Periodic Control: A Frequency Domain Approach

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Abstract—The problem is discussed of establishing whether the optimal constant control of a given plant (with respect to a specified performance index and subject to equality and/or inequality constraints) can be improved by cycling. A frequency domain approach is followed in order to establish a general condition, which, in the particular case of single-input systems, admits a simple geometric interpretation. Connections with high frequency and slowly varying periodic control problems are also pointed out.

I. Introduction

OPTIMAL STEADY-STATE control of industrial plants has now reached a well-established theoretical development, and a great number of important applications are reported in the literature.

In recent years, the opportunity of allowing the control of the plant to be periodically varied has begun to be investigated. A survey of the main theoretical results achieved in periodic optimization may be found in [1], while a number of important applications are reported in [2] and [3]. To be more specific, a conspicuous number of works, mostly in applications, have been devoted to the somewhat preliminary question whether the optimal steady-state operation could be improved by cycling. Till now, the problem has been approached in two ways: 1) by determining a (possibly suboptimal) solution of the optimal periodic control problem and then comparing it with the optimal steady-state solution; and 2) by a local analysis at the optimal steady-state without solving any further dynamic optimization problems. Since the first approach requires a solution of the periodic optimization problem, it takes over the particular question under consideration and does not look conceptually relevant by itself; thus, it will not be discussed in this paper. As for the second approach, a sufficient condition has been first pointed out by Horn and Lin [4] and subsequently strenghtened by Bailey and Horn [5] by means of the maximum principle. Specifically, the so-called relaxed control theory enabled one to prove that, whenever the optimal steady-state violates the maximum principle, the optimal constant operation can be improved by a suitable periodic bang-bang control, whose frequency must be sufficiently large with respect to the dynamics of the system ("high-frequency" control [6], [7]). The aim of the present paper is to give a frequency domain criterion for the possibility of improving the optimal steady-state control by a cyclic operation. Through a second variation analysis, it is shown that the problem here is equivalent to the unboundedness of the supremum of a quadratic criterion for a linear system under a periodicity constraint. Properties of this kind play a fundamental role not only in optimal control theory [8], [9], where they are connected, for instance, with the solvability of an algebraic Riccati equation, but also in stability [10] and sensitivity [11] analysis. As for optimal control, the status of the theory has been extensively reviewed by Willems in [12], where relevant contributions to the subject are also given. The frequency domain criterion given in this paper may also be viewed as a proper generalization of the result obtained in [4] in a way different from the one followed in [5]. The paper is organized into five sections. In Section II, the problem is stated in detail and proper notations and terminology are introduced. The frequency domain criterion is proved in Section III, and its connections with previous results are pointed out. Two particular cases, where this criterion can be given a significant graphical interpretation are analyzed in Section IV, while Section V supplies a pair of illustrative examples.

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II. Problem Statement

An optimal periodic control problem can be stated in the following way. Consider the time-invariant system

$$\dot{x} = f(x,u) \tag{1}$$

$$y = h(x) \tag{2}$$

where

$$u(t) \in R^m, \quad x(t) \in R^n, \quad y(t) \in R^p$$

$$u(\cdot) \in \Omega = \{\text{piecewise continuous functions}\}.$$

The performance index to be maximized is of the form

$$J[u(\cdot);\tau] \triangleq \frac{1}{\tau}\int_0^\tau g(y,u)\,dt, \quad \tau > 0 \tag{3}$$

subject to

$$x(\tau) = x(0) \tag{4}$$

$$\frac{1}{\tau}\int_0^\tau v(y,u)\,dt = 0 \tag{5}$$

$$\frac{1}{\tau}\int_0^\tau w(y,u)\,dt \leqslant 0 \tag{6}$$

where

$$v(\cdot,\cdot): R^p \times R^m \rightarrow R^q, \qquad w(\cdot,\cdot): R^p \times R^m \rightarrow R^r.$$

Moreover, let $\bar{u}^o$, $\bar{x}^o$, $\bar{y}^o$ be optimal within the class of constant solutions of problems (1)–(6) (optimal steady state); assume that $f(\cdot,\cdot)$, $h(\cdot)$, $g(\cdot,\cdot)$, $v(\cdot,\cdot)$, and $w(\cdot,\cdot)$ are twice differentiable at the optimal steady state; and define

$$H(x,u,\lambda,\mu) \triangleq g(h(x),u) + \lambda' f(x,u) + \mu' v(h(x)u). \quad (7)$$

