this class includes the origin. Hence, if a certain periodic subspace $\mathbf{W}(t)$ is selected, it is possible to recursively find the maximal $(A(\cdot), B(\cdot))$ -invariant subspace, say $\mathbf{V}^*(t)$, included in $\mathbf{W}(t)$.

³ A periodic subspace $\mathbf{V}(t) \subseteq \mathbb{R}^n$ is said to be $(A(\cdot), B(\cdot))$ -invariant if $A(t)\mathbf{V}(t) \subseteq \mathbf{V}(t+1) + \text{Im}[B(t)]$.

⁴ An $(A(\cdot), B(\cdot))$ -invariant subspace $\mathbf{V}(t)$ is said to be a $(A(\cdot), B(\cdot))$ -reachability subspace if, for each t and each $\bar{x} \in \mathbf{V}$, there exists a positive integer h and a control function such that $x(t-h) = 0$, $x(t-h+1) \in \mathbf{V}(t+h-1)$, $x(t-h+2) \in \mathbf{V}(t+h-2)$, \dots , $x(t) = \bar{x}$.

To be precise, taking $\mathbf{W}(t) = \text{Ker}[C(t)]$, of it is possible to write⁵:

$$\begin{aligned}\mathbf{V}^0(t) &= \text{Ker}[C(t)] \\ \mathbf{V}^i(t) &= \text{Ker}[C(t)] \cap A(t)^- \left(\mathbf{V}^{i-1}(t+1) + \text{Im}[B(t)] \right), \quad i = 1, 2, \dots\end{aligned}$$

The above iterative formulas define a sequence of periodic subspaces such that $\mathbf{V}(t) \subset \mathbf{V}^{i+1}(t) \subset \mathbf{V}^i(t)$, where $\mathbf{V}(t)$ is any $(A(\cdot), B(\cdot))$ -invariant subspace included in $\text{Ker}[C(t)]$. An easy verification can be carried out by induction and by noticing that the $(A(\cdot), B(\cdot))$ -invariant subspace $\mathbf{V}(t)$ satisfies

$$\mathbf{V}(t) = \text{Ker}[C(t)] \cap A(t)^- (\mathbf{V}(t+1) + \text{Im}[B(t)]).$$

Consequently, there exists an integer $r \leq \sum_{i=0}^{T-1} \dim[\text{Ker}[C(i)]]$ such that

$$\mathbf{V}^{r+j}(t) = \mathbf{V}^r(t), \quad \forall j \geq 0.$$

Of course $\mathbf{V}^*(t) = \mathbf{V}^r(t)$.

Analogous considerations hold for the class of $(A(\cdot), B(\cdot))$ -reachability subspaces. Therefore, the maximal $(A(\cdot), B(\cdot))$ -reachability subspace, $\mathbf{V}_{\mathbf{R}}^*(t)$, included in $\text{Ker}[C(t)]$ can be found as the final subspace of the sequence

$$\begin{aligned}\mathbf{V}_{\mathbf{R}}^0(t) &= \{0\} \\ \mathbf{V}_{\mathbf{R}}^i(t+1) &= \text{Ker}[C(t+1)] \cap \left(A(t)\mathbf{V}_{\mathbf{R}}^{i-1}(t) + \text{Im}[B(t)] \right), \quad i = 1, 2, \dots\end{aligned}$$

The subspaces of these two sequences are such that $\mathbf{V}_{\mathbf{R}}^i(t) \subset \mathbf{V}_{\mathbf{R}}^{i+1}(t)$ and there exists an integer $r_r \leq \text{Ker}[C(i)]$ such that

$$\mathbf{V}_{\mathbf{R}}^{r_r+j}(t) = \mathbf{V}_{\mathbf{R}}^{r_r}(t), \quad \forall j \geq 0.$$

Of course $\mathbf{V}_{\mathbf{R}}^*(t) = \mathbf{V}_{\mathbf{R}}^{r_r}(t)$.

Now, let a T -periodic gain $F(\cdot)$ be such that $\mathbf{V}^*(\cdot)$ is an $(A(\cdot) + B(\cdot)F(\cdot))$ -invariant subspace and define

$$A_{st}(\cdot) = (A(\cdot) + B(\cdot)F(\cdot))|_{\bar{\mathbf{V}}^*(\cdot)}, \quad \bar{\mathbf{V}}^*(t) = \mathbf{V}^*(t) \pmod{\mathbf{V}_{\mathbf{R}}^*(t)}. \quad (8.2)$$

Hence, $\bar{\mathbf{V}}^*(t)$ is the quotient space $\mathbf{V}^*(t)$ over $\mathbf{V}_{\mathbf{R}}^*(t)$ and $A_{st}(\cdot)$ is the restriction of $(A(\cdot) + B(\cdot)F(\cdot))$ to $\mathbf{V}^*(t)$ having $\bar{\mathbf{V}}^*(t+1)$ as image space. The characteristic multipliers of $A_{st}(\cdot)$ do not depend upon the choice of $F(\cdot)$ yielding $\mathbf{V}^*(t)$ an $(A(\cdot) + B(\cdot)F(\cdot))$ -invariant subspace. They are called the *structural zeros* of the periodic system. Interestingly, such zeros coincide with the lifted invariant zeros defined in the previous section.

⁵ If X is a subspace of \mathbb{R}^n , by F^-X we mean the counter image, i.e., the subspace $Y = \{y : x = Fy, x \in X\}$.

Example 8.5 Let $T = 2$ and

$$A(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The supremal elements of the sequences are

$$\mathbf{V}^*(t) = \mathbf{V}_{\mathbf{C}}^*(t) = \begin{cases} \text{Im} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & t = 0 \\ \text{Im} \begin{bmatrix} -1 \\ 1 \end{bmatrix} & t = 1 \end{cases}, \quad \mathbf{V}_{\mathbf{R}}^*(t) = \begin{cases} \text{Im} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & t = 0 \\ \{0\} & t = 1 \end{cases}.$$

Moreover, the periodic gain $F(\cdot)$ such that $\mathbf{V}^*(t)$ is $A(\cdot) + B(\cdot)F(\cdot)$ invariant is

$$F(0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad F(1) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

and

$$A_{st}(t) = \begin{cases} 0, & t = 0 \\ 1, & t = 1 \end{cases}.$$

Since $\Psi_{A_{st}}(\tau) = 0$, $\tau = 0, 1$, the conclusion follows that there is only one structural zero at $\lambda = 0$. ■

Structural zeros in the SISO case

The computation of the zeros is much easier in the case of SISO systems with a *uniform relative degree*. The relative degree of a SISO system is a T -periodic integer $r(\cdot)$ such that

$$0 = C(t+i+1)\Phi_A(t+i+1, t+1)B(t), \quad 0 \leq i < r(t)-1, \quad \forall t$$

$$0 \neq C(t+r(t))\Phi_A(t+r(t), t+1)B(t), \quad \forall t.$$

If such an integer does not exist, the system is said to have an infinite relative degree. The interpretation of $r(\cdot)$ in terms of the impulse response is immediate. Indeed, if $x(t) = 0$ and the input is an impulse at time t , then

$$y(t) = 0, \quad y(t+1) = 0, \quad \dots, \quad y(t+r(t)-1) = 0,$$

$$y(t+r(t)) = C(t+r(t))\Phi_A(t+r(t), t+1)B(t).$$

Hence, $r(t)$ corresponds to the delay between the input and the output. When $r(t) = r$, $\forall t$, the system is said to have a *uniform relative degree*. Of course, $r \leq n$. It is possible to show that

$$\mathbf{V}^*(t) = \text{Ker}[O_r(t)], \quad \mathbf{V}_{\mathbf{R}}^*(t) = \{0\},$$

where $O_r(t)$ is the observability matrix of length r defined in Eq. (4.33) and reported here for ease of reference, namely

$$O_r(t) = \begin{bmatrix} C(t) \\ C(t+1)A(t) \\ C(t+2)A(t+1)A(t) \\ \vdots \\ C(t+r-1)\Phi_A(t+r-1, t) \end{bmatrix}.$$

The T -periodic gain $F(\cdot)$ such that $\text{Ker}[O_r(t)]$ is $(A(\cdot) + B(\cdot)F(\cdot))$ invariant is

$$F(t) = -\frac{C(t+r)\Phi_A(t+r, t)}{C(t+r)\Phi_A(t+r, t+1)B(t)}.$$

Letting

$$R(t) = A(t) - B(t) \frac{C(t+r)\Phi_A(t+r, t)}{C(t+r)\Phi_A(t+r, t+1)B(t)},$$

the structural zeros are the characteristic multipliers of

$$A_r(t) = R(t)|\mathbf{V}^*(t)(\text{mod } \mathbf{V}_R^*(t)) = R(t)|\text{Ker}[O_r(t)].$$

Hence, the structural zeros are the unobservable eigenvalues of the time-invariant pair $(\Psi_R(t), O_r(t))$. Since the rank of $O_r(t)$ is equal to r , it follows that the number of the structural zeros is $n - r$. On the other hand, the unobservable eigenvalues of $(\Psi_R(t), O_r(t))$ are a subset of the characteristic multipliers of observable eigenvalues of $R(\cdot)$, and finally, the observable eigenvalues of $(\Psi_R(t), O_r(t))$ are all concentrated in the origin. In conclusion, if $R(\cdot)$ has q characteristic multipliers at the origin, the structural zeros are given by the $n - q$ non-zero characteristic multipliers of $R(\cdot)$ and by the $q - r$ zero characteristic multipliers.

It is interesting to interpret the above results when the SISO system is written in an input–output formulation. For instance, consider a T -periodic system in the right fractional representation

$$y(t) = n(\sigma, t)d(\sigma, t)^{-1} \cdot u(t),$$

where

$$\begin{aligned} d(\sigma, t) &= \sigma^n + \alpha_1(t)\sigma^{n-1} + \alpha_2(t)\sigma^{n-2} + \cdots + \alpha_n(t) \\ n(\sigma, t) &= \gamma_0(t)\sigma^n + \gamma_1(t)\sigma^{n-1} + \gamma_2(t)\sigma^{n-2} + \cdots + \gamma_n(t). \end{aligned}$$

The assumption of a uniform relative degree r means that all coefficients $\gamma_0(t), \gamma_1(t), \dots, \gamma_{n-r+1}(t)$ are all identically zero and $\gamma_r(t) \neq 0, \forall t$, i.e.,

$$n(\sigma, t) = \gamma_r(t)\sigma^{n-r} + \gamma_{r+1}(t)\sigma^{n-r-1} + \gamma_{r+2}(t)\sigma^{n-r-2} + \cdots + \gamma_n(t).$$

Assume that $r > 0$ and take a realization in the reachability canonical form (see Sect. 7.2):

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n(t) & -\alpha_{n-1}(t) & -\alpha_{n-2}(t) & \cdots & -\alpha_1(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}$$

$$C(t) = [\gamma_n(t) \cdots \gamma_r(t) \ 0 \cdots 0], \quad D(t) = 0.$$

It is a matter of simple computations to show that

$$C(t+r)\Phi_A(t+r, t+1)B(t) = \gamma_r(t)$$

and

$$R(t) = A(t) - B(t) \frac{C(t+r)\Phi_A(t+r, t)}{C(t+r)\Phi_A(t+r, t+1)B(t)}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -\delta_n(t) & -\delta_{n-1}(t) & -\delta_{n-2}(t) & \cdots & -\delta_2(t) & -\delta_1(t) \end{bmatrix},$$

where $\delta_n(t), \delta_{n-1}(t), \dots, \delta_{n-r+1}(t)$ are identically zero and

$$\begin{aligned} \delta_{n-r}(t) &= -\frac{\gamma_n(t)}{\gamma_r(t)} \\ \delta_{n-r-1}(t) &= -\frac{\gamma_{n-1}(t)}{\gamma_r(t)} \\ &\vdots \\ \delta_1(t) &= -\frac{\gamma_{r+1}(t)}{\gamma_r(t)}. \end{aligned}$$

Hence, the characteristic multipliers of $R(\cdot)$ are the singularities of the T -periodic polynomial

$$p(\sigma, t) = \sigma^n + \frac{\gamma_{r+1}(t)}{\gamma_r(t)} \sigma^{n-1} + \frac{\gamma_{r+2}(t)}{\gamma_r(t)} \sigma^{n-2} + \cdots + \frac{\gamma_n(t)}{\gamma_r(t)} \sigma^r.$$

On the other hand, as it is easy to verify that

$$p(\sigma, t) = \frac{1}{\gamma_r(t)} n(\sigma, t) \sigma^r,$$

so that, obviously, the structural zeros coincide with the singularities of the periodic polynomial $n(\sigma, t)$. Analogous reasoning could be pursued for SISO systems in the left fractional representation.

8.4 Delay Structure and Interactor Matrix

The structure of the zeros at infinity for multivariable systems generalizes the concept of relative degree for single-input single-output systems. Again, for periodic systems, the distinction between the cyclic and lifted structure at infinity at a given time point can be made. Consider first a cyclic transfer function $\hat{W}_\tau(z)$ at τ . It is well-known that it is possible to find two bi-causal transfer functions (proper with proper inverses) $\hat{S}_\tau(z)$ and $\hat{T}_\tau(z)$ such that

$$\hat{W}_\tau(z) = \hat{S}_\tau(z) \begin{bmatrix} z^{-\hat{n}_1(\tau)} & 0 & \dots & 0 & 0 \dots 0 \\ 0 & z^{-\hat{n}_2(\tau)} & \dots & 0 & 0 \dots 0 \\ 0 & 0 & \ddots & 0 & 0 \dots 0 \\ 0 & 0 & \dots & z^{-\hat{n}_{\hat{r}(\tau)}(\tau)} & 0 \dots 0 \\ 0 & 0 & \dots & 0 & 0 \dots 0 \\ 0 & 0 & \dots & 0 & 0 \ddots 0 \\ 0 & 0 & \dots & 0 & 0 \dots 0 \end{bmatrix} \hat{T}_\tau(z).$$

The integer $\hat{r}(\tau)$ is the rank of $\hat{W}_\tau(z)$ and the integers $0 \leq \hat{n}_1(\tau) \leq \hat{n}_2(\tau) \leq \dots \leq \hat{n}_{\hat{r}(\tau)}(\tau)$ constitute the *cyclic zero structure at infinity at time τ* . The number of cyclic zeros at infinity at τ is the number of non-zero indices, $\hat{n}_i(\tau)$ is the order of the i -th cyclic zero at infinity and the sum of all indices is the total order of the cyclic zeros at infinity at τ .

In view of the recursive formula (6.30), it is apparent that the cyclic structure at infinity does not depend upon τ , i.e., $\hat{r}(\tau) = r$ and $\hat{n}_i(\tau) = \hat{n}_i, \forall \tau$.

Turning now to the lifted reformulation $W_\tau(z)$, there exist two bi-causal transfer functions $S_\tau(z)$ and $T_\tau(z)$ such that

$$W_\tau(z) = S_\tau(z) \begin{bmatrix} z^{-n_1(\tau)} & 0 & \dots & 0 & 0 \dots 0 \\ 0 & z^{-n_2(\tau)} & \dots & 0 & 0 \dots 0 \\ 0 & 0 & \ddots & 0 & 0 \dots 0 \\ 0 & 0 & \dots & z^{-n_{r(\tau)}(\tau)} & 0 \dots 0 \\ 0 & 0 & \dots & 0 & 0 \dots 0 \\ 0 & 0 & \dots & 0 & 0 \ddots 0 \\ 0 & 0 & \dots & 0 & 0 \dots 0 \end{bmatrix} T_\tau(z).$$

Analogously to the previous case, the integer $r(\tau)$ is the rank of $W_\tau(z)$ and the integers $0 \leq n_1(\tau) \leq n_2(\tau) \leq \dots \leq n_{r(\tau)}(\tau)$ are the *lifted zero structure at infinity at time τ* .

In view of the recursive formula (6.12) and the periodicity of $W_\tau(z)$ with respect to τ , it is easy to understand that the rank $r(\tau)$ of $W_\tau(z)$ is indeed independent of τ . However, the indices of the structure at infinity can change in time.

Example 8.6 Consider the periodic System (2.19) with period $T = 2$ and

$$\begin{aligned} A(0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & A(1) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & B(0) &= B(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C(0) &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & C(1) &= \begin{bmatrix} 0 & 1 \end{bmatrix}, & D(0) &= D(1) = 0. \end{aligned}$$

The transfer functions of the cyclic reformulation are

$$\hat{W}_0(z) = \frac{1}{z^4 - 1} \begin{bmatrix} 1 & z^3 \\ z & 1 \end{bmatrix}, \quad \hat{W}_1(z) = \frac{1}{z^4 - 1} \begin{bmatrix} 1 & z \\ z^3 & 1 \end{bmatrix}.$$

As evident, the associated structures at infinite coincide in $\text{diag}\{1, 3\}$, i.e., $n_1(0) = \hat{n}_1(1) = 1$, $\hat{n}_2(0) = \hat{n}_2(1) = 3$. On the other hand, the transfer functions of the lifted reformulation at $\tau = 0$ and $\tau = 1$ are

$$W_0(z) = \frac{1}{z^2 - 1} \begin{bmatrix} 1 & z \\ z & 1 \end{bmatrix}, \quad W_1(z) = \frac{1}{z^2 - 1} \begin{bmatrix} 1 & 1 \\ z^2 & 1 \end{bmatrix},$$

so that the associated structures at infinite are $\text{diag}\{1, 1\}$ and $\text{diag}\{0, 2\}$, i.e., $n_1(0) = n_1(1) = 1$, $n_2(0) = 1$, $n_2(1) = 2$. ■

A knowledgeable characterization of the delay structure of a linear time-invariant system includes the *interactor matrix*, the original introduction of which goes back to the early seventies. Such a matrix acts as a “delay eraser”, enabling one to convert a given system into an invertible (bi-causal) one. Among the many forms in which the basic definition can be displayed, the so-called *spectral interactor matrix* (also named *unitary interactor matrix*) will be of interest for the developments to follow, see Sect. 10.2.2.

We will now introduce the notion of interactor matrix and spectral interactor matrix for periodic systems. With this aim, we will restrict ourselves to the case of square invertible systems, i.e., $m = p$ and $\text{rank } [G(\cdot, t)] = m$, $\forall t$, where the periodic transfer operator $G(\sigma, t)$ is related to a state-space description as follows:

$$G(\sigma, t) = D(t) + C(t)(\sigma I - A(t))^{-1} B(t). \quad (8.3)$$

Definition 8.1 A periodic interactor matrix operator of System (2.19) is any operator matrix $L(\sigma, t)$, polynomial in the one step ahead operator σ and periodic in t :

$$L(\sigma, t) = L_0(t) + L_1(t)\sigma + L_2(t)\sigma^2 + \dots + L_q(t)\sigma^q, \quad L_i(t) = L_i(t + T), \quad (8.4)$$

such that

$$\lim_{z \rightarrow \infty} [G(\sigma, t) L(\sigma, t)]|_{\sigma=z} = \Lambda(t), \quad \det(\Lambda(t)) \neq 0, \quad \forall t \in [0, T-1].$$

■

Obviously, this definition coincides with the classical one for time-invariant systems. As already said, among the infinity of interactor matrices, we will focus on the so-called *Periodic Spectral Interactor Matrix Operator* (PSIMO). If compared with the periodic interactor matrix, the PSIMO has the additional feature of preserving the spectral properties of the system. The concept of spectral equivalence is precisely stated in the following definition, which requires the notion of adjoint system, introduced in Sect. 2.5.

Definition 8.2 *We will say that two periodic systems with transfer operators $G(\sigma, t)$ and $\hat{G}(\sigma, t)$, respectively, are spectrally equivalent if the identity*

$$G(\sigma, t) G^\sim(\sigma, t) = \hat{G}(\sigma, t) \hat{G}^\sim(\sigma, t), \quad \forall t. \quad (8.5)$$

holds, for each points.

■

We now need the definition of adjoint polynomial operator $L^\sim(\sigma, t)$, which mimics the one given for the transfer function operator. Precisely, the adjoint of $L(\sigma, t)$ is

$$L^\sim(\sigma, t) = L_0(t)' + L_1(t-1)' \sigma^{-1} + L_2(t-2)' \sigma^{-2} + \cdots + L_q(t-q)' \sigma^{-q}.$$

Notice in passing that this adjoint operator corresponds to a periodic MA (Moving Average) system, namely a PARMA system where the autoregressive components are missing (Eq. (2.11) with $r = 0$). This particular system with all poles at the origin was also considered in Sect. 5.5, where the acronym PFIR (Periodic finite impulse response) was used.

Thanks to the invertibility assumption, Eq. (8.5) is equivalent to:

$$L(\sigma, t) L^\sim(\sigma, t) = I, \quad \forall t. \quad (8.6)$$

Any operator satisfying (8.6) will be referred to as a *periodic paraunitary matrix operator*.

Definition 8.3 *A periodic spectral interactor matrix operator of System (2.19) is any interactor matrix operator $L(\sigma, t)$ satisfying (8.6).*

Remark 8.2 *Recalling that $\sigma^{-k} L(\sigma, t) = L(\sigma, t-k) \sigma^{-k}$, a q -th order polynomial matrix operator $L(\sigma, t)$ is paraunitary if and only if*

$$\sum_{i=0}^{q-k} L_i(t) L_{i+k}(t-k)' = \begin{cases} I, & k = 0, \\ 0, & k = 1, 2, \dots, q. \end{cases}$$

■

Obviously, whenever the input–output matrix $D(t)$ is invertible $\forall t \in [0, T - 1]$, the identity is a PSIMO, and $\Lambda(t) = D(t)$. In such a case, the system is said to be “delay-free”.

The system associated with the transfer operator

$$\bar{G}(\sigma, t) = G(\sigma, t)L(\sigma, t) \quad (8.7)$$

is a delay-free version of the original system. Moreover, (8.6) implies that

$$\bar{G}(\sigma, t)\bar{G}^\sim(\sigma, t) = G(\sigma, t)G^\sim(\sigma, t).$$

In order to find a PSIMO for the original periodic system, the obvious idea is to first find a Spectral Interactor Matrix $P(z)$ for the associated lifted transfer function $W_\tau(z)$ and then to recover $L(\sigma, t)$ from $P(z)$. For this procedure to make sense, however, its delay-free version $\bar{W}_\tau(z) = W_\tau(z)P(z)$ must be interpretable as the lifted reformulation of a periodic system. This means that $\bar{W}_\tau(z)$ must be m -block lower triangular when $z = \infty$ (periodic realizability condition, see Chap. 7). This major issue is dealt with in the following result.

Lemma 8.1 *There exists a polynomial matrix $P(z)$, with*

$$P(z) = P_0 + P_1z + P_2z^2 + \cdots + P_rz^r \quad (8.8)$$

such that

1. $\det[W_\tau(z)P(z)]_{z=\infty} \neq 0$
2. $P(z)P(z^{-1})' = I, \forall z$
3. $[W_\tau(z)P(z)]_{z=\infty}$ is an m -block lower triangular matrix.

Proof Let $\hat{P}(z)$ be a spectral interactor matrix of $W_\tau(z)$. It obviously satisfies conditions (1) and (2). Now apply a QR decomposition to $[W_\tau(z)\hat{P}(z)]_{z=\infty}$, yielding $[W_\tau(z)\hat{P}(z)]_{z=\infty} = RQ'$, where R is lower triangular and Q is orthonormal. Letting $P(z) = \hat{P}(z)Q$, it is apparent that $P(z)$ meets (3) besides (1) and (2).

Remark 8.3 *For a given lifted transfer function $W_\tau(z)$, define as Lifted Spectral Interactor Matrix (LSIM) any polynomial matrix satisfying the conditions of Lemma 8.1. An LSIM $P(z)$ is a polynomial matrix which renders $W_\tau(z)$ delay-free (condition (1)), without altering its spectrum (condition (2)) and is such that $W_\tau(z)P(z)$ is the lifted reformulation of a periodic system (condition (3)). The class of LSIM's can be constructed from any particular element by post-multiplying the latter by any m -block diagonal orthonormal matrix. ■*

The issue is now how to deduce a PSIMO from the LSIM. This can be done along two different paths. The first one is related to the realization theory for periodic systems presented in Chap. 7. Indeed, since $P(z)$ is an LSIM, both $(W_\tau(z)P(z))^{-1}$ and $W_\tau(z)$ are periodically realizable, and $P(z^{-1})' = (W_\tau(z)P(z))^{-1}W_\tau(z)$ is periodically realizable too (P'_0 is lower m -block triangular). As such, from the realization theory of periodic systems it is possible to obtain a periodic system whose input–output

operator is exactly $L^\sim(\sigma, t)$, which is actually a periodic delay operator. The second possibility pursued here consists in exploiting the polynomial structure of $P(z)$ directly, as will be clarified in Proposition 8.1. The first point of this statement is devoted to the relationship between $P(z)$ and $L(\sigma, t)$. The second point supplies a parametrization result for the equivalence class of all PSIMO's of a periodic system. Finally, it is addressed the issue of building a periodic realization of the delay-free periodic system. As in the time-invariant case, only matrices $B(\cdot)$ and $D(\cdot)$ actually change.

In Proposition 8.1, symbol $[P_k]_{i,j}$ will denote the i, j block entry of dimension $m \times m$ of the $mT \times mT$ matrix P_k of Expression (8.8).

Proposition 8.1

(i) A PSIMO of $G(\sigma, t)$ can be obtained from an LSIM of $W_\tau(z)$ by letting

$$L_{sT+k}(i+\tau) = [P_{s+d}]_{i+1, k+i+1-dT},$$

where

$$d = \begin{cases} 0, & k+1+i \leq T \\ 1, & \text{otherwise} \end{cases}$$

$$s = 0, 1, \dots, r$$

$$i, k = 0, 1, \dots, T-1.$$

(ii) The PSIMO is unique up to a right (operatorial) multiplication by a periodic matrix $V(t)$, orthonormal for each t .

(iii) A realization of $\bar{G}(\sigma, t) = G(\sigma, t)L(\sigma, t)$ is given by

$$\bar{S} \equiv \bar{S}(A(\cdot), \bar{B}(\cdot), C(\cdot), \bar{D}(\cdot)),$$

where

$$\bar{B}(t) = \sum_{i=0}^q \Phi_A(t+1, t+1-i)B(t-i)L_i(t-i)$$

$$\bar{D}(t) = D(t)L_0(t) + \sum_{i=1}^q C(t)\Phi_A(t, t+1-i)B(t-i)L_i(t-i)$$

$$q = rT + T - 1.$$

Proof Since the proof consists of simple but cumbersome algebraic passages, we will only give a sketch of the main steps.

(i) To obtain $L(\sigma, t)$ from $P(z)$ we first consider the inverse of these two operators, respectively: $L^\sim(\sigma, t)$ and $P^\sim(z)$ (note that the latter is indeed the transfer

function of a causal system). In order to analyze the input–output behavior of these two systems, notice that while σ is the unit forward operator in time t , σ^T represents the unit forward operator in the “sample time” k . Now, compare the signal

$$s(t) = L^\sim(\sigma, t) \cdot v(t) \quad (8.9)$$

with its corresponding “lifted version”

$$s_\tau(k) = P^\sim(\sigma^T) v_\tau(k), \quad (8.10)$$

where

$$s_\tau(k) = \begin{bmatrix} s(kT + \tau) \\ s(kT + \tau + 1) \\ \vdots \\ s(kT + \tau + T - 1) \end{bmatrix}, \quad v_\tau(k) = \begin{bmatrix} v(kT + \tau) \\ v(kT + \tau + 1) \\ \vdots \\ v(kT + \tau + T - 1) \end{bmatrix} \quad (8.11)$$

are the lifted representations of signals $s(\cdot)$ and $v(\cdot)$. It is left to the reader to derive $L_i(t)$ by means of (8.9)-(8.11).

(ii) The proof of this point follows from Remark 8.3 and from the lifting isomorphism between the periodic $m \times m$ matrices and the m -block diagonal $mT \times mT$ matrices.

(iii) This last statement is derived by lengthy calculations from (8.7) exploiting (8.3), (8.4) and (8.6). ■

The PSIMO operator is actually an anticipative operator and there exists an intimate relationship between the transfer operator of system S and the one of its “delay-free” image \bar{S} , respectively $G(\sigma, t)$ and $\bar{G}(\sigma, t)$. It is now important to clarify those assertions in a rigorous manner.

The original system S and its image \bar{S} can be compared by analyzing their invariant zero structure. With this aim we first state the definition of invariant zero of a square (more in general left-invertible) T -periodic system.

While it is obvious that systems S and \bar{S} have the same characteristic multipliers, we now show that they share the same finite zeros with the exception of some zeros in the origin.

Proposition 8.2 *The periodic systems \bar{S} and S have coincident non-zero invariant zeros.*

Proof It is not difficult to show that

$$\begin{bmatrix} \lambda \sigma I - A(t) & -B(t) \\ C(t) & D(t) \end{bmatrix} \begin{bmatrix} I & f(\lambda \sigma, t) \\ 0 & g(\lambda \sigma, t) \end{bmatrix} = \begin{bmatrix} \lambda \sigma I - A(t) & -\bar{B}(t) \\ C(t) & \bar{D}(t) \end{bmatrix},$$

where

$$f(\lambda \sigma, t) = \sum_{i=1}^{rT+T-1} \sum_{k=0}^{i-1} \Phi_A(t, t+1-i+k) B(t-i+k) L_i(t-i+k) \lambda^k \sigma^k$$

$$g(\lambda \sigma, t) = \sum_{i=0}^{rT+T-1} L_i(t) \lambda^i \sigma^i$$

are two polynomial operators. Since $g(\lambda \sigma, t) = L(\lambda \sigma, t)$, then $g(\lambda \sigma, t)$ is invertible if $\lambda \neq 0$; in this way the claim is evident.

8.5 Bibliographical Notes

The characterization of the zeros of periodic systems in terms of the lifted time-reformulation can be found in [83]. The geometric structure of the zeros has been developed in [169] where Example 8.5 was taken from. The idea underlying the computation of the structural zeros for a uniform relative degree SISO system was provided in [125].

The periodic version of the invariant subspace algorithm has been proposed in [168] and then used in [115] to establish the relationships between the periodic structure at infinity and the structure at infinity of the time-lifted reformulations. Geometric methods in control go back to the pioneer papers [21] and [308], see also [22] for a reference textbook.

Finally, the concept of interactor matrix can be found in classical texts as [307]. For square time-invariant systems the notion and characterization of the spectral interactor matrix has been provided in [246] and [57]. This notion has been extended to periodic systems in [58].

Chapter 9

Norms of Periodic Systems

This chapter is devoted to the characterization of the “size” of a periodic system. To be precise, we will introduce the notion of norm in a variety of facets: the L_2 norm, the L_∞ norm and the Hankel norm. Moreover, as a trade-off between the L_2 and the L_∞ norm, we also introduce the concept of “entropy”. As pointed out herein and in subsequent chapters, each of introduced system measures plays a specific role in diverse classical problems of analysis, filtering and control.

9.1 L_2 Norm

In this section, the notion of the L_2 -norm for a periodic system is introduced in the frequency domain. Later on, we will also provide the characterization in the time-domain, based on a periodic Lyapunov equation.

For a time-invariant SISO system with transfer function $W(z)$, the L_2 norm is defined as follows:

$$\|W(z)\|_2^2 = \frac{1}{2\pi} \left[\int_{-\pi}^{+\pi} |W(e^{j\theta})|^2 d\theta \right].$$

Notice that if one focuses on finite-dimensional systems, the above integral exists if and only if the transfer function $W(z)$ does not have unit modulus poles. In general, the vector space constituted by all functions $W(z)$ for which the integral exists is denoted by L_2 . Such a space can be put in a one-to-one correspondence with the vector space l_2 of all signals $w(t)$ which are square summable, namely for which the l_2 norm

$$\|w\|_{l_2}^2 = \sum_{t=-\infty}^{\infty} |w(t)|^2$$

exists. Indeed, if $W(z)$ is the bilateral z-transform of $w(t)$, then

$$\|W(z)\|_2 = \|w\|_{l_2}.$$

This is nothing but the celebrated Parseval rule. For a stable system, the L_2 norm coincides with the variance of the stationary process generated at the output when the input is a unit-variance white-noise. Therefore, one can see the L_2 norm of a stable SISO time-invariant system as the “the input–output variance gain”.

Turning to MIMO systems, the definition of L_2 norm takes the following form:

$$\|W(z)\|_2^2 = \frac{1}{2\pi} \text{trace} \left[\int_{-\pi}^{+\pi} W(e^{-j\theta})' W(e^{j\theta}) d\theta \right].$$

In the finite-dimensional case, this formula makes sense provided that $W(z)$ does not possess unit modulus poles. For stable MIMO systems, such a quantity coincides with the sum of the variances of the stationary processes at the various output channels produced by uncorrelated white-noise input signals with unit variances. Of course, the vector space of all stable transfer functions is a proper subspace of L_2 . In the relevant literature, this subspace is referred to as H_2 .

Turning now to the L_2 norm of a periodic system, some preliminary observations are in order. As seen in Chap. 6, there is a bijective correspondence between periodic systems and a subset of time-invariant ones, a correspondence which can be made explicit through the transfer function of the time-lifted and cyclic reformulations $W_\tau(z)$ and $\hat{W}_\tau(z)$, respectively. Whichever the adopted reformulation be, the obtained L_2 norm turns out to be the same. Indeed, from (6.31) it follows

$$\begin{aligned} \|\hat{W}_\tau(z)\|_2^2 &= \frac{1}{2\pi} \text{trace} \left[\int_{-\pi}^{+\pi} \hat{W}_\tau(e^{-j\theta})' \hat{W}_\tau(e^{j\theta}) d\theta \right] \\ &= \frac{1}{2\pi} \text{trace} \left[\int_{-\pi}^{+\pi} \nabla_m(e^{j\theta}) W_\tau(e^{-j\theta T})' W_\tau(e^{j\theta T}) \nabla_m(e^{-j\theta}) d\theta \right] \\ &= \frac{1}{2\pi} \text{trace} \left[\int_{-\pi}^{+\pi} W_\tau(e^{-jT\theta})' W_\tau(e^{jT\theta}) d\theta \right] \\ &= \frac{1}{2\pi T} \text{trace} \left[\int_{-\pi T}^{+\pi T} W_\tau(e^{-j\theta})' W_\tau(e^{j\theta}) d\theta \right] \\ &= \frac{1}{2\pi} \text{trace} \left[\int_{-\pi}^{+\pi} W_\tau(e^{-j\theta})' W_\tau(e^{j\theta}) d\theta \right] \\ &= \|W_\tau(z)\|_2^2. \end{aligned}$$

The second observation concerns the dependence of the norm of the lifted or cyclic reformulations upon the particular time tag τ . In view of the structure of $\Delta_k(z)$ in (6.11), $\hat{\Delta}_k$ in (6.29) and the recursive formulas (6.12) and (6.30), one obtains

$$\begin{aligned} \text{trace} \left[W_{\tau+1}(e^{-j\theta})' W_{\tau+1}(e^{j\theta}) \right] &= \text{trace} \left[W_{\tau}(e^{-j\theta})' W_{\tau}(e^{j\theta}) \right] \\ \text{trace} \left[\hat{W}_{\tau+1}(e^{-j\theta})' \hat{W}_{\tau+1}(e^{j\theta}) \right] &= \text{trace} \left[\hat{W}_{\tau}(e^{-j\theta})' \hat{W}_{\tau}(e^{j\theta}) \right], \end{aligned}$$

so that both $\|W_{\tau}(z)\|_2$ and $\|\hat{W}_{\tau}(z)\|_2$ are in fact independent of τ . This allows us to define the L_2 norm of a system whose periodic transfer function operator is $G(\sigma, t)$.

Definition 9.1 *The quantity*

$$\|G(\sigma, \cdot)\|_2 := \|W_{\tau}(z)\|_2 = \|\hat{W}_{\tau}(z)\|_2$$

(which is independent of τ) is the L_2 norm of the periodic system. ■

Obviously, the above norm turns out to be bounded if and only if the periodic system does not have unit modulus poles.

The L_2 norm can be expressed in terms of the transfer functions $G(\sigma, 0)$, $G(\sigma, 1)$, \dots , $G(\sigma, T-1)$. Indeed, from Eq. (6.19),

$$\begin{aligned} \|G(\sigma, \cdot)\|_2^2 &= \frac{1}{2\pi} \text{trace} \left[\int_{-\pi}^{+\pi} W_{\tau}(e^{-j\theta})' W_{\tau}(e^{j\theta}) d\theta \right] = \\ &= \frac{T}{2\pi} \text{trace} \left[\int_{-\pi/T}^{+\pi/T} W_{\tau}(e^{-j\theta T})' W_{\tau}(e^{j\theta T}) d\theta \right] \\ &= \frac{T}{2\pi} \sum_{r=0}^{T-1} \sum_{p=0}^{T-1} \sum_{k=0}^{T-1} \text{trace} \left[\int_{-\pi/T}^{+\pi/T} G(e^{-j\theta} \phi^k, r)' \sum_{q=1}^T \phi^{-(k+p)q} G(e^{j\theta} \phi^p, r) d\theta \right] \\ &= \frac{1}{2\pi} \sum_{r=0}^{T-1} \sum_{k=0}^{T-1} \text{trace} \left[\int_{-\pi/T}^{+\pi/T} G(e^{-j\theta} \phi^k, r)' G(e^{j\theta} \phi^{-k}, r) d\theta \right] \\ &= \frac{1}{2\pi} \sum_{r=0}^{T-1} \text{trace} \left[\int_{-\pi}^{+\pi} G(e^{-j\theta}, r)' G(e^{j\theta}, r) d\theta \right], \end{aligned}$$

so that the L_2 squared norm of the periodic system is given by the sum of the squared L_2 norms of the T rational functions $G(\sigma, t)$, i.e.,

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\sum_{i=0}^{T-1} \|G(\sigma, i)\|_2^2}. \quad (9.1)$$

It is worth noting that the L_2 norm of an unstable system $G(\sigma, t)$ can be computed by taking a new system $\hat{G}(\sigma, t)$, which is stable, and satisfying the operatorial identity $\hat{G}^\sim(\sigma, t)\hat{G}(\sigma, t) = G^\sim(\sigma, t)G(\sigma, t)$.

Example 9.1 Consider the periodic system of period $T = 2$ in Example 2.3. The transfer operator is

$$G(\sigma, t) = \frac{a(t)\sigma + b(t)}{\sigma^2 + 10},$$

with $a(0) = -1$, $a(1) = 3$, $b(0) = -2.5$, $b(1) = -12$. Notice that $G(\sigma, t)$ corresponds to an unstable system. It is easy to see that the adjoint operator $G^\sim(\sigma, t)$ is

$$G^\sim(\sigma, t) = \frac{\sigma^{-1}a(t) + b(t)}{\sigma^{-2} + 10} = \frac{a(t-1)\sigma^{-1} + b(t)}{\sigma^{-2} + 10},$$

so that

$$G^\sim(\sigma, t)G(\sigma, t) = \frac{a(t-1)^2 + b(t)^2 + a(t-1)b(t-1)\sigma^{-1} + a(t)b(t)\sigma}{101 + 10\sigma^2 + 10\sigma^{-2}}.$$

Now, notice that the operator

$$H(\sigma, t) = \frac{\sigma^2 + 10}{1 + 10\sigma^2}$$

is inner, i.e., $H^\sim(\sigma, t)H(\sigma, t) = 1$, and define

$$\hat{G}(\sigma, t) = H(\sigma, t)G(\sigma, t) = \frac{0.1a(t)\sigma + 0.1b(t)}{\sigma^2 + 0.1}.$$

The new system with transfer operator $\hat{G}(\sigma, t)$ is stable. Moreover, it turns out that

$$\hat{G}^\sim(\sigma, t)\hat{G}(\sigma, t) = G^\sim(\sigma, t)G(\sigma, t).$$

Therefore, the L_2 norm can be computed on the basis of $\hat{G}(\sigma, t)$, since

$$\|G(\sigma, \cdot)\|_2 = \|\hat{G}(\sigma, \cdot)\|_2.$$

Remark 9.1 (Modal decomposition and Pythagorean theorem)

The L_2 norm of an unstable system can be computed on the basis of its stable and anti-stable parts, by exploiting their orthogonality (Pythagorean theorem). Precisely, consider a transfer operator $G(\sigma, \cdot)$ without poles on the unit circle. Then, it is possible to write

$$G(\sigma, t) = (G(\sigma, t))_- + (G(\sigma, t))_+,$$

where $(G(\sigma, t))_-$ and $(G(\sigma, t))_+$ have all poles in or outside the unit disc, respectively. Moreover, $(G(\sigma, t))_+$ is chosen as strictly proper. To illustrate the way in which such modal decomposition can be performed, we adopt a geometric approach in state-space. Indeed, consider a realization $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of $G(\sigma, \cdot)$ and take a matrix basis $X(t)$ for the characteristic multipliers of $A(\cdot)$ inside the unit disk,

i.e.,

$$A(t)X(t) = X(t+1)A_1(t),$$

where $A_1(t)$ is the restriction of $A(t)$ with respect to the $A(\cdot)$ -invariant subspace $\text{Im}[X(t)]$. Of course $A_1(\cdot)$ is stable.

Analogously, take a matrix basis $Y(t)$ for the characteristic multipliers of $A(\cdot)$ outside the unit disk, i.e.,

$$A(t)Y(t) = Y(t+1)A_2(t),$$

where $A_2(\cdot)$ is anti-stable. Finally, letting $\Gamma(t)^{-1} = [X(t) \ Y(t)]$, it follows

$$\begin{aligned}\Gamma(t+1)A(t)\Gamma(t)^{-1} &= \begin{bmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{bmatrix} \\ \Gamma(t+1)B(t) &= \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} \\ C(t)\Gamma(t)^{-1} &= [C_1(t) \ C_2(t)],\end{aligned}$$

so that $G(\sigma, t)$ can be written as the parallel composition of a stable system and an anti-stable one, i.e.,

$$\begin{aligned}(G(\sigma, t))_- &= C_1(t)(\sigma I - A_1(t))^{-1}B_1(t) + D(t) \\ (G(\sigma, t))_+ &= C_2(t)(\sigma I - A_2(t))^{-1}B_2(t).\end{aligned}$$

In conclusion, due to the orthogonality of the stable and anti-stable parts, the L_2 norm of $G(\sigma, \cdot)$ obeys to:

$$\|G(\sigma, \cdot)\|_2^2 = \|(G(\sigma, t))_-\|_2^2 + \|(G(\sigma, t))_+\|_2^2.$$

This is just the expression of the celebrated Pythagorean theorem in the L_2 space. ■

For a stable system, the previously defined L_2 norm can be given interesting time-domain interpretations in terms of Lyapunov equations and impulse response.

9.1.1 Lyapunov Equation Interpretation

Consider the periodic Lyapunov equations already studied in Chap. 4:

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' \quad (9.2a)$$

$$Q(t) = A(t)'Q(t+1)A(t) + C(t)'C(t). \quad (9.2b)$$

As extensively discussed in Sect. 4.5, each of these equations admits a unique periodic solution (which is positive semi-definite) if $A(\cdot)$ is stable. In this case, the L_2

norm (also referred to as the H_2 norm) can be computed by means of $P(\cdot)$ or $Q(\cdot)$ as follows:

$$\|G(\sigma, \cdot)\|_2^2 = \text{trace} \left[\sum_{i=0}^{T-1} C(i)P(i)C(i)' + D(i)D(i)' \right] \quad (9.3a)$$

$$= \text{trace} \left[\sum_{i=0}^{T-1} B(i)'Q(i+1)B(i) + D(i)'D(i) \right]. \quad (9.3b)$$

Let's show the correctness of the second equality, the proof of the first one being completely analogous. This can be done, for instance, by resorting to the cyclic reformulation of the periodic system, which is characterized by the quadruplet $(\hat{F}_\tau, \hat{G}_\tau, \hat{H}_\tau, \hat{E}_\tau)$. In fact, from the time-invariant theory,

$$\|G(\sigma, \cdot)\|_2^2 = \|\hat{W}_\tau(z)\|_2^2 = \text{trace} [\hat{G}_\tau' \hat{Q}_\tau \hat{G}_\tau + \hat{E}_\tau' \hat{E}_\tau],$$

where \hat{Q}_τ is the unique solution of the algebraic Lyapunov equation:

$$\hat{F}_\tau' \hat{Q}_\tau \hat{F}_\tau + \hat{H}_\tau' \hat{H}_\tau = \hat{Q}_\tau. \quad (9.4)$$

Due to the particular structure of the matrices in the cyclic reformulation, it is easy to recognize that

$$\hat{Q}_\tau = \text{diag} \{ Q(\tau), Q(\tau+1), \dots, Q(\tau+T-1) \}$$

which immediately leads to our assertion. Analogously, it can be easily checked that

$$\hat{P}_\tau = \text{diag} \{ P(\tau), P(\tau+1), \dots, P(\tau+T-1) \}$$

is the unique solution of

$$\hat{F}_\tau \hat{P}_\tau \hat{F}_\tau' + \hat{G}_\tau \hat{G}_\tau' = \hat{P}_\tau. \quad (9.5)$$

Notice that the values at $t = \tau$ of the periodic solution of Eqs. (9.2) are the constant solutions of the Lyapunov equations associated with the lifted reformulation of the periodic system at $t = \tau$:

$$F_\tau P(\tau) F_\tau' + G_\tau G_\tau' = P(\tau) \quad (9.6a)$$

$$F_\tau' Q(\tau) F_\tau + H_\tau' H_\tau = Q(\tau). \quad (9.6b)$$

The matrices $(F_\tau, G_\tau, H_\tau, E_\tau)$ are those defining the lifted system.

To end this section, notice that, thanks to (9.1), the L_2 norm of a periodic system can be computed starting from the stable T rational functions $G(\sigma, t), t = 0, 1, \dots, T-1$.

Indeed, if $(A_t, B_t, C_t, D_t), t = 0, 1, \dots, T-1$, are T -associated state-space realizations, then

$$\|G(\sigma, \cdot)\|_2^2 = \sum_{t=0}^{T-1} \text{trace} [B_t' X_t B_t + D_t' D_t] = \sum_{i=0}^{T-1} \text{trace} [C_i Y_i C_i' + D_i D_i'] ,$$

where X_t, Y_t are the unique positive semi-definite solutions of the Lyapunov equations

$$\begin{aligned} X_t &= A_t' X_t A_t + C_t' C_t \\ Y_t &= A_t Y_t A_t' + B_t B_t', \end{aligned}$$

respectively.

Example 9.2 Consider again Example 2.3 and the computations carried out in Example 9.1. For $t = 0, 1$ let

$$A_t = \begin{bmatrix} 0 & 1 \\ -0.1 & 0 \end{bmatrix}, \quad B_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_t = [0.1b(t) \quad 0.1a(t)]$$

be state-space realizations of the rational functions $\hat{G}(\sigma, t), t = 0, 1$. Therefore, the solution of the Lyapunov equation $Y_t = A_t Y_t A_t' + B_t B_t'$ is $Y_t = 0.0101 I_2$ so that

$$\|G(\sigma, \cdot)\|_2^2 = \text{trace} [C_0 Y_0 C_0' + C_1 Y_1 C_1'] = 1.0101 [C_0 C_0' + C_1 C_1'] = 1.6187.$$

9.1.2 Impulse Response Interpretation

It is well-known that the L_2 norm of a stable, single-input, time-invariant system with transfer function $W(z)$, coincides with the (signal) l_2 norm of the output $h(t)$ of the system produced by a unitary impulse, i.e.,

$$\|W(z)\|_2 = \sqrt{\|h(t)\|_2^2} = \sqrt{\sum_{i=0}^{\infty} \|h(t)\|^2} = \sqrt{\sum_{t=0}^{\infty} h(t)' h(t)}.$$

If there are more input variables, say m , then the norm is

$$\|W(z)\|_2 = \sqrt{\sum_{j=1}^m \sum_{t=0}^{\infty} h^j(t)' h^j(t)},$$

where $h^j(t)$ denotes the forced output of the system produced by a unitary impulse at the j -th input channel.

For stable periodic systems, the interpretation of the L_2 norm in terms of impulse response is as follows. Take an arbitrary time tag τ and define $h^{i,j}(\tau)$ as the response

of the system to an impulse applied to the j -th input variable at time $\tau + i$ (with initial condition $x(\tau) = 0$). Thus

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\sum_{j=1}^m \sum_{i=0}^{T-1} \sum_{t=\tau}^{\infty} \|h^{i,j}(t)\|^2}. \quad (9.7)$$

To prove (9.7), recall that the impulse response $h^{i,j}(t)$ is related to the Markov parameters as follows

$$h^{i,j}(\tau + i + k) = \begin{cases} D(\tau + i)e_j & k = 0 \\ C(\tau + i + k)\Phi_A(\tau + i + k, \tau + i + 1)B(\tau + i)e_j & k > 0, \end{cases}$$

where e_j denotes the j -th column of the identity, whose dimensions are clear from the context. Now,

$$\begin{aligned} \|G(\sigma, \cdot)\|_2 &= \sqrt{\sum_{j=1}^m \sum_{i=0}^{T-1} \sum_{k=1}^{\infty} \|C(\tau + i + k)\Phi_A(\tau + i + k, \tau + i + 1)B(\tau + i)e_j\|^2} \\ &= \sqrt{\text{trace} \left[\sum_{i=0}^{T-1} B(\tau + i)'Q(\tau + i + 1)B(\tau + i) + D(\tau + i)'D(\tau + i) \right]}, \end{aligned}$$

where

$$Q(\tau) = \sum_{k=1}^{\infty} \Phi_A(\tau + k - 1, \tau)'C(\tau + k - 1)'C(\tau + k - 1)\Phi_A(\tau + k - 1, \tau).$$

Hence, as already seen, the L_2 norm does not depend upon the time tag τ and it is given by

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\text{trace} \left[\sum_{t=0}^{T-1} B(t)'Q(t+1)B(t) + D(t)'D(t) \right]}.$$

Finally, it is easy to check that $Q(\tau + i + 1)$ is the periodic solution of (9.2b), so that the proof of (9.7) is completed.

Notice that it is possible to rewrite (9.7) in terms of the periodic solution of (9.2a). Indeed,

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\sum_{j=1}^m \sum_{i=0}^{T-1} \sum_{t=\tau}^{\infty} h^{i,j}(t)'h^{i,j}(t)} = \sqrt{\sum_{j=1}^m \sum_{i=0}^{T-1} \sum_{t=\tau}^{\infty} \text{trace} [h^{i,j}(t)h^{i,j}(t)']}.$$

In view of the above expressions of $h^{i,j}(t)$, it turns out that

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\text{trace} \left[\sum_{i=0}^{T-1} C(\tau+i)P(\tau+i)C(\tau+i)' + D(\tau+i)D(\tau+i)' \right]},$$

where

$$P(\tau+i) = \sum_{k=1}^{\infty} \Phi_A(\tau+i-1, \tau+i-k)B(\tau+i-k)B(\tau+i-k)'\Phi_A(\tau+i-1, \tau+i-k)'.$$

Hence $P(\cdot)$ is the periodic solution of (9.2a) and

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\text{trace} \left[\sum_{t=0}^{T-1} C(t)P(t)C(t)' + D(t)D(t)' \right]}.$$

9.1.3 Stochastic Interpretation

Besides the deterministic interpretation concerning the impulse response of the system, the L_2 norm lends itself to an interpretation in terms of power spectral density, as already pointed out at the beginning of the chapter. Precisely, assume that the system is stable and subject to an input signal given by a white-noise with identity covariance. Moreover, denote by $E[\cdot]$ the expectation operator. It turns out that

$$\|G(\sigma, \cdot)\|_2 = \sqrt{\lim_{k \rightarrow \infty} \sum_{t=0}^{T-1} E[y(kT+t)'y(kT+t)]} \quad (9.8a)$$

$$= \sqrt{\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \sum_{t=0}^{T-1} E[y(kT+t)'y(kT+t)]}. \quad (9.8b)$$

To prove (9.8) just notice that

$$\begin{aligned} E[y(t+kT)'y(t+kT)] &= E[\text{trace}[y(t+kT)y(t+kT)']] \\ &= \text{trace}[D(t)D(t)' + C(t)X(t+kT)C(t)'], \end{aligned}$$

where

$$X(t+kT) = \sum_{i=0}^{t+kT-1} \Phi_A(t+kT, i+1)B(i)B(i)'\Phi_A(t+kT, i+1)'$$

is the solution of the difference Eq. (9.2a) at time $t+kT$, with initial condition $X(0) = 0$. Thanks to the stability of $A(\cdot)$, it follows that $X(t+kT)$ converges to the T -periodic solution $P(t)$ of (9.2a), $\forall t \in [0, T-1]$. Therefore (9.8a) holds.

As for (9.8b), notice that

$$Y_N(t) = \frac{1}{N} \sum_{k=0}^{N-1} X(t + kT)$$

again satisfies the difference Eq. (9.2a), so that (9.8b) follows from

$$\lim_{N \rightarrow \infty} Y_N(t) = P(t),$$

where $P(\cdot)$ is the T -periodic solution of (9.2a).

9.2 L_∞ Norm

In this section, we define the concept of L_∞ norm of a periodic system. Before entering this context, first recall the meaning of the L_∞ norm of a SISO time-invariant system with transfer function $W(z)$. Assume that $W(z)$ does not have poles on the unit circle and define the L_∞ norm as the maximum value of the modulus of the frequency response, i.e.,

$$\|W(z)\|_\infty = \max_{\theta} |G(e^{j\theta})|.$$

If the system is multivariable, then the L_∞ norm is defined as

$$\|W(z)\|_\infty = \max_{\theta} \bar{\sigma} \left[G(e^{j\theta}) \right] = \max_{\theta} \sqrt{\lambda_{\max} [G(e^{-j\theta}) G(e^{j\theta})]},$$

where $\lambda_{\max}[\cdot]$ denotes the maximum eigenvalue.

In order to define the L_∞ norm of a periodic system with transfer operator $G(\sigma, \cdot)$, observe first that the L_∞ norm of the transfer function $W_\tau(z)$ associated with the time-lifted reformulation coincides with the L_∞ norm of the transfer function $\hat{W}_\tau(z)$ associated with the cyclic reformulations. Indeed, from (6.31) it follows that

$$\begin{aligned} \|\hat{W}_\tau(z)\|_\infty^2 &= \left[\max_{\vartheta} \lambda_{\max} \left[\hat{W}_\tau(e^{-j\vartheta})' \hat{W}_\tau(e^{j\vartheta}) \right] \right] \\ &= \left[\max_{\vartheta} \lambda_{\max} \left[\nabla_m(e^{j\vartheta}) W_\tau(e^{-j\vartheta T})' \nabla_p(e^{-j\vartheta}) \nabla_p(e^{j\vartheta}) W_\tau(e^{j\vartheta T}) \nabla_m(e^{-j\vartheta}) \right] \right]. \end{aligned}$$

Considering (6.26), this expression can be further elaborated yielding

$$\begin{aligned}\|\hat{W}_\tau(z)\|_\infty^2 &= \left[\max_{\vartheta} \lambda_{\max} \left[\nabla_m(e^{j\vartheta}) W_\tau(e^{-j\vartheta T})' W_\tau(e^{j\vartheta T}) \nabla_m(e^{-j\vartheta}) \right] \right] \\ &= \left[\max_{\vartheta} \lambda_{\max} \left[W_\tau(e^{-j\vartheta})' W_\tau(e^{j\vartheta}) \right] \right] \\ &= \|W_\tau(z)\|_\infty^2.\end{aligned}$$

Moreover, both norms do not depend upon τ . To prove this fact, focus on the time-lifted reformulation. From (6.12), it follows that

$$\begin{aligned}\|W_{\tau+1}(z)\|_\infty^2 &= \left[\max_{\vartheta} \lambda_{\max} \left[W_{\tau+1}(e^{-j\vartheta})' W_{\tau+1}(e^{j\vartheta}) \right] \right] \\ &= \left[\max_{\vartheta} \lambda_{\max} \left[\Delta_m(e^{-j\vartheta})' W_\tau(e^{-j\vartheta})' \Delta_p(e^{j\vartheta}) \Delta_p(e^{-j\vartheta})' W_\tau(e^{j\vartheta}) \Delta_m(e^{j\vartheta}) \right] \right] \\ &= \left[\max_{\vartheta} \lambda_{\max} \left[\Delta_m(e^{-j\vartheta})' W_\tau(e^{-j\vartheta})' W_\tau(e^{j\vartheta}) \Delta_m(e^{j\vartheta}) \right] \right] \\ &= \left[\max_{\vartheta} \lambda_{\max} \left[W_\tau(e^{-j\vartheta})' W_\tau(e^{j\vartheta}) \right] \right] \\ &= \|W_\tau(z)\|_\infty^2.\end{aligned}$$

Therefore, if $G(\sigma, \cdot)$ is the transfer function operator of the periodic system, the following definition makes sense.

Definition 9.2 *The quantity*

$$\|G(\sigma, \cdot)\|_\infty = \|W_\tau(z)\|_\infty = \|\hat{W}_\tau(z)\|_\infty, \quad \forall \tau$$

is the L_∞ norm of the periodic system. ■

Obviously, this norm is bounded if the periodic system does not have unit modulus poles.

9.2.1 Input–Output Interpretation

The L_∞ norm above defined admits a most useful interpretation as an input–output “gain”. This can be shown by making reference to the space $l_2(\tau, \infty)$ of sequences $v(k)$ such that

$$\sum_{k=\tau}^{\infty} v(k)'v(k) < \infty.$$

Now consider again the time-lifted reformulation described by the input–output equation $Y_{\tau}(z) = W_{\tau}(z)U_{\tau}(z)$, where $U_{\tau}(z)$ and $Y_{\tau}(z)$ are the Z -transforms of the lifted signals at τ corresponding to $u(\cdot)$ and $y(\cdot)$, respectively. Precisely,

$$u_{\tau}(k) = [u(kT + \tau)' u(kT + \tau + 1)' \cdots u(kT + \tau + T - 1)']',$$

and analogously for $y_{\tau}(k)$. It is immediate to verify that

$$\|u\|_{l_2(\tau, \infty)}^2 = \sum_{t=\tau}^{\infty} u(t)'u(t) = \sum_{k=0}^{\infty} u_{\tau}(k)'u_{\tau}(k) = \|u_{\tau}\|_{l_2(0, \infty)}^2,$$

so that $u_{\tau}(k) \in l_2(0, \infty)$ if and only if $u(t) \in l_2(\tau, \infty)$. Obviously, the same can be said for $y_{\tau}(k)$ and $y(t)$. Then, assume that the system is stable and recall the interpretation of the L_{∞} norm (or, equivalently, the H_{∞} norm) of a time-invariant system as l_2 gain,

$$\|W_{\tau}(z)\|_{\infty} = \sup_{\substack{u_{\tau} \neq 0 \\ u_{\tau} \in l_2(0, \infty)}} \frac{\|y_{\tau}\|_{l_2(0, \infty)}}{\|u_{\tau}\|_{l_2(0, \infty)}} = \sup_{\substack{u \neq 0 \\ u \in l_2(\tau, \infty)}} \frac{\|y\|_{l_2(\tau, \infty)}}{\|u\|_{l_2(\tau, \infty)}}.$$

Therefore, in view of Definition 9.2,

$$\|G(\sigma, \cdot)\|_{\infty} = \sup_{\substack{u \neq 0 \\ u \in l_2(\tau, \infty)}} \frac{\|y\|_{l_2(\tau, \infty)}}{\|u\|_{l_2(\tau, \infty)}}.$$

This formula provides the *gain interpretation* of the H_{∞} norm.

In Sect. 2.3 we have introduced the concept of transfer operator in frequency domain. Indeed, in an EMP regime, the values of the input and output signals over one period are related as

$$y(t) = G_{\lambda}(\sigma, t) \cdot u(t), \quad t = 0, 1, \dots, T-1,$$

where the frequency transfer operator $G_{\lambda}(\sigma, t)$ is determined by the state–space matrices as follows (recall Eq. (2.26))

$$G_{\lambda}(\sigma, t) = D(t) + C(t)(\lambda^T I - \Psi_A(t))^{-1} \sum_{j=0}^{T-1} \Phi_A(t, t+j-T+1)B(t+j)\sigma^j.$$

It is easy to see that the L_{∞} norm of the periodic system can be given an interpretation in terms of this operator. In this connection, define the norm of $G_{\lambda}(\sigma, \cdot)$ as the induced norm

$$\|G_\lambda(\sigma, \cdot)\| = \sup_{\substack{u \neq 0 \\ u \in l_2(0, T)}} \frac{\|y\|_{l_2(0, T)}}{\|u\|_{l_2(0, T)}},$$

where the norm of a (complex) signal $v(\cdot)$ in $l_2(0, T)$ is defined as

$$\|v(\cdot)\|_{l_2[0, T]}^2 = \sum_{i=0}^{T-1} v(t)^* v(t).$$

If the periodic system is stable, it is possible to compute the supremum value of the norm of $G_\lambda(\sigma, t)$ as λ varies outside the open unit disk. This supremum value exists and coincides with the L_∞ norm. In conclusion, for a stable system

$$\|G(\sigma, \cdot)\|_\infty = \sup_{|\lambda| \geq 1} \|G_\lambda(\sigma, \cdot)\|.$$

9.2.2 Riccati Equation Interpretation

An important question is whether the infinite norm of a periodic system is bounded by some positive value γ . In terms of the cyclic reformulation, the problem amounts to finding conditions under which $\|\hat{W}_\tau(z)\|_\infty < \gamma$. For this purpose, we apply the time-invariant theory to the system $(\hat{F}_\tau, \hat{G}_\tau, \hat{H}_\tau, \hat{E}_\tau)$, supplying a realization of the cyclic transfer function

$$\hat{W}_\tau(z) = \hat{H}_\tau (zI - \hat{F}_\tau)^{-1} \hat{G}_\tau + \hat{E}_\tau.$$

This requires introducing the algebraic Riccati equation

$$\hat{Q}_\tau = \hat{F}_\tau' \hat{Q}_\tau \hat{F}_\tau + \hat{H}_\tau' \hat{H}_\tau + \hat{K}_\tau' \hat{V}_\tau \hat{K}_\tau, \quad (9.9)$$

with

$$\hat{K}_\tau = \hat{V}_\tau^{-1} [\hat{G}_\tau' \hat{Q}_\tau \hat{F}_\tau + \hat{E}_\tau' \hat{H}_\tau]$$

$$\hat{V}_\tau = \gamma^2 I - \hat{E}_\tau' \hat{E}_\tau - \hat{G}_\tau' \hat{Q}_\tau \hat{G}_\tau.$$

The Bounded Real Lemma is as follows. Let \hat{F}_τ be stable. Then, $\|\hat{W}_\tau(z)\|_\infty < \gamma$ if and only if there exists a positive semi-definite solution \hat{Q}_τ to the above Riccati equation, such that $\hat{V}_\tau > 0$ and $\hat{F}_\tau + \hat{G}_\tau \hat{K}_\tau$ is stable.

Of course the condition given above is also a necessary and sufficient condition for a periodic system to have an L_∞ norm less than γ . However, it is possible to cast the problem in a purely periodic realm, thus leading to the Periodic Bounded Real Lemma.

In mathematical terms this corresponds to finding conditions under which the following operator inequality is satisfied

$$G^\sim(\sigma, \cdot)G(\sigma, \cdot) < \gamma^2 I, \quad (9.10)$$

where $G^\sim(\sigma, \cdot)$ is the adjoint operator of $G(\sigma, \cdot)$. Indeed, if the system is

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned}$$

then

$$G(\sigma, t) = C(t)(\sigma I - A(t))^{-1}B(t) + D(t)$$

and

$$G^\sim(\sigma, t) = B(t)'(\sigma^{-1}I - A(t)')^{-1}C(t)' + D(t)'.$$

It is possible to write a state-space realization for the adjoint operator in terms of the descriptor system (see also Sect. 2.5):

$$\begin{aligned} A(t)'\lambda(t+1) &= \lambda(t) - C(t)'y(t) \\ w(t) &= B(t)'\lambda(t+1) + D(t)'y(t). \end{aligned}$$

It is easy to verify that the transfer functions of the lifted and cyclic reformulations at τ of the adjoint system are given by $W_\tau(z^{-1})'$ and $\hat{W}_\tau(z^{-1})'$, respectively. Now, in

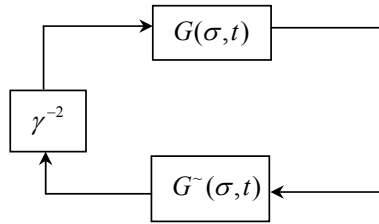


Fig. 9.1 The feedback configuration for the H_∞ norm

order to give a state-space interpretation to Eq. (9.10) in terms of Riccati equation, consider the interconnected system depicted in Fig. 9.1. A cumbersome computation shows that the state $x(t)$ and the co-state $\lambda(t)$ are related by $\lambda(t) = Q(t)x(t)$, where $Q(\cdot)$ is a solution of the T -periodic Riccati equation

$$Q(t) = A(t)'Q(t+1)A(t) + C(t)'C(t) + K(t)'V(t)K(t) \quad (9.11a)$$

$$K(t) = V(t)^{-1} [B(t)'Q(t+1)A(t) + D(t)'C(t)] \quad (9.11b)$$

$$V(t) = \gamma^2 I - D(t)'D(t) - B(t)'Q(t+1)B(t). \quad (9.11c)$$

As easily verified by inspection, the solution \hat{Q}_τ of (9.9) is related to the periodic solution $Q(\cdot)$ of (9.11a) in the following way

$$\hat{Q}_\tau = \text{diag}(Q(\tau) \ Q(\tau+1) \ \cdots \ Q(\tau+T-1)) .$$

Analogously,

$$\begin{aligned} \hat{V}_\tau &= \text{diag}(V(\tau) \ V(\tau+1) \ \cdots \ V(\tau+T-1)) \\ \hat{K}_\tau &= \text{diag}(K(\tau) \ K(\tau+1) \ \cdots \ K(\tau+T-1)) . \end{aligned}$$

Finally, notice that \hat{Q}_τ satisfies (9.9) and

$$\begin{aligned} \hat{Q}_\tau \geq 0 &\iff Q(\cdot) \geq 0 \\ \hat{V}_\tau \geq 0 &\iff V(\cdot) \geq 0 \\ \hat{F}_\tau + \hat{G}_\tau \hat{K}_\tau \text{ stable} &\iff A(\cdot) + B(\cdot)K(\cdot) \text{ stable} . \end{aligned}$$

We are now in the position to state the following theorem, whose proof directly is demonstrated from the considerations above.

Proposition 9.1 (Periodic Bounded Real Lemma)

Let $A(\cdot)$ be stable. Then, the L_∞ norm of the periodic system is less than γ if and only if there exists a positive semi-definite T -periodic solution $Q(\cdot)$ of (9.11) so that

- i) $V(t) > 0, \forall t$
- ii) $A(t) + B(t)K(t)$ is stable. ■

A solution of (9.11) satisfying condition ii) is said to be *stabilizing*, and it is said to be *feasible* if it satisfies condition i). It can be proven that, if there exists such a solution, it is the unique stabilizing solution.

The Riccati Equation (9.11) lends itself to a direct computation of the spectral factor of

$$\gamma^2 I - G^\sim(\sigma, \cdot) G(\sigma, \cdot)$$

Indeed, if $\gamma^2 I - G^\sim(\sigma, \cdot) G(\sigma, \cdot)$ is a positive definite operator, then it is possible to find a bi-causal rational operator $T(\sigma, \cdot)$ so that

$$\gamma^2 I - G^\sim(\sigma, \cdot) G(\sigma, \cdot) = T^\sim(\sigma, \cdot) T(\sigma, \cdot) . \quad (9.12)$$

The relevant computation makes use of the following operator identities.

$$\begin{aligned} Q(t) - A(t)' Q(t+1) A(t) &= (I - A(t)' \sigma) Q(t) + A(t)' Q(t+1) (\sigma I - A(t)) \\ \sigma Q(t) (\sigma I - A(t))^{-1} &= Q(t+1) + Q(t+1) A(t) (\sigma I - A(t))^{-1} . \end{aligned}$$

The conclusion follows by left [right] multiplying Eq. (9.11a) by $B(t)'(\sigma^{-1}I - A(t)')[(\sigma I - A(t))^{-1}B(t)]$ and by completing the squares. It turns out that

$$\gamma^2 I - G^\sim(\sigma, \cdot) G(\sigma, \cdot) = \bar{T}^\sim(\sigma, \cdot) V(t) \bar{T}(\sigma, \cdot) , \quad (9.13)$$

where $V(\cdot)$ was defined in (9.11c) and

$$\bar{T}(\sigma, t) = I - K(t)(\sigma I - A(t))^{-1}B(t).$$

Hence, if $Q(\cdot)$ is feasible, $V(\cdot) > 0$ so that the H_∞ norm of the system is less than γ and a spectral factor $T(\sigma, \cdot)$ in (9.12) is

$$T(\sigma, t) = V(t)^{1/2} (I - K(t)(\sigma I - A(t))^{-1}B(t)) .$$

Moreover, if $Q(\cdot)$ is stabilizing, then $T(\sigma, \cdot)$ has a stable inverse. Indeed, exploiting the matrix inversion lemma, the dynamic matrix of $T(\sigma, \cdot)^{-1}$ is $A(t) + B(t)K(t)$.

