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Frequency Response of Linear Time Periodic Systems *

Norman M. Wereley and Steven R. Hall
Dept. of Aeronautics and Astronautics
Massachusetts Institute of Technology
77 Massachusetts Ave., Rm 33-105
Cambridge, MA 02139

Abstract

A frequency response notion comparable to the classical Bode gain and phase response for linear time invariant (LTI) systems has not been developed for linear time periodic (LTP) systems. In this paper, fundamental input and output signal spaces are identified that lead to a one-to-one map and a linear operator (transfer function). The LTP frequency response, including a characterization of gain, phase and their directional properties, is then presented in terms familiar to the multivariable LTI control theory.

Introduction

The fundamental notion behind Bode plots for LTI systems is that a complex exponential (sinusoidal) test input signal at a given frequency is mapped by the LTI transfer function into a complex exponential output signal of the same frequency, but with possibly different amplitude and phase. In contrast, if a complex exponential is input to an LTP system, possibly several (or an infinite number of) harmonics will appear in the output signal, each with possibly different gain and phase. As a result, the notion of a transfer function for LTP systems has been elusive.

This one-to-many map for complex exponentials is well understood and has been one motivation for the development of the describing function (DF) [1] or equivalently, the harmonic balance technique [2] as applied to LTP systems. Both the DF and harmonic balance approaches enforce the one-to-one map, so that higher harmonics can be neglected. However, this can lead to grossly inaccurate results. Several authors [1,3] have dealt with this problem by including as many harmonics as influence the fundamental harmonic. However, an infinite number of harmonics may influence the fundamental. To date, no comparable notion to the LTI transfer function has been presented for LTP systems. Therefore, in this paper, a systematic approach is taken to identify the linear operator that explicitly describes the input–output relationship between signal spaces of fundamental importance to LTP systems.

Linear Time Periodic Systems

LTP systems are described by a state space model of the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{1}$$

where $x(t) \in R^n$, $y(t) \in R^m$, $u(t) \in R^m$, and $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are real-valued matrices of appropriate dimension. $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are periodic with period T. That is,

$$A(t + NT) = A(t) \tag{2}$$

and similarly for $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$. (In the sequel, the period will be assumed to be T throughout, hence the term T-periodic). The state space model (1) will be denoted by the 4-tuple $S = [A(t),B(t),C(t),D(t)]$, and the set of all such systems with n finite will be denoted by $P^{m\times m}$. The state space model S is *strictly proper* iff $D(t) \equiv 0$.

Let $A(t)$ be an $n \times n$ matrix whose elements are piecewise continuous functions of time with a finite number of discontinuities in $t \in [0.T]$. Then, the $n \times n$ matrix $\Phi(t,\tau)$ is the *state transition matrix* [4] that satisfies the differential equation

$$\frac{\partial}{\partial t}\Phi(t,\tau) = A(t)\Phi(t,\tau); \quad \Phi(t,t) = I \tag{3}$$

Since $A(\cdot)$ is periodic, then $\Phi(\cdot,\cdot)$ is also periodic, so that

$$\Phi(t + NT, \tau + NT) = \Phi(t,\tau) \tag{4}$$

This leads to the well known Floquet Theorem.

Theorem 1 (Floquet) *Consider the LTP state space model (1), then:*

1. **State transition matrix.** *The state transition matrix of (1) can always be expressed as*

$$\Phi(t,t_0) = P(t,t_0)e^{Q(t-t_0)} \tag{5}$$

where $P(t,t_0)$ is a nonsingular $n \times n$ T-periodic matrix and Q is a constant matrix.

2. **Similarity transformation.** *The state transformation*

$$x(t) = P(t,t_0)v(t) \tag{6}$$

transforms $x(t)$ into a periodically time varying system of coordinates, $v(t)$, such that the dynamics matrix in the new state space is time invariant:

$$\begin{aligned}\dot{v}(t) &= Qv(t) + \bar{B}(t)u(t),\\ y(t) &= \bar{C}(t)v(t) + D(t)u(t).\end{aligned} \tag{7}$$

where $\bar{B}(t) = P^{-1}(t,t_0)B(t)$ and $\bar{C}(t) = C(t)P(t,t_0)$.

3. **Stability.** *An LTP system is stable iff all eigenvalues of the* **monodromy matrix,** *$\Phi(t_0 + T, t_0)$, lie on the open unit disk,*

$$\lambda\{\Phi(t_0 + T, t_0)\} \in D_o, \tag{8}$$

where $D_o = \{z\,;\,|z| < 1\}$.

Proof: The proof is straightforward. For details, see [5]. □

It will be assumed that $t_0 = 0$ in the sequel, such that $P(t,t_0) = P(t)$. The Floquet stability theorem has played a pervasive role in the analysis of LTP systems [2,6], and will be referred to repeatedly in the sequel.

Fundamental Signal Spaces

For LTI systems the test input signal of interest is the complex exponential,

$$u(t) = u_0e^{st}, \quad s \in C, \tag{9}$$

which is inappropriate for LTP systems because it results in a one to many map [7]. However, Floquet theory can be viewed as the search for signals that increase *geometrically* from period to period. This leads to the concept of a *geometrically periodic* test signal for LTP systems

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Definition 2 (Geometrically periodic signals) *A geometrically periodic (GP) signal, $u(t)$, with fundamental frequency, ω_p, and corresponding period T, has the property*

$$u(t+NT) = z^N u(t), \tag{10}$$

where $z \in C$. □

Since the LTP system maps a single harmonic into many harmonics, then it makes sense that the test signal should include all harmonics as well. Moreover, the GP signal can be expressed as a complex Fourier series of the periodic portion of the GP signal, modulated by a complex exponential signal, which leads to the concept of the *exponentially modulated periodic* test signal.

