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Continuous time frequency domain LPV state space identification via periodic time-varying input-output modeling

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***Abstract*— We aim to identify a parameter-varying state space model that is suited for control design. Current LPV controller synthesis tools usually require a state space formulation that is affine in the scheduling parameters. We therefore present a frequency domain state space identification method for periodic parameter variation, in continuous time. First, we identify a periodic time-varying input-output differential equation. Next, this representation is transformed into a time-varying state space form. We use a closed-form expression for the states, consisting of binomial coefficients and derivatives of the original differential equation coefficients. Finally, an affine LPV state space model is fitted. The difficulty is to select the proper basis functions, but in this routine, we have an educated guess. Special attention is given to the sparsity and structure in the frequency domain calculations.**

I. INTRODUCTION

We are studying the class of Linear Parameter-Varying (LPV) systems [1], where the system dynamics change according to some external signals $p(t)$, called the scheduling parameters. We assume that the trajectory of $p(t)$ is known exactly in real-time. In control literature, the basic LPV State Space (SS) equations are defined in continuous time by

$$\dot{x}(t) = A\big(p(t)\big)x(t) + B\big(p(t)\big)u(t) \tag{1}$$
$$y(t) = C\big(p(t)\big)x(t) + D\big(p(t)\big)u(t) \tag{2}$$

As in the Linear Time-Invariant (LTI) case, we have input signals $u(t) \in \mathbb{R}^{n_u \times 1}$, the states $x(t) \in \mathbb{R}^{n_x \times 1}$ and output signals $y(t) \in \mathbb{R}^{n_y \times 1}$. The scheduling parameter $p(t) \in \mathbb{R}^{n_p \times 1}$ is not simply another input to the model, but directly affects the dynamics via the matrix coefficients.

Most current LPV controller synthesis tools [1], [2] require a state space representation that is affine in the scheduling parameters $p(t)$. Equations (1)-(2) then read

$$\dot{x}(t) = \sum_{i=1}^{n_p} A_i\, p_i(t)\, x(t) + \sum_{i=1}^{n_p} B_i\, p_i(t)\, u(t) \tag{3}$$
$$y(t) = \sum_{i=1}^{n_p} C_i\, p_i(t)\, x(t) + \sum_{i=1}^{n_p} D_i\, p_i(t)\, u(t) \tag{4}$$

with A_i, B_i, C_i and D_i constant matrices, and where the time-variation is due to the scheduling signals $p_i(t)$. Usually, the first scheduling sequence is kept constant : $p_1(t) = 1$. If the mean values of the $p_i(t), i > 1$ are zero, then the corresponding LTI system (A_1,B_1,C_1,D_1) can be interpreted as a mean model, around which the variations take place.

In previous work [3] we proposed a non-linear optimization scheme for the Output Error (OE) maximum likelihood function, using an affine periodic parameter-varying state space model. It was shown on an example that this method is consistent and asymptotically efficient [4], but (as all non-linear optimization techniques) requires initial estimates of the matrix coefficients. Originally, we started by estimating the Best Linear Time-Invariant (BLTI) approximation [5], [6], and setting the time-varying coefficients to zero.

$$\begin{aligned} &A_1 = A_{\text{BLTI}} && A_{i>1} = 0 && B_i = B_{\text{BLTI}} \\ &C_1 = C_{\text{BLTI}} && C_{i>1} = 0 && D_i = D_{\text{BLTI}} \end{aligned}$$

In the case of slow parameter variations, the BLTI approximation already captures a lot of the dynamics, and the initialization appeared to be "good enough". To make our identification method more robust and applicable to fast time-varying systems, we investigate another initialization procedure, via the identification of a Linear Time-Varying Input-Output (LTV IO) model.

$$\sum_{i=1}^{n_a} a_i(t) y^{(i)}(t) = \sum_{i=1}^{n_b} b_i(t) u^{(i)}(t) \tag{5}$$

Recently, some frequency domain identification methods were developed for the estimation of LTV IO models [7], [8]. Starting values for these IO models are easily obtained via (weighted generalized total) least squares estimates. We will constrain ourselves to periodically time-varying systems [8]. For example, mechanical systems with rotating parts [9] fall within this category. Also, if we have full control over input and scheduling, we can impose synchronized periodicity between the input and the scheduling parameters. The drawback is that the input $u(t)$ and the time variation should be periodic and synchronized, meaning an integer number of periods of both $u(t)$ and the time (parameter) variation have to be observed. This is not be possible for every application. However, the important advantage is that the resulting representations are very structured and sparse in the frequency domain, which can be exploited to increase the speed of all calculations. Section III will explain briefly how the model (5) can be estimated when the time-variation is periodic, taking the sparse structure into account.