Then, there must exist $\bar{\lambda}^o$, $\bar{\mu}^o$, $\bar{\nu}^o$ such that

$$H_x(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o) + (\bar{\nu}^o)' w_x(h(\bar{x}^o),\bar{u}^o) = 0 \quad (8)$$

$$H_u(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o) + (\bar{\nu}^o)' w_u(h(\bar{x}^o),\bar{u}^o) = 0 \quad (9)$$

$$f(\bar{x}^o,\bar{u}^o) = 0 \quad (10)$$

$$v(h(\bar{x}^o),\bar{u}^o) = 0 \quad (11)$$

$$\begin{array}{ll} \bar{\nu}_i^o \leq 0, & w_i(h(\bar{x}^o),\bar{u}^o) = 0 \\ \bar{\nu}_i^o = 0, & w_i(h(\bar{x}^o),\bar{u}^o) < 0, \end{array} \qquad i = 1,2,\cdots,r.$$

Finally, let $\bar{J}^o$ be the value of the performance index at the optimal steady state.

Definition 1: An optimal periodic control problem is said to be proper if there exist $\hat{\tau}$ and an admissible control $\hat{u}(\cdot)$ such that

$$J[\hat{u}(\cdot);\hat{\tau}] > \bar{J}^o.$$

Definition 2: An optimal periodic control problem is said to be locally proper if there exist $\hat{\tau}$ and an admissible weak control variation $\hat{\delta u}(\cdot)$ such that

$$J[\bar{u}^o + \hat{\delta u}(\cdot);\hat{\tau}] > \bar{J}^o. \quad (12)$$

Conditions establishing whether an optimal periodic control problem is proper or not will hereafter be called proper periodicity conditions. From a general point of view, the problem of establishing proper periodicity conditions consists in finding conditions under which the optimal constant control can be proved not to be an optimal one, within the class of the piecewise continuous functions. Furthermore, a second variation analysis will be employed in Section III in order to derive a frequency domain proper periodicity condition.

III. Proper Periodicity Conditions

Before discussing proper periodicity conditions, it would be useful to introduce the following terminology and notations. A $(k \times k)$-Hermitian matrix M will be called partially positive if there exists $\chi \in \mathcal{C}^k$ such that $\chi^* M \chi > 0$, where the asterisk denotes conjugate transpose. Let

$$A \triangleq f_x(\bar{x}^o,\bar{u}^o), \qquad B \triangleq f_u(\bar{x}^o,\bar{u}^o),$$

$$P \triangleq H_{xx}(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o), \qquad Q \triangleq H_{xu}(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o),$$

$$R \triangleq H_{uu}(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o)$$

and assume that no eigenvalue of A has zero real part.

Theorem 1: If the $(n \times n)$-Hermitian matrix

$$\Pi(\omega) \triangleq G'(-j\omega)PG(j\omega) + Q'G(j\omega) + G'(-j\omega)Q + R, \quad (13)$$

where $G(s) \triangleq (sI - A)^{-1}B$, is partially positive for some $\omega > 0$, then the optimal periodic control problem (1)–(6) is proper. Conversely, if problem (1)–(6) is locally proper, then there exists $\omega > 0$ such that $\Pi(\omega)$ is not negative definite.

Proof: Consider the first and second (weak) variations of the performance index (3) around the optimal constant solution (see Appendix)

$$\delta^1 J = -\frac{(\bar{\nu}^o)'}{\tau}\int_0^\tau (w_x(h(\bar{x}^o),\bar{u}^o)\,\delta x + w_u(h(\bar{x}^o),\bar{u}^o))\,\delta u\,dt$$

$$\delta^2 J = \frac{1}{2\tau}\int_0^\tau \begin{vmatrix}\delta x' & \delta u'\end{vmatrix} \begin{vmatrix} P & Q \\ Q' & R\end{vmatrix} \begin{vmatrix}\delta x \\ \delta u\end{vmatrix} dt$$

where

$$\delta\dot{x} = A\,\delta x + B\,\delta u, \qquad \delta x(\tau) = \delta x(0)$$