Definition 3 (Exponentially modulated periodic signal) *A (complex) exponentially modulated periodic (EMP) signal can be expressed as the complex Fourier series of a periodic signal of frequency, ω_p, modulated by a complex exponential signal,*

$$u(t) = \sum_{n\in Z} u_n e^{s_n t} \tag{11}$$

where $t \geq 0$, $s_n = s + jn\omega_p$, and $s \in C$. □

A strong analogy will be proposed at this point. *GP (or EMP) signals are to LTP systems, what complex exponentials (sinusoids) are to LTI systems.* One difficulty introduced by the EMP signal is the infinite number of Fourier coefficients required to describe an arbitrary periodic signal. This is a drawback that is inherent to the analysis of LTP systems and cannot be avoided. Clearly, bounded GP (or EMP) signals are $L_2[0,T]$ signals, and have bounded energy. In addition, an orthonormal basis for GP signals in $L_2[0,T]$ consists of sinusoids that are harmonics of the fundamental or pump frequency, ω_p.

Integral Operator Approach

Two closely related representations of an LTP system will be introduced. The first representation is based on an integral operator approach.

Theorem 4 (LTP system response to GP signals) *Consider the state space model of an asymptotically stable LTP system in (1). If the input to the LTP system is a GP signal, where z is not an eigenvalue of the monodromy matrix, then the state response consists of a GP* **steady state response**

$$\begin{aligned} x_{ss}(t) &= \Phi(t,0)[zI - \Phi(t,0)]^{-1} \int_0^T \Phi(t,\tau)B(\tau)u(\tau)d\tau \\ &\quad + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau; \quad t > \tau \end{aligned} \tag{12}$$

and a **transient response** *of the form*

$$x_{tr}(t) = \Phi(t,0)\left\{\xi_0 - [zI - \Phi(t,0)]^{-1}\int_0^T \Phi(t,\tau)B(\tau)u(\tau)d\tau\right\} \tag{13}$$

which vanishes as $t \to \infty$. Also, the GP **steady state output response** *is given by*

$$\begin{aligned} y_{ss}(t) &= C(t)\Phi(t,0)[zI - \Phi(t,0)]^{-1} \int_0^T \Phi(t,\tau)B(\tau)u(\tau)d\tau \\ &\quad + C(t)\int_0^t \Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t); \quad t > \tau \end{aligned} \tag{14}$$

and the **transient output response** *is given by*

$$y_{tr}(t) = C(t)\Phi(t,0)\left\{\xi_0 - [zI - \Phi(t,0)]^{-1}\int_0^T \Phi(t,\tau)B(\tau)u(\tau)d\tau\right\} \tag{15}$$

Proof: The proof is omitted for brevity. For details, see [5]. □

Note that the steady state response is a one–to–one map from GP input signals to GP output signals. Thus, the GP steady state output response is analogous to the LTI transfer function.

Definition 5 (LTP integral operator representation) *The steady state output response can be expressed as*

$$y(t) = \int_0^T \hat{G}(z;t,\tau)u(\tau)d\tau \tag{16}$$

where the integral operator kernel $\hat{G}(z;t,\tau)$ is defined as

$$\begin{aligned} \hat{G}(z;t,\tau) &= C(t)\Phi(T,0)[zI - \Phi(T,0)]^{-1}\Phi(T,\tau)B(\tau) \\ &\quad + D(t)\delta(t-\tau) + \begin{cases} 0; & t < \tau \\ \frac{1}{2}C(t)B(t); & t = \tau \\ C(t)\Phi(t,\tau)B(\tau); & t > \tau \end{cases} \end{aligned} \tag{17}$$

The integral operator defined by (16–17) will be denoted by $\hat{G}(z)$ so that (16) can be expressed in a more compact form,

$$y(t) = \hat{G}(z)u(t). \tag{18}$$

$\hat{G}(z)$ is the **integral operator transfer function.** □

Note that $\hat{G}(z;t,\tau)$ has been defined, for $t = \tau$, as the average of the integral operator kernel across the discontinuity. Since this is a set of measure zero, it does not affect the value of the integral. However, $\hat{G}(z;t,t)$ is chosen such that correct answers are obtained in certain limiting procedures not discussed here [5]. The integral operator $\hat{G}(z)$ plays a similar role in the study of LTP systems to the transfer function matrix in the LTI control theory.

The state space model appropriate to the description of LTP systems evolving from period to period is defined.