We know from [10], [11] that it is possible to transform

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a parameter-varying differential equation into state space form. However, the resulting coefficients have a dynamic dependence on $p(t)$, and they become rational functions of the original a_i and b_i defined in (5). We extend this transformation to the time-varying case. Section IV gives a closed form expression for the transformation of an arbitrary LTV IO model into a SS representation. In correspondence with the literature, we find rational forms of a_i and b_i and their derivatives.

In a final step, we discuss the relation between the parameter variation $p(t)$ and the obtained matrix coefficients. By studying the transformation on a fourth order simulation example, we can propose a possibly big, but finite set of basis functions. In this fitting step, we learn that experiment design is crucial for both the input and the scheduling parameter.

Summarized, we want to identify an affine LPV state space model. To this end, we are developing a procedure to generate improved initial estimates for a non-linear optimization routine [3]. We first identify a periodic time-varying input-output model, and transform this into a canonical state space form. The main contribution of this paper is the closed form expression for this transformation. Finally, we discuss the connection between the obtained time-varying state space and an affine LPV state space model via a simulation example in V.

II. Linear Periodic Time-Varying (LPTV) system output response to a synchronized input

Before we dive into the identification algorithms, we will illustrate the output behavior of an LPTV system when excited with a synchronized input and scheduling. This will help the reader understand why we propose these specific experimental conditions. In the following sections, we will show where the sparsity and structure come from, and how it can be used to speed up the computations. For the sake of simplicity, we will assume that the scheduling is a single sine wave : $p(t) = \sin(2\pi f_p t)$

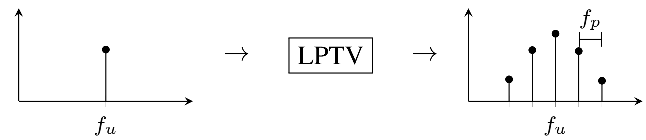
Note that all the computations [3], [7], [8] take place in the frequency domain. If the input and scheduling are periodic and synchronized, the state space equations (1)-(2) and the differential equation (5) can be transformed to the frequency domain without leakage errors [8]. In the frequency domain, we can select only the frequency band of interest, and also estimate the noise level non-parametrically.

By definition, the response of a linear time-invariant system to a single frequency tone will be an amplified, time-shifted output signal.

$$\sin(2\pi f_u t) \;\rightarrow\; \boxed{\text{LTI}} \;\rightarrow\; A(f_u)\sin(2\pi f_u t + \phi(f_u)) \tag{6}$$

Signals at different frequencies are thus decoupled in the LTI framework. This is no longer the case when dealing with LPV systems. Extra spectral lines, called harmonics, will appear at harmonic frequencies $f_u \pm l f_p$: integer multiples of the scheduling frequency f_p away from the excited frequencies f_u.

Although there is extra frequency content, and a correlation over the frequencies, the output spectrum is very



structured. We can predict exactly where the harmonics will appear. If there is energy in between the harmonics, it can only stem from a noise source. We can therefore estimate a non-parametric noise model, if multiple periods are measured.

III. Linear Periodic Time-Varying IO equation

This section will briefly discuss the identification technique for LPTV differential equations [8]. First we transform (5) into the frequency domain by means of the Fourier transform. The multiplication of two signals in the time domain corresponds to a convolution in the frequency domain.

$$\sum_{i=1}^{n_a} \mathcal{F}\{a_i(t)\} * s^i Y(s) = \sum_{i=1}^{n_b} \mathcal{F}\{b_i(t)\} * s^i U(s) \tag{7}$$

with $s = j\omega = j2\pi f$, and $U(s)$ & $Y(s)$ the Fourier transformed input and output. Take N equidistant samples in time, with a sampling frequency f_s. We call $f_0 = f_s/N$ the base frequency of the measurements, and $b_p = f_p/f_0$ the base bin of the time variation. This means b_p periods of the time variation were measured.