$$\frac{1}{\tau}\int_0^\tau (v_x(h(\bar{x}^o),\bar{u}^o)\,\delta x + v_u(h(\bar{x}^o),\bar{u}^o)\,\delta u)\,dt = 0 \quad (14)$$

$$\frac{\Psi}{\tau}\int_0^\tau (w_x(h(\bar{x}^o),\bar{u}^o)\,\delta x + w_u(h(\bar{x}^o),\bar{u}^o)\,\delta u)\,dt \leqslant 0 \quad (15)$$

$$\psi_{ij} = \begin{cases} 0, & i \neq j \\ 0, & i = j, \quad w_i(h(\bar{x}^o),\bar{u}^o) < 0, \\ 1, & i = j, \quad w_i(h(\bar{x}^o),\bar{u}^o) = 0, \end{cases} \qquad i,j = 1,2,\cdots,r.$$

Let

$$\delta u(t) = \sum_{-\infty}^{+\infty}{}_k\, U_k \exp(jk\Omega t), \qquad \text{a.e. in } (0,\tau), \quad \Omega \triangleq \frac{2\pi}{\tau}$$

be the Fourier-series expansion of any (periodically extended) weak control variation $\delta u(\cdot)$. Correspondingly, recalling that no eigenvalue of A has zero real part, standard Fourier analysis computations yield

$$\delta^1 J = -(\bar{\nu}^o)'[w_x(h(\bar{x}^o),\bar{u}^o)G(j0) + w_u(h(\bar{x}^o),\bar{u}^o)]U_0 \quad (16)$$

$$\delta^2 J = \frac{1}{2}\sum_{-\infty}^{+\infty}{}_k\, U_k^* \Pi(k\Omega) U_k \quad (17)$$

where $\Pi(\cdot)$ is given by (13); (14) and (15) become

$$[v_x(h(\bar{x}^o),\bar{u}^o)G(j0) + v_u(h(\bar{x}^o),\bar{u}^o)]U_0 = 0 \quad (18)$$

$$\Psi[w_x(h(\bar{x}^o),\bar{u}^o)G(j0) + w_u(h(\bar{x}^o),\bar{u}^o)]U_0 \leq 0. \quad (19)$$

Then, the first part of the theorem can be proved in the following way. If, for some $\omega > 0$, $\Pi(\omega)$ is partially positive, then there exists χ such that $\chi^*\,\Pi(\omega)\,\chi > 0$. Hence, taking the weak control variation $\delta u(t) = \epsilon\,(\chi \exp(j\omega t) + \bar{\chi}\exp(-j\omega t)$, where $\bar{\chi}$ denotes the conjugate of χ, it turns out that conditions (18) and (19) are obviously satisfied ($U_0 = 0$), while, in view of (16) and (17), $\delta^1 J = 0$ and $\delta^2 J = \epsilon^2 \chi^*\,\Pi(\omega)\,\chi > 0$. Thus, the optimal periodic control problem (1)–(6) is proper.

As for the second part of the theorem, assume that problem (1)–(6) is locally proper. Then, in view of (12), note that in correspondence with $\hat{\delta u}(\cdot)$ it cannot be $\delta^1 J < 0$; neither can it be $\delta^1 J > 0$, since in this case the assumption of $\bar{u}^o$ being an optimal constant control would be contradicted [in view of (16), $\bar{u}^o + \hat{U}_0$ would be a constant control better than $\bar{u}^o$]. Hence, it must be $\delta^1 J = 0$. This and the local property assumption allow the conclusion that $\delta^2 J \geq 0$. On the other hand, the optimality of $\bar{u}^o$ within the class of constant controls also implies

$$U_0' \, \Pi(0) U_0 \leqslant 0, \qquad \forall U_0: \delta^1 J = 0. \tag{20}$$

Hence, in view of (17), there must exist $\omega > 0$ such that $\Pi(\omega)$ is not negative definite. □

Remark 1: Despite the assumption that $\bar{u}^o$ is the optimal constant control, $\Pi(0)$ may be partially positive provided that the problem is affected by nontrivial inequality constraints. In fact, as shown above, the only condition to be satisfied by $\Pi(0)$ is (20). So, in view of (16), there might exist $U_0 \not\in \mathfrak{N}[\Psi(w_x(\bar{y}^o,\bar{u}^o)G(j0) + w_u(\bar{y}^o,\bar{u}^o))]$ ($\mathfrak{N}$ denotes nullspace) such that $U_0' \, \Pi(0) \, U_0 > 0$. If this is the case, an improved proper control may be found by the continuity of $\Pi(\cdot)$, within the class of slowly varying controls [13].