The periodic bounded real lemma (Proposition 9.1) can be equivalently formulated in terms of the dual Riccati equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' + Y(t)W(t)Y(t)' \quad (9.14a)$$

$$Y(t) = [A(t)P(t)C(t)' + B(t)D(t)'] W(t)^{-1} \quad (9.14b)$$

$$W(t) = \gamma^2 I - D(t)D(t)' - C(t)P(t)C(t)' . \quad (9.14c)$$

Therefore, it can be concluded that, if $A(\cdot)$ is stable, then the L_∞ norm of the periodic system is less than γ if and only if there exists a positive semi-definite T -periodic solution of (9.14) so that $W(t) > 0$, $\forall t$ and $A(t) + Y(t)C(t)$ is stable. In this case, the equivalent of (9.13) takes the form

$$\gamma^2 I - G(\sigma, \cdot)G^\sim(\sigma, \cdot) = \hat{T}(\sigma, \cdot)W(t)\hat{T}^\sim(\sigma, \cdot), \quad (9.15)$$

where

$$\hat{T}(\sigma, t) = I - C(t)(\sigma I - A(t))^{-1}Y(t).$$

Remark 9.2 (LMI formulation)

Proposition 9.1 is based on the stabilizing solution of the Riccati Equation (9.11). Instead of using the Riccati equation, it is possible to formulate the bounded real lemma on the basis of a Riccati inequality, which in turn, can be equivalently rewritten in terms of a periodic LMI. Indeed, recall the definitions of $Y(\cdot)$ and $W(t)$ given in equations (9.14b) and (9.14c), and consider, for $t = 0, 1, \dots, T-1$, the matrix

$$M(t) = \begin{bmatrix} P(t+1) - A(t)P(t)A(t)' - B(t)B(t)' & Y(t)W(t) \\ W(t)Y(t)' & W(t) \end{bmatrix},$$

where $P(\cdot)$ is constrained to adhere to the periodicity condition $P(T) = P(0)$. Hence, the inequality $M(t) > 0$ corresponds to a set of T LMIs'. Such a set is also referred to as a Periodic Linear Matrix Inequality (PLMI). On the other hand, in view of the Schur complement lemma, the condition $M(t) > 0$ is equivalent to the T -periodic inequalities

$$P(t+1) > A(t)P(t)A(t)' + Y(t)W(t)Y(t)' + B(t)B(t)', \quad W(t) > 0, \quad (9.16)$$

where $W(t)$ and $Y(t)$ have been already defined in (9.14b), (9.14c). It is possible to prove that, if $P(\cdot) \geq 0$ is a T -periodic solution of the inequality $M(\cdot) > 0$, then the inequality version of Eq. (9.15) holds, namely

$$\gamma^2 I - G(\sigma, \cdot) G^\sim(\sigma, \cdot) > \hat{T}(\sigma, \cdot) W(t) \hat{T}^\sim(\sigma, \cdot).$$

This means that $M(\cdot) > 0$ is a sufficient condition for the L_∞ norm of the system to be less than γ .

The converse result is also true, namely under the stability condition of matrix $A(\cdot)$, the norm less than γ implies the existence of a periodic positive semi-definite (definite) matrix $P(\cdot)$ satisfying the Inequalities (9.16). As a matter of fact, define the transfer operator

$$\bar{G}(\sigma, t) = [G(\sigma, t) \quad \varepsilon \Xi(\sigma, t)], \quad \Xi(\sigma, t) = C(t) (\sigma I - A(t))^{-1}.$$

Then

$$\|\bar{G}(\sigma, \cdot)\|_\infty^2 \leq \|G(\sigma, \cdot)\|_\infty^2 + \varepsilon^2 \|\Xi(\sigma, \cdot)\|_\infty^2.$$

Now, letting

$$\varepsilon = \frac{\gamma^2 - \|G(\sigma, \cdot)\|_\infty^2}{\|\Xi(\sigma, \cdot)\|_\infty^2},$$

it follows

$$\|\bar{G}(\sigma, \cdot)\|_\infty < \gamma.$$

Finally, notice that

$$\bar{G}(\sigma, \cdot) = C(t) (\sigma I - A(t))^{-1} [B(t) \sqrt{\varepsilon} I] + [D(t) \quad 0].$$

This means that there exists the stabilizing T -periodic solution $P(\cdot)$ of

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' + \varepsilon I + Y(t)W(t)Y(t)',$$

where $Y(\cdot)$ and $W(\cdot)$ are the same as in (9.14b), (9.14c). This proves the converse result.

In conclusion, if $A(\cdot)$ is stable, it is possible to state that the L_∞ norm of the system is less than γ if and only if there exists a T -periodic $P(\cdot) \geq 0$ so that $M(\cdot) > 0$. ■

The interpretation of the H_∞ norm in terms of stability of feedback systems can be useful in characterizing the quadratic stability of norm-bounded uncertain systems (recall Chap. 3.3.6).

Proposition 9.2 Consider the uncertain system

$$x(t+1) = (A_n(t) + L(t)\Delta(t)N(t))x(t),$$

with $A_n(\cdot)$ stable, and $\Delta(t)$ is any (complex) matrix and therefore $\|\Delta(t)\| \leq 1$. This norm-bounded uncertain system is quadratically stable if, for each real number $\beta >$

0, the L_∞ norm of the T -periodic system

$$x(t+1) = A_n(t)x(t) + \beta L(t)w(t) \quad (9.17a)$$

$$y(t) = \frac{N(t)}{\beta}x(t) \quad (9.17b)$$

is lower than 1.

Proof As seen in Remark 9.2, if the System (9.17) has L_∞ norm lower than one, then there exists a T -periodic positive semi-definite solution $P(\cdot)$ of the inequality

$$\begin{bmatrix} P(t+1) - A_n(t)P(t)A_n(t)' - \beta^2 L(t)L(t)' & \beta^{-1}A_n(t)P(t)N(t)' \\ \beta^{-1}N(t)P(t)A_n(t)' & I - \beta^{-2}N(t)P(t)N(t)' \end{bmatrix} > 0.$$

This inequality can be equivalently rewritten as

$$\begin{aligned} P(t+1) &> (A_n(t) + L(t)\Delta(t)N(t))P(t)(A_n(t) + L(t)\Delta(t)N(t))' + R(t) \\ R(t) &= X(t)S(t)^{-1}X(t)' \\ X(t) &= A_n(t)P(t)N(t)' - L(t)\Delta(t)S(t) \\ S(t) &= \beta^2 I - N(t)P(t)N(t)' > 0. \end{aligned}$$

Of course, matrix $S(\cdot)$ is positive semi-definite due to the positive definiteness of $R(\cdot)$. Hence,

$$P(t+1) > (A_n(t) + L(t)\Delta(t)N(t))P(t)(A_n(t) + L(t)\Delta(t)N(t))',$$

so that $A_n(t) + L(t)\Delta(t)N(t)$ is stable for each $\Delta(t)$ with $\|\Delta(t)\| < 1$. In view of Remark 3.3, $x(t)'Q(t)x(t)$ with $Q(t) = P(t)^{-1}$ is a common Lyapunov function. This implies quadratic stability. ■

In Proposition 3.6 it was stated that the norm-bounded uncertain system is quadratically stable if the H_∞ norm of the underlying periodic system is less than 1, or, equivalently, if the associated PLMI admits a T -periodic positive definite solution. As anticipated in Sect. 3.3.7, the converse statement is generally false, as witnessed by the following simple counterexample.

Example 9.3 Consider the scalar periodic system of period $T = 2$

$$x(t+1) = (A(t) + L(t)\Delta(t)N(t))x(t),$$

where

$$A(0) = A(1) = 0, \quad L(0) = 2, \quad L(1) = 1/3, \quad N(0) = N(1) = 1$$

and $\Delta(t)$ is any periodic function with $|\Delta(t)| \leq 1$. Notice that this system is quadratically stable. Indeed, take $V(x, t) = x'Q(t)x$ as a Lyapunov function with

$$Q(0) = 4.5, \quad Q(1) = 0.6.$$

We have:

$$\begin{aligned} 4.5 &= Q(0) > (A(0) + L(0)\Delta(0)N(0))Q(1)(A(0) + L(0)\Delta(0)N(0))' = 4\Delta(0)^2 \\ 0.6 &= Q(1) > (A(1) + L(1)\Delta(1)N(1))Q(0)(A(1) + L(1)\Delta(1)N(1))' = 0.5\Delta(1)^2, \end{aligned}$$

so that the system is quadratically stable. However, as we now show, the H_∞ norm of $(A(\cdot), L(\cdot), N(\cdot))$ is far from being less than 1.

To compute this norm, consider the system rewritten in feedback form:

$$\begin{aligned} x(t+1) &= A(t)x(t) + L(t)u(t) \\ y(t) &= N(t)x(t) \\ u(t) &= \Delta(t)y(t). \end{aligned}$$

The cyclic representation of the feedback system is

$$\begin{aligned} \hat{x}(t+1) &= \hat{F}\hat{x}(t) + \hat{G}\hat{u}(t) \\ \hat{y}(t) &= \hat{H}\hat{x}(t) \\ \hat{u}(t) &= \hat{\Delta}\hat{y}(t), \end{aligned}$$

where \hat{F} is the 2-dimensional zero matrix, \hat{H} is the 2-dimensional identity matrix and

$$\hat{G} = \begin{bmatrix} 0 & 1/3 \\ 2 & 0 \end{bmatrix}, \quad \hat{\Delta} = \begin{bmatrix} \Delta(0) & 0 \\ 0 & \Delta(1) \end{bmatrix}.$$

Now, the H_∞ norm of the periodic system $(A(\cdot), L(\cdot), N(\cdot))$ coincides with the H_∞ norm of the time-invariant system $(\hat{F}, \hat{G}, \hat{H})$. A simple computation shows that this norm is 2, which is greater than 1.

Notice that, since the periodic system has a norm greater than 1, and in view of the small-gain theorem, there exists a T -periodic system, whose H_∞ norm is less than 1, which destabilizes the closed-loop system when connected in feedback configuration. Indeed, it suffices to take the periodic dynamic feedback

$$u(t+1) = \alpha(t)y(t), \quad \alpha(0) = 1/9, \quad \alpha(1) = 2/3.$$

This feedback dynamic system has H_∞ norm equal to $2/3 < 1$ and is capable of destabilizing the closed-loop system. Indeed, as it can be easily verified, the closed-loop system has a characteristic multiplier equal to $4/3 > 1$. ■

As seen above, in the periodic case, quadratic stability (with $\|\Delta(\cdot)\| \leq 1$) does not imply the H_∞ norm being less than 1. The reason is that the matrix Δ , even if complex, is reformulated in the feedback configuration as a periodic system with a specific block-diagonal form.

Remark 9.3 (A simpler form of the Riccati equation)

If the T -periodic solution is positive definite, then the Riccati Equation (9.11) can also be written in the compact way:

$$Q(t) = \begin{bmatrix} A(t)' & C(t)' \end{bmatrix} \begin{bmatrix} Q(t+1)^{-1} - \frac{1}{\gamma^2} B(t)B(t)' & -\frac{1}{\gamma^2} B(t)D(t)' \\ -\frac{1}{\gamma^2} D(t)B(t)' & I - \frac{1}{\gamma^2} D(t)D(t)' \end{bmatrix}^{-1} \begin{bmatrix} A(t) \\ C(t) \end{bmatrix}.$$

Moreover, if the usual assumptions $D(t)'D(t) = I$, $\forall t$ and $D(t)'B(t) = 0$, $\forall t$ are made, then the equation simplifies to

$$Q(t) = A(t)' \left[Q(t+1)^{-1} - \frac{1}{\gamma^2} B(t)B(t)' \right]^{-1} A(t) + C(t)'C(t).$$

Remark 9.4 (Difference game)

The solution $Q(\cdot)$ of (9.11) at $t = \tau$ (with properties i) and ii)), can be also related to the following optimization problem for the system with non-zero initial condition $x(\tau)$:

$$\sup_{u \in L_2[\tau, \infty)} \|y\|_2^2 - \gamma^2 \|u\|_2^2.$$

It is easily seen that such a problem has the solution

$$\sup_{u \in L_2[\tau, \infty)} \|y\|_2^2 - \gamma^2 \|u\|_2^2 = x(\tau)' P(\tau) x(\tau).$$

Indeed, from (9.11) it follows that

$$\begin{aligned} & x(t)' Q(t) x(t) - x(t+1)' Q(t+1) x(t+1) \\ &= x(t)' [Q(t) - A(t)' Q(t+1) A(t)] x(t) + 2x(t)' Q(t+1) B(t) u(t) \\ &= y(t)' y(t) - \gamma^2 u(t)' u(t) + q(t)' q(t), \end{aligned}$$

where

$$q(t) = V(t)^{-1/2} [B(t)' Q(t+1) A(t) + D(t)' C(t)] x(t) - V(t)^{1/2} u(t)$$

and $V(\cdot)$ was defined in (9.11c). By taking the sum of both members from $t = \tau$ to $t = \infty$, we have

$$x(\tau)' Q(\tau) x(\tau) = \|y\|_2^2 - \gamma^2 \|u\|_2^2 + \|q\|_2^2,$$

so that the conclusion easily follows by noticing that $q \equiv 0$ corresponds to the optimal input

$$u(t) = V(t)^{-1} [B(t)' Q(t+1) A(t) + D(t)' C(t)] x(t)$$

belonging to $l_2(\tau, \infty)$, in view of the stabilizing property of $Q(\cdot)$.

9.3 Entropy

The Riccati Equation (9.11) can also be given an interpretation in terms of the so-called γ -entropy of the system, a notion associated with the feedback system with an uncertain return transfer function. To be precise, the entropy is connected with the expected value of the L_2 norm of a closed-loop transfer function when the return element has a frequency response seen as a random variable taking independent values at different frequencies. We decline here to discuss this issue in detail and limit ourselves to the problem of entropy computation in terms of Riccati equations.

Consider a periodic system with transfer operator $G(\sigma, \cdot)$ and take the transfer function $\hat{W}_\tau(z)$ of its associated cyclic reformulation at τ . Moreover, consider the index

$$I_\gamma = -\frac{\gamma^2}{2\pi} \int_{-\pi}^{\pi} \log \det \left[I - \frac{\hat{W}_\tau(e^{-j\theta})' \hat{W}_\tau(e^{j\theta})}{\gamma^2} \right] d\theta.$$

This index is well defined if $\|\hat{W}_\tau(z)\|_\infty < \gamma$. It is easy to check that, thanks to the recursive formula (6.30), I_γ is independent of τ . It can be named the γ -entropy of the periodic system.

Hereafter, we first derive the general relationship between the γ -entropy and the L_2 norm, valid for any $\gamma > \|G(\sigma, \cdot)\|_\infty$. From this relationship, it will be seen that, for γ tending to infinity, the difference between the γ -entropy and the squared L_2 norm vanishes.

We start the derivation by first noticing that I_γ can be rewritten as

$$I_\gamma = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_i f(\sigma_i^2 [\hat{W}_\tau(e^{j\theta})]) d\theta,$$

where $\sigma_i(\cdot)$ denotes the i -th singular value and

$$f(x) = -\gamma^2 \log(1 - \gamma^{-2}x).$$

In Fig. 9.2 the function $f(x)$ for $\gamma = 1$ is shown. It is apparent that $f(x) \geq x$ for each $0 \leq x \leq \gamma$ so that

$$\begin{aligned} I_\gamma &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_i f(\sigma_i^2 [\hat{W}_\tau(e^{j\theta})]) d\theta \\ &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_i \sigma_i^2 [\hat{W}_\tau(e^{j\theta})] d\theta = \|G(\sigma, \cdot)\|_2^2. \end{aligned}$$

On the other hand, let

$$r_i = \frac{\gamma^2}{\sigma_i^2 [\hat{W}_\tau(e^{j\theta})]}, \quad \beta = \frac{\gamma^2}{\|\hat{W}_\tau(z)\|_\infty^2}$$

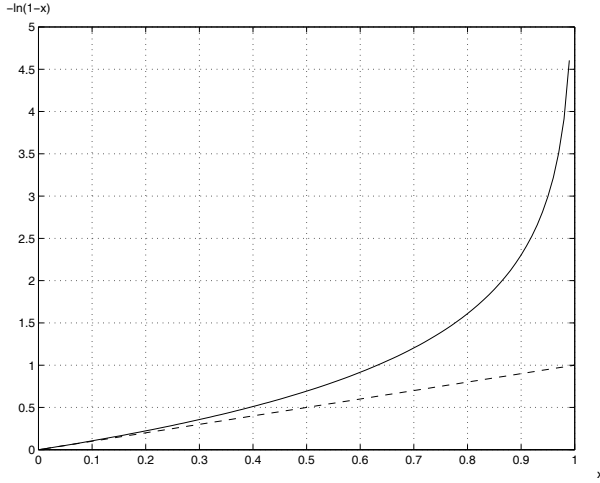


Fig. 9.2 The functions $y = -\gamma^2 \log(1 - \frac{x}{\gamma^2})$ for $\gamma = 1$ (continuous-line) and $y = x$ (dotted-line)

and notice that

$$r_i \geq \beta \geq 1, \quad \log [1 - r_i^{-1}] \geq \frac{\beta}{r_i} \log [1 - \beta^{-1}].$$

Therefore

$$\begin{aligned} I_\gamma &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_i -\gamma^2 \log [1 - r_i^{-1}] d\theta \\ &\leq -\beta \log [1 - \beta^{-1}] \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_i \sigma_i^2 [\hat{W}_\tau(e^{j\theta})] d\theta \\ &\leq \frac{\gamma^2}{\|G(\sigma, \cdot)\|_\infty^2} \log \left[\frac{\gamma^2}{\gamma^2 - \|G(\sigma, \cdot)\|_\infty^2} \right] \|G(\sigma, \cdot)\|_2^2. \end{aligned}$$

In conclusion, the γ -entropy satisfies

$$\|G(\sigma, \cdot)\|_2^2 \leq I_\gamma \leq \alpha(\gamma) \|G(\sigma, \cdot)\|_2^2, \quad (9.18)$$

where

$$\alpha(\gamma) = \frac{\gamma^2}{\|G(\sigma, \cdot)\|_\infty^2} \log \left[\frac{\gamma^2}{\gamma^2 - \|G(\sigma, \cdot)\|_\infty^2} \right].$$

The function $\alpha(\gamma)$ tends to 1 as γ increases. Hence, from (9.18), it turns out that the γ -entropy approaches the squared L_2 norm, i.e.,

$$\lim_{\gamma \rightarrow \infty} I_\gamma = \|G(\sigma, \cdot)\|_2^2.$$

We pass now to the interpretation of the γ -entropy in terms of the T -periodic solution of the Riccati Equation (9.11).

To this purpose, consider the cyclic Riccati Equation (9.9) and note that the solution \hat{Q} generates a spectral factor $\hat{T}_\tau(z)$ of $\gamma^2 I - \hat{W}_\tau(z^{-1})' \hat{W}_\tau(z)$ as follows

$$\begin{aligned} I - \gamma^{-2} \hat{W}_\tau(z^{-1})' \hat{W}_\tau(z) &= \hat{T}_\tau(z^{-1})' \hat{T}_\tau(z) \\ \hat{T}_\tau(z) &= \gamma^{-1} V_\tau^{1/2} (I - K_\tau(zI - \hat{F}_\tau)^{-1} \hat{G}_\tau) . \end{aligned}$$

Hence,

$$\begin{aligned} I_\gamma &= -\frac{\gamma^2}{2\pi} \int_{-\pi}^{\pi} \log \det [\hat{T}_\tau(e^{-j\theta})' \hat{T}_\tau(e^{j\theta})] d\theta \\ &= \lim_{r \rightarrow \infty} \frac{-\gamma^2}{2\pi} \int_{-\pi}^{\pi} \log |\det [\hat{T}_\tau(e^{-j\theta})' T_\tau(e^{j\theta})]| \Big|_{\frac{r^2-1}{|r-e^{j\theta}|^2}} d\theta \\ &= \lim_{r \rightarrow \infty} \frac{-\gamma^2}{\pi} \int_{-\pi}^{\pi} \log |\det [\hat{T}_\tau(e^{j\theta})]| \Big|_{\frac{r^2-1}{|r-e^{j\theta}|^2}} d\theta . \end{aligned}$$

In view of the Poisson–Jensen formula¹

$$\begin{aligned} I_\gamma &= -2\gamma^2 \log |\det [\hat{T}_\tau(\infty)]| = \\ &= -2\gamma^2 \log \det [\gamma^{-1} V_\tau^{1/2}] \\ &= -\gamma^2 \log \det [I - \gamma^{-2} \hat{E}_\tau' \hat{E}_\tau - \gamma^{-2} \hat{G}_\tau' \hat{Q}_\tau \hat{G}_\tau] . \end{aligned}$$

Thanks to the structure of the cyclic matrices \hat{E}_τ , \hat{Q}_τ , \hat{G}_τ , it finally follows that

$$I_\gamma = \sum_{i=0}^{T-1} -\gamma^2 \log \det [I - \gamma^{-2} D(i)' D(i) - \gamma^{-2} B(i)' Q(i+1) B(i)] .$$

It turns out that the γ entropy is a non-decreasing function of the T -periodic (positive semi-definite) solution $Q(\cdot)$ of (9.11). This is why, as we will see in a subsequent chapter, the H_∞ central controller coincides with the minimum entropy controller.

Again, one can show that the γ entropy tends to the squared L_2 norm as γ tends to infinity.

¹ For a rational function $g(z)$, with all zeros and poles belonging to the open unit disk, the Poisson–Jensen formula specializes as follows:

$$\ln |g(r)| = \frac{1}{2\pi} \int_0^{2\pi} P_{1,1/r}(\theta) \ln |g(e^{j\theta})| d\theta ,$$

where the Poisson kernel $P_{\rho,\beta}(x)$ is defined as:

$$P_{\rho,\beta}(x) = \frac{\rho^2 - \beta^2}{\rho^2 - 2\beta\rho \cos(x) + \beta^2} .$$

Indeed, for a certain matrix R so that $I - \gamma^{-2}R > 0$, it turns out that

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} -\gamma^2 \log \det [I - \gamma^{-2}R] &= \\ \lim_{\gamma \rightarrow \infty} -\gamma^2 \sum_i \log(1 - \gamma^{-2}\lambda_i(R)) &= \\ \lim_{\gamma \rightarrow \infty} \sum_i \log(1 - \gamma^{-2}\lambda_i(R))^{-\gamma^2} &= \\ \lim_{\gamma \rightarrow \infty} \sum_i \lambda_i(R) &= \text{trace}[R]. \end{aligned}$$

Hence,

$$\lim_{\gamma \rightarrow \infty} I_\gamma = \text{trace}[D(i)'D(i) + B(i)'Q(i+1)B(i)] = \|G(\sigma, \cdot)\|_2^2.$$

9.4 Hankel Norm and Order Reduction

As is well-known, the Hankel operator of a stable system links the past input to the future output through the initial state of the system. Assume that the periodic system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ is stable, and consider the input

$$u(t) = 0, \quad t > \tau - 1, \quad u(\cdot) \in l_2(-\infty, \tau - 1), \quad (9.19)$$

where $l_2(-\infty, \tau - 1)$ is the space of square summable signals over $(-\infty, \tau - 1)$. Now assume that the state of the system is 0 at $t = -\infty$. Therefore, the state at $t = \tau$ is

$$x(\tau) = \sum_{k=-\infty}^{\tau-1} \Phi_A(\tau, k+1)B(k)u(k)$$

and the output for $t \geq \tau$ is

$$y(t) = C(t)\Phi_A(t, \tau) \sum_{k=-\infty}^{\tau-1} \Phi_A(\tau, k+1)B(k)u(k). \quad (9.20)$$

The system stability entails $y(\cdot) \in l_2(\tau, \infty)$.

The Hankel operator at time τ of the periodic system is defined as the operator mapping the input over $(-\infty, \tau - 1)$ defined by (9.19) into the output over (τ, ∞) given by (9.20).

Such an operator can be related to the infinite Hankel matrix of the lifted reformulation of the periodic system at time τ . As a matter of fact, recall the definition of the lifted input and output signals $u_\tau(k)$ and $y_\tau(k)$, and the associated lifted system (see Sect. 6.2).

From (9.20) a simple computation shows that

$$\begin{bmatrix} y_\tau(0) \\ y_\tau(1) \\ y_\tau(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} H_\tau G_\tau & H_\tau \Phi_A(\tau) G_\tau & H_\tau \Phi_A(\tau)^2 G_\tau & \cdots \\ H_\tau \Phi_A(\tau) G_\tau & H_\tau \Phi_A(\tau)^2 G_\tau & H_\tau \Phi_A(\tau)^3 G_\tau & \cdots \\ H_\tau \Phi_A(\tau)^2 G_\tau & H_\tau \Phi_A(\tau)^3 G_\tau & H_\tau \Phi_A(\tau)^4 G_\tau & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} u_\tau(-1) \\ u_\tau(-2) \\ u_\tau(-3) \\ \vdots \end{bmatrix}.$$

The previous expression points out how the past of $u_\tau(\cdot)$ determines the future of $y_\tau(\cdot)$ through the Hankel operator at τ , say $T_{yu}(\tau)$.

From the above considerations, it makes sense to define the Hankel norm at τ of the periodic system as the Hankel norm of its lifted reformulation at τ . Notice that $T_{yu}(\tau)$ is independent of the input–output matrix E_τ . From the time-invariant case, it is known that the Hankel norm of the lifted reformulation at τ can be computed as the square root of the largest eigenvalue of the product of the unique solutions of the Lyapunov Equations (9.6). On the other hand such solutions coincide with the periodic solutions at τ of (9.2). As such, the Hankel norm at τ of the system (assumed to be stable) is given by

$$\lambda_{\max}(P(\tau)Q(\tau))^{1/2},$$

where, again, $P(\tau)$ and $Q(\tau)$ are the solutions of (9.2).

Analogous considerations can be developed in order to introduce the Hankel norm of the cyclic reformulation, where the relevant Lyapunov equations are given by Eqs. (9.4), (9.5). Hence, the associated Hankel norm at τ is given by

$$\lambda_{\max}(\hat{P}_\tau \hat{Q}_\tau)^{1/2} = \max_{\tau} \lambda_{\max}(P(\tau)Q(\tau))^{1/2}.$$

Obviously, in contrast to the lifted Hankel norm, such a norm is independent of τ . This is why it is advisable to define the Hankel norm of the periodic system as the one induced from its *cyclic reformulation*, i.e.,

$$\|T_{yu}\|_H = \max_{\tau} \lambda_{\max}(P(\tau)Q(\tau))^{1/2}.$$

Model Reduction

A classical problem in system theory is the determination of a reduced-order system which approximates, in some sense to be specified, the original dynamical system. An effective way of performing such a reduction is to analyze the Hankel singular values of the system in order to proceed to a suitable truncation.

To be specific, consider the periodic system $\Sigma = (A(\cdot), B(\cdot), C(\cdot))$ and assume that $A(\cdot)$ is stable and the system is reachable and observable for each t . The dimension $n(\cdot)$ of the system is possibly time-periodic (recall Chap. 7 on the minimal realization problem). Hence,

$$A(t) \in \mathbb{R}^{n(t+1) \times n(t)}, \quad B(t) \in \mathbb{R}^{n(t+1) \times m}, \quad C(t) \in \mathbb{R}^{p \times n(t)}.$$

Now, consider the Lyapunov equations (9.2) and note that, thanks to system stability, such equations admit the unique T -periodic solutions $P(t) \in \mathbb{R}^{n(t) \times n(t)}$ and $Q(t) \in \mathbb{R}^{n(t) \times n(t)}$. In view of the minimality assumption, these solutions are positive definite for each t . The eigenvalues $\sigma_i^2(\tau)$ of their product $P(\tau)Q(\tau)$ are positive, for each τ , and can be ordered as follows:

$$\sigma_1^2(\tau) \geq \sigma_2^2(\tau) \geq \cdots \geq \sigma_n^2(\tau) > 0$$

These positive numbers are called *Hankel singular values* at τ of the system, and coincide with the singular values of the Hankel matrix associated with the lifted reformulation at time τ .

In order to construct a periodic system of reduced order $r(\cdot)$, one can split the singular values in two sets, i.e., $\forall \tau$,

$$\begin{aligned} \sigma_1^2(\tau) &\geq \sigma_2^2(\tau) \geq \cdots \geq \sigma_{r(\tau)}^2(\tau) \\ \sigma_{r(\tau)+1}^2(\tau) &\geq \sigma_{r(\tau)+2}^2(\tau) \geq \cdots \geq \sigma_{n(\tau)}^2(\tau). \end{aligned}$$

The idea is then to eliminate the lower singular values and obtain a simplified system by truncation. The easiest way to do this is to resort to the change of basis

$$\tilde{A}(t) = T(t+1)A(t)T(t)^{-1}, \quad \tilde{B}(t) = T(t+1)B(t), \quad \tilde{C}(t) = C(t)T(t)^{-1},$$

where the transformation $T(\cdot)$ can be found following the arguments presented in Sect. 7.3 in order to obtain a balanced realization $\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot)$. Then the so-obtained balanced system matrices are decomposed according to the selected order $r(\cdot)$, i.e.,

$$\tilde{A}(t) = \begin{bmatrix} A_{1,1}(t) & A_{1,2}(t) \\ A_{2,1}(t) & A_{2,2}(t) \end{bmatrix}, \quad \tilde{B}(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}, \quad \tilde{C}(t) = [C_1(t) \ C_2(t)],$$

where

$$A_{1,1}(t) \in \mathbb{R}^{r(t+1) \times r(t)}, \quad B_1(t) \in \mathbb{R}^{r(t+1) \times m}, \quad C_1(t) \in \mathbb{R}^{p \times r(t)}.$$

Finally, the truncated system Σ_r is defined by means of this triplet, i.e.,

$$\Sigma_r = (A_{1,1}(t), B_1(t), C_1(t)).$$

Exploiting the isomorphism between the periodic system and its cyclic reformulation, it can be proven that the reduced system $\Sigma_r = (A_{1,1}(\cdot), B_1(\cdot), C_1(\cdot))$ is stable, in minimal form (reachable and observable for each t) and, moreover,

$$\|\Sigma - \Sigma_r\|_\infty \leq \sum_{t=0}^{T-1} \sum_{i=r(t)+1}^{n(t)} \sigma_i(t).$$

9.5 Time-domain Specifications

In this section, we characterize the so-called l_∞/l_2 gain of a periodic system, namely the largest peak value of the forced output of the system produced by any input signal in the unit l_2 ball. To be precise, consider the periodic system

$$\begin{aligned}x(t+1) &= A(t)x(t) + B(t)w(t) \\ z(t) &= C(t)x(t) \\ x(0) &= 0\end{aligned}$$

and assume that

- (i) $A(\cdot)$ is stable.
- (ii) The pair $(A(\cdot), B(\cdot))$ is reachable.
- (iii) The system is reversible, i.e., $\det[A(t)] \neq 0, \forall t$.

The input $w(\cdot)$ is assumed to be in $l_2(0, \infty)$, the space of all square summable signals in $(0, \infty)$ with norm less than one. To shorten, we denote this set as \mathcal{W} , i.e.,

$$\mathcal{W} = \{w(\cdot) \mid w(\cdot) \in l_2(0, \infty), \|w(\cdot)\| \leq 1\}.$$

The problem is to compute the maximum value of the l_∞ norm of the output $z(\cdot)$ over all bounded square summable signals $w(\cdot)$. Two kinds of l_∞ norms are considered, namely

$$\begin{aligned}J_1(w) &= \sup_{t \geq 0} \sqrt{z(t)'z(t)} \\ J_2(w) &= \sup_{t \geq 0} \max_i |z_i(t)|.\end{aligned}$$

Denote by $d_{\max}(\cdot)$ the maximum diagonal element of a matrix. The following result can now be stated.

Proposition 9.3 *Under the assumptions (i)–(iii), the following expressions hold true:*

$$\sup_{w \in \mathcal{W}} J_1(w) = \max_{t \in [0, T-1]} \sqrt{\lambda_{\max}[C(t)P(t)C(t)']}\tag{9.21a}$$

$$\sup_{w \in \mathcal{W}} J_2(w) = \max_{t \in [0, T-1]} \sqrt{d_{\max}[C(t)P(t)C(t)']},\tag{9.21b}$$

where $P(\cdot)$ is the unique (positive definite) periodic solution of the Lyapunov equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)'.$$

Proof First, notice that in view of stability, the Lyapunov equation admits only one periodic solution $P(\cdot)$ and that such a solution is positive definite for each t thanks to the reachability assumption. Being $A(t)$ invertible, for each t , the equation can be

written in the unknown $P(t)^{-1}$ as follows

$$\begin{aligned} P(t)^{-1} &= A(t)P(t+1)^{-1}A(t)' + \\ &\quad - A(t)P(t+1)^{-1}B(t) [I - B(t)'P(t+1)^{-1}B(t)']^{-1} P(t+1)^{-1}A(t)'. \end{aligned}$$

Now, define the function $V(x, t) = x'P(t)^{-1}x$ and compute its variation along the trajectories of the system. It turns out that

$$\begin{aligned} V(x(i+1), i+1) - V(x(i), i) &= w(i)'w(i) - w(i)'B(i)'P(i+1)^{-1}B(i)w(i) \\ &\quad - [w(i) - \tilde{w}(i)]'[w(i) - \tilde{w}(i)], \end{aligned}$$

where

$$\tilde{w}(i) = (I - B(i)'P(i+1)^{-1}B(i))^{-1}B(i)'P(i+1)^{-1}A(i)'x(i).$$

Therefore

$$V(x(i+1), i+1) - V(x(i), i) \leq w(i)'w(i).$$

Taking into account that $x(0) = 0$ and summing up both sides from $i = 0$ up to $i = t - 1$, it turns out that

$$x(t)'P(t)^{-1}x(t) \leq \sum_{i=0}^{t-1} w(i)'w(i) \leq \sum_{i=0}^{\infty} w(i)'w(i) \leq 1,$$

so that

$$x(t)'P(t)^{-1}x(t) \leq 1, \quad \forall t \geq 0.$$

This means that the trajectories of the system's state satisfy $x(t)P(t)^{-1}x(t) \leq 1$ whenever $w(\cdot)$ remains bounded by $\|w\|_2 \leq 1$. Now, let $\bar{x}(t) = P(t)^{-1/2}x(t)$ so that $\bar{x}(t)'\bar{x}(t) \leq 1, \forall t$.

Thanks to this inequality, it follows that

$$\begin{aligned} \sup_{w \in \mathcal{W}} J_1^2(w) &= \sup_{w \in \mathcal{W}} \max_{t \geq 0} z(t)'z(t) \\ &\leq \max_{t \geq 0} \{x(t)'C(t)'C(t)x(t), \quad x(t)'P(t)^{-1}x(t) \leq 1\} \\ &\leq \max_{t \geq 0} \{\bar{x}(t)'P(t)^{1/2}C(t)'C(t)P(t)^{1/2}\bar{x}(t), \quad \bar{x}(t)'\bar{x}(t) \leq 1\} \\ &\leq \max_{t \geq 0} \max_{\|\bar{x}(t)\| \leq 1} \bar{x}(t)'P(t)^{1/2}C(t)'C(t)P(t)^{1/2}\bar{x}(t) \\ &\leq \max_{t \geq 0} \lambda_{\max}[C(t)P(t)C(t)'] \\ &\leq \max_{t \in [0, T-1]} \lambda_{\max}[C(t)P(t)C(t)']. \end{aligned}$$

As for $J_2(w)$, denoting by $C_i(t)$ the i -th row of $C(t)$, it is possible to write

$$\begin{aligned}
 \sup_{w \in W} J_2^2(w) &= \sup_{w \in W} \max_{t \geq 0} \max_i |z_i(t)|^2 \\
 &\leq \max_{t \geq 0} \max_i \{x(t)' C_i(t)' C_i(t) x(t), \quad x(t)' P(t)^{-1} x(t) \leq 1\} \\
 &\leq \max_{t \geq 0} \max_i \{\bar{x}(t)' P(t)^{1/2} C_i(t)' C_i(t) P(t)^{1/2} \bar{x}(t), \quad \bar{x}(t)' \bar{x}(t) \leq 1\} \\
 &\leq \max_{t \geq 0} \max_i \max_{\|\bar{x}(t)\| \leq 1} \bar{x}(t)' P(t)^{1/2} C_i(t)' C_i(t) P(t)^{1/2} \bar{x}(t) \\
 &\leq \max_{t \geq 0} \max_i C_i(t)' P(t) C_i(t) \\
 &\leq \max_{t \in [0, T-1]} \max_i C_i(t)' P(t) C_i(t) \\
 &\leq \max_{t \in [0, T-1]} d_{\max}[C(t) P(t) C(t)'].
 \end{aligned}$$

To finalize the proof of the proposition, what is missing is to show that there exists an input $w(\cdot)$ such that the indexes J_1 and J_2 of (9.21) are arbitrarily closely approached. Let's start with index J_1 . Denote by $\Pi(t)$ the solution of the Lyapunov equation with initial condition $\Pi(0) = 0$. It can be easily written as

$$\Pi(t) = \sum_{k=0}^{t-1} \Phi_A(t, k+1) B(k) B(k)' \Phi_A(t, k+1)$$

Of course such a solution asymptotically converges to the periodic solution $P(\cdot)$, i.e.,

$$\lim_{k_1 \rightarrow \infty} \Pi(k_1 T + k_2) = P(k_2), \quad \forall k_2 \in [0, T-1]$$

Now take a sufficiently large time instant l and define

$$w(t) = \begin{cases} B(t)' \Phi_A(l, t+1)' \Pi(l)^{-1/2} y, & 0 \leq t \leq l-1 \\ 0, & t \geq l, \end{cases}$$

where y is a unit-norm vector. Then,

$$\begin{aligned}
 \sum_{i=0}^{\infty} w(i)' w(i) &= \sum_{i=0}^{l-1} w(i)' w(i) \\
 &= y' \Pi(l)^{-1/2} \sum_{i=0}^{l-1} \Phi_A(l, i+1) B(i) B(i)' \Phi_A(l, i+1) \Pi(l)^{-1/2} y \\
 &= y' y = 1,
 \end{aligned}$$

and

$$z(l) = C(l) \sum_{i=0}^{l-1} \Phi_A(l, i+1) B(i) w(i) = C(l) \Pi(l)^{1/2} y.$$

If y is chosen as the unit-modulus eigenvector associated with the maximum eigenvalue of $C(l)P(t)C(l)'$, it follows that

$$z(l)'z(l) = \lambda_{\max}[C(l)P(l)C(l)'].$$

Finally, letting $l = k_1T + k_2$, $k_2 \in [0, T - 1]$, one obtains

$$\begin{aligned} \max_t z(t)'z(t) &\geq \max_{k_2} z(k_1T + k_2)'z(k_1T + k_2) \\ &\geq \lambda_{\max}[C(k_2)\Pi(k_1T + k_2)C(k_2)']. \end{aligned}$$

This last expression completes the proof (9.21a) since $\Pi(k_1T + k_2)$ becomes arbitrarily close to $P(k_2)$ as k_1 increases.

Now, turn to index J_2 . To prove that there exists an input $w(\cdot)$ so that J_2 is eventually approached, it suffices to change the definition of vector y . To be precise, take $y^{(i)}$ as the unit-norm vector associated with the maximum eigenvalues of $\Pi(l)^{1/2}C_i(l)'C_i(l)\Pi(l)^{1/2}$ and select the integer r so that, $\forall i$,

$$C_i(l)\Pi(l)C_i(l)' \leq C_r(l)\Pi(l)C_r(l)'.$$

Finally, set $y = y^{(r)}$. Hence,

$$\begin{aligned} \max_{t \geq 0} \max_i \|z_i(t)\|^2 &\geq \max_{k_2} \max_i |z_i(k_1T + k_2)|^2 \\ &\geq \max_{k_2} \max_i y' \Pi(k_1T + k_2)^{1/2} C_i(k_2)' C_i(k_2) \Pi(k_1T + k_2)^{1/2} y \\ &\geq \max_l y' \Pi(k_1T + k_2)^{1/2} C_r(k_2)' C_r(k_2) \Pi(k_1T + k_2)^{1/2} y \\ &\geq \max_{k_2} C_r(k_2) \Pi(k_1T + k_2) C_r(k_2)'. \end{aligned}$$

Again, the result (9.21b) follows from the fact that $\Pi(k_1T + k_2)$ converges to $P(k_2)$ as $k_1 \rightarrow \infty$.

9.6 Bibliographical Note

Most material concerning the definition and characterization of the L_2 and L_∞ norms can be found in [42]. In this regard, the PhD thesis [305] contains a number of results relative to the quadratic stability of time-varying and periodic systems. The characterization of the entropy for periodic systems in discrete-time is presented here for the first time. However, the underlying theory for general time-varying discrete-time systems is developed in [193, 247]. The order reduction in the Hankel norm has been treated in [287, 289].

Chapter 10

Factorization and Parametrization

In the analysis and design of linear systems, factorization problems arise in a variety of contexts. Herein we focus on the factorization algorithms involved in the H_2 and H_∞ filtering problems, known in the literature as *spectral factorization* and *J-spectral factorization*, respectively. By duality, the corresponding techniques for control can be easily derived. For the introduction to factorization methods, the extension of a few concepts of the linear systems theory is necessary. In particular, we will dwell upon the concept of *adjoint system*, already introduced in Sect. 2.5, and that of *symplectic pencil*. Taking advantage of all these items, the important issue of the parametrization of stabilizing controllers will be treated in the last part of the chapter. The parametrization problem will be tackled with reference to both the polynomial and rational descriptions of periodic systems.

10.1 Periodic Symplectic Pencil

Consider the T -periodic system

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (10.1a)$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad (10.1b)$$

with transfer operator $G(\sigma, t) = C(t)(\sigma I - A(t))^{-1}B(t) + D(t)$ and the adjoint system

$$A(t)'\lambda(t+1) = \lambda(t) - C(t)'\nu(t) \quad (10.2a)$$

$$\zeta(t) = B(t)'\lambda(t+1) + D(t)'\nu(t), \quad (10.2b)$$

whose transfer operator is $G^\sim(\sigma, t) = B(t)'(\sigma^{-1}I - A(t)')^{-1}C(t)' + D(t)'$.

Moreover, consider the series connection depicted in Fig. 10.1, where R is a constant symmetric matrix whose role is to select among different specific factorization problems. Our aim is now to write the dynamics of the periodic system whose transfer

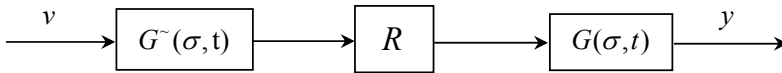


Fig. 10.1 The cascade connection between the system and its adjoint

operator is

$$\bar{G}(\sigma, t) = (G(\sigma, t)RG^{\sim}(\sigma, t))^{-1}.$$

A little thought reveals that $\bar{G}(\sigma, t)$ exists if and only if the matrix $D(t)RD(t)'$ is invertible for each t . This assumption will be implicitly made throughout the derivation of the factorization results.

We now derive the description of the cascade system $G(\sigma, t)RG^{\sim}(\sigma, t)$ of Fig. 10.1. To this purpose, let $\xi = [x' \ \lambda']'$ be the overall state, y the output and v the input. It follows that

$$\begin{bmatrix} I & -B(t)RB(t)' \\ 0 & A(t)' \end{bmatrix} \xi(t+1) = \begin{bmatrix} A(t) & 0 \\ 0 & I \end{bmatrix} \xi(t) + \begin{bmatrix} B(t)RD(t)' \\ -C(t)' \end{bmatrix} v(t)$$

$$y(t) = [C(t) \ 0] \xi(t) + [0 \ D(t)RB(t)'] \xi(t+1) + D(t)RD(t)'v(t).$$

Obviously for the inverse system $\bar{G}(\sigma, t)$, the signal $y(t)$ acts as input. Therefore, the free dynamics of $\bar{G}(\sigma, t) = (G(\sigma, t)RG^{\sim}(\sigma, t))^{-1}$ is obtained by setting $y(t) = 0$ and solving then the latter equation with respect to $v(t)$.

By replacing the so obtained expression in the first equation, the free dynamics of $G(\sigma, t)$ can be given the following explicit form:

$$L(t)\xi(t+1) = M(t)\xi(t) \tag{10.3a}$$

$$L(t) = \begin{bmatrix} I - B(t)RB(t)' + B(t)RD(t)'(D(t)RD(t)')^{-1}D(t)RB(t)' \\ 0 \ A(t)' - C(t)'(D(t)RD(t)')^{-1}D(t)RB(t)' \end{bmatrix} \tag{10.3b}$$

$$M(t) = \begin{bmatrix} A(t) - B(t)RD(t)'(D(t)RD(t)')^{-1}C(t) & 0 \\ C(t)'(D(t)RD(t)')^{-1}C(t) & I \end{bmatrix}. \tag{10.3c}$$

Equation (10.3) defines a system in so-called *pencil form*, characterized by the pair $(L(\cdot), M(\cdot))$. Such a pair is named a T -periodic pencil-pair.

The characteristic values of $(L(\cdot), M(\cdot))$ can be obtained from the corresponding time-lifted reformulation at a given time tag τ :

$$\begin{aligned}
 L_\tau \bar{\xi}(k+1) &= M_\tau \bar{\xi}(k) \\
 L_\tau &= \begin{bmatrix} I - G_\tau R_\tau G'_\tau + G_\tau R_\tau E'_\tau (E_\tau R_\tau E'_\tau)^{-1} E_\tau R_\tau G'_\tau & \\ 0 & F'_\tau - H'_\tau (E_\tau R_\tau E'_\tau)^{-1} E_\tau R_\tau G'_\tau \end{bmatrix} \\
 M_\tau &= \begin{bmatrix} F_\tau - G_\tau R_\tau E'_\tau (E_\tau R_\tau E'_\tau)^{-1} H_\tau & 0 \\ H'_\tau (E_\tau R_\tau E'_\tau)^{-1} H_\tau & I \end{bmatrix} \\
 R_\tau &= \text{diag}\{R\},
 \end{aligned}$$

where F_τ , G_τ , H_τ and E_τ are the lifted matrices as defined in Chap. 6. The characteristic polynomial associated with the pencil $(L(\cdot), M(\cdot))$ is defined as

$$\Psi(\lambda) = \det[\lambda L_\tau - M_\tau]. \quad (10.4)$$

The singularities of such a polynomial are the so-called characteristic multipliers at τ of the periodic pencil $(L(\cdot), M(\cdot))$. In this connection, it is worthwhile pointing out that, if $\lambda \neq 0$ is a singularity for the characteristic polynomial equation at τ , it is also a singularity for the characteristic polynomial equation at $\tau + 1$, and therefore at any other time point. Indeed, it is easy to show that $\lambda \neq 0$ is a solution of the above equation if and only if

$$\det [W_\tau(\lambda) R_\tau W_\tau(\lambda^{-1})'] = 0,$$

where $W_\tau(z)$ is the transfer function of the lifted reformulation of the periodic system with transfer operator $G(\sigma, t)$. In view of Eq. (6.12), the structure of $\Delta_k(z)$ in (6.11) and the block diagonal structure of R_τ , it follows

$$\begin{aligned}
 W_{\tau+1}(\lambda) R_\tau W_{\tau+1}(\lambda^{-1})' &= \\
 &= \Delta_p(\lambda^{-1})' W_\tau(\lambda) \Delta_m(\lambda) R_\tau \Delta_m(\lambda^{-1})' W_\tau(\lambda^{-1})' \Delta_p(\lambda) = \\
 &= \Delta_p(\lambda^{-1})' W_\tau(\lambda) R_\tau W_\tau(\lambda^{-1})' \Delta_p(\lambda),
 \end{aligned}$$

so that

$$\det [W_{\tau+1}(\lambda) R_\tau W_{\tau+1}(\lambda^{-1})'] = 0.$$

Hence, λ is also a characteristic multiplier at $\tau + 1$ of the periodic pencil. In conclusion, for the non-zero characteristic multipliers of the symplectic pencil, it is not necessary to specify an associated time point. Moreover, as is well-known due to the

symplectic property¹, if $\lambda \neq 0$ is a characteristic multiplier of the symplectic pencil, then λ^{-1} is a characteristic multiplier as well.

On the other hand, the pencil $(L(\cdot), M(\cdot))$ is intimately related to a T -periodic Riccati equation in the unknown $P(\cdot)$. This equation can be obtained by letting

$$x(t) = P(t)\lambda(t).$$

Indeed, from (10.3), and letting

$$\begin{aligned}\alpha(t) &= (D(t)RD(t))^{-1} (D(t)RB(t)'\lambda(t+1) + C(t)x(t)) \\ \beta(t) &= (A(t)P(t)C(t)' + B(t)RD(t)') \alpha(t),\end{aligned}$$

one obtains

$$\begin{aligned}(P(t+1) - B(t)RB(t)' - A(t)P(t)A(t)') \lambda(t+1) + \beta(t) &= 0 \\ (D(t)RD(t)' + C(t)P(t)C(t)') \alpha(t) &= (D(t)RB(t)' + C(t)P(t)A(t)') \lambda(t+1).\end{aligned}$$

From these last equations, the following periodic Riccati equation is derived

$$P(t+1) = A(t)P(t)A(t)' + B(t)RB(t)' - S_R(t)W_R(t)S_R(t)' \quad (10.5a)$$

$$S_R(t) = A(t)P(t)C(t)' + B(t)RD(t)' \quad (10.5b)$$

$$W_R(t) = (D(t)RD(t)' + C(t)P(t)C(t)')^{-1}. \quad (10.5c)$$

We now elaborate further the above Riccati equation in an operatorial description. By adopting a line of reasoning that is analogous to the one used in the derivation of the factorization (9.15) in Sect. 9.2.2, from (10.5), one obtains

$$G(\sigma, t)RG^\sim(\sigma, t) = T_R(\sigma, t)W_R(t)T_R^\sim(\sigma, t) \quad (10.6a)$$

$$T_R(\sigma, t) = W_R(t)^{-1} + C(t)(\sigma I - A(t))^{-1}S_R(t). \quad (10.6b)$$

Notice that the $T_R(\sigma, t)$ shares the same dynamic matrix of the original system. Moreover, the dynamic matrix of $T_R(\sigma, t)^{-1}$ is

$$\hat{A}_R(t) = A(t) - S_R(t)W_R(t)C(t).$$

Hence $T_R(\sigma, t)^{-1}$ is stable if the T -periodic solution of the Riccati Equation (10.5) is *stabilizing*. The characteristic multipliers of $\hat{A}_R(\cdot)$ are also characteristic multipliers of the pencil $(L(\cdot), M(\cdot))$, which in turn, are reciprocal to each other. This means that a necessary condition for the existence of the T -periodic stabilizing solution of

¹ The periodic pair $L(\cdot), M(\cdot)$ is said to be symplectic if

$$L(t)J_sL(t)' = M(t)J_sM(t)', \forall t, \quad J_s = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

As a consequence, its characteristic values, namely the characteristic values of its time-lifted reformulation, are reciprocal.

(10.5) is that the pencil does not have characteristic multipliers on the unit circle. The necessary and sufficient condition for the existence of the T -periodic stabilizing solution will be briefly discussed when the particular factorization problems are tackled. However, notice that the T -periodic stabilizing solution (if any) is unique².

Remark 10.1 (H_∞ analysis)

The general discussion carried out so far can be particularly applied to the study of the H_∞ norm (Sect. 9.2). Let

$$\bar{G}(\sigma, t) = \bar{D}(t) + \bar{C}(t) (\sigma I - \bar{A}(t))^{-1} \bar{B}(t)$$

be the transfer operator of a stable periodic system and consider a positive scalar γ . Moreover, in (10.6) let

$$\begin{aligned} G(\sigma, t) &= [\bar{G}(\sigma, t) \ I] \\ A(t) &= \bar{A}(t) \\ C(t) &= \bar{C}(t) \\ B(t) &= [\bar{B}(t) \ 0] \\ D(t) &= [\bar{D}(t) \ I] \\ R &= \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}. \end{aligned}$$

Then, the factorization (10.6) coincides with (9.15) and the solution $P(\cdot)$ of the periodic Riccati Equation (9.14) coincides with $P(\cdot)$ in (10.5).

Finally, note that if parameter γ in the above expression of matrix R tends to infinity, then Eqs. (10.5) reduce to the Lyapunov Equation (9.2a).

10.2 Spectral Factorization

Consider a square rational T -periodic operator $\Phi(\sigma, t)$, $t \in [0, T - 1]$ and assume that such an operator satisfies³

$$\begin{aligned} \Phi(\sigma, t) &= \Phi^\sim(\sigma, t), \\ \Phi(\sigma, t) &> 0. \end{aligned}$$

² The uniqueness of the T -periodic stabilizing solution can be proved by contradiction, assuming that there exist two T -periodic stabilizing solutions $P_1(\cdot)$ and $P_2(\cdot)$. It follows that the difference $\Delta(t) = P_1(t) - P_2(t)$ satisfies the difference equation $\Delta(t+1) = \Delta(t)A_{R1}(t)' + A_{R2}(t)\Delta(t)$, where $A_{R1}(\cdot)$ and $A_{R2}(\cdot)$ are the closed-loop matrices associated with $P_1(\cdot)$ and $P_2(\cdot)$, respectively. Since they are both stable, the difference equation admits only one T -periodic solution, namely $\Delta(\cdot) = 0$.

³ A periodic operator $S(\sigma, t)$, in the one shift ahead operator σ , is said to be positive if, for every non-identically zero $u(\cdot)$ in the considered signal space, it turns out that the scalar product $\langle y', u \rangle$ is positive, where $y(t) = S(\sigma, t) \cdot u(t)$.

This operator can be considered the spectrum of a discrete-time T -periodic system, as will be explained in Sect. 11.3. The spectral factorization problem consists in finding a T -periodic square invertible system with transfer operator $G_m(\sigma, t)$ so that

$$G_m(\sigma, t)G_m^\sim(\sigma, t) = \Phi(\sigma, t). \quad (10.7)$$

Notice first that the solution of the problem is not unique. Indeed, if $G_m(\sigma, t)$ is a solution, then also $G_m(\sigma, t)U(\sigma, t)$ is a solution, provided that the periodic system with transfer operator $U(\sigma, t)$ is *co-inner*, i.e., $U(\sigma, t)U^\sim(\sigma, t) = I$. Therefore, if the problem admits a solution, it admits infinitely many solutions. Typically, when dealing with filtering (control) problems, the issue is to find a factor with stable inverse. Indeed, such a factor guarantees the stability of the filter (controller). This particular factor is named *canonical spectral factor*. If in addition it is stable, then it is the *minimal spectral factor*.

The spectral factorization problem can be tackled in different ways. For instance, by using polynomial tools and the associated diophantine analysis. Here the state–space approach is preferred, since it brings into light the relationship with the underlying periodic Riccati equation.

We actually subdivide the study into two parts. In the first, we assume that the spectrum $\Phi(\sigma, t)$ is given, whereas in the second only a (possibly non-square) factor is available.

10.2.1 From the Spectrum to the Minimal Spectral Factor

Let the spectrum $\Phi(\sigma, t)$ be realized by the state–space model (10.1). In view of the assumption upon $\Phi(\sigma, t)$, the periodic matrix $D(t)$ is square and non-singular for each t .

The first step consists in performing a modal decomposition of $\Phi(\sigma, t)$ (see Remark 9.1). Indeed, it is always possible to write

$$\Phi(\sigma, t) = \Phi_1(\sigma, t) + \Phi_2(\sigma, t) + D(t),$$

where $\Phi_1(\sigma, t)$ is stable and $\Phi_2(\sigma, t)$ is anti-stable, i.e., all the characteristic multipliers of the system with transfer operator $\Phi_2(\sigma, t)$ are greater than one in modulus. Therefore from $\Phi(\sigma, t) = \Phi^\sim(\sigma, t)$ it follows

$$\Phi_1(\sigma, t) - \Phi_2^\sim(\sigma, t) = \Phi_1^\sim(\sigma, t) - \Phi_2(\sigma, t).$$

The first member of this equation is stable, whereas the second member is anti-stable. Hence,

$$\Phi_2(\sigma, t) = \Phi_1^\sim(\sigma, t).$$

Now, let $\Phi_1(\sigma, t)$ be realized by

$$\begin{aligned} x_1(t+1) &= A_1(t)x_1(t) + B_1(t)u(t) \\ y_1(t) &= C_1(t)x_1(t) \end{aligned}$$

and write its adjoint $\Phi_1^\sim(\sigma, t)$ according to (10.2)

$$\begin{aligned} A_1(t)' \lambda(t+1) &= \lambda(t) - C_1(t)' u(t) \\ y_2(t) &= B_1(t)' \lambda(t+1). \end{aligned}$$

Accordingly, the spectrum can be rewritten as

$$\Phi(\sigma, t) = \Phi_1(\sigma, t) + \Phi_1^\sim(\sigma, t) + D(t).$$

The second step is to compute a realization of the inverse spectrum, i.e., of $\Phi(\sigma, t)^{-1}$. An easy computation shows that such an inverse is described by

$$\begin{aligned} L(t) \begin{bmatrix} x_1(t+1) \\ \lambda(t+1) \end{bmatrix} &= M(t) \begin{bmatrix} x_1(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B_1(t)D(t)^{-1} \\ -C_1(t)'D(t)^{-1} \end{bmatrix} y(t) \\ u(t) &= -D(t)^{-1}C_1(t)x_1(t) - D(t)^{-1}B_1(t)'\lambda(t+1) + D(t)^{-1}y(t), \end{aligned}$$

where

$$\begin{aligned} L_1(t) &= \begin{bmatrix} I & B_1(t)D(t)^{-1}B_1(t)' \\ 0 & A_1(t)' - C_1(t)'D(t)^{-1}B_1(t)' \end{bmatrix} \\ M_1(t) &= \begin{bmatrix} A_1(t) - B_1(t)D(t)^{-1}C_1(t) & 0 \\ C_1(t)'D(t)^{-1}C_1(t) & I \end{bmatrix}. \end{aligned}$$

The pair $(L_1(\cdot), M_1(\cdot))$ describes a symplectic periodic pencil associated with the periodic Riccati equation

$$P(t+1) = A_1(t)P(t)A_1(t)' + S_1(t)W_1(t)S_1(t)' \quad (10.8a)$$

$$S_1(t) = A_1(t)P(t)C_1(t)' - B_1(t) \quad (10.8b)$$

$$W_1(t) = (D(t) - C_1(t)P(t)C_1(t)')^{-1}. \quad (10.8c)$$

Assume for the moment that there exists a T -periodic solution $P(\cdot)$ and let $\Psi_1(\sigma, t) = (\sigma I - A_1(t))^{-1}$. From (10.8) it follows (the computations are left to the reader)

$$\begin{aligned} \Phi(\sigma, t) &= \Phi_1(\sigma, t) + \Phi_1^\sim(\sigma, t) + D(t) \\ &= (C_1(t)\Psi_1(\sigma, t)S_1(t) - W_1(t)^{-1})W_1(t)(S_1(t)'\Psi_1^\sim(\sigma, t)C_1(t)' - W_1(t)^{-1}). \end{aligned}$$

Hence, if

$$W_1(t) > 0, \forall t \quad (10.9a)$$

$$\hat{A}_1(\cdot) = A_1(\cdot) + S_1(\cdot)W_1(\cdot)C_1(\cdot) \text{ is stable}, \quad (10.9b)$$

the minimal spectral factor $G_m(\sigma, t)$ in (10.7) is given by the T -periodic system with transfer operator

$$G_m(\sigma, t) = C_1(t) (\sigma I - A_1(t))^{-1} S_1(t) W_1(t)^{1/2} - W_1(t)^{-1/2}. \quad (10.10)$$

Indeed, $W_1(t)^{1/2}$ exists thanks to (10.9a). Moreover, $G_m(t)$ is stable since it shares the same dynamic matrix $A_1(\cdot)$ as $\Phi_1(\sigma, t)$. Finally, its inverse is stable since the dynamic matrix of $G_m(\sigma, t)^{-1}$ is $\hat{A}_1(\cdot)$ the stability of which is guaranteed by (10.9b).

Remark 10.2 *Let*

$$\begin{aligned} \bar{A}_1(t) &= A_1(t) - B_1(t)D(t)^{-1}C_1(t) \\ \bar{C}_1(t) &= D(t)^{-1/2}C_1(t) \\ \bar{B}_1(t) &= B_1(t)D(t)^{-1/2}. \end{aligned}$$

With these positions, the Riccati Equation (10.8) can be equivalently rewritten as

$$\begin{aligned} P(t+1) &= \bar{A}_1(t)P(t)\bar{A}_1(t)' + \bar{B}_1(t)\bar{B}_1(t)' \\ &\quad + \bar{A}_1(t)P(t)\bar{C}_1(t)' (I - \bar{C}_1(t)P(t)C_1(t)')^{-1} \bar{C}_1(t)P(t)\bar{A}_1(t)'. \end{aligned}$$

This shows that the Riccati equation under consideration is the one relevant to the H_∞ analysis for the system with transfer operator (see Sect. 9.2.2 for more details)

$$\bar{G}_1(\sigma, t) = \bar{C}_1(t) (\sigma I - \bar{A}_1(t))^{-1} \bar{B}_1(t).$$

Notice now that

$$\begin{aligned} \bar{G}_1(\sigma, t) &= D(t)^{-1/2}C_1(t) (\sigma I - A_1(t) + B_1(t)D(t)^{-1}C_1(t))^{-1} B_1(t)D(t)^{-1/2} \\ &= D(t)^{-1/2}C_1(t) (I + \Psi_1(\sigma, t)B_1(t)D(t)^{-1}C_1(t))^{-1} \Psi_1(\sigma, t)B_1(t)D(t)^{-1/2} \\ &= D(t)^{1/2} (D(t) + C_1(t)\Psi_1(\sigma, t)B_1(t))^{-1} C_1(t)\Psi_1(\sigma, t)B_1(t)D(t)^{-1/2} \\ &= D(t)^{1/2} (D(t) + \Phi_1(\sigma, t))^{-1} \Phi_1(\sigma, t)D(t)^{-1/2}, \end{aligned}$$

so that a simple computation shows that

$$\begin{aligned} I - \bar{G}_1(\sigma, t)\bar{G}_1^\sim(\sigma, t) &= \bar{\Phi}_1(\sigma, t)\Phi(\sigma, t)\bar{\Phi}_1^\sim(\sigma, t) > 0 \\ \bar{\Phi}_1(\sigma, t) &= D(t)^{1/2} (D(t) + \Phi_1(\sigma, t))^{-1}. \end{aligned}$$

This means that the system $\bar{G}_1(\sigma, t)$ has L_∞ norm less than 1. This would imply the existence of the T -positive semi-definite stabilizing solution $P(\cdot)$ of (10.8) provided that $\bar{G}_1(\sigma, t)$ is stable (recall the dual result of Proposition 9.1 discussed

after Eq. (9.14)). Unfortunately, matrix $A_1(\cdot)$ is generally unstable. However, it is still possible to prove that the positiveness of the so-called Popov operator $I - \bar{G}_1(\sigma, t) \bar{G}_1^\sim(\sigma, t)$ suffices to guarantee the existence of the T -periodic stabilizing solution $P(\cdot)$ of (10.8) satisfying conditions (10.9). This proof, which is here omitted for the sake of simplicity, passes through the lifted reformulation of $\bar{G}_1(\sigma, t)$ and an analogous result for time-invariant discrete-time systems via the Cayley–Hamilton transformation.

10.2.2 From a Spectral Factor to the Minimal Spectral Factor

Assume now that a (non-minimal) factor $G(\sigma, t)$ of the spectrum $\Phi(\sigma, t)$ is given, i.e.,

$$\Phi(\sigma, t) = G(\sigma, t) G^\sim(\sigma, t),$$

where

$$G(\sigma, t) = C(t) (\sigma I - A(t))^{-1} B(t) + D(t)$$

is the transfer operator of the periodic System (10.1). Notice that since $\Phi(\sigma, \cdot)$ is required to be positive, the periodic system is constrained to be full-row rank. In particular, $D(t)D(t)' > 0, \forall t$. We aim to find the minimal spectral factor $G_m(\sigma, t)$ in (10.7). The solution of this problem can be viewed as a sub-product of the solution of the general factorization problem presented in Sect. 10.1. Indeed, by letting $R = I$, see (10.6a), the Riccati Eq. (10.5) becomes

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - S_I(t)W_I(t)S_I(t)' \quad (10.11a)$$

$$S_I(t) = A(t)P(t)C(t)' + B(t)D(t)' \quad (10.11b)$$

$$W_I(t) = (D(t)D(t)' + C(t)P(t)C(t)')^{-1}. \quad (10.11c)$$

This is the celebrated periodic Kalman filter Riccati equation. Under mild conditions (namely, stabilizability and detectability of the underlying system) this equation admits a unique positive semi-definite solution $P(\cdot)$. This solution is stabilizing, in that the periodic matrix

$$\hat{A}_I(t) = A(t) - S_I(t)W_I(t)C(t)$$

is stable. Turning to the spectral factor, from (10.6) it follows that

$$G(\sigma, t) G^\sim(\sigma, t) = G_m(\sigma, t) G_m^\sim(\sigma, t) \quad (10.12a)$$

$$G_m(\sigma, t) = W_I(t)^{-1/2} + C(t) (\sigma I - A(t))^{-1} S_I(t) W_I(t)^{1/2}. \quad (10.12b)$$

Therefore, $G_m(\sigma, t)$ is a canonical spectral factor of $\Phi(\sigma, t)$, since, in view of the stabilizing property of $P(\cdot)$, it turns out that $G_m(\sigma, t)^{-1}$ has $\hat{A}_I(\cdot)$ as dynamic matrix and therefore is stable. Of course, such a factor is not necessarily minimal, unless $G(\sigma, t)$ is stable per-se. Otherwise, in order to find the minimal factor, one can resort to the results of the previous Sect. 10.2.1.

Remark 10.3 So far, we have assumed that $D(t)D(t)' > 0$, for each t . When this assumption is not met with, one can resort to the spectral interactor matrix introduced in Sect. 8.4. Although the theory developed therein is based on the assumption that $D(t)$ is a square matrix, the extension to the general case is a simple matter of output transformation and is therefore taken for granted. By resorting to the interactor polynomial matrix

$$L(\sigma, t) = \sum_{i=0}^q L_i(t) \sigma^i,$$

one comes up with a new delay-free system with transfer operator

$$\bar{G}(\sigma, t) = C(t)(\sigma I - A(t))^{-1} \bar{B}(t) + \bar{D}(t).$$

This system shares the same spectrum of the original system, i.e.,

$$\bar{G}(\sigma, t) \bar{G}^\sim(\sigma, t) = G(\sigma, t) G^\sim(\sigma, t)$$

and it is such that $\bar{D}(t)\bar{D}(t)' > 0$, for each t . Associated with the new system we can introduce the periodic Riccati equation

$$\begin{aligned} \bar{P}(t+1) &= A(t)P(t)A(t)' + \bar{B}(t)\bar{B}(t)' - \bar{S}_I(t)\bar{W}_I(t)\bar{S}_I(t)' \\ \bar{S}_I(t) &= A(t)\bar{P}(t)C(t)' + \bar{B}(t)\bar{D}(t)' \\ \bar{W}_I(t) &= (\bar{D}(t)\bar{D}(t)' + C(t)\bar{P}(t)C(t)')^{-1}. \end{aligned}$$

The periodic solution $\bar{P}(\cdot)$ of the above Riccati equation furnishes the spectral factor

$$\bar{G}_m(\sigma, t) = \bar{W}_I(t)^{-1/2} + C(t)(\sigma I - A(t))^{-1} \bar{S}_I(t)\bar{W}_I(t)^{1/2}$$

that obviously coincides with $G_m(\sigma, t)$, i.e.,

$$\bar{G}_m(\sigma, t) = G_m(\sigma, t).$$

Notice, however, that $P(\cdot)$ and $\bar{P}(\cdot)$ do not coincide in general. To be precise, they are related as follows (the proof is left to the reader)

$$P(t) = \bar{P}(t) + P_d(t), \quad (10.13)$$

where $P_d(t)$ reflects the delay structure of $G(\sigma, t)$ and is given by

$$P_d(t) = \sum_{i=1}^{q-1} H_i(t)H_i(t)',$$

with

$$H_i(t) = \sum_{j=1}^{q-i} \Phi_A(t+1, t-j+2)B(t+1-j)L_{i+j}(t+1-j).$$

The Expression (10.13) brings into light the effects of unstable finite zeros of $G(\sigma, t)$ that are responsible for $\bar{P}(\cdot) \neq 0$ and that of the infinite zeros (delays) accounting for $P_d(\cdot) \neq 0$. In this way, the role of the invariant zeros in the shaping of the solution of the Riccati equation is pointed out. ■

We end this section by applying the spectral factorization theory to the so-called inner–outer factorization, as discussed in the subsequent remark.