Definition 6 (Integral operator state space model) *An LTP system (1) can be represented by an* **integral operator state space model** *of the form*

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k \\ \tilde{y}_k &= \tilde{C}\tilde{x}_k + \tilde{D}\tilde{u}_k \end{aligned} \tag{19}$$

where $\tilde{x}_k \in C^n$, and $\tilde{u}_k \in L_2^m[0,T]$ defined over the subinterval k as

$$\tilde{u}_k = u(t-kT), \quad k \in Z,\ t > kT. \tag{20}$$

Here: $\tilde{A} : C^n \to C^n$, $\tilde{B} : L_2^m[0,T] \to C^n$, $\tilde{C} : C^n \to L_2^m[0,T]$, $\tilde{D} : L_2^m[0,T] \to L_2^m[0,T]$. The operators are defined below:

$$\begin{aligned} \tilde{A} &= \Phi(T,0), \\ \tilde{B}\tilde{u}_k &= \int_0^T \Phi(t,\tau)B(\tau)u(\tau)d\tau, \\ \tilde{C} &= C(t)\Phi(t,0), \\ \tilde{D}\tilde{u}_k &= \int_0^t \{C(t)\Phi(t,\tau)B(\tau) + D(t)\delta(t-\tau)\}\, u(\tau)d\tau. \end{aligned} \tag{21}$$

The integral operator state space model will be denoted by the 4–tuple $\tilde{S} = [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}] = [\Phi(T,0), \tilde{B}, \tilde{C}, \tilde{D}]$. □

Using the above definition, the integral operator can be expressed in a compact form.

Lemma 7 (Integral operator transfer function) *The integral operator transfer function, $\hat{G}(z)$, explicitly describes the relationship between the input, $u(t) \in L_2^m[0,T]$ and the output, $y(t) \in L_2^p[0,T]$, as*

$$\hat{G}(z) = \tilde{C}\left(zI - \tilde{A}\right)\tilde{B} + \tilde{D} \tag{22}$$

Proof: The above expression follows directly from Definition 6. □

Although the notation, $\tilde{A}$, has been introduced for the monodromy matrix, the notation $\Phi(T,0)$ will be retained for clarity.

The integral operator approach was useful in the development of a generalized Nyquist criterion for LTP systems [5,8], and is useful primarily in an analytical context. However, the integral operator approach does not lead to useful numerical procedures, so that alternatives must be developed.

Harmonic Balance Approach

The time periodic parametric excitation associated with most physical systems can be expressed by a sum of sinusoids of relatively low harmonic number. From an engineering standpoint, a sensible numerical procedure would exploit this tendency, so that an approach based on harmonic balance is developed [2]. Harmonic balance offers two advantages: (1) low frequency contributions to the parametric excitation can be more easily captured than with a time discretization approach, (2) time periodic parametric excitation and EMP signals are naturally described by the complex Fourier series. Here, the frequency response (or the steady state forced response) of the LTP state space model (1) to an EMP signal (11) is determined.

Theorem 8 (LTP system response to EMP signals) *Consider the LTP state space model in (1), which is assumed to be asymptotically stable. The T-periodic matrix $\bar{B}(t)$ can be expanded in a complex Fourier series,*

$$\bar{B}(\tau) = \sum_{l\in Z} \bar{B}_l e^{jl\omega_p\tau} \tag{23}$$

and similarly for $\bar{C}(t)$ and $D(t)$. If the input to the LTP system above is an EMP signal (11), then the state response consists of an EMP **steady state forced response,**

$$x_{ss}(t) = \sum_{n,l,m\in Z} P_{n-l}\left(s_l I - Q\right)^{-1} \bar{B}_{l-m} u_m e^{s_n t}, \tag{24}$$

and a **transient state response,**

$$x_{tr}(t) = \Phi(t,0)\left\{\xi_0 - \sum_{l,m\in Z}\left(s_l I - Q\right)^{-1}\bar{B}_{l-m}u_m\right\}, \tag{25}$$

which vanishes as $t\to\infty$. The **steady state output response** *is an EMP signal of the form*

$$y_{ss}(t) = \sum_{n\in Z}\left\{\sum_{l,m\in Z}\bar{C}_{n-l}\left(s_l I - Q\right)^{-1}\bar{B}_{l-m}u_m + \sum_{m\in Z} D_{n-m}u_m\right\}e^{s_n t}, \tag{26}$$

and the **transient output response** *is given by*

$$y_{tr}(t) = C(t)\Phi(t,0)\left\{\xi_0 - \sum_{l,m\in Z}\left(s_l I - Q\right)^{-1}\bar{B}_{l-m}u_m\right\}.. \tag{27}$$

Proof: The proof is omitted for brevity. For details, see [5] □

Thus, an LTP system induces a one-to-one map from EMP inputs to EMP outputs at steady state. Hence, the LTP frequency response notion can be stated: *at steady state, an asymptotically stable LTP system maps an EMP input signal, to an EMP output signal of the same frequency, but with possibly different amplitude and phase (as long as s is not an eigenvalue of Q).* The amplitude and phase refers to the amplitude and phase of all the harmonics in the input and output signals. This *spatial* dependency will be quantified in the sequel.