A. Sparse band structure in the frequency domain

The continuous Fourier transform of a band-limited signal at the frequencies $f_k = f_s k/N = k f_0$ can be reconstructed exactly from the discrete Fourier transform. Further more, the coefficients of the differential equation can be represented by their (truncated) Fourier series, which is equivalent to a sum of dirac impulses in the frequency domain.

$$\mathcal{F}\{a_i(t)\} = \sum_{l=-N/2+1}^{N/2-1} A_{[i,l]}\delta(j2\pi f_0 l) \tag{8}$$

$$\mathcal{F}\{b_i(t)\} = \sum_{l=-N/2+1}^{N/2-1} B_{[i,l]}\delta(j2\pi f_0 l) \tag{9}$$

Let's fix the truncation of all the $\mathcal{F}\{a_i(t)\}$ to N_a components in (8), and likewise $\mathcal{F}\{b_i(t)\}$ to N_b in (9), so only a few selected spectral lines are selected out of the possible $N/2-1$. This is a realistic approximation, because in parameter-varying systems, the bandwidth of the parameter variation is usually smaller than the bandwidth of the system. Then (7) can be written as

$$\sum_{i=0}^{n_a} \sum_{l=-N_a}^{N_a} A_{[i,l]}\Big(j2\pi f_0(k-lb_p)\Big)^i Y(k-lb_p) \tag{10}$$
$$= \sum_{i=0}^{n_b} \sum_{l=-N_b}^{N_b} B_{[i,l]}\Big(j2\pi f_0(k-lb_p)\Big)^i Y(k-lb_p)$$

Again, we find that only certain frequencies contribute to the output response $Y(k)$ at frequency f_k. These are the harmonics $f_u + l f_p = k f_0 + l b_p f_0 = (k + l b_p) f_0$.

The convolution can also be written as a matrix product, using a Toeplitz structure

$$\mathcal{F}\left\{a_i(t)\right\} * s^i Y(s) \tag{11}$$
$$= \text{Toeplitz}\left(A_{[i,l]}, A^H_{[i,l]}\right) \text{diag}\left(j2\pi f_0 l\right)^i Y(l)$$
$$\mathcal{F}\left\{b_i(t)\right\} * s^i Y(s) \tag{12}$$
$$= \text{Toeplitz}\left(B_{[i,l]}, B^H_{[i,l]}\right) \text{diag}\left(j2\pi f_0 l\right)^i Y(l)$$

After the truncation, the Toeplitz matrices in (11)-(12) have a band structure, and we can rewrite the sum of convolutions (7) in a concise way using two band-limited (but no longer Toeplitz) matrices

$$\boldsymbol{\mathcal{A}} Y = \boldsymbol{\mathcal{B}} U \tag{13}$$

Given and input $u(t)$ the system coefficients a_i and b_i, the system response Y can be calculated in the frequency domain by solving (13) without the need for a computationally expensive ODE solver. Because $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are sparse and band structured, (13) can be solved time-efficiently.

B. LPTV IO model identification

The goal of the paper, however, is to estimate the model parameters from input-output data. The identification scheme in [8] computes the complex coefficients in (10). Recall that the input U and the output Y are measured, and the $s = j2\pi f$ terms can be computed.

In the behavioral setting [12], input $u(t)$ and output $y(t)$ are viewed as a single trajectory $w(t)$ of the system. The input-output relation (10) is written as

$$\boldsymbol{R} W = \begin{bmatrix}\boldsymbol{\mathcal{A}} & -\boldsymbol{\mathcal{B}}\end{bmatrix} \begin{bmatrix} Y \\ U \end{bmatrix} = e \approx 0 \tag{14}$$

In [8], the difference between the left and the right hand of (10) is minimized. In the noiseless case, the residual e in (14) is exactly zero. In the noisy case however, there will always be some error. Then a cost function must be selected which, when minimized, yields the "best" model. As a first guess, we can simply calculate the regular Linear least Squares (LS) solution to (14). This minimizes the squared error.

$$\hat{\theta}_{\text{LS}} = \underset{\theta}{\text{argmin}}\, e(\theta)^H e(\theta)$$

If there is input noise, or if the equation error is non-white, this approach does not result in a consistent estimate.