Remark 10.4 (Inner–outer factorization)

Consider a stable periodic system with transfer operator

$$G(\sigma, t) = C(t)(\sigma I - A(t))^{-1}B(t) + D(t)$$

and assume that the system is full-row rank without zeros in the unit circle. The inner–outer factorization amounts to expressing $G(\sigma, t)$ in the following form

$$G(\sigma, t) = G_-(\sigma, t)G_+(\sigma, t),$$

where $G_-(\sigma, t)$ is stable with stable inverse (outer) and $G_+(\sigma, t)$ is co-inner, i.e., $G_+(\sigma, t)G_+^\sim(\sigma, t)$ is the identity operator. This factorization enables one to subdivide a system as the cascade connection between a minimum phase system and an all pass system. In this way, all unstable zeros are isolated into $G_+(\sigma, t)$.

To solve the inner–outer factorization problem, take the minimal spectral factor $G_m(\sigma, t)$ of $G(\sigma, t)G^\sim(\sigma, t)$, i.e., $G(\sigma, t)G^\sim(\sigma, t) = G_m(\sigma, t)G_m^\sim(\sigma, t)$, as explained in Sect. 10.2.2. Of course, $G_m(\sigma, t)$ is stable with stable inverse. On the other hand

$$G_m(\sigma, t)^{-1}G(\sigma, t)G^\sim(\sigma, t)G_m^\sim(\sigma, t)^{-1} = I.$$

Therefore, the solution of the inner–outer factorization problem is

$$G_-(\sigma, t) = G_m(\sigma, t), \quad G_+(\sigma, t) = G_m(\sigma, t)^{-1}G(\sigma, t).$$

It goes without saying that a similar technique can be used to work out an inner–outer factorization of a full column transfer operator by using the dual spectral factorization result. ■

The Riccati Equation (10.11) will play a main role in Sect. 12.1, where the periodic Kalman filtering problem will be considered. As will be pointed out there, such a problem is closely connected to the periodic H_2 filtering problem, where the designer aims to minimize the H_2 norm of the transfer operator between the disturbances and the estimation error.

10.3 J-Spectral Factorization

In this section, we give a concise treatment of a novel factorization problem called the J -spectral factorization problem. To this purpose, consider the periodic system

$$\begin{aligned} x(t+1) &= \bar{A}(t)x(t) + \bar{B}(t)w(t) \\ y(t) &= \bar{C}(t)x(t) + \bar{D}(t)w(t) \\ z(t) &= \bar{L}(t)x(t), \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are the state and the measurement output vector, respectively and $w(t) \in \mathbb{R}^m$ is the disturbance vector. Signal $z(t) \in \mathbb{R}^q$ is then an inaccessible linear combination of the state. Therefore, the given system is characterized by two transfer operators, relating the disturbance input $w(t)$ to the measurement output $y(t)$ and the remote signal $z(t)$, namely

$$\bar{G}_1(\sigma, t) = \bar{C}(t) (\sigma I - \bar{A}(t))^{-1} \bar{B}(t) + \bar{D}(t), \quad \bar{G}_2(\sigma, t) = \bar{L}(t) (\sigma I - \bar{A}(t))^{-1} \bar{B}(t).$$

Now, introduce the square matrix of dimensions $(a+b) \times (a+b)$

$$J_{a,b} = \begin{bmatrix} I_a & 0 \\ 0 & -\gamma^2 I_b \end{bmatrix}.$$

We consider now the factorization problem stated in Sect. 10.1, with the positions:

$$\begin{aligned} G(\sigma, t) &= \begin{bmatrix} \bar{G}_1(\sigma, t) & 0 \\ \bar{G}_2(\sigma, t) & I \end{bmatrix} \\ A(t) &= \bar{A}(t) \\ C(t) &= \begin{bmatrix} \bar{C}(t) \\ \bar{L}(t) \end{bmatrix} \\ B(t) &= \begin{bmatrix} \bar{B}(t) & 0 \end{bmatrix} \\ D(t) &= \begin{bmatrix} \bar{D}(t) & I \end{bmatrix} \\ R &= J_{m,q}. \end{aligned}$$

The Riccati Equation (10.5) becomes:

$$P(t+1) = \bar{A}(t)P(t)\bar{A}(t)' + \bar{B}(t)B(t)' - S_J(t)W_J(t)S_J(t)' \quad (10.14a)$$

$$S_J(t) = [\bar{A}(t)P(t)\bar{C}(t)' + \bar{B}(t)\bar{D}(t)' \quad \bar{A}(t)P(t)\bar{L}(t)'] \quad (10.14b)$$

$$W_J(t) = \begin{bmatrix} D(t)D(t)' + \bar{C}(t)P(t)\bar{C}(t)' & \bar{C}(t)P(t)\bar{L}(t)' \\ \bar{L}(t)'P(t)C(t) & -\gamma^2 I + \bar{L}(t)P(t)\bar{L}(t)' \end{bmatrix}^{-1}. \quad (10.14c)$$

Therefore, if $P(\cdot)$ is a T -periodic solution, then

$$G(\sigma, t)J_{m,q}G^\sim(\sigma, t) = T_J(\sigma, t)W_J(t)T_J^\sim(\sigma, t) \quad (10.15a)$$

$$T_J(\sigma, t) = W_J(t)^{-1} + \begin{bmatrix} \bar{C}(t) \\ \bar{L}(t) \end{bmatrix} (\sigma I - A(t))^{-1} S_J(t). \quad (10.15b)$$

The transfer operator $G(\sigma, t)J_{m,q}G^\sim(\sigma, t)$ is called the *J-spectrum*. The factorization problem consists in finding a square invertible transfer operator $G_m(\sigma, t)$ (of dimension $p + q$) so that

$$G(\sigma, t)J_{m,q}G^\sim(\sigma, t) = G_m(\sigma, t)J_{p,q}G_m^\sim(\sigma, t). \quad (10.16)$$

Again, we are interested in a factor $G_m(\sigma, t)$ with stable inverse. This factor is named *canonical J-spectral factor*. If, in addition $G_m(\sigma, t)$ is stable per-se, then it is the *minimal J-spectral factor*.

The existence of a canonical *J-spectral factor* $G_m(\sigma, t)$ is related to the existence of a T -periodic solution $P(\cdot)$ of the Riccati Equation (10.14). Notice that if $G(\sigma, t)$ is full-row rank, Eq. (10.15) implies that the matrices $J_{m,q}$ and $W_J(t)$ have the same inertia. Therefore, there exists an invertible matrix $\Gamma(t)$ in such a way that $W_J(t) = \Gamma(t)J_{p,q}\Gamma(t)'$. Hence

$$G(\sigma, t)J_{m,q}G^\sim(\sigma, t) = G_m(\sigma, t)J_{p,q}G_m^\sim(\sigma, t) \quad (10.17a)$$

$$G_m(\sigma, t) = T_J(\sigma, t)\Gamma(t). \quad (10.17b)$$

Of course, the factor $G_m(\sigma, t)$ in (10.17) has a stable inverse if $P(\cdot)$ is the T -periodic stabilizing solution of the Riccati Equation (10.14).

Remark 10.5 In order to clarify the role of the parameter γ , sometimes named attenuation level, rewrite the Riccati Equation (10.14) in the following way

$$\begin{aligned} P(t+1) &= \bar{A}(t)P(t)\bar{A}(t)' + \bar{B}(t)\bar{B}(t)' - \bar{S}_J(t)\bar{W}_J(t)\bar{S}_J(t)' \\ \bar{S}_J(t) &= [\bar{A}(t)P(t)\bar{C}(t)' + \bar{B}(t)\bar{D}(t)'\gamma^{-1}\bar{A}(t)P(t)\bar{L}(t)'] \\ \bar{W}_J(t) &= \begin{bmatrix} \bar{D}(t)\bar{D}(t)' + \bar{C}(t)P(t)\bar{C}(t)' & \gamma^{-1}\bar{C}(t)P(t)\bar{L}(t)' \\ \gamma^{-1}\bar{L}(t)'P(t)\bar{C}(t) & -I + \gamma^{-2}\bar{L}(t)P(t)\bar{L}(t)' \end{bmatrix}^{-1}. \end{aligned}$$

Therefore, as can be easily verified, as $\gamma \rightarrow \infty$, this Riccati equation becomes the Riccati Equation (10.11).

Remark 10.6 The Riccati Equation (10.14) is rather involved. However, it takes a simple expression if some standard assumptions are made on the original system. For instance, if

$$\bar{D}(t)\bar{D}(t)' = I, \quad \bar{D}(t)\bar{B}(t)' = 0$$

and the solution $P(\cdot)$ is invertible, then it is possible to rewrite (10.11) as

$$P(t+1) = \bar{A}(t) (P(t)^{-1} + \bar{C}(t)' \bar{C}(t) - \gamma^{-2} \bar{L}(t)' \bar{L}(t))^{-1} \bar{A}(t)' + \bar{B}(t) \bar{B}(t)'.$$

This expression clearly points out the sign-indefiniteness of the term appearing in the bracketed inverse. This fact can be interpreted in a game theoretic framework, as will be discussed in Chap. 12. ■

The Riccati Equation (10.14) is associated with the \mathcal{H}_∞ estimation problem, to be treated in Sect. 12.4.

10.4 Parametrization of Stabilizing Controllers

In this section, we discuss the problem of parametrization of stabilizing controllers for periodic systems. Reference will be made to both the parametrization in the field of polynomial operators and in the field of rational operators. The main idea, inherited from the time-invariant theory, is to resort to a fractional description for both the plant and the controller. This means that the plant and the controller are written as the ratio of stable polynomial operators (rational operators). The periodicity of the plant renders practically immaterial the distinction between the scalar case and the multivariable case. As a matter of fact, both the left and the right coprime factorization are needed even in the scalar case. Therefore when dealing with polynomials, we make reference to SISO systems without any loss of generality.

10.4.1 Polynomial Formulation

Assume that a transfer function $W(z)$ of a SISO time-invariant system is given:

$$W(z) = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}.$$

Obviously, one can rewrite this expression as

$$W(z) = \frac{W_1(z)}{W_2(z)},$$

with

$$\begin{aligned} W_1(z) &= \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{z^n} \\ W_2(z) &= \frac{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}{z^n}. \end{aligned}$$

This means that a SISO discrete-time system can be always described as the ratio of two rational functions having all poles in the origin (FIR systems). Observe also that the two transfer functions $W_1(z)$ and $W_2(z)$ are two polynomials in the variable z^{-1} (unitary delay).

Turning to periodic systems, we can still allow ourselves to write the system as the ratio of two polynomial operators in the *delay operator* σ^{-1} (PFIR systems). Indeed, its transfer operator can always be written as a function of the delay operator σ^{-1} in fractional form

$$g(\sigma^{-1}, t) = \bar{a}(\sigma^{-1}, t)^{-1} \bar{b}(\sigma^{-1}, t) = \tilde{b}(\sigma^{-1}, t) \tilde{a}(\sigma^{-1}, t)^{-1}, \quad (10.18)$$

where the pair $(\bar{a}(\sigma^{-1}, t), \bar{b}(\sigma^{-1}, t))$ is strongly right coprime and, moreover, the pair $(\tilde{a}(\sigma^{-1}, t), \tilde{b}(\sigma^{-1}, t))$ is strongly left coprime. Consequently, it is possible to work out T -periodic polynomials $\bar{x}(\sigma^{-1}, t)$, $\bar{w}(\sigma^{-1}, t)$, $\tilde{x}(\sigma^{-1}, t)$ and $\tilde{w}(\sigma^{-1}, t)$ in such a way that

$$\bar{a}(\sigma^{-1}, t) \bar{x}(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t) \bar{w}(\sigma^{-1}, t) = 1 \quad (10.19a)$$

$$\tilde{x}(\sigma^{-1}, t) \tilde{a}(\sigma^{-1}, t) + \tilde{w}(\sigma^{-1}, t) \tilde{b}(\sigma^{-1}, t) = 1 \quad (10.19b)$$

$$\bar{x}(\sigma^{-1}, t) \bar{w}(\sigma^{-1}, t) = \tilde{w}(\sigma^{-1}, t) \tilde{x}(\sigma^{-1}, t). \quad (10.19c)$$

Notice that the above characterization is the so-called *double-coprime factorization* of the system, in that

$$\begin{bmatrix} \bar{a}(\sigma^{-1}, t) & \bar{b}(\sigma^{-1}, t) \\ -\bar{w}(\sigma^{-1}, t) & \bar{x}(\sigma^{-1}, t) \end{bmatrix} \begin{bmatrix} \tilde{x}(\sigma^{-1}, t) & -\tilde{b}(\sigma^{-1}, t) \\ \tilde{w}(\sigma^{-1}, t) & \tilde{a}(\sigma^{-1}, t) \end{bmatrix} = I.$$

It is worth pointing out that Eqs. (10.19) can be alternatively written in terms of lifted polynomials.

All that above said, consider a periodic controller for the system above, described by the transfer operator

$$k(\sigma^{-1}, t) = \bar{r}(\sigma^{-1}, t)^{-1} \bar{s}(\sigma^{-1}, t) = \tilde{s}(\sigma^{-1}, t) \tilde{r}(\sigma^{-1}, t)^{-1}, \quad (10.20)$$

where pair $(\bar{r}(\sigma^{-1}, t), \bar{s}(\sigma^{-1}, t))$ is strong left coprime and pair $(\tilde{r}(\sigma^{-1}, t), \tilde{s}(\sigma^{-1}, t))$ is strong right coprime. The overall control scheme is depicted in Fig. 10.2. The ob-

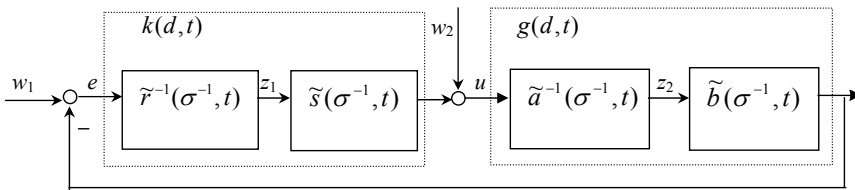


Fig. 10.2 Fractional representation of the control scheme

jective of the following considerations is to show that the class of periodic stabilizing controllers can be fully described on the basis of the quadruplet

$$\bar{x}(\sigma^{-1}, t), \bar{w}(\sigma^{-1}, t), \tilde{a}(\sigma^{-1}, t), \tilde{b}(\sigma^{-1}, t),$$

or, alternatively,

$$\tilde{x}(\sigma^{-1}, t), \tilde{w}(\sigma^{-1}, t), \bar{a}(\sigma^{-1}, t), \bar{b}(\sigma^{-1}, t)$$

provided that further free periodic polynomial $q(\sigma^{-1}, t)$ is introduced.

Before proceeding with the parametrization result, we need a preliminary lemma concerning the internal stability of a control system characterized in terms of periodic transfer operators. To this purpose, we define the transfer operator $C(\sigma^{-1}, t)$ from the inputs w_1, w_2 to the outputs u and e in Fig. 10.2. Letting

$$\begin{aligned} c_1(\sigma^{-1}, t) &= 1 + \tilde{s}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)^{-1}\tilde{b}(\sigma^{-1}, t)\bar{a}(\sigma^{-1}, t)^{-1} \\ c_2(\sigma^{-1}, t) &= 1 + \tilde{b}(\sigma^{-1}, t)\bar{a}(\sigma^{-1}, t)^{-1}\tilde{s}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)^{-1}, \end{aligned}$$

such an operator takes the expression

$$\begin{aligned} C(\sigma^{-1}, t) &= \\ &= \begin{bmatrix} c_1(\sigma^{-1}, t) & 0 \\ 0 & c_2(\sigma^{-1}, t) \end{bmatrix}^{-1} \begin{bmatrix} \tilde{r}(\sigma^{-1}, t)^{-1}\tilde{s}(\sigma^{-1}, t) & 1 \\ 1 & -\bar{a}(\sigma^{-1}, t)^{-1}\bar{b}(\sigma^{-1}, t) \end{bmatrix}. \end{aligned}$$

Moreover, define the transfer operator $Z(\sigma^{-1}, t)$ from the inputs w_1, w_2 to the outputs z_1, z_2 of Fig. 10.2, which takes the expression

$$Z(\sigma^{-1}, t) = \begin{bmatrix} \tilde{r}(\sigma^{-1}, t) & \tilde{b}(\sigma^{-1}, t) \\ -\tilde{s}(\sigma^{-1}, t) & \bar{a}(\sigma^{-1}, t) \end{bmatrix}^{-1}. \quad (10.21)$$

The rational operators $C(\sigma^{-1}, t)$ and $Z(\sigma^{-1}, t)$ can be related to each other through the solution $(t_1(\sigma^{-1}, t), t_2(\sigma^{-1}, t))$ of the Bezout equation

$$t_1(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t) + t_2(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t) = 1.$$

Indeed, it turns out that:

$$\begin{aligned} Z(\sigma^{-1}, t) &= \begin{bmatrix} t_1(\sigma^{-1}, t) & t_2(\sigma^{-1}, t) \\ \tilde{x}(\sigma^{-1}, t) & -\tilde{w}(\sigma^{-1}, t) \end{bmatrix} C(\sigma^{-1}, t) + \\ &+ \begin{bmatrix} 0 & -t_1(\sigma^{-1}, t) \\ \tilde{w}(\sigma^{-1}, t) & 0 \end{bmatrix} \end{aligned} \quad (10.22a)$$

$$C(\sigma^{-1}, t) = \begin{bmatrix} 0 & \bar{a}(\sigma^{-1}, t) \\ \tilde{r}(\sigma^{-1}, t) & 0 \end{bmatrix} Z(\sigma^{-1}, t). \quad (10.22b)$$

These last expressions show that the rational operators $C(\sigma^{-1}, t)$ and $Z(\sigma^{-1}, t)$ are related to each other through polynomial matrices. Therefore, $C(\sigma^{-1}, t)$ is stable if and only if $Z(\sigma^{-1}, t)$ is stable.

Lemma 10.1 *Consider the control system depicted in Fig. 10.2. This system is stable if and only if one of the two following equivalent conditions holds:*

- (i) *The input–output operator $C(\sigma^{-1}, t)$ from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} u \\ e \end{bmatrix}$ is stable.*
- (ii) *The input–output operator $Z(\sigma^{-1}, t)$ from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is stable.*

Proof It suffices to show that the internal stability is equivalent to the stability of $C(\sigma^{-1}, t)$, since the equivalence between the stability of $C(\sigma^{-1}, t)$ and the stability of $Z(\sigma^{-1}, t)$ has been already established in view of (10.22).

Obviously, if the system is internally stable then $C(\sigma^{-1}, t)$ is stable. To prove the converse statement, we only have to prove that the overall system is reachable and observable. To this end, consider a (minimal) state–space realization of the system

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ w_1(t) - e(t) &= C(t)x(t) + D(t)u(t), \end{aligned}$$

and a (minimal) state–space realization of the controller

$$\begin{aligned} \xi(t+1) &= F(t)\xi(t) + G(t)e(t) \\ u(t) - w_2(t) &= H(t)\xi(t) + E(t)e(t). \end{aligned}$$

Let $\Delta_{12}(t) = (I + D(t)E(t))$, $\Delta_{21}(t) = (I + E(t)D(t))$ and assume that they are invertible (notice that $\Delta_{12}(t)$ is invertible if and only if $\Delta_{21}(t)$ is invertible). This condition guarantees that the closed-loop system is proper. The feedback system is described by the quadruplet $(\bar{A}(t), \bar{B}(t), \bar{C}(t), \bar{D}(t))$ where

$$\begin{aligned} \bar{A}(t) &= \begin{bmatrix} A(t) - B(t)\Delta_{21}(t)^{-1}E(t)C(t) & B(t)\Delta_{21}(t)^{-1}H(t) \\ -G(t)\Delta_{12}(t)^{-1}C(t) & F(t) - G(t)\Delta_{12}(t)^{-1}D(t)H(t) \end{bmatrix} \\ \bar{B}(t) &= \begin{bmatrix} B(t)\Delta_{21}(t)^{-1}E(t) & B(t)\Delta_{21}(t)^{-1} \\ G(t)\Delta_{21}(t)^{-1} & -G(t)\Delta_{12}(t)^{-1}D(t) \end{bmatrix} \\ \bar{C}(t) &= \begin{bmatrix} -\Delta_{21}(t)^{-1}E(t)C(t) & \Delta_{21}(t)^{-1}H(t) \\ -\Delta_{12}(t)^{-1}C(t) & -\Delta_{12}(t)^{-1}D(t)H(t) \end{bmatrix} \\ \bar{D}(t) &= \begin{bmatrix} \Delta_{21}(t)^{-1}E(t) & \Delta_{21}(t)^{-1} \\ \Delta_{12}(t)^{-1} & -\Delta_{12}(t)^{-1}D(t) \end{bmatrix}. \end{aligned}$$

Now introduce the following matrices (which are invertible for all t)

$$T_C(t) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ C(t) & 0 & I & D(t) \\ 0 & -H(t) & -E(t) & I \end{bmatrix}$$

$$T_O(t) = \begin{bmatrix} I & 0 & -B(t) & 0 \\ 0 & I & 0 & -G(t) \\ 0 & 0 & I & -E(t) \\ 0 & 0 & D(t) & I \end{bmatrix}.$$

Recall the modal test for reachability, see Sect. 4.1.6 and assume, by contradiction, that the closed-loop system is not reachable, i.e., that there exists a nonzero periodic solution of

$$\begin{bmatrix} \lambda \sigma I - \bar{A}(t)' \\ -\bar{B}(t)' \end{bmatrix} \cdot \theta(t) = 0.$$

By premultiplying by $T_C(t)'$, simple computations show that

$$\begin{bmatrix} \lambda \sigma I - A(t)' & 0 \\ 0 & \lambda \sigma I - F(t)' \\ 0 & -G(t)' \\ -B(t)' & 0 \end{bmatrix} \cdot \theta(t) = 0,$$

so that reachability of the system and/or the controller is violated.

Analogously, recall the modal test for observability, see Sect. 4.2.7, and assume, by contradiction, that the closed-loop system is not observable, i.e., that there exists a non-zero periodic solution of

$$\begin{bmatrix} -\lambda \sigma I + \bar{A}(t) \\ \bar{C}(t) \end{bmatrix} \cdot \theta(t) = 0.$$

By premultiplying by $T_O(t)$, simple computations show that

$$\begin{bmatrix} -\lambda \sigma I + A(t) & 0 \\ 0 & -\lambda \sigma I + F(t) \\ 0 & H(t) \\ -C(t) & 0 \end{bmatrix} \cdot \theta(t) = 0,$$

so that observability of the system and/or the controller is violated. ■

As in all parametrization results, a free parameter $q(\sigma^{-1}, \cdot)$ enters the game. This free parameter corresponds to a stable T -periodic causal system. Therefore, it is not required that $q(\sigma^{-1}, \cdot)$ be a finite-length polynomial in σ^{-1} , and hence a PFIR system. Precisely, the following result holds.

Proposition 10.1 Assume that the periodic system with transfer operator (10.18) is reachable and observable for each t . Let $\bar{x}(\sigma^{-1}, t)$, $\bar{\tilde{x}}(\sigma^{-1}, t)$, $\bar{w}(\sigma^{-1}, t)$ and $\bar{\tilde{w}}(\sigma^{-1}, t)$ be particular polynomial solutions of the operator Eqs. (10.19). Then, all linear T -periodic stabilizing controllers for System (10.18) are given by (10.20) with

$$\tilde{r}(\sigma^{-1}, t) = \bar{x}(\sigma^{-1}, t) + \tilde{b}(\sigma^{-1}, t)q(\sigma^{-1}, t) \quad (10.23a)$$

$$\tilde{s}(\sigma^{-1}, t) = \bar{w}(\sigma^{-1}, t) - \tilde{a}(\sigma^{-1}, t)q(\sigma^{-1}, t), \quad (10.23b)$$

or, equivalently

$$\bar{r}(\sigma^{-1}, t) = \tilde{x}(\sigma^{-1}, t) + q(\sigma^{-1}, t)\bar{b}(\sigma^{-1}, t) \quad (10.24a)$$

$$\bar{s}(\sigma^{-1}, t) = \tilde{w}(\sigma^{-1}, t) - q(\sigma^{-1}, t)\bar{a}(\sigma^{-1}, t), \quad (10.24b)$$

where $q(\sigma^{-1}, \cdot)$ is a free stable periodic operator such that $\tilde{r}(\sigma^{-1}, \cdot)$ and $\bar{r}(\sigma^{-1}, \cdot)$ are not identically zero.

Proof The proof relies on Lemma 10.1. To be precise, consider the feedback connection of $g(\sigma^{-1}, t) = \tilde{b}(\sigma^{-1}, t)\tilde{a}(\sigma^{-1}, t)^{-1}$ and $k(\sigma^{-1}, t) = \tilde{s}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)^{-1}$ in Fig. 10.2. Now, assume that a stable $q(\sigma^{-1}, \cdot)$ be given and consider the controller $k(\sigma^{-1}, t) = \tilde{s}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)^{-1} = \bar{r}(\sigma^{-1}, t)^{-1}\bar{s}(\sigma^{-1}, t)$. Then

$$\begin{aligned} c_1(\sigma^{-1}, t) &= 1 + \bar{r}(\sigma^{-1}, t)^{-1}\bar{s}(\sigma^{-1}, t)\tilde{b}(\sigma^{-1}, t)\tilde{a}(\sigma^{-1}, t)^{-1} \\ &= \bar{r}(\sigma^{-1}, t)^{-1}(\bar{r}(\sigma^{-1}, t)\tilde{a}(\sigma^{-1}, t) + \bar{s}(\sigma^{-1}, t)\tilde{b}(\sigma^{-1}, t))\tilde{a}(\sigma^{-1}, t)^{-1} \\ &= \bar{r}(\sigma^{-1}, t)^{-1}(\tilde{x}(\sigma^{-1}, t)\tilde{a}(\sigma^{-1}, t) + \tilde{w}(\sigma^{-1}, t)\tilde{b}(\sigma^{-1}, t))\tilde{a}(\sigma^{-1}, t)^{-1} \\ &= \bar{r}(\sigma^{-1}, t)^{-1}\tilde{a}(\sigma^{-1}, t)^{-1} \end{aligned}$$

$$\begin{aligned} c_2(\sigma^{-1}, t) &= 1 + \bar{a}(\sigma^{-1}, t)^{-1}\bar{b}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)^{-1} \\ &= \bar{a}(\sigma^{-1}, t)^{-1}(\bar{a}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t))\tilde{r}(\sigma^{-1}, t)^{-1} \\ &= \bar{a}(\sigma^{-1}, t)^{-1}(\bar{a}(\sigma^{-1}, t)\bar{x}(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t)\bar{w}(\sigma^{-1}, t))\tilde{r}(\sigma^{-1}, t)^{-1} \\ &= \bar{a}(\sigma^{-1}, t)^{-1}\tilde{r}(\sigma^{-1}, t)^{-1}. \end{aligned}$$

Hence,

$$C(\sigma^{-1}, t) = \begin{bmatrix} \tilde{a}(\sigma^{-1}, t)\bar{s}(\sigma^{-1}, t) & \tilde{a}(\sigma^{-1}, t)\bar{r}(\sigma^{-1}, t) \\ \tilde{r}(\sigma^{-1}, t)\bar{a}(\sigma^{-1}, t) & -\tilde{r}(\sigma^{-1}, t)\bar{b}(\sigma^{-1}, t) \end{bmatrix}$$

is a polynomial matrix and thus the underlying system is stable.

Conversely, assume that the closed-loop is stable with a certain controller $k(\sigma^{-1}, t) = \tilde{s}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)^{-1}$ and let

$$\begin{aligned} q(\sigma^{-1}, t) &= \\ &(\tilde{w}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t) - \tilde{x}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t))(\bar{a}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t))^{-1}. \end{aligned}$$

The function $q(\sigma^{-1}, t)$ can also be written as

$$q(\sigma^{-1}, t) = (\tilde{x}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t) - \tilde{w}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t)) \begin{bmatrix} 1 & 0 \end{bmatrix} Z(\sigma^{-1}, t) \begin{bmatrix} -\bar{x}(\sigma^{-1}, t) \\ \bar{w}(\sigma^{-1}, t) \end{bmatrix},$$

where $Z(\sigma^{-1}, t)$ has been defined in (10.21) (the simple check is left to the reader). Since all elements in the above expression are stable, it turns out that $q(\sigma^{-1}, t)$ is stable as well. Finally,

$$\begin{aligned} \bar{x}(\sigma^{-1}, t) + \tilde{b}(\sigma^{-1}, t)q(\sigma^{-1}, t) &= \\ &= \tilde{r}(\sigma^{-1}, t) (\bar{a}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t))^{-1}, \end{aligned}$$

$$\begin{aligned} \bar{w}(\sigma^{-1}, t) - \tilde{b}(\sigma^{-1}, t)q(\sigma^{-1}, t) &= \\ &= \tilde{s}(\sigma^{-1}, t) (\bar{a}(\sigma^{-1}, t)\tilde{r}(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t)\tilde{s}(\sigma^{-1}, t))^{-1} \end{aligned}$$

which show that Eqs. (10.23), hold since

$$k(\sigma^{-1}, t) = (\bar{w}(\sigma^{-1}, t) - \tilde{b}(\sigma^{-1}, t)q(\sigma^{-1}, t)) (\bar{x}(\sigma^{-1}, t) + \tilde{b}(\sigma^{-1}, t)q(\sigma^{-1}, t))^{-1}.$$

The proof of Eqs. (10.24) is completely similar and therefore omitted.

10.4.2 Rational Formulation

In this section, the parametrization problem in the field of rational operators is addressed. The discussion follows the same line of reasoning previously adopted. However, here the focus is centered on the description of periodic systems in the forward shift operator σ and on a state-space representation:

$$\mathcal{G} : \begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}.$$

The associated transfer operator is:

$$G(\sigma, t) = D(t) + C(t)(\sigma I - A(t))^{-1}B(t).$$

To avoid possible misunderstandings, we specify from the beginning that in this section we consider controllers fed by the feedback signal $y(t)$ without additional change of sign. To guarantee the existence of at least one stabilizing controller, it is assumed that the system is stabilizable and detectable.

The following preliminary result (double coprime factorization) is in order.

Lemma 10.2 *There exist eight stable periodic rational operators $S(\sigma, t)$, $N(\sigma, t)$, $\hat{S}(\sigma, t)$, $\hat{N}(\sigma, t)$, $X(\sigma, t)$, $Y(\sigma, t)$, $\hat{X}(\sigma, t)$ and $\hat{Y}(\sigma, t)$, such that*

(i)

$$G(\sigma, t) = N(\sigma, t)S(\sigma, t)^{-1} = \hat{S}(\sigma, t)^{-1}\hat{N}(\sigma, t)$$

(ii)

$$\begin{bmatrix} \hat{X}(\sigma, t) & -\hat{Y}(\sigma, t) \\ -\hat{N}(\sigma, t) & \hat{S}(\sigma, t) \end{bmatrix} \begin{bmatrix} S(\sigma, t) & Y(\sigma, t) \\ N(\sigma, t) & X(\sigma, t) \end{bmatrix} = I.$$

These operators can be expressed in terms of the matrices of the original system as follows:

$$\begin{aligned} S(\sigma, t) &= I + K(t)(\sigma I - A(t) - B(t)K(t))^{-1}B(t) \\ N(\sigma, t) &= D(t) + (C(t) + D(t)K(t))(\sigma I - A(t) - B(t)K(t))^{-1}B(t) \\ X(\sigma, t) &= I - (C(t) + D(t)K(t))(\sigma I - A(t) - B(t)K(t))^{-1}L(t) \\ Y(\sigma, t) &= -K(t)(\sigma I - A(t) - B(t)K(t))^{-1}L(t) \\ \hat{S}(\sigma, t) &= I + C(t)(\sigma I - A(t) - L(t)C(t))^{-1}L(t) \\ \hat{N}(\sigma, t) &= D(t) + C(t)(\sigma I - A(t) - L(t)C(t))^{-1}(B(t) + L(t)D(t)) \\ \hat{X}(\sigma, t) &= I - K(t)(\sigma I - A(t) - L(t)C(t))^{-1}(B(t) + L(t)D(t)) \\ \hat{Y}(\sigma, t) &= -K(t)(\sigma I - A(t) - L(t)C(t))^{-1}L(t), \end{aligned}$$

where $K(\cdot)$ and $L(\cdot)$ are two T -periodic matrices such that $A(\cdot) + B(\cdot)K(\cdot)$ and $A(\cdot) + L(\cdot)C(\cdot)$ are stable, respectively.

Proof Preliminarily recall that, in view of the assumed stabilizability and detectability of \mathcal{G} , there exist two periodic matrices $K(\cdot)$ and $L(\cdot)$ such that $A(\cdot) + L(\cdot)C(\cdot)$ and $A(\cdot) + B(\cdot)K(\cdot)$ are stable.

The proof can be completely carried out in the input–output context. However, here we prefer to use the state–space formulation, since it clarifies the structure of the parametrization in terms of state-feedback and output injection. Thus, consider the four stable periodic systems $\mathcal{F}_i, i = 1, \dots, 4$ given by

$$\begin{aligned} \mathcal{F}_1 : \quad & \begin{cases} x(t+1) = (A(t) + B(t)K(t))x(t) + B(t)v(t) \\ y(t) = (C(t) + D(t)K(t))x(t) + D(t)v(t) \\ u(t) = K(t)x(t) + v(t) \end{cases} \\ \mathcal{F}_2 : \quad & \begin{cases} \hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + L(t)\eta(t) \\ \eta(t) = C(t)\hat{x}(t) + D(t)u(t) - y(t) \end{cases} \\ \mathcal{F}_3 : \quad & \begin{cases} \mu(t+1) = (A(t) + L(t)C(t))\mu(t) + (B(t) + L(t)D(t))u(t) - y(t) \\ g(t) = u(t) - K(t)\mu(t) \end{cases} \\ \mathcal{F}_4 : \quad & \begin{cases} p(t+1) = (A(t) + B(t)K(t))p(t) + L(t)d(t) \\ y(t) = (C(t) + D(t)K(t))p(t) - d(t) \\ u(t) = K(t)p(t). \end{cases} \end{aligned}$$

The reason why certain input (output) variables of these systems are denoted with the same symbols is that, in the proof, some of these systems will be considered in an interconnected configuration, so that the input of one system may coincide with the output of another one. In particular, notice that system \mathcal{F}_1 is nothing but system \mathcal{G} equipped with the periodic feedback control law $u(t) = K(t)x(t) + v(t)$, whereas system \mathcal{F}_2 is a periodic output injection for system \mathcal{G} .

We will also assume that all the initial conditions of all four systems are zero.

From \mathcal{F}_1 , it follows:

$$u(t) = S(\sigma, t) \cdot v(t), \quad y(t) = N(\sigma, t) \cdot v(t).$$

By comparing with the expression $y(t) = G(\sigma, t)u(t)$ and noticing that $S(\sigma, t)$ is invertible, it follows:

$$G(\sigma, t) = N(\sigma, t)S(\sigma, t)^{-1}. \quad (10.25)$$

From the expression of system \mathcal{F}_2 it is easy to see that:

$$\eta(t) = \hat{N}(\sigma, t) \cdot u(t) - \hat{S}(\sigma, t) \cdot y(t). \quad (10.26)$$

Consider now the cascade connection $\mathcal{F}_2\mathcal{F}_1$:

$$\begin{aligned} (\hat{x}(t+1) - x(t+1)) &= (A(t) + L(t)C(t))(\hat{x}(t) - x(t)) \\ \eta(t) &= C(t)(\hat{x}(t) - x(t)), \end{aligned}$$

so that (being all initial conditions at zero), $\hat{x}(t) = x(t)$ and therefore $\eta(t) = 0, \forall t$. Equation (10.26), the invertibility of $\hat{S}(\sigma, t)$ and the expression $y(t) = G(\sigma, t) \cdot u(t)$ imply that:

$$G(\sigma, t) = \hat{S}(\sigma, t)^{-1} \hat{N}(\sigma, t). \quad (10.27)$$

Equations (10.25) and (10.27) prove point (i) and entail

$$\hat{S}(\sigma, t)N(\sigma, t) = \hat{N}(\sigma, t)S(\sigma, t). \quad (10.28)$$

Moreover, the definition of system \mathcal{F}_3 leads to:

$$g(t) = \hat{X}(\sigma, t) \cdot u(t) - \hat{Y}(\sigma, t) \cdot y(t). \quad (10.29)$$

But the cascade connection $\mathcal{F}_3\mathcal{F}_1$ is described by

$$\begin{aligned} (\mu(t+1) - x(t+1)) &= (A(t) + L(t)C(t))(\mu(t) - x(t)) \\ g(t) &= v(t) - K(t)(\mu(t) - x(t)), \end{aligned}$$

so that $\mu(t) = x(t)$ and $g(t) = v(t), \forall t$. From $u(t) = S(\sigma, t)v(t)$ and (10.29), it then follows

$$\hat{X}(\sigma, t)S(\sigma, t) - \hat{Y}(\sigma, t)N(\sigma, t) = I. \quad (10.30)$$

Analogously, from the definition of \mathcal{F}_4 one obtains

$$y(t) = -X(\sigma, t) \cdot d(t), \quad u(t) = -Y(\sigma, t) \cdot d(t). \quad (10.31)$$

In addition, the cascade connection $\mathcal{F}_2\mathcal{F}_4$ implies

$$\begin{aligned} (p(t+1) - \hat{x}(t+1)) &= (A(t) + L(t)C(t))(p(t) - \hat{x}(t)) \\ \eta &= d(t) - C(t)(p(t) - \hat{x}(t)), \end{aligned}$$

so that $p(t) = \hat{x}(t)$ and $\eta(t) = d(t)$, $\forall t$. From (10.26), (10.31) it then follows

$$\hat{S}(\sigma, t)X(\sigma, t) - \hat{N}(\sigma, t)Y(\sigma, t) = I. \quad (10.32)$$

Finally, the cascade connection $\mathcal{F}_3\mathcal{F}_4$ entails that

$$\begin{aligned} (p(t+1) - \mu(t+1)) &= (A(t) + L(t)C(t))(p(t) - \mu(t)) \\ g(t) &= K(t)(p(t) - \mu(t)). \end{aligned}$$

Hence $p(t) = \mu(t)$ and $g(t) = 0$, $\forall t$. Then, from (10.29) and (10.31)

$$\hat{X}(\sigma, t)Y(\sigma, t) - \hat{Y}(\sigma, t)X(\sigma, t) = 0 \quad (10.33)$$

follows. Equations (10.25), (10.27), (10.28), (10.30), (10.32), (10.33) prove point (ii). \blacksquare

We are now in the position to state the main parametrization result. The class of periodic stabilizing controllers can be fully described by the quadruplet of periodic operators

$$S(\sigma, t), N(\sigma, t), X(\sigma, t), Y(\sigma, t),$$

or, alternatively,

$$\hat{S}(\sigma, t), \hat{N}(\sigma, t), \hat{X}(\sigma, t), \hat{Y}(\sigma, t),$$

with the additional use of a free rational operator $Q(\sigma, t)$. The proof is omitted since it is similar to that of Proposition 10.1.

Proposition 10.2 *Consider the periodic system \mathcal{G} with transfer operator $G(\sigma, t)$. All the periodic controllers that internally stabilize the system have transfer operator:*

$$\begin{aligned} K(\sigma, t) &= (Y(\sigma, t) + S(\sigma, t)Q(\sigma, t))(X(\sigma, t) + N(\sigma, t)Q(\sigma, t))^{-1} \\ &= (\hat{X}(\sigma, t) + Q(\sigma, t)\hat{N}(\sigma, t))^{-1}(\hat{Y}(\sigma, t) + Q(\sigma, t)\hat{S}(\sigma, t)), \end{aligned}$$

where $Q(\sigma, t)$ is the transfer operator of any stable periodic system so that $(I + G(\sigma, t)Q(\sigma, t))$ is invertible. \blacksquare

The structure of the stabilizing controllers is depicted in Fig. 10.3, where

$$K_0(\sigma, t) = Y(\sigma, t)X(\sigma, t)^{-1} = \hat{X}(\sigma, t)^{-1}\hat{Y}(\sigma, t).$$

In this structure it is possible to enucleate a subsystem (encircled by the dotted line) having two input vectors (y, y_1) and two output vectors (u, u_1) . Such a subsystem is characterized by the transfer operator

$$R_F(\sigma, t) = \begin{bmatrix} K_0(\sigma, t) & -\hat{X}(\sigma, t)^{-1} \\ X(\sigma, t)^{-1} & X(\sigma, t)^{-1}N(\sigma, t) \end{bmatrix},$$

and corresponds to the controller associated with the choice of free parameter $Q(\sigma, t) = 0$. For this reason, it is called *central controller*.

A state-space realization of the central controller $R_F(\sigma, t)$ is

$$\hat{x}(t+1) = \bar{A}(t)\hat{x}(t) + L(t)y(t) + \bar{B}(t)y_1(t) \quad (10.34a)$$

$$u(t) = -K(t)\hat{x}(t) - y_1(t) \quad (10.34b)$$

$$u_1(t) = \bar{C}(t)\hat{x}(t) + y(t) + D(t)y_1(t), \quad (10.34c)$$

where

$$\bar{C}(t) = C(t) + D(t)K(t)$$

$$\bar{A}(t) = A(t) + B(t)K(t) + L(t)\bar{C}(t)$$

$$\bar{B}(t) = B(t) + L(t)D(t).$$

Notice that the controller described by equations (10.34) is constituted of a feedback acting on the state-observer. Therefore, the closed-loop system includes both the dynamics of $A(\cdot) + B(\cdot)K(\cdot)$ and $A(\cdot) + L(\cdot)C(\cdot)$. Any stabilizing controller is closed around the central controller through the free parameter $Q(\sigma, t)$. It is easy to show

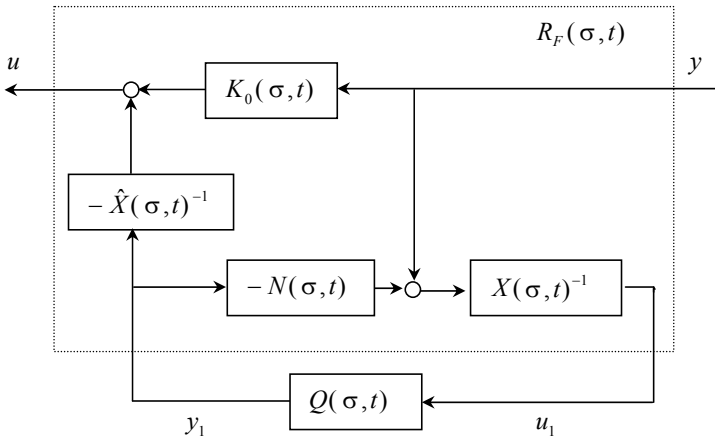


Fig. 10.3 The structure of the stabilizing controllers

that, letting $A_q(\cdot)$ be the dynamic matrix of $Q(\sigma, t)$, in a suitable state-space basis, the dynamic matrix $A_{cl}(t)$ of the overall closed-loop system is

$$A_{cl}(t) = \begin{bmatrix} A(t) + L(t)C(t) & 0 & 0 \\ \star & A_q(t) & 0 \\ \star & \star & A(t) + B(t)K(t) \end{bmatrix}.$$

This means that the characteristic multipliers of the closed-loop system are constituted by three groups; one given by the chosen stabilizing state-feedback control law, one by the chosen state-observer, and one by the chosen free parameter.

10.5 Bibliographical Notes

The parametrization of stabilizing controllers is a classical topic in systems and control, see [212, 314]. For periodic systems, it has been approached by means of a polynomial representation in [110], and via lifting in [151]. The parametrization via polynomial description calls for the solution of periodic diophantine equations. These equations can be equivalently studied by lifting as in [197]. As for the Riccati equation, the topic covered in Remark 10.2 can be found in [253] by extending the results in the time-invariant discrete-time case. Finally, the role of the system delays in the derivation of the stabilizing solution of the periodic Riccati equation (Remark 10.3) is extensively discussed in [58].

Chapter 11

Stochastic Periodic Systems

In this chapter, we treat periodic systems subject to stochastic external actions. Systems of this type are useful not only to account for random effects but also to describe the uncertainties affecting the model at hand, as an imperfect, approximated description of part of the real world. As such, stochastic models represent a cornerstone in the developments of systems and control. In particular, a fundamental problem of our days, dynamic filtering, is effectively tackled starting from such models. Dynamic filtering is the problem of estimating the latent (state) variables of a system from the available measurement of (possibly few) observable signals. On-line filtering algorithms, Kalman filtering techniques *in primis*, have become a key ingredient in modern communication and control systems. The knowledge of the state supplied by the estimator is the basic information to decode a noisy signal or to decide the best action in a control chain.

Periodic stochastic models arise whenever the system presents some intrinsic periodicity. In this connection, it is worthy mentioning that many natural phenomena have some form of repetition. For instance, the celebrated Wolf series of sunspot numbers indicate that the solar activity is subject to a sort of seasonality of about 11 years.

The subject of deseasoning has been studied as an attempt of pre-treatment of seasonal data to convert them into a stationary time series, to which existing modeling and prediction theories could be applied. Deseasoning is often limited to a removal of a periodic mean or, at most, of a periodic variance. Hidden periodicity, however, can involve the entire correlation structure of data. This has led to the definition of periodically correlated or cyclostationary processes, as a stochastic process having not only periodic mean value but also a periodic correlation structure.

Periodic stochastic models are also widely used in multirate systems for communication and control.

The purpose of this chapter is to develop a theory of stochastic periodic systems and clarify its role in the treatment of cyclostationarity in data analysis. We plan to fulfill this objective according to the following steps. First, we introduce the notion of cyclostationary process in Sect. 11.1. Then, we turn to the frequency domain description in Sect. 11.2, and we see how to pass from a time domain representation

to the corresponding spectral representation, Sect. 11.3. Finally, in Sect. 11.4, we consider state–space stochastic models (*Markov models*). It is shown that, under mild assumptions, there exists a unique cyclostationary process compatible with a given Markov model. Moreover, under the stability assumption, the state and output variables tend to this unique cyclostationary process whatever the initial condition be. The variance of such a process can be computed from the periodic solution of a Lyapunov equation with periodic coefficients.

This chapter provides preliminary tools to deal with the periodic filtering problem, treated in the subsequent Chap. 12.

11.1 Cyclostationary Processes

Consider a discrete-time stochastic process $y(t)$. We will focus on the first and second-order moments, namely the mean value and the autocovariance function, given by

$$\begin{aligned} m^y(t) &= \mathcal{E}[y(t)] \\ g^{yy}(t, \tau) &= \mathcal{E}[(y(t) - m_y(t))(y(\tau) - m_y(\tau))'], \end{aligned}$$

where $\mathcal{E}[\cdot]$ denotes the expectation operator.

The process is said to be Gaussian if, for any integer k , the joint distribution of any set of random vectors $\{y(t_1), y(t_2), \dots, y(t_k)\}$ is Gaussian. As is well-known, a description of the stochastic process based on the first- and second-order moments is a complete stochastic description in the Gaussian case.

The family of stochastic processes we deal with is the family of *periodically correlated* or *cyclostationary* processes. A process of this family is characterized by the existence of an integer $T > 0$ such that the following two conditions are satisfied for each pair of time points t and τ :

$$m^y(t + T) = m^y(t), \quad g^{yy}(t + T, \tau + T) = g^{yy}(t, \tau).$$

The smallest T for which these conditions hold true is named the *process period*. Obviously, a stationary process can be seen as a cyclostationary process for any T . Therefore, the period of a stationary process is $T = 1$.

In conclusion, a cyclostationary process of period T is described by a function for the mean value and T functions for the covariance. Of course, according to the current terminology, this is a second-order description only. In our treatment, we will not consider higher-order moments.

In many circumstances, an effective way of capturing this situation is to introduce a *seasonal* suffix $v \in [0, T - 1]$, a *year index* k and a *lag* variable h . Letting $t = kT + v$ and $\tau = t - h$, the mean value and the autocovariance function can be written as

$$\mu_v^y = m^y(kT + v), \quad \gamma_v^{yy}(h) = g^{yy}(kT + v, kT + v - h).$$

In the stationary case, v can take only a single value and therefore the mean value is constant, while the autocovariance is a function of the lag h only, i.e., $\mu_v^y = \mu^y$, $\gamma_v^{yy}(h) = \gamma^{yy}(h)$. Moreover, the covariance function enjoys the even-symmetry property $\gamma^{yy}(-h)' = \gamma^{yy}(h)$.

In a cyclostationary process, the mean value depends on a particular seasonal suffix, and the autocovariance function depends both on the suffix and the lag. Note that the even-symmetry property is no longer valid. Rather, from the obvious observation that $g^{yy}(t, \tau) = g^{yy}(\tau, t)'$ it follows that

$$\gamma_v^y(-h)' = \gamma_{v+h}^y(h).$$

The autocovariance function $g(t, \tau)$ of the cyclostationary process $y(\cdot)$ satisfies a notable inequality, namely

$$g^{yy}(t, t) - g^{yy}(t, \tau) g^{yy}(\tau, \tau)^{-1} g^{yy}(\tau, t)' \geq 0,$$

or, equivalently

$$\gamma_v^{yy}(0) - \gamma_v^{yy}(h) \gamma_{v-h}^{yy}(0)^{-1} \gamma_v^{yy}(h)' \geq 0.$$

These inequalities hold whenever the inverse matrices exist. They are easily derived, in view of the Schur complement lemma, by exploiting the positive semi-definiteness of the covariance matrix of the random vector $[y(t)' y(\tau)']'$. In the case of a univariate (scalar) cyclostationary process, the above inequality becomes

$$\gamma_v^{yy}(h)^2 \leq \gamma_v^{yy}(0) \gamma_{v-h}^{yy}(0).$$

This is the periodic version of the celebrated inequality $|\gamma^{yy}(h)| \leq \gamma^{yy}(0)$, valid in the stationary case.

A cyclostationary process $y(t)$ generates T stationary processes $y^{(i)}(k)$, $i = 0, 1, \dots, T-1$. Indeed it is easy to see that the process $y^{(i)}(k) = y(kT + i)$ as a function of time k , is stationary. By piling up these stationary processes into a unique vector, one obtains the so-called *lifted* process

$$y_L(k) = \begin{bmatrix} y^{(0)}(k) \\ y^{(1)}(k) \\ \vdots \\ y^{(T-1)}(k) \end{bmatrix}. \quad (11.1)$$

Conversely, consider a multivariate stationary process $v(t)$ of dimension pT , partitioned in T vectors of dimension p :

$$v(t) = \begin{bmatrix} v^{(0)}(t) \\ v^{(1)}(t) \\ \vdots \\ v^{(T-1)}(t) \end{bmatrix}.$$

From such a process, it is possible to extract a multivariate process of dimension p as follows

$$y(t) = \begin{cases} v^{(0)}(t) & t = 0, \pm T, \pm 2T, \dots \\ v^{(1)}(t) & t = 1, 1 \pm T, 1 \pm 2T, \dots \\ \vdots & \\ v^{(T-1)}(t) & t = T-1, T-1 \pm T, T-1 \pm 2T, \dots \end{cases}$$

One can immediately verify that $v(\cdot)$ is indeed a cyclostationary process.

A simple subclass of cyclostationary processes is made up of the so-called *periodically modulated series*. Precisely, given a zero-mean stationary process $v(t)$, a T -periodic deterministic function $m(t)$ and a T -periodic square matrix function $S(t)$, a periodically modulated series is obtained as

$$y(t) = m(t) + S(t)v(t).$$

It is not difficult to see that

$$\mathcal{E}[y(t)] = m(t), \quad \mathcal{E}[(y(t) - m(t))(y(\tau) - m(\tau))'] = S(t)\gamma^{vv}(t - \tau)S(\tau)',$$

where $\gamma^{vv}(\cdot)$ is the autocovariance function of the stationary process $v(t)$. Obviously, $y(t)$ is cyclostationary. Notice that not all cyclostationary processes can be represented as periodically modulated series as shown in the example below.

Example 11.1 Consider two scalar zero-mean stationary processes $v(t)$ and $w(t)$ with the same variance, $\text{Var}[v(t)] = \text{Var}[w(t)] = \lambda^2$, and different autocovariance functions $\gamma^{vv}(h)$ and $\gamma^{ww}(h)$. Moreover, assume that $v(t)$ and $w(t)$ are independent of each other. Now, interweave these processes so as to form

$$y(t) = \begin{cases} v(t) & t \text{ even} \\ w(t) & t \text{ odd}. \end{cases}$$

Obviously, $y(t)$ is cyclostationary of period $T = 2$. Notice that both the mean value and the variance of $y(t)$ are constant: $\mathcal{E}[y(t)] = 0$ and $\text{Var}[y(t)] = \lambda^2$. The autocovariance function $\gamma_v^{yy}(\cdot)$, $v = 0, 1$, of $y(t)$ is given by

$$\gamma_0^{yy}(h) = \begin{cases} \gamma^{vv}(h) & h \text{ even} \\ 0 & h \text{ odd} \end{cases}, \quad \gamma_1^{yy}(h) = \begin{cases} \gamma^{ww}(h) & h \text{ even} \\ 0 & h \text{ odd}. \end{cases}$$

Suppose now, by contradiction, that it is possible to represent $y(t)$ as a periodically modulated series, namely $y(t) = S(t)n(t)$, for some 2-periodic function

$S(t)$ and stationary process $n(t)$ with zero mean value and variance μ^2 . Then, $\lambda^2 = \text{Var}[y(t)] = S(t)^2 \mu^2$. Therefore, $S(t)$ must be constant and $y(t) = S(t)n(t)$ turns out to be stationary, a contradiction. ■

A cyclostationary process can be given a frequency-domain representation as shown in the subsequent section.

11.2 Spectral Representation

The complex spectrum of a cyclostationary signal $y(\cdot)$ is defined on the basis of the correlation function $\gamma_v^{yy}(h)$ as follows:

$$\Gamma_v^{yy}(\sigma) = \sum_{h=-\infty}^{+\infty} \sigma^{-h} \gamma_v^{yy}(h), \quad v = 0, 1, \dots, T-1.$$

Contrary to the stationary case, such a spectrum depends on the seasonal suffix v besides the delay operator σ^{-1} . As already said, there is a one-to-one correspondence between the cyclostationary process $y(\cdot)$ and its associated lifted (stationary) process $y_L(k)$. In order to relate the spectral characteristics of the cyclostationary signal to those of its lifted version, let $\gamma^{y_L y_L}(r)$ be the autocovariance function of $y_L(k)$. Moreover, let

$$\Gamma^{y_L y_L}(z) = \sum_{r=-\infty}^{+\infty} z^{-r} \gamma^{y_L y_L}(r)$$

be the complex spectrum of $\gamma^{y_L y_L}(\cdot)$. It turns out that this spectrum can be computed from the complex spectrum of $y_L(\cdot)$ as follows

$$\Gamma_v^{yy}(\sigma) = e'_{v+1} \Gamma^{y_L y_L}(\sigma^T) \begin{bmatrix} \sigma^{-v} I \\ \sigma^{-v+1} I \\ \vdots \\ \sigma^{-v+T-1} I \end{bmatrix}, \quad (11.2)$$

where e_i denotes the i -th (T -dimensional) block column of the identity matrix of appropriate dimensions. Vice versa, the spectrum of $y_L(\cdot)$ can be computed from the spectrum of $y(\cdot)$. Indeed, the (i, j) element of matrix $\Gamma^{y_L y_L}(\cdot)$ is given by:

$$e'_i \Gamma^{y_L y_L}(\sigma^T) e_j = \frac{\sigma^{|i-j|}}{T} \sum_{k=0}^{T-1} \Gamma_{i-1}^{yy}(\sigma \phi^k) \phi^{k(i-j)}, \quad i = j = 1, 2, \dots, T.$$

Remark 11.1 In order to clarify the meaning of the lifted output spectrum, consider the T stationary processes $y^{(i)}(k)$, $i = 0, 1, \dots, T-1$, obtained by sampling the cyclostationary process $y(t)$, according to definition (11.1). For example $y^{(0)}(k) = y(kT)$ is given by the samples of $y(t)$ at the instants $0, \pm T, \pm 2T, \dots$. To each of these stationary processes one can associate the corresponding spectrum $\Gamma^{y^{(i)} y^{(i)}}(z)$.

Notice that the spectra obtained in this way are the block diagonal elements of the lifted spectrum $\Gamma^{yLyL}(z)$. The out-of-diagonal block element in position (i, j) is the cross-spectrum of the processes $y^{(i-1)}(k)$ and $y^{(j-1)}(k)$, $i = j = 1, 2, \dots, T$.

11.3 Spectrum from the Transfer Operator

A main achievement of periodic system theory is that the spectrum of a cyclostationary process can be computed from the transfer function operator defined in Sect. 2.1. Among the different paths that can be pursued to work out this result, we will adopt here the one relying on the spectrum of the lifted process.

Consider a periodic system, described by the transfer operator $G(\sigma, t)$:

$$y(t) = G(\sigma, t) \cdot \xi(t).$$

Under the BIBO stability assumption, the output $y(\cdot)$ tends to a cyclostationary process whenever the input $\xi(\cdot)$ is cyclostationary. The characteristics of the cyclostationary output can be easily worked out by considering Expression (2.1) of Sect. 2.1, i.e.,

$$y(t) = \sum_{i=0}^{\infty} M_i(t) \xi(t-i).$$

Hence, the mean values of the input and the output are related to each other by

$$m^y(t) = \sum_{i=0}^{\infty} M_i(t) m^{\xi}(t-i).$$

Passing to the autocovariance function, in agreement with the symbols introduced above, denote by

$$\begin{aligned} g^{\xi\xi}(t, \tau) &= \mathcal{E}[(\xi(t) - m^{\xi}(t))(\xi(\tau) - m^{\xi}(\tau))'] \\ g^{y\xi}(t, \tau) &= \mathcal{E}[(y(t) - m^y(t))(\xi(\tau) - m^{\xi}(\tau))'] \\ g^{\xi y}(t, \tau) &= \mathcal{E}[(\xi(t) - m^{\xi}(t))(y(\tau) - m^y(\tau))'] \\ g^{yy}(t, \tau) &= \mathcal{E}[(y(t) - m^y(t))(y(\tau) - m^y(\tau))']. \end{aligned}$$

The following expression for the output-input cross-covariance and the output autocovariance can be easily worked out:

$$\begin{aligned} g^{y\xi}(t, \tau) &= \sum_{i=0}^{\infty} M_i(t) g^{\xi\xi}(t-i, \tau) \\ g^{yy}(t, \tau) &= \sum_{i=0}^{\infty} M_i(t) g^{\xi y}(t-i, \tau). \end{aligned}$$

The input autocovariance function being bounded and the system BIBO stable, the cross covariance $g^{y\xi}(t, \tau)$ turns out to be bounded too. Moreover, for the same reason, and taking into account that $g^{\xi y}(t, \tau) = g^{y\xi}(\tau, t)'$, the output autocovariance is bounded. Finally, the fact that $g^{\xi\xi}(t, \tau) = g^{\xi\xi}(t+T, \tau+T)$ and $M_i(t+T) = M_i(t)$ entails that $g^{y\xi}(t, \tau) = g^{y\xi}(t+T, \tau+T)$ and, in turn, $g^{yy}(t+T, \tau+T) = g^{yy}(t, \tau)$. In conclusion, the output is cyclostationary and its autocovariance function is given by

$$g^{yy}(t, \tau) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} M_i(t) g^{\xi\xi}(t-i, \tau-k) M_k(t)'$$

We turn now to the computation of the complex spectrum $\Gamma_v^{yy}(r)$ of the cyclostationary process $y(t)$ in terms of the periodic transfer operator $G(\sigma, \nu)$. We consider first the case when the input $\xi(\cdot)$ is a white-noise with zero mean value and identity autocovariance matrix. Then, we will pass to the general case.

The starting point is the lifted operator-expression of the system output:

$$y_L(k) = W_0(z) \xi_L(k),$$

where $W_0(z)$ is the lifted transfer function of the periodic system (see Sect. 6.2.2) and $y_L(k)$ and $\xi_L(k)$ are the associated input and output lifted signals. Assuming that the input $\xi(\cdot)$ is a white-noise with zero mean value and identity autocovariance matrix, its lifted counterpart $\xi_L(k)$ is also a zero mean white-noise with identity covariance matrix. Now, $y_L(\cdot)$ is stationary. Therefore, its complex spectrum can be computed by means of the celebrated expression

$$\Gamma^{y_L y_L}(z) = W_0(z) W_0(z^{-1})'.$$

Then, from (11.2) and the structure (6.14) of $W_0(z)$, it follows

$$\begin{aligned} \Gamma_v^{yy}(\sigma) &= e'_{v+1} W_0(\sigma^T) W_0(\sigma^{-T})' \begin{bmatrix} \sigma^{-\nu} I \\ \sigma^{-\nu+1} I \\ \vdots \\ \sigma^{-\nu+T-1} I \end{bmatrix} \\ &= e'_{v+1} W_0(\sigma^T) \begin{bmatrix} I & \sigma^{-1} I & \dots & \sigma^{-T+1} I \\ \sigma I & I & \dots & \sigma^{-T+2} I \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{T-1} I & \sigma^{T-2} I & \dots & I \end{bmatrix} W_0(\sigma^{-T})' e_{v+1}, \end{aligned}$$

Now, defining

$$V(\sigma) = \begin{bmatrix} I \\ \sigma I \\ \vdots \\ \sigma^{T-1} I \end{bmatrix}, \quad (11.3)$$

the above expression takes the form

$$\begin{aligned}\Gamma_v^{yy}(\sigma) &= e'_{v+1} W_0(\sigma^T) V(\sigma) V(\sigma^{-1})' W_0(\sigma^{-1})' e_{v+1} \\ &= \sum_{i=0}^{T-1} e'_{v+1} W_0(\sigma^T) e_{i+1} \sigma^{i-v} \sum_{j=1}^{T-1} \sigma^{v-j} e'_{j+1} W_0(\sigma^{-T})' e_{v+1},\end{aligned}$$

so that, taking into account Expression (6.18), we have

$$\Gamma_v^{yy}(\sigma) = G(\sigma, v) G(\sigma, v)^\sim. \quad (11.4)$$

This formula points out that the spectrum can be easily computed from the transfer operator of the system and its adjoint.

Example 11.2 Consider the transfer operator

$$G(\sigma, t) = (1 - a(t)\sigma^{-1})^{-1} b(t)\sigma^{-1},$$

and assume that the coefficients $a(t)$ and $b(t)$ are periodic of period $T = 2$. First, notice that

$$(1 - a(t)\sigma^{-1})^{-1} = (\sigma - a(t+1))^{-1}\sigma,$$

and

$$\sigma^2 - a(0)a(1) = (\sigma + a(t))(\sigma - a(t+1)).$$

Hence,

$$\begin{aligned}G(\sigma, t) &= (\sigma^2 - a(0)a(1))^{-1} (\sigma^2 - a(0)a(1)) (1 - a(t)\sigma^{-1})^{-1} b(t)\sigma^{-1} \\ &= \frac{b(t)\sigma + a(t)b(t+1)}{\sigma^2 - a(0)a(1)}.\end{aligned}$$

In view of (11.4) the spectrum is

$$\begin{aligned}\Gamma_v^{yy}(\sigma) &= \frac{(b(v)\sigma + a(v)b(v+1))(\sigma^{-1}b(v) + a(v)b(v+1))}{(\sigma^2 - a(v)a(v+1))(\sigma^{-2} - a(v)a(v+1))} \\ &= \frac{b(v)^2 + a(v)^2 b(v+1)^2 + b(v)^2 a(v+1)\sigma + a(v)b(v+1)^2 \sigma^{-1}}{(\sigma^2 - a(0)a(1))(\sigma^{-2} - a(0)a(1))}.\end{aligned}$$

The same conclusion can be drawn by the lifting approach. Precisely, it is possible to compute $W_0(z)$ from $G(\sigma, t)$ by means of (6.19), so obtaining

$$W_0(z) = \frac{1}{d(z)} \begin{bmatrix} a(0)^2 b(1)^2 + b(0)^2 & a(0)b(1)^2 z^{-1} + a(1)b(0)^2 \\ b(1)^2 a(0)z + a(1)b(0)^2 & b(1)^2 + a(1)^2 b(0)^2 \end{bmatrix},$$

where

$$d(z) = (z - a(1)a(0))(z^{-1} - a(1)a(0)).$$

Then, the spectrum $\Gamma_v^{yy}(\sigma)$ can be also found by resorting to (11.2) where $\Gamma^{yLyL}(z) = W_0(z)W_0(z^{-1})'$.

When the input signal $\xi(t)$ is a colored cyclostationary signal with zero mean value and spectrum $\Gamma_v^{\xi\xi}(\sigma)$, the spectrum $\Gamma_v^{yy}(\sigma)$ of the cyclostationary output can be derived following an analogous procedure. Indeed, letting $\Gamma^{\xi_L\xi_L}(z)$ be the spectrum of the lifted input $\xi_L(k)$, and taking into account Expression (11.3) it follows

$$\begin{aligned}\Gamma_v^{yy}(\sigma) &= e'_{v+1} W_0(\sigma^T) \Gamma^{\xi_L\xi_L}(\sigma^T) W_0(\sigma^{-T})' \begin{bmatrix} \sigma^{-v} I \\ \sigma^{-v+1} I \\ \vdots \\ \sigma^{-v+T-1} I \end{bmatrix} \\ &= e'_{v+1} W_0(\sigma^T) \Gamma^{\xi_L\xi_L}(\sigma^T) \begin{bmatrix} I & \sigma^{-1} I & \dots & \sigma^{-T+1} I \\ \sigma I & I & \dots & \sigma^{-T+2} I \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{T-1} I & \sigma^{T-2} I & \dots & I \end{bmatrix} W_0(\sigma^{-T})' e_{v+1} \\ &= e'_{v+1} W_0(\sigma^T) \Gamma^{\xi_L\xi_L}(\sigma^T) V(\sigma) \sigma^{-v} \sigma^v V(\sigma^{-1})' W_0(\sigma^{-T})' e_{v+1}.\end{aligned}$$

Moreover, from (11.2), it results

$$\begin{aligned}\Gamma_v^{yy}(\sigma) &= e'_{v+1} W_0(\sigma^T) \begin{bmatrix} \Gamma_0^{\xi\xi}(\sigma) \\ \Gamma_1^{\xi\xi}(\sigma) \sigma \\ \vdots \\ \Gamma_{T-1}^{\xi\xi}(\sigma) \sigma^{T-1} \end{bmatrix} \sigma^{-v} G(\sigma, v)^\sim \\ &= \sum_{i=0}^{T-1} e'_{v+1} W_0(\sigma^T) e_{i+1} \Gamma_i^{\xi\xi}(\sigma) \sigma^{i-v} G(\sigma, v)^\sim \\ &= \sum_{i=0}^{T-1} e'_{v+1} W_0(\sigma^T) e_{i+1} \sigma^{i-v} \Gamma_v^{\xi\xi}(\sigma) G(\sigma, v)^\sim.\end{aligned}$$

Finally, taking again into account formula (6.18), the spectrum of the output can be computed from the complex spectrum of the input via the operational form

$$\Gamma_v^{yy}(\sigma) = G(\sigma, v) \Gamma_v^{uu}(\sigma) G(\sigma, v)^\sim. \quad (11.5)$$

Example 11.3 Consider again the system defined in Example 11.2 with $b(0) = 0$. Moreover, let the input $\xi(t)$ be a cyclostationary process with zero mean and spectrum given by

$$\Gamma_v^{\xi\xi}(\sigma) = \begin{cases} 2 - \sigma^{-1}, & v = 0 \\ 1 - \sigma, & v = 1 \end{cases}.$$

The complex spectrum of the cyclostationary process of the output $y(t)$ can be computed by means of formula (11.5) as $\Gamma_v^{yy}(\sigma) = G(\sigma, v) \Gamma_v^{\xi\xi}(\sigma) G^\sim(\sigma, v)$ with

$$\Gamma_0^{yy}(\sigma) = \frac{2a(0)^2b(1)^2 + 2b(1)^2\sigma^{-1}}{(\sigma^{-2} - a(1)a(0)(\sigma^2 - a(0)a(1)))}$$

$$\Gamma_1^{yy}(\sigma) = \frac{2b(1)^2 + 2a(0)b(1)^2\sigma}{(\sigma^{-2} - a(1)a(0)(\sigma^2 - a(0)a(1)))}.$$

This result can be obtained following different routes. In particular, observe that $\xi(t)$ is generated by the periodic moving average system of period $T = 2$ described by

$$\xi(t) = \alpha(t)\xi(t-1) + w(t),$$

where $\alpha(0) = 1$ and $\alpha(1) = 0$ and $w(t)$ is a white-noise with zero mean and unit variance. Indeed the transfer function operator of such a system is

$$F(\sigma, t) = 1 - \alpha(t)\sigma^{-1},$$

so that

$$\Gamma_v^{\xi\xi}(\sigma) = F(\sigma, t)F^\sim(\sigma, t) = 1 + \alpha(t)^2 - \alpha(t)\sigma^{-1} - \alpha(t+1)\sigma,$$

which obviously coincides with the given input spectrum. In this way, the signal $y(t)$ is generated by the white-noise $w(t)$ through the filter constituted by the cascade of the operators $F(\sigma, t)$ and $G(\sigma, t)$. By letting $H(\sigma, t) = G(\sigma, t)F(\sigma, t)$, the complex spectrum of $y(t)$ can be computed from the spectral formula for white-noise input as $\Gamma_v^{yy}(\sigma, t) = H(\sigma, t)H(\sigma, t)^\sim$.