The steady state response can also be obtained by direct application of harmonic balance. Consider the LTP state space model (1). The dynamics matrix can be expanded in a complex Fourier series

$$A(t) = \sum_{m\in Z} A_m e^{jm\omega_p t}, \tag{28}$$

and similarly for $B(t)$, $C(t)$, $D(t)$. An EMP test signal, $u(t)$, implies that the steady state response is also an EMP signal,

$$x(t) = \sum_{n\in Z} x_n e^{s_n t}, \tag{29}$$

$$\dot{x}(t) = \sum_{n\in Z} s_n x_n e^{s_n t}, \tag{30}$$

and that the steady state output, $y(t)$, is an EMP signal,

$$y(t) = \sum_{n\in Z} y_n e^{s_n t}. \tag{31}$$

Expanding (1) in terms of these Fourier series':

$$\begin{aligned} 0 &= \sum_{n\in Z}\left\{s_n x_n - \sum_{m\in Z} A_{n-m}x_m - \sum_{m\in Z} B_{n-m}u_m\right\}e^{s_n t}, \\ 0 &= \sum_{n\in Z}\left\{y_n - \sum_{m\in Z} C_{n-m}x_m - \sum_{m\in Z} D_{n-m}u_m\right\}e^{s_n t}. \end{aligned}$$

Now, the complex exponentials, $\{e^{jn\omega_p t} \mid n\in Z\}$, form an orthonormal basis in $L_2[0,T]$, so that the terms enclosed by braces must vanish. This procedure is referred to as the *principle of harmonic balance.* Hence, the two equations below hold $\forall n\in Z$:

$$\begin{aligned} s_n x_n &= \sum_{m\in Z} A_{n-m}x_m + \sum_{m\in Z} B_{n-m}u_m, \\ y_n &= \sum_{m\in Z} C_{n-m}x_m + \sum_{m\in Z} D_{n-m}u_m. \end{aligned} \tag{32}$$

Although the above equations are a concise representation of the input–output relationship between the Fourier coefficients of the input and output signals, manipulating summations can be tedious. Therefore, a Toeplitz form notation will be utilized.

Definition 9 (Harmonic state space model) *The system of equations (32) can be expressed as the infinite dimensional matrix equation,*

$$\begin{aligned} s\mathcal{X} &= (\mathcal{A}-\mathcal{N})\mathcal{X} + \mathcal{B}\mathcal{U}, \\ \mathcal{Y} &= \mathcal{C}\mathcal{X} + \mathcal{D}\mathcal{U}. \end{aligned} \tag{33}$$

Define doubly infinite vectors representing the harmonics of the state,

$$\mathcal{X}^T = \left[\cdots, x_{-2}^T, x_{-1}^T, x_0^T, x_1^T, x_2^T, \cdots\right], \tag{34}$$

and similarly for the control signal, $\mathcal{U}$, and output signal, $\mathcal{Y}$. The T-periodic dynamics matrix, $A(t)$, is expressed in terms of its harmonics, $\{A_n | n\in Z\}$, as a doubly infinite block Toeplitz matrix called a **Toeplitz form** *[10],*

$$\mathcal{A} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & A_0 & A_{-1} & A_{-2} & A_{-3} & A_{-4} & \cdots \\ \cdots & A_1 & A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ \cdots & A_2 & A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ \cdots & A_3 & A_2 & A_1 & A_0 & A_{-1} & \cdots \\ \cdots & A_4 & A_3 & A_2 & A_1 & A_0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{35}$$

with a similar definition for $\mathcal{B}$ in terms of $\{B_n | n\in Z\}$, $\mathcal{C}$ in terms of $\{C_n | n\in Z\}$, and $\mathcal{D}$ in terms of $\{D_n | n\in Z\}$. Finally, define the modulation frequency matrix,

$$\mathcal{N} = \text{blkdiag}\{jn\omega_p I\}; \quad \forall n\in Z. \tag{36}$$

This infinite dimensional matrix equation (33) is called the **harmonic state space model,** *denoted by the 4-tuple $\mathcal{S} = [(\mathcal{A}\text{-}\mathcal{N}), \mathcal{B}, \mathcal{C}, \mathcal{D}]$.* □

The harmonic state space model is similar to Toeplitz systems [11]. In fact, if the modulation frequency matrix is set to zero, $\mathcal{N}=0$, then the harmonic state space model is identical to the Toeplitz system. Unfortunately, the Toeplitz system results are not directly applicable to the LTP case since the harmonic state space model is *quasi-Toeplitz.* Also, the harmonic state space model has more structure than the general Toeplitz system. The Toeplitz forms in the harmonic state space model are Hermitian, and since the harmonics associated with a given system matrix generally grow small as the harmonic number grows large, the Toeplitz forms are also effectively banded. However, this additional structure is not exploited by the Toeplitz system results in [11].

Lemma 10 (Harmonic transfer functions) *The* **harmonic transfer function** *(HTF), $\hat{\mathcal{G}}(s)$, describes the input-output relationship between the harmonics of the input signal, and those of the output signal, such that*

$$\mathcal{Y} = \hat{\mathcal{G}}(s)\mathcal{U}, \tag{37}$$

where

$$\hat{\mathcal{G}}(s) = \mathcal{C}\left[s\mathcal{I} - (\mathcal{A}-\mathcal{N})\right]^{-1}\mathcal{B} + \mathcal{D}. \tag{38}$$

Proof: Eliminating $\mathcal{X}$ from (33), in favor of $\mathcal{Y}$ and $\mathcal{U}$, results in the desired input—output relationship. □

However, there are two problems associated with the harmonic transfer function. First, it is not clear that the harmonic transfer function, which requires the inversion of a doubly infinite matrix, will always exist. Second, the harmonic transfer function described above is a doubly infinite matrix operator, which cannot be implemented on the computer. The first problem will be dealt with, in general, by application of the second Floquet result. The second problem will be mitigated by truncating the HTF in order to implement analyses on the computer.