Remark 2: If the matrix R is partially positive, then the optimal periodic control problem is proper. In fact, since, for any purely dynamic system,

$$\lim_{\omega \to \infty} G(j\omega) = 0,$$

then $\Pi(\omega)$ must be partially positive, at least for ω large enough. The partial positivity of R was first taken as a sufficient condition for proper periodicity by Horn and Lin [4], who also pointed out its compatibility with the necessary condition (20).

Remark 3: While the present paper was under review, there appeared an important work by Willems [12], the conclusions of which might usefully be employed to give, under the additional assumption that (A,B) is a controllable pair, an alternative proof of Theorem 1 in the unconstrained case (see also [14]). The results reported in [12] are also of interest from a computational viewpoint, since they suggest many different ways of testing the condition given by Theorem 1.

Finally it may be worthwhile to discuss the connections existing between the sufficient condition given in Theorem 1 and the one introduced by Bailey and Horn in [5]. First, note that the former takes into account weak variations only, while the latter, based on the maximum principle, includes strong variations too. On the other hand, the condition given in [5] applies only to problems where the optimal constant control can be improved by a special class of high-frequency periodic controls (bang-bang controls infinitely fast switching between two suitable levels). This is not the case, of course, for the present condition, which may also be satisfied in problems where no high-frequency control better than the optimal constant one exists.

IV. Particular Cases

For a special class of systems, i.e., single-input systems, the general condition stated in Theorem 1 can be given a simple geometric interpretation. Moreover, a further restriction to single-output linear systems leads to a circle criterion quite similar to the well-known one from the theory of absolute stability in the sense of Lur'e.

A. Single-Input Systems

Assume that the control $u(t)$ of system (1) is a scalar ($m = 1$). Then, $\Pi(\omega)$ turns out to be a scalar too, and Theorem 1 can be rephrased in the following way.

Corollary 1: If there exists $\omega > 0$ such that $\Pi(\omega) > 0$, then the optimal periodic control problem (1)–(6) is proper. Conversely, if the problem is locally proper, then there exists $\omega > 0$ such that $\Pi(\omega) \geq 0$. □

Corollary 1 can be given a useful interpretation through the following elementary lemmas.

Lemma 1: When $m = 1$, by a suitable choice of the state variables ($\tilde{x} = T^{-1}x, \tilde{G}(j\omega) = T^{-1}G(j\omega)$), it is always possible to give $\Pi(\omega)$ the following form:

$$\Pi(\omega) = \sum_{1}^{n}{}_i \, \Pi_i(\omega)$$

$$\Pi_i(\omega) \triangleq \alpha_i |\tilde{G}_i(j\omega)|^2 + 2\beta_i \operatorname{Re} \tilde{G}_i(j\omega) + \gamma \tag{21}$$

where α_i, β_i, and γ are real numbers depending upon P, Q, and R only.

Proof: Since P is symmetric, there exists an orthogonal real matrix $T(T^{-1} = T')$ so $\tilde{P} = T^{-1}PT$ is diagonal. Then, assuming new state variables $\tilde{x} = T^{-1}x$, and letting α_i be the ith eigenvalue of P, β_i the ith entry of $T^{-1}Q$, $\gamma = (R/n)$, the conclusion follows immediately from (13). □

Lemma 2: Any Π_i-constant locus in the plane of the complex variable $\tilde{G}_i$ is a circle with center on the real axis at $-\beta_i/\alpha_i$ and radius $\rho_i = \sqrt{(\beta_i/\alpha_i)^2 - (\gamma - a)/\alpha_i}$, where a denotes the constant value of Π_i. When $\alpha_i = 0$, the circle degenerates into a straight-line parallel to the imaginary axis and crossing the real axis at $-(\gamma - a)/2\beta_i$. The case $\alpha_i = \beta_i = 0$ is trivial. □

This property supplies a quick way to compute $\Pi_i(\omega)$ simply by drawing the polar plot of $\tilde{G}(j\omega)$. In view of Lemma 1, the test of Corollary 1 can subsequently be performed very easily.