A further possibility is to resort to the lifting procedure. This amounts to computing $\Gamma_v^{yy}(\sigma)$ from Eq. (11.4) by using the expression

$$\Gamma^{yLyL}(z) = W_0(z)\Gamma^{\xi_L\xi_L}(z)(W_0(z)^{-1})',$$

where $\Gamma^{\xi_L\xi_L}(z)$ is the spectrum of the lifted input and $W_0(z)$ is the lifted transfer function of the system. ■

One of the most classical representations of cyclostationary processes is supplied by a PARMA model:

$$y(t) = a_1(t)y(t-1) + a_2(t)y(t-2) + \dots + a_r(t)y(t-r) \\ + b_0(t)\xi(t) + b_1(t)\xi(t-1) + \dots + b_s(t)\xi(t-s),$$

where $\xi(\cdot)$ is a white-noise with a periodically varying mean value and a unit variance:

$$\xi(t) \sim WN(\bar{\xi}(t), I).$$

This last assumption about the variance of $\xi(t)$ can be taken for granted without loss of generality. Indeed, it is always possible to recover this case even if a more general model with a periodically varying covariance matrix were considered. This would simply require us to re-scale the coefficients of the MA part (coefficients $b_i(t)$) of the model according to a factorization of the covariance matrix.

Notice that the use of the acronym PARMA, originated in the area of stochastic time-series analysis, finds in this chapter the appropriate collocation. However, in the previous parts of this book, we took the liberty of using the same terminology in the deterministic framework as well, see Sects. 2.2 and 7.2.

The spectrum of a PARMA-generated process can be obviously computed from its associated transfer operator by means of Eq. (11.4)

11.4 Periodic State–Space Stochastic Models

In this section, we study the state–space dynamic modelization of cyclostationary processes. A typical Markovian representation corresponds to seeing the process as the output $y(\cdot)$ of the dynamical system

$$x(t+1) = A(t)x(t) + w(t) \quad (11.6a)$$

$$y(t) = C(t)x(t) + v(t), \quad (11.6b)$$

where $v(\cdot)$ and $w(\cdot)$ are white-noise signals

$$\begin{aligned} \mathcal{E} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} &= \begin{bmatrix} m^v(t) \\ m^w(t) \end{bmatrix} \\ \mathcal{E} \begin{bmatrix} w(t) - m^w(t) \\ v(t) - m^v(t) \end{bmatrix} \begin{bmatrix} (v(t) - m^v(t))' & (w(t) - m^w(t))' \end{bmatrix} &= \begin{bmatrix} Q(t) & S(t) \\ S(t)' & R(t) \end{bmatrix}, \end{aligned}$$

and all elements are periodic of period T . Moreover, the covariance matrices of $v(t)$ and $w(t)$, respectively $Q(t)$ and $R(t)$, are symmetric and positive semi-definite matrices, $\forall t$.

The mean values $m^x(\cdot)$ of $x(\cdot)$ and $m^y(\cdot)$ of $y(\cdot)$ are easily computed by applying the expectation operator to (11.6a)

$$m^x(t+1) = A(t)m^x(t) + m^w(t). \quad (11.7)$$

Hence, from the Lagrange Equation (2.20)

$$m^x(t) = \Phi_A(t, \tau)m^x(\tau) + \sum_{j=\tau}^{t-1} \Phi_A(t, j+1)m^w(j). \quad (11.8)$$

To find the variance of $x(\cdot)$, i.e.,

$$\Sigma(t) = \mathcal{E} \left[(x(t) - m^x(t))(x(t) - m^x(t))' \right],$$

subtract (11.7) from (11.6a) and obtain

$$x(t+1) - m^x(t+1) = A(t)(x(t) - m^x(t)) + w(t) - m^w(t)$$

Thus, by post-multiplying by $(x(t+1) - m^x(t+1))'$ and taking expectation,

$$\begin{aligned}\Sigma(t+1) &= A(t)\mathcal{E}[(x(t) - m^x(t))(x(t+1) - m^x(t+1))'] + \\ &\quad + \mathcal{E}[(w(t) - m^w(t))(x(t+1) - m^x(t+1))'] = \\ &= A(t)\Sigma(t)A(t)' + Q(t) + A(t)\mathcal{E}[(x(t) - m^x(t))(w(t) - m^w(t))'] + \\ &\quad + \mathcal{E}[(w(t) - m^w(t))(x(t) - m^x(t))']A(t)'.\end{aligned}$$

Now, notice that the state may depend on past values of the input only, i.e.,

$$x(t) = f(w(t-1), w(t-2), \dots)$$

Then, taking into account the fact that $w(\cdot)$ is a white-noise, it follows that

$$\mathcal{E}[(x(t) - m^x(t))(w(t) - m^w(t))'] = 0,$$

so that

$$\Sigma(t+1) = A(t)\Sigma(t)A(t)' + Q(t). \quad (11.9)$$

This is a difference filtering Lyapunov equation (see Sect. 4.5), whose solution is

$$\Sigma(t) = \Phi_A(t, \tau)\Sigma(\tau)\Phi_A(t, \tau)' + W(t, \tau)$$

where

$$W(\tau, t) = \sum_{j=\tau+1}^t \Phi_A(t, j)Q(j-1)\Phi_A(t, j)'$$

is the Reachability Grammian.

Once the variance $\Sigma(t)$ of the state has been determined, the autocovariance function $g^{xx}(t, \tau)$ of the state, defined as

$$g^{xx}(t, \tau) = \mathcal{E}[(x(t) - m^x(t))(x(\tau) - m^x(\tau))']$$

can be easily derived:

$$g^{xx}(t, \tau) = \begin{cases} \Phi_A(t, \tau)\Sigma(\tau) & t \geq \tau \\ \Sigma(t)\Phi_A(\tau, t)' & t \leq \tau. \end{cases}$$

We are now in the position to investigate the T -periodic solutions of the above equations. Therefore, impose $m^x(t) = m^x(\tau)$ with $t = \tau + T$ in (11.8) to obtain

$$m^x(\tau) = \Psi_A(\tau)m^x(\tau) + \sum_{i=\tau}^{\tau+T-1} \Phi_A(\tau+T, i+1)m^w(i),$$

from which

$$m^x(\tau) = (I - \Psi_A(\tau))^{-1} \sum_{i=\tau}^{\tau+T-1} \Phi_A(\tau+T, i+1)m^w(i).$$

Therefore, $m^x(\tau)$ is the periodic generator for the periodic function $\bar{m}^x(t)$. Notice that the inverse appearing in the equation above does exist if and only if $A(\cdot)$ does not have characteristic multipliers equal to one.

Analogously, let $\Sigma(t) = \Sigma(\tau)$ with $t = \tau + T$ in (11.9) to obtain the algebraic Lyapunov equation

$$\Sigma(\tau) = \Psi_A(\tau)\Sigma(\tau)\Psi_A(\tau)' + \sum_{i=\tau}^{\tau+T-1} \Phi_A(\tau+T, i+1)Q(i)\Phi_A(\tau+T, i+1)'.$$

If $\Sigma(\tau)$ is a positive semi-definite solution of such an equation, then it is also the periodic generator of the T -periodic solution $\bar{\Sigma}(\cdot)$ of the periodic Lyapunov Equation (11.9).

Associated with $\bar{\Sigma}(\tau)$, the autocovariance function of the state is given by

$$g^{xx}(t, \tau) = \Phi_A(t, \tau)\bar{\Sigma}(\tau), \quad t \geq \tau.$$

Therefore:

$$g^{xx}(t+T, \tau+T) = \Phi_A(t+T, \tau+T)\bar{\Sigma}(\tau+T) = \Phi_A(t, \tau)\bar{\Sigma}(\tau) = g^{xx}(t, \tau).$$

In conclusion, the process $x(\cdot)$ generated by the Markovian model with initial condition $x(\tau)$ with mean value $\bar{m}^x(\tau)$ and variance $\bar{\Sigma}(\tau)$ is cyclostationary.

As for the system output, if $x(\cdot)$ is cyclostationary, then $y(\cdot)$ is cyclostationary as well, as it appears in (11.6b). Its mean value is given by

$$m^y(t) = C(t)m^x(t) + m^v(t).$$

As for the autocovariance function

$$g^{yy}(t, \tau) = \mathcal{E} [(y(t) - m^y(t))(y(\tau) - m^y(\tau))'],$$

from (11.6) it follows

$$g^{yy}(t, \tau) = \begin{cases} C(t)\Phi_A(t, \tau)\bar{\Sigma}(\tau)C(\tau)' + C(t)\Phi_A(t, \tau+1)S(\tau) & t > \tau \\ C(t)\bar{\Sigma}(t)C(t)' + R(t) & t = \tau \\ C(t)\bar{\Sigma}(t)\Phi_A(\tau, t)'C(\tau)' + S(t)'\Phi_A(\tau, t+1)'C(\tau)' & t \leq \tau. \end{cases}$$

In conclusion, the process $y(\cdot)$ obtained from the Markovian model for the initial condition $x(\tau)$ characterized by mean value $\bar{m}^x(\tau)$ and variance $\bar{\Sigma}(\tau)$ is cyclostationary. Obviously, if the system is stable, this cyclostationary process is asymptotically obtained from any initial condition.

11.5 Stochastic Realization

In this section, we face the so-called stochastic realization problem from input–output data. Roughly speaking, this is the problem of finding a minimal stable periodic system from the covariance function of the output signal. Recall that a minimal (reachable and observable at each time point) realization of a discrete-time periodic system has in general time-periodic dimensions of the state–space (see Chap. 7). As such, we are well advised to introduce, from the very beginning, a state–space model of a periodic system with time-periodic dimensions. For simplicity, and without loss of generality, we consider the system in the form:

$$x(t+1) = A(t)x(t) + B(t)w(t) \quad (11.10a)$$

$$y(t) = C(t)x(t) + D(t)w(t), \quad (11.10b)$$

where $x(t) \in \mathbb{R}^{n(t)}$ is the state vector, $A(t) \in \mathbb{R}^{n(t+1) \times n(t)}$ the dynamic matrix, $B(t) \in \mathbb{R}^{n(t+1) \times m}$, $C(t) \in \mathbb{R}^{p \times n(t)}$ are the input and output matrices, $D(t) \in \mathbb{R}^{p \times m}$ is the direct feed-through term and $w(t) \in \mathbb{R}^m$ is a vector noise whose characteristics will be specified later on. The state–space dimension $n(\cdot)$ is periodic of period T and all elements of $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ are periodic functions of the same period.

A first important observation is that there are many models (11.10) which can provide a given output covariance function. Indeed, take a T -periodic matrix $\Omega(t) \in \mathbb{R}^{n(t) \times n(t)}$, invertible for each t , and perform a periodic change of basis:

$$z(t) = \Omega(t)x(t).$$

In the new coordinates, the system becomes

$$z(t+1) = \tilde{A}(t)z(t) + \tilde{B}(t)w(t) \quad (11.11a)$$

$$y(t) = \tilde{C}(t)z(t) + \tilde{D}(t)w(t), \quad (11.11b)$$

where

$$\tilde{A}(t) = \Omega(t+1)A(t)\Omega(t)^{-1}$$

$$\tilde{B}(t) = \Omega(t+1)B(t)$$

$$\tilde{C}(t) = C(t)\Omega(t)^{-1}.$$

Systems (11.10) and (11.11) are equivalent with each other and enjoy the same structural properties, namely stability, reachability, observability, etc. Moreover, they share the same input–output properties. In particular the output covariance function is the same.

Stability of System (11.10) can be equivalently studied via Lyapunov theory, and in particular by resorting to the (filtering-type) difference periodic Lyapunov equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)'. \quad (11.12)$$

Note that we already encountered such an equation in Sect. 4.5; however, in that section reference was made to the case of constant dimension only ($n(t) = n, \forall t$). The results stated in Proposition 4.15 hold true even in the time-varying dimension case, as it is easy to verify. Therefore we can conclude that if the pair $(A(\cdot), B(\cdot))$ is reachable for any time-point, then the periodic System (11.10) is stable if and only if the Eq. (11.12) admits a T -periodic positive definite solution $P(\cdot)$. Notice that the solution $P(t)$ is square for each t , that is $P(t) \in \mathbb{R}^{n(t) \times n(t)}$. If such a positive definite solution $P(\cdot)$ exists, then a change of basis is possible in the state-space such that, in the new coordinates, the solution of the Lyapunov equation is the identity. Indeed, letting

$$\tilde{A}(t) = P(t+1)^{-1/2} A(t) P(t)^{1/2}, \quad \tilde{B}(t) = P(t+1)^{-1/2} B(t),$$

the Lyapunov equation becomes

$$I_{n(t+1)} = \tilde{A}(t) \tilde{A}(t)' + \tilde{B}(t) \tilde{B}(t)'. \quad (11.13)$$

11.5.1 Problem Formulation

The input-output description of the periodic system can be expressed through the Markov parameters $h_i(t)$ as follows:

$$y(t) = \sum_{i=0}^{\infty} h_i(t) w(t-i).$$

As seen in Sect. 2.4, the Markov parameters are related to the state-space matrices as follows:

$$h_0(t) = D(t), \quad h_i(t) = C(t) \Phi_A(t, t-i+1) B(t-i), \quad i = 1, 2, \dots$$

Of course, the Markov parameters do not depend upon the specific basis chosen in the state-space basis to describe the system.

If the system is stable and $w(\cdot)$ is a white-noise with zero mean and unit variance, for any initial condition the state of the system converges to a cyclostationary process $x(\cdot)$ with variance given by $P(t)$, the unique solution of the Lyapunov Equation (11.12). The output also converges to a cyclostationary process $y(\cdot)$, whose autocovariance matrices are defined as:

$$r_i(t) := E [y(t+i) y(t)'] .$$

Being the process $y(\cdot)$ cyclostationary, the functions $r_i(\cdot)$ are T -periodic. As seen in Sect. 11.4, these matrices are given by

$$r_i(t) = \begin{cases} C(t) P(t) C(t)' + D(t) D(t)', & i = 0 \\ C(t+i) \Phi_A(t+i, t+1) [A(t) P(t) C(t)' + B(t) D(t)'], & i > 0. \end{cases} \quad (11.14)$$

We now construct the Hankel matrix of the output autocovariances as

$$\mathcal{R}_q(t) = \begin{bmatrix} r_0(t) & r_1(t) & \cdots & r_q(t) \\ r_1(t) & r_0(t+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_1(t+q-1) \\ r_q(t) & \cdots & r_1(t+q-1) & r_0(t+q) \end{bmatrix}.$$

By introducing the matrices $\mathcal{O}_q(t) \in \mathbb{R}^{(q+1)p \times n(t)}$ and $\mathcal{H}_q(t) \in \mathbb{R}^{(q+1)p \times (q+1)m}$ as

$$\mathcal{O}_q(t) = \begin{bmatrix} C(t) \\ C(t+1)A(t) \\ \vdots \\ C(t+q)\Phi_A(t+q, t) \end{bmatrix}$$

$$\mathcal{H}_q(t) = \begin{bmatrix} h_0(t) & 0 & \cdots & 0 \\ h_1(t+1) & h_0(t+1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_q(t+q) & \cdots & h_1(t+q) & h_0(t+q) \end{bmatrix},$$

the Hankel matrix $\mathcal{R}_q(t)$ can be given a useful expression. Indeed, from (11.14) it turns out that

$$\mathcal{R}_q(t) = \mathcal{O}_q(t)P(t)\mathcal{O}_q(t)' + \mathcal{H}_q(t)\mathcal{H}_q(t)'. \quad (11.15)$$

An important consequence of (11.15) is a *data consistency* condition to be satisfied in the realization problem. To be precise, if the system is stable, $P(t) \geq 0$ for each t . Therefore $\mathcal{O}_q(t)P(t)\mathcal{O}_q(t)' \geq 0$ for each t as well. Then, letting

$$\mathcal{Z}_q(t) = \mathcal{R}_q(t) - \mathcal{H}_q(t)\mathcal{H}_q(t)',$$

from (11.15) it turns out that

$$\mathcal{Z}_q(t) \geq 0, \quad \forall t. \quad (11.16)$$

This is the consistency condition that must be satisfied by the autocovariances in $\mathcal{R}_q(t)$ and the Markov parameters in $\mathcal{H}_q(t)$.

11.5.2 Realization for SISO Periodic Systems

From now on, the attention is restricted to *SISO periodic systems*, i.e., systems with $m = 1$ and $p = 1$. In this case, a state-space realization from the periodic matrices $\mathcal{R}_q(t)$ and $\mathcal{Z}_q(t)$ can be found as described below. First of all, formula (11.13) implies that there is no loss of generality to look for a realization with $P(t) = I_{n(t)}$.

Now, let

$$\mathcal{Z}_q(t) = G(t)G(t)',$$

where $G(t) \in \mathbb{R}^{(q+1) \times n(t)}$, $n(t)$ being the rank of $\mathcal{Z}_q(t)$. We assume that $n(t) \leq q$, $\forall t$.

Matrix $G(t)$ can be given the following partitioned expression

$$G(t) = \begin{bmatrix} G_1(t) \\ G_2(t) \end{bmatrix} = \begin{bmatrix} G_3(t) \\ G_4(t) \\ G_5(t) \end{bmatrix}, \quad (11.17)$$

where

$$\begin{aligned} G_1(t) &\in \mathbb{R}^{n(t) \times n(t)}, & G_2(t) &\in \mathbb{R}^{(q+1-n(t)) \times n(t)}, & G_3(t) &\in \mathbb{R}^{1 \times n(t)} \\ G_4(t) &\in \mathbb{R}^{n(t+1) \times n(t)} & G_5(t) &\in \mathbb{R}^{(q-n(t+1)) \times n(t)} \end{aligned}$$

Moreover, select from $\mathcal{H}_q(t)$ the vector

$$M_1(t) = \begin{bmatrix} h_1(t+1) \\ h_2(t+2) \\ \vdots \\ h_{n(t+1)}(t+n(t+1)) \end{bmatrix}. \quad (11.18)$$

The state–space realization is derived as described in the result below.

Proposition 11.1 *Let $q \geq 0$, $\mathcal{R}_q(t)$ and $\mathcal{H}_q(t)$ be given for $t \in [0, T-1]$ and assume that, for each t :*

$$\mathcal{Z}_q(t) := \mathcal{R}_q(t) - \mathcal{H}_q(t)\mathcal{H}_q(t)' \geq 0.$$

Moreover, let, for each t , $G(t)$ be a full rank factor of $\mathcal{Z}_q(t)$. Define $G_1(t)$, $G_3(t)$, $G_4(t)$ and $M_1(t)$ as in (11.17), (11.18). Then, the T -periodic quadruplet

$$\tilde{A}(t) = G_1(t+1)^{-1}G_4(t) \quad (11.19a)$$

$$\tilde{B}(t) = G_1(t+1)^{-1}M_1(t) \quad (11.19b)$$

$$\tilde{C}(t) = G_3(t) \quad (11.19c)$$

$$\tilde{D}(t) = h_0(t) \quad (11.19d)$$

defines a T -periodic realization of the given Markov coefficients and covariance data. Moreover, if $(\tilde{A}(\cdot), \tilde{B}(\cdot))$ is reachable for each t , $\tilde{A}(\cdot)$ is stable.

Proof (sketch) The choice of (11.19c) and (11.19d) is self-evident. The verification of choices (11.19a) and (11.19b) is based on the special structure of $\mathcal{R}_q(t)$ and $\mathcal{H}_q(t)$. In particular,

$$G_4(t)G_4(t)' + M_1(t)M_1(t)' = G_1(t+1)G_1(t+1)'.$$

The proof of the fact that $G_1(t+1)$ is indeed invertible is left to the reader. This, together with (11.19a), (11.19b), leads to (11.13). The fact that System (11.19) is

indeed a realization of the input–output data can be proved by inspection. Finally, the reachability of $(A(\cdot), \tilde{B}(\cdot))$ implies the stability of $A(\cdot)$.

11.5.3 Realization from Normalized Data

In the previous section it was assumed that the variance of $w(\cdot)$ was known. Turn now to the case when such variance is unknown. We define then the *normalized Markov parameters*:

$$\hat{h}_i(t) = \frac{h_i(t)}{h_0(t-i)}, \quad \hat{r}_i(t) = \frac{r_i(t)}{\sqrt{r_0(t)r_0(t+i)}}. \quad (11.20)$$

According to (11.20), one can introduce the normalized matrices

$$\begin{aligned} \hat{\mathcal{H}}_q(t) &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \hat{h}_1(t+1) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \hat{h}_q(t+q) & \cdots & \hat{h}_1(t+q) & 1 \end{bmatrix} \\ \hat{\mathcal{R}}_q(t) &= \begin{bmatrix} 1 & \hat{r}_1(t) & \cdots & \hat{r}_q(t) \\ \hat{r}_1(t) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \hat{r}_q(t) & \cdots & \hat{r}_1(t+q-1) & 1 \end{bmatrix}. \end{aligned}$$

The obvious relation with $\mathcal{H}_q(t)$ and $\mathcal{R}_q(t)$ is:

$$\begin{aligned} H_q(t) &= \hat{\mathcal{H}}_q(t) \Delta_{H,q}(t) \\ R_q(t) &= \Delta_{R,q} \hat{\mathcal{R}}_q(t) \Delta_{R,q}(t), \end{aligned}$$

where

$$\begin{aligned} \Delta_{H,q}(t) &= \begin{bmatrix} h_0(t) & 0 & \cdots & 0 \\ 0 & h_0(t+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_0(t+q) \end{bmatrix} \\ \Delta_{R,q}(t) &= \begin{bmatrix} \sqrt{r_0(t)} & 0 & \cdots & 0 \\ 0 & \sqrt{r_0(t+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{r_0(t+q)} \end{bmatrix}. \end{aligned}$$

The data consistency condition (11.16) can be rewritten as

$$\mathcal{Z}_q(t) = \Delta_{R,q}(t) \hat{\mathcal{H}}_q(t) \Delta_{R,q}(t) - \hat{\mathcal{H}}_q(t) \Delta_{H,q}(t)^2 \hat{\mathcal{H}}_q(t)' \geq 0, \quad \forall t. \quad (11.21)$$

If we compare this condition with the previous one, we see that in the present situation there are more degrees of freedom in that matrices $\Delta_{H,q}(t)$ and $\Delta_{R,q}(t)$ are unknown and therefore have to be found in order to build a state–space realization. To be precise, the problem can be formulated as that of finding

$$\bar{\Delta} = [h_0(0) \ h_0(1) \ \cdots \ h_0(T-1) \ r_0(0) \ r_0(1) \ \cdots \ r_0(T-1)]$$

such that, for each t , $\mathcal{Z}_q(t)$ satisfies (11.21) and has minimum rank. Notice that the minimality of the rank corresponds to the natural request of minimality of the state–space realization.

To cope with the excessive number of degrees of freedom, the general problem above can be simplified by considering two cases. In the first one, the instantaneous impulse response $h_0(t)$ is fixed and the rank of the positive semi-definite matrix $\mathcal{Z}_q(t)$ is minimized with respect to $r_0(t)$. In the second one, the converse situation is undertaken, namely the values of $r_0(t)$ are fixed and the rank is minimized with respect to $h_0(t)$. This latter case is elaborated further as follows. Chose a positive definite diagonal matrix $\Delta_{R,q}(t)$ and let

$$M(t) = \hat{\mathcal{H}}_q(t)^{-1} \Delta_{R,q}(t) \hat{\mathcal{H}}_q(t) \Delta_{R,q}(t) \hat{\mathcal{H}}_q(t)'^{-1}.$$

The rank minimization problem can be written as:

$$\min_{\bar{\Delta}} \text{Rank diag}\{M(i) - D(t), i = 0, 1, \dots, T-1\}$$

$$\begin{aligned} \text{s.t.} \quad & M(t) - D(t) \geq 0, \quad \forall t \\ & D(t) \geq 0, \quad \forall t \\ & D(t) \text{ diagonal, } \forall t. \end{aligned}$$

The solution is then $\Delta_{H,q}(i)^2 = D(i)$. This is a slightly modified version of the well-known Frisch problem, for which no analytical solution seems to exist. However, it is always possible to find a solution via heuristic methods.

To end the section, notice that, once the diagonal matrices $\Delta_{H,q}(t)$ and $\Delta_{R,q}(t)$ have been determined, the state–space realization can be worked out as in the unnormalized case (Proposition 11.1) from $\mathcal{Z}_q(t)$ in (11.21).

Example 11.4 Consider $q = 1$, $T = 3$ and the normalized data:

$$\hat{r}_1(0) = 0.5, \hat{r}_1(1) = -0.2, \hat{r}_1(2) = 0.75, \hat{h}_1(0) = 5, \hat{h}_1(1) = 2, \hat{h}_1(2) = 1.$$

It is easy to see that $\hat{R}_q(t)$ is positive definite for each t . To proceed, we fix the values of the variances of $y(0), y(1), \dots, y(T-1)$. To be precise we take $r_0(t) = 1, \forall t$. Hence, the problem is to find $\alpha := h_0(0)^2$, $\beta := h_0(1)^2$, $\gamma := h_0(2)^2$ such that

the three matrices

$$Z_q(0) = \begin{bmatrix} 1-\alpha & 0.5-2\alpha \\ 0.5-2\alpha & 1-4\alpha-\beta \end{bmatrix}, \quad Z_q(1) = \begin{bmatrix} 1-\beta & -0.2-\beta \\ -0.2-\beta & 1-\beta-\gamma \end{bmatrix}$$

$$Z_q(2) = \begin{bmatrix} 1-\gamma & 0.75-5\gamma \\ 0.75-5\gamma & 1-25\gamma-\alpha \end{bmatrix}$$

are positive semi-definite with minimum rank.

The only solution is

$$\alpha = 0.136, \beta = 0.3959, \gamma = 0.0164,$$

so that, from Proposition 11.1, it follows that

$$G(0) = \begin{bmatrix} -0.9295 \\ -0.2453 \end{bmatrix}, \quad G(1) = \begin{bmatrix} -0.7773 \\ 0.7666 \end{bmatrix}$$

$$G(2) = \begin{bmatrix} -0.9917 \\ -0.6738 \end{bmatrix}$$

and

$$\begin{aligned} \tilde{A}(0) &= 0.3156 & \tilde{A}(1) &= -0.773 & \tilde{A}(2) &= 0.724 \\ \tilde{B}(0) &= -2.5730 & \tilde{B}(1) &= -1.0084 & \tilde{B}(2) &= -5.3792 \\ \tilde{C}(0) &= -0.9295 & \tilde{C}(1) &= -0.7773 & \tilde{C}(2) &= -0.9917 \\ \tilde{D}(0) &= 0.3688 & \tilde{D}(1) &= 0.6292 & \tilde{D}(2) &= 0.128 \end{aligned}$$

11.6 Bibliographical Notes

Cyclostationary processes (also called periodically correlated processes) have been investigated since the 1960s, see the pioneering papers [157, 159, 166, 199, 242, 244, 281]. Further contributions are [6, 30, 40, 60, 61, 98, 156, 160, 191, 192, 258–261, 292]. All these papers have been originated in various fields, such as statistics, signal processing and control science. Periodically correlated time series have been extensively used in environmetrics, see [19, 79, 178, 222–225, 262, 268, 293], and econometrics, [146–148, 161, 199, 240, 241]. The problem of parameter identification has been dealt with in several papers, such as [2, 9, 40, 98, 199, 229]. The stochastic realization problem for SISO periodic systems has been investigated in [112] and extends previous results for time-invariant systems, see [133]. This problem is reminiscent of the well-known Frisch problem, for which no analytical solution seems to exist, and only heuristic methods are available in general as discussed in [137].

Chapter 12

Filtering and Deconvolution

Estimating a remote signal from the measurements of another observed signal is a classical problem of today. Among the possible facets the problem can take, we will assume herein that the two signals are related each other by a dynamical system in state-space form. To be precise assume that the observed signal $y(\cdot)$ and the remote signal $z(\cdot)$ obey the following equations:

$$\begin{aligned}x(t+1) &= A(t)x(t) + B(t)w(t) \\y(t) &= C(t)x(t) + D(t)w(t) \\z(t) &= E(t)x(t) + N(t)w(t),\end{aligned}$$

where $w(\cdot)$ is a disturbance. The dimensions are: $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $w(t) \in \mathbb{R}^m$ and $z(t) \in \mathbb{R}^q$. It is assumed that all matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$, $E(\cdot)$ and $N(\cdot)$ are known and T -periodic.

The estimation problem consists in finding a dynamical system that, fed by $y(\cdot)$, generates an estimate $\hat{z}(\cdot)$ of $z(\cdot)$. Three typical subproblems are in order (the symbol h denotes a positive integer):

- (i) *filtering*: Estimate $z(t)$ by exploiting $y(\cdot)$ up to time t .
- (ii) *h -step prediction*: Estimate $z(t)$ by exploiting $y(\cdot)$ up to time $t - h$.
- (iii) *h -step smoothing*: Estimate $z(t)$ by exploiting $y(\cdot)$ up to time $t + h$.

There is a further important problem which can be encountered, the so-called *deconvolution* problem, namely estimating the input signal of the system by observing the output signal. More in general, one can pose it as the question of estimating a linear combination of the input variables from the snapshots of the output. Such a problem is encompassed in the formulation above, by considering $w(\cdot)$, or a linear combination of its elements, as unknown. This corresponds to assuming $E(t) = 0$ in the equation of the remote signal, and selecting an appropriate matrix $N(t)$, the matrix providing the linear combination of interest of the input variables, to obtain $z(\cdot)$.

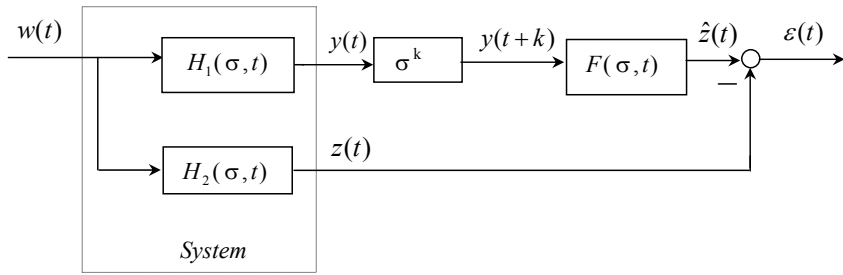


Fig. 12.1 The estimation problem

In the sequel we denote by

$$\begin{aligned} H_1(\sigma, t) &= C(t) (\sigma I - A(t))^{-1} B(t) + D(t) \\ H_2(\sigma, t) &= E(t) (\sigma I - A(t))^{-1} B(t) + N(t) \end{aligned}$$

the transfer operators relating the disturbance $w(t)$ to the measurements $y(t)$ and the remote signal $z(t)$, respectively.

To write in a unified way the estimator, we are well advised to introduce the integer k defined as follows:

- (i) $k = h = 0$ for the filtering problem.
- (ii) $k = -h \leq 0$ for the h -step prediction problem.
- (iii) $k = h \geq 0$ for the h -step smoothing problem.

Thanks to such definitions, the estimator (assumed to be linear) can be given the form

$$\hat{x}(t+1) = A_F(t)\hat{x}(t) + B_F(t)y(t+k) \quad (12.1a)$$

$$\hat{z}(t) = C_F(t)\hat{x}(t) + D_F(t)y(t+k). \quad (12.1b)$$

The corresponding transfer operator is denoted by

$$F(\sigma, t) = C_F(t) (\sigma I - A_F(t))^{-1} B_F(t) + D_F(t).$$

The system, the estimator and the corresponding estimation error $\varepsilon(t)$ are schematically represented in Fig. 12.1.

The objective of the estimator is that of guaranteeing a limited estimation error. The specification on the error can be set by considering different input–output norms, the most celebrated of them being the H_2 norm and the H_∞ norm. Such norms for periodic systems have been introduced in Chap. 9. Notice that the choice of the adopted norm corresponds to a specific interpretation of the class which the signal $w(\cdot)$ belongs to. For instance, the minimization of the H_2 norm is strictly related to

the celebrated Kalman filter, where the disturbance $w(\cdot)$ is seen as a white-noise and one has to minimize the variance of the estimation error.

Notice that the dimension of the state vector of the estimator may not coincide with that of the system. There is a notable exception, as pointed out in the subsequent observation.

Remark 12.1 *In the case of the filtering problem ($k = 0$), it is often assumed from the very beginning that the filter has the same dimension as the given system with the following structure (named observer form);*

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)e(t) \quad (12.2a)$$

$$\hat{z}(t) = E(t)\hat{x}(t) + D_F(t)e(t) \quad (12.2b)$$

$$e(t) = y(t) - C(t)\hat{x}(t). \quad (12.2c)$$

As we will see, there are many important situations of practical interest where the optimal filter has indeed such an expression.

Once the observer form is adopted, the filter (12.1) is determined by only two matrices, i.e., $B_F(\cdot)$ and $D_F(\cdot)$. Correspondingly:

$$A_F(t) = A(t) - B_F(t)C(t), \quad C_F(t) = E(t) - D_F(t)C(t).$$

Then, the associated estimation error

$$\varepsilon(t) = z(t) - \hat{z}(t)$$

obeys the following equations

$$\eta(t+1) = (A(t) - B_F(t)C(t))\eta(t) + (B(t) - B_F(t)D(t))w(t) \quad (12.3a)$$

$$\varepsilon(t) = (E(t) - D_F(t)C(t))\eta(t) + (N(t) - D_F(t)D(t))w(t). \quad (12.3b)$$

If such an error system is stable, one can define the H_2 and H_∞ norms from $w(\cdot)$ to $\varepsilon(\cdot)$, and accordingly pose the filtering problem, so assessing the quality of the obtained estimates. If $w(\cdot) = 0$, then, for any initial state $\hat{x}(0)$, the error goes asymptotically to zero, and the filter can be interpreted as an asymptotic state reconstructor. If instead $w(\cdot)$ is a white-noise, then the error tends to a cyclostationary process whose statistics could be controlled through the choice of the design matrices $B_F(\cdot)$ and $D_F(\cdot)$.

12.1 Filtering and Deconvolution in the H_2 Setting

In this section the filtering and deconvolution problems are treated in a unified way. As already pointed out, this corresponds to the case $k = 0$, namely the problem is to elaborate the data up to time t in order to estimate the unknown signal $z(\cdot)$ at time t .

According to the expression $z(t) = E(t)x(t) + N(t)w(t)$, the filtering problem corresponds to the case $N(\cdot) = 0$, so that the unknown is a linear combination of the state variables. Instead, the deconvolution problem is associated with the choice $E(\cdot) = 0$ and the corresponding unknown is a linear combination of the input variables. As for the approach, the entity of the error $\varepsilon(\cdot)$ is assessed by resorting to the H_2 norm of a periodic system, introduced in Chap. 9. As already said, we focus on the observer form structure. Therefore, our aim is to minimize the H_2 norm of the transfer operator from the disturbance $w(\cdot)$ to the estimation error $\varepsilon(\cdot)$, being such error generated by System (12.3). This amounts to finding two T -periodic matrices $B_F(\cdot)$ and $D_F(\cdot)$ such as to guarantee the stability of the error System (12.3), and to minimize the H_2 norm of such a system. In this respect, we will resort to the characterization of this norm in terms of a periodic Lyapunov equation. In conclusion, the problem is to minimize the index

$$J_2 = \text{trace} \sum_{i=0}^{T-1} (E(i) - D_F(i)C(i))P(i)(E(i) - D_F(i)C(i))' + \text{trace} \sum_{i=0}^{T-1} (N(i) - D_F(i)D(i))(N(i) - D_F(i)D(i))', \quad (12.4)$$

where $P(\cdot)$ is a positive semi-definite T -periodic matrix satisfying the periodic Lyapunov equation

$$P(i+1) = (A(i) - B_F(i)C(i))P(i)(A(i) - B_F(i)C(i))' + (B(i) - B_F(i)D(i))(B(i) - B_F(i)D(i))'. \quad (12.5)$$

Notice that the performance index (12.4) can be equivalently rewritten in the following way

$$J_2 = \text{trace} \sum_{i=0}^{T-1} (D_F(i) - D_F^o(i))W_I(i)^{-1}(D_F(i) - D_F^o(i))' + \text{trace} \sum_{i=0}^{T-1} \begin{bmatrix} E(i) & N(i) \end{bmatrix} (R(i) - S(i)W_I(i)S(i)') \begin{bmatrix} E(i)' \\ N(i)' \end{bmatrix}, \quad (12.6)$$

where

$$\begin{aligned} W_I(i) &= (C(i)'P(i)C(i) + D(i)D(i)')^{-1} \\ D_F^o(i) &= (E(i)P(i)C(i)' + N(i)D(i)')W_I(i) \\ R(i) &= \begin{bmatrix} P(i) & 0 \\ 0 & I \end{bmatrix}, \quad S(i) = \begin{bmatrix} P(i)C(i)' \\ D(i)' \end{bmatrix}. \end{aligned}$$

On the other hand, the Lyapunov Equation (12.5) can be rewritten as

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - S_I(t)W_I(t)S_I(t)' + \tilde{W}_I(t) \quad (12.7a)$$

$$S_I(t) = A(t)P(t)C(t)' + B(t)D(t)' \quad (12.7b)$$

$$\tilde{W}_I(t) = (B_F(i) - B_F^o(i))W_I(t)(B_F(i) - B_F^o(i))' \quad (12.7c)$$

$$B_F^o(i) = S_I(i)W_I(i). \quad (12.7d)$$

Since we are looking for a positive semi-definite T -periodic solution $P(\cdot)$, it follows that $D(t)D(t)' + C(t)P(t)C(t)'$ is positive semi-definite. By assuming that $D(t)D(t)' > 0$ for all t , it follows that $W_I(\cdot)$ is positive definite. In light of all these observations, let us reconsider the expression of the cost to be minimized (12.6). Here we see two non-negative terms. The first one can be independently minimized by the obvious choice $D_F(\cdot) = D_F^o(\cdot)$. As for the second term, it depends on $P(\cdot)$ which in turn depends on $B_F(\cdot)$ through $\tilde{W}_I(\cdot)$. On the other hand, $\tilde{W}_I(\cdot)$ in (12.7c) appears as the “known term” of a periodic Riccati equation. In order to minimize its stabilizing solution as a function of $\tilde{W}_I(\cdot)$, the best choice is to minimize the known term $\tilde{W}_I(\cdot)$ itself. In view of (12.7c), this is achieved by letting $B_F(\cdot) = B_F^o(\cdot)$. With this choice, the following Riccati equation enters the game.

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - S_I(t)W_I(t)S_I(t)' \quad (12.8a)$$

$$S_I(t) = A(t)P(t)C(t)' + B(t)D(t)' \quad (12.8b)$$

$$W_I(t) = (D(t)D(t)' + C(t)P(t)C(t'))^{-1}. \quad (12.8c)$$

This is exactly the same equation encountered in Sect. 10.2.2 when dealing with the spectral factorization problem. Notice also that the filter dynamical matrix is

$$A_F(t) = A(t) - B_F(t)C(t) = A(t) - S_I(t)W_I(t)C(t) = \hat{A}_I(t),$$

where $\hat{A}_I(t)$ is the closed-loop matrix associated with the Riccati equation, compare (12.8) with (10.11). The following result summarizes the previous considerations and provides the proof of the existence of the stabilizing solution to (12.8).

Proposition 12.1 *Assume that*

- (i) $D(t)D(t)' > 0, \forall t$.
- (i) *The pair $(A(\cdot), C(\cdot))$ is detectable.*
- (ii) *The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle.*

Then, the Riccati equation (12.8) admits the T -periodic (positive semi-definite) stabilizing solution $P(\cdot)$ and the solution of the optimal H_2 filtering/deconvolution problem is given by

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)e(t)$$

$$\hat{z}(t) = E(t)\hat{x}(t) + D_F(t)e(t)$$

$$e(t) = y(t) - C(t)\hat{x}(t),$$

with

$$B_F(t) = (A(t)P(t)C(t)' + B(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t)')^{-1} \quad (12.9a)$$

$$D_F(t) = (E(t)P(t)C(t)' + N(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t)')^{-1}. \quad (12.9b)$$

Proof We already discussed the fact that if the T -periodic stabilizing solution $P(\cdot)$ exists, then the optimal solution of the H_2 filtering or deconvolution problem is given by the expressions $B_F(\cdot)$ and $D_F(\cdot)$ above. We only miss to show that the stabilizing solution exists. First of all we show that the T -periodic pencil associated with (12.8) does not have characteristic multipliers on the unit circle. Indeed, consider the pencil (10.3) with $R = I$. In this way, the pencil corresponds to the H_2 problem considered herein. Let

$$\begin{aligned} \bar{A}(t) &= A(t) - B(t)D(t)'(D(t)D(t)')^{-1}C(t) \\ \bar{B}(t) &= B(t) \left(I - D(t)'(D(t)D(t)')^{-1}D(t) \right) \\ \bar{C}(t) &= (D(t)D(t)')^{-1/2}C(t), \end{aligned}$$

and note that the detectability of $(A(\cdot), C(\cdot))$ is equivalent to that of $(\bar{A}(\cdot), \bar{C}(\cdot))$. Then, the pencil becomes

$$L(t) = \begin{bmatrix} I & -\bar{B}(t)\bar{B}(t)' \\ 0 & \bar{A}(t)' \end{bmatrix}, \quad M(t) = \begin{bmatrix} \bar{A}(t) & 0 \\ \bar{C}(t)'C(t) & I \end{bmatrix}.$$

Assume by contradiction that λ is a characteristic multiplier of the pencil on the unit circle. Then, there exists a periodic vector $\bar{x}(t) = [x_1(t)' \ x_2(t)']'$ such that

$$(L(t)\sigma\lambda - M(t)) \cdot \bar{x}(t) = 0. \quad (12.10)$$

Splitting this equation into two parts, after some computation it follows

$$\begin{aligned} 0 &= (I + \bar{G}(\sigma\lambda, t)\bar{G}^\sim(\sigma\lambda, t))\bar{B}(t)'\sigma\lambda \cdot x_2(t) \\ 0 &= (I + \bar{G}^\sim(\sigma\lambda, t)\bar{G}(\sigma\lambda, t))\bar{C}(t) \cdot x_1(t), \end{aligned}$$

where

$$\bar{G}(\sigma, t) = \bar{C}(t)(\sigma I - \bar{A}(t))^{-1}\bar{B}(t).$$

Hence, from these equations and (12.10) it follows

$$\begin{aligned} 0 &= \bar{B}(t)'x_2(t+1) \\ 0 &= \bar{C}(t)x_1(t) \\ 0 &= \lambda x_1(t+1) - \bar{A}(t)x_1(t) \\ 0 &= \lambda^{-1}x_2(t) - \bar{A}(t)'x_2(t+1). \end{aligned}$$

Recalling the definitions of $\bar{A}(\cdot)$, $\bar{B}(\cdot)$ and $\bar{C}(\cdot)$, we have

$$\begin{aligned}
0 &= C(t)x_1(t) \\
0 &= \lambda x_1(t+1) - A(t)x_1(t) \\
0 &= \begin{bmatrix} \sigma^{-1}\lambda^{-1}I - A(t)' & -C(t)' \\ B(t)' & D(t)' \end{bmatrix} \begin{bmatrix} I \\ -(D(t)D(t)')^{-1}D(t)B(t)' \end{bmatrix} \cdot x_2(t+1).
\end{aligned}$$

These last equations contradict either the detectability of $((A(\cdot), C(\cdot)))$ or the assumptions about the system's zeros, or both. In conclusion, the pencil $(L(\cdot), M(\cdot))$ does not have characteristic multipliers on the unit circle. As such, there exists a n -dimensional $(L(\cdot), M(\cdot))$ -invariant subspace associated with the characteristic multipliers inside the open unit circle, i.e., it is possible to compute $[X(t)' Y(t)']'$ so that

$$L(t) \begin{bmatrix} X(t+1) \\ Y(t+1) \end{bmatrix} = M(t) \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \Gamma(t), \quad (12.11)$$

where the periodic matrix $\Gamma(\cdot)$ has all characteristic multipliers inside the open unit disk. Letting $S(t) = Y^*(t)X(t)$, from this equation it follows

$$\begin{aligned}
S(t+1) &= \Gamma(t)S(t)\Gamma^*(t) + Y^*(t+1)B(t)B(t)'\Gamma(t) \\
&\quad + \Gamma^*(t)X^*(t)C(t)'C(t)X(t)\Gamma(t).
\end{aligned} \quad (12.12)$$

This is a periodic Lyapunov equation. Since $\Gamma(\cdot)$ is stable, the solution $S(t) = Y^*(t)X(t)$ is positive semi-definite. Notice that this conclusion implies that, if $Y(t)$ is invertible for each t , then $X(t)Y(t)^{-1}$ is Hermitian and positive semi-definite. On the other hand, under the same assumption it turns out that $\bar{C}(t)'\bar{C}(t)X(t)Y(t)^{-1} + I$ is invertible as well, so that

$$X(t)Y(t)^{-1} = \bar{B}(t)\bar{B}(t)'\bar{A}(t)X(t)Y(t)^{-1} (\bar{C}(t)'\bar{C}(t))^{-1} \bar{A}(t)'.$$

The above equation coincides with (12.8) by letting $P(t) = X(t)Y(t)^{-1}$ and moreover,

$$\hat{A}_I(t) = \bar{A}(t) (I + Q(t)\bar{C}(t)'\bar{C}(t))^{-1} = Y^*(t)^{-1}\Gamma^*(t)Y^*(t).$$

In conclusion, assuming that $Y(\cdot)$ is invertible, $P(t) = X(t)Y(t)^{-1}$ is indeed the stabilizing solution. Finally, we have to show that $Y(t)$ is invertible for each t . Assume by contradiction that there exists a T -periodic vector $\xi(t)$ so that $Y(t)\xi(t) = 0, \forall t$. Then, from (12.11) and (12.12) it is

$$\begin{aligned}
0 &= Y(t)\Gamma(t)\xi(t+1) \\
0 &= \bar{C}(t)X(t)\Gamma(t)\xi(t+1) \\
X(t+1)\xi(t+1) &= \bar{A}(t)X(t)\Gamma(t)\xi(t+1).
\end{aligned}$$

By iterating this reasoning and letting $\bar{\Gamma}(t) = \Gamma(t)\Gamma(t+1) \cdots \Gamma(t+T-1)$, we have

$$0 = Y(t)\bar{\Gamma}(t)\xi(t) \quad (12.13a)$$

$$0 = \bar{C}(t)X(t)\bar{\Gamma}(t)\xi(t) \quad (12.13b)$$

$$X(t)\xi(t) = \Psi_A^-(t)X(t)\bar{\Gamma}(t)\xi(t), \quad (12.13c)$$

where $\Psi_A^-(t)$ is the monodromy matrix of $\bar{A}(\cdot)$. Notice that $\bar{\Gamma}(t)$ is the transpose of the monodromy matrix of $\Gamma(t)'$. As such, the eigenvalues of $\bar{\Gamma}(t)$ do not depend on t and are inside the open unit disk. Notice also that the three equations (12.13) hold if $\xi(t)$ is substituted by $v(\bar{\Gamma}(t))\xi(t)$, where $v(\bar{\Gamma}(t))$ is any polynomial matrix in the powers of $\bar{\Gamma}(t)$. Now, let $v(\bar{\Gamma}(t))$ be the minimum degree polynomial such that $v(\bar{\Gamma}(t))\xi(t) = 0$ and let $v(\bar{\Gamma}(t)) = (\lambda I - \bar{\Gamma}(t))\hat{v}(\bar{\Gamma}(t))$, where, of course, the polynomial $\hat{v}(\bar{\Gamma}(t))$ is such that $z(t) = \hat{v}(\bar{\Gamma}(t))\xi(t) \neq 0$. Notice that $\bar{\Gamma}(t)z(t) = \lambda z(t)$ so that λ is an eigenvalue of $\Gamma(t)$. If $\lambda = 0$, from (12.13c) it follows that

$$0 = \Psi_A^-(t)X(t)(-\lambda I + \bar{\Gamma}(t))z(t) = X(t)z(t),$$

and this conclusion, together with $Y(t)z(t) = 0$ and the n -dimensionality of the subspace $\text{Im}[X(t)' Y(t)']$, implies that $z(t) = 0$, which is a contradiction. On the other hand, if $\lambda \neq 0$, then, from (12.13b), (12.13c) it follows that

$$0 = \Psi_A^-(t)X(t)(-I + \lambda^{-1}\bar{\Gamma}(t))z(t) = (\lambda^{-1}I - \Psi_A^-(t))X(t)z(t)$$

$$0 = \bar{C}(t)X(t)(I - \lambda^{-1}\bar{\Gamma}(t))z(t) = \bar{C}(t)X(t)z(t).$$

A moment's thought reveals that the last equation can be generalized to $\bar{C}(t+i)\Phi_A^-(t+i,t)X(t)z(t) = 0$, $i \geq 0$, where $\Phi_A^-(\cdot, \cdot)$ denotes the transition matrix of $A(\cdot)$. Therefore, the conditions above, together with $|\lambda| \leq 1$, contradict the detectability of $(A(\cdot), \bar{C}(\cdot))$ (recall the PBH test in Sects. 4.1.6, 4.2.7) and hence the detectability of $(A(\cdot), C(\cdot))$, unless $X(t)z(t) = 0$. Again this last condition and $Y(t)z(t) = 0$ lead to a contradiction. ■

In conclusion, the solution of the H_2 filtering/deconvolution problem in a state-space context calls for the Riccati Equation (12.8). This equation depends upon the matrices entering the state and measurement equations, namely $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$. Remarkably, the matrices $E(\cdot), N(\cdot)$ entering the remote signal $z(\cdot)$ do not play any role in the computation of the gains $B_F(\cdot)$ and $D_F(\cdot)$. As a consequence, the state variable $\hat{x}(\cdot)$ in (12.8) corresponds to the optimal H_2 filtered estimate of the system's state regardless of the structure of the remote signal $z(\cdot)$.

12.1.1 Kalman Filtering

The H_2 filtering problem has been settled in a complete deterministic context. However, it can be re-stated in a stochastic framework by assuming that the disturbances acting on the given plant are suitable stochastic noises with known statistics. Specifically, assume that the system is given by

$$\begin{aligned}x(t+1) &= A(t)x(t) + \xi_1(t) \\ y(t) &= C(t)x(t) + \xi_2(t) \\ z(t) &= E(t)x(t),\end{aligned}$$

where $\xi(t) = [\xi_1(t)' \ \xi_2(t)']'$ is a white-noise with zero mean and T -periodic covariance matrix

$$W(t) = \begin{bmatrix} W_{11}(t) & W_{12}(t) \\ W_{12}(t)' & W_{22}(t) \end{bmatrix}.$$

We assume that the noise $\xi_2(\cdot)$ affects all the measurements $\bar{y}(\cdot)$, i.e., $W_{22}(t) > 0$, $\forall t$. Moreover, $W(\cdot) \geq 0$ implies that

$$\hat{W}(t) = W_{11}(t) - W_{12}(t)W_{22}(t)^{-1}W_{12}(t)'$$

is positive semi-definite for each t . Now, let

$$\begin{aligned}B(t) &= [\hat{W}(t)^{1/2} \ W_{12}(t)W_{22}(t)^{-1/2}] \\ D(t) &= [0 \ W_{22}(t)^{1/2}],\end{aligned}$$

so that the system can be written as

$$\begin{aligned}x(t+1) &= A(t)x(t) + B(t)w(t) \\ y(t) &= C(t)x(t) + D(t)w(t) \\ z(t) &= E(t)x(t),\end{aligned}$$

where

$$w(t) = \begin{bmatrix} \hat{W}(t)^{-1/2} & -\hat{W}(t)^{-1/2}W_{12}(t)W_{22}(t)^{-1} \\ 0 & W_{22}(t)^{-1/2} \end{bmatrix} \xi(t)$$

is a white-noise with zero mean and an identity covariance matrix. Hence, the optimal stochastic filtering problem consists in minimizing

$$J_{2s} = \lim_{k \rightarrow \infty} \sum_{t=0}^{T-1} \mathcal{E} [\varepsilon(kT+t)' \varepsilon(kT+t)],$$

where $\varepsilon(t)$ is the estimation error $z(t) - \hat{z}(t)$. Except for the lack of specifications regarding the initial state, this is the celebrated Kalman filtering problem. It is worthy pointing out that, under the assumption of Proposition 12.1, the solution of the optimal stochastic filtering problem coincides with that of the H_2 filtering problem. This follows from the stochastic interpretation of the H_2 norm provided in Sect. 9.1.3.

The theory of Kalman filtering (including the role of the initial state) can be developed provided that suitable assumptions are made on the structural characteristics of the system. As a matter of fact, it is possible to show that a necessary and sufficient condition for the existence of a T -periodic positive semi-definite solution of the periodic Riccati equation is the stability of the controllable and unreconstructable part of the pair constituted by matrix $(A(\cdot) - B(\cdot)D(\cdot)'(D(\cdot)D(\cdot)')^{-1}C(\cdot))$ and matrix

$B(\cdot)(I - D(\cdot)'(D(\cdot)D(\cdot)')^{-1})$. Correspondingly, one can work out a Kalman filter under the assumption that the initial state estimation error $\eta(0)$ in Eqs. (12.3) does not affect the estimation error $\varepsilon(t)$ of the remote signal $z(t)$. In this way, the obtained Kalman filter may not be internally stable.

12.1.2 A Factorization View

Above we have provided the solution of the H_2 filtering problem in a state-space framework. Alternatively, the problem can be posed in an input-output context. Precisely, recall that $H_1(\sigma, t)$ is the transfer operator from the disturbance $w(\cdot)$ to the measured signal $y(\cdot)$ and $H_2(\sigma, t)$ the transfer operator from the disturbance $w(\cdot)$ to the remote signal $z(\cdot)$. As outlined in Fig. 12.1, the problem consists in finding a causal T -periodic stable filter $F(\sigma, t)$ such that $T_F(\sigma, t) = F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)$ is stable and the H_2 norm of $T_F(\sigma, t)$ is minimized. As a matter of fact, if one relaxes the condition of stability of $T_F(\sigma, t)$, the problem can be generalized in a L_2 context, and given a closed-form solution in a general input-output formulation. In this section we adopt the latter point of view, and we will rely on a factorization approach.

Following the rationale in Sect. 10.2, let $\Omega(\sigma, t)$ be the minimal spectral factor of $H_1(\sigma, t)$ (the existence conditions have been already discussed therein). Hence, $H_1(\sigma, t)H_1^\sim(\sigma, t) = \Omega(\sigma, t)\Omega^\sim(\sigma, t)$ where $\Omega(\sigma, \cdot)$ is stable with stable inverse. Now, standard computations show that

$$\begin{aligned} T_F(\sigma, t)T_F^\sim(\sigma, t) &= \bar{T}_F(\sigma, t)\bar{T}_F^\sim(\sigma, t) \\ &\quad + H_2(\sigma, t)H_1^\sim(\sigma, t)(H_1(\sigma, t)H_1^\sim(\sigma, t))^{-1}H_1(\sigma, t)H_2(\sigma, t), \end{aligned}$$

where

$$\bar{T}_F(\sigma, t) = F(\sigma, t)\Omega(\sigma, t) - H_2(\sigma, t)H_1^\sim(\sigma, t)\Omega^\sim(\sigma, t)^{-1}.$$

If the second term $H_2(\sigma, t)H_1^\sim(\sigma, t)\Omega^\sim(\sigma, t)^{-1}$ is free of unit modulus poles, it can be decomposed in the stable and unstable parts as discussed in Remark 9.1. Considering that $F(\sigma, t)$ is required to be stable and that $\Omega(\sigma, t)$ and $\Omega(\sigma, t)^{-1}$ are already stable, the solution of the L_2 optimal filtering problem is given by

$$F_{fil}(\sigma, t) = (H_2(\sigma, t)H_1^\sim(\sigma, t)\Omega^\sim(\sigma, t)^{-1})_- \Omega(\sigma, t)^{-1}, \quad (12.14)$$

where $(\cdot)_-$ denotes the stable part.

Filter (12.14) can be compared with the one above derived in the state-space framework. This is done under the assumption that both transfer operators $H_1(\sigma, t)$ and $H_2(\sigma, t)$ are stable. Should these assumptions be violated, the input-output L_2 optimal filter may not coincide with the state-space H_2 optimal filter.

Assume now that $H_1(\sigma, t)$ and $H_2(\sigma, t)$ are stable and then take the factorization of $H_1(\sigma, t)$ induced by the stabilizing solution of the Riccati Equation (12.8) as in Eq.

(10.12), namely

$$\Omega(\sigma, t) = W_I(t)^{-1/2} + C(t) (\sigma I - A(t))^{-1} S_I(t) W_I(t)^{1/2}. \quad (12.15)$$

Now, let

$$\begin{aligned} \Phi(\sigma, t) &= (\sigma I - A(t))^{-1} \\ \hat{\Phi}(\sigma, t) &= (\sigma I - A(t) + S_I(t) W_I(t) C(t))^{-1} = (\sigma I - A_F(t))^{-1}, \end{aligned}$$

and notice that

$$\Omega(\sigma, t) = (I + C(t) \Phi(\sigma, t) S_I(t) W_I(t)) W_I(t)^{-1/2} \quad (12.16a)$$

$$\Omega(\sigma, t)^{-1} = W_I(t)^{1/2} (I - C(t) \hat{\Phi}(\sigma, t) S_I(t) W_I(t)) \quad (12.16b)$$

$$\Phi(\sigma, t) S_I(t) = \hat{\Phi}(\sigma, t) (I - W_I(t) C(t) \hat{\Phi}(\sigma, t) S_I(t))^{-1} \quad (12.16c)$$

$$\begin{aligned} \Phi(\sigma, t) B(t) B(t)' \Phi^\sim(\sigma, t) &= \Phi(\sigma, t) S_I(t) S_I(t)' \Phi^\sim(\sigma, t) \\ &\quad + \Phi(\sigma, t) A(t) Q(t) A(t)' \Phi^\sim(\sigma, t) + Q(t). \end{aligned} \quad (12.16d)$$

We now develop some computations where the arguments σ and t are omitted here for the sake of conciseness:

$$\begin{aligned} (H_2 H_1^\sim (\Omega^\sim)^{-1})_- &= \left((N + E \Phi B) (D' + B' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2} \right)_- \\ &= N D' W^{1/2} + \left(E \Phi (S_I - A Q C' + B B' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2} \right)_- \\ &= N D' W_I^{1/2} + \left((E Q C' + E Q A' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2} \right)_- \\ &\quad + \left(E \Phi S_I + E \Phi S_I W_I S_I' \Phi^\sim C' \right) (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2} \Big)_- \\ &= (N D' + E Q C') W_I^{1/2} + E \Phi S_I \left((I + W_I S_I' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2} \right)_- \\ &= (N D' + E Q C') W_I^{1/2} + E \Phi S_I W_I^{1/2} \\ &= (N D' + E Q C') W_I^{1/2} + E \hat{\Phi} (I - W_I C \hat{\Phi} S_I)^{-1} W_I^{1/2}, \end{aligned}$$

so that

$$F_{fil} = (H_2 H_1^\sim (\Omega^\sim)^{-1})_- \Omega^{-1} = D_F + (E - D_F C) \hat{\Phi} B_F,$$

i.e.,

$$F_{fil}(\sigma, t) = D_F(t) + (E(t) - D_F(t) C(t)) (\sigma I - A(t) + S_I(t) W_I(t) C(t))^{-1} B_F(t).$$

This is exactly the solution worked out in Proposition 12.1.

Remark 12.2 *The above derivations are based on the assumption of invertibility of $D(t)D(t)'$, for each t . When this assumption is not satisfied (singular filtering problem), one can resort to the spectral interactor matrix introduced in Sect. 8.4*

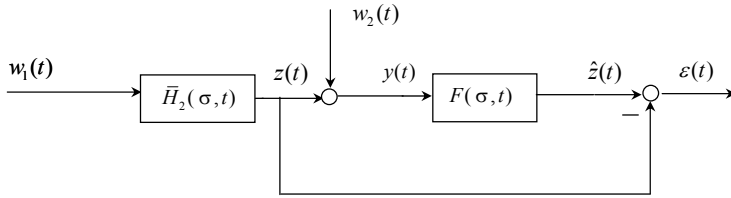


Fig. 12.2 The Wiener filter problem

and applied for spectral factorization in Remark 10.3. In this way, the singular problem can be transformed into a non-singular one.

12.1.3 Wiener Filtering

We now assume that the remote signal to be estimated $z(t)$ coincides with the uncorrupted output of a periodic system with transfer operator $H_2(\sigma, t)$, and its estimation has to be performed from the signal $y(t)$ which is a noise-corrupted measurement of $z(t)$. This situation, schematically represented in Fig. 12.2, can be cast in the general framework by letting $z(t) = C(t)x(t)$ so that

$$E(t) = C(t), \quad N(t) = 0. \quad (12.17)$$

The corresponding estimation problem is often referred to as a Wiener filtering problem. The solution of this problem has already been obtained. Indeed, Wiener filtering is just a special case of the general estimation problem previously dealt with, obtained from the setting outlined in Fig. 12.1 by letting $w = [w_1' \ w_2']'$, $H_2(\sigma, t) = [\bar{H}_2(\sigma, t) \ 0]$ and $H_1(\sigma, t) = [\bar{H}_2(\sigma, t) \ D(t)]$.

Precisely, the transfer operator between the disturbance $w(t)$ and the filtering error $\varepsilon(t)$ has an attenuation level (evaluated in terms of H_∞ norm) lower than a pre-computable value.

To derive this result on robustness, consider the filter in observer form (12.2) with

$$\begin{aligned} B_F(t) &= (A(t)P(t)C(t)' + B(t)D(t)'(C(t)P(t)C(t)' + D(t)D(t)')^{-1} \\ D_F(t) &= C(t)P(t)C(t)'(C(t)P(t)C(t)' + D(t)D(t)')^{-1}, \end{aligned}$$

where $P(\cdot)$ is the T -periodic positive semi-definite stabilizing solution of the periodic Riccati equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - S_I(t)W_I(t)S_I(t)' + \varepsilon I \quad (12.18a)$$

$$S_I(t) = A(t)P(t)C(t)' + B(t)D(t)' \quad (12.18b)$$

$$W_I(t) = (D(t)D(t)' + C(t)P(t)C(t)')^{-1}. \quad (12.18c)$$

Notice that such a filter coincides with the optimal H_2 filter when $\varepsilon \rightarrow 0$, since in this case (12.18) becomes (12.8). Now, consider the error System (12.2). In view of (12.17) the matrices of such a system are:

$$\begin{aligned} A_F(t) &= A(t) - B_F(t)C(t) \\ \hat{B}_F(t) &= B(t) - B_F(t)D(t) \\ C_F(t) &= (I - D_F(t))C(t) \\ \hat{D}_F(t) &= -D_F(t)D(t). \end{aligned}$$

Notice that the Riccati Equation (12.18) can be rewritten as

$$P(t+1) = A_F(t)P(t)A_F(t)' + \hat{B}_F(t)\hat{B}_F(t)' + \varepsilon I. \quad (12.19)$$

We now compute a guaranteed value γ for the attenuation level of the H_∞ norm of the transfer operator from $w(\cdot)$ to the estimation error $\hat{z}(t) - z(t)$. This computation is carried over by resorting to the results provided in Sect. 9.2 for the characterization of the H_∞ norm in terms of the bounded-real lemma. Precisely, according to Remark 9.2, the optimal H_2 Wiener filter ensures an attenuation level γ if and only if there exists a T -periodic positive semi-definite matrix $\hat{P}(\cdot)$ satisfying

$$\hat{P}(t+1) > A_F(t)\hat{P}(t)A_F(t)' + \hat{B}_F(t)\hat{B}_F(t)' + \hat{Y}(t)\hat{W}(t)\hat{Y}(t)' \quad (12.20a)$$

$$\hat{Y}(t) = (A_F(t)\hat{P}(t)C_F(t)' + \hat{B}_F(t)\hat{D}_F(t)')\hat{W}(t)^{-1} \quad (12.20b)$$

$$\hat{W}(t) = \gamma^2 I - \hat{D}_F(t)\hat{D}_F(t)' - C_F(t)\hat{P}(t)C_F(t)' > 0. \quad (12.20c)$$

Starting from the solution $P(\cdot)$ of (12.18), or equivalently of Eq. (12.19), we look for a solution of this inequality by letting

$$\hat{P}(t) = \alpha^{-1}P(t), \quad \alpha \in (0, 1). \quad (12.21)$$

First, compare Inequality (12.20a) with Eq. (12.19). Once $\hat{P}(t)$ has been replaced by $\alpha^{-1}P(t)$, Inequality (12.20a) is equivalent to

$$(1 - \alpha)B_F(t)B_F(t)' - \alpha\hat{Y}(t)\hat{W}(t)\hat{Y}(t)' > -\varepsilon I. \quad (12.22)$$

On the other hand, by using (12.21) in Eq. (12.20b), after some computations, one obtains

$$\begin{aligned} \hat{Y}(t)\hat{W}(t) &= \alpha^{-1}B_F(t)D(t)'Z(t) \\ Z(t) &= \alpha I + (1 - \alpha)T_P(t)^{-1}D(t)D(t)' \\ T_P(t) &= D(t)D(t)' + C(t)P(t)C(t)'. \end{aligned}$$

Using again Eq. (12.21), Eq. (12.20c) becomes

$$\hat{W}(t) = \alpha^{-1}(1 - \alpha)^{-1}(\gamma^2\alpha(1 - \alpha)I + (Z(t) - I)D(t)D(t)'Z(t)).$$

Hence, Inequality (12.22) is verified if

$$\begin{aligned} \hat{B}_F(t)R(t)\hat{B}_F(t)' &\geq 0 \\ R(t) &= I - D(t)'Z(t) \left(\alpha(1 - \alpha)\gamma^2 I + (Z(t)' - I)D(t)D(t)'Z(t) \right)^{-1} Z(t)'D(t). \end{aligned}$$

Thanks to the matrix inversion lemma, notice that

$$R(t)^{-1} = I + D(t)'Z(t) \left(\alpha(1 - \alpha)\gamma^2 I - D(t)D(t)'Z(t) \right)^{-1} Z(t)'D(t).$$

The H_∞ robustness result is therefore obtained by guaranteeing that matrix

$$\alpha(1 - \alpha)\gamma^2 I - D(t)D(t)'Z(t)$$

is positive definite for some $\alpha \in (0, 1)$. This is equivalent to imposing, $\forall t$

$$\gamma^2 I > \frac{D(t)D(t)'Z(t)}{\alpha(1 - \alpha)}.$$

On the other hand, in order to minimize the value of the attenuation level, one is led to choose $\alpha = 0.5$ (so that $\alpha^{-1}(1 - \alpha)^{-1}$ is minimized). Correspondingly, $\forall t$

$$\gamma^2 I > 4D(t)D(t)'Z(t).$$

Noticing that $D(t)D(t)'Z(t) < D(t)D(t)'$, it finally follows that

$$\gamma \geq 2 \max_t \|D(t)\|.$$

Summing up, the given filter for the uncorrupted output, although not designed for obtaining particular robustness features, at least guarantees a disturbance-to-error H_∞ norm strictly less than $2 \max_t \|D(t)\|$. By continuity, as $\varepsilon \rightarrow 0$, the conclusion is drawn that the optimal H_2 Wiener filter guarantees a disturbance-to-error H_∞ norm less than or equal to $2 \max_t \|D(t)\|$.

12.2 H_2 Prediction

The prediction problem consists in finding an estimator characterized by the periodic transfer operator $F(\sigma, t)$ which, fed by the measurements $y(\cdot)$ up to time $t - h$, yields the optimal estimate $\hat{z}(t)$ of the remote signal $z(\cdot)$ at time t in the H_2 sense. In other words, $F(\sigma, t)$ is designed so as to minimize the H_2 norm between the disturbance $w(\cdot)$ and the prediction error $\hat{z}(t) - z(t)$. The positive integer h is the prediction horizon.