In section II, we briefly explained how a non-parametric noise model can be obtained by inspecting the frequencies between the harmonics. This can be done by measuring multiple periods. For the full details, we refer to [8]. By weighting the residuals with the estimated output error covariance matrix C_e, the Maximum Likelihood (ML) cost function is obtained.

$$\hat{\theta}_{\text{ML}} = \underset{\theta}{\text{argmin}}\, e(\theta)^H C_e^{-1}(\theta) e(\theta)$$

This cost function is minimal only in the real system parameters, which is a necessary condition for consistency. Also, the ML estimator is efficient, meaning it reaches the Cramér-Rao lower bound on the variance of the residuals.

By only taking the main diagonal of C_e into account, we find the Weighted Non-linear Least Squares (WNLS) estimator [13], which approximates the maximum likelihood estimator. This is computationally less expensive, because we can avoid inverting the (band dominant) C_e matrix. The efficiency is lost, but we stay close to the Cramér-Rao lower bound, and the estimation is still consistent!

$$\hat{\theta}_{\text{WNLS}} = \underset{\theta}{\text{argmin}}\, e(\theta)^H \text{diag}\left\{C_e(\theta)\right\}^{-1} e(\theta)$$

The chosen (WNLS) cost function is minimized using an iterative Gauss-Newton optimization approach. The free variables θ are the complex $A_{[i,l]}$ and $B_{[i,l]}$ in (10). Note that these Fourier coefficients all appear linearly in the equation. We can thus calculate the Jacobian of the residuals (16) analytically. Again, because of the sparse band structure, the computations are time-efficient.

The user of the identification routine has to make sure the input is periodic and synchronized with the time variation. Before using the algorithm, we have to choose appropriate orders n_a and n_b for the differential equation (5). Finally, the amount of harmonics N_a and N_b have to be selected. Obviously, prior knowledge is very helpful in this stage. The model structure can also be determined by means of a validation data set [8], [13].

IV. From input-output to state space

The previous sections showed how to identify an LPTV differential equation with the use of periodic input, synchronized with the time variation. However, the goal is identification for (state space) controller synthesis. In this section, we investigate the transformation from a time-varying input-output equation to a canonical time-varying state space representation. We return to the behavioral setting, to ease the notation. The differential equation (5) becomes

$$\sum_{i=1}^{n_r} \begin{bmatrix} a_i(t) & -b_i(t) \end{bmatrix} \begin{bmatrix} y^{(i)}(t) \\ u^{(i)}(t) \end{bmatrix} = 0 \tag{15}$$

$$\sum_{i=1}^{n_r} r_i(t) w^{(i)}(t) = 0 \tag{16}$$

where $n_r = \max\{n_a, n_b\}$. We assume that $a_{n_a} \neq 0$, so there is no loss in degree. Additionally, in case of a strictly proper system $n_a > n_b$. It then holds that $b_{n_r} = b_{n_a} = 0$.

A. Closed form expression for the state equations

We then want to find state equations for $\dot{x}_i(t)$, that only depend on other states $x_j(t)$ and $w(t)$, and not on its derivatives $w^{(i)}(t)$. Based on [10], [11], we have found a closed form expression for the transformation from an arbitrary time-varying input-output equation to a time-varying state space model using binomial coefficients. We omit the dependency on the continuous time t to avoid clutter.

$$x_0 = 0 \tag{17}$$

$$x_k = \sum_{i=0}^{n-k} \sum_{j=0}^{n-k-i} (-1)^j C_{j+k-1}^{k-1} r_{j+k}^{(j)} w^{(i)} \tag{18}$$

$$\dot{x}_k = x_{k-1} + \