B. Single-Input–Single-Output Linear Systems

In the case of linear systems (first considered in [14], without integral constraints) described by

$$\dot{x} = Ax + Bu \qquad y = Cx,$$

first define $K(y,u,\mu) \triangleq g(y,u) + \mu' v(y,u)$; then, letting

$$\check{p} = K_{yy}(\bar{y}^o,\bar{u}^o,\bar{\mu}^o)$$
$$\check{q} = K_{yu}(\bar{y}^o,\bar{u}^o,\bar{\mu}^o)$$
$$\check{r} = K_{uu}(\bar{y}^o,\bar{u}^o,\bar{\mu}^o),$$

it turns out to be

$$P = \check{p}C'C \qquad Q = \check{q}C \qquad R = \check{r}.$$

So, we easily see that $\Pi(\omega)$ can be expressed as

$$\Pi(\omega) = \check{p}|\check{G}(j\omega)|^2 + 2\check{q} \operatorname{Re} \check{G}(j\omega) + \check{r}$$

where $\check{G}(s) \triangleq C(sI - A)^{-1}B$. Furthermore, if the observation is restricted to single-input–single-output systems, then proper periodicity can be ascertained by checking whether the polar plot of $\check{G}(s)$ has points in common with a suitable circular region. More specifically, consider the following definition.

Definition 3: The critical region $\mathfrak{L}$ in the plane of the complex variable s is as follows:

a) $\mathfrak{L} = \{s: (|s - c| - \rho) \operatorname{sgn} \check{p} > 0\}$, if $\check{p} \neq 0$
b) $\mathfrak{L} = \{s: (\operatorname{Re} s - d) \operatorname{sgn} \check{q} > 0\}$, if $\check{p} = 0$, $\hat{q} \neq 0$

where

$$c = \frac{\check{q}}{\check{p}} + j0, \qquad \rho = \sqrt{\left(\frac{\check{q}}{\check{p}}\right)^2 - \frac{\check{r}}{\check{p}}}, \qquad d = -\frac{\check{r}}{2\check{q}}. \qquad \square$$

In view of Lemma 2, denoting the closure of $\mathfrak{L}$ by $\bar{\mathfrak{L}}$, Corollary 1 can easily be given the following form.

Corollary 2 (Circle Criterion): If there exists $\omega > 0$ such that $\check{G}(j\omega) \in \bar{\mathfrak{L}}$, then the optimal periodic problem is proper. Conversely, if the problem is locally proper, then there exists $\omega > 0$ such that $\check{G}(j\omega) \in \bar{\mathfrak{L}}$. $\square$

Corollary 1 and Corollary 2 will be used in the two examples discussed in the following section.

V. Examples

Example 1

First, consider the elementary distillation system of Fig. 1, consisting of a reboiler $\mathfrak{R}$ and a condenser $\mathfrak{C}$. Let F, B, V, R, and D be, respectively, the feed, bottom, vapor, reflux, and distillate flow rates; and let x_F, x_B, x_C, and y_B be, respectively, the feed, reboiler, condenser, and vapor concentration of the light component. Moreover, let H_C and H_B (considered as constant) be the condenser and reboiler liquid holdups, and u the heating fluid flow rate, which is taken as the system input. The shape of the equilibrium curve $\varphi(x)$ of the mixture is shown in Fig. 2; by a linearization in the region of low concentrations, it may be assumed that $y_B \simeq Kx_B$.