As we will prove shortly, the optimal H_2 predictor can be easily obtained from the H_2 optimal filter, based on the unique T -periodic stabilizing solution $P(\cdot)$ of the periodic Riccati Equation (12.8). We will supply two derivations of the resulting

predictor. The first one is worked out in a fully state–space environment by adding h slack state variables to the original ones. These extra variables are the delayed disturbance variables $w(\cdot)$ from $w(t-h)$ to $w(t-1)$. The second line of reasoning relies on the spectral factorization approach and will be postponed to the next point.

In order to work out the solution via the state–space approach, we introduce an augmented system for which the original prediction problem is transformed in a filtering problem. With this objective, let

$$r(t) = x(t-h), \quad \xi_i(t) = w(t-h+i-1), \quad i = 1, \dots, h, \quad \eta(t) = y(t-h),$$

and define the augmented state

$$x_a(t) = \begin{bmatrix} r(t) \\ \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_h(t) \end{bmatrix}.$$

It is easy to recognize that the h -step prediction problem can be formulated as a filtering problem for the augmented system

$$x_a(t+1) = A_a(t)x_a(t) + B_a(t)w(t) \quad (12.23a)$$

$$\eta(t) = C_a(t)x_a(t) \quad (12.23b)$$

$$z(t) = E_a(t)x_a(t) + N(t)w(t), \quad (12.23c)$$

where

$$A_a(t) = \begin{bmatrix} A(t-h) & B(t-h) & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & I \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B_a(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$$C_a(t) = [C(t-h) \ D(t-h) \ 0 \ \dots \ 0]$$

$$E_a(t) = \left[E(t)\Phi_A(t, t-h) \ E_a^{(1)}(t) \ E_a^{(2)}(t) \ \dots \ E_a^{(h)}(t) \right]$$

$$E_a^{(i)}(t) = E(t)\Phi_A(t, t-h+i)B(t-h+i-1), \quad i = 1, 2, \dots, h.$$

We can now resort to the filtering Riccati Equation (12.8) for System (12.23)

$$P_a(t+1) = A_a(t)P_a(t)A_a(t)' + B_a(t)B_a(t)' - S_{al}(t)W_{al}(t)S_{al}(t)' \quad (12.24a)$$

$$S_{al}(t) = A_a(t)P_a(t)C_a(t)' \quad (12.24b)$$

$$W_{al}(t) = (C_a(t)P_a(t)C_a(t)')^{-1}. \quad (12.24c)$$

Exploiting the structure of the matrices, it can be seen that the unique stabilizing T -periodic solution of (12.24) is

$$P_a(t) = \begin{bmatrix} P(t-h) & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix},$$

where $P(\cdot)$ is the T -periodic stabilizing solution of (12.8). Therefore, according to the optimal H_2 filtering theory, the optimal H_2 filter for the augmented system is

$$\hat{x}_a(t+1) = A_a(t)\hat{x}(t) + B_{Fa}(t)(\eta(t) - C_a(t)\hat{x}_a(t)) \quad (12.25a)$$

$$\hat{z}(t) = E_a(t)\hat{x}_a(t) + D_{Fa}(t)(\eta(t) - C_a(t)\hat{x}(t)), \quad (12.25b)$$

where

$$B_{Fa}(t) = A_a(t)P_a(t)C_a(t)'(C_a(t)P_a(t)C_a(t)')^{-1}$$

$$D_{Fa}(t) = C_a(t)P_a(t)C_a(t)'(C_a(t)P_a(t)C_a(t)')^{-1}.$$

In view of the peculiar structure of all matrices entering these expressions, it follows that

$$B_{Fa}(t) = \begin{bmatrix} B_F(t-h) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$D_{Fa}(t) = E(t)\Phi_A(t, t-h+1)B_F(t-h),$$

where $B_F(\cdot)$ is the gain matrix already introduced for the filtering problem, i.e.,

$$B_F(t) = A(t)P(t)C(t)'(C(t)P(t)C(t)' + D(t)D(t)')^{-1}.$$

Obviously, by isolating the first n -dimensional state-vector components $\hat{r}(t)$ of $\hat{x}_a(t)$, from Eqs. (12.25) one can work out the optimal H_2 predictor. To be precise, letting $\hat{x}(t) = \hat{r}(t+h)$, the following result holds:

Proposition 12.2 *Under the assumptions of Proposition 12.1, the h -step optimal H_2 predictor is given by*

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)e(t) \quad (12.26a)$$

$$\hat{z}(t) = E(t)\Phi_A(t, t+1-h)\hat{x}(t+1-h) \quad (12.26b)$$

$$e(t) = y(t) - C(t)\hat{x}(t). \quad (12.26c)$$

Remark 12.3 For $h = 1$, Eqs. (12.26) lead to

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)e(t)$$

$$\hat{z}(t) = E(t)\hat{x}(t)$$

$$e(t) = y(t) - C(t)\hat{x}(t).$$

If one takes $E(t) = I$, so that the variable to be predicted is the full state vector ($z(t) = x(t)$), then one obtains $\hat{z}(t) = \hat{x}(t)$. In other words, $\hat{x}(t)$ is the optimal 1-step predictor of the state vector. The expression above for the predictor can be compared with that of the filter as given in Proposition 12.1, namely

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)e(t)$$

$$\hat{z}(t) = E(t)\hat{x}(t) + D_F(t)e(t)$$

$$e(t) = y(t) - C(t)\hat{x}(t).$$

It is apparent that vector $\hat{x}(t)$ appearing in the filter formulas is still the optimal 1-step predictor. Moreover, the filtered estimate of $z(t)$ (namely the estimate obtained by the measurements $y(\cdot)$ up to time t) is given by the 1-step predictor of $z(t)$ (namely the estimate obtained by the measurements up to time $t-1$) plus the additional term $D_F(t)e(t)$ that takes into account the information conveyed by the last measurement $y(t)$.

Remark 12.4 Notice that the h -step predictor, $h \geq 1$, does not depend upon matrix $N(t)$. This fact is no longer valid in the H_∞ approach, see Sect. 12.5.

Remark 12.5 As in the case of filtering, the optimal predictor can be given a stochastic interpretation in terms of the minimization of the prediction error variance. In this setting, the stabilizing solution of the periodic Riccati equation is the variance of the optimal 1-step prediction error.

12.2.1 A Factorization View

The considerations developed in Sect. 12.1.2 for the L_2 filtering problem can be adapted to the optimal L_2 prediction problem. This simply requires replacing the operator $H_2(\sigma, t)$ with $\sigma^h H_2(\sigma, t)$. Consequently, the optimal L_2 h -step predictor is

$$F_{pred}(\sigma, t) = \sigma^{-h} \left(\sigma^h H_2(\sigma, t) H_1^\sim(\sigma, t) \Omega^\sim(\sigma, t)^{-1} \right)_- \Omega(\sigma, t)^{-1}, \quad (12.27)$$

where $\Omega(\sigma, t)$ is the minimal spectral factor of $H_1(\sigma, t)$ and, as already said, $(\cdot)_-$ denotes the stable part. Again under the assumption of stability of $H_1(\sigma, t)$ and $H_2(\sigma, t)$, letting

$$\begin{aligned}\Phi(\sigma, t) &= (\sigma I - A(t))^{-1} \\ \hat{\Phi}(\sigma, t) &= (\sigma I - A(t) + B_F(t)C(t))^{-1} = (\sigma I - A_F(t))^{-1},\end{aligned}$$

and recalling Expressions (12.16), it follows that (the time-dependence of the operators is omitted)

$$\begin{aligned}\left(\sigma^h H_2 H_1^\sim (\Omega^\sim)^{-1}\right)_- &= \left(\sigma^h (N + E \Phi B) (D' + B' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2}\right)_- \\ &= \left(\sigma^h E \Phi (S_I - A Q C' + B B' \Phi C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2}\right)_- \\ &= \left(\sigma^h (E Q C' + E Q A' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2}\right)_- \\ &\quad + \left(\sigma^h (E \Phi S_I + E \Phi S_I W_I S_I' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2}\right)_- \\ &= \left(\sigma^h L \Phi S_I (I + W_I S_I' \Phi^\sim C') (I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2}\right)_- \\ &= \left(\sigma^h E \Phi S_I W_I^{1/2}\right)_- = \left(\sigma^h E \Phi\right)_- S_I W_I^{1/2}.\end{aligned}$$

Now, note that matrix $\Phi(\sigma, t)$ can be expressed in terms of the transition matrix $\Phi_A(t, \tau)$:

$$\Phi(\sigma, t) = \sum_{k=0}^{\infty} \Phi_A(t, t-k) \sigma^{-k-1}. \quad (12.28)$$

By exploiting this formula one obtains:

$$\begin{aligned}F_{pred}(\sigma, t) &= \sigma^{-h} \left(\sigma^h E(t) \Phi(\sigma, t) S_I(t) W_I(t)^{1/2} \right)_- \Omega(\sigma, t)^{-1} \\ &= \sigma^{-h} \left(\sigma^h E(t) \sum_{k=0}^{\infty} \Phi_A(t, t-k) \sigma^{-k-1} S_I(t) W_I(t)^{1/2} \right)_- \Omega(\sigma, t)^{-1} \\ &= \sigma^{-h} \left(\sigma^h E(t) \sum_{k=h-1}^{\infty} \Phi_A(t, t-k) \sigma^{-k-1} S_I(t) W_I(t)^{1/2} \right)_- \Omega(\sigma, t)^{-1} \\ &= E(t) \sum_{k=h-1}^{\infty} \Phi_A(t, t-k) \sigma^{-k-1} S_I(t) W_I(t)^{1/2} \Omega(\sigma, t)^{-1} \\ &= E(t) \sum_{i=0}^{\infty} \Phi_A(t, t+1-i-h) \sigma^{-i-h} S_I(t) W_I(t)^{1/2} \Omega(\sigma, t)^{-1}.\end{aligned}$$

We now observe that Eq. (12.28) can be generalized as:

$$\begin{aligned}
\sum_{i=0}^{\infty} \Phi_A(t, t+1-i-h) \sigma^{-i-h} &= \\
&= \Phi_A(t, t+1-h) \sigma^{-h} \sum_{k=0}^{\infty} \Phi_A(t+1, t+1-k) \sigma^{-k} \\
&= \Phi_A(t, t+1-h) \sigma^{-h} \left(I + \sum_{k=1}^{\infty} \Phi_A(t+1, t+1-k) \sigma^{-k} \right) \\
&= \Phi_A(t, t+1-h) \sigma^{-h} (I + A(t) \Phi(\sigma, t)) \\
&= \Phi_A(t, t+1-h) \sigma^{-h+1} \Phi(\sigma, t).
\end{aligned}$$

Thanks to such an expression, one can write:

$$\begin{aligned}
F_{pred}(\sigma, t) &= E(t) \Phi_A(t, t+1-h) \sigma^{-h+1} \Phi(\sigma, t) S_I(t) W_I(t)^{1/2} \Omega(\sigma, t)^{-1} \\
&= E(t) \Phi_A(t, t+1-h) \sigma^{-h+1} \hat{\Phi}(\sigma, t) B_F(t).
\end{aligned}$$

In conclusion, the transfer operator of the optimal h -step ahead predictor is:

$$F_{pred}(\sigma, t) = E(t) \Phi_A(t, t+1-h) \sigma^{-h} \left(I + A_F(t) (\sigma I - A_F(t))^{-1} \right) B_F(t).$$

Not surprisingly, this expression coincides with the transfer operator induced by the optimal predictor in state-space form (12.26).

12.3 H_2 Smoothing

This section is devoted to the fixed-lag smoothing problem in the H_2 setting. Here, the measurements are available until time $t+h$, $h > 0$, and the designer aims at estimating the variable $z(t)$ so as to minimize the H_2 norm between the disturbance $w(t)$ and the estimation error. Equivalently, we can assume that the measurements are available up to time t and the remote variable to be estimated is $z(t-h)$.

As in the previous section, the state-space approach is pursued first. Again we reformulate the problem as a filtering problem. However, the computations are more involved than in the prediction problem. Therefore we are well advised to first consider the simplest case of 1-step smoothing ($h = 1$). Subsequently, the method will be generalized to any smoothing horizon h .

Let

$$x_a(t) = \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix}, \quad \zeta(t) = z(t-1)$$

be the augmented state consisting of the original state variable along with the variable to be estimated. The corresponding augmented system is given by:

$$\begin{aligned}
x_a(t+1) &= A_a(t)x_a(t) + B_a(t)w(t) \\
y(t) &= C_a(t)x_a(t) + D(t)w(t) \\
\zeta(t) &= E_a(t)x_a(t),
\end{aligned}$$

where

$$\begin{aligned}
A_a(t) &= \begin{bmatrix} A(t) & 0 \\ E(t) & 0 \end{bmatrix}, & B_a(t) &= \begin{bmatrix} B(t) \\ N(t) \end{bmatrix} \\
C_a(t) &= \begin{bmatrix} C(t) & 0 \end{bmatrix}, & E_a(t) &= \begin{bmatrix} 0 & I \end{bmatrix}.
\end{aligned}$$

With this reformulation, we can resort to the filtering Riccati Equation (12.8) for the augmented system:

$$P_a(t+1) = A_a(t)P_a(t)A_a(t)' + B_a(t)B_a(t)' - S_{al}(t)W_{al}(t)S_{al}(t)' \quad (12.29a)$$

$$S_{al}(t) = A_a(t)P_a(t)C_a(t)' + B_a(t)D(t)' \quad (12.29b)$$

$$W_{al}(t) = (C_a(t)P_a(t)C_a(t)' + D(t)D(t)')^{-1}. \quad (12.29c)$$

By partitioning $P_a(t)$ according to the structure of the augmented state,

$$P_a(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t)' & P_{22}(t) \end{bmatrix},$$

an easy computation reveals that the solution of the augmented Riccati Equation (12.29) can be written in terms of the solution $P(t)$ of the standard Riccati Equation (12.8).

To be precise, the following relations hold:

$$\begin{aligned}
P_{11}(t) &= P(t) \\
P_{12}(t+1) &= A(t)P(t)E(t)' + B(t)N(t)' - S_I(t)W_I(t)U_I(t)' \\
P_{22}(t+1) &= E(t)P(t)E(t)' + N(t)N(t)' - U_I(t)W_I(t)U_I(t)' \\
U_I(t) &= E(t)P(t)C(t)' + N(t)D(t)'.
\end{aligned} \quad (12.30)$$

The optimal H_2 filter for the augmented system is given by:

$$\begin{aligned}
\hat{x}_a(t+1) &= A_a(t)\hat{x}_a(t) + B_{Fa}(t)(y(t) - C_a(t)\hat{x}_a(t)) \\
\hat{\zeta}(t) &= E_a(t)\hat{x}_a(t) + D_{Fa}(t)(y(t) - C_a(t)\hat{x}_a(t)),
\end{aligned}$$

where, by again exploiting the structure of the matrices of the augmented system,

$$B_{Fa}(t) = \begin{bmatrix} S_I(t) \\ U_I(t) \end{bmatrix} W_I(t).$$

Finally, letting $\hat{z}(t) = \hat{\zeta}(t+1)$ be the optimal estimate of $z(t)$ based on the measurements $y(\cdot)$ up to $t+1$, it is possible to express the optimal 1-step H_2 smoother in terms of the innovation $e(\cdot)$ at time t and at time $t+1$. To be precise, define two

periodic gains

$$K_0(t) = U_I(t)W_I(t), \quad K_1(t) = P_{12}(t)'C(t)'W_I(t).$$

Then the optimal H_2 smoother, with smoothing horizon $h = 1$, can be written as

$$\begin{aligned}\hat{x}(t+1) &= A(t)\hat{x}(t) + B_F(t)e(t) \\ \hat{z}(t) &= E(t)\hat{x}(t) + K_0(t)e(t) + K_1(t+1)e(t+1) \\ e(t) &= y(t) - C(t)\hat{x}(t).\end{aligned}$$

Notice that only the first block column of the solution of the augmented Riccati equation is required in order to compute the smoother gain matrices $K_0(t)$ and $K_1(t)$. As for the conditions of existence of the stabilizing periodic solution of the augmented periodic Riccati equation, it is easy to see that they coincide with those allowing the existence of the stabilizing periodic solution of the standard filtering Riccati equation.

By iteratively applying the previous rationale to the cases $h = 2, h = 3, \dots$, one can obtain the optimal H_2 smoother for any smoothing horizon. To be precise, define the smoother gains:

$$\begin{aligned}K_0(t) &= U_I(t)W_I(t) \\ K_{i-1}(t) &= P_{1i}(t)'C(t)'W_I(t), \quad i = 2, \dots, h+1,\end{aligned}$$

where

$$\begin{aligned}P_{12}(t+1) &= A(t)P(t)E(t)' + B(t)N(t)' - S_I(t)W_I(t)U_I(t)' \\ P_{1i}(t+1) &= (A(t) - B_F(t)C(t))P_{1i-1}(t) \quad i = 3, 4, \dots, h+1.\end{aligned}$$

The following result provides the optimal H_2 smoother.

Proposition 12.3 *Under the assumptions of Proposition 12.1, the optimal H_2 smoother, with smoothing horizon h , is given by*

$$\begin{aligned}\hat{x}(t+1) &= A(t)\hat{x}(t) + S_I(t)W_I(t)e(t) \\ \hat{z}(t) &= E(t)\hat{x}(t) + \sum_{i=0}^h K_i(t+i)e(t+i) \\ e(t) &= y(t) - C(t)\hat{x}(t).\end{aligned}$$

Remark 12.6 *From (12.8a), (12.8b), it is easily seen that $S_I(t)W_I(t) = B_F(t)$. Therefore, the equations*

$$\begin{aligned}
\hat{x}(t+1) &= A(t)\hat{x}(t) + S_I(t)W_I(t)e(t) \\
\hat{z}(t) &= E(t)\hat{x}(t) + \sum_{i=0}^h K_i(t+i)e(t+i) \\
e(t) &= y(t) - C(t)\hat{x}(t)
\end{aligned}$$

are the usual 1-step prediction formulae. Then, for a while, let's use the symbol $\hat{z}(t|i)$ to denote the optimal estimate of $z(t)$ based on observations $y(\cdot)$ up to time i . In Remark 12.3, we have seen that the optimal filter estimate and the optimal predictor are linked as follows:

$$\hat{z}(t|t) = \hat{z}(t|t-1) + D_F(t)e(t).$$

We now see that the optimal smoother with smoothing horizon h is given by

$$\hat{z}(t|t+h) = \hat{z}(t|t-1) + K_0(t)e(t) + K_1(t)e(t+1) + \cdots + K_h(t)e(t+h)$$

The additional terms $K_j(t)e(t+j)$, $j = 0, 1, \dots, h$, exploit the information contained in the current snapshot $y(t)$ and in the future snapshots $y(t+1)$, \dots , $y(t+h)$ to improve the estimate of $z(t)$. Note that, from (12.8c) and (12.30), $K_0(t)$ can be given the expression

$$K_0(t) = (E(t)P(t)E(t)' + N(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t'))^{-1}.$$

By comparison with Eq. (12.9b), it follows

$$K_0(t) = D_F(t).$$

Hence, the current snapshot appears with the same weighting coefficient in the filter and smoother formulas.

12.3.1 A Factorization View

Following the factorization approach worked out for the filtering and prediction problems, it is apparent that the optimal smoother is given by

$$F_{smo}(\sigma, t) = \sigma^h \left(\sigma^{-h} H_2(\sigma, t) H_1^{\sim}(\sigma, t) \Omega^{\sim}(\sigma, t)^{-1} \right)_{-} \Omega(\sigma, t)^{-1}. \quad (12.31)$$

The subsequent derivation relies on the same rationale previously used. To be precise, under the stability assumption and again omitting the dependence upon t , from (12.15) and (12.31) one obtains

$$\begin{aligned}
F_{smo}\Omega &= \sigma^h \left(\sigma^{-h} H_2 H_1^\sim (\Omega^\sim)^{-1} \right)_- \\
&= \sigma^h \left(\sigma^{-h} (N + E\Phi B)(D' + B'\Phi^\sim C')(\Omega^\sim)^{-1} \right)_- \\
&= \sigma^h \left(\sigma^{-h} [N(D' + B'\Phi^\sim C') + E\Phi(S_I - APC' + BB'\Phi^\sim C')] (\Omega^\sim)^{-1} \right)_- \\
&= E\Phi S_I W_I^{1/2} + \sigma^h \left(\sigma^{-h} ((ND' + EPC') + (NB' + EPA')\Phi^\sim C')(\Omega^\sim)^{-1} \right)_-.
\end{aligned}$$

Since

$$\Phi^\sim C' = \hat{\Phi}^\sim C' W_I^{1/2} \Omega^\sim,$$

it follows

$$\begin{aligned}
F_{smo}\Omega &= E\Phi S_I W_I^{1/2} \\
&\quad + \sigma^h \left(\sigma^{-h} (NB' + EPA') \hat{\Phi}^\sim C' W_I^{1/2} + (ND' + EPC')(\Omega^\sim)^{-1} \right)_- \\
&= E\Phi S_I W_I^{1/2} \\
&\quad + \sigma^h \left(\sigma^{-h} (NB' + EPA') \hat{\Phi}^\sim C' W_I^{1/2} \right)_- \\
&\quad + \sigma^h \left(\sigma^{-h} (ND' + EPC')(I - W_I S_I' \hat{\Phi}^\sim C') W_I^{1/2} \right)_- \\
&= (E\Phi S_I + ND' + EPC') W_I^{1/2} \\
&\quad + \sigma^h \left(\sigma^{-h} (NB' + EPA' - (ND' + EPC') W_I S_I') \hat{\Phi}^\sim C' W_I^{1/2} \right)_-.
\end{aligned}$$

Now, notice that

$$\hat{\Phi}^\sim(\sigma, t) = \sum_{i=0}^{\infty} \sigma^{i+1} \hat{\Phi}_A(t, t-i)'.$$

Therefore

$$\begin{aligned}
F_{smo}\Omega &= (E\Phi S_I + ND' + EPC') W_I^{1/2} \\
&\quad + (NB' + EPA' - (ND' + EPC') W_I S_I') \sum_{i=0}^{h-1} \sigma^{i+1} \hat{\Phi}(t, t-i) C' W_I^{1/2}.
\end{aligned}$$

We now introduce the gain matrices

$$K_0(t) = N(t)D(t)' + E(t)P(t)C(t)',$$

and, for $i = 1, 2, \dots, h$

$$\begin{aligned}
K_i(t+i) &= (E(t)P(t)A(t)' + N(t)B(t)') \hat{\Phi}(t+i, t+1)' C(t+i)' W_I(t+i) \\
&\quad - (N(t)D(t)' + E(t)P(t)C(t)') \hat{\Phi}(t+i, t+1)' C(t+i)' W_I(t+i).
\end{aligned}$$

Then, the optimal H_2 smoother turns out to be given by

$$F_{smo}(\sigma, t) = E(t)\hat{\Phi}(\sigma, t)S_I(t)W_I(t) + \sum_{i=0}^h \sigma^i K_i(t)(I - C(t)\hat{\Phi}(\sigma, t)S_I(t)W_I(t)). \quad (12.32)$$

As a matter of fact, the gain matrices previously defined and entering this expression do coincide with the smoother gains introduced in the state-space approach. Therefore, Expression (12.32), suitably realized in the state-space context, coincides with the already obtained formula given in Proposition 12.3.

Remark 12.7 All H_2 problems formulated in the state-space domain seen so far have been tackled under the assumption that $D(t)D(t)' > 0$, for each t . If this assumption does not hold, one can resort to the delay structure concept developed in Sect. 8.4 and to the associated generalization of the spectral factor seen in Remark 10.3.

12.4 Filtering and Deconvolution in the H_∞ Setting

In this section, we pursue the objective of minimizing the H_∞ norm of the transfer operator from the disturbance $w(\cdot)$ to the estimation error $\varepsilon(\cdot)$. More precisely, given $\gamma > 0$, the usual requirement is to render such a norm bounded from below by γ . We focus on filters in *observer form*

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)(y(t) - C(t)\hat{x}(t)) \quad (12.33a)$$

$$\hat{z}(t) = E(t)\hat{x}(t) + D_F(t)(y(t) - C(t)\hat{x}(t)) \quad (12.33b)$$

characterized by the two unknown matrices $B_F(\cdot)$ and $D_F(\cdot)$, only. In order to formulate the first existence result, we have to resort to the H_∞ analysis carried out in Sect. 9.2, and precisely, to the LMI formulation in Remark 9.2. Letting

$$A_F(t) = A(t) - B_F(t)C(t)$$

$$\hat{B}_F(t) = B(t) - B_F(t)D(t)$$

$$C_F(t) = E(t) - D_F(t)C(t)$$

$$\hat{D}_F(t) = N(t) - D_F(t)D(t),$$

the Riccati inequality for the error System (12.3) is

$$P(t+1) > A_F(t)P(t)A_F(t)' + \hat{B}_F(t)\hat{B}_F(t)' + Y_F(t)W_F(t)Y_F(t)' \quad (12.34a)$$

$$Y_F(t) = [A_F(t)P(t)C_F(t)' + \hat{B}_F(t)\hat{D}_F(t)'] W_F(t)^{-1} \quad (12.34b)$$

$$W_F(t) = \gamma^2 I - \hat{D}_F(t)\hat{D}_F(t)' - C_F(t)P(t)C_F(t)'. \quad (12.34c)$$

Hence, the problem is to find $B_F(\cdot)$ and $D_F(\cdot)$ such that (12.34) admits a T -periodic solution $P(\cdot)$ which is positive semi-definite and *feasible*, i.e.,

$$W_F(t) > 0, \quad \forall t. \quad (12.35)$$

Notice that the Inequality (12.34a) together with condition (12.35) can be set in a PLMI form:

$$\begin{bmatrix} P(t+1) - A_F(t)P(t)A_F(t)' - \hat{B}_F(t)\hat{B}_F(t)' & Y_F(t) \\ Y_F(t)' & W_F(t) \end{bmatrix} > 0.$$

At this point it is convenient to rewrite the Inequalities (12.34) in a way that puts in evidence the structure of the so-called *central solution*:

$$\begin{aligned} B_F^0(t) &= (A(t)P(t)C(t)' + B(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t)')^{-1} \\ D_F^0(t) &= (E(t)P(t)C(t)' + N(t)D(t)')(C(t)'P(t)C(t) + D(t)D(t)')^{-1}. \end{aligned}$$

Taking into account the definitions of $A_F(\cdot)$, $\hat{B}_F(\cdot)$, $C_F(\cdot)$, $\hat{D}_F(\cdot)$ and repeatedly using the matrix inversion lemma, cumbersome computations show that Inequalities (12.34) can be equivalently rewritten as

$$P(t+1) > A(t)P(t)A(t)' + B(t)B(t)' - S_J(t)W_J(t)S_J(t)' + E_F(t), \quad (12.36)$$

where

$$\begin{aligned} S_J(t) &= [A(t)P(t)C(t)' + B(t)D(t)' \quad A(t)P(t)E(t)' + B(t)N(t)'] \\ W_J(t) &= \begin{bmatrix} D(t)D(t)' + C(t)P(t)C(t)' & C(t)P(t)E(t)' + D(t)N(t)' \\ E(t)'P(t)C(t) + N(t)D(t)' & -\gamma^2 I + E(t)P(t)E(t)' + N(t)N(t)' \end{bmatrix}^{-1} \\ E_F(t) &= E_1(t)\Xi(t)^{-1}E_1(t)' \\ E_1(t) &= S_1(t)S_3(t)^{-1}(D_F^0(t) - D_F(t)) + B_F^0(t) - B_F(t) \\ \Xi(t) &= T_P(t)^{-1} - (D_F^0(t) - D_F(t))'S_3(t)^{-1}(D_F^0(t) - D_F(t)) \\ S_1(t) &= A(t)P(t)E(t)' + B(t)N(t)' - B_F^0(t)T_P(t)D_F^0(t)' \\ S_3(t) &= \gamma^2 I - E(t)P(t)E(t)' - N(t)N(t)' + D_F^0(t)T_P(t)D_F^0(t)' \\ T_P(t) &= C(t)P(t)C(t)' + D(t)D(t)' \\ W_F(t) &> 0. \end{aligned}$$

Notice that the two design matrices $B_F(\cdot)$ and $D_F(\cdot)$ enter Inequality (12.36) only through matrix $E_F(\cdot)$. On the other hand, every feasible solution is such that $E_F(\cdot) \geq 0$. Indeed, should $\Xi(\cdot)$ be invertible, then

$$\Xi(t)^{-1} = T_P(t) + T_P(t)(D_F^0(t) - D_F(t))'W_F(t)^{-1}(D_F^0(t) - D_F(t))T_P(t).$$

Therefore, the conditions $W_F(\cdot) > 0$ (feasibility), and $T_P(\cdot) > 0$ (inherited by $P(\cdot) \geq 0$ and $D(\cdot)D(\cdot)' > 0$) imply that $\Xi(\cdot) > 0$ so that $E_F(\cdot) \geq 0$. In conclusion, we have verified that the additional term $E_F(t)$ is positive semi-definite, for each feasible choice of $D_F(t)$ and $B_F(t)$.

Notice in passing that if $B_F(t) = B_F^o(t)$ and $D_F(t) = D_F^o(t)$, then $E_1(t) = 0$. Hence the central solution leads to $E_F(t) = 0$.

We are now in a position to give the solution of the H_∞ filtering problem. Here reference is made again to Fig. 12.1, where filtering corresponds to $k = 0$.

Proposition 12.4 *Assume that*

- (i) $D(t)D(t)' > 0, \forall t$.
- (i) The pair $(A(\cdot), C(\cdot))$ is detectable.
- (ii) The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle.

Then, there exist two T -periodic matrices $B_F(t)$ and $D_F(t)$ such that the T -periodic filter with transfer operator $F(\sigma, t)$ is stable and such that

$$\|F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma$$

if and only if the T -periodic Riccati inequality

$$\begin{aligned} P(t+1) &> A(t)P(t)A(t)' + B(t)B(t)' - S_J(t)W_J(t)S_J(t)' \\ S_J(t) &= [A(t)P(t)C(t)' + B(t)D(t)' \quad A(t)P(t)E(t)' + B(t)N(t)'] \\ W_J(t) &= \begin{bmatrix} D(t)D(t)' + C(t)P(t)C(t)' & C(t)P(t)E(t)' + D(t)N(t)' \\ E(t)'P(t)C(t) + N(t)D(t)' & -\gamma^2 + E(t)P(t)E(t)' + N(t)N(t)' \end{bmatrix}^{-1} \end{aligned}$$

admits a T -periodic positive semi-definite solution $P(\cdot)$ such that

$$S_3(t) = \gamma^2 I - E(t)P(t)E(t)' - N(t)N(t)' + D_F^o(t)T_P(t)D_F^o(t)' > 0.$$

In this case, a solution is given by $B_F(t) = B_F^o(t)$ and $D_F(t) = D_F^o(t)$.

Proof Consider a T -periodic positive semi-definite solution $P(\cdot)$ satisfying the inequality, and let $B_F(t) = B_F^o(t)$, $D_F(t) = D_F^o(t)$ be the filter matrices. With this choice it obviously results $E_F(\cdot) = 0$ so that $P(\cdot)$ satisfies (12.36). Moreover, under the same positions, one can also show that $W_F(\cdot) = S_3(\cdot)$. This means that the same matrix $P(\cdot)$ satisfies (12.34) as well. In view of the periodic Lyapunov Lemma, the filter matrix $A_F(\cdot)$ is stable. Moreover, in agreement with Remark 9.2 the chosen filter ensures an H_∞ norm less than γ .

Vice versa, if for some $B_F(\cdot)$ and $D_F(\cdot)$ the filter is stable and the norm is less than γ , then (12.34) are satisfied by some T -periodic $P(\cdot)$. Such a solution also solves inequality (12.36) being $E_F(\cdot)$ positive semi-definite. ■

The result above makes reference to possible H_∞ filters in the family defined by Eqs. (12.33) (observer form). However, its significance is more general: indeed it can be shown that the existence of a feasible and positive semi-definite solution of the Riccati inequality is a necessary and sufficient condition for the existence of a stable H_∞ filter without any a-priori restriction on its structure. A second observation concerns the possibility of finding the minimal solution (in the lattice of positive semi-definite

matrices) of the Riccati Inequality (12.36). A classical result of the Riccati equation theory is that the stabilizing solution is monotonic with respect to the known term. This means that the solution of Inequality (12.36) is “minimized” by taking $E_F(t) = 0$ and forcing the inequality into an equality. Taking into account the characterization of the entropy measure in terms of the Riccati equation (Sect. 9.3), this corresponds to the so-called minimum entropy solution. These observations lead to the following result, whose proof is only sketched.

Proposition 12.5 *Assume that*

- (i) $D(t)D(t)' > 0, \forall t$.
- (ii) *The pair $(A(\cdot), C(\cdot))$ is detectable.*
- (iii) *The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle.*

Then, there exists a causal stable T -periodic filter with transfer operator $F(\sigma, t)$ such that

$$\|F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma$$

if and only if the Riccati equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - S_J(t)W_J(t)S_J(t)', \quad (12.37)$$

where

$$\begin{aligned} S_J(t) &= [A(t)P(t)C(t)' + B(t)D(t)' \quad A(t)P(t)E(t)' + B(t)N(t)'] \\ W_J(t) &= \begin{bmatrix} D(t)D(t)' + C(t)P(t)C(t)' & C(t)P(t)E(t)' + D(t)N(t)' \\ E(t)'P(t)C(t) + N(t)D(t)' & -\gamma^2 I + E(t)P(t)E(t)' + N(t)N(t)' \end{bmatrix}^{-1} \end{aligned}$$

admits a T -periodic positive semi-definite satisfying the following two properties:

(a) *(Stabilizing property)*

$$\hat{A}_J(t) = A(t) - S_J(t)W_J(t) \begin{bmatrix} C(t) \\ E(t) \end{bmatrix}$$

is stable.

(b) *(Feasibility property)*

$$\begin{aligned} S_3(t) &= \gamma^2 I - E(t)P(t)E(t)' - N(t)N(t)' \\ &\quad + (E(t)P(t)C(t)' + N(t)D(t)') \times \\ &\quad \times (D(t)D(t)' + C(t)P(t)C(t))^{-1} (C(t)'P(t)E(t)' + D(t)N(t)') \end{aligned} \quad (12.38)$$

is positive definite.

In this case a filter is given by (12.33) with

$$\begin{aligned} B_F(t) &= (A(t)P(t)C(t)' + B(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t)')^{-1} \\ D_F(t) &= (E(t)P(t)C(t)' + N(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t)')^{-1}. \end{aligned}$$

Proof Assume that a T -periodic positive semi-definite stabilizing and feasible solution $P(\cdot)$ exists. Then, with the given choice of $B_F(\cdot)$ and $D_F(\cdot)$, the Riccati equation can be written as

$$\begin{aligned} P(t+1) &= A_F(t)P(t)A_F(t)' + \hat{B}_F(t)\hat{B}_F(t)' + Y_F(t)W_F(t)Y_F(t)' \\ Y_F(t) &= [A_F(t)P(t)C_F(t)' + \hat{B}_F(t)\hat{D}_F(t)'] W_F(t)^{-1} \\ W_F(t) &= \gamma^2 I - \hat{D}_F(t)\hat{D}_F(t)' - C_F(t)P(t)C_F(t)', \end{aligned}$$

with

$$\begin{aligned} A_F(t) &= A(t) - B_F(t)C(t) \\ \hat{B}_F(t) &= B(t) - B_F(t)D(t) \\ C_F(t) &= E(t) - D_F(t)C(t) \\ \hat{D}_F(t) &= N(t) - D_F(t)D(t). \end{aligned}$$

Obviously, $A_F(\cdot)$ is stable and $W_F(\cdot)$ is positive definite. Hence, in view of the (dual version of) periodic real lemma (Proposition (9.1)), the filter is stable and the norm of the error system is less than γ . Notice in passing that the stabilizing solution of the Riccati equation is unique (the proof is omitted). Conversely, if such a filter exists, one can prove the existence of a T -periodic solution $P(\cdot)$ with the given properties. A direct proof of this part of the statement is too lengthy and hence omitted. Notice, however, that an indirect proof can be worked out by resorting to the isomorphism between periodic systems and time-invariant systems via time-invariant reformulations.

Remark 12.8 *For a sufficiently high value of the attenuation level γ , the H_∞ filtering problem admits infinitely many solutions. The set of all solutions can be described starting from the stabilizing solution of the Riccati Equation (12.37) via a free parameter. The filter provided in Proposition 12.5 is the so-called central H_∞ filter, since it corresponds to a specific choice of the free parameter. A second observation concerns the entropy of the error system associated with the central filter. As a matter of fact, the central filter was found by minimizing the solution of the Riccati equation, and hence, in view of the characterization given in Sect. 9.3, such a filter minimizes the γ entropy of the error system, and is therefore referred to as minimum entropy filter. ■*

The Riccati equation in the form (12.37) is rather complex. However, it may be given alternative formulations, that better evidenciate the interplay between the measurements (represented by matrix $C(\cdot)$) and the variable to be estimated (represented by matrix $E(\cdot)$). By using the definition of matrices $T_P(t)$, $S_1(t)$, $S_3(t)$ and $B_F^o(t)$ above, one formulation is

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - B_F^o(t)T_P(t)B_F^o(t)' + S_1(t)S_3(t)^{-1}S_1(t)',$$

with the feasibility condition $S_3(\cdot) > 0$. This last expression has the distinctive feature of evidentiating the two terms with opposite sign having $T_P(t)$ and $S_3(t)^{-1}$ as kernel matrices.

Moreover, under some further assumptions, the Riccati equation takes on a particularly expressive form. To be precise, if

$$D(t)D(t)' = I, \quad D(t)B(t)' = 0, \quad N(t) = 0, \quad (12.39)$$

then the equation reduces to

$$P(t+1) = A(t) [P(t)^{-1} + C(t)'C(t) - \gamma^{-2}E(t)'E(t)]^{-1} A(t)' + B(t)B(t)'$$

and the feasibility condition becomes

$$P(t)^{-1} + C(t)'C(t) - \gamma^{-2}E(t)'E(t) > 0.$$

The gain matrices of the *central H_∞ filter* become

$$B_F(t) = A(t)K_0(t) \quad (12.40a)$$

$$D_F(t) = L(t)K_0(t) \quad (12.40b)$$

$$K_0(t) = P(t)C(t)' (I + C(t)P(t)C(t)')^{-1}. \quad (12.40c)$$

Notice that Assumptions (12.39) are quite standard, since refer to the pure filtering problem for full row rank systems with orthogonal disturbances in the state and output dynamics. Of course, in order to the Riccati equation above make sense, one has to assume that the T -periodic solution $P(\cdot)$ be invertible. In this regard, it is not difficult to prove that if $(A(\cdot), B(\cdot))$ is reachable at each time-instant, then any feasible T -periodic solution $P(\cdot)$ is invertible.

Remark 12.9 (Difference game)

The H_∞ filtering problem has a natural interpretation in terms of a difference game between “man and nature”, in the sense that the designer (“man”) aims at minimizing a quadratic performance index, as in the usual optimal H_2 filtering problem, whereas “nature” (disturbances and initial state) tends to maximize it. This game is well formalized in terms of a quadratic sign-indefinite performance index constrained by the difference equations of the underlying dynamical system. To be precise, consider a finite horizon $[\tau+1, \tau+N]$, where τ is an initial time-instant, and an a priori estimate $\hat{x}(\tau)$ of the initial state $x(\tau)$. The finite horizon H_∞ filtering problem consists in finding a linear causal filter based on the measurements $y(\tau+1), y(\tau+2), \dots, y(\tau+N)$ yielding an estimate $\hat{z}(\tau+i)$, $i = 1, 2, \dots, N$, which guarantees an attenuation level $\gamma > 0$, i.e., such that, for any $(w(\cdot), x(\tau) - \hat{x}(\tau)) \neq 0$, $J_\infty < 0$ where

$$J_{\infty} = \sum_{t=\tau+1}^{\tau+N} \|z(t) - \hat{z}(t)\|^2 - \gamma^2 \left[(x(\tau) - \hat{x}(\tau))' Z (x(\tau) - \hat{x}(\tau)) + \sum_{t=\tau+1}^{\tau+N} \|w(t)\|^2 \right].$$

The matrix $Z = Z' > 0$ in the above formula is a given weighting matrix, which plays a role similar to the one of the inverse of the initial state-covariance in standard Kalman filtering.

For simplicity, we make the assumptions that $D(t)B(t)' = 0$, $N(t) = 0$ (pure filtering) and $D(t)D(t)' = I$. Under such assumptions, it can be proven that the above problem has a solution if and only if there exist two sequences of matrices $P(t)$ and $S(t)$, $t = \tau, \tau+1, \dots, \tau+N$, positive definite at each time point, satisfying the matrix recursions

$$\begin{aligned} P(t+1) &= A(t)S(t)^{-1}A(t)' + B(t)B(t)' \\ S(t) &= P(t)^{-1} + C(t)'C(t) - \gamma^{-2}E(t)'E(t), \end{aligned}$$

with initial condition $S(0) = Z$. In this case, a filter guaranteeing the prescribed attenuation level is

$$\begin{aligned} \xi(t+1) &= A(t)\xi(t) + K_0(t+1)(y(t+1) - C(t+1)A(t)\xi(t)) \\ \hat{z}(t) &= E(t)\xi(t+1), \end{aligned}$$

with initial state $\xi(\tau) = \hat{x}(\tau)$ and

$$K_0(t) = P(t)C(t)' [I + C(t)P(t)C(t)]^{-1}.$$

Alternatively, the filter can be written as

$$\begin{aligned} \hat{x}(t+1) &= A(t)\hat{x}(t) + A(t)K_0(t)(y(t) - C(t)\hat{x}(t)) \\ \hat{z}(t) &= E(t)\hat{x}(t) + E(t)K_0(t)(y(t) - C(t)\hat{x}(t)), \end{aligned}$$

which coincides with (12.33) through (12.40). The proof of this statement can be worked out on the basis of the error equation (12.3), along the same lines used to derive the conclusion of Remark 9.4. Notice that the existence of the filter calls, at each step, for the fulfillment of the matrix inequality $S(t) > 0$ (feasibility condition). Furthermore, the convergence of the solution $P(\cdot)$ to the T -periodic solution is not guaranteed by the standard structural assumptions on the underlying system. In other words, the transient behavior and the convergence properties strongly depend on the choice of the weighting matrix Z and the attenuation level γ .

Remark 12.10 (H_2 vs H_{∞})

As apparent, there is a major difference between the solution of the H_2 filtering problem and the H_{∞} filtering problem. As a matter of fact, the solution of the former does not depend on the “deconvolution matrix” $N(t)$, whereas the solution of the latter does. This is not surprising since in H_{∞} the noise acts as a player in the

difference game operating against the designer choices. Of course if γ increases, the H_2 filter is eventually recovered.

Remark 12.11 (A factorization view)

The H_∞ filtering problem can be given an interesting interpretation in terms of J -spectral factorization, discussed in Sect. 10.3. The problem is to find a causal stable T -periodic filter $F(\sigma, t)$ such that $F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)$ is stable with norm less than γ . By letting

$$H(\sigma, t) = \begin{bmatrix} H_1(\sigma, t) & 0 \\ H_2(\sigma, t) & I \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} < 0,$$

the above condition can be written as

$$\begin{bmatrix} F(\sigma, t) & -I \end{bmatrix} H(\sigma, t) J H^\sim(\sigma, t) \begin{bmatrix} F^\sim(\sigma, t) \\ -I \end{bmatrix}.$$

As discussed in Sect. 10.3, the J -spectral factorization problem consists in finding a square bi-causal transfer operator $\Omega(\sigma, t)$, such that

$$H(\sigma, t) J H^\sim(\sigma, t) = \Omega(\sigma, t) J \Omega^\sim(\sigma, t) \quad (12.41a)$$

$$\Omega(\sigma, t) \text{ is stable} \quad (12.41b)$$

$$\Omega(\sigma, t)^{-1} \text{ is stable} \quad (12.41c)$$

$$\Omega_{11}(\sigma, t)^{-1} \text{ is stable}, \quad (12.41d)$$

where $\Omega_{11}(\sigma, t)^{-1}$ represents the $(1,1)$ -block entry of $\Omega(\sigma, t)$. We limit the discussion to the case $H(\sigma, t)$ stable. Notice, however, that it is possible to equip the original system with a preliminary output injection in order to match this stability requirement. The relation between the J -spectral factorization theory and the H_∞ filtering theory can then be posed as follows: there exists a stable filter such that $F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)$ is stable and its norm is less than γ if and only if there exists a J -spectral factor $\Omega(\sigma, t)$ satisfying (12.41). In one direction, this statement is straightforward in light of the results in Sect. 10.3 and Proposition 12.5. In this remark we adopt the symbols used in this last statement. Notice that the Riccati equation (10.14) coincides with (12.37) so that, in view of (10.15), if a T -periodic solution $P(\cdot)$ exists, a J -spectral factor is

$$\begin{aligned} \Omega(\sigma, t) &= \left[W_J(t)^{-1} + \begin{bmatrix} C(t) \\ E(t) \end{bmatrix} (\sigma I - A(t))^{-1} S_J(t) \right] \Gamma(t) \\ \Gamma(t) &= \begin{bmatrix} T_P(t)^{1/2} & T_P(t)^{-1/2} (C(t)P(t)E(t)' + D(t)N(t)') \\ 0 & -\gamma S_3(t)^{1/2} \end{bmatrix}^{-1}, \end{aligned}$$

where again $T_P(t) = C(t)P(t)C(t)' + D(t)D(t)'$. Therefore, if a filter with the given properties exists, then, in view of Proposition 12.5, there exists a positive semi-definite T -periodic solution $P(\cdot)$, which is feasible and stabilizing. Hence, $\Omega(\sigma, t)$

can be constructed as above. Notice that $\Omega(\sigma, t)$ is stable (in view of the assumption that $A(\cdot)$ is stable). The dynamic matrix of $\Omega(\sigma, t)^{-1}$ is $\hat{A}_J(\cdot)$, which is stable and the dynamic matrix of $\Omega_{11}(\sigma, t)^{-1}$ is $A_F(\cdot)$, which is stable as well. Hence, $\Omega(\sigma, t)$ satisfies all requirements expressed in (12.41). We miss to show how to construct a filter out of a J -spectral factor. This will also allow for a parametrization of all filters with the required properties. As a matter of fact, if $\Omega_{ij}(\sigma, t)$, $i, j = 1, 2$, denotes the (i, j) block entry of $\Omega(\sigma, t)$, then the class of all stable filters $F(\sigma, t)$ such that

$$\begin{aligned} F(\sigma, t) \text{ is stable} \\ F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t) \text{ is stable} \\ \|F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma \end{aligned}$$

is given by

$$F(\sigma, t) = (\Omega_{21}(\sigma, t) - \Omega_{22}(\sigma, t)U(\sigma, t))(\Omega_{11}(\sigma, t) - \Omega_{12}(\sigma, t)U(\sigma, t))^{-1}, \quad (12.42)$$

where $U(\sigma, t)$ is the transfer operator of any stable T -periodic system with

$$\|U(\sigma, t)\|_\infty < \gamma.$$

Expression (12.42) can be derived from the small gain theorem via some lengthy computations.

12.5 H_∞ 1-step Prediction

In this section, we tackle the problem of 1-step prediction in the H_∞ setting. To start with, we again assume that the estimator (in this case the predictor) takes the observer form

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_P(t)(C(t)\hat{x}(t) - y(t)) \quad (12.43a)$$

$$\hat{z}(t) = E(t)\hat{x}(t). \quad (12.43b)$$

The form of the predictor coincides with that of the filter given in (12.33), with $D_F(t) = 0$. In this way, the estimator is constrained to be strictly proper and is at all effect a 1-step predictor. The only unknown is the T -periodic gain matrix $B_P(t)$.

By letting

$$\begin{aligned} A_P(t) &= A(t) - B_P(t)C(t) \\ \hat{B}_P(t) &= B(t) - B_P(t)D(t), \end{aligned}$$

the error system becomes

$$\begin{aligned} \eta(t+1) &= A_P(t)\eta(t) + \hat{B}_P(t)w(t) \\ \varepsilon(t) &= E(t)\eta(t) + N(t)w(t), \end{aligned}$$

and the Riccati inequality for such error system is

$$P(t+1) > A_P(t)P(t)A_P(t)' + \hat{B}_P(t)\hat{B}_P(t)' + Y_P(t)W_P(t)Y_P(t)' \quad (12.44a)$$

$$Y_P(t) = [A_P(t)P(t)E(t)' + \hat{B}_P(t)N(t)'] W_P(t)^{-1} \quad (12.44b)$$

$$W_P(t) = \gamma^2 I - N(t)N(t)' - E(t)P(t)E(t)'. \quad (12.44c)$$

Hence, the problem is to find $B_P(\cdot)$ such that (12.44) admits a T -periodic solution $P(\cdot)$ which is positive semi-definite and *feasible* for each t , i.e.,:

$$W_P(t) > 0.$$

Now, we adopt the same rationale used in Sect. 12.4. This amounts to consider again matrices

$$\begin{aligned} B_P^0(t) &= (A(t)P(t)C(t)' + B(t)D(t)')(C(t)P(t)C(t)' + D(t)D(t)')^{-1} \\ D_P^0(t) &= (E(t)P(t)C(t)' + N(t)D(t)')(C(t)'P(t)C(t) + D(t)D(t)')^{-1}, \end{aligned}$$

and rewrite (12.44) as

$$P(t+1) > A(t)P(t)A(t)' + B(t)B(t)' - S_J(t)W_J(t)S_J(t)' + E_P(t), \quad (12.45)$$

where

$$\begin{aligned} S_J(t) &= [A(t)P(t)C(t)' + B(t)D(t)' \quad A(t)P(t)E(t)' + B(t)N(t)'] \\ W_J(t) &= \begin{bmatrix} D(t)D(t)' + C(t)P(t)C(t)' & C(t)P(t)E(t)' + D(t)N(t)' \\ E(t)'P(t)C(t) + N(t)D(t)' & -\gamma^2 I + E(t)P(t)E(t)' + N(t)N(t)' \end{bmatrix}^{-1} \\ E_P(t) &= E_1(t)\Xi(t)^{-1}E_1(t)' \\ E_1(t) &= S_1(t)S_3(t)^{-1}D_P^0(t) + B_P^0(t) - B_P(t) \\ \Xi(t) &= T_P(t)^{-1} - D_P^0(t)'S_3(t)^{-1}D_P^0(t) \\ S_1(t) &= A(t)P(t)E(t)' + B(t)N(t)' - B_P^0(t)T_P(t)D_P^0(t)' \\ S_3(t) &= \gamma^2 I - E(t)P(t)E(t)' - N(t)N(t)' + D_P^0(t)T_P(t)D_P^0(t)' \\ T_P(t) &= C(t)P(t)C(t)' + D(t)D(t)' \\ W_P(t) &> 0. \end{aligned}$$

Again it is possible to verify that $E_P(t)$ is positive semi-definite for any feasible choice of $B_P(t)$. Moreover, $E_P(t)$ can be zeroed by taking $E_1(t) = 0$, which means

$$B_P(t) = B_P^0(t) + S_1(t)S_3(t)^{-1}D_P^0(t).$$

We are now in a position to state the following result.

Proposition 12.6 *Assume that*

- (i) $D(t)D(t)' > 0, \forall t$.
- (ii) *The pair $(A(\cdot), C(\cdot))$ is detectable.*

(iii) The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle.

Then, there exists a sequence of T -periodic matrices $B_P(t)$ such that the T -periodic 1-step predictor with transfer operator $F(\sigma, t)$ is stable and

$$\|F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma$$

if and only if the T -periodic Riccati inequality

$$\begin{aligned} P(t+1) &> A(t)P(t)A(t)' + B(t)B(t)' - S_J(t)W_J(t)S_J(t)' \\ S_J(t) &= [A(t)P(t)C(t)' + B(t)D(t)' \quad A(t)P(t)E(t)' + B(t)N(t)'] \\ W_J(t) &= \begin{bmatrix} D(t)D(t)' + C(t)P(t)C(t)' & C(t)P(t)E(t)' + D(t)N(t)' \\ E(t)'P(t)C(t) + N(t)D(t)' & -\gamma^2 + E(t)P(t)E(t)' + N(t)N(t)' \end{bmatrix}^{-1} \end{aligned}$$

admits a T -periodic positive semi-definite solution $P(\cdot)$ such that

$$W_P(t) = \gamma^2 I - N(t)N(t)' - E(t)P(t)E(t)' > 0.$$

In this case, a solution is given by

$$B_P(t) = B_P^o(t) + S_1(t)S_3(t)^{-1}D_P^o(t).$$

Proof If there exists a T -periodic positive semi-definite $P(\cdot)$ satisfying the inequality, then the same $P(\cdot)$ satisfies (12.44) with $B_P(t) = B_P^o(t) + S_1(t)S_3(t)^{-1}D_P^o(t)$. In view of the periodic Lyapunov Lemma, the filter matrix $A_P(\cdot)$ is stable. Moreover, the predictor ensures an H_∞ norm less than γ . Vice versa, if for some $B_P(\cdot)$ the predictor is stable and the norm is less than γ , then (12.44) are satisfied by some T -periodic $P(\cdot)$. Such $P(\cdot)$ satisfies Inequality (12.45) too. The conclusion follows from the fact that $E_P(\cdot)$ is positive semi-definite. ■

As is done for the filtering problem, it is also possible to state a necessary and sufficient condition based on a Riccati equation rather than a Riccati inequality.

Proposition 12.7 Assume that

- (i) $D(t)D(t)' > 0, \forall t$.
- (ii) The pair $(A(\cdot), C(\cdot))$ is detectable.
- (iii) The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle.

Then, there exists a causal stable T -periodic 1-step predictor with transfer operator $F(\sigma, t)$ such that

$$\|F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma$$

if and only if the Riccati equation

$$P(t+1) = A(t)P(t)A(t)' + B(t)B(t)' - S_J(t)W_J(t)S_J(t)',$$

where

$$S_J(t) = [A(t)P(t)C(t)' + B(t)D(t)' \quad A(t)P(t)E(t)' + B(t)N(t)']$$

$$W_J(t) = \begin{bmatrix} D(t)D(t)' + C(t)P(t)C(t)' & C(t)P(t)E(t)' + D(t)N(t)' \\ E(t)'P(t)C(t) + N(t)D(t)' & -\gamma^2 I + E(t)P(t)E(t)' + N(t)N(t)' \end{bmatrix}^{-1}$$

admits a T -periodic positive semi-definite satisfying the following two properties:

(a) (Stabilizing property)

$$\hat{A}_J(t) = A(t) - S_J(t)W_J(t) \begin{bmatrix} C(t) \\ E(t) \end{bmatrix}$$

is stable.

(b) (Feasibility property)

$$W_P(t) = \gamma^2 I - E(t)P(t)E(t)' - N(t)N(t)'$$

is positive definite.

In this case a 1-step H_∞ predictor is given by (12.43) with

$$B_P(t) = B_P^0(t) + S_1(t)S_3(t)^{-1}D_P^0(t).$$

Remark 12.12 (Feasibility conditions)

The feasibility condition for the predictor, i.e., $W_P(\cdot) > 0$, is more restrictive than the feasibility condition of the filter, i.e., $W_F(\cdot) > 0$. Indeed, these two matrices are related each other by $W_F(t) = W_P(t) + D_F^0(t)T_P(t)D_F^0(t)'$. This means that the existence of a 1-step H_∞ predictor for a given γ entails also the existence of a filter for the same attenuation level γ , but the converse is not true in general. ■

The treatment of the prediction problem in H_∞ for a general number of steps is rather complex and goes beyond the scope of this volume.

12.6 H_∞ Smoothing

We end this chapter devoted to the estimation problems in H_∞ by discussing the smoothing problem. In analogy with the H_2 smoothing (Sect. 12.3), we will assume available the measurements $y(\cdot)$ up to time $t + h$ and we will denote as $z(\cdot)$ as the remote variable to be estimated at time t .

A natural possibility to face the smoothing problem is to formulate it as a filtering problem for a system with augmented dimension of the state vector. Following the derivation in Sect. 12.3, we focus first on the 1-step smoothing problem ($h = 1$). The generalization to higher lags will follow.

For $h = 1$ the augmented system can be defined by introducing the augmented state:

$$x_a(t) = \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix}, \quad \zeta(t) = z(t-1).$$

The corresponding system is:

$$x_a(t+1) = \begin{bmatrix} A(t) & 0 \\ E(t) & 0 \end{bmatrix} x_a(t) + \begin{bmatrix} B(t) \\ N(t) \end{bmatrix} w(t) \quad (12.46a)$$

$$y(t) = \begin{bmatrix} C(t) & 0 \end{bmatrix} x_a(t) + D(t)w(t) \quad (12.46b)$$

$$z(t-1) = \begin{bmatrix} 0 & I \end{bmatrix} x_a(t). \quad (12.46c)$$

Let

$$\begin{aligned} \bar{A}(t) &= \begin{bmatrix} A(t) & 0 \\ E(t) & 0 \end{bmatrix}, & \bar{B}(t) &= \begin{bmatrix} B(t) \\ N(t) \end{bmatrix} \\ \bar{C}(t) &= \begin{bmatrix} C(t) & 0 \end{bmatrix}, & \bar{E}(t) &= \begin{bmatrix} 0 & I \end{bmatrix}. \end{aligned}$$

Then, the Riccati equation associated with the filtering problem for the augmented system is

$$\bar{P}(t+1) = \bar{A}(t)\bar{P}(t)\bar{A}(t)' + \bar{B}(t)\bar{B}(t)' - \bar{S}_J(t)\bar{W}_J(t)\bar{S}_J(t)' \quad (12.47a)$$

$$\bar{S}_J(t) = \begin{bmatrix} \bar{A}(t)\bar{P}(t)\bar{C}(t)' + \bar{B}(t)D(t)' & \bar{A}(t)\bar{P}(t)\bar{E}(t)' \end{bmatrix} \quad (12.47b)$$

$$\bar{W}_J(t) = \begin{bmatrix} D(t)D(t)' + \bar{C}(t)\bar{P}(t)\bar{C}(t)' & \bar{C}(t)\bar{P}(t)\bar{E}(t)' \\ \bar{E}(t)'P(t)C(t) & -\gamma^2 I + \bar{E}(t)P(t)\bar{E}(t)' \end{bmatrix}^{-1} \quad (12.47c)$$

This solution can be split as follows

$$\bar{P}(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t)' & P_{22}(t) \end{bmatrix}.$$

Lemma 12.1 *Let $\bar{P}(\cdot)$ be the T -periodic stabilizing solution of the H_∞ filtering Riccati equation associated with the augmented system and assume that $\gamma^2 I - P_{22}(t)$ and $T_P(t) = D(t)D(t)' + C(t)P(t)C(t)'$ are invertible for each t . Moreover, let*

$$P(t) = P_{11}(t) + P_{12}(t) (\gamma^2 I - P_{22}(t))^{-1} P_{12}(t)'.$$

Then, $P(\cdot)$ is the T -periodic stabilizing solution of (12.37) and

$$P_{11}(t+1) = A(t)P(t)A(t)' + B(t)B(t)' \quad (12.48a)$$

$$- (B(t)D(t)' + A(t)P(t)C(t)') T_P(t)^{-1} (B(t)D(t)' + A(t)P(t)C(t)')'$$

$$P_{12}(t+1) = A(t)P(t)E(t)' + B(t)N(t)' \quad (12.48b)$$

$$- (B(t)D(t)' + A(t)P(t)C(t)') T_P(t)^{-1} (N(t)D(t)' + E(t)P(t)C(t)')'$$

$$P_{22}(t+1) = E(t)P(t)E(t)' + N(t)N(t)' \quad (12.48c)$$

$$- (N(t)D(t)' + E(t)P(t)C(t)') T_P(t)^{-1} (N(t)D(t)' + E(t)P(t)C(t)')'.$$

Moreover, $\bar{A}_{sl}(\cdot)$ is stable if and only if $A_{sl}(\cdot)$ is stable, where

$$\bar{A}_{sl}(t) = \bar{A}(t) - (\bar{A}(t)\bar{P}(t)\bar{C}(t)' + \bar{B}(t)D(t)') T_{\bar{P}}(t)^{-1} \bar{C}(t) \quad (12.49a)$$

$$A_{sl}(t) = A(t) - (A(t)P_{11}(t)C(t)' + B(t)D(t)') T_{P_{11}}(t)^{-1} C(t), \quad (12.49b)$$

with

$$T_{\bar{P}}(t) = \bar{C}(t)\bar{P}(t)\bar{C}(t)' + D(t)D(t)'$$

$$T_{P_{11}}(t) = C(t)P_{11}(t)C(t)' + D(t)D(t)'.$$

Proof First of all, define

$$\hat{C}(t) = \begin{bmatrix} C(t) & 0 \\ 0 & I \end{bmatrix}, \quad \bar{D}(t) = \begin{bmatrix} D(t) \\ 0 \end{bmatrix}, \quad \bar{J}(t) = \begin{bmatrix} D(t)D(t)' & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

Then, $\bar{P}(t+1)$ is the T -periodic solution of the pencil

$$\begin{bmatrix} \bar{A}(t)' & 0 & \hat{C}(t)' \\ -\bar{B}(t)\bar{B}(t)' & I & -\bar{B}(t)D(t)' \\ \bar{D}(t)\bar{B}(t)' & 0 & J(t) \end{bmatrix} \begin{bmatrix} I \\ \bar{P}(t+1) \\ X(t+1) \end{bmatrix} =$$

$$= \begin{bmatrix} I & 0 & 0 \\ 0 & \bar{A}(t) & 0 \\ 0 & -\hat{C}(t) & 0 \end{bmatrix} \begin{bmatrix} I \\ \bar{P}(t) \\ X(t) \end{bmatrix} \hat{A}(t),$$

where $\hat{A}(\cdot)$ is a suitable stable T -periodic matrix and $X(\cdot)$ is a suitable T -periodic matrix. Notice that:

$$\begin{bmatrix} 0 & I \end{bmatrix} X(t+1) = \begin{bmatrix} 0 & I \end{bmatrix} \hat{A}(t).$$

Moreover,

$$\gamma^2 \begin{bmatrix} 0 & I \end{bmatrix} X(t+1) = \begin{bmatrix} 0 & I \end{bmatrix} \bar{P}(t) \hat{A}(t)$$

$$= P_{21}(t) \begin{bmatrix} I & 0 \end{bmatrix} \hat{A}(t) + P_{22}(t) \begin{bmatrix} 0 & I \end{bmatrix} X(t+1),$$

so that

$$\begin{bmatrix} 0 & I \end{bmatrix} X(t+1) = (\gamma^2 I - P_{22}(t))^{-1} P_{21}(t) \begin{bmatrix} I & 0 \end{bmatrix} \hat{A}(t).$$

Hence,

$$\begin{aligned}
 [I \ 0] \bar{P}(t) \hat{A}(t) &= P_{11}(t) [I \ 0] \hat{A}(t) + P_{12}(t) [0 \ I] \hat{A}(t) \\
 &= P_{11}(t) [I \ 0] \hat{A}(t) + P_{12}(t) [0 \ I] X(t+1) \\
 &= (P_{11}(t) + P_{12}(t)(\gamma^2 I - P_{22}(t))^{-1} P_{21}(t)) [I \ 0] \hat{A}(t) \\
 &= P(t) [I \ 0] \hat{A}(t).
 \end{aligned}$$

From the last block row equation of the pencil it follows

$$\begin{aligned}
 D(t)' [B(t)' N(t)'] + D(t)D(t)' [I \ 0] X(t+1) &= \\
 &= -[I \ 0] \bar{C}(t) \bar{P}(t) \hat{A}(t)' \\
 &= -C(t) [I \ 0] \bar{P}(t) \hat{A}(t) \\
 &= C(t)P(t) [I \ 0] \hat{A}(t) \\
 &= -C(t)P(t) [A(t)' E(t)'] - C(t)P(t)C(t)' [I \ 0] X(t+1),
 \end{aligned}$$

so that

$$\begin{aligned}
 [I \ 0] X(t+1) &= \\
 &= -T_P(t)^{-1} (C(t)P(t) [A(t)' E(t)'] + D(t)' [B(t)' N(t)']) .
 \end{aligned}$$

Finally, from the second block row equation

$$\begin{aligned}
 -\bar{B}(t)\bar{B}(t)' + \bar{P}(t+1) - \bar{B}(t)D(t)' [I \ 0] X(t+1) &= \\
 &= \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} [I \ 0] \bar{P}(t) \hat{A}(t) = \\
 &= \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t) [I \ 0] \hat{A}(t) = \\
 &= \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t) [A(t)' E(t)'] + \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t)C(t)' [I \ 0] X(t+1).
 \end{aligned}$$

This means that

$$\begin{aligned}
 \bar{P}(t+1) &= \begin{bmatrix} B(t) \\ N(t) \end{bmatrix} [B(t)' N(t)'] + \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t) [A(t)' E(t)'] \\
 &\quad + \left(\begin{bmatrix} B(t) \\ N(t) \end{bmatrix} D(t)' + \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t)C(t)' \right) [I \ 0] X(t+1).
 \end{aligned}$$

Taking into account the above expression for $[I \ 0]X(t+1)$, the link between $P(t)$ and $\bar{P}(t)$ turns out to be

$$\begin{aligned} \bar{P}(t+1) = & \begin{bmatrix} B(t) \\ N(t) \end{bmatrix} [B(t)' N(t)'] + \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t) [A(t)' E(t)'] \\ & - \left(\begin{bmatrix} B(t) \\ N(t) \end{bmatrix} D(t)' + \begin{bmatrix} A(t) \\ E(t) \end{bmatrix} P(t) C(t)' \right) T_P(t)^{-1} \\ & \times (C(t)P(t) [A(t)' E(t)'] + D(t)' [B(t)' N(t)']) . \end{aligned}$$

Therefore, the formulae (12.48) for $P_{ij}(t+1)$ eventually follow.

The proof that $P(t)$ satisfies the n -dimensional Riccati equation is a matter of matrix manipulations and hence is omitted. The fact that this solution is stabilizing follows from the stabilizing property of $\bar{P}(t)$. Finally, the relation between $A_{sl}(t)$ and $\bar{A}_{sl}(t)$ is

$$\bar{A}_{sl}(t) = \begin{bmatrix} A_{sl}(t) & 0 \\ \star & 0 \end{bmatrix},$$

so that also the last conclusion follows. ■

From (12.38) and (12.48c) it is clear that the feasibility condition for the H_∞ filtering problem can be equivalently rewritten as

$$\gamma^2 I - P_{22}(t) > 0.$$

Moreover, the feasibility condition for the 1-step smoother can be easily derived from the augmented Riccati Equation (12.47a) and from (12.38) by substituting $\bar{P}(t)$ in place of $P(t)$,

$$\gamma^2 I - P_{22}(t) + P_{12}(t)' C(t)' (D(t) D(t)' + C(t) P_{11}(t) C(t)')^{-1} C(t)' P_{12}(t) > 0.$$

Matrices $P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$ are functions of $P(t)$, given by expressions (12.48). By comparing the two feasibility conditions, it is apparent that, given a value of the attenuation level γ , if the filtering problem has a solution, then the smoothing problem has a solution as well. The converse is not true in general.

We are now in a position to provide the result for the 1-step smoother.

Proposition 12.8 *Assume that*

- (i) $D(t) D(t)' > 0, \forall t$.
- (ii) The pair $(A(\cdot), C(\cdot))$ is detectable.
- (iii) The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle.