Let us consider some examples. Consider the LTP state space model with time invariant dynamics, $S = [\boldsymbol{Q}, \bar{\boldsymbol{B}}(t), \bar{\boldsymbol{C}}(t), \boldsymbol{D}(t)]$. Here, $\boldsymbol{Q}$, is a constant matrix, so that $\mathcal{A} = \text{blkdiag}\{\boldsymbol{Q}\}$. Following the procedure outlined above yields

$$\boldsymbol{y}_m = \sum_{n\in Z}\left\{\sum_{l\in Z}\boldsymbol{C}_{m-l}(s_l - \boldsymbol{Q})^{-1}\boldsymbol{B}_{l-n} + \boldsymbol{D}_{m-n}\right\}\boldsymbol{u}_n \quad \forall m \in Z. \quad (39)$$

The HTF is the terms in braces:

$$\hat{\mathcal{G}}_{m,n}(s) = \sum_{l\in Z}\boldsymbol{C}_{m-l}\,(s_l\boldsymbol{I} - \boldsymbol{Q})^{-1}\,\boldsymbol{B}_{l-n} + \boldsymbol{D}_{m-n}. \quad (40)$$

Direct application of harmonic balance produces the same result as in Theorem 8.

A large class of problems that are important in the study of LTP systems are those represented by LTI plants with input or output amplitude modulations. The HTF for both these cases simplifies markedly. Consider the state space model of an amplitude modulated input to an LTI plant, $S = [\boldsymbol{Q}, \boldsymbol{B}(t), \boldsymbol{C}, \boldsymbol{D}(t)]$. The input–output relationship (39) can be further simplified,

$$\boldsymbol{y}_m = \sum_{n\in Z}\left\{\boldsymbol{C}(s_m - \boldsymbol{Q})^{-1}\boldsymbol{B}_{m-n} + \boldsymbol{D}_{m-n}\right\}\boldsymbol{u}_n \quad (41)$$

Hence, the HTF is the terms in braces:

$$\hat{\mathcal{G}}_{m,n}(s) = \boldsymbol{C}\,(s_m\boldsymbol{I} - \boldsymbol{Q})^{-1}\,\boldsymbol{B}_{m-n} + \boldsymbol{D}_{m-n}. \quad (42)$$

Instead, consider the case of an amplitude modulated output signal from an LTI plant, $S = [\boldsymbol{Q}, \boldsymbol{B}, \bar{\boldsymbol{C}}(t), \boldsymbol{D}(t)]$, then the HTF is

$$\hat{\mathcal{G}}_{m,n}(s) = \boldsymbol{C}_{m-n}\,(s_n\boldsymbol{I} - \boldsymbol{Q})^{-1}\,\boldsymbol{B} + \boldsymbol{D}_{m-n}. \quad (43)$$

In the latter two cases, no infinite sums need to be computed.

Two different forms of the HTF were presented. The first form of the HTF in Theorem 8, corresponds to the harmonic state space model $\mathcal{S} = [(\mathcal{Q}\text{-}\mathcal{N}), \bar{\mathcal{B}}, \bar{\mathcal{C}}, \mathcal{D}]$, where $(\mathcal{Q} - \mathcal{N})$ is a block diagonal quasi–Toeplitz form. The second form of the HTF in Lemma 10, corresponds to the harmonic state space model $\mathcal{S} = [(\mathcal{A}\text{-}\mathcal{N}), \bar{\mathcal{B}}, \bar{\mathcal{C}}, \mathcal{D}]$, where $(\mathcal{A} - \mathcal{N})$ a full quasi–Toeplitz form. However, both representations are related by a similarity transformation.

Lemma 11 (A similarity transformation) *Consider the LTP state space model,* $S = [\boldsymbol{A}(t), \boldsymbol{B}(t), \boldsymbol{C}(t), \boldsymbol{D}(t)]$. *Its harmonic state space model, denoted by the 4-tuple* $\mathcal{S}_1 = [(\mathcal{A}\text{-}\mathcal{N}), \mathcal{B}, \mathcal{C}, \mathcal{D}]$, *where* $(\mathcal{A}\text{-}\mathcal{N})$ *is a full quasi–Toeplitz form. The similarity transformation given by* $\mathcal{X} = \mathcal{P}\,\mathcal{V}$, *where* $\mathcal{P}$ *is the Toeplitz form corresponding to the periodic portion of the Floquet solution,* $\boldsymbol{P}(t)$, *transforms the above harmonic state space model to a harmonic state space model, denoted by the 4-tuple* $\mathcal{S}_2 = [(\mathcal{Q} - \mathcal{N}), \bar{\mathcal{B}}, \bar{\mathcal{C}}, \mathcal{D}]$, *where* $(\mathcal{Q}\text{-}\mathcal{N})$ *is a doubly infinite block diagonal matrix. Here:*

$$\begin{aligned}\mathcal{Q} - \mathcal{N} &= \mathcal{P}^{-1}(\mathcal{A} - \mathcal{N})\mathcal{P}, \\ \bar{\mathcal{B}} &= \mathcal{P}^{-1}\mathcal{B}, \\ \bar{\mathcal{C}} &= \mathcal{C}\mathcal{P}. \end{aligned} \quad (44)$$

Proof: For details, see [5]. □

This similarity transformation has the virtue that it is an algebraic similarity transformation, instead of a time varying one requiring the solution of an ODE [4,5]. Unfortunately, the similarity transformation requires the infinite dimensional Toeplitz form $\mathcal{P}$.