Choose the state and output variables

$$x_C \triangleq x_1 \triangleq y_1$$
$$x_B \triangleq x_2$$
$$V \triangleq x_3 \triangleq y_2$$

and assume vapor production dynamics of the kind (first-order heat exchanger) $\dot{V} = -aV + bu$. By considering condenser and reboiler mass balances (vapor holdups negligible) and recalling that $B = F + R - V$, $D = V - R$, the system is described by

$$\dot{x}_1 = \frac{x_3}{H_C}(Kx_2 - x_1)$$

$$\dot{x}_2 = \frac{1}{H_B}\{Rx_1 - (F + R)x_2 - (K - 1)x_2x_3 + Fx_2\}$$

$$\dot{x}_3 = -ax_3 + bu, \qquad R \leq x_3 \leq R + F.$$

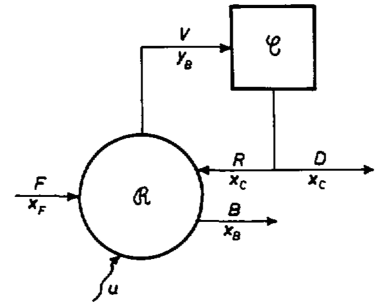

Fig. 1. Elementary distillation system.

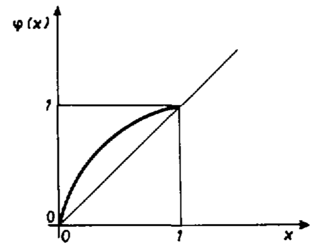

Fig. 2. A binary equilibrium curve.

Take $F = R = 0.2$; $x_F = 0.5$; $H_C = 0.7$; $H_B = 1.8$; $a = 3$; $b = 1$; and assume $K = 2$. The problem is to maximize the overall average return

$$J \triangleq \frac{1}{\tau}\int_0^\tau g(x)\, dt$$

where

$$g(x) \triangleq g_r(x) - g_c(x)$$
$$g_r(x) \triangleq (x_3 - R)(2x_1 - 1)$$
$$g_c(x) \triangleq 0.005 + 0.06x_3 - 0.05x_3^2$$

accounts for both the profit from product quality and the vapor cost. By standard computation, the optimal steady-state solution is

$$\bar{u}^o = 0.831; \quad \bar{x}_1^o = 0.724; \quad \bar{x}_2^o = 0.362; \quad \bar{x}_3^o = 0.277.$$

The procedure described in the preceding section supplies $\Pi(\omega)$, as drawn in Fig. 3. Hence, a sinusoidal weak control variation of the kind

$$\delta u(t) = \epsilon \sin \omega t$$

improves the optimal steady state for all $\omega > \sim 0.4$. However, it is interesting to note that in this case no periodic "relaxed control" [5], improving the optimal steady state, exists.

Example 2

This example is derived from [5]. Consider the single-input–single-output linear system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - \alpha x_2 + u, \qquad \alpha = \text{real parameter}$$
$$y = x_1.$$

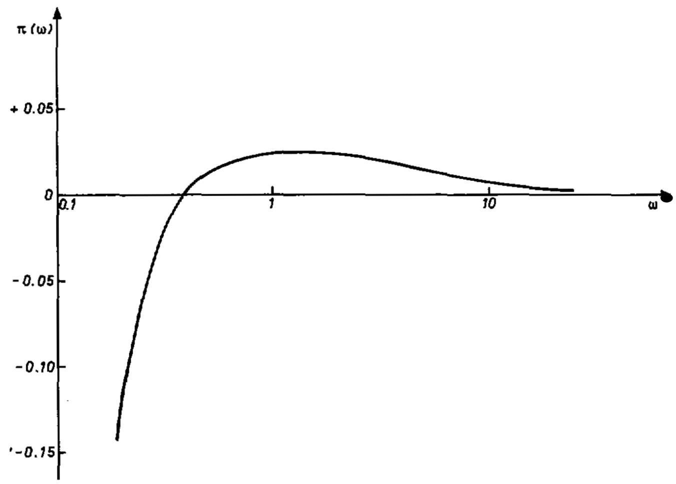

Fig. 3. The function $\pi(\cdot)$.

Let

$$J \triangleq \frac{1}{T}\int_0^T g(y,u)\,dt$$

$$g(y,u) \triangleq \phi(u) - \frac{1}{2}\phi(y)$$

$$\phi(u) \triangleq 2u^2 - u^4 - \frac{1}{4}(u-1)^2.$$

It is easy to see that, for any α, the sufficient condition in [5], based on the maximum principle, is never effective. As for the circle criterion, since $\check{p} = -\frac{1}{2}\check{n}\check{q}\check{r}, \check{q} = 0$, in this particular case, the critical region turns out to be independent of $\bar{u}^o$, $\bar{x}_1{}^o$, $\bar{x}_2{}^o$. The frequency response of the system for two different values of α is illustrated in Fig. 4. For $\alpha = 3$, the circle criterion shows the existence of a periodic control better than the steady-state one; while, for $\alpha = 0.6$, it is only possible to state that no weak variation of the optimal steady-state control exists that improves the performance index. On the other hand, it can be seen that, for any α, there exists a periodic "relaxed control" that is better than the constant one [5]. This shows that the condition given by the circle criterion is only locally necessary.