Then, there exists a 1-step periodic smoother $F(\sigma, t)$ such that

$$\|F(\sigma, t) H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma$$

if and only if the T -periodic Riccati Equation (12.37) admits a T -periodic positive semi-definite solution $P(\cdot)$ such that

$$\gamma^2 I - P_{22}(t) + P_{12}(t)' C(t)' (D(t) D(t)' + C(t) P_{11}(t) C(t)')^{-1} C(t)' P_{12}(t) > 0,$$

and $A_{sl}(\cdot)$, defined in (12.49a) is stable. In this case a solution is given by

$$\begin{aligned}\hat{x}(t+1) &= A(t)\hat{x}(t) + B_S(t)e(t) \\ \hat{z}(t) &= E(t)\hat{x}(t) + D_{S0}(t)e(t) + D_{S1}(t+1)e(t+1) \\ e(t) &= y(t) - C(t)\hat{x}(t),\end{aligned}$$

where

$$\begin{aligned}B_S(t) &= (A(t)P_{11}(t)C(t)' + B(t)D(t)')(C(t)P_{11}(t)C(t)' + D(t)D(t)')^{-1} \\ D_{S0}(t) &= (E(t)P_{11}(t)C(t)' + N(t)D(t)')(C(t)P_{11}(t)C(t)' + D(t)D(t)')^{-1} \\ D_{S1}(t) &= P_{12}(t)'C(t)'(C(t)P_{11}(t)C(t)' + D(t)D(t)')^{-1}.\end{aligned}$$

Proof The proof can be derived from the main result on filtering stated in Proposition 12.5 applied to the augmented System (12.46), by using Lemma 12.1. ■

This result can be generalized to cope with the smoothing problem with any lag. To be precise, the smoother with lag $h+1$ can be constructed from the equations of the h -lag smoother. Hence, an iterative scheme can be worked out. Indeed, notice that the augmented system associated with the i -th lag is characterized by a state vector of dimensions $n+iq$. The corresponding solution $\bar{P}(t)$ of the associated Riccati equation can be split as

$$\bar{P}(t) = \begin{bmatrix} P_{11}^{(i)}(t) & P_{12}^{(i)}(t) & \cdots & P_{1i+1}^{(i)}(t) \\ P_{12}^{(i)}(t)' & P_{22}^{(i)}(t) & \cdots & P_{2i+1}^{(i)}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{1i+1}^{(i)}(t)' & P_{2i+1}^{(i)}(t)' & \cdots & P_{i+1i+1}^{(i)}(t) \end{bmatrix}, \quad i = 1, 2, \dots, h.$$

The details of this generalization are a matter of cumbersome computations and are omitted for the sake of conciseness. The iterations are initialized by

$$\begin{aligned}P_{11}^{(1)}(t) &= P_{11}(t) \\ P_{12}^{(1)}(t) &= P_{12}(t) \\ P_{22}^{(1)}(t) &= P_{22}(t)\end{aligned}$$

where $P_{11}(t)$, $P_{12}(t)$ and $P_{22}(t)$, which satisfy (12.48), depend on the solution $P(t)$ of (12.37).

The iterative scheme is:

$$\begin{aligned}
 P_{11}^{(i+1)}(t+1) &= A(t)P_{11}^{(i)}(t)A(t)' + B(t)B(t)' - S^{(i)}(t)T_{P_{11}^{(i)}}(t)^{-1}S^{(i)}(t)' \\
 T_{P_{11}^{(i)}}(t) &= C(t)P_{11}^{(i)}C(t)' + D(t)D(t)' \\
 S^{(i)}(t) &= B(t)D(t)' + A(t)P_{11}^{(i)}(t)C(t)' \\
 P_{12}^{i+1}(t+1) &= U^{(i)}(t) - S^{(i)}(t)T_{P_{11}^{(i)}}(t)^{-1} \left(C(t)P_{11}^{(i)}E(t)' + D(t)N(t)' \right) \\
 U^i(t) &= B(t)N(t)' + A(t)P_{11}^{(i)}(t)E(t)' \\
 A_{sl}^{(i)}(t) &= A(t) - S^{(i)}(t)T_{P_{11}^{(i)}}(t)^{-1}C(t) \\
 P_{1k}^{(i+1)}(t+1) &= A_{sl}^{(i)}(t)P_{1k-1}^{(i)}(t), \quad 0 < k-2 \leq i \\
 P_{kk}^{(i+1)}(t+1) &= P_{k-1k-1}^{(i)}(t) - P_{1k-1}^{(i)}(t)'T_{P_{11}^{(i)}}(t)^{-1}P_{1k-1}^{(i)}(t) \quad 0 < k-2 \leq i,
 \end{aligned}$$

We can now state the following result.

Proposition 12.9 Assume that

- (i) $D(t)D(t)' > 0, \forall t$.
- (ii) The pair $(A(\cdot), C(\cdot))$ is detectable.
- (iii) The system $(A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ does not have invariant zeros on the unit circle

and that the recursion of periodic matrices $P_{11}^{(i)}(\cdot)$, $i = 1, 2, \dots, h$ admits a solution. Then, there exist a h -step periodic smoother $F(\sigma, t)$ such that

$$\|F(\sigma, t)H_1(\sigma, t) - H_2(\sigma, t)\|_\infty < \gamma$$

if and only if the T -periodic Riccati equation (12.37) admits a T -periodic positive semi-definite solution $P(\cdot)$ such that $A_{sl}^{(h)}(\cdot)$ is stable and $\gamma^2 I - P_{22}^{(h+1)}(t) > 0, \forall t$. In this case a solution is given by

$$\begin{aligned}
 \hat{x}(t+1) &= A(t)\hat{x}(t) + B_S(t)e(t) \\
 \hat{z}(t) &= E(t)\hat{x}(t) + \sum_{k=0}^h D_{Sk}(t+k)e(t+k) \\
 e(t) &= y(t) - C(t)\hat{x}(t),
 \end{aligned}$$

where

$$\begin{aligned}
 B_S(t) &= (A(t)P_{11}^{(h)}(t)C(t)' + B(t)D(t)')(C(t)P_{11}^{(h)}C(t)' + D(t)D(t)')^{-1} \\
 D_{S0}(t) &= \left(E(t)P_{11}^{(h)}(t)C(t)' + N(t)D(t)' \right) (C(t)P_{11}^{(h)}C(t)' + D(t)D(t)')^{-1} \\
 D_{Si}(t) &= P_{1i+1}^{(h)}(t)'C(t)'(C(t)P_{11}^{(h)}C(t)' + D(t)D(t)')^{-1}, \quad i = 1, 2, \dots, h.
 \end{aligned}$$

■

As a final remark, notice that when γ tends to infinity, the H_2 smoother described in Proposition 12.3 is eventually recovered. Indeed, in this case, the Riccati equation reduces to Equation (12.8) with unknown $P(t)$ and $P_{11}^{(i)}(t) = P(t)$, $\forall i$. Easy computations show that $D_{Si}(t+i) = K_i(t)$ and $B_S(t) = S_I(t)W_I(t)$ so that the gains of the H_2 smoother are recovered.

12.7 Bibliographical Notes

In this chapter, the filtering problem has been addressed in the H_2 framework and H_∞ framework. In both cases, a periodic Riccati equation comes into the stage, see the reference paper [51]. In the H_2 setting, there are a number of numerical algorithms to find the periodic solution of the relevant Riccati equation. Among them the lifting technique and the Kleinman (quasi-linearization) technique, see [50, 70, 180]. The H_∞ smoothing problem for time-invariant discrete-time systems has been studied following different methods. The one pursued here for periodic systems traces the arguments in [82, 279] (for time-invariant discrete-time systems) and in [103, 104] (for periodic continuous-time systems).

Chapter 13

Stabilization and Control

A control problem can be formulated according to different objectives and design criteria. In the first part of the chapter, we will consider state-feedback control systems, for which it is assumed that the state variables can be measured and therefore fed back to the controller. In such a framework, we preliminarily deal with the problem of stabilization, namely the possibility of ensuring stability with a periodic control law. We first characterize the class of stabilizing controllers when the system matrices are fully known, or subject to uncertainty (of norm-bounded or polytopic type). Then we pass to control design. After pole assignment and model matching, we discuss the H_2 and H_∞ design techniques. The second part of the chapter is devoted to output-feedback control problems, where only the output variable can be used for control design. Both static and dynamic controllers are considered. In the latter case, the polynomial approach is undertaken for the design of a stabilizing controller under hard input constraints. Finally, the classical linear quadratic Gaussian (LQG) controller is studied, as well as the extension to the H_∞ framework.

13.1 The Class of State-feedback Stabilizing Controllers

Stability is a basic requirement of any control system. In the first part of this chapter, we address the problem of stabilizing a T -periodic system

$$x(t+1) = A(t)x(t) + B(t)u(t), \quad (13.1)$$

with the control law

$$u(t) = K(t)x(t). \quad (13.2)$$

$K(t) \in \mathbb{R}^{n \times m}$ is the controller gain, T -periodic too. From (13.1), (13.2) the closed-loop dynamic matrix is given by

$$A(\cdot) + B(\cdot)K(\cdot). \quad (13.3)$$

Gain $K(\cdot)$ is said to be *stabilizing* when the characteristic multipliers of this matrix lie in the open unit disk of the complex plane.

A first basic question is to find the class of all stabilizing gains $K(\cdot)$. In this regard, recall the *Lyapunov inequality condition* seen in Sect. 3.3.3, enabling one to conclude that the closed-loop system associated with a periodic gain $K(\cdot)$ is stable, if and only if there exists a positive definite periodic matrix $P(\cdot)$ satisfying the inequality:

$$P(t+1) > (A(t) + B(t)K(t))P(t)(A(t) + B(t)K(t))'. \quad (13.4)$$

A sensible way of approaching this problem is to write $K(\cdot)$ in the following factorized form

$$K(t) = W(t)'P(t)^{-1}, \quad (13.5)$$

where $W(\cdot)$ is a periodic matrix to be determined. Such position does not imply any lack of generality. Indeed, with (13.5), the Inequality (13.4) becomes

$$\begin{aligned} P(t+1) &> A(t)P(t)A(t)' + B(t)W(t)'A(t)' + \\ &+ A(t)W(t)B(t)' + B(t)W(t)'P(t)^{-1}W(t)B(t)'. \end{aligned} \quad (13.6)$$

On the basis of the periodic Lyapunov Lemma, it is obvious that, if there exists a pair of matrices $W(\cdot)$ and $P(\cdot)$ such that, for all t ,

1. $W(t+T) = W(t)$
2. $P(t+T) = P(t)$
3. $P(t) > 0$
4. $W(\cdot)$ and $P(\cdot)$ satisfy Inequality (13.6),

then $K(t) = W(t)'P(t)^{-1}$ is a stabilizing gain.

Conversely, if $K(\cdot)$ is a T -periodic stabilizing gain, then the periodic Lyapunov Lemma guarantees that there exists a periodic and positive definite $P(\cdot)$ satisfying (13.4). Then, letting $W(t)' = K(t)P(t)$ it is obvious that the gain $K(t)$ takes the factorized form $K(t) = W(t)'P(t)^{-1}$.

In conclusion, the class of stabilizing gains is generated by all periodic pairs $W(\cdot)$ and $P(\cdot)$, with $P(t) > 0$, $\forall t$, satisfying Eq. (13.6).

Interestingly enough, by resorting to the so-called Schur complement Lemma 3.1, Inequality (13.6) can be further elaborated giving rise to the Periodic Linear Matrix Inequality (PLMI) condition

$$\begin{bmatrix} -P(t+1) & B(t)W(t)' + A(t)P(t) \\ W(t)B(t)' + P(t)A(t)' & -P(t) \end{bmatrix} < 0. \quad (13.7)$$

Note that the right-hand side of Inequality (13.6) is *non-linear* in $P(\cdot)$ due to the presence of the inverse term $P(t)^{-1}$. On the opposite, the left hand side of (13.7) is linear in the pair $(P(\cdot), W(\cdot))$. So, Inequality (13.7) is by all means a LMI in the unknowns $W(\cdot)$ and $P(\cdot)$.

All previous discussions lead to the following result.

Proposition 13.1 *All stabilizing T -periodic gains $K(\cdot)$ are given by*

$$K(t) = W(t)'P(t)^{-1},$$

where $W(\cdot)$ and $P(\cdot)$ are T -periodic solutions of the LMI (13.7). ■

The class of solutions of a PLMI constitutes reportedly a convex set. So, the family of generating pairs $(W(\cdot), P(\cdot))$ with $P(t) > 0$, $\forall t$, is convex. However, by no means, this entails that the set of gains $K(\cdot) = W(\cdot)'P(\cdot)^{-1}$ is convex.

Example 13.1 *Consider the time-invariant scalar pair $A = 0$, $B = 1$, and a 2-periodic control gain $K(t)$. The class of stabilizing gains is given by*

$$|K(0)K(1)| < 1.$$

The set of pairs $K(0)$, $K(1)$ satisfying the above inequality is not convex. Indeed consider the two stabilizing gains $K_1(0) = 5$, $K_1(1) = 1/6$, and $K_2(0) = 1/6$, $K_2(1) = 5$. Now take the third controller as a linear combination of the previous ones, i.e., $K(t) = \alpha K_1(t) + (1 - \alpha)K_2(t)$ with $\alpha = 0.9$. It turns out that $K(\cdot)$ is not stabilizing since $K(0)K(1) = 2.9358$.

13.2 Robust Stabilization

In the previous section, we have considered a given system, with known matrices $A(\cdot)$ and $B(\cdot)$. Here we pass to consider the stabilization problem in the case in which both $A(\cdot)$ and $B(\cdot)$ in (13.1) are uncertain matrices belonging to given classes: $A(\cdot) \in \mathcal{A}$ and $B(\cdot) \in \mathcal{B}$ (*robust stabilization problem*). To be precise, we consider first the class of *norm-bounded* perturbations, namely we assume that

$$\mathcal{A} = \{A(\cdot) \mid A(t) = A_n(t) + L(t)\Delta(t)N_a(t)\} \quad (13.8a)$$

$$\mathcal{B} = \{B(\cdot) \mid B(t) = B_n(t) + L(t)\Delta(t)N_b(t)\}, \quad (13.8b)$$

where $A_n(\cdot)$, $B_n(\cdot)$, $L(\cdot)$, $N_a(\cdot)$, $N_b(\cdot)$ are known T -periodic matrices while $\Delta(\cdot)$ is an unstructured norm-bounded perturbation satisfying for each t the inequality

$$\|\Delta(t)\| \leq 1.$$

Consequently, the whole uncertainty is concentrated in matrix $\Delta(\cdot)$.

The second class under consideration is characterized by a *polytopic* uncertainty, namely the families \mathcal{A} and \mathcal{B} are given by

$$\mathcal{A} = \{A(\cdot) = \sum_{i=1}^N \alpha_i A_i(t), \quad \alpha_i > 0, \quad \sum_{i=1}^N \alpha_i = 1\} \quad (13.9a)$$

$$\mathcal{B} = \{B(\cdot) = \sum_{i=1}^M \beta_i B_i(t), \quad \beta_i > 0, \quad \sum_{i=1}^M \beta_i = 1\}, \quad (13.9b)$$

where $A_i(\cdot)$ and $B_i(\cdot)$, $i = 1, 2, \dots, N$ are periodic matrices.

We exploit the results of Chap. 3 in order to design a T -periodic state-feedback controller (13.2) which provides a stable closed-loop system for any possible choice of $A(\cdot)$ and $B(\cdot)$ in the considered uncertain sets \mathcal{A} and \mathcal{B} .

Proposition 13.2 *The T -periodic linear system with matrices $A(\cdot)$ and $B(\cdot)$ belonging to the norm-bounded sets (13.8) is quadratically stabilizable by a T -periodic state-feedback control law if and only if there exist two T -periodic matrices $W(\cdot)$ and $P(\cdot)$ such that for each t*

$$\begin{bmatrix} -P(t+1) + L(t)L(t)' & 0 & A_n(t)P(t) + B_n(t)W(t)' \\ 0 & -I & N_a(t)P(t) + N_b(t)W(t)' \\ P(t)A_n(t)' + W(t)B_n(t)' & P(t)N_a(t)' + W(t)N_b(t)' & -P(t) \end{bmatrix} < 0$$

All quadratically stabilizing T -periodic gains can be obtained from $W(\cdot)$ and $P(\cdot)$ according to (13.5).

Proof This result is easily obtained from (3.25) by taking $K(t)P(t) = W(t)'$ and recognizing that the closed-loop system can be written as

$$x(t+1) = (A_n(t) + B_n(t)K(t))x(t) + L(t)\Delta(t)(N_a(t) + N_b(t)K(t))x(t).$$

Indeed (3.25) transforms into

$$\begin{aligned} 0 &> \begin{bmatrix} -P(t+1) + L(t)L(t)' & 0 \\ 0 & -I \end{bmatrix} \\ &+ \begin{bmatrix} A_n(t)P(t) + B_n(t)W(t)' \\ N_a(t)P(t) + N_b(t)W(t)' \end{bmatrix} P(t)^{-1} \begin{bmatrix} A_n(t)P(t) + B_n(t)W(t)' \\ N_a(t)P(t) + N_b(t)W(t)' \end{bmatrix}'. \end{aligned}$$

In view of the Schur complement lemma, this condition is equivalent to the one given in the statement. ■

Turning to the case of polytopic uncertainty, as defined in (13.9), we work out two results. The first one concerns the quadratic stabilization problem, which aims at finding a stabilizing T -periodic gain $K(\cdot)$ by constructing a *unique* quadratic Lyapunov function associated with $A(t) + B(t)K(t)$, valid for any $A(\cdot) \in \mathcal{A}$ and any $B(\cdot) \in \mathcal{B}$. The second result provides a sufficient condition of robust stabilization based on a parameter dependent Lyapunov function. Obviously, the latter condition is less conservative.

Proposition 13.3 *The T -periodic linear system with matrices $A(\cdot)$ and $B(\cdot)$ belonging to the polytopic families (13.9) is quadratically stabilizable by a T -periodic state-feedback control law if and only if there exist T -periodic matrices $W(\cdot)$ and $P(\cdot)$ such that, for each $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$, and for each t , the following matrix inequalities hold:*

$$\begin{bmatrix} -P(t+1) & B_j(t)W(t)' + A_i(t)P(t) \\ W(t)B_j(t)' + P(t)A_i(t) & -P(t) \end{bmatrix} < 0. \quad (13.10)$$

Proof If the system is quadratically stabilizable then there exists a T -periodic gain $K(\cdot)$ such that $A(\cdot) + B(\cdot)K(\cdot)$ is quadratically stable. This obviously implies the stability for any pair $(A_i(\cdot), B_j(\cdot))$, so that there exists a positive definite T -periodic matrix $P(\cdot)$ and $W(\cdot) = P(\cdot)K(\cdot)'$ such that (13.10) is satisfied. Conversely, assume that (13.10) is met with for some T -periodic $W(\cdot)$ and for some positive definite T -periodic matrix $P(\cdot)$. Then, multiply (13.10) by $\alpha_i\beta_j$ and sum with respect to i , $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$ to obtain

$$\begin{bmatrix} -P(t+1) & (A(t) + B(t)K(t))P(t) \\ P(t)(A(t) + B(t)K(t))' & -P(t) \end{bmatrix} < 0,$$

where

$$K(t) = W(t)'P(t)^{-1}.$$

The result finally follows from Lemma 3.3. ■

The next result provides a less conservative sufficient robust stabilizability condition based on a parameter-dependent Lyapunov function.

Proposition 13.4 *The T -periodic linear system with matrices $A(\cdot)$ and $B(\cdot)$ belonging to the polytopic families (13.9) is robustly stabilizable with a T -periodic state-feedback if there exist T -periodic matrices $Z(\cdot)$, $W(\cdot)$ and T -periodic matrices $P_{ij}(\cdot)$, such that, for each $i = 1, 2, \dots, N$, $j = 1, 2, \dots, M$, and for each t , the following inequalities hold:*

$$\begin{bmatrix} -P_{ij}(t+1) & A_i(t)Z(t) + B_j(t)W(t)' \\ Z(t)'A_i(t)' + W(t)B_j(t)' & P_{ij}(t) - Z(t) - Z(t)' \end{bmatrix} < 0. \quad (13.11)$$

If (13.11) is satisfied, then a robustly stabilizing T -periodic control law is given by $K(t) = W(t)'Z(t)^{-1}$.

Proof Multiplying (13.11) by $\alpha_i\beta_j$ and sum with respect to i , $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$, one has

$$\begin{bmatrix} -P(t+1) & (A(t) + B(t)K(t))Z(t) \\ Z(t)'(A(t) + B(t)K(t))' & P(t) - Z(t) - Z(t)' \end{bmatrix} < 0,$$

where

$$P(t) = \sum_{i=1}^N \alpha_i \sum_{j=1}^M \beta_j P_{ij}(t), \quad K(t) = W(t)'Z(t)^{-1}.$$

Then Lemma 3.3 leads to the statement.

13.3 Pole Assignment for Control

Pole assignment (by state-feedback) is one of the typical problems studied in linear system theory. In general terms, the problem is that of imposing a desired dynamics by means of state feedback. We will explore two ways to put into practice this objective. The first one refers to the possibility of shaping a new dynamics by means of a *sample and hold strategy* of the type:

$$u(t) = \bar{K}(t)x(iT + \tau), \quad t \in [iT + \tau, iT + \tau + T - 1], \quad (13.12)$$

where the gain $\bar{K}(t)$ is a T -periodic function to be suitably designed and τ is a fixed tag in time. The advantage of this approach is that the pole assignment algorithm developed for time-invariant systems can be eventually applied to the periodic case too, as will be explained in the sequel. However, in this sampled feedback strategy, the input signal is updated only at the beginning of each period, so that the system is operated in open loop in the interperiod instants. This may be a serious drawback when important information is lost due to rare sampling. In particular, in the interperiod instants the performance may deteriorate due to the prolonged action of a persistent disturbance. To prevent performance deterioration in the interperiod intervals, continuous monitoring of the state would be preferable. This leads to the alternative feedback control law:

$$u(t) = K(t)x(t), \quad (13.13)$$

where $K(\cdot)$ is a T -periodic gain matrix. This control strategy will be referred to as *instantaneous feedback*. The purpose of this section is to present the design techniques which can be used in the two approaches now outlined.

13.3.1 Pole Assignment via Sampled Feedback

For time-invariant systems, it is well-known that reachability is necessary and sufficient for arbitrary pole-assignability. Starting from this observation, consider the system with feedback (13.12). Then it is apparent that

$$x(iT + \tau + T) = [\Psi_A(\tau) + \sum_{j=\tau+1}^{\tau+T} \Phi_A(\tau + T, j)B(j-1)\bar{K}(j-1)]x(iT + \tau).$$

Hence, the monodromy matrix at τ of the closed-loop system is

$$\hat{\Psi}_A(\tau) = [\Psi_A(\tau) + \sum_{j=\tau+1}^{\tau+T} \Phi_A(\tau+T, j)B(j-1)\bar{K}(j-1)] = F_\tau + G_\tau\tilde{K},$$

where F_τ and G_τ are the lifted matrices introduced in (6.20a), (6.20b) and

$$\tilde{K} = \begin{bmatrix} \bar{K}(\tau) \\ \bar{K}(\tau+1) \\ \vdots \\ \bar{K}(\tau+T-1) \end{bmatrix}.$$

Therefore, should the pair (F_τ, G_τ) be reachable (i.e., the periodic system reachable at time τ), it is possible to select \tilde{K} so as to impose the prespecified set of eigenvalues of $\hat{\Psi}_A(\tau)$.

In adopting this rationale, one should be advised that the system dynamics is also influenced by the latent variables of the sample and hold device yielding (13.12). As a matter of fact, this device is governed by the periodic state equations:

$$\begin{aligned} v(t+1) &= S(t)v(t) + (I - S(t))x(t) \\ u(t) &= \bar{K}(t)S(t)v(t) + \bar{K}(t)(I - S(t))x(t), \end{aligned}$$

where

$$S(\tau) = 0, \quad S(t) = I, \quad t = \tau+1, \tau+2, \dots, \tau+T-1.$$

In other words, the sample and hold mechanism has its own dynamics, with a n -dimensional state vector $v(t)$. Correspondingly, the full dynamics in closed-loop is described by the $2n$ -dimensional system

$$\begin{bmatrix} x(t+1) \\ v(t+1) \end{bmatrix} = \begin{bmatrix} A(t) + B(t)K(t)(I - S(t)) & B(t)K(t)(I - S(t)) \\ I - S(t) & S(t) \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}.$$

The associated monodromy matrix is (the simple check is left to the reader)

$$\begin{bmatrix} F_\tau + G_\tau\tilde{K} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, n characteristic multipliers are given by the eigenvalues of $F_\tau + G_\tau\tilde{K}$, and these can be freely assigned due the reachability assumption, and the remaining n , evoked by the sampling mechanism, are located at zero.

13.3.2 Pole Assignment via Instantaneous Feedback

Consider now the memoryless feedback law (13.13) applied to the system. The associated closed-loop dynamic matrix is

$$\hat{A}(t) = A(t) + B(t)K(t).$$

The question is whether it is possible to find, for each set Λ of n complex numbers (in conjugate pairs), a periodic gain $K(\cdot)$ in such a way that $\hat{A}(\cdot)$ has all its characteristic multipliers coincident with the elements of Λ .

A first idea is to explore the possibility of inferring a memoryless feedback law (13.13) from the sampled feedback law (13.12) in such a way that the characteristic multipliers of $\hat{A}(\cdot)$ coincide with the eigenvalues of $\hat{\Psi}_A(\tau)$. The simplest way to pursue this objective is to impose the coincidence of the two state evolutions associated with the control action (13.13) and the control action (13.12). To be precise, this amounts to equating the two expressions of $x(iT + \tau + k)$, starting from $k = 1$ and prosecuting up to $k = T - 1$. It is easy to conclude that, if

$$\bar{K}(\tau) = K(\tau) \tag{13.14a}$$

$$\bar{K}(\tau + 1) = K(\tau + 1)(A(\tau) + B(\tau)K(\tau)) \tag{13.14b}$$

$$\bar{K}(\tau + 2) = K(\tau + 2)(A(\tau + 1) + B(\tau + 1)K(\tau + 1))(A(\tau) + B(\tau)K(\tau)) \tag{13.14c}$$

$$\vdots,$$

then the two state evolutions coincide. Notice that this set of equations can be given the compact form

$$\bar{K}(\tau + i) = K(\tau + i)\Phi_{\hat{A}}(\tau + i, \tau), \quad i = 0, 1, \dots, T - 1.$$

It is worth noticing that this relation establishes a “triangular dependence” between $\bar{K}(\cdot)$ and $K(\cdot)$, in the sense that $\bar{K}(t)$ depends on $K(\tau)$, $\tau \leq t$. Obviously, once $K(\cdot)$ is given, the computation of $\bar{K}(\cdot)$ is straightforward. The converse problem requires more attention. A close inspection of equations (13.14) reveals that the computation of $K(\cdot)$ from $\bar{K}(\cdot)$ can indeed be performed provided that matrix $\tilde{A}(t) = A(t) + B(t)K(t)$ is invertible for each time point. In such a case

$$K(\tau + i) = \bar{K}(\tau + i)[\Phi_{\tilde{A}}(\tau + i, \tau)]^{-1}, \quad i = 0, 1, \dots, T - 1.$$

Although this pole assignment procedure via an instantaneous feedback law is simple, there is no guarantee that the invertibility condition on $\tilde{A}(\cdot)$ is satisfied for a certain $\bar{K}(\cdot)$. An interesting question is then whether, among all possible $\bar{K}(\cdot)$ which assign the prescribed characteristic multipliers, there is one for which the invertibility condition is met. However, finding an algorithm of wide applicability which exploits the time-invariant view point is still an open question.

We turn then to the design of an instantaneous feedback law for pole assignment in a direct way, namely without going through $\bar{K}(\cdot)$. By recalling Eqs. (4.29), a first answer to pole assignability can be easily given. In fact, from that decomposition into reachable/unreachable parts, it clearly appears that the characteristic multipliers of the unreachable part $A_r(\cdot)$ cannot be moved by any (periodic) feedback gain $K(\cdot)$. Hence the problem can be faced by restraining to the reachable pair $(A_r(\cdot), B_r(\cdot))$ only. Note that the dimensions of the involved matrices may be time varying since $A_r(t) \in \mathbb{R}^{n_r(t+1) \times n_r(t)}$ and $B_r(t) \in \mathbb{R}^{n_r(t+1) \times m}$. Correspondingly, one is led to consider a control gain $K_r(t)$ with time varying dimensions: $K_r(t) \in \mathbb{R}^{m \times n_r(t)}$. This matrix gain is obtained from the original “full gain” $K(t)$ by canonical decomposition, i.e., $K_r(t) = K(t)Q_r(t)$, where $Q_r(t)$ is the basis matrix of the reachability subspace $X_r(t)$ (see (4.27)). The relevant part of the closed-loop system is governed by matrix

$$\hat{A}_r(t) = A_r(t) + B_r(t)K_r(t)$$

of dimension $n_r(t+1) \times n_r(t)$. The associated monodromy matrix $\Psi_{\hat{A}_r}(t)$ is a $n_r(t)$ -dimensional square matrix. In view of Remark 3.1

$$\det(\lambda I - \Psi_{\hat{A}_r}(t)) = \lambda^{n_r(t) - n_r(\tau)} \det(\lambda I - \Psi_{\hat{A}_r}(\tau)).$$

Therefore, letting

$$n_{rm} = \min_t \{n_r(t)\},$$

it is clear that $n_r(t) - n_{rm}$ characteristic multipliers of $A_r(\cdot)$ are always equal to zero. These considerations can be restated in terms of the closed-loop characteristic multipliers. A first important fact is that $n - n_{rm}$ characteristic multipliers are independent of $K(\cdot)$. They can be further distinguished into two categories:

- (i) $n - n_r(t)$ characteristic multipliers are those of the unreachable part of the system.
- (ii) $n_r(t) - n_{rm}$ characteristic multipliers are in the origin.

What it is really interesting is that the remaining n_{rm} characteristic multipliers can be freely placed. Indeed, the following result, reported without proof, holds.

Proposition 13.5 *Consider a T -periodic pair $(A(\cdot), B(\cdot))$ with constant dimensions, i.e., $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$, $\forall t$. Let n_{rm} be the minimum dimension of the reachability subspace and define arbitrarily a set Λ of n_{rm} complex numbers (in conjugate pairs). Then, there exists a T -periodic matrix gain $K(\cdot)$ such that n_{rm} characteristic multipliers of $\hat{A}(\cdot)$ are elements of Λ . It is not possible to freely assign more than n_{rm} characteristic multipliers.*

Proposition 13.5 above leads to an obvious necessary and sufficient condition for the solution of the pole-placement problem for a T -periodic discrete-time system with time-invariant state-space dimension n . To be precise, it is possible to freely assign the n characteristic multipliers of $\hat{A}(\cdot)$ (in conjugate pairs) if and only if the system is reachable for all t . Notice that this condition (reachability for each t) is no

longer necessary when the state–space of the system has time-varying dimensions, as witnessed by the following simple example.

Example 13.2 Let $T = 2$, $n(0) = 1$, $n(1) = 2$, and

$$A(0) = B(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 2 & 3 \end{bmatrix}, \quad B(1) = 1.$$

As can be easily verified, the system is reachable at $t = 0$ since $\dim[X_r(0)] = n(0) = 1$ whereas is not reachable at $t = 1$ since $\dim[X_r(1)] = 1 < n(1) = 2$. However, being $n_{rm} = 1$, it is possible to freely assign one characteristic multiplier of $A(\cdot) + B(\cdot)K(\cdot)$ by properly selecting matrix $K(t)$. Indeed, letting λ such (real) characteristic multiplier, it suffices to select

$$K(0) = 0, \quad K(1) = [\lambda - 5 \ 0]. \quad \blacksquare$$

An important corollary of the pole-placement problem is the so-called *dead-beat* control. A periodic state-feedback gain $K(\cdot)$ is said to be a dead-beat state-feedback periodic controller if all the characteristic multipliers of the closed-loop system $\hat{A}(\cdot)$ lie in the origin. Thanks to Proposition 13.5, it is easy to check that a dead-beat state-feedback gain $K(\cdot)$ exists if and only if all the characteristic multipliers of the unreachable part are all zero. This is true, see Eq. (4.31) if and only if $A_c(t)$ vanishes, i.e the system is controllable.

Interestingly, if $K(\cdot)$ is a state-feedback dead-beat control gain, and $y(t) = C(t)x(t) + D(t)u(t)$ is a measurement vector, the feedback $u(t) = K(t)x(t) + v(t)$ results in

$$y(t) = (C(t) + D(t)K(t)) \sum_{k=t}^{t+nT-1} \Phi_{\hat{A}}(t+nT, k+1)B(k)v(k) + D(t)v(t+nT).$$

In this way, the output depends upon a *finite number* of previous samples of the input function. This implies that the output reaches its regime in a *finite number* of periods. Notice in passing that the input–output relation above is a *periodic moving-average* model, see Sect. 2.2.

13.4 Exact Model Matching

Roughly speaking, the problem of model matching consists in finding a compensator for a given system so as to obtain an overall dynamics as close as possible to an assigned input–output map. Herein we restrict attention to the exact model matching for periodic systems via state feedback controllers. To be precise, given a periodic input–output map S

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned}$$

we aim at finding a state-feedback controller

$$u(t) = K(t)x(t) + G(t)v(t), \quad (13.15)$$

with $K(t)$ and $G(t)$ T -periodic design matrices, in such a way that the closed-loop input–output map coincides with the assigned one.

A special case arises when the assigned map is actually time-invariant, so that the controller task is that of “invariantize” the input–output closed-loop behavior.

To simplify the solution of the main problem, we make the assumption that $G(t)$ is square and non-singular for each t . The overall scheme is depicted in Fig. 13.1.

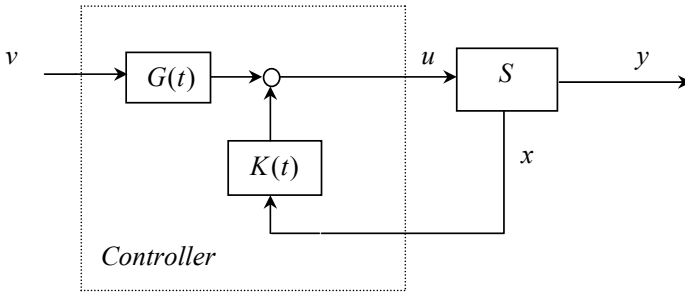


Fig. 13.1 Exact model matching scheme

The solution of the problem passes through the cyclic reformulation of the system and the controller, see Sect. 6.3. For simplicity, the time-tag τ of the cyclic reformulation is set to zero. In the realm of the cyclic reformulation, the feedback (13.15) is equivalent to

$$\hat{u}(t) = \hat{K}\hat{x}(t) + \hat{G}\hat{v}(t),$$

where $\hat{u}(t) \in \mathbb{R}^{mT}$, $\hat{x}(t) \in \mathbb{R}^{nT}$, and $\hat{v}(t) \in \mathbb{R}^{mT}$ are the cycled vectors associated with $u(t)$, $x(t)$ and $v(t)$. Correspondingly, $\hat{K} \in \mathbb{R}^{mT \times nT}$ and $\hat{G} \in \mathbb{R}^{mT \times mT}$ are block diagonal matrices with entries $K(i)$ and $G(i)$, $i = 0, 1, \dots, T-1$, respectively (as usual, m denotes the number of input variables and n the number of state variables). The periodic feedback control system can be redrawn in terms of the cyclic reformulation too. To this purpose, let

$$\hat{T}(z) = (zI - \hat{A})^{-1}\hat{B}, \quad \hat{W}(z) = \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D}$$

denote the cyclic transfer functions from the input \hat{u} to the state \hat{x} and to the output \hat{y} , respectively. Moreover, let us denote by $\hat{W}_m(z) \in \mathbb{C}^{mT \times pT}$ the transfer function (from \hat{v} to \hat{y}) of the cyclic reformulation associated with the target closed-loop (periodic) system (with input v and output y). Notice the special case when $\hat{W}_m(z)$ represents the cyclic reformulation of a time-invariant system. In such a case, the aim of the

designer is to work out a periodic feedback such that the closed-loop system turns out to be time-invariant.

In general, one has to find T -periodic matrices $K(\cdot)$ and $G(\cdot)$ such that the following equation is met with:

$$\hat{W}_m(z) = \hat{W}(z)(I - \hat{K}\hat{T}(z))^{-1}\hat{G}. \quad (13.16)$$

The solution to such a problem is provided in the subsequent result.

Proposition 13.6 *There exist a periodic feedback matrix $K(\cdot)$ and a periodic matrix $G(\cdot)$, the latter non-singular for all t , such that the closed-loop system cyclic transfer function coincides with $\hat{W}_m(z)$ if and only if the following conditions are satisfied*

- (i) *The two transfer functions $[\hat{W}(z) \hat{W}_m(z)]$ and $\hat{W}_m(z)$ have the same finite zeros structure.*
- (ii) *The two transfer functions $[\hat{W}(z) \hat{W}_m(z)]$ and $\hat{W}(z)$ have the same infinite zeros structure.*
- (iii) *The two transfer functions $[\hat{W}(z) \hat{W}_m(z)]$ and $\hat{W}_m(z)$ have the same infinite zeros structure.*

Proof (Sketch) Notice first that Eq. (13.16) can be rewritten as $\hat{W}_m(z) = \hat{W}(z)\hat{R}(z)$, where $\hat{R}(z)$ corresponds to an invertible dynamic system thanks to the invertibility of \hat{G} . As for sufficiency, conditions (i)-(iii) indeed ensure the existence of a proper $mT \times mT$ transfer matrix $\hat{R}(z)$ such that $\hat{R}(z)^{-1}$ is still proper and $\hat{W}_m(z) = \hat{W}(z)\hat{R}(z)$ so that $\hat{W}(z) = \hat{W}_m(z)\hat{R}(z)^{-1}$. $\hat{R}(z)$ can be considered as the transfer function of the cyclic reformulation of a suitable periodic system. Therefore $\hat{R}(\infty)^{-1}$ is block diagonal and $\hat{R}(z)^{-1} - \hat{R}(\infty)^{-1}$ is strictly proper. Thanks again to the assumptions, $\hat{R}(z)^{-1} - \hat{R}(\infty)^{-1}$ and $\hat{T}(z)$ share the same structure in the sense that the equation

$$\hat{R}(z)^{-1} - \hat{R}(\infty)^{-1} = \hat{H}\hat{T}(z) \quad (13.17)$$

admits a block-diagonal constant solution matrix $\hat{H} \in \mathbb{R}^{mT \times nT}$. Then, let

$$\hat{G} = \hat{R}(\infty), \quad \hat{K} = -\hat{G}\hat{H}.$$

Being \hat{G} block diagonal, \hat{K} is block diagonal as well. The block entries of the matrices \hat{K} and \hat{G} yield the solution of the problem.

The necessity of conditions (i) – (iii) is easily derived by inverse considerations. ■

Obviously, if $\hat{W}_m(z)$ is stable, then the solution of the exact model matching problem given in Proposition 13.6 supplies a BIBO stable closed-loop system. However, there is no guarantee of internal stability.

Example 13.3 *Consider the single-input single-output periodic system with period $T = 2$ defined by matrices:*

$$\begin{aligned} A(0) &= \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C(0) &= [1 \ -1], \quad C(1) = [1/3 \ 2], \quad D(0) = 0, \quad D(1) = 0. \end{aligned}$$

Then $\hat{T}(z)$ and $\hat{W}(z)$ are as follows:

$$\hat{T}(z) = \begin{bmatrix} z^2 & z^3 \\ 9 & 9z \\ 3z & 3z^2 \\ z^3 & 18 \end{bmatrix} \frac{1}{(z^4 - 18)}$$

$$\hat{W}(z) = \begin{bmatrix} z^2 - 9 & z^3 - 9z \\ 2z^3 + z & z^2 + 36 \end{bmatrix} \frac{1}{(z^4 - 18)}.$$

Suppose we require the closed-loop system to be time-invariant with

$$W_m(z) = \frac{z-3}{z^2} \quad (13.18)$$

as target transfer function. Its cyclic version is

$$\hat{W}_m(z) = \frac{1}{z^2} \begin{bmatrix} -3 & z \\ z & -3 \end{bmatrix}.$$

Then,

$$\hat{R}(z)^{-1} - \hat{R}(\infty)^{-1} = \begin{bmatrix} 22z^4 + 9z^2 - 324 & 4z^5 + 9z^3 \\ 7z^5 - 6z^3 & 3z^4 + 126z^2 - 162 \end{bmatrix} \frac{z^2}{(z^2 - 9)(z^4 - 18)}.$$

Direct verification shows that Eq. (13.17) does not admit any constant solution \hat{H} . This is due to the fact that the condition (i) of Proposition 13.6 is not satisfied, even though $\hat{W}_m(z)$ and $\hat{W}(z)$ have the same finite zero at $\lambda = 3$. In conclusion the reference model (13.18) is not achievable.

As a second target function, consider:

$$W_m(z) = \frac{1}{z}.$$

Then

$$\hat{W}_m(z) = \frac{1}{z^2} \begin{bmatrix} 0 & z \\ z & 0 \end{bmatrix}$$

$$\hat{R}(z)^{-1} - \hat{R}(\infty)^{-1} = \frac{1}{z^4 - 18} \begin{bmatrix} z^2 + 36 & z^3 + 36z \\ z^3 - 9z & -9z^2 + 180 \end{bmatrix}.$$

Now Eq. (13.17) has a block diagonal solution

$$\hat{H} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}.$$

Hence

$$\hat{G} = \hat{R}(\infty) = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{K} = -\hat{G}\hat{H} = \begin{bmatrix} -0.5 & -2 & 0 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}.$$

The periodic control law is then

$$u(t) = K(t)x(t) + G(t)v(t),$$

where

$$G(0) = 0.5, \quad G(1) = 1 \\ K(0) = \begin{bmatrix} -0.5 & -2 \end{bmatrix}, \quad K(1) = \begin{bmatrix} 3 & -1 \end{bmatrix}.$$

Notice, however, that in this case, the resulting closed-loop system characteristic multipliers are 0 and 9 and therefore the system is internally unstable. ■

The next result provides a necessary and sufficient condition for the exact model matching with internal stability.

Proposition 13.7 *There exist a periodic feedback matrix $K(\cdot)$ and a periodic matrix $G(\cdot)$, the latter non-singular for each time point, such that the closed-loop system cyclic transfer function coincides with $\hat{W}_m(z)$ and the closed-loop system is stable if and only if besides conditions (i)–(iii) of Proposition 13.6, the following requirement is satisfied*

(iv) *The two transfer functions $[\hat{W}(z) \hat{W}_m(z)]$ and $\hat{W}(z)$ have the same finite unstable zeros structure.*

Proof (Sketch) The proof follows from Proposition 13.6 by noticing that the additional condition (iv) avoids cancellations between $\hat{R}(z)$ and $\hat{W}(z)$ that would result in a loss of internal stability. ■

If the design procedure carried out in Proposition 13.6 leads to a internally stable time-invariant closed-loop system, then it is possible to resort to the variety of design methods known in the time-invariant case to work out a suitable controller. The resulting controller (consisting of the composition of the model matching loop and the time-invariant controller) is obviously periodic.

Example 13.4 *Consider the system with $T = 2$, and*

$$A(0) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C(1) = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D(0) = 0, \quad D(1) = 0.$$

Then $\hat{T}(z)$ and $\hat{W}(z)$ are as follows:

$$\hat{T}(z) = \begin{bmatrix} z^2 - 1 & 0 \\ 1 & z^3 - z \\ z^3 - z & 0 \\ z & z^2 - 1 \end{bmatrix} \frac{1}{(z^2 - 1)^2}, \quad \hat{W}(z) = \begin{bmatrix} z^2 - 1 & 0 \\ z & z^2 - 1 \end{bmatrix} \frac{1}{(z^2 - 1)^2}.$$

Suppose that the target transfer function in closed-loop is

$$W_m(z) = \frac{1}{z^2},$$

the cyclic version of which is

$$\hat{W}_m(z) = \frac{1}{z^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\hat{R}^{-1}(z) = \begin{bmatrix} z^2(z^2 - 1) & 0 \\ z^3 & z^2(z^2 - 1) \end{bmatrix} \frac{1}{(z^2 - 1)^2}, \quad \hat{G} = \hat{R}(\infty) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Equation (13.17) is satisfied by

$$\hat{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

so that

$$\hat{K} = -\hat{G}\hat{H} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}.$$

The periodic control law is then

$$u(t) = K(t)x(t) + G(t)v(t),$$

where

$$\begin{aligned} G(0) &= 1, \quad G(1) = 1 \\ K(0) &= \begin{bmatrix} -1 & 0 \end{bmatrix}, \quad K(1) = \begin{bmatrix} -1 & -1 \end{bmatrix}. \end{aligned}$$

All conditions (i)–(iv) of Propositions 13.7 are satisfied and the closed-loop system is stable and dead-beat, i.e., all the characteristic multipliers of the closed-loop system are in the origin. ■

The model-matching procedure introduced so far is based on a state-feedback control law. However, it can be easily extended to the case when the full state is not available for feedback but only the variable

$$y(t) = C(t)x(t) + D(t)u(t)$$

can be measured. In that case, one can resort to the control scheme depicted in Fig. 13.2, where \hat{x} indicates the estimate of the system state x carried out by a state observer of the form

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B(t)u(t) + L(t)(C(t)\hat{x}(t) - y(t)).$$

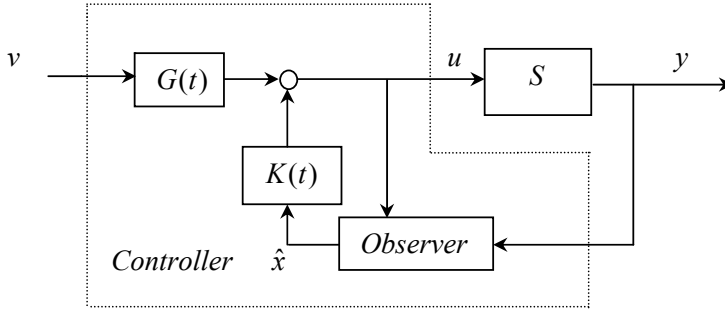


Fig. 13.2 Model matching via observer design

If the system is detectable, then a T -periodic matrix $L(\cdot)$ exists such that $A(\cdot) + L(\cdot)C(\cdot)$ is stable. Hence, due to the unbiased form¹ of the observer, the input–output map from v to y coincides with the input–output map in Fig. 13.1.

13.5 H_2 Full-information Optimal Control

We now pass to the realm of optimal control methods. In this section, we consider the so-called full-information problem in H_2 . In such a problem, it is assumed that, for a system described in state–space form, both the state-variable $x(\cdot)$ and the disturbance $w(\cdot)$ are available for feedback and the issue is to minimize the effect of the disturbance over a performance variable $z(\cdot)$. We limit the attention to memoryless control laws, namely

$$u(t) = K(t)x(t) + W(t)w(t). \quad (13.19)$$

In mathematical terms, the problem can be formulated as follows. Given the periodic system

$$x(t+1) = A(t)x(t) + \bar{B}(t)w(t) + B(t)u(t) \quad (13.20a)$$

$$z(t) = E(t)x(t) + \bar{N}(t)w(t) + N(t)u(t), \quad (13.20b)$$

find a stabilizing periodic state control law of the type (13.19) so as to minimize the H_2 norm from w to z of the corresponding closed-loop system. We shall denote by $G_{z,w}(\sigma, \cdot)$ the associated input–output operator.

For a given stabilizing periodic $K(\cdot)$ and a given periodic $W(\cdot)$, we exploit the characterization of the H_2 norm in Eq. (9.3). To be precise, if $Q(\cdot)$ is the unique T -

¹ Notice indeed that the observation error $e(t) = \hat{x}(t) - x(t)$ obeys to the unforced dynamical system difference equation

$$e(t+1) = (A(t) + L(t)C(t))e(t).$$

periodic solution of the Lyapunov Equation,

$$\begin{aligned} Q(t) = & [A(t) + B(t)K(t)]' Q(t+1) [A(t) + B(t)K(t)] \\ & + [E(t) + N(t)K(t)]' [E(t) + N(t)K(t)], \end{aligned}$$

the squared H_2 norm of $G_{z,w}(\sigma, \cdot)$ can be written as:

$$\begin{aligned} \|G_{z,w}(\sigma, \cdot)\|_2^2 = & \text{trace} \sum_{i=0}^{T-1} [\bar{B}(i) + B(i)W(i)]' Q(i+1) [\bar{B}(i) + B(i)W(i)] \\ & + \text{trace} \sum_{i=0}^{T-1} [\bar{N}(i) + N(i)W(i)]' [\bar{N}(i) + N(i)W(i)]. \end{aligned}$$

We leave it to the reader to verify that the above Lyapunov equation can be equivalently written as:

$$\begin{aligned} Q(t) = & A(t)' Q(t+1) A(t) + E(t)' E(t) \\ & - K^0(t)' [N(t)' N(t) + B(t)' Q(t+1) B(t)] K^0(t) + \nabla(Q(t)), \end{aligned}$$

with

$$\nabla(Q(t)) = [K(t) - K^0(t)]' [N(t)' N(t) + B(t)' Q(t+1) B(t)] [K(t) - K^0(t)],$$

and

$$K^0(t) = - [N(t)' N(t) + B(t)' Q(t+1) B(t)]^{-1} [B(t)' Q(t+1) A(t) + N(t)' E(t)].$$

On the other hand, $\|G_{z,w}(\sigma, \cdot)\|_2^2$ can be rewritten as

$$\begin{aligned} \|G_{z,w}(\sigma, \cdot)\|_2^2 = & \text{trace} \sum_{i=0}^{T-1} (W(i) - W^o(i))' T_Q(i) (W(i) - W^o(i)) \\ & + \text{trace} \sum_{i=0}^{T-1} [\bar{B}(i)' \bar{N}(i)'] (\bar{R}(i) - \bar{R}(i) \bar{S}(i) T_Q(i)^{-1} \bar{S}(i)' \bar{R}(i)) \begin{bmatrix} \bar{B}(i) \\ \bar{N}(i) \end{bmatrix}, \end{aligned}$$

where

$$T_Q(i) = (B(i)' Q(i+1) B(i) + N(i)' N(i))^{-1} \quad (13.21a)$$

$$\bar{R}(i) = \begin{bmatrix} Q(i+1) & 0 \\ 0 & I \end{bmatrix} \quad (13.21b)$$

$$\bar{S}(i) = \begin{bmatrix} B(i) \\ N(i) \end{bmatrix} \quad (13.21c)$$

$$W^o(i) = -T_Q(i) (B(i)' Q(i+1) \bar{B}(i) + N(i)' \bar{N}(i)). \quad (13.21d)$$

So far, the derivations required to work out the solution of the H_2 optimal control problem can be deduced by duality from those used in Sect. 12.1 for the determi-

nation of the optimal H_2 filter. As for duality, the transformation of the involved system matrices was already discussed in Sect. 4.2.2, whereas the correspondence between the solution of the Riccati equation $P(t)$ for the filter and the solution $Q(t)$ for the controller is given by $Q(t) = P(-t)$. In this way, it can be established that the optimal control gains are $K(t) = K^o(t)$ and $W(t) = W^o(t)$ and the relevant Riccati equation is:

$$Q(t) = A(t)'Q(t+1)A(t) + E(t)'E(t) - T_N(t)T_Q(t)T_N(t)' \quad (13.22a)$$

$$T_N(t) = [A(t)'Q(t+1)B(t) + E(t)'N(t)] \quad (13.22b)$$

$$T_Q(t) = [N(t)'N(t) + B(t)'Q(t+1)B(t)]^{-1}. \quad (13.22c)$$

In conclusion, we can formulate the main result on control in H_2 .

Proposition 13.8 *Assume that*

- (i) *For each t , $N(t)'N(t) > 0$.*
- (ii) *The pair $(A(\cdot), B(\cdot))$ is stabilizable.*
- (iii) *The system $(A(\cdot), B(\cdot), E(\cdot), N(\cdot))$ does not have invariant zeros on the unit circle.*

Then, the Riccati Equation (13.22) admits the T -periodic (positive semi-definite) stabilizing solution $Q(\cdot)$. Moreover, the solution of the optimal H_2 control problem is given by (13.19) with

$$K(t) = -(B(t)'Q(t+1)B(t) + N(t)'N(t))^{-1} (B(t)'Q(t+1)A(t) + N(t)'E(t))$$

$$W(t) = -(B(t)'Q(t+1)B(t) + N(t)'N(t))^{-1} (B(t)'Q(t+1)\bar{B}(t) + N(t)'\bar{N}(t)).$$

13.5.1 H_2 Optimal State-feedback Control

Above we have studied the full-information control problem where the control variable is designed on the basis of the knowledge of both the state x and the disturbance w . Here we focus on pure state-feedback where only the state is available for control, namely the control law is

$$u(t) = K(t)x(t).$$

At first sight, one might think that the solution $K(\cdot)$ coincides with the one obtained above for the full-information problem. As a matter of fact, it turns out that this is not correct². Finding the solution of this new optimal control problem requires further elaboration. A possibility is to resort to the 1-step prediction problem in H_2 , discussed in Sect. 12.2. Indeed, this optimal control problem is just the dual of that prediction problem. Consequently, from the statement of Proposition 12.2

² This fact is a typical aspect of optimal control in discrete-time. Interestingly, in continuous-time the optimal solution of the full-information problem yields $W(t) = 0$.

with prediction horizon $h = 1$ the optimal solution of the state-feedback H_2 optimal control problem is straightforwardly derived as

$$K(t) = -[N(t)'N(t) + B(t)'Q(t+1)B(t)]^{-1} B(t)'Q(t+1)A(t),$$

where $Q(\cdot)$ is the unique positive periodic semi-definite solution of the Riccati Equation (13.22).

To complete the picture one can also consider the delayed state-feedback control law

$$u(t) = K(t)x(t-h), \quad h > 0. \quad (13.23)$$

Again, by duality, this corresponds to the $h+1$ step ahead prediction problem. Hence the optimal solution can be derived by appropriately dualizing the general solution given by Proposition 12.2, yielding

Proposition 13.9 *Assume that*

- (i) *For each t , $N(t)'N(t) > 0$.*
- (ii) *The pair $(A(\cdot), B(\cdot))$ is stabilizable.*
- (iii) *The system $(A(\cdot), B(\cdot), E(\cdot), N(\cdot))$ does not have invariant zeros on the unit circle.*

Then, the Riccati Equation (13.22) admits the T -periodic (positive semi-definite) stabilizing solution $Q(\cdot)$ and the solution of the optimal H_2 delayed state-feedback control problem is given by (13.23) with

$$K(t) = -(B(t)'Q(t+1)B(t) + N(t)'N(t))^{-1} B(t)'Q(t+1)A(t)\Phi_A(t, t-h). \quad \blacksquare$$

Analogous considerations apply to the case of H_2 optimal state-feedback control with preview of length h :

$$u(t) = K(t)x(t) + \sum_{i=0}^h W_i(t+i)w(t+h), \quad h > 0. \quad (13.24)$$

The solution to this problem can be worked out by dualizing Proposition 12.3, so obtaining

Proposition 13.10 *Assume that*

- (i) *For each t , $N(t)'N(t) > 0$.*
- (ii) *The pair $(A(\cdot), B(\cdot))$ is stabilizable.*
- (iii) *The system $(A(\cdot), B(\cdot), E(\cdot), N(\cdot))$ does not have invariant zeros on the unit circle.*

Then, the Riccati Equation (13.22) admits the T -periodic (positive semi-definite) stabilizing solution $Q(\cdot)$ and the solution of the optimal H_2 control problem with preview length h is given by (13.24) with

$$\begin{aligned}
K(t) &= -T_Q(t) (B(t)'Q(t+1)A(t) + N(t)'E(t)) \\
W_0(t) &= -T_Q(t) (B(t)'Q(t+1)\bar{B}(t) + N(t)'\bar{N}(t)) \\
W_{i-1}(t+i) &= -T_Q(t)B(t)Q_{1i}(t+1), \quad i = 2, 3, \dots, h+1,
\end{aligned}$$

where

$$\begin{aligned}
Q_{12}(t) &= A(t)'Q(t+1)\bar{B}(t) + (A(t)'Q(t+1)B(t) + E(t)'N(t))W_0(t) + E(t)'\bar{N}(t) \\
Q_{1i}(t) &= (A(t) + B(t)K(t))'Q_{1i-1}(t+1),
\end{aligned}$$

and $T_Q(t)$ and $W_0(t)$ were defined in (13.21a) and (13.21d), respectively.

13.6 State-feedback H_∞ Control

In this chapter, we tackle the control design in the state-feedback H_∞ framework. The mathematical derivation is only outlined since the relevant results can be obtained by duality from the corresponding ones concerning H_∞ filtering discussed in Sect. 12.4. As in the previous section, we will start with the full-information problem and then move to the problems of pure state-feedback, delayed state-feedback and state-feedback with preview.

In the full-information H_∞ problem the issue is to design a controller in order to bound the effect of disturbances. To be precise, consider again the periodic System (13.20) with the periodic control law (13.19). The problem is to find a stabilizing control law so as to bound the H_∞ norm from $w(\cdot)$ to $z(\cdot)$ of the corresponding closed-loop system. Denoting by $G_{z,w}(\sigma, \cdot)$ the associated input-output operator, this requirement can be expressed via inequality

$$\|G_{z,w}(\sigma, t)\|_\infty < \gamma.$$

Here γ is a positive parameter, chosen by the designer, characterizing the acceptable level of influence of the disturbance $w(\cdot)$ (*attenuation parameter*). Of course, should γ be too small, no controller might exist satisfying the performance requirement.

To tackle the design problem, we exploit the characterization of the H_∞ norm in Eqs. (9.11). Therefore, for a given periodic $K(\cdot)$ and a given periodic $W(\cdot)$, define the closed-loop matrices

$$\begin{aligned}
A_C(t) &= A(t) + B(t)K(t) \\
B_C(t) &= \bar{B}(t) + B(t)W(t) \\
E_C(t) &= E(t) + N(t)K(t) \\
D_C(t) &= \bar{N}(t) + N(t)W(t),
\end{aligned}$$

and consider the associated H_∞ Riccati equation (in the unknown $Q(\cdot)$):

$$Q(t) = A_C(t)'Q(t+1)A_C(t) + E_C(t)'E_C(t) + Y_C(t)'V_C(t)Y_C(t) \quad (13.25a)$$

$$Y_C(t) = V_C(t)^{-1} [B_C(t)'Q(t+1)A_C(t) + D_C(t)'E_C(t)] \quad (13.25b)$$

$$V_C(t) = \gamma^2 I - D_C(t)'D_C(t) - B_C(t)'Q(t+1)B_C(t). \quad (13.25c)$$

Then the problem becomes that of finding matrices $K(\cdot)$ and $W(\cdot)$ in such a way that Eq. (13.25) admits a T -periodic positive semi-definite solution $Q(\cdot)$ which is stabilizing and enjoys the feasibility property, i.e., $V_C(t) > 0$, $\forall t$. As a matter of fact, the computation of $K(\cdot)$ and $W(\cdot)$ starting from this Riccati equation can be performed via duality by mimicking the steps followed in Sect. 12.4 for the filtering problem. This leads to the following conclusion, the dual of Proposition 12.5.

Proposition 13.11 *Assume that*

- (i) *For each t , $N(t)'N(t) > 0$.*
- (ii) *The pair $(A(\cdot), B(\cdot))$ is stabilizable.*
- (iii) *The system $(A(\cdot), B(\cdot), E(\cdot), N(\cdot))$ does not have invariant zeros on the unit circle.*

Then, there exists a causal T -periodic controller of the form (13.19) such such that the input–output operator of the closed-loop system satisfies the condition

$$\|G_{z,w}(\sigma, t)\|_\infty < \gamma$$

if and only if the Riccati equation

$$Q(t) = A(t)'Q(t+1)A(t) + E(t)'E(t) - Z_J(t)V_J(t)Z_J(t)', \quad (13.26)$$

where

$$Z_J(t) = [A(t)'Q(t+1)B(t) + E(t)'N(t) \quad A(t)'Q(t+1)\bar{B}(t) + E(t)'\bar{N}(t)]$$

$$V_J(t) = \begin{bmatrix} N(t)'N(t) + B(t)'Q(t+1)B(t) & B(t)'Q(t+1)\bar{B}(t) + N(t)'\bar{N}(t) \\ \bar{B}(t)'Q(t+1)B(t) + \bar{N}(t)'N(t) & -\gamma^2 I + \bar{B}(t)'Q(t+1)\bar{B}(t) + \bar{N}(t)'\bar{N}(t) \end{bmatrix}^{-1}$$

admits a T -periodic positive semi-definite satisfying the following two properties:

(a) *(Stabilizing property)*

$$\hat{A}_J(t) = A(t) - [B(t) \quad \bar{B}(t)] V_J(t) Z_J(t)'$$

is stable.

(b) *(Feasibility property)*

$$\hat{S}_3(t) = \gamma^2 I - \bar{B}(t)'Q(t+1)\bar{B}(t) - \bar{N}(t)'\bar{N}(t)$$

$$+ (\bar{B}(t)'Q(t+1)B(t) + \bar{N}(t)'N(t)) \times$$

$$\times (N(t)'N(t) + B(t)'Q(t+1)B(t))^{-1} (B(t)'Q(t+1)\bar{B}(t) + N(t)'\bar{N}(t))$$

is positive definite.

In this case, a full-information memoryless controller is given by (13.19) with

$$\begin{aligned} K(t) &= (B(t)'Q(t+1)B(t) + N(t)'N(t))^{-1} (B(t)'Q(t+1)A(t) + N(t)'E(t)) \\ W(t) &= (B(t)'Q(t+1)B(t) + N(t)'N(t))^{-1} (B(t)'Q(t+1)\bar{B}(t) + N(t)'\bar{N}(t)). \end{aligned}$$

Remark 13.1 (Difference game)

The H_∞ control problem can be interpreted as a game between the disturbance $w(\cdot)$ and the designer handling the variable $u(\cdot)$. Correspondingly, one can define over an interval of n steps, say from a certain time point τ to a time point $\tau + N - 1$, the performance index

$$J_\infty = \sum_{t=\tau}^{\tau+N-1} \|z(t)\|^2 - \gamma^2 \|w(t)\|^2 + x(\tau+N)'Sx(\tau+N).$$

The third term in this expression takes into consideration the energy of the final state through a weighting matrix $S = S' > 0$. Its introduction in the performance index is dictated by the fact that the considered horizon is finite.

By imposing that $J_\infty < 0$ one obtains the design objective of attenuating the effect of $w(\cdot)$ on $z(\cdot)$ over the interval from τ to $\tau + N - 1$. Thus, the problem can be stated as that of finding the design matrices $K(\cdot)$ and $W(\cdot)$ of controller (13.19) so that, for any disturbance $w(\cdot)$, performance J_∞ be negative. Of course, matrices $K(\cdot)$ and $W(\cdot)$ solving the problem are not necessarily periodic: they might tend to become periodic as $N \rightarrow \infty$.

Under the assumptions stated in Proposition 13.11, the finite horizon problem now stated admits indeed a solution. To express it by avoiding overcomplex formulae, we make the simplifying assumption on the performance variable $z(\cdot)$ that, for each t , $N(t)'E(t) = 0$, $N(t) = 0$ and $N(t)'N(t) = I$. To be precise, it can be proven that the above problem has a solution if and only if, for all $t = \tau, \tau + 1, \dots, \tau + N$, there exist positive definite matrices $Q(t)$ and $V(t)$, which satisfy the matrix recursions

$$\begin{aligned} Q(t) &= A(t)'V(t)^{-1}A(t) + E(t)'E(t) \\ V(t) &= [Q(t+1)^{-1} + B(t)B(t)' - \gamma^{-2}\bar{B}(t)\bar{B}(t)'] , \end{aligned}$$

with final condition $V(\tau + N) = S$. In this case, a control law guaranteeing the prescribed attenuation level is

$$u(t) = (I + B(t)'Q(t+1)B(t))^{-1}B(t)'Q(t+1)(A(t)x(t) + B(t)w(t)).$$

The convergence of the solution $Q(\cdot)$ to the T -periodic solution is not guaranteed by the standard structural assumptions on the underlying system. In other words, the transient behavior and the convergence properties strongly depend on the choice of the weighting matrix S and the attenuation level γ .

13.7 Static Output-feedback

In the previous sections of this chapter, we have considered feedback control laws exploiting the measurement of the state. In the remaining ones, we study the control design when only the measurement of the output signal is available. The output-feedback problem is tackled first by static controllers (this section) and then by dynamic controllers (Sect. 13.8).

The problem of stabilization via static output-feedback (SOF) is probably one of the main open issues in systems and control. The research is also spurred by the fact that in some cases a periodic control law can stabilize a time-invariant system which cannot be stabilized by constant feedback, see e.g., the subsequent Example 13.5. Throughout the section, reference is made to the usual periodic system

$$x(t+1) = A(t)x(t) + B(t)u(t) \quad (13.27a)$$

$$y(t) = C(t)x(t). \quad (13.27b)$$

13.7.1 Sampled-data Approach

The first idea to solve the SOF problem is to resort to the sampled-data control law

$$u(t) = F(t)y(kT + \tau), \quad t \in [kT + \tau, kT + T - 1 + \tau]. \quad (13.28)$$

With this position, one obtains

$$x(kT + \tau + T) = [\Psi_A(\tau) + \sum_{j=\tau}^{\tau+T-1} \Phi_A(\tau + T - 1, j)B(j)F(j)C(j)]x(kT + \tau).$$

Hence, the monodromy matrix at τ of the closed-loop system is

$$\hat{\Psi}_A(\tau) = [\Psi_A(\tau) + \sum_{j=\tau}^{\tau+T-1} \Phi_A(\tau + T - 1, j)B(j)F(j)C(j)].$$

Assume now that the Grammian matrix

$$\Sigma(\tau) = \sum_{j=\tau}^{\tau+T-1} \Phi_A(\tau + T - 1, j)B(j)B(j)'\Phi_A(\tau + T - 1, j)' \quad (13.29)$$

is invertible and take as gain matrix:

$$F(j) = B(j)'\Phi_A(\tau + T - 1, j)'\Sigma(\tau)^{-1}\Omega. \quad (13.30)$$

The closed-loop monodromy matrix can be written as

$$\hat{\Psi}_A(\tau) = \Psi_A(\tau) + \Omega C(\tau).$$

Therefore, if the pair $(\Psi_A(\tau), C(\tau))$ is detectable, it is possible to select Ω , and hence thereby the T -periodic gain $F(\cdot)$ given by (13.30), in order to stabilize the closed-loop system. If in addition $(\Psi_A(\tau), C(\tau))$ is observable, it is possible to arbitrarily assign the closed-loop poles of the system.

Notice that the invertibility of matrix $\Sigma(\tau)$ in (13.29) corresponds to the assumption of reachability in one period of the periodic system at time τ . Thus the discussion above is summarized as follows.

Proposition 13.12 *If the periodic system is reachable in one period at some point τ and the pair $(\Psi_A(\tau), C(\tau))$ is detectable, then it is possible to find a stabilizing sampled-data control law (13.28). In such a case, a sampled-data stabilizing gain $F(\cdot)$ is given by*

$$F(j) = B(j)' \Phi_A(\tau + T - 1, j)' \Sigma(\tau)^{-1} \Omega,$$

where Ω is such that $\Psi_A(\tau) + \Omega C(\tau)$ is stable. ■

Obviously, the gain $F(\cdot)$ is computed over a period, say from τ to $\tau + T - 1$, and then repeats periodically.

As already seen in Sect. 13.3.1, a sampled-data mechanism introduces extra dynamics into the system. Indeed, the sample-and-hold mechanism is governed by the periodic state equations:

$$\begin{aligned} v(t+1) &= S(t)v(t) + (I - S(t))y(t) \\ u(t) &= F(t)S(t)v(t) + F(t)(I - S(t))y(t), \end{aligned}$$

where

$$S(\tau) = 0, \quad S(t) = I, \quad t = \tau + 1, \tau + 2, \dots, \tau + T - 1.$$

Correspondingly, the full dynamics in closedloop is described by the system

$$\begin{bmatrix} x(t+1) \\ v(t+1) \end{bmatrix} = \begin{bmatrix} A(t) + B(t)F(t)(I - S(t)) & B(t)F(t)(I - S(t)) \\ I - S(t) & S(t) \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix},$$

and the associated monodromy matrix is (the simple check is left to the reader)

$$\begin{bmatrix} \Psi_A(\tau) + \Omega C(\tau) & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, n characteristic multipliers are given by the eigenvalues of $\Psi_A(\tau) + \Omega C(\tau)$, whereas the remaining ones are null eigenvalues.