Principal Gains and Directions

Directional properties of LTP systems can be generalized as any property of the integral operator transfer function (harmonic transfer function) or GP (EMP) signals that exhibits a *spatial* dependency, in addition to the frequency dependency shared with scalar LTI transfer functions and scalar sinusoidal signals. This spatial dependency is manifested in the infinite number of Fourier coefficients required to characterize EMP signals, as well as the multi–input multi–output nature of general LTP systems.

Clearly, the one-to-one map induced by GP signals, that is, the integral operator transfer function, is linear since the underlying dynamics were linear. Thus, the singular value decomposition of the integral operator transfer function will provide useful interpretations of domain and range spaces, and directions of maximal amplification for this linear map. The singular values of the integral operator, and their associated directions, are well defined [9], and are parameterized by z on the unit circle in the z plane. However, the integral operator approach provides little insight into how to compute the singular values since the integral operator is a functional operating on vector functions (signals).

On the other hand, the harmonic balance approach involves the multiplication of complex matrices and vectors and can be easily be implemented on the computer. The harmonic transfer function maps an EMP input into an EMP output according to

$$\hat{\mathcal{G}}(s) : \mathcal{U} \to \mathcal{Y}; \quad \hat{\mathcal{G}}(s) \in P^{m\times m} \quad (45)$$

that is, the plant has m independent inputs and m independent outputs. The input signal $\boldsymbol{u}(t)$ is an EMP signal, and provided the internal dynamics represented by $\boldsymbol{A}(t)$ are asymptotically stable (the eigenvalues of the monodromy matrix are on the unit disk, or the eigenvalues of Q are in the LHP), the steady state output signal, $\boldsymbol{y}(t)$, will also be an EMP signal.

The harmonic transfer function is a complex matrix that changes value with frequency, ω, where the frequency range of interest is given by

$$\omega \in \Omega_0 = [-\frac{\omega_p}{2}, \frac{\omega_p}{2}], \quad (46)$$

that is, the unit circle in the z-plane is mapped to the imaginary axis in the fundamental strip in the s–plane. For any value of frequency, $\omega \in \Omega_0$, the singular value decomposition (SVD) of the HTF function can be computed,

$$\hat{\mathcal{G}}(j\omega) = \mathbf{U}(j\omega)\Sigma(j\omega)\mathbf{V}^*(j\omega), \quad (47)$$

where the superscript $*$ denotes the Hermitian or complex conjugate. Each quantity in the SVD is parameterized by frequency.

Since the harmonic transfer function is infinite dimensional, the singular values will be studied by examining the singular values of the truncation of the HTF. Thus, N positive harmonics (as well as N negative harmonics and the zeroth harmonic so that the HTF is symmetric) are included and the truncated HTF is denoted by $\hat{\mathcal{G}}_N(j\omega)$. The implication is that if enough harmonics are included in the truncated HTF, then correct answers will be obtained in SVD analyses. Assume that $\hat{\mathcal{G}}_N(j\omega)$ is invertible. Then;

(a) the quantity $\Sigma_N(j\omega)$ is a $m(2N+1)\times m(2N+1)$ complex matrix consisting of the singular values of the truncated HTF:

$$\sigma_{max}(\omega) = \sigma_1(\omega) \geq \sigma_2(\omega) \geq \cdots \geq \sigma_{2N+1}(\omega) = \sigma_{min}(\omega). \quad (48)$$

The minimum singular value $\sigma_{min}(\omega)$ is the gain associated with the minimum amplification direction. However if more harmonics are included in the analysis, a smaller minimum singular value may be found. These singular values are known as the *principal gains*.

(b) $\mathbf{V}_N(j\omega)$ is a complex matrix $m(2N+1)\times m(2N+1)$ whose column vectors $\left\{v^{(n)}(\omega)\right\}$ are the *principal input directions* or right singular vectors of $\hat{\mathcal{G}}_N(j\omega)$ and form a basis for the *domain space*.

(c) $\mathbf{U}_N(j\omega)$ is a complex matrix $m(2N+1) \times m(2N+1)$ whose column vectors $\left\{u^{(n)}(\omega)\right\}$ are the *principal output directions* or left singular vectors of $\hat{\mathcal{G}}_N(j\omega)$ and form a basis for the *range space*.

This singular value analysis is carried out for $\omega \in \Omega_0$, so that the principal gains can be plotted versus frequency as a Bode gain plot (that is, decibels versus frequency, $\omega \in \Omega_0$). This *principal gain diagram* is analogous in many respects to the singular value plot for multivariable LTI systems, although the specific interpretation must be carefully worked. Also, the principal gain diagram repeats itself in the nth complementary strip in the s-plane, that is,

$$\omega \in \Omega_n = \left[\frac{(2n-1)\omega_p}{2}, \frac{(2n+1)\omega_p}{2}\right]. \tag{49}$$