VI. Concluding Remarks

The problem of establishing whether the optimal steady-state control can be improved by a suitable periodic operation has been analyzed in this paper. A new sufficient condition has been obtained through a frequency domain approach, and a comparison with previous results has been made. In the particular case of single-input–single-output linear system, the condition given in this paper results in a circle criterion quite similar to the one which applies to the Lur'e problem. An open question for consideration is how to exploit all information supplied by the frequency domain analysis in order to get as close as possible to the optimal periodic operation. Finally, the theory presented here could be extended to other classes of systems, such as discrete, distributed parameter, or stochastic systems.

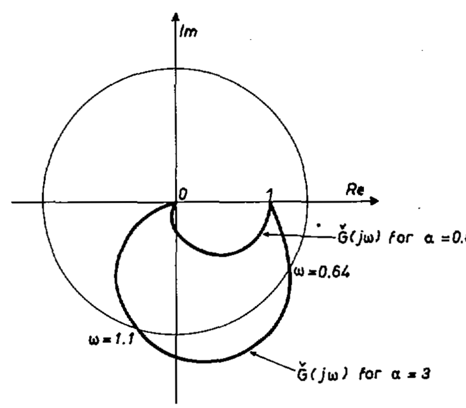

Fig. 4. The circle criterion.

Appendix

Here the first and second variation of the performance index J around the optimal steady state are derived.

Note that, in view of (7), (10), and (11),

$$\frac{1}{T}\int_0^T (H(x,u,\bar{\lambda}^o,\bar{\mu}^o) - (\bar{\lambda}^o)'f(x,u) - (\bar{\mu}^o)'v(x,u))\,dt = \frac{1}{T}\int_0^T g(x,u)\,dt = J$$

$$\frac{1}{T}\int_0^T H(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o)\,dt = g(\bar{x}^o,\bar{u}^o) = \bar{J}^o.$$

By taking (1), (4), and (5) into account,

$$\begin{aligned} J - \bar{J}^o &= \frac{1}{T}\int_0^T (H(x,u,\bar{\lambda}^o,\bar{\mu}^o) - H(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o))\,dt \\ &\quad + \frac{1}{T}\left[(\bar{\lambda}^o)'(x(T) - x(0)) + (\bar{\mu}^o)'\int_0^T v(x,u)\,dt\right] \\ &= \frac{1}{T}\int_0^T (H(x,u,\bar{\lambda}^o,\bar{\mu}^o) - H(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o))\,dt. \end{aligned}$$

Hence, according to the definition of P, Q, and R,

$$\delta^1 J + \delta^2 J = \frac{1}{\tau}\int_0^\tau (H_x(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o)\,\delta x + H_u(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o) \cdot \delta u)\,dt + \frac{1}{2\tau}\int_0^\tau \left|\delta x' \; \delta u'\right| \begin{vmatrix} P & Q \\ Q' & R \end{vmatrix} \begin{vmatrix} \delta x \\ \delta u \end{vmatrix} dt.$$

But, (8) and (9) require that

$$\delta^1 J = \frac{1}{\tau}\int_0^\tau (H_x(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o)\,\delta x + H_u(\bar{x}^o,\bar{u}^o,\bar{\lambda}^o,\bar{\mu}^o)\,\delta u)\,dt$$

$$= -\frac{(\bar{\nu}^o)'}{\tau}\int_0^\tau (w_x(h(\bar{x}^o),\bar{u}^o)\,\delta x + w_u(h(\bar{x}^o),\bar{u}^o)\,\delta u)\,dt.$$

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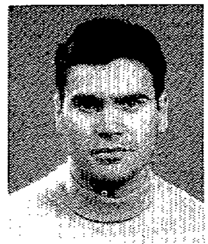

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