Remark 13.2 *Observe that if the sampled-data control law (13.28) is replaced by*

$$u(t) = F(t)y(nkT + \tau), \quad t \in [kT + \tau, kT + nT - 1 + \tau], \quad (13.31)$$

then one can drop the 1-period reachability assumption and replace it with the standard reachability assumption at τ . Indeed, in this case, all the previous computations can be rewritten in terms of the extended Grammian

$$\hat{\Sigma}(\tau) = \sum_{j=\tau}^{\tau+nT-1} \Phi_A(\tau+nT-1, j)B(j)B(j)'\Phi_A(\tau+nT-1, j)',$$

so that it is possible to find a stabilizing sampled-data control law (13.31) if the periodic system is reachable at some point τ and the pair $(\Psi_A(\tau), C(\tau))$ is detectable. In this case a sampled-data stabilizing gain $F(\cdot)$ is given by

$$F(j) = B(j)'\Phi_A(\tau+nT-1, j)'\Sigma_n(\tau)^{-1}\Omega$$

$$\Sigma_n(\tau) = \sum_{j=\tau}^{\tau+nT-1} \Phi_A(\tau+nT-1, j)B(j)B(j)'\Phi_A(\tau+nT-1, j)',$$

where Ω is such that $\Psi_A(\tau) + \Omega C(\tau)$ is stable. In this case the gain $F(\cdot)$ is computed over n periods, say from τ to $\tau + nT - 1$, and then repeats periodically with period nT . ■

An important application of Proposition 13.12 concerns the case when the system is time-invariant. Here, the period T is a design variable and the choice of τ becomes immaterial so that we set $\tau = 0$. Taking $T = n$ (the system order), it follows that

$$\Sigma = \sum_{j=0}^{n-1} A^{n-j-1}BB'(A')^{n-j-1}$$

is the reachability Grammian. Hence, if (A, B) is reachable, this Grammian is invertible. Then, the gain over the interval from 0 to $n-1$ is given by

$$F(t) = B'(A')^{n-t-1}\Sigma^{-1}\Omega, \quad t \in [0, n-1].$$

By periodic extension of such function, a periodic controller is obtained. The associated closed-loop matrix turns out to be

$$\hat{\Psi}_A(\tau) = A^n + \Omega C.$$

Therefore, stabilization is possible if the pair (A^n, C) is detectable. In turn, this pair is detectable if (A, C) is detectable and no pair of distinct eigenvalues λ_1 and λ_2 of A is such that $\lambda_1^n = \lambda_2^n$.

Example 13.5 Consider System (13.27) with A, B, C constant and given by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 1].$$

This unstable system cannot be stabilized by any constant output-feedback control law, as it can be easily verified. However, following the rationale previously pursued,

stabilization can be achieved by means of a sampled-data hold control of period 2:

$$u(2k) = F(0)y(2k), \quad u(2k+1) = F(1)y(2k).$$

Indeed, take Ω such that $A^2 + \Omega C$ is stable. Since (A^2, C) is observable, the poles can be arbitrarily placed. For instance, one can adopt a dead-beat control strategy and impose that both closed-loop characteristic multipliers be located in the origin. This can be obtained by taking

$$\Omega = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Correspondingly,

$$\begin{aligned} F(0) &= B'A'\Sigma^{-1}\Omega = 5 \\ F(1) &= B'\Sigma^{-1}\Omega = -3. \end{aligned}$$

Remark 13.3 The sampled-data hold function technique is very powerful from a theoretical point of view. For instance, as we have just seen, it is possible to stabilize by output-feedback systems that cannot be stabilized by a constant output-feedback. This is achieved, however, at the expenses of possible deterioration of the system overall dynamic behavior. In this respect, recall that a sampled-data hold control is not really a static output-feedback, since the sample and hold device introduces extra (null) dynamics into the closed-loop system and any disturbance acting inside the intersample instants can lead to severe deterioration of the closed-loop system performances.

13.7.2 A Riccati Equation Approach

In this section, we aim at characterizing the static output-feedback (SOF) problem as a constrained periodic linear matrix inequality problem.

System (13.27) is said to be *output-feedback stabilizable* if there exists a T -periodic memoryless control law

$$u(t) = F(t)y(t),$$

such that all the characteristic multipliers of $\hat{A}(\cdot) = A(\cdot) + B(\cdot)F(\cdot)C(\cdot)$ are inside the open unit disc, i.e., the closed-loop periodic system is stable. If such an output-feedback does exist, we say that the system is *output stabilizable* and that $F(\cdot)$ is a solution of the problem.

Throughout the section we make the assumption that matrix $C(t)$ is full row rank for each t . This assumption allows us to define the projection matrix

$$V(t) = I - C(t)'(C(t)C(t'))^{-1}C(t).$$

Proposition 13.13 *The system is output-feedback stabilizable if and only if*

- (i) *The system is stabilizable and detectable.*
- (ii) *There exist a T -periodic positive semi-definite matrix $Q(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ and a T -periodic matrix $G(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ such that, $\forall t$*

$$Q(t) = A'(t)Q(t+1)A(t) + C'(t)C(t) + G'(t)G(t) \quad (13.32a)$$

$$\begin{aligned} & -A'(t)Q(t+1)B(t)(I + B'(t)Q(t+1)B(t))^{-1}B'(t)Q(t+1)A(t) \\ 0 = & [G(t) - (I + B'(t)Q(t+1)B(t))^{-1/2}B'(t)Q(t+1)A(t)]V(t). \end{aligned} \quad (13.32b)$$

Proof If $F(\cdot)$ is a T -periodic stabilizing gain, then the periodic Lyapunov equation

$$\begin{aligned} Q(t) = & (A(t) + B(t)F(t)C(t))'Q(t+1)(A(t) + B(t)F(t)C(t)) \\ & + C'(t)C(t) + C'(t)F'(t)F(t)C(t) \end{aligned} \quad (13.33)$$

admits a positive semi-definite T -periodic solution $Q(\cdot)$. The thesis follows by recognizing that the same $Q(\cdot)$ solves (13.32a) with

$$\begin{aligned} G(t) = & (I + B'(t)Q(t+1)B(t))^{1/2}F(t)C(t) + \\ & + (I + B'(t)Q(t+1)B(t))^{-1/2}B'(t)Q(t+1)A(t). \end{aligned} \quad (13.34)$$

Conversely, if the periodic positive semi-definite matrix $Q(\cdot)$ and the periodic matrix $G(\cdot)$ satisfy (13.32), consider a T -periodic orthogonal matrix $T(\cdot)$ such that, $\forall t$

$$T'(t)G(t)V(t) = (I + B'(t)Q(t+1)B(t))^{-1/2}B'(t)Q(t+1)A(t)V(t),$$

and define

$$\begin{aligned} F(t) = & (I + B'(t)Q(t+1)B(t))^{-1/2}T'(t)G(t)C'(t)(C(t)C'(t))^{-1} \\ & - (I + B'(t)Q(t+1)B(t))^{-1}B'(t)Q(t+1)A(t)C'(t)(C(t)C'(t))^{-1}. \end{aligned}$$

Then, it is a fact of cumbersome computation to show that $Q(\cdot)$ satisfies

$$\begin{aligned} Q(t) = & (A(t) + B(t)F(t)C(t))'Q(t+1)(A(t) + B(t)F(t)C(t)) \\ & + C'(t)C(t) + C'(t)F'(t)F(t)C(t), \end{aligned}$$

so that the stability of $A(\cdot) + B(\cdot)F(\cdot)C(\cdot)$ comes from the Lyapunov Lemma. ■

The proof of the previous result allows us to provide the parametrization of all stabilizing periodic SOF gains as follows.

Corollary 13.1 *Consider the System (13.27). The family of all output T -periodic feedback stabilizing gains $F(\cdot)$ is given by*

$$\begin{aligned} F(t) = & (I + B'(t)Q(t+1)B(t))^{-1/2}T'(t)G(t)C'(t)(C(t)C'(t))^{-1} \\ & - (I + B'(t)Q(t+1)B(t))^{-1}B'(t)P(t+1)A(t)C'(t)(C(t)C'(t))^{-1}, \end{aligned}$$

where $Q(\cdot) \geq 0$ and $G(\cdot)$ solve (13.32), and $T(\cdot)$ is any orthogonal matrix satisfying

$$T'(t)G(t)V(t) = (I + B'(t)Q(t+1)B(t))^{-1/2}B'(t)Q(t+1)A(t)V(t).$$

Example 13.6 Consider again Example 13.5, i.e., System (13.27) with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 1].$$

As already seen, the system can be stabilized with a periodic SOF of period 2, for instance by taking $F(0) = -1, F(1) = 3$. Correspondingly, the closed-loop characteristic multipliers are both located in the origin of the complex plane (dead-beat controller). In the Riccati equation formulation of the SOF problem, associated with such a choice of $F(\cdot)$, the solution of (13.32) is

$$\begin{aligned} Q(0) &= \begin{bmatrix} 330 & 178 \\ 178 & 100 \end{bmatrix}, \quad Q(1) = \begin{bmatrix} 2 & 2 \\ 2 & 20 \end{bmatrix} \\ G(0) &= [18.11 \ 9.82], \quad G(1) = [-0.10 \ -2.29]. \end{aligned}$$

■

It is worth pointing out that it is possible to remove the unknown $G(t)$ in Proposition 13.13 by replacing the Riccati equality with a Riccati inequality, as stated in the following

Proposition 13.14 System (13.27) is output-feedback stabilizable if and only if there exists a symmetric positive T -periodic semi-definite matrix $Q(t)$ such that

$$Q(t) \geq A'(t)Q(t+1)A(t) + C'(t)C(t) \quad (13.35a)$$

$$-A'(t)Q(t+1)B(t)(I + B'(t)Q(t+1)B(t))^{-1}B'(t)Q(t+1)A(t)$$

$$0 = V(t)(Q(t) - A'(t)Q(t+1)A(t))V(t). \quad (13.35b)$$

Proof If the system is SOF stabilizable, then there exists a periodic matrix $F(\cdot)$ and a positive semi-definite matrix $Q(\cdot)$ satisfying the periodic Lyapunov Equation (13.33). Pre-multiplying and post-multiplying the two members of this equation by the projection matrix $V(t)$, equality (13.35b) is straightforwardly derived. Moreover, the periodic Lyapunov equation can be equivalently rewritten in the form (13.32a) with $G(t)$ given by (13.34). This obviously leads to Inequality (13.35a). Conversely, assume that $Q(t) \geq 0$ satisfies (13.35). If the rank of

$$\begin{aligned} S_Q(t) &= Q(t) - A'(t)Q(t+1)A(t) + C'(t)C(t) \\ &\quad - A'(t)Q(t+1)B(t)(I + B'(t)Q(t+1)B(t))^{-1}B'(t)Q(t+1)A(t) \end{aligned}$$

is not greater than m , for each t , then the proof follows again from Proposition 13.13 by taking $G(t) \in \mathbb{R}^{m \times n}$ such that $G'(t)G(t) = S_Q(t)$, for each t . Otherwise, define the matrix with a time-varying number of columns $\tilde{B}(t) = [B(t) \ 0]$ where the zero elements are such that the number of columns $\tilde{m}(t)$ of $\tilde{B}(t)$ equals the rank of $S_Q(t)$

whenever this rank is greater than m . Notice that replacing $B(t)$ with $\tilde{B}(t)$ does not affect Inequality (13.35a). Now, taking $\tilde{G}(t) \in \mathbb{R}^{\tilde{m} \times n}$ such that $\tilde{G}'(t)\tilde{G}(t) = S_Q(t)$, it is possible to select

$$\begin{aligned} \tilde{F}(t) = & (I + \tilde{B}'(t)Q(t+1)\tilde{B}(t))^{-1/2}T'(t)\tilde{G}(t)C'(t)(C(t)C'(t))^{-1} \\ & - (I + \tilde{B}'(t)Q(t+1)\tilde{B}(t))^{-1}\tilde{B}'(t)Q(t+1)A(t)C'(t)(C(t)C'(t))^{-1}, \end{aligned}$$

where $T(t)$ is any orthogonal matrix satisfying

$$T'(t)\tilde{G}(t)V(t) = (I + \tilde{B}'(t)Q(t+1)\tilde{B}(t))^{-1/2}\tilde{B}'(t)Q(t+1)A(t)V(t).$$

As proved in Proposition 13.13, such a gain $\tilde{F}(t)$ is such that $A(\cdot) + \tilde{B}(\cdot)\tilde{F}(\cdot)C(\cdot)$ is stable. Denoting by $F(\cdot)$ the first m rows of $\tilde{F}(\cdot)$, the conclusion is drawn that $A(\cdot) + B(\cdot)F(\cdot)C(\cdot)$ is stable as well, so concluding the proof.

Remark 13.4 *If the periodic system is observable (for each time point), it is possible to prove that $Q(t)$ is invertible for each t . Then, Inequality (13.35a) can be transformed into a periodic LMI in the unknown $X(t) = Q(t)^{-1}$ as follows*

$$0 \leq \begin{bmatrix} X(t) & X(t+1)A'(t) & X(t+1)C'(t) \\ A(t)X(t+1) & X(t+1) + B(t)B'(t) & 0 \\ C(t)X(t+1) & 0 & I \end{bmatrix}. \quad (13.36)$$

Hence, in principle, the SOF problem can be seen as the periodic LMI (13.36) complemented with the (non-linear) coupling condition $Q(t)X(t) = I$, for each t . ■

The main problem concerning the search of a feasible solution $Q(\cdot)$ generating a stabilizing gain $F(\cdot)$ is that such a solution can be, generally speaking, arbitrarily large (in some norm to be defined). However, $Q(\cdot)$ can be given a system-theoretic interpretation in terms of a closed-loop norm. To be precise, consider the system

$$x(t+1) = A(t)x(t) + B(t)u(t) + \bar{B}(t)w(t) \quad (13.37a)$$

$$y(t) = C(t)x(t) \quad (13.37b)$$

$$z(t) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \quad (13.37c)$$

where $w(\cdot)$ is a disturbance. Given a periodic stabilizing gain $F(\cdot)$, consider the closed-loop system with $u(t) = F(t)y(t)$ and denote by $\|T_{zw}\|_2$ the H_2 norm from the disturbance w to the performance signal z . In view of the Lyapunov characterization of the H_2 norm given in Sect. 9.1.1, it turns out that

$$\|T_{zw}\|_2^2 = \sum_{i=0}^{T-1} \text{trace}(\bar{B}'(t)Q(t+1)\bar{B}(t)),$$

where $Q(\cdot)$ satisfies

$$Q(t) = (A(t) + B(t)F(t)C(t))'Q(t+1)(A(t) + B(t)F(t)C(t)) \\ + C'(t)C(t) + C'(t)F'(t)F(t)C(t).$$

The following result relate the “size” of $Q(\cdot)$ to the H_2 norm of the closed-loop system. The proof is omitted since it is analogous to that of Proposition 13.13.

Proposition 13.15 *Consider System (13.37). There exists a T -periodic matrix $F(\cdot)$ such that $A(\cdot) + B(\cdot)F(\cdot)C(\cdot)$ is stable and $\|T_{zw}\|_2 \leq \gamma$ if and only if there exists a T -periodic symmetric positive semi-definite matrix $Q(\cdot)$ and a matrix $G(\cdot)$ such that (13.32) are met with and*

$$\sum_{i=0}^{T-1} \text{trace}(\bar{B}'(t)Q(t+1)\bar{B}(t)) \leq \gamma^2.$$

13.8 Dynamic Output-feedback

In this final section, we consider a few stabilization and control problems based on the dynamic elaboration of the measurements of the output. Initially, in Sects. 13.8.1 and 13.8.2, we deal with SISO systems in a polynomial description and consider the assignment problem together with its robust version. Then we pass to the problem of stabilization with input constraint (Sect. 13.8.3). Finally, we remove the SISO assumption and return to the state-space description in order to tackle the so-called Linear Quadratic Gaussian (LQG) control problem in Sect. 13.8.4 and its robust version (H_∞) in Sect. 13.8.5.

13.8.1 Dynamic Polynomial Assignment

In this section, we consider the pole assignment problem, based on the polynomial description of the periodic system and the periodic controller. To be precise, consider a SISO T -periodic given by its transfer operator

$$g(\sigma, t) = b(\sigma, t)a(\sigma, t)^{-1}, \quad (13.38)$$

where

$$a(\sigma, t) = a_0(t)\sigma^n + a_1(t)\sigma^{n-1} + \dots + a_n(t) \\ b(\sigma, t) = b_0(t)\sigma^{n-1} + b_1(t)\sigma^{n-2} \dots + b_{n-1}(t).$$

Assume that $a(\sigma, t)$, $b(\sigma, t)$ are strongly right coprime and let the periodic controller be described by

$$k(\sigma, t) = -p(\sigma, t)^{-1}q(\sigma, t), \quad (13.39)$$

where $p(\sigma, t)$ and $q(\sigma, t)$ are the periodic polynomials

$$\begin{aligned} p(\sigma, t) &= p_0(t)\sigma^{n-1} + p_1(t)\sigma^{n-2} + \cdots + p_{n-1}(t) \\ q(\sigma, t) &= q_0(t)\sigma^{n-1} + q_1(t)\sigma^{n-2} + \cdots + q_{n-1}(t). \end{aligned}$$

Given $a(\sigma, t)$, $b(\sigma, t)$, the issue is to find the polynomials $p(\sigma, t)$, $q(\sigma, t)$ so as to meet some closed-loop requirements. In this connection, observe first that, from (13.38) and (13.39), the closed-loop polynomial is given by

$$c(\sigma, t) = p(\sigma, t)a(\sigma, t) + q(\sigma, t)b(\sigma, t). \quad (13.40)$$

This is a polynomial of degree $2n - 1$, which can be written as

$$c(\sigma, t) = c_0(t)\sigma^{2n-1} + c_1(t)\sigma^{2n-2} + \cdots + c_{2n-1}(t).$$

In order to find the relation between the parameters of such polynomials and those of the controller, define the vectors

$$\begin{aligned} c(t) &= [c_0(t) \ c_1(t) \ \cdots \ c_{2n-1}(t)]' \\ a(t) &= [a_0(t) \ a_1(t) \ \cdots \ a_n(t)]' \\ b(t) &= [b_0(t) \ b_1(t) \ \cdots \ b_{n-1}(t)]' \\ p(t) &= [p_0(t) \ p_1(t) \ \cdots \ p_{n-1}(t)]' \\ q(t) &= [q_0(t) \ q_1(t) \ \cdots \ q_{n-1}(t)]' \end{aligned}$$

It is possible to rewrite (13.40) in the following compact way

$$c(t) = G_1(t)p(t) + G_2(t)q(t),$$

where

$$G_1(t) = \begin{bmatrix} a_0(t+n-1) & 0 & \cdots & 0 \\ a_1(t+n-1) & a_0(t+n-2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1}(t+n-1) & a_{n-2}(t+n-2) & \cdots & a_0(t) \\ a_n(t+n-1) & a_{n-1}(t+n-2) & \cdots & a_1(t) \\ 0 & a_n(t+n-2) & \cdots & a_2(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1}(t) \\ 0 & 0 & \cdots & a_n(t) \end{bmatrix}$$

and

$$G_2(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_0(t+n-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-2}(t+n-1) & b_{n-3}(t+n-2) & \cdots & 0 \\ b_{n-1}(t+n-1) & b_{n-2}(t+n-2) & \cdots & b_0(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n-2}(t) \\ 0 & 0 & \cdots & b_{n-1}(t) \end{bmatrix}$$

have dimensions $2n \times n$. By letting

$$\xi(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}, \quad G(t) = \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix},$$

it is possible to summarize the relationships between the coefficients of the closed-loop polynomial and the coefficients of the controller polynomial as follows:

$$c(t) = G(t)\xi(t). \quad (13.41)$$

Matrices $G_1(t)$ and $G_2(t)$ and $G(\cdot)$ are known as Sylvester matrices. Since $G(t)$ is square of dimension $2n$, the solution of the polynomial assignment problem is straightforward if such a matrix is invertible. In this regard, the following proposition holds.

Proposition 13.16 *The Sylvester matrix $G(t)$ is invertible for each t if and only if the polynomials $a(\sigma, \cdot)$ and $b(\sigma, \cdot)$ are strongly right coprime.*

Proof If $G(t)$ is invertible for each t , then, choosing

$$c(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

the solution of Eq. (13.41) is $\xi(t) = G(t)^{-1}c(t)$. Associated with this solution, one can form the periodic polynomials $p(\sigma, t)$ and $q(\sigma, t)$ solving the Bezout equation $p(\sigma, t)a(\sigma, t) + q(\sigma, t)b(\sigma, t) = 1$, which is equivalent to right strong coprimeness. Conversely, assume that, for some time point \bar{t} , matrix $G(\bar{t})$ is not invertible, so that there exists a vector $\bar{c} \neq 0$ such that $G(\bar{t})'\bar{c} = 0$. Now, take a periodic vector $c(\cdot)$ such that $c(\bar{t}) = \bar{c}$. Let $c(\sigma, t)$ be the polynomial associated with $c(t)$. If, by contradiction, the pair $(a(\sigma, t), b(\sigma, t))$ were strongly right coprime, there would exist $p(\sigma, t)$ and $q(\sigma, t)$ such that $p(\sigma, t)a(\sigma, t) + q(\sigma, t)b(\sigma, t) = c(\sigma, t)$. Letting $x(t)$ be the vector of the coefficients of $p(\sigma, t)$ and $q(\sigma, t)$, one would obtain $c(t) = G(t)x(t)$. Premultiplying by $c(t)'$ and evaluating the result at $t = \bar{t}$, this would lead to $0 = \|\bar{c}\|^2$. This contradicts the assumption.

Remark 13.5 Notice that the “closed-loop poles”, namely the singularity of the closed-loop periodic polynomial, are given by the roots of

$$\det(P(\sigma^T, t)A(\sigma^T, t) + Q(\sigma^T, t)B(\sigma^T, t)),$$

where $P(\sigma^T, t)$, $A(\sigma^T, t)$, $Q(\sigma^T, t)$ and $B(\sigma^T, t)$ are the lifted version of polynomials $p(\sigma, t)$, $a(\sigma, t)$, $q(\sigma, t)$ and $b(\sigma, t)$, respectively, see Sect. 5.2. ■

As seen, the condition of strong right coprimeness is necessary and sufficient in order to guarantee the existence of a periodic controller of order $n - 1$ capable of arbitrarily impose the $2n$ closed-loop periodic coefficients. In particular, one can assign a closed-loop polynomial with constant coefficients.

When $a(\sigma, t)$ and $b(\sigma, t)$ are not strong right coprime, the Sylvester matrix $G(t)$ is singular at some instant, say \bar{t} . The previous line of reasoning can still be adopted provided that vector $c(\bar{t})$ belongs to the range of $G(\bar{t})$. The example below illustrates this point.

Example 13.7 Consider the 2-periodic polynomials

$$a(\sigma, t) = \sigma^2 + 2\sigma + a_2(t), \quad b(\sigma, t) = 0.5\sigma + 1, \quad a_2(0) = 0, \quad a_2(1) = 3.$$

Notice that the singularity of $a_2(0)$ prevents the strong right coprimeness of $a(\sigma, t)$ and $b(\sigma, t)$. This reflects in the fact that the associated Sylvester matrix $G(t)$ is not invertible for each t . Indeed,

$$G(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0.5 & 0 \\ 3 & 2 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G(1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0.5 & 0 \\ 0 & 2 & 1 & 0.5 \\ 0 & 3 & 0 & 1 \end{bmatrix},$$

so that $G(0)$ is singular. Nevertheless, one can assign a closed-loop polynomial $c(t)$ provided that $c(0) \in \text{Im}[G(0)]$. For instance, take the constant (stable) polynomial

$$c(\sigma, t) = \sigma^3 + 0.5\sigma^2.$$

Notice that

$$c(0) = \begin{bmatrix} 1 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

belongs to the range of $G(0)$. Then, the equation $c(t) = G(t)\xi(t)$ is solvable and a solution is

$$\xi(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad \xi(1) = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 6 \end{bmatrix},$$

so yielding

$$p(\sigma, t) = \sigma + p_1(t), \quad p_1(0) = -1, \quad p_1(1) = -2$$

$$q(\sigma, t) = q_0(t)\sigma + q_1(t), \quad q_0(0) = -1, \quad q_0(1) = 1, \quad q_1(0) = 0, \quad q_1(1) = 6.$$

■

We end this section by studying the pole assignment problem for time-invariant plants. It is well-known that with a constant output-feedback control law it is not possible to freely assign the closed-loop poles. We will now discuss the action of a periodic static output-feedback control law on the closed-loop periodic polynomial. It will be shown that with a suitable choice of the period, it is generically possible to assign the closed-loop poles. Therefore, let

$$k(\sigma, t) = -p(t)^{-1}q(t), \quad p(t) \neq 0, \forall t$$

be the periodic output-feedback gain. The period T of the periodic coefficients $p(t)$ and $q(t)$ is a design parameter. Taking

$$T = n + 1,$$

the closed-loop polynomial becomes

$$c(\sigma, t) = p(t)(a_0 + a_1\sigma + \cdots + a_n\sigma^n) + q(t)(b_0 + b_1\sigma + \cdots + b_{n-1}\sigma^{n-1}). \quad (13.42)$$

The lifted polynomial associated with (13.42) takes on the form

$$C(\sigma^T, t) = \begin{bmatrix} p(t)a_0 + q(t)b_0 & \cdots & p(t)a_n \\ \vdots & \ddots & \vdots \\ (p(t+n)a_1 + q(t+n)b_1)\sigma^T & \cdots & p(t+n)a_0 + q(t+n)b_0 \end{bmatrix}.$$

Recall that the closed-loop characteristic multipliers are the zeros of the determinant of $C(\sigma^T, t)$ and that the determinant of such a matrix does not depend on t . On the other hand, the structure of $C(\sigma^T, t)$ is such that the coefficients of its determinant can be generically assigned by choosing the pair $(p(t), q(t))$ for $t = 0, 1, \dots, n = T - 1$.

13.8.2 Dynamic Robust Polynomial Assignment

We now assume that the plant is uncertain in the sense that the coefficients of the system belong to a polytopic region. To be precise, letting

$$g(t) = [a(t)' \ b(t)']',$$

we consider the plant's parameters as unknowns belonging to the set

$$g(t) \in \mathcal{P}_G(t) = \text{convex hull} \{g(t)^1, g(t)^2, \dots, g(t)^k\}.$$

The set $\mathcal{P}_G(t)$ will be referred to as the plant polytope.

Obviously, due to plant uncertainties, the pole assignment problem makes sense only if some freedom in the choice of the closed-loop polynomial is allowed. Here, such flexibility is captured by introducing a bounded convex set of the coefficients of the closed-loop characteristic polynomial. To be precise, we assume that the set is a polytope defined as follows. Denoting again as $c(t)$ the vector of all coefficients of the closed-loop polynomial, this polytope can be written as

$$\mathcal{P}_C = \{c(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{2n}, \sum_{t=0}^{T-1} H(t)c(t) \geq h\},$$

where $H(t)$ are real matrices, and the symbol \geq is to be interpreted element-wise. Note that the set \mathcal{P}_C is defined as the intersection of a number of semi-planes in \mathbb{R}^{2nT} , each of which given by the condition $\sum_{t=0}^{T-1} H_i(t)c(t) \geq h_i \geq 0$, where $H_i(t)$ is the i -th row of $H(t)$ and h_i is the i -th element of vector h .

The problem is to design a controller such that the closed-loop polynomial associated with $\{c(0), c(1), \dots, c(T-1)\}$, given by (13.41), belongs to \mathcal{P}_C . As in the previous section, the controller parameters are organized in two vectors, $p(t)$ and $q(t)$, forming the global controller vector $\xi(t)$. Now, given $\xi(t)$, the feasible set of closed-loop polynomials is given by

$$\text{convex hull } \{G^1(t)\xi(t), G^2(t)\xi(t), \dots, G^k(t)\xi(t)\},$$

where $G^i(t)$ is the Sylvester matrix associated with the vector $g^i(t)$, $t = 0, 1, \dots, T-1$ of the plant polytope $\mathcal{P}_G(t)$.

Therefore, thanks to convexity, we can say that $\xi(t)$ results into a robust controller if and only if

$$\sum_{i=0}^{T-1} H(t)G^j(t)\xi(t) \geq h, \quad j = 1, 2, \dots, k. \quad (13.43)$$

Any solution $\xi(\cdot)$ of the set of linear inequalities (13.43) represents a robust controller. The set of such solutions is a convex polytope is denoted with the symbol \mathcal{P}_ξ . Any solution $\{\xi(0), \xi(1), \dots, \xi(T-1)\}$ can be parametrized as a linear combination of its vertices, although these vertices can be hardly computed in advance. Of course, if some plant coefficients were known, the complexity of the linear programming problem (13.43) reduces accordingly.

The robust polynomial assignment problem considered so far can be complemented with some further requirements concerning a target closed-loop polynomial defined by $c^0(\cdot)$. To be precise, the designer may aim at selecting, among the feasible robust controllers, the one minimizing the distance, in some sense to be specified, from the target closed-loop polynomial defined by $c^0(t)$. Such distance can be characterized in a variety of ways. A simple choice is the quadratic cost

$$J_1 = \min_{\xi(\cdot) \in \mathcal{P}_\xi} \sum_{t=0}^{T-1} \sum_{i=1}^k \|G^i(t)\xi(t) - c^0(t)\|^2. \quad (13.44)$$

Such minimum is a measure of the distance between the target polynomial and the achievable ones. Another possibility is the following one. Associated with the vector of the coefficients of the target polynomial $c^0(t)$, consider the vector c_L^0 of the coefficients of the lifted polynomial. Analogously vector $(G\xi)_L$ denotes the vector of the coefficients of the lifted closed-loop polynomial associated with the controller defined by $\xi(t)$ and the system model defined by the Sylvester matrix $G(t)$. Then, one can consider the cost

$$J_2 = \min_{\xi(\cdot) \in \mathcal{P}_\xi} \max_{g(t) \in \mathcal{P}_{G(t)} \forall t} \|(G\xi)_L - c_L^0\|^2. \quad (13.45)$$

The minimization problem (13.44) is a convex optimization problem, since the cost is quadratic with respect to the unknown and the set \mathcal{P}_ξ is characterized by linear constraints. Conversely, (13.45) is highly non-linear and finding its solution is a numerical complex problem.

We illustrate these ideas by means of a simple numerical example relative to a time-invariant system. In particular, we show that a periodic controller can reduce the spread around the target closed-loop polynomial.

Example 13.8 *Consider the scalar time-invariant system*

$$x(t+1) = -a_1x(t) + u(t), \quad y(t) = x(t),$$

and assume that

$$a_1 \in \mathcal{A}, \quad \mathcal{A} = [-6, -5].$$

We take as a target closed-loop pole the origin (dead-beat control). Due to the uncertainties in the system's parameters, one cannot impose a sharp closed-loop polynomial. Rather, we require that the closed-loop poles be located in a certain range. In this regard, we will first study the achievable performances with a time-invariant controller and then we will discuss the robustness improvements achievable with a periodic control law. In the time-invariant case, we consider the static output-feedback control law $u(t) = -q_0y(t)$. The correspondent closed-loop polynomial is $\sigma + a_1 + q_0 = \sigma + c_1$ and it is possible to tolerate a maximum mismatch in the pole position not larger than $|c_1| = 0.5$. Indeed

$$\min_{q_0} \max_{a_1(\cdot) \in \mathcal{A}} |c_1| = 0.5.$$

The value of q_0 such that $\max_{a_1 \in \mathcal{A}} |c_1| = 0.5$ is $q_0 = 5.5$.

Turn now to a periodic static output-feedback controller with period $T = 2$. The closed-loop polynomial is then $\sigma + a_1 + q_0(t) = \sigma + c_1(t)$. Its lifted reformulation is $\sigma^2 - c_1(0)c_1(1)$. Obviously, if $q_0(0) = q_0(1)$, then $c_1(0) = c_1(1)$ and the previous case is recovered yielding $\max_{a_1 \in \mathcal{A}} |c_1(0)c_1(1)| = 0.25$. This result can be improved

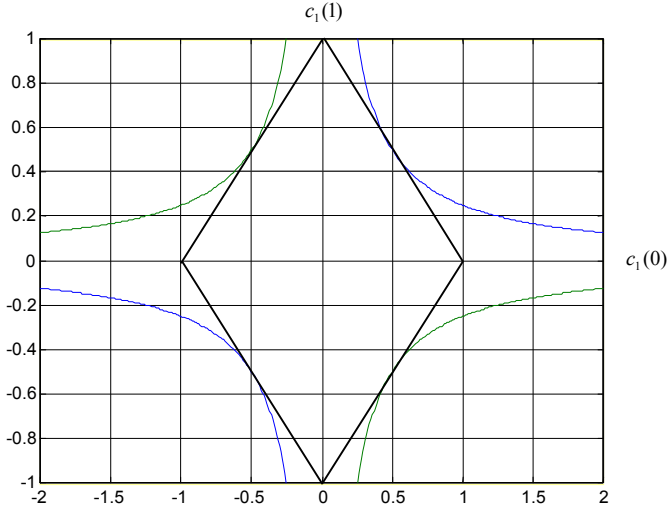


Fig. 13.3 Region of coefficients $c_1(0)$ and $c_1(1)$ satisfying Inequality (13.46) (thin line) and an approximating polytope

by letting $q_0(0) \neq q_0(1)$. Unfortunately, the region described by

$$|c_1(0)c_1(1)| \leq 0.25 \quad (13.46)$$

is not convex (see Fig. 13.3). An easy way to approximate this region with a polytope is illustrated in the same figure. Letting

$$c(t) = \begin{bmatrix} 1 \\ c_0(t) \end{bmatrix},$$

this polytope is characterized by the condition

$$H(0)c(0) + H(1)c(1) \geq h,$$

where

$$H(0) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad H(1) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad h = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

The Sylvester matrix is

$$G(t) = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix}, \quad t = 0, 1,$$

whose vertices are

$$G^1 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}, \quad G^2 = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}.$$

Letting

$$\xi(t) = \begin{bmatrix} 1 \\ q_0(t) \end{bmatrix},$$

the robust constraint (13.43) on the controller parameters is

$$H(0)G^1\xi(0) + H(1)G^1\xi(1) \geq h, \quad H(0)G^2\xi(0) + H(1)G^2\xi(1) \geq h.$$

The set of feasible controllers is characterized by the conditions

$$5 \leq q_0(0) \leq 6, \quad 5 \leq q_0(1) \leq 6, \quad q_0(0) + q_0(1) = 11.$$

We now consider the objective of obtaining a closed-loop behavior close a dead-beat one. Taking $c^0(t) = [1 \ 0]'$, $\forall t$, the solution of (13.44) leads again to the time-invariant controller $q_0(0) = q_0(1) = 5.5$ yielding $J_1 = 0.25$. On the other hand, notice that the min-max problem defined in (13.44) takes now the simple form

$$J_2 = \min_{5 \leq q_0(0) \leq 6} \max_{a_1 \in \mathcal{A}} |(a_1 + q_0(0))(a_1 + 11 - q_0(0))|.$$

The one obtains two possible controllers, one given by

$$q_0(0) = \frac{11}{2} - \frac{\sqrt{2}}{4}, \quad q_0(1) = \frac{11}{2} + \frac{\sqrt{2}}{4},$$

and the other one by

$$q_0(0) = \frac{11}{2} + \frac{\sqrt{2}}{4}, \quad q_0(1) = \frac{11}{2} - \frac{\sqrt{2}}{4}.$$

The corresponding cost is $J_2 = \frac{1}{8}$.

13.8.3 Stabilization with Input Constraints and Uncertain Initial State

In this section, we address the problem of stabilizing a system with uncertain initial state and constraints on the overshoot of the control action. It will be seen how this leads to a linear programming problem. Again, we refer to SISO systems in the input–output description; however the initial state $x(0)$ plays a major role, and therefore must be explicitated in the model. Correspondingly, we consider the output $y(t)$ of a strictly proper discrete-time periodic SISO system with initial state $x(0)$ as described by the model

$$\begin{aligned}
a_0(t)y(t) &= a_1(t)y(t-1) + a_2(t)y(t-2) + \cdots + a_n(t)y(t-n) + \\
&+ b_1(t)u(t-1) + b_2(t)u(t-2) + \cdots + b_n(t)u(t-n) + \\
&+ [c_0\delta(t) + c_1\delta(t-1) + \cdots + c_{n-1}\delta(t-n+1)]x(0), \quad (13.47)
\end{aligned}$$

where all coefficients $a_i(\cdot)$ and $b_j(\cdot)$ are T -periodic, whereas the c_k 's, are constant row vectors. Finally, $\delta(t-i)$ is the (discrete-time) impulse function applied at time $t=i$. The linear regression (13.47) can be written in a compact form by introducing the unit delay operator σ^{-1} operating on the signals $u(\cdot)$ and $\delta(\cdot)$. Therefore

$$y(t) = \bar{a}(\sigma^{-1}, t)^{-1} \bar{b}(\sigma^{-1}, t) \cdot u(t) + \bar{a}(\sigma^{-1}, t)^{-1} c(\sigma^{-1}) \cdot \delta(t)x(0), \quad (13.48)$$

where $\bar{a}(\sigma^{-1}, t)$, $\bar{b}(\sigma^{-1}, t)$ and $c(\sigma^{-1}, t)$ are the polynomials

$$\begin{aligned}
\bar{a}(\sigma^{-1}, t) &= a_0(t) - a_1(t)\sigma^{-1} - \cdots - a_n(t)\sigma^{-n} \\
\bar{b}(\sigma^{-1}, t) &= b_1(t)\sigma^{-1} + b_2(t)\sigma^{-2} + \cdots + b_n(t)\sigma^{-n} \\
c(\sigma^{-1}) &= c_0 + c_1\sigma^{-1} + \cdots + c_{n-1}\sigma^{-n+1}.
\end{aligned}$$

In the developments to follow we need the results concerning the parametrization of the class of stabilizing controllers, worked out in Sect. 10.4.1. Therefore, we assume that the state-space system behind (13.48) is reachable and observable for each time point. Correspondingly, the transfer operator $g(\sigma^{-1}, t)$ can be written as

$$g(\sigma^{-1}, t) = \bar{a}(\sigma^{-1}, t)^{-1} \bar{b}(\sigma^{-1}, t) = \tilde{b}(\sigma^{-1}, t) \tilde{a}(\sigma^{-1}, t)^{-1},$$

where the pair $(\bar{a}(\sigma^{-1}, t), \bar{b}(\sigma^{-1}, t))$ is strongly right coprime and, moreover, the pair $(\tilde{a}(\sigma^{-1}, t), \tilde{b}(\sigma^{-1}, t))$ is strongly left coprime (see Chap. 5). Consequently, it is possible to work out T -periodic polynomials $x(\sigma^{-1}, t)$, $w(\sigma^{-1}, t)$, $\tilde{x}(\sigma^{-1}, t)$ and $\tilde{w}(\sigma^{-1}, t)$ such that

$$\bar{a}(\sigma^{-1}, t)x(\sigma^{-1}, t) + \bar{b}(\sigma^{-1}, t)w(\sigma^{-1}, t) = 1 \quad (13.49a)$$

$$\tilde{x}(\sigma^{-1}, t)\tilde{a}(\sigma^{-1}, t) + \tilde{w}(\sigma^{-1}, t)\tilde{b}(\sigma^{-1}, t) = 1 \quad (13.49b)$$

$$\tilde{x}(\sigma^{-1}, t)w(\sigma^{-1}, t) = \tilde{w}(\sigma^{-1}, t)x(\sigma^{-1}, t). \quad (13.49c)$$

The system is assumed to be driven by a T -periodic causal controller fed by $y(\cdot)$ and yielding $u(\cdot)$. The controller is characterized by the associated transfer function operator $k(\sigma^{-1}, t)$, such that $u(t) = -k(\sigma^{-1}, t)y(t)$. By considering $k(\sigma^{-1}, t)$ in its right and left strong coprime factorizations

$$k(\sigma^{-1}, t) = \bar{r}(\sigma^{-1}, t)^{-1} \bar{s}(\sigma^{-1}, t) = \tilde{s}(\sigma^{-1}, t) \tilde{r}(\sigma^{-1}, t)^{-1},$$

the control law can be given the form:

$$u(t) = -\bar{r}(\sigma^{-1}, t)^{-1} \bar{s}(\sigma^{-1}, t)y(t) = -\tilde{s}(\sigma^{-1}, t) \tilde{r}(\sigma^{-1}, t)^{-1}y(t). \quad (13.50)$$

It is also assumed that the input of the system is constrained to satisfy the hard bound

$$-u^- \leq u(t) \leq u^+, \quad \forall t \geq 0 \quad (13.51)$$

and the initial state $x(0)$ is uncertain, in the sense that it belongs to a given polyhedron \mathcal{X}_N , i.e.,

$$x(0) \in \mathcal{X}_N = \{x(0) : Nx(0) \leq v\}. \quad (13.52)$$

Obviously, the polyhedron is defined by matrix N and vector v . With all these notations the problem is as follows.

Problem 13.1. Find a causal dynamic T -periodic controller (13.50) such that the closed-loop System (13.48), (13.50) is stable and satisfies the input constraints (13.51) for all initial states $x(0)$ belonging to the uncertainty set (13.52). ■

In order to solve it we need two preliminary results. The first one is the parametrization of all periodic stabilizing controllers, already discussed in Chap. 10, see Proposition 10.1. To be precise, let $x(\sigma^{-1}, t)$, $\tilde{x}(\sigma^{-1}, t)$, $\bar{x}(\sigma^{-1}, t)$ and $\tilde{w}(\sigma^{-1}, t)$ be particular solutions of the operator equations (13.49). Then, all linear T -periodic stabilizing controllers for (13.48) are given by (13.50) with

$$\begin{aligned} \tilde{r}(\sigma^{-1}, t) &= x(\sigma^{-1}, t) + \tilde{b}(\sigma^{-1}, t)q(\sigma^{-1}, t) \\ \tilde{s}(\sigma^{-1}, t) &= w(\sigma^{-1}, t) - \tilde{a}(\sigma^{-1}, t)q(\sigma^{-1}, t), \end{aligned}$$

or, equivalently,

$$\bar{r}(\sigma^{-1}, t) = \tilde{x}(\sigma^{-1}, t) + q(\sigma^{-1}, t)b(\sigma^{-1}, t) \quad (13.53a)$$

$$\bar{s}(\sigma^{-1}, t) = \tilde{w}(\sigma^{-1}, t) - q(\sigma^{-1}, t)a(\sigma^{-1}, t), \quad (13.53b)$$

where $q(\sigma^{-1}, \cdot)$ is a free stable periodic operator such that $\tilde{r}(\sigma^{-1}, \cdot)$ and $\bar{r}(\sigma^{-1}, \cdot)$ are not identically zero.

The second preliminary result refers to the so-called *Farkas Lemma* :

Lemma 13.1 A polyhedron $\mathcal{X}_N = \{x(0) : Nx(0) \leq v\}$ is included in a polyhedron $\mathcal{X}_M = \{x(0) : Mx(0) \leq \mu\}$ if and only if there exists a matrix P with non-negative entries such that

$$PN = M, \quad Pv \leq \mu, \quad (13.54)$$

where the symbol \leq is to be interpreted elementwise. ■

Now, from equations (13.48), (13.49), (13.50) it follows

$$\begin{aligned} u(t) &= \\ &= -\bar{r}(\sigma^{-1}, t)^{-1}\bar{s}(\sigma^{-1}, t) (\bar{a}(\sigma^{-1}, t)^{-1}\bar{b}(\sigma^{-1}, t)u(t) + \bar{a}(\sigma^{-1}, t)^{-1}c(\sigma^{-1})\delta(t)x(0)) \\ &= -\bar{r}(\sigma^{-1}, t)^{-1}\bar{s}(\sigma^{-1}, t) (\tilde{b}(\sigma^{-1}, t)\tilde{a}(\sigma^{-1}, t)^{-1}u(t) + a(\sigma^{-1}, t)^{-1}c(\sigma^{-1})\delta(t)x(0)), \end{aligned}$$

so that, from (13.53) and (13.49b),

$$\begin{aligned} u(t) &= -\tilde{a}(\sigma^{-1}, t) \tilde{s}(\sigma^{-1}, t) \bar{a}(\sigma^{-1}, t)^{-1} c(\sigma^{-1}) \delta(t) x(0) \\ &= -\tilde{a}(\sigma^{-1}, t) (\tilde{w}(\sigma^{-1}, t) \bar{a}(\sigma^{-1}, t)^{-1} - q(\sigma^{-1}, t)) c(\sigma^{-1}) \delta(t) x(0). \end{aligned}$$

From (13.49) notice that

$$\begin{aligned} w(\sigma^{-1}, t) &= \tilde{a}(\sigma^{-1}, t) \tilde{x}(\sigma^{-1}, t) w(\sigma^{-1}, t) + \\ &\quad + \tilde{a}(\sigma^{-1}, t) \tilde{w}(\sigma^{-1}, t) \bar{a}(\sigma^{-1}, t)^{-1} b(\sigma^{-1}, t) w(\sigma^{-1}, t) \\ &= \tilde{a}(\sigma^{-1}, t) \tilde{x}(\sigma^{-1}, t) w(\sigma^{-1}, t) + \\ &\quad + \tilde{a}(\sigma^{-1}, t) \tilde{w}(\sigma^{-1}, t) \bar{a}(\sigma^{-1}, t)^{-1} (1 - \bar{a}(\sigma^{-1}, t) x(\sigma^{-1}, t)) \\ &= \tilde{a}(\sigma^{-1}, t) \tilde{w}(\sigma^{-1}, t) \bar{a}(\sigma^{-1}, t)^{-1} + \\ &\quad + \tilde{a}(\sigma^{-1}, t) (\tilde{x}(\sigma^{-1}, t) w(\sigma^{-1}, t) - \tilde{w}(\sigma^{-1}, t) x(\sigma^{-1}, t)) \\ &= \tilde{a}(\sigma^{-1}, t) \tilde{w}(\sigma^{-1}, t) \bar{a}(\sigma^{-1}, t)^{-1}. \end{aligned}$$

Hence

$$u(t) = (\tilde{a}(\sigma^{-1}, t) q(\sigma^{-1}, t) - w(\sigma^{-1}, t)) c(\sigma) \delta(t) x(0).$$

It is possible to write

$$(\tilde{a}(\sigma^{-1}, t) q(\sigma^{-1}, t) - w(\sigma^{-1}, t)) c(\sigma^{-1}) = \sum_{k=0}^{\infty} \bar{M}_k(q) \sigma^{-k},$$

where the elements $\bar{M}_k(q)$ are row vectors of dimension n . They depend affinely on the parameters of the polynomial

$$q(\sigma^{-1}, t) = q_0(t) + q_1(t) \sigma^{-1} + q_2(t) \sigma^{-2} + \dots.$$

It is then apparent that any value of the input at time $i \geq 0$ can be written as a linear function of the coefficients $q_k(j)$, $j = 0, 1, \dots, i$, $k = 0, 1, \dots, j$. Hence, with a little abuse of notation, we can write

$$u(i) = \bar{M}_i(\bar{q}) x(0),$$

where

$$\bar{q} = [q_0(0) \dots q_0(T-1) \ q_1(0) \dots q_1(T-1) \dots]'$$

The input $u(i)$ satisfies constraint (13.51) if and only if

$$x(0) \in \mathcal{X}_M = \{x(0) : M(\bar{q})x(0) \leq \mu\},$$

where

$$M(\bar{q}) = \begin{bmatrix} \bar{M}_0(q) \\ -\bar{M}_0(q) \\ \bar{M}_1(q) \\ -\bar{M}_1(q) \\ \vdots \end{bmatrix}, \quad \mu = \begin{bmatrix} \bar{u}^+ \\ \bar{u}^- \\ \bar{u}^+ \\ \bar{u}^- \\ \vdots \end{bmatrix}.$$

By recalling Lemma 13.1, we are now in the position to state the following result, whose proof is immediate and therefore omitted.

Proposition 13.17 *Problem 13.1 is solved if and only if there exists a matrix P with non-negative entries and a stable $q(\sigma^{-1}, t)$ such that the following condition*

$$PN = M(\bar{q}), \quad Pv \leq \mu$$

is satisfied. ■

Notice that, in order to find the free stable operator $q(\sigma^{-1}, t)$, one has to solve, in principle, an infinite linear programming problem. When $q(\sigma^{-1}, t)$ is taken as a finite-degree polynomial, however, the problem becomes finite-dimensional and one has to find a non-negative P and the finite set of coefficients $q_i(t)$ which solve the equations in Proposition 13.17. Notice also that the case when the free-parameter operator is indeed a finite-degree periodic polynomial corresponds to the choice of a dead-beat periodic controller, forcing the input variable $u(t)$ to be zeroed in finite time.

The above procedure is illustrated in the following simple example, which refers to a time-invariant system with a 2-periodic controller.

Example 13.9 *Consider the following time-invariant system:*

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned}$$

with

$$A = \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0].$$

The input of the system is constrained to

$$-7 \leq u(t) \leq 7,$$

and the initial state $x(0)$ is assumed to belong to the polyhedron

$$\mathcal{X}_N = \{x(0) : Nx(0) \leq v\},$$

where

$$N = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ -1.5 & 2 \\ 1.5 & -2 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 5 \\ 10 \\ 10 \end{bmatrix}.$$

Our aim is to find a stabilizing periodic controller. We begin by considering the polynomial representation of the plant and of a stabilizing controller. It follows

$$\begin{aligned}\bar{a}(\sigma^{-1}, t) &= \tilde{a}(\sigma^{-1}, t) = \sigma^{-2} - 1.7241\sigma + 0.8621 \\ \bar{b}(\sigma^{-1}, t) &= \tilde{b}(\sigma^{-1}, t) = 0.431\sigma^{-2} \\ \bar{c}(\sigma^{-1}, t) &= [0.8621 - 1.0345\sigma^{-1} \quad 0.431\sigma^{-1}] \\ x(\sigma^{-1}, t) &= \tilde{x}(\sigma^{-1}, t) = 1.16 + 2.32\sigma^{-1} \\ w(\sigma^{-1}, t) &= \tilde{w}(\sigma^{-1}, t) = 6.5888 - 5.3824\sigma^{-1}.\end{aligned}$$

The minimum degree of a fixed-order constant polynomial $q(\sigma^{-1})$ such that Problem 13.1 admits a solution is 3. Hereafter, we show that the problem is indeed solvable by looking at a periodic polynomial

$$q(\sigma^{-1}, t) = q_0(t) + q_1(t)\sigma^{-1}$$

of period $T = 2$ and degree 1. Numerical computations on the polynomial

$$(\tilde{a}(\sigma^{-1})q(\sigma^{-1}, t) - w(\sigma^{-1}))c(\sigma^{-1}),$$

were first performed in order to obtain the 10×2 -dimensional matrices R_i , $i = 0, 1, 2, 3, 4$, yielding

$$M(\bar{q}) = R_0 + R_1 q_0(0) + R_2 q_0(1) + R_3 q_1(0) + R_4 q_1(1).$$

Then, the linear programming toolbox of MATLAB[®] has been used to find matrix P and vector \bar{q} satisfying Proposition 13.17. The matrices are

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.6967 & 0 & 0.3517 \\ 0.6967 & 0 & 0.3517 & 0 \\ 0.7871 & 0 & 0.3065 & 0 \\ 0 & 0.7871 & 0.3065 & 0 \\ 0.3035 & 0 & 0 & 0 \\ 0 & 0.3035 & 0 & 0 \\ 0 & 0.976 & 0.1715 & 0 \\ 0.0976 & 0 & 0 & 0.1715 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.1691 & -2.0967 \\ 0.1691 & 2.0967 \\ 1.2467 & -1.2467 \\ -1.2467 & 1.2467 \\ 0.3036 & 0.6070 \\ -0.3036 & -0.6070 \\ -0.3549 & 0.1479 \\ 0.3549 & -0.1479 \end{bmatrix},$$

and the polynomial $q(\sigma^{-1}, t)$ reads

$$q(\sigma^{-1}, t) = \begin{cases} 7.6429 + 0.3431\sigma^{-1}, & t = 0 \\ 2 + 2.042\sigma^{-1}, & t = 1. \end{cases}$$

The 2-periodic controller turns out to be of third order. It has been found by solving (13.53), i.e.,

$$\begin{aligned} r(\sigma^{-1}, t) &= \begin{cases} 1.16 + 2.32\sigma^{-1} + 3.294\sigma^{-2} + 0.148\sigma^{-3} & t = 0 \\ 1.16 + 2.32\sigma^{-1} + 0.862\sigma^{-2} + 0.88\sigma^{-3} & t = 1 \end{cases} \\ \bar{s}(\sigma^{-1}, t) &= \begin{cases} 7.499\sigma^{-1} - 7.05\sigma^{-2} - 0.343\sigma^{-3} & t = 0 \\ 4.8648 - 3.695\sigma^{-1} + 1.521\sigma^{-2} - 2.042\sigma^{-3} & t = 1 \end{cases}. \end{aligned}$$

In conclusion, the controller can be given the recursive representation

$$\begin{aligned} u(t) &= -\alpha_1(t)u(t-1) - \alpha_2(t)u(t-2) - \alpha_3(t-3)u(t-3) \\ &\quad + \beta_0(t)y(t) + \beta_1(t)y(t-1) + \beta_2y(t-2) \\ &\quad + \beta_3(t)y(t-3), \end{aligned}$$

where

$$\alpha_1(0) = \alpha_1(1) = 2, \alpha_2(0) = 2.8397, \alpha_2(1) = 0.743, \alpha_3(0) = 0.1275, \alpha_3(1) = 0.7587,$$

and

$$\begin{aligned} \beta_0(0) &= 0, \beta_0(1) = 4.1938, \beta_1(0) = 6.4646, \beta_1(1) = -3.1853, \\ \beta_2(0) &= -6.0788, \beta_2(1) = 1.311, \beta_3(0) = -0.2958, \beta_3(1) = -1.7603. \end{aligned}$$

Equivalently, the periodic controller can be given a state-space description in canonical reachability form as follows

$$\begin{aligned} x_r(t+1) &= A_r(t)x_r(t) + B_r(t)y(t) \\ u(t) &= C_r(t)x_r(t) + D_r(t)y(t), \end{aligned}$$

with

$$\begin{aligned} A_r(t) &= \begin{bmatrix} 0 & 0 & \alpha_3(t+3) \\ 1 & 0 & \alpha_2(t+2) \\ 0 & 1 & \alpha_1(t+1) \end{bmatrix}, \\ B_r(t) &= \begin{bmatrix} \beta_3(t+3) - \alpha_3(t+3)\beta_0(t) \\ \beta_2(t+2) - \alpha_2(t+2)\beta_0(t) \\ \beta_1(t+1) - \alpha_1(t+1)\beta_0(t) \end{bmatrix} \\ C_r(t) &= [0 \ 0 \ 1], \quad D_r(t) = \beta_0(t). \end{aligned}$$

The closed-loop system is a 5th order 2-periodic stable system. The input variable $u(t)$ associated with the initial state

$$x(0) = \begin{bmatrix} 6 \\ -0.5 \end{bmatrix}$$

is

$$u(t) = \begin{cases} 0 & t = 0 \\ 0.03 & t = 1 \\ 7 & t = 2 \\ 1.53 & t = 3 \\ -2.2 & t = 4 \\ 0 & t \geq 5 \end{cases}.$$

As can be seen, the input variable satisfies the given constraints and goes to zero in finite time, thanks to the choice of a finite-degree free parameter $q(\sigma^{-1}, t)$.

13.8.4 LQG Control

This section is devoted to the classical Linear Quadratic Gaussian (LQG) problem. This problem refers to systems subject to additive stochastic disturbances, with known mean and known covariance matrices. The objective is to find an output-feedback controller minimizing the expected value of a non-negative quadratic index in the state and input variables. As such, the LQG problem can be considered as the stochastic version of the H_2 output-feedback problem.

The system under consideration is described by:

$$x(t+1) = A(t)x(t) + \bar{B}(t)w_1(t) + B(t)u(t) \quad (13.55a)$$

$$y(t) = C(t)x(t) + \bar{N}(t)w_2(t), \quad (13.55b)$$

where $w_1(\cdot)$ and $w_2(\cdot)$ are disturbances described as uncorrelated white Gaussian stochastic processes, with zero mean and identity covariance matrices. Notice that, without any loss of generality, the input–output matrix relating $u(t)$ with $y(t)$ has been set to zero, since both signals are measurable. The performance index is

$$J = \lim_{\substack{t_0 \rightarrow -\infty \\ t_1 \rightarrow \infty}} \mathcal{E} \left[\frac{1}{t_1 - t_0} \sum_{t=t_0}^{t_1-1} x(t)' \Theta(t) x(t) + u(t)' \Lambda(t) u(t) \right], \quad (13.56)$$

where \mathcal{E} denotes expected value. All matrices are T -periodic; moreover, $\Theta(t) \geq 0$, $\bar{N}(t)\bar{N}(t)' > 0$ and $\Lambda(t) > 0$, for each t . We make the following assumptions:

- (i) The pairs $(A(\cdot), B(\cdot))$ and $(A(\cdot), \bar{B}(\cdot))$ are stabilizable.
- (ii) The pairs $(A(\cdot), C(\cdot))$ and $(A(\cdot), \Theta(\cdot))$ are detectable.

The solution is based on the following Riccati equations:

$$P(t+1) = A(t)P(t)A(t)' + \bar{B}(t)\bar{B}(t)' \quad (13.57a)$$

$$-A(t)P(t)C(t)'(\bar{N}(t)\bar{N}(t)' + C(t)P(t)C(t))^{-1}C(t)P(t)A(t)'$$

$$Q(t) = A(t)'Q(t+1)A(t) + \Theta(t) \quad (13.57b)$$

$$-A(t)'Q(t+1)B(t)(\Lambda(t) + B(t)'Q(t+1)B(t))^{-1}B(t)'Q(t+1)A(t).$$

Equation (13.57a) refers to the H_2 filtering problem for the estimation of the state variable from the measurements, whereas (13.57b) is associated to the state-feedback control problem.

Proposition 13.18 *Under assumptions (i) and (ii), the optimal LQG controller is given by*

$$\hat{x}(t+1) = A(t)\hat{x}(t) + B_F(t)(y(t) - C(t)\hat{x}(t)) + B(t)u(t) \quad (13.58a)$$

$$u(t) = K(t)\hat{x}(t) + D_F(t)(y(t) - C(t)\hat{x}(t)), \quad (13.58b)$$

where

$$\begin{aligned} B_F(t) &= A(t)P(t)C(t)'(\bar{N}(t)\bar{N}(t)' + C(t)P(t)C(t)')^{-1} \\ D_F(t) &= K(t)P(t)C(t)'(C(t)P(t)C(t)' + \bar{D}(t)\bar{D}(t)')^{-1} \\ K(t) &= -(\Lambda(t) + B(t)'Q(t+1)B(t))^{-1}B(t)'Q(t+1)A(t), \end{aligned}$$

and $P(\cdot)$ and $Q(\cdot)$ are the T -periodic stabilizing (positive semi-definite) solutions of the Riccati Equations (13.57). Moreover, the optimal closed-loop system Σ_{LQG} given by (13.55), (13.58) is stable and the corresponding value of the performance index (13.56) is given by

$$\begin{aligned} J^0 &= \\ &= \frac{1}{T} \sum_0^{T-1} \text{trace} [K(t)'(\Lambda(t) + B(t)'Q(t+1)B(t)')K(t)P(t) + Q(t+1)\bar{B}(t)\bar{B}(t)'] \\ &= \frac{1}{T} \sum_0^{T-1} \text{trace} [Q(t+1)B_F(t)(\bar{N}(t)\bar{N}(t)' + C(t)P(t)C(t)')B_F(t)' + P(t)\Theta(t)]. \end{aligned}$$

Proof First, let

$$z(t) = E(t)x(t) + N(t)u(t) := \begin{bmatrix} \Theta(t)^{1/2} \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \Lambda(t)^{1/2} \end{bmatrix} u(t). \quad (13.59)$$

Then, the performance index can be compactly written as

$$J = \lim_{\substack{t_0 \rightarrow -\infty \\ t_1 \rightarrow \infty}} \mathcal{E} \left[\frac{1}{t_1 - t_0} \sum_{t=t_0}^{t_1-1} z(t)'z(t) \right].$$

Thanks to the assumptions, the two Riccati equations admit their respective periodic positive semi-definite stabilizing solutions. Hence, controller (13.58) makes sense and it is possible to define the closed-loop system Σ_{LQG} . Letting $\eta(t) = x(t) - \hat{x}(t)$ this system takes on the form

$$\begin{aligned}
\begin{bmatrix} x(t+1) \\ \eta(t+1) \end{bmatrix} &= \begin{bmatrix} A(t) + B(t)K(t) & -B(t)(K(t) - D_F(t)C(t)) \\ 0 & A(t) - B_F(t)C(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \\
&\quad + \begin{bmatrix} \bar{B}(t) & B(t)D_F(t)\bar{D}(t) \\ \bar{B}(t) & -B_F(t)N(t) \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \\
z(t) &= \begin{bmatrix} E(t) + N(t)K(t) & -N(t)(K(t) - D_F(t)C(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix}.
\end{aligned}$$

The dynamic matrix of this system is block-triangular, with $A(t) + B(t)K(t)$ and $A(t) - B_F(t)C(t)$ as block diagonal elements. These periodic matrices are stable since the gains $K(\cdot)$ and $B_F(\cdot)$ are computed on the basis of stabilizing solutions of Riccati equations. Consequently, the overall system Σ_{LQG} (defined by (13.55) and (13.58)) is stable. Therefore $z(\cdot)$ asymptotically converges to a cyclostationary process.

So far, recall the stochastic interpretation of the L_2 norm discussed in Chap. 9 (see Eq. (9.8b)). Then, one can write:

$$J = \frac{1}{T} \lim_{N \rightarrow \infty} \mathcal{E} \left[\frac{1}{N} \sum_{t=t_0}^{NT-1} z(t)' z(t) \right] = \frac{1}{T} \|\Sigma_{LQG}\|_2^2.$$

Therefore the original LQG problem is equivalent to that of minimizing the H_2 norm of system Σ_{LQG} . To solve this minimization problem, we replace the input variable $u(t)$ appearing in the original expression of the system with the new input variable

$$v(t) = u(t) - K(t)x(t).$$

By also defining

$$w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix},$$

we can formally write

$$z(t) = R(\sigma, t)v(t) + S(\sigma, t)w(t), \quad (13.60)$$

where $R(\cdot, \cdot)$ is the transfer operator of the system

$$\begin{aligned}
x_R(t+1) &= (A(t) + B(t)K(t))x_R(t) + B(t)v(t) \\
z_R(t) &= (E(t) + N(t)K(t))x_R(t) + N(t)v(t),
\end{aligned}$$

and $S(\cdot, \cdot)$ is the transfer operator of the system

$$\begin{aligned}
x_S(t+1) &= (A(t) + B(t)K(t))x_S(t) + [\bar{B}(t) \ 0]w(t) \\
z_S(t) &= (E(t) + N(t)K(t))x_S(t).
\end{aligned}$$

Focus now on the transfer operators $R(\cdot, \cdot)$ and $S(\cdot, \cdot)$ just defined.

We preliminary prove that $R^\sim(\sigma, t)R(\sigma, t)$ is algebraic (i.e., it does not depend upon σ). With this objective in mind, consider a state-space realization of the adjoint

$R^\sim(\sigma, t)$ of $R(\sigma, t)$, i.e.,

$$\begin{aligned} (A(t) + B(t)K(t))'\lambda_R(t+1) &= \lambda_R(t) - (E(t) + N(t)K(t))'\tilde{v}(t) \\ \tilde{z}_R(t) &= B(t)'\lambda_R(t+1) + N(t)'\tilde{v}(t). \end{aligned}$$

Notice that the Riccati equation in the unknown $Q(\cdot)$ can be equivalently rewritten as

$$\begin{aligned} Q(t) &= (A(t) + B(t)K(t))'Q(t+1)(A(t) + B(t)K(t)) + \\ &\quad + (E(t) + N(t)K(t))'(E(t) + N(t)K(t)). \end{aligned}$$

Now consider the cascade $R^\sim(\sigma, t)R(\sigma, t)$ obtained by imposing $\tilde{v}(t) = z_R(t)$. Exploiting the above expression of the Riccati equation, it turns out that a state-space realization of $R^\sim(\sigma, t)R(\sigma, t)$ is

$$\begin{aligned} (A(t) + B(t)K(t))'\mu_R(t+1) &= \mu_R(t) \\ \tilde{z}_R(t) &= B(t)'\mu_R(t+1) + (N(t)'N(t) + B(t)'Q(t+1)B(t))v(t), \end{aligned}$$

where $\mu_R(t) = \lambda_R(t) - Q(t)((A(t-1) + B(t-1)K(t-1))'x_R(t-1) + B(t-1)v(t-1))$. The equation in the variable $\mu_R(\cdot)$ is autonomous, so that, in the input-output framework we can set $\mu_R(t) = 0$. This leads to the following conclusion

$$R^\sim(\sigma, t)R(\sigma, t) = N(t)'N(t) + B(t)'Q(t+1)B(t),$$

namely $R^\sim(\sigma, t)R(\sigma, t)$ does not depend upon σ .

Secondly, one can show that $R^\sim(\sigma, t)S(\sigma, t)$ is anti-stable. Indeed, a procedure, analogous to the one used to show that $R^\sim(\sigma, t)R(\sigma, t)$ is algebraic, can be applied to the cascade of $R^\sim(\sigma, t)$ and $S(\sigma, t)$ (so letting $\tilde{v}(t) = z_S(t)$). In this way, it results that the state-space realization for $R^\sim(\sigma, t)S(\sigma, t)$ is

$$\begin{aligned} (A(t) + B(t)K(t))'\mu_S(t+1) &= \mu_S(t) - Q(t)\bar{B}(t-1)w_1(t-1) \\ \tilde{z}_R(t) &= B(t)'\mu_S(t+1), \end{aligned}$$

where $\mu_S(t) = \lambda_R(t) - Q(t)((A(t-1) + B(t-1)K(t-1))'x_S(t-1))$. Obviously, the stability of $A(\cdot) + B(\cdot)K(\cdot)$ implies that $R^\sim(\sigma, t)S(\sigma, t)$ is anti-stable.

So far, denote by $T_{vw}(\sigma, t)$ the transfer operator from w to v . One can rewrite Eq. (13.60) as

$$z(t) = (R(\sigma, t)T_{vw}(\sigma, t) + S(\sigma, t)) \cdot w(t).$$

The stability of the closed-loop system entails the stability of $T_{vw}(\sigma, t)$. Since $R^\sim(\sigma, t)S(\sigma, t)$ is unstable and strictly proper, from the Pythagorean rule (see Remark 9.1) one obtains

$$\|\Sigma_{LQG}\|_2^2 = \|R(\sigma, t)T_{vw}(\sigma, t)\|_2^2 + \|S(\sigma, t)\|_2^2.$$

Then, thanks to the fact that $R^\sim(\sigma, t)R(\sigma, t) = N(t)'N(t) + B(t)'Q(t+1)B(t)$

$$\|\Sigma_{LQG}\|_2^2 = \|T_{\bar{v}w}(\sigma, t)\|_2^2 + \|S(\sigma, t)\|_2^2,$$

where $\bar{v}(t)$ is the signal obtained from $v(t)$ according to the rule:

$$\bar{v}(t) = (N(t)'N(t) + B(t)'Q(t+1)B(t))^{1/2} v(t),$$

and $T_{\bar{v}w}(\sigma, t)$ is the transfer operator from $w(\cdot)$ to $\bar{v}(\cdot)$. Notice that the minimization, with respect to all possible state-feedback control laws of $\|S(\sigma, t)\|_2^2$ brings to the Riccati equation (13.57b) and the associate gain $K(t)$, see Sect. 13.5. On the other hand, consider the system

$$\begin{aligned} x(t+1) &= A(t)x(t) + \bar{B}(t)w_1(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + \bar{N}(t)w_2(t) + D(t)u(t) \\ \eta(t) &= K(t)x(t) \\ \bar{v}(t) &= (N(t)'N(t) + B(t)'Q(t+1)B(t))^{1/2} (u(t) - \eta(t)). \end{aligned}$$

The minimization (in the H_2 sense) of the transfer operator $T_{\bar{v}w}(\sigma, t)$ consists in finding the H_2 estimate $u(t) = \hat{\eta}(t)$ of the remote variable $\eta(t)$. Hence, in view of the results of Sect. 12.1, the solution to this last problem is actually given by the filter gains $B_F(t)$ and $D_F(t)$ as stated in Proposition 13.18. Correspondingly, the optimal filter coincides with controller (13.58), which in turn, supplies the optimal LQG solution. As for the expression of the optimal index J^0 , it can be derived on the basis of the L_2 norm computation via the Lyapunov technique presented in Sect. 9.1.1. The Lyapunov equations are those associated with $T_{\bar{v}w}(\sigma, t)$ and $S(\sigma, t)$ and coincide with the two Riccati equations in $P(\cdot)$ and $Q(\cdot)$. In this way, the second expression of J^0 directly follows. As for the first one, it can be verified via algebraic manipulations over the second expression by again making use of the Lyapunov equations.

Remark 13.6 *In case the state and output disturbances are correlated with each other, the solution takes a slightly more complex expression. A way to find the LQG solution is to reformulate the problem through output injection in order to obtain a noise-decoupled system. This idea is well developed in the literature for time-invariant systems and periodicity does not require particular extensions. Moreover, the basic assumption that $N(t)N(t)' > 0$ for each t can be removed by using the concept of spectral interactor as discussed in Remark 12.7. The same considerations can be applied in the control counterpart of the LQG result. To be precise, if the performance index contains a cross input-state term, the control problem can be cast in the cross-term free case by a preliminary state-feedback transformation. As for the case when the input weighting matrix $\Lambda(t)$ is singular for some t , one can resort to a dual version of the spectral interactor matrix concept developed in Sect. 8.4.*

13.8.5 Output-feedback H_∞ Control

In this section, we briefly consider the H_∞ output-feedback control design problem, for the periodic system

$$x(t+1) = A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) \quad (13.61a)$$

$$z(t) = C_1(t)x(t) + D_{11}(t)w(t) + D_{12}(t)u(t) \quad (13.61b)$$

$$y(t) = C_2(t)x(t) + D_{21}(t)w(t). \quad (13.61c)$$

The meaning of the signals is standard: x is the system's state, u the control input, z the performance variable and y the measurement vector. All matrices are periodic with period T . As in the LQG problem, without any loss of generality, the input–output matrix relating $u(t)$ with $y(t)$ has been set to zero, since both signals are measurable.