At each frequency it is assumed that the input to the asymptotically stable system is a *unit complex periodic signal* of the form

$$u(t) = \sum_{n \in Z} u_n e^{j(\omega + n\omega_p)t}; \tag{50}$$

where $\|u_N\|_2 = 1$. Then assuming that the system is in steady state, the resulting output will also be a complex periodic signal,

$$y(t) = \sum_{n \in Z} y_n e^{j(\omega + n\omega_p)t}, \tag{51}$$

where

$$\|y_N\|_2 = \left\|\hat{\mathcal{G}}_N(j\omega) u_N\right\|_2. \tag{52}$$

Consider a maximum direction analysis at a specified frequency $\omega \in \Omega_0$. The SVD produces a maximum singular value, $\bar{\sigma}(\omega)$, with a corresponding complex valued right singular vector, $\bar{v}(\omega)$, with $m(2N+1)$ elements, and left singular vector, $\bar{u}(\omega)$ with $m(2N+1)$ elements. The maximum right and left singular vectors correspond to the first columns of $\mathbf{V}_N(j\omega)$ and $\mathbf{U}_N(j\omega)$, respectively. The block vector elements of $\bar{v}(\omega)$ can be expressed in polar form as

$$\bar{v}_n(\omega) = \bar{a}_n \circ e^{j\bar{\psi}_n}; \quad \forall n \in Z. \tag{53}$$

Here, the $\circ$ denotes the *Schur product* or an element by element vector multiplication. The quantities $\bar{a}_n$ and $\bar{\psi}_n$ are parameterized by frequency, although this frequency dependence will be implicit to simplify notation. Also, n denotes the block vector of $\bar{v}(\omega)$ corresponding to the nth harmonic. Each block vector element of $\bar{v}(\omega)$ contributes a different amplitude and phase. The block vector elements of $\bar{u}(\omega)$ can also be written in polar form as

$$\bar{u}_n(\omega) = \bar{b}_n \circ e^{j\bar{\phi}_n}; \quad \forall n \in Z. \tag{54}$$

Again, the quantities $\bar{u}_n$ and $\bar{\phi}_n$ depend implicitly on frequency.

At steady state, the input and output signals that correspond to the direction of maximum amplification can be reconstructed from the information contained in the maximum direction analysis. First, the input signal corresponding to the maximum amplification direction can be reconstructed from the quantities $\bar{a}_n$ and $\bar{\psi}_n$ for all $n \in Z$ as below:

$$\bar{u}(t) = \sum_{n \in Z} \bar{a}_n \circ e^{j\bar{\psi}_n} e^{j(\omega + n\omega_p)t}. \tag{55}$$

The steady state output signal corresponding to the maximum amplification direction, $\bar{y}(t)$, can be reconstructed from $\bar{\sigma}$, $\bar{u}_n$, and $\bar{\phi}_n$:

$$\bar{y}(t) = \bar{\sigma} \sum_{n \in Z} \bar{b}_n \circ e^{j\bar{\phi}_n} e^{j(\omega + n\omega_p)t}. \tag{56}$$

Finally, the amplitude and phase of any periodic input to the LTP system at a frequency $\omega \in \Omega_0$ can be expressed as a linear combination of the principal input directions (right singular vectors) at that frequency. The amplitude and phase of the corresponding output can be predicted at steady state by the same linear combination of the principal output directions (left singular vectors) weighted by their corresponding principal gains (singular values).

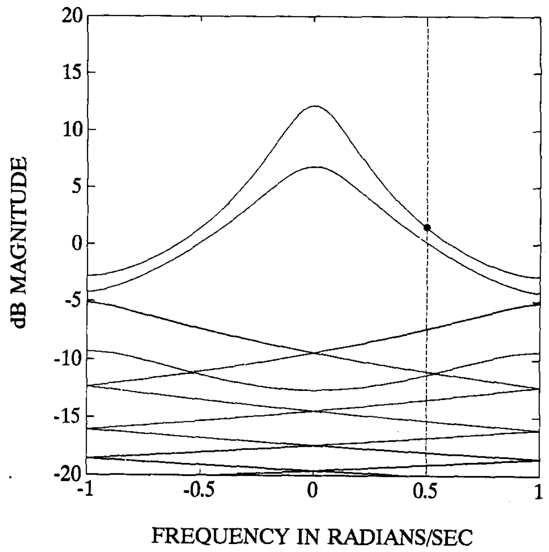

Figure 1: Principal gain diagram for lossy Mathieu equation example. The gain of the LTP system at a given frequency is characterized by the principal gains (singular values) of the harmonic transfer function. The principal gain diagram is a graph of these singular values over the frequency range $\omega \in [-\omega_p/2, \omega_p/2]$. The shaded circle on the maximum principal gain locus corresponds to the frequency of the maximum direction analysis shown in Figure 2.

The Lossy Mathieu Equation

Here, the time periodic dynamics of the Lossy Mathieu equation [7] are considered. Defining the state vector as

$$x^T = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}, \tag{57}$$

leads to the system matrices:

$$\begin{aligned} A(t) &= \begin{bmatrix} 0 & 1 \\ -(1 - 2\beta\cos\omega_p t) & -2\zeta \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 1 \end{bmatrix}. \end{aligned} \tag{58}$$

The parameter values selected for this example are $\omega_p = 2$, $\zeta = 0.2$, and $\beta = 0.2$. For most of the analyses performed in this section, truncation of the harmonic transfer function to $N = 10$ harmonics was sufficient to obtain reasonable results.

The principal gain diagram of this system is shown in Figure 1. There are an infinite number of principal gain loci corresponding to the infinite dimensional domain and range spaces of the harmonic transfer function, and only a small number are shown in this diagram. The largest gains occur for the zeroth, and first harmonics, which can be concluded by examination of the singular vectors associated with each gain plot.

A maximum amplification analysis is performed for $\omega = \omega_p/4 = 0.5$rad/sec as indicated by the shaded circle in Figure 1. The maximal input (principal) direction is given by the right singular vector associated with the maximum singular value,

$$\bar{u}(t) = e^{jt/2} \begin{bmatrix} \vdots \\ 0.8227 - j0.1088 \\ 0.3497 + j0.0572 \\ 0.3319 + j0.2692 \\ \vdots \end{bmatrix}^T \begin{bmatrix} \vdots \\ e^{-j\omega_p t} \\ 1 \\ e^{j\omega_p t} \\ \vdots \end{bmatrix}$$

It is instructive to express the input direction in polar form,

$$\bar{u}(t) = e^{jt/2} \begin{bmatrix} \vdots \\ 0.8299e^{-j7.534^\circ} \\ 0.3544e^{j9.292^\circ} \\ 0.4274e^{j39.05^\circ} \\ \vdots \end{bmatrix}^T \begin{bmatrix} \vdots \\ e^{-j\omega_p t} \\ 1 \\ e^{j\omega_p t} \\ \vdots \end{bmatrix}. \tag{59}$$

Note that all of the harmonics enter with a specific phase. The maximum principal gain is the maximum singular value, $\bar{\sigma} = 1.1798$. The maximum output (principal) direction is given by the left singular vector associated with the maximum singular value,

$$\bar{y}(t) = 1.1798e^{jt/2} \begin{bmatrix} \vdots \\ 0.5594 - j0.5626 \\ 0.1664 + j0.0646 \\ 0.2647 - j0.4809 \\ \vdots \end{bmatrix}^T \begin{bmatrix} \vdots \\ e^{-j\omega_p t} \\ 1 \\ e^{j\omega_p t} \\ \vdots \end{bmatrix}.$$

In polar form,

$$\bar{y}(t) = 1.1798e^{jt/2} \begin{bmatrix} \vdots \\ 0.7934e^{-j45.17^\circ} \\ 0.1784e^{j21.21^\circ} \\ 0.5490e^{-j61.17^\circ} \\ \vdots \end{bmatrix}^T \begin{bmatrix} \vdots \\ e^{-j\omega_p t} \\ 1 \\ e^{j\omega_p t} \\ \vdots \end{bmatrix} \tag{60}$$

Note that the direction associated with the maximum principal gain consists predominantly of three harmonics, and that each harmonic has a corresponding associated phase change. Thus, even for single–input single–output LTP systems, phase is a directional property of the harmonic transfer function.

The maximum input direction is shown in Figure 2a, and the corresponding output direction is shown in Figure 2b, both as predicted from the singular value analysis. The input signal in Figure 2a was used as the input to a simulation of the LTP system with zero initial conditions. The resulting output signal is shown in Figure 2c. At steady state, the simulated system output and the output as predicted from the singular value analysis coincide, illustrating the LTP interpretation of the Bode frequency response.

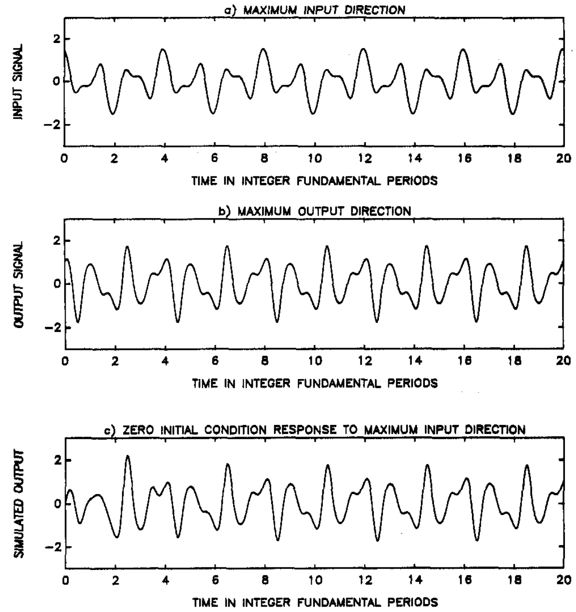

Figure 2: Maximum direction analysis for lossy Mathieu equation example. The directions of maximum amplification are the input and output directions associated with the largest principal gain (singular value) of the harmonic transfer function at a particular frequency.

Summary

A frequency response for linear time periodic systems was developed that exploited the one–to–one map induced by geometrically periodic signals. This map was described by an integral operator, based on the GP test input, and a generalized harmonic balance approach, based on an EMP input. The singular values or principal gains of the LTP operator were discussed and the LTP principal gain diagram described. Directional properties of the LTP operator were discussed, and notions of the domain and range spaces are presented.

The framework of linear operators described in this paper have lead to the development of a comprehensive open loop analysis theory for linear time periodic systems, including a characterization of poles, transmission zeroes and their directional properties [5]. A generalized Nyquist criterion has also been developed using this framework, as well as application of stability robustness and small gain notions to LTP systems. This linear operator framework has also led to the development of a comprehensive frequency domain interpretation for LTP systems.

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