Given a positive number γ , the output-feedback problem in the H_∞ framework consists in finding a causal T -periodic controller

$$\xi(t+1) = A_C(t)\xi(t) + B_C(t)y(t) \quad (13.62a)$$

$$u(t) = C_C(t)\xi(t) + D_C(t)y(t) \quad (13.62b)$$

in such a way that the input–output T -periodic operator from w to z has H_∞ norm less than γ . We shall denote by $G_{z,w}(\sigma, \cdot)$ the input–output operator of the global System (13.61), (13.62).

The main result is based on the solution of the H_∞ filtering problem and the solution of the full-information H_∞ control problem. Therefore, we are well advised to rewrite the two relevant Riccati Equations (12.37) and (13.26), respectively, associated with System (13.61).

H_∞ Filtering Riccati Equation (HFRE)

The filtering Riccati equation for System (13.61) is

$$P(t+1) = A(t)P(t)A(t)' + B_1(t)B_1(t)' - S_{J_\infty}(t)W_{J_\infty}(t)^{-1}S_{J_\infty}(t)'$$

$$S_{J_\infty}(t) = A(t)P(t) \begin{bmatrix} C_2(t)' & C_1(t)' \end{bmatrix} + B_1(t) \begin{bmatrix} D_{21}(t)' & D_{11}(t)' \end{bmatrix}$$

$$W_{J_\infty}(t) = \begin{bmatrix} D_{21}(t)D_{21}(t)' & D_{21}(t)D_{11}(t)' \\ D_{11}(t)D_{21}(t)' & D_{11}(t)D_{11}(t)' - \gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2(t) \\ C_1(t) \end{bmatrix} P(t) \begin{bmatrix} C_2(t)' & C_1(t)' \end{bmatrix}.$$

Correspondingly, it is possible to define the closed-loop matrix

$$\hat{A}_{JF}(t) = A(t) - S_{J_\infty}(t)W_{J_\infty}(t) \begin{bmatrix} C_2(t) \\ C_1(t) \end{bmatrix},$$

and the Schur complement

$$\begin{aligned}
S_3(t) = & \gamma^2 I - C_1(t)P(t)C_1(t)' - D_{11}(t)D_{11}(t)' \\
& + (C_1(t)P(t)C_2(t)' + D_{11}(t)D_{21}(t)') \times \\
& \times (D_{21}(t)D_{21}(t)' + C_2(t)P(t)C_2(t)')^{-1} (C_2(t)'P(t)C_1(t)' + D_{21}(t)D_{11}(t)').
\end{aligned}$$

We will say that $P(t)$ is stabilizing and feasible if

- (a) $P(\cdot) \geq 0$
- (b) $\hat{A}_{JF}(\cdot)$ is stable
- (c) $S_3(\cdot) > 0$.

H_∞ Control Riccati Equation (HCRE)

The control Riccati equation for System (13.61) is

$$\begin{aligned}
Q(t) &= A(t)'Q(t+1)A(t) + C_1(t)'C_1(t) - Z_{J\infty}(t)V_{J\infty}(t)^{-1}Z_{J\infty}(t)' \\
Z_{J\infty}(t) &= A(t)'Q(t+1) \begin{bmatrix} B_2(t) & B_1(t) \end{bmatrix} + C_1(t)' \begin{bmatrix} D_{12}(t) & D_{11}(t) \end{bmatrix} \\
V_{J\infty}(t) &= \begin{bmatrix} D_{12}(t)'D_{12}(t) & D_{12}(t)'D_{11}(t) \\ D_{11}(t)'D_{12}(t) & D_{11}(t)'D_{11}(t) - \gamma^2 I \end{bmatrix} + \begin{bmatrix} B_2(t)' \\ B_1(t)' \end{bmatrix} Q(t+1) \begin{bmatrix} B_2(t) & B_1(t) \end{bmatrix}.
\end{aligned}$$

Correspondingly it is possible to define the closed-loop matrix

$$\hat{A}_{JC}(t) = A(t) - \begin{bmatrix} B_2(t) & B_1(t) \end{bmatrix} V_{J\infty}(t)Z_{J\infty}(t)',$$

and the Schur complement

$$\begin{aligned}
\hat{S}_3(t) = & \gamma^2 I - B_1(t)'Q(t+1)B_1(t) - D_{11}(t)'D_{11}(t) \\
& + (B_1(t)'Q(t+1)B_2(t) + D_{11}(t)'D_{12}(t)) \times \\
& \times (D_{12}(t)'D_{12}(t) + B_2(t)'Q(t+1)B_2(t))^{-1} (B_2(t)'Q(t+1)B_1(t) + D_{12}(t)'D_{11}(t)).
\end{aligned}$$

We will say that $Q(t)$ is stabilizing and feasible if

- (a) $Q(\cdot) \geq 0$
- (b) $\hat{A}_{JC}(\cdot)$ is stable
- (c) $\hat{S}_3(\cdot) > 0$.

We are now in the position to provide the main result of this section. The proof of the subsequent theorem is quite complex and is therefore left out.

Proposition 13.19 *Assume that*

- (i) *For each t , $D_{21}(t)D_{21}(t)' > 0$.*
- (ii) *For each t , $D_{12}(t)'D_{12}(t) > 0$.*
- (iii) *The pair $(A(\cdot), C_2(\cdot))$ is detectable*
- (iv) *The pair $(A(\cdot), B_2(\cdot))$ is stabilizable.*
- (v) *The system $(A(\cdot), B_1(\cdot), C_2(\cdot), D_{21}(\cdot))$ does not have invariant zeros on the unit circle.*
- (vi) *The system $(A(\cdot), B_2(\cdot), C_1(\cdot), D_{12}(\cdot))$ does not have invariant zeros on the unit circle.*

Then, there exists a causal T -periodic stabilizing controller such that $\|G_{z,w}(\sigma, \cdot)\|_\infty < \gamma$ if and only if

(α) The HFRE admits the stabilizing and feasible T -periodic solution $P(\cdot)$.

(β) The HCRE admits the stabilizing and feasible T -periodic solution $Q(\cdot)$.

(δ) The eigenvalues of $P(t)Q(t)$ are less than γ , for each t .

A controller guaranteeing $\|G_{z,w}(\sigma, \cdot)\|_\infty < \gamma$ is described by (13.62) with

$$D_C(t) = -\hat{T}_Q(t)^{-1} (C_{1Q}(t)Y(t)C_{2Q}(t)' + W_{12}(t)\hat{S}_3(t)^{-1}D_{21}(t)') T_{NQ}(t) \quad (13.63a)$$

$$C_C(t) = -(\hat{T}_Q(t)^{-1}C_{1Q}(t) + D_C(t)C_{2Q}(t)) \quad (13.63b)$$

$$B_C(t) = B_2(t)D_C(t) + (A_Q(t)Y(t)C_{2Q}(t)' + B_1(t)\hat{S}_3(t)^{-1}D_{21}(t)') T_{NQ}(t) \quad (13.63c)$$

$$A_C(t) = \hat{A}_J(t) - B_C(t)C_{2Q}(t), \quad (13.63d)$$

where

$$\hat{T}_Q(t) = D_{12}(t)'D_{12}(t) + B_2(t)'Q(t+1)B_2(t)$$

$$Z_{J1}(t) = A(t)'Q(t+1)B_2(t) + C_1(t)'D_{12}(t)$$

$$Z_{J2}(t) = A(t)'Q(t+1)B_1(t) + C_1(t)'D_{11}(t)$$

$$W_{12}(t) = B_2(t)'Q(t+1)B_1(t) + D_{12}(t)'D_{11}(t)$$

$$Y(t) = P(t) (\gamma^2 I - Q(t)P(t))^{-1}$$

$$Z(t) = Z_{J2}(t)' - W_{12}(t)'\hat{T}_Q(t)^{-1}Z_{J1}(t)'$$

$$A_Q(t) = A(t) + B_1(t)\hat{S}_3(t)^{-1}Z(t)$$

$$C_{2Q}(t) = C_2(t) + D_{21}(t)\hat{S}_3(t)^{-1}Z(t)$$

$$C_{1Q}(t) = Z_{J1}(t)' + W_{12}(t)\hat{S}_3(t)^{-1}Z(t)$$

$$T_{NQ}(t) = (D_{21}(t)\hat{S}_3(t)^{-1}D_{21}(t)' + C_{2Q}(t)Y(t)C_{2Q}(t)')^{-1}.$$

■

As γ tends to infinity, the stabilizing solutions of HFRE and HCRE approach the stabilizing solution of the H_2 Riccati Equations (13.57a) and (13.57b), respectively. Moreover, notice that

$$\lim_{\gamma \rightarrow \infty} \frac{\hat{S}_3(t)}{\gamma^2} = I$$

$$\lim_{\gamma \rightarrow \infty} Y(t)\gamma^2 = P(t)$$

$$\lim_{\gamma \rightarrow \infty} T_{NQ}(t)\gamma^2 = C_2(t)P(t)C_2(t)' + D_{21}(t)D_{21}(t)'$$

Thanks to these expressions, and comparing (13.61) with (13.55), (13.59), it is easy to check that the H_∞ controller defined by matrices (13.63) reduces to the LQG controller (13.58), as γ tends to infinity.

13.9 Bibliographical Notes

The idea of working out a convex set generator for the class of robust stabilizing controllers has been explored in a number of papers such as [46, 129, 158] and in the PhD thesis [135].

The first approach for pole assignment has been carried out by exploiting the potentialities offered by the so-called generalized sampled-data hold functions, [200, 201]. When dealing with pole assignment of periodic systems the classical reference is [89] where the continuous-time case is treated by means of a suitable point-wise controllability concept. Classical basic references in discrete-time are [101, 171, 210] and, more recently, [3, 56, 219, 288]. The stabilization problem with input constraints has been tackled in [110] for time-invariant systems. The Farkas Lemma can be found in [136]. For the polynomial description of periodic systems via lifted reformulation see also [197]. Example 13.9 is taken from [110].

In the time-invariant case, the model matching problem was originally solved in [296], see also [213] for an updated view. The use of cyclic reformulation is inherited from [108].

The pure stabilization via static output-feedback has been tackled in [109]. A different insight in the output stabilization problem for SISO systems is based on the solution of non-linear equations, see [4].

The solution of the LQG problem is rooted in [59, 113] for discrete-time multi-rate systems and in [105] for continuous-time multi-rate systems, see also the preliminary results in [52] and the application in [69].

The extension of periodic Hamiltonian methods to H_2 and H_∞ problems can be traced back to [74, 100, 106, 295].

A number of further issues, including sensitivity analysis and receding horizon regulators are dealt with in [313].

References

1. P. Abbati, S. Bittanti, F. Previdi, "Forcing pulsatile behavior in semiconductor lasers by periodic control", *14th IFAC World Congress*, Beijing, P. R. China, 1999.
2. G. J. Adams, G. C. Goodwin, "Parameter estimation for periodic ARMA models", *Journal of Time Series Analysis*, Vol. 16, pp. 127–145, 1995.
3. D. Aeyels, J. L. Willems, "Pole assignment for linear time-invariant systems by periodic memoryless output-feedback", *Automatica*, Vol. 28, No. 6, pp. 1159–1168, 1992.
4. D. Aeyels, J. L. Willems, "Pole assignment for linear periodic systems by memoryless output-feedback", *IEEE Trans. on Automatic Control*, Vol. 40, No. 4, pp. 735–739, 1995.
5. H. M. Al-Rahmani, G. F. Franklin, "Linear periodic systems: eigenvalues assignment using discrete periodic feedback", *IEEE Trans. on Automatic Control*, Vol. 34, No. 1, pp. 99–103, 1989.
6. J. Andel, "On periodic autoregression with unknown mean", *Applied Mathematics*, Vol. 30, pp. 126–139, 1985.
7. B. D. O. Anderson, J. Moore, "Optimal Control: Linear Quadratic Methods", Prentice Hall, Englewood Cliffs, USA, 1990.
8. B. D. O. Anderson, J. Moore, "Time-varying feedback laws for decentralized control", *IEEE Trans. on Automatic Control*, Vol. 26, No. 5, pp. 1133–1139, 1981.
9. P. L. Anderson, A. J. Vecchia, "Asymptotic results for periodic autoregressive-moving average processes", *Journal of Time Series Analysis*, Vol. 14, No. 1, pp. 1–18, 1993.
10. M. Araki, "Recent developments in digital control theory", *12th IFAC World Congress*, Sidney, Vol. IX, pp. 251–260, 1993.
11. P. Arcara, S. Bittanti, M. Lovera, "Periodic control of helicopter rotors for attenuation of vibrations in forward flight", *IEEE Trans. on Control Systems Technology*, Vol. 8, No. 6, pp. 883–894, 2000.
12. C. Arduini, P. Baiocco, "Active magnetic damping attitude control for gravity gradient stabilized space-craft", *Journal of Guidance and Control*, Vol. 20, No. 1, pp. 117–122, 1997.
13. K. J. Åström, "Limitations on control system performance", *European Journal Control*, Vol. 6, No. 1, pp. 2–20, 2000.
14. A. Bacciotti, L. Rosier, *Lyapunov functions and stability in control theory*, Springer-Verlag, Vol. 267, 2001.
15. J. E. Bailey, "The importance of relaxed control in the theory of periodic optimization", *Periodic Optimization*, A. Marzollo Ed., Springer-Verlag, pp. 35–61, 1972.
16. J. E. Bailey, "Improvement of an optimal steady-state process by dynamic periodic optimization", A. Marzollo Ed., Springer-Verlag, pp. 153–182, 1972.
17. J. E. Bailey, "Periodic operation of chemical reactor: a review", *Chemical Engineering Communication*, No. 1, pp. 111–124, 1973.
18. J. E. Bailey, F. J. Horn, "Improvement of the performances of a fixed-bed catalytic reactor by relaxed steady-state operation", *AIChE Journal*, Vol. 17, No. 3, pp. 550–553, 1971.
19. T. P. Barnett, H. D. Heinz, K. Hasselman, "Statistical prediction of seasonal air temperature over Eurasia", *Tellus Series A*, Vol. 36, pp. 132–146, 1984.

20. T. Basar, P. Bernhard, *H_∞ optimal control and related minimax design problems*, Birkhäuser, 1991.
21. G. Basile, G. Marro, "Controlled and conditioned invariant subspace in linear systems theory", *Journal of Optimization Theory and Applications*, Vol. 3, No. 5, pp. 306–315, 1969.
22. G. Basile, G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*, Prentice Hall, 1992.
23. E. C. Bekir, R. S. Bucy, "Periodic equilibria for matrix Riccati equations, *Stochastics*, Vol. 2, pp. 1–104, 1976.
24. R. Bellman, J. Bentsman, S. M. Meerkov, "Stability of fast periodic systems", *IEEE Trans. on Automatic Control*, Vol. 30, pp. 289–291, 1985.
25. D. S. Bernstein, E. G. Gilbert, "Optimal periodic control: the π -test revisited", *IEEE Trans. on Automatic Control*, Vol. 25, 673–684, 1980.
26. V. Bertogalli, S. Bittanti, M. Lovera, "Simulation and identification of helicopter rotor dynamics using a general purpose multi-bode code", *Journal of the Franklin Institute*, Vol. 336, pp. 783–797, 1999.
27. R. R. Bitmead, M. R. Gevers, I. R. Petersen, R. J. Kaye, "Monotonicity and stabilizability properties of solutions of the Riccati difference equation: propositions, lemmas, theorems, fallacious conjectures and counterexamples", *Systems and Control Letters*, Vol. 5, No. 5, pp. 309–315, 1986.
28. S. Bittanti, *Sulla controllabilit  ed osservabilit  dei sistemi lineari periodici*, Master thesis, Politecnico di Milano, 1970.
29. S. Bittanti, "Deterministic and stochastic linear periodic systems", in *Time Series and Linear Systems*, S. Bittanti (Ed.), Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, Vol. 86, pp. 141–182, 1986.
30. S. Bittanti, "The periodic prediction problem for cyclostationary processes", *NATO Adv. Res. Workshop*, Groningen, The Netherlands, pp. 239–249, 1987.
31. S. Bittanti, P. Bolzern, "Four equivalent notions of stabilizability of periodic linear systems", *American Control Conference*, San Diego, USA, pp. 1321–1323, 1984.
32. S. Bittanti, P. Bolzern, "Can the Kalman canonical decomposition be performed for a discrete-time linear periodic system?", *1st Latin-American Conference on Automatica*, Campina Grande, Brazil, 1984.
33. S. Bittanti, P. Bolzern, "Canonical decomposition of discrete-time linear systems", *23rd IEEE Conference on Decision and Control*, Las Vegas, USA, pp. 1737–1738, 1984.
34. S. Bittanti, P. Bolzern, "Discrete-time linear periodic systems: Gramian and modal criteria for reachability and controllability", *International Journal of Control*, Vol. 41, pp. 909–928, 1985.
35. S. Bittanti, P. Bolzern, "Reachability and controllability of discrete-time linear periodic systems", *IEEE Trans. on Automatic Control*, Vol. 30, No. 4, pp. 399–401, 1985.
36. S. Bittanti, P. Bolzern, "Stabilizability and detectability of linear periodic systems", *Systems and Control Letters*, Vol. 6, pp. 141–145, 1985; with an addendum *Systems and Control Letters*, Vol. 7, p. 73, 1986.
37. S. Bittanti, P. Bolzern, "On the structure theory of discrete-time linear systems", *International Journal of Systems Science*, Vol. 17, pp. 33–47, 1986.
38. S. Bittanti, P. Bolzern, P. Colaneri, "Inertia theorems for Lyapunov and Riccati equations", in *Linear Algebra in Signals, Systems and Control*, B. N. Datta, C. R. Johnson, M. A. Kaashoek, R. J. Plemmons, E. D. Sontag (Eds.), SIAM Pub., 1988.
39. S. Bittanti, P. Bolzern, P. Colaneri, G. Guardabassi, "H and K controllability of linear periodic systems", *22nd IEEE Conference on Decision and Control*, pp. 1376–1379, 1983.
40. S. Bittanti, P. Bolzern, G. De Nicolao, L. Piroddi, "Representation, prediction and identification of cyclostationary processes", in *Cyclostationarity in Communications and Signal processing*, W. A. Gardner (Ed.), IEEE press, pp. 267–294, 1994.
41. S. Bittanti, P. Bolzern, G. Guardabassi, "Some critical issues concerning the state- representation of time-varying ARMA models", *7th IFAC Symposium on Identification and System Parameter Estimation*, York, UK, pp. 1479–1484, 1985.

42. S. Bittanti, P. Colaneri, "Analysis of discrete-time linear periodic systems", in *Control and Dynamic Systems*, Vol. 78, Academic Press, C.T. Leondes (Ed.), 1996.
43. S. Bittanti and P. Colaneri, "Periodic Control", in *Encyclopedia of Electrical and Electronic Engineering*, J. G. Webster (Ed.), Vol. 16, pp. 59–74, J. Wiley, 1999.
44. S. Bittanti, P. Colaneri, "Invariant representations of discrete-time periodic systems – a survey", *Automatica*, Vol. 36, No. 12, pp. 1777–1793, 2000.
45. S. Bittanti, P. Colaneri (Eds.), *Periodic Control Systems*, Pergamon Press, 2002 (Proceedings of the 1st IFAC Workshop on Periodic Control Systems, Cernobbio - Como, Italy, 2001).
46. S. Bittanti, P. Colaneri, "Stabilization of periodic systems: overview and advances", *1st IFAC Workshop on Periodic Control Systems*, Cernobbio - Como, Italy, pp. 157–164, 2001.
47. S. Bittanti, P. Colaneri, "Companion forms and cyclic matrices for discrete-time periodic systems", *15th IFAC World Congress*, Barcelona, Spain, 2002.
48. S. Bittanti, P. Colaneri, "The odd world of periodic polynomials", *16th IFAC World Congress*, Prague, Czech Republic, 2005.
49. S. Bittanti, P. Colaneri, G. De Nicolao, "Discrete-time linear periodic systems: a note on the reachability and controllability interval length", *Systems and Control Letters*, Vol. 8, pp. 75–78, 1986.
50. S. Bittanti, P. Colaneri, G. De Nicolao, "Two techniques for the solution of the discrete-time periodic Riccati equation", in *Linear Algebra in Signals, Systems and Control*, SIAM pub., Philadelphia, pp. 315–331, 1988.
51. S. Bittanti, P. Colaneri, G. De Nicolao, "The periodic Riccati equation", in *The Riccati equation*, S. Bittanti, A. J. Laub, J. C. Willems (Eds.), Springer-Verlag, pp. 127–162, 1990.
52. S. Bittanti, P. Colaneri, G. De Nicolao, "Periodic regulators with incomplete and noisy measurements", *29th IEEE Conference on Decision and Control*, Haway, USA, pp. 3648–3649, 1990.
53. S. Bittanti, P. Colaneri, G. Guardabassi, "H-controllability and observability of linear periodic systems", *SIAM Journal on Control and Optimization*, Vol. 22, No. 6, pp. 889–893, 1984.
54. S. Bittanti, P. Colaneri, G. Guardabassi, "Periodic solutions of periodic Riccati equations", *IEEE Trans. on Automatic Control*, Vol. 29, pp. 665–667, 1984.
55. S. Bittanti, P. Colaneri, G. Guardabassi, "Analysis of the periodic Riccati equation via canonical decomposition", *SIAM Journal on Control and Optimization*, Vol. 24, No. 6, pp. 1138–1149, 1986.
56. S. Bittanti, P. Colaneri, V. Kucera, "Robust polynomial assignment for uncertain periodic discrete-time systems", *3rd IFAC Symposium on Robust Control Design*, Prague, Czech Republic, 2000.
57. S. Bittanti, P. Colaneri, and M. F. Mongioví, "The spectral interactor matrix in the singular filtering problem", *IEEE Trans. on Automatic Control*, Vol. 40, No. 8, pp. 1492–1497, 1995.
58. S. Bittanti, P. Colaneri, M. F. Mongioví, "From singular to non-singular filtering for periodic systems: filling the gap with the spectral interactor matrix", *IEEE Trans. on Automatic Control*, Vol. 44, No. 1, pp. 222–226, 1999.
59. S. Bittanti, P. Colaneri, S. Strada, "Towards robust control of periodic systems via an LQG/LTR strategy", *12th IFAC World Congress*, Sydney, Australia, Vol. 8, pp. 149–1522, 1993.
60. S. Bittanti, G. De Nicolao, "Markovian representation of cyclostationary processes", in *Topics in Stochastic Systems*, L. Gerencser, P. E. Caines (Eds.), Lecture Notes in Control and Information Sciences, Springer-Verlag, pp. 31–46, 1991.
61. S. Bittanti, G. De Nicolao, "Spectral factorization of linear periodic systems with application to the optimal prediction of periodic ARMA models", *Automatica*, Vol. 29, No. 2, pp. 517–522, 1993.
62. S. Bittanti, G. Fronza, G. Guardabassi, "Periodic control: a frequency domain approach", *IEEE Trans. on Automatic Control*, Vol. 18, No. 1, pp. 33–38, 1973.
63. S. Bittanti, G. Fronza, G. Guardabassi, "Periodic optimization of linear systems under control power constraints", *Automatica*, Vol. 9, 1, pp. 269–271, 1973.

64. S. Bittanti, G. Fronza, G. Guardabassi, "Necessary optimality conditions in a discrete periodic optimization problem", *Automation and Remote Control*, No. 10, pp. 69–76, 1974.
65. S. Bittanti, G. Fronza, G. Guardabassi, "Optimal steady-state versus periodic operation in discrete systems", *Journal of Optimization Theory and Applications*, Vol. 18, No. 4, pp. 521–536, 1976.
66. S. Bittanti, G. Guardabassi, "Optimal process cycling for parameter identification", *4th IFAC Symposium on Identification and System Parameter Estimation*, Tbilisi, URSS, pp. 589–603, 1976.
67. S. Bittanti, G. Guardabassi, "Optimal cyclostationary control: a parameter-optimization frequency domain approach", *8th IFAC World Congress*, Kyoto, Japan, Vol. 6, pp. 191–196, 1981.
68. S. Bittanti, G. Guardabassi, C. Maffezzoni, L. Silverman, "Periodic systems: controllability and the matrix Riccati equation", *SIAM Journal on Control and Optimization*, Vol. 16, pp. 37–40, 1978.
69. S. Bittanti, D. B. Hernandez, G. Zerbi, "The simple pendulum and the periodic LQG control problem", *Journal of the Franklin Institute*, Vol. 328, No. 2/3, pp. 299–316, 1991.
70. S. Bittanti, A. J. Laub, J. C. Willems (Eds.), *The Riccati Equation*, Springer Verlag, 1990.
71. S. Bittanti, F. Lorito, S. Strada, "An LQ approach to active control of vibrations in helicopters", *ASME Transactions on Dynamic Systems, Measurement and Control*, Vol. 118, No. 3, pp. 482–488, 1996.
72. S. Bittanti, A. Locatelli, C. Maffezzoni, "Second variation methods in periodic optimization", *Journal of Optimization Theory and Applications*, Vol. 14, pp. 31–49, 1974.
73. S. Bittanti, M. Lovera, "On the zero dynamics of helicopter rotors loads", *European Journal of Control*, Vol. 2, No. 1, pp. 57–68, 1996.
74. S. Bittanti, C. Maffezzoni, "Structural properties of Hamiltonian system in a periodic optimization problem", *Automation and remote Control*, Vol. 36, No. 6, pp. 877–884, 1975.
75. S. Bittanti, L. Moiraghi, "Active control of vibrations in helicopters via pole assignment techniques", *IEEE Trans. on Control Systems Technology*, Vol. 2, pp. 343–351, 1994.
76. M. Bitzer, *Control of Periodically Operated Adsorption Processes*, PhD thesis, Stuttgart, 2004.
77. V. Blondel, *Simultaneous Stabilization of Linear Systems*, Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, Vol. 191, 1994.
78. V. Blondel, M. Gevers, R. Mortini, R. Rupp, "Simultaneous stabilization of three or more systems: conditions on the real axis do not suffice", *SIAM Journal on Control and Optimization*, Vol. 32, No. 2, pp. 572–590, 1994.
79. P. Bloomfield, H. Hurd, R. B. Lund, "Periodic correlation in stratospheric ozone data", *Journal of Time Series Analysis*, Vol. 15, pp. 127–150, 1994.
80. P. Bolzern, P. Colaneri, "Inertia theorems for the periodic Lyapunov difference equation and periodic Riccati difference equation", *Linear Algebra and its Application*, Vol. 85, pp. 249–265, 1987.
81. P. Bolzern, P. Colaneri, "The periodic Lyapunov equation", *SIAM Journal on Matrix Analysis and Application*, No. 4, pp. 499–512, 1988.
82. P. Bolzern, P. Colaneri, G. De Nicolao, "On discrete-time H_∞ fixed-lag smoothing", *IEEE Transactions on Signal Processing*, Vol. 52, pp. 132–141, 2004.
83. P. Bolzern, P. Colaneri, R. Scattolini, "Zeros of discrete-time linear periodic systems", *IEEE Transactions on Automatic Control*, Vol. 31, No. 11, pp. 1057, 1058, 1986.
84. G. Box, G. M. Jenkins, G. Reinsel, *Time Series Analysis: Forecasting and Control*, Prentice Hall (3rd edition), 1994.
85. R. W. Brockett, *Finite-Dimensional Linear Systems*, Wiley, New York, 1970.
86. R. W. Brockett, "Asymptotic stability and feedback stabilization", in *Differential Geometry Control Theory*, Progress in Mathematics, Vol. 27, Birkhäuser, pp. 181–208, 1983.
87. R. Brockett, "A stabilization problem", in *Open Problems in Mathematical Systems and Control theory*, V. D. Blondel, E. D. Sontag, M. Vidyasagar, J. C. Willems (Eds.), Springer-Verlag, London, p. 78, 1999.

88. P. Brunovski, "Controllability and linear closed-loop controls in linear periodic systems", *Journal of Differential Equations*, Vol. 6, pp. 296–313, 1969.
89. P. Brunovski, "A classification of linear controllable systems", *Kybernetika*, Vol. 3, pp. 173–187, 1970.
90. J. V. Burke, D. Henrion, A. S. Lewis, M. L. Overton, "Analysis of a Belgian chocolate stabilization problem", *LAAS-CNRS Research Report*, 05164, Toulouse, France, 2005.
91. J. R. Canabal, "Periodic geometry of the Riccati equation", *Stochastics*, Vol. 1, pp. 432–435, 1974.
92. L. Chai, L. Qiu, "Multirate periodic systems and constrained analytic function interpolation problems", *SIAM Journal of Control and Optimization*, Vol. 43, No. 6, pp. 1972–1986, 2005.
93. A. B. Chammas, C. T. Leondes, "On the design of linear time-invariant systems by periodic output-feedback", *International Journal of Control*, Vol. 27, pp. 885–903, 1978.
94. A. B. Chammas, C. T. Leondes, "Pole assignment by piecewise constant output -feedback", *International Journal of Control*, Vol. 44, pp. 1661–1673, 1986.
95. H. Chapellat, M. Dahleh, "Analysis of time-varying control strategies for optimal disturbance rejection and robustness", *IEEE Transactions on Automatic Control*, Vol. 37, pp. 1734–1745, 1992.
96. C. T. Chen, *Introduction to Linear System Theory*, Holt, Rinehard and Winston, New York, USA, 1970.
97. C. C. Chung, J. Hauser, "Nonlinear control of a swinging pendulum", *Automatica*, Vol. 31, No. 6, pp. 851–862, 1995.
98. T. Cipra, "Periodic moving average process", *Applied Mathematics*, Vol. 30, No. 33, pp. 218–229, 1985.
99. D. W. Clarke, C. Mohtadi, P. C. Tuffs, "Generalized predictive control – Part I – the basic algorithm – Part II – extensions and interpolators", *Automatica*, Vol. 23, No. 2, pp. 137–160, 1987.
100. P. Colaneri, "Hamiltonian matrices for lifted systems and periodic Riccati equations in H_2/H_∞ analysis and control", *30th IEEE Conference on Decision and Control*, Brighton, UK, pp. 1914–1919, 1991.
101. P. Colaneri, "Output stabilization via pole-placement of discrete-time linear periodic systems", *IEEE Transactions on Automatic Control*, Vol. 36, No. 6, pp. 739–742, 1991.
102. P. Colaneri, "Theoretical aspects of continuous-time periodic systems", *Annual Reviews in Control*, pp. 206–215, 2005.
103. P. Colaneri, "State-feedback problems in H_2 and H_∞ for continuous-time periodic systems. Part I: theoretical aspects", *Kybernetika*, Vol. 36, No. 2, pp. 211–242, 2000.
104. P. Colaneri, "State-feedback problems in H_2 and H_∞ for continuous-time periodic systems. Part II: feedback problems", *Kybernetika*, Vol. 12, No. 2, pp. 1777–1793, 2000.
105. P. Colaneri, G. De Nicolao, "Multirate LQG control of continuous-time stochastic systems", *Automatica*, Vol. 31, No. 4, pp. 591–596, 1995.
106. P. Colaneri, C. E. de Souza, "The H_∞ control problem for continuous-time linear periodic systems", *2nd IFAC Workshop on System Structure and Control*, Prague, Czech Republic, pp. 292–295, 1992.
107. P. Colaneri, D. Henrion, "Switching and periodic control of the Belgian chocolate problem", *5th IFAC Symposium on Robust Control Design*, Toulouse, France, 2006.
108. P. Colaneri, V. Kucera, "The model matching problem for discrete-time periodic systems", *IEEE Transactions on Automatic Control*, Vol. 42, No. 10, pp. 1472–1476, 1997.
109. P. Colaneri, V. Kucera, C. E. de Souza, "Output stabilization for periodic systems: necessary and sufficient conditions", *American Control Conference*, Philadelphia, USA, pp. 2795–2796, 1998.
110. P. Colaneri, V. Kucera, S. Longhi, "Polynomial approach to the control of SISO periodic systems subject to input constraint", *Automatica*, Vol. 39, No. 8, pp. 1417–1424, 2003.
111. P. Colaneri, S. Longhi, "The minimal realization problem for discrete-time periodic systems", *Automatica*, Vol. 31, No. 5, pp. 779–783, 1995.
112. P. Colaneri, S. Pinzoni, "Realization from covariances and Markov parameters of a discrete-time periodic system", *23rd American Control Conference*, Boston, USA, 2004.

113. P. Colaneri, R. Scattolini, N. Schiavoni, "LQG optimal control for multirate sampled-data systems", *IEEE Transactions on Automatic Control*, Vol. 37, No. 5, pp. 675–682, 1992.
114. F. Colonius, *Optimal Periodic Control*, Lecture Notes in Mathematics, Springer-Verlag, Vol. 1313, 1988.
115. G. Conte, A.M. Perdon, S. Longhi, "Zero structures at infinity of linear periodic systems", *Circuits, Systems, and Signal Processing*, Vol. 10, No. 1, pp. 91–100, 1991.
116. J. M. Coron, "Stabilization in finite time of locally controllable systems by means of continuous-time-varying feedback law", *SIAM Journal on Control and Optimization*, Vol. 33, pp. 804–833, 1995.
117. J. M. Coron, "On the stabilization of some non-linear control systems: results, tools and applications", in *Nonlinear Analysis, Differential Equations and Control*, F. H. Clarke, G. Sabidussi, R. J. Stern (Eds.), Kluwer Academic Publication, Dordrecht, The Netherlands, pp. 307–363, 1999.
118. J. M. Coron, L. Rosier, "A relation between continuous-time-varying and discontinuous feedback stabilization", *Mathematical Systems, Estimation and Control*, Vol. 1, pp. 67–84, 1994.
119. R. E. Crochiere, L. R. Rabiner, *Multirate Digital Signal Processing*, Prentice Hall, Englewood Cliffs, USA, 1983.
120. J. Daafouz, J. Bernussou, "Poly-quadratic stability and H_∞ performance for discrete systems with time varying uncertainties", *40th IEEE Conference on Decision and Control*, pp. 267–272, 2001.
121. P. D'Alessandro, A. Isidori, A. Ruberti, "A new approach to the theory of canonical decomposition of linear dynamical systems", *SIAM Journal on Control*, Vol. 11, No. 1, pp. 148–158, 1973.
122. M. A. Dahleh, P. G. Voulgaris, L. S. Valavani, "Optimal and robust controllers for periodic and multirate systems", *IEEE Transactions on Automatic Control*, Vol. 37, No. 1, pp. 90–99, 1992.
123. H. D'Angelo, *Linear Time-Varying Systems: Analysis and Synthesis*, Allyn and Bacon, Boston, USA, 1970.
124. G. De Nicolao, "Cyclomonotonicity, Riccati equations and periodic receding horizon control", *Automatica*, Vol. 30, pp. 1375–1388, No. 9, 1994.
125. G. De Nicolao, G. Ferrari Trecate, "On the zeros of discrete-time linear periodic systems", *Circuits, Systems, and Signal Processing*, Vol. 16, No. 6, pp. 703–718, 1997.
126. G. De Nicolao, G. Ferrari Trecate, S. Pinzoni, "Zeros of continuous-time linear periodic systems", *Automatica*, Vol. 34, pp. 1651–1655, 1998.
127. C. A. Desoer, "Slowly varying systems $\dot{x} = A(t)x$ ", *IEEE Transactions on Automatic Control*, Vol. 14, pp. 780–781, 1969.
128. C. E. de Souza, "Riccati differential equation in optimal filtering of periodic non-stabilizing systems", *International Journal of Control*, Vol. 46, pp. 1235–1250, 1987.
129. C. E. de Souza, A. Trofino, "An LMI approach to the stabilization of linear discrete-time periodic systems", *International Journal of Control*, Vol. 73, No. 8, pp. 696–703, 2000.
130. V. Dolezal, *Dynamics of Linear Systems*, Academia, Prague, Czechoslovak Republic, 1967.
131. P. Dorato, H. K. Knudsen, "Periodic optimization with applications to solar energy control", *Automatica*, Vol. 15, pp. 673–679, 1979.
132. Y. El Mrabet, H. Bourles, "Periodic polynomial interpretation for structural properties of linear periodic discrete-time systems", *Systems and Control Letters*, Vol. 33, pp. 241–251, 1998.
133. P. Enqvist, "Pole-zero identification based on simultaneous realization of normalized covariances and Markov parameters", *41st IEEE Conference on Decision and Control*, Las Vegas, USA, 2002.
134. M. Faraday, "On a peculiar class of acoustical figures and on certain forms assumed by a group of particles upon vibrating elastic surfaces", *Phil. Transactions Royal Society*, London, UK, pp. 299–318, 1831.
135. C. Farges, *Méthode d'Analyse et de Synthèse Robustes pour les Systèmes Linéaires Périodiques*, PhD thesis, LAAS-CNRS Toulouse, France, 2007.

136. M. Farkas, *Periodic Motion*, Springer-Verlag, New York, USA, 1994.
137. M. Fazel, *Matrix Rank Minimization with Applications*, PhD Thesis, Stanford University, USA, 2002.
138. G. Feichtinger, "Limit cycles in dynamic economic systems", *Annals of Operations Research*, Vol. 37, pp. 313–344, 1992.
139. G. Feichtinger, A. Novak, "Optimal pulsing in advertising diffusion models", *Optimal Control Applications and Methods*, Vol. 15, pp. 267–276, 1994.
140. A. Feuer, G. C. Goodwin, *Sampling in Digital Signal Processing and Control*, Birkhäuser, 1996.
141. A. Feuer, G. C. Goodwin, "Generalized sampled-data functions: frequency domain analysis of robustness, sensitivity, and intersample difficulties", *IEEE Transactions on Automatic Control*, Vol. 39, pp. 1042–1047, 1994.
142. A. Feuer, G. C. Goodwin, M. Salgado, "Potential benefits of hybrid control for linear time-invariant plants", *26th American Control Conference*, Albuquerque, USA, 1997.
143. D. S. Flamm, A new shift-invariant representation for periodic systems, *Systems and Control Letters*, Vol. 17, No. 1, pp. 9–14, 1991.
144. N. J. Fliege, *Multirate Digital Signal Processing*, J. Wiley and Sons, 1984.
145. G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques, *Annales de l'École Normale Supérieure*, Vol. 12, 47–89, 1883.
146. P. H. Franses, Multi-step forecast error variances for periodically integrated time-series", *Journal of Forecasting*, Vol. 15, pp. 83–95, 1996.
147. P. H. Franses, R. Paap, Modeling day-of-the-week seasonality in the S&P500 index", *Applied Financial Economics*, No. 10, pp. 483–488, 2000.
148. P. H. Franses, R. Paap, *Periodic Time Series Models*, Oxford University Press, UK, 2004.
149. B. A. Francis, T. T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers", *IEEE Transactions on Automatic Control*, Vol. 33, pp. 820–832, 1988.
150. L. Freidovich, A. Robertson, A. Shiriaev, R. Johansson, "Periodic motions of the Pendubot via virtual holonomic constraints: Theory and experiments", *Automatica*, Vol. 44, No. 3, pp. 785791, 2008.
151. J. S. Freudenberg, J. W. Grizzle, "An observation on the parametrization of causal stabilizing controllers for lifted systems", *Control Theory and Advanced Tech.* Vol. 5, No. 3, pp. 367–372, 1989.
152. B. Friedland, "Sampled-data control systems containing periodically varying members", *1st IFAC World Congress*, Moscow, URSS, pp. AX1–AX8, 1960.
153. M. Fjeld, *Stability and Control of Periodic Processes*, PhD thesis, University of Trondheim, Sweden, 1971.
154. F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsea, New York, USA, 1959.
155. W. A. Gardner (Ed.), *Cyclostationarity in Communications and Signal Processing*, IEEE Press, New York, USA, 1994.
156. W. A. Gardner (Guest Ed.), "Special issue on cyclostationary signals", *IEEE Signal Processing Magazine*, 1991.
157. W. A. Gardner, L. E. Franks, "Characterization of cyclostationary random processes", *IEEE Transactions on Information Theory*, Vol. 2, pp. 1–24, 1975.
158. J. C. Geromel, P. L. D. Peres and J. Bernussou, "On a convex parameter space method for linear control design of uncertain systems", *SIAM Journal Control and Optimization*, Vol. 29, No. 2, pp. 381–402, 1991.
159. E. G. Gladyshev, "Periodically correlated random sequences", *Soviet Mathematics*, No. 2, pp. 385–388, 1961.
160. N. L. Gerr, H. Hurd, "Graphical Methods for Determining the Presence of Periodic Correlation in Time Series", *Journal of Time Series Analysis*, Vol. 12, No. 4, pp. 337–350, 1991.
161. E. Ghysels, B. Osborn, *The Econometric Analysis of Seasonal Time-series*, Cambridge University Press, UK, 2001.
162. E. G. Gilbert, "Periodic control of vehicle cruise: improved fuel economy by relaxed steady-state, quasi steady-state and quasi relaxed steady-state control", *Automatica*, Vol. 12, No. 2, pp. 159–166, 1976.

163. E. G. Gilbert, "Optimal periodic control: a general theory of necessary conditions", *SIAM Journal on Control and Optimization*, Vol. 15, No. 5, pp. 717–746, 1977.
164. I. Gohberg, M. A. Kaashoek, L. Lerer, "Minimality and realization of discrete-time-varying systems", *Operator Theory: Advances and Application*, Vol. 56, pp. 261–296, 1992.
165. R. Granat, *Algorithms and Library Software for Periodic and Parallel Eigenvalue Reordering and Sylvester-Type Matrix Equations with Condition Estimation Robert*, PhD thesis, Umeå University, Sweden, 2007.
166. C. W. J. Granger, "Some new time-series models: non-linear, bilinear and non-stationary", *Statistician*, Vol. 27, No. 3–4, pp. 237–254, 1978.
167. O. M. Grasselli, "A canonical decomposition of linear periodic discrete-time systems", *International Journal of Control*, Vol. 40, No. 1, pp. 201–214, 1984.
168. O. M. Grasselli, S. Longhi, "Disturbance localization with dead-beat control for linear periodic discrete-time systems", *International Journal of Control*, Vol. 44, No. 5, pp. 1319–1347, 1988.
169. O. M. Grasselli, S. Longhi, "Zeros and poles of linear periodic multivariable discrete-time systems", *Circuits, Systems, and Signal Processing*, Vol. 7, No. 3, pp. 361–380, 1988.
170. O. M. Grasselli, S. Longhi, "Robust tracking and regulation of linear periodic discrete-time systems", *International Journal of Control*, Vol. 54, No. 3, pp. 613–633, 1991.
171. O. M. Grasselli, S. Longhi, "Pole placement for non-reachable periodic discrete-time systems", *Mathematics of Control, Signal and Systems*, Vol. 4, pp. 437–453, 1991.
172. G. Guardabassi, "Optimal steady-state versus periodic operation: a circle criterion", *Ricerche di Automatica*, Vol. 2, No. 3, pp. 240–252, 1971.
173. G. Guardabassi, "The optimal periodic control problem", *Journal A*, Vol. 17, pp. 75–83, 1976.
174. G. Guardabassi, A. Locatelli, "Periodic optimization: a pricing problem", *2nd Polish-Italian Conference on Application of System Theory to Economy, Management and Technology*, Pugnochiuso, Italy, 1974.
175. A. Halanay, *Differential Equations - Stability, Oscillations, Time-lags*, Academic Press, New York, USA, 1966.
176. D. Ham, "Helicopter individual blade control research at MIT: 1977–1985", *Vertica*, Vol. 11, No. 1/2, pp. 109–122, 1987.
177. L. P. Hansen, T. J. Sargent, "Seasonality and approximation errors in rational expectations models", *Journal of Econometrics*, Vol. 55, pp. 21–56, 1993.
178. K. Hasselman, T. P. Barnett, "Techniques of linear prediction for systems with periodic statistics", *Journal of Atmospheric Sciences*, Vol. 28, No. 10, pp. 2275–2283, 1981.
179. J. Hauser, C. Chung, "Converse Lyapunov functions for exponentially stable periodic orbits", *Systems and Control Letters*, Vol. 22, pp. 2734, 1994.
180. J. J. Hench, A. J. Laub, "Numerical solution of the discrete-time periodic Riccati equation", *IEEE Transactions on Automatic Control*, Vol. 39, No. 6, pp. 1197–1210, 1994.
181. V. Hernandez, L. Jodar, "Boundary problems for periodic Riccati equations", *IEEE Transactions on Automatic Control*, Vol. 30, pp. 1131–1133, 1985.
182. V. Hernandez, A. Urbano, "Pole-placement problem for discrete-time linear periodic systems", *International Journal of Control*, Vol. 50, No. 2, pp. 361–371, 1989.
183. G. A. Hewer, "Periodicity, detectability and the matrix Riccati equation", *SIAM Journal on Control*, Vol. 13, pp. 1235–1251, 1975.
184. G. W. Hill, "On the part of the lunar perigee which is a function of the mean motion of the sun and moon", *Acta Math.*, Vol. 8, pp. 1–36, 1886.
185. B. L. Ho and R. E. Kalman, "Efficient construction of linear state variable models from input/output functions", *Regelungstechnik*, Vol. 14, pp. 545–548, 1966.
186. H. K. Khalil, *Nonlinear Systems*, Prentice Hall, 1996.
187. E. Kreindler, P. E. Sarachik, "On the concept of controllability and observability of linear systems", *IEEE Transactions on Automatic Control*, Vol. 9, pp. 129–136, 1964.
188. S. Hylleberg (Ed.), *Modeling Seasonality*, Oxford University Press, UK, 1952.
189. F. J. Horn, R. C. Lin, "Periodic processes: a variational approach", *I and EC Proc. Des. Development*, 6, 21–30, 1967.

190. H. L. Hurd, *An Investigation of Periodically Correlated Stochastic Processes*, PhD thesis, Duke University, Durham, USA, 1969.
191. H. Hurd, "Nonparametric time series analysis for periodically correlated processes", *IEEE Transactions on Information Theory*, Vol. 35, No. 2, pp. 350–359, 1989.
192. H. Hurd, "Correlation theory of almost periodically correlated processes", *Journal of Multivariate Analysis*, Vol. 37, No. 1, pp. 24–45, 1991.
193. P. Iglesias, "An entropy formula for time-varying discrete-time control systems", *SIAM Journal on Control and Optimization*, Vol. 34, No. 5, pp. 1691–1706, 2002.
194. A. Ilchman, M. Mueller, "Time-varying linear systems: relative degree and normal form", *IEEE Transactions on Automatic Control*, Vol. 52, No. 5, pp. 840–851, 2007.
195. A. Isidori, *Nonlinear Control Systems*, Springer-Verlag, 1985.
196. S. A. Jacklin, K. Q. Nguyen, A. Blaas, P. Richter, "Full-scale wind tunnel test of an helicopter IBC system", *50th Annual Forum American Helicopter Society*, Washington DC, pp. 579–596, 1994.
197. L. Jetto, S. Longhi, "Assigning closed-loop invariant polynomials over polytopes for linear periodic systems", *Automatica*, Vol. 35, No. 1, pp. 49–58, 1999.
198. W. Johnson, *Helicopter Theory*, Princeton University Press, USA, 1981.
199. R. H. Jones, W. M. Brelsford, "Time-series with periodic structure", *Biometrika*, Vol. 54, pp. 403–407, 1967.
200. P. T. Kabamba, "Monodromy Eigenvalue Assignment in Linear Periodic Systems", *IEEE Transactions on Automatic Control*, Vol. 31, No. 10, pp. 950–952, 1986.
201. P. T. Kabamba, "Control of linear systems using generalized sampled-data hold functions", *IEEE Transactions on Automatic Control*, Vol. 32, pp. 772–783, 1987.
202. T. Kaczorek, "Pole placement for linear discrete-time systems by periodic output-feedbacks", *Systems and Control Letters*, Vol. 6, pp. 267–269, 1985.
203. T. Kailath, *Linear System Theory*, Prentice Hall, Englewood Cliffs, USA, 1980.
204. R. E. Kalman, P. L. Falb, M. A. Arbib, *Topics in Mathematical System Theory*, Prentice Hall, Englewood Cliffs, USA, 1969.
205. E. W. Kamen, K. M. Hafez, "Algebraic theory of linear time-varying systems", *SIAM Journal on Control and Optimization*, Vol. 17, No. 4, pp. 500–510, 1979.
206. E. W. Kamen, P. P. Khargonekar, K. R. Poolla, "A transfer function approach to linear time-varying discrete-time systems", *SIAM Journal on Control and Optimization*, Vol. 23, No. 4, pp. 550–565, 1985.
207. H. Kano, T. Nishimura, "Periodic solutions of matrix Riccati equations with detectability and stabilizability", *International Journal of Control*, Vol. 29, No. 3, pp. 471–481, 1979.
208. P. P. Khargonekar, K. Poolla, A. Tannembaum, "Robust control of linear time-invariant plants using periodic compensation", *IEEE Transactions on Automatic Control*, Vol. 30, No. 11, pp. 1088–1096, 1985.
209. P. V. Kokotovic, H. K. Khalil, J. O'Reilly, *Singular Perturbation Methods in control analysis and design*, Academic Press, 1986.
210. M. Kono, Eigenvalue assignment in linear periodic discrete-time systems, *International Journal of Control*, Vol. 32, No. 1, pp. 149–158, 1980.
211. G. M. Krank, "Input-output analysis of multirate feedback systems", *IRE Transactions Automatic Control*, Vol. 32, pp. 21–28, 1957.
212. V. Kučera, *Discrete Linear Control: the Polynomial Approach*, John Wiley and Sons, Chichester, UK, 1979.
213. V. Kucera, *Analysis and Design of Discrete Linear Control Systems*. Prentice-Hall, London, UK, 1991.
214. M. Kuijper, "A periodically time-varying minimal partial realization algorithm based on twisting", *Automatica*, Vol. 35, No. 9, pp. 1543–1548, 1999.
215. M. Kuijper, J. C. Willems, "A behavioral framework for periodically time-varying systems", *36th IEEE Conference of Decision and Control*, San Diego, USA, pp. 2013–2016, 1997.
216. W. H. Kwon, A. E. Pearson, "Linear systems with two-point boundary Lyapunov and Riccati equations", *IEEE Transactions on Automatic Control*, Vol. 27, No. 2, pp. 436–441, 1982.

217. G. Laroque, *Analyse d'une Méthode de Désaisonnalisation: le Programme X 11, du US Bureau of Census*, Annales de l'INSEE 28, pp. 105127, 1977.
218. S. Lee, S. Meerkov, T. Runolfsson, "Vibrational feedback control: zero placement capabilities", *IEEE Transactions on Automatic Control*, Vol. 32, pp. 604–611, 1987.
219. S. Longhi, R. Zulli, "A robust periodic pole assignment algorithm", *IEEE Transactions on Automatic Control*, Vol. 40, No. 5, pp. 890–894, 1995.
220. M. Lovera, "Optimal magnetic momentum control for inertially pointing spacecraft", *European Journal of Control*, Vol. 7, No. 1, pp. 30–39, 2001.
221. M. Lovera, E. De Marchi, S. Bittanti, "Periodic attitude control techniques for small satellites with magnetic actuators", *IEEE Transactions on Control System Technology*, Vol. 10, No. 1, pp. 90–95, 2002.
222. R. B. Lund, I. V. Basawa, "Modeling for periodically correlated time series", in *Asymptotics, Nonparametrics, and Time Series*, S. Ghosh (Ed.), pp. 37–62, 1999.
223. R. B. Lund, I. V. Basawa, "Recursive prediction and likelihood evaluation for periodic ARMA models", *Journal of Time Series Analysis*, Vol. 21, pp. 75–93, 2000.
224. R. B. Lund, H. Hurd, P. Bloomfield, R. L. Smith, "Climatological time series with periodic correlation", *Journal of Climate*, Vol. 11, pp. 2787–2809, 1995.
225. R. B. Lund, P. L. Seymour, "Periodic time series", *Encyclopedia of Environmetrics*, A. H. El-Shaarawi and W. W. Piegorsch (Eds.), pp. 2204–2210, 2002.
226. A. M. Lyapunov, *Problème Général de la Stabilité du Mouvement*, Annals of Mathematics Studies, Vol. 17, Princeton, New York, 1947 (original version in Russian published in Kharkov in 1892).
227. A. M. Lyapunov, "Sur une série relative à la théorie des équations différentielles linéaires à coefficients périodiques", *Compt. Rend.*, 123, No. 26, pp. 1248–1252, 1896 (also in *Collected works*, Vol. 2, pp. 387–390, Moscow, 1956).
228. A. Marzollo (Ed.), *Periodic Optimization*, Springer-Verlag, New York, 1972.
229. A. I. McLeod, "Diagnostic checking of periodic autoregression models with application", *Journal of Time Series Analysis*, Vol. 15, No. 2, pp. 221–233, 1994.
230. E. Mathieu, "Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique", *Journal Math.* Vol 13, pp. 137–203, 1868.
231. R. A. Meyer, C. S. Burrus, "A unified analysis of multirate and periodically time-varying digital filters", *IEEE Transactions on Circuits and Systems*, Vol. 22, pp. 162–167, 1975.
232. F. C. Mills, *Statistical Methods*, Pittman, London, UK, 1924.
233. K.L. Moore, S. P. Bhattacharyya, M. Dahleh, "Capabilities and limitations of multirate control schemes", *Automatica*, Vol. 29, pp. 941–951, 1993.
234. L. Moreau, D. Aeyels, "Periodic output-feedback stabilization of single-input single-output continuous-time systems with odd relative degree", *Systems and Control Letters*, Vol. 51, pp. 395–406, 2004.
235. P. Morin, C. Samson, J. B. Pomet, Z. P. Jiang, "Time-varying feedback stabilization of the attitude of a rigid spacecraft with two controls", *Systems and Control Letters*, Vol. 25, pp. 375–385, 1995.
236. P. V. Mouche, W. Heijman, "Floquet theory and economic dynamics", *Wageningen University Economic Papers*, Vol. 6, 1997.
237. A. H. Nayfeh, D. T. Mook, *Nonlinear Oscillations*, J. Wiley and Sons, New York, USA, 1979.
238. A. W. Olbrot, "Robust stabilization of uncertain systems by periodic feedback", *International Journal of Control*, Vol. 45, No. 3, pp. 747–758, 1987.
239. R. Ortega, G. Kreisselmeier, "Discrete-time model reference adaptive control for continuous-time systems using generalized sample data hold functions", *IEEE Transactions on Automatic Control*, Vol. 35, pp. 334–338, 1990.
240. D. Osborn, "Seasonality and habit persistence in a life-cycle model of consumption", *Journal of Applied Econometrics*, No. 3, pp. 255–266, 1988.
241. D. Osborn, J. R. Smith, "The performance of periodic autoregressive models in forecasting UK consumption", *Journal of Business and Economic Statistics*, No. 7, pp. 117–127, 1989.

242. M. Pagano, "On periodic and multiple autoregressions", *The Annals of Statistics*, Vol. 6, pp. 1310–1317, 1978.
243. B. Park, E. I. Verriest, "Canonical forms for discrete-linear periodically time-varying systems and a control application", *28th IEEE Conference on Decision and Control*, Tampa, USA, pp. 1220–1225, 1989.
244. E. Parzen, M. Pagano, "An approach to modeling seasonally stationary time-series", *Journal of Econometrics*, Vol. 9, pp. 137–153, 1979.
245. V. V. Patel, G. Deodhare, T. Viswanath, "Some applications of randomized algorithms for control system design", *Automatica*, Vol. 38, No. 12, pp. 2085–2092, 2002.
246. Y. Peng, M. Kinnaert, "Explicit solutions to the singular LQ regulation problem", *IEEE Transactions on Automatic Control*, Vol. 37, No. 5, pp. 633–636, 1992.
247. M. A. Peters, P. A. Iglesias, "Minimum entropy control for discrete-time time-varying systems", *Automatica*, Vol. 33, No. 4, pp. 591–605, 1997.
248. S. Pinzoni, *Stabilization and Control of Linear Time-Varying Systems*, PhD thesis, Arizona State University, Tempe, USA, 1989.
249. M. L. Psiaki, "Magnetic torquer attitude control via asymptotic periodic linear quadratic regulation", *Journal of Guidance, Control and Dynamics*, Vol. 24, No. 2, pp. 386–394, 2001.
250. J. S. Rayleigh, "On the crispation of fluid resting upon vibrating support", *Phil. Mag.*, Vol. 16, pp. 50–58, 1883.
251. J. S. Rayleigh, *The Theory of Sound*, Vol. 1, MacMillan, London, UK, 1877.
252. J. A. Richards, *Analysis of Periodically Time-varying Systems*, Springer-Verlag, 1983.
253. L. Rodman, P. Lancaster, *Algebraic Riccati Equation*, Oxford University Press, 1995.
254. H. H. Rosenbrock, "The stability of time-dependent control systems", *Journal of Electronics and Control*, Vol. 15, No. 1, pp. 73–80, 1963.
255. H. H. Rosenbrock, *State-space and Multivariable Theory*, John Wiley, New York, 1970.
256. T. L. Saaty, J. Bram, *Nonlinear Mathematics*, General Publishing Company Ltd., Toronto, Canada, 1964.
257. G. Sachs, T. Christotoulou, "Reducing fuel consumption of subsonic aircraft by optimal cycling cruise", *AIAA Journal*, Vol. 24, pp. 616–622, 1987.
258. H. Sakai, "Circular lattice filtering using Pagano's method", *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Vol. 30, No. 2, pp. 279–287, 1982.
259. H. Sakai, "Covariance matrices characterization by a set of scalar partial autocorrelation coefficients", *The Annals of Statistics*, Vol. 11, pp. 337–340, 1983.
260. H. Sakai, "Further results on circulant Levinson algorithm", *IEICE Transactions*, Vol. 74, No. 12, pp. 2962–2967, 1991.
261. H. Sakai, S. Ohno, "On backward periodic autoregressive processes", *Journal Time-Series Analysis*, Vol. 18, No. 4, pp. 415–427, 1997.
262. J. D. Salas, D.C. Boes, R.A. Smith, "Estimation of ARMA models with seasonal parameters", *Water Resources Research*, Vol. 18, No. 44, pp. 106–1010, 1982.
263. J. Salt, P. Albertos, "Model-based multirate controllers design", *IEEE Transactions on Control Systems Technology*, Vol. 13, No. 6, pp. 988–997, 2005.
264. C. Samson, "Time-varying feedback stabilization of car-like wheeled mobile robot", *International Journal of Robotics Research*, Vol. 12, No. 1, 1993.
265. H. Sandberg, *Model Reduction for Linear Time-varying Systems*, PhD thesis, Dept. of Automatic Control, Lund Institute of Technology, Sweden, 2004.
266. H. Sandberg, E. Möllerstedt, "Periodic modeling of power systems", *1st IFAC Workshop on Periodic Control Systems*, Cernobbio-Como, Italy, pp. 91–96, 2001.
267. H. Sandberg, E. Möllerstedt, "Harmonic modeling of the motor side of an inverter locomotive", *9th IEEE Conf. on Control Applications*, Anchorage (USA), pp. 918–923, 2000.
268. Q. Shao, R. B. Lund, "Computation and characterization of autocorrelations and partial autocorrelations in periodic ARMA models", *Journal of Time Series Analysis*, Vol. 25, No. 3, pp. 1–14, 2004.
269. A. Serrani, "A taxonomy for time-varying immersions in periodic internal-model control", in *Analysis and Design of Non-linear Control Systems*, A. Astolfi, L. Marconi (Eds.), Springer, pp. 303–318, 2008.

270. M. A. Shayman, "On the phase portrait of the matrix Riccati equation arising from the periodic control problem", *SIAM Journal Control and Optimization*, Vol. 23, No. 5, pp. 717–751, 1985.
271. L. M. Silverman, H. E. Meadows, "Controllability and time-variable unilateral networks", *IEEE Transactions on Circuit Theory*, Vol. 12, pp. 308–314, 1965.
272. L. M. Silverman, H. E. Meadows, "Controllability and observability in time-variable linear systems", *SIAM Journal on Control*, Vol. 5, pp. 64–73, 1967.
273. V. Solo, "On the stability of slowly time-varying linear systems", *Mathematics of Control, Signals, and Systems*, Vol. 7, No. 4, pp. 331–350, 1994.
274. J. L. Speyer, "On the fuel optimality of cruise", *AIAA Journal*, Vol. 10, 763–764, 1973.
275. J. Speyer, "Non-optimality of steady-state cruise for aircraft", *AIAA Journal*, Vol. 14, pp. 1604–1610, 1976.
276. J. Speyer, R. T. Evans, "A second variational theory of optimal periodic processes", *IEEE Transactions on Automatic Control*, Vol. 29, No. 22, pp. 138–148, 1984.
277. J. Sreedhar, P. Van Dooren, "Periodic descriptor systems: solvability and conditionability", *IEEE Transactions on Automatic Control*, Vol. 44, No. 2, pp. 310–313, 1999.
278. V. L. Syrmos, C. T. Abdallah, P. Dorato, K. Grigoriadis, "Static output-feedback- a survey", *Automatica*, Vol. 33, pp. 125–137, 1997.
279. G. Tadmor, L. Mirkin, " H_∞ control and estimation with preview - Part I: Matrix ARE solutions in continuous-time; Part II: Fixed-size ARE solutions in discrete-time," *IEEE Transactions on Automatic Control*, Vol. 50, No. 1, pp. 1940, 2005.
280. D. Teves, G. Niesl, A. Blaas, S. Jacklin, "The role of active control in future rotorcraft", *21st European Rotorcraft Forum*, St. Petersburg, Russia, III.10.1–17, 1995.
281. B. M. Troutman, "Some results in periodic autoregression", *Biometrika*, Vol. 6, pp. 219–228, 1979.
282. P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice-Hall, Englewood Cliffs, NJ., 1999.
283. M. Vaishya, R. Singh, "Analysis of periodically varying gear mesh systems with Coulomb friction using Floquet theory", *Journal of Sound and Vibration*, Vol. 243, No. 3, pp. 525–545, 2001.
284. P. Van Dooren, J. Sreedhar, "When is a periodic discrete-time system equivalent to a time-invariant one?", *Linear Algebra and its Applications*, Vol. 212/213, pp. 131–151, 1994.
285. A. Varga, "Periodic Lyapunov equation: some application and new algorithms", *International Journal of Control*, Vol. 67, pp. 69–87, 1997.
286. A. Varga, "Balancing related methods for minimal realization of periodic systems", *Systems and Control Letters*, Vol. 36, pp. 339–349, 1999.
287. A. Varga, "Balanced truncation model reduction of periodic systems", *39th IEEE Conference on Decision and Control*, Sydney, Australia, 2000.
288. A. Varga, "Robust and minimum norm pole assignment with periodic state feedback", *IEEE Transactions on Automatic Control*, Vol. 45, pp. 1017–1022, 2000.
289. A. Varga, "On balancing and order reduction of unstable periodic systems", *1st IFAC Workshop on Periodic Control Systems*, Como, Italy, pp. 177–182, 2001.
290. A. Varga, "Computation of minimal periodic realization of transfer matrices", *22nd American Control Conference*, Denver, USA, 2003.
291. A. Varga, "Computation of Kalman decompositions of periodic systems", *European Journal of Control*, Vol. 10, No. 1, pp. 1–8, 2004.
292. A. V. Vecchia, "Maximum likelihood estimation for periodic autoregressive moving average models", *Technometrics*, Vol. 27, pp. 375–384, 1985.
293. A. V. Vecchia, "Periodic autoregressive-moving average (PARMA) modeling with applications to water resources", *Journal of the American Water Resources Association*, Vol. 21, No. 5, pp. 721–730, 1985.
294. E. I. Verriest, "The operational transfer function and parametrization of N -periodic systems", *27th IEEE Conference on Decision and Control*, Austin, USA, pp. 1994–1999, 1988.
295. P. G. Voulgaris, M. A. Dahleh, L. Valavani, " H_∞ and H_2 optimal controllers for periodic and multirate systems", *Automatica*, Vol. 30, pp. 251–263, 1994.

296. S. H. Wang and C. A. Desoer, "The exact model matching of linear multivariable systems", *IEEE Transactions on Automatic Control*, Vol. 17, pp. 347349, 1972.
297. Q. Wang, J. L. Speyer, "Necessary and sufficient conditions for local optimality of a periodic process", *SIAM Journal of Control and Optimization*, Vol. 28, No. 2, pp. 482–497, 1990.
298. Q. Wang, J. Speyer, L. D. Dewell, "Regulators for optimal periodic processes", *IEEE Transactions on Automatic Control*, Vol. 40, No. 10, pp. 1777–1778, 1995.
299. Y. Wang, "An analytical solution for periodic response of elastic-friction damper systems", *Journal of Science and Vibration*, Vol. 189, No. 3, pp. 209–313, 1996.
300. L. Weiss, "On the structure theory of linear differential systems", *SIAM Journal on Control and Optimization*, Vol. 6, pp. 659–667, 1984.
301. N. M. Wereley, S. R. Hall, "Frequency response of linear time periodic systems", *29th IEEE Conference on Decision and Control*, Honolulu, USA, pp. 3650–3655, 1990.
302. J. Wertz, *Spacecraft Attitude Determination and Control*, D. Reidel Publishing Company, Dordrecht, Holland, 1978.
303. J. C. Willems, *The Analysis of Feedback Systems*, The MIT Press, Research Monograph No. 62, 1971.
304. J. L. Willems, V. Kucera, P. Brunovski, "On the assignment of invariant factors by time-varying feedback strategies", *Systems and Control Letters*, Vol. 5, No. 2, pp. 75–80, 1984.
305. F. Wirth, *Robust Stability of Discrete-time Systems under Time-varying Perturbation*, PhD thesis, University of Bremen, Germany, 1995.
306. R. Wisniewski, M. Blanke, "Fully magnetic attitude control for spacecraft subject to gravity gradient", *Automatica*, Vol. 35, N. 7, pp. 1201–1214, 1999.
307. W. A. Wolovich, *Linear Multivariable Systems*, Springer-Verlag, New York, USA, 1974.
308. W. M. Wonham, A. S. Morse, "Decoupling and pole assignment in linear systems: a geometric approach", *SIAM Journal on Control*, Vol. 8, No. 1, pp. 1–18, 1970.
309. V. A. Yakubovich, "Dichotomy and absolute stability of non-linear systems with periodically non-stationary linear part", *Systems and Control Letters*, Vol. 11, pp. 221–228, 1988.
310. V. A. Yakubovich, "Linear quadratic optimization problem and the frequency theorem for periodic systems, II", *Siberian Mathematical Journal*, Vol. 31, pp. 1027–1039, 1991 (original Russian version in *Sibirsk Math. Zh.*, Vol. 31, pp. 176–191, 1990).
311. V. A. Yakubovich, "Stability of non-linear systems with periodically non-stationary linear part", *Zeitschrift für Analysis und ihre Anwendungen*, Vol. 10, pp. 345–355, 1991.
312. V. A. Yakubovich, V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients*, J. Wiley and Sons, New York, USA, 1975.
313. W.-Y. Yan, *Robust Control and Periodic Control*, PhD thesis, The Australian national University, Canberra, Australia, 1990.
314. D. C. Youla, A. Jabr, J. J. Bongiorno, "Modern Wiener-Hopf design of optimal controller. Part II: the multivariable case", *IEEE Transactions on Automatic Control*, Vol. 21, No. 3, pp. 319–338, 1976.
315. L. A. Zadeh, "Frequency analysis of variable networks", *Proc. IRE*, Vol. 38, pp. 291–299, 1950.
316. L. A. Zadeh, C. Desoer, *Linear Systems Theory: the State–space Approach*, Mc Graw-Hill, New York, USA, 1963.
317. C. Zhang, J. Zhang, K. Furuta, "Performance analysis of periodically time-varying controllers", *13th IFAC World Congress*, San Francisco, USA, pp. 207–212, 1996.
318. Z. Zhang, A. Serrani, "The linear periodic output regulation problem", *System and Control letters*, Vol. 55, pp. 518–529, 2006.
319. J. Zhou, *Harmonic Analysis of Linear Continuous-time Periodic Systems*, PhD thesis, Kyoto University, Japan, 2001.
320. J. Zhou, T. Hagiwara, " H_2 and H_∞ norm computations of linear continuous-time periodic systems via the skew analysis of frequency response operators", *Automatica*, Vol. 38, No. 8, pp. 1381–1387, 2002.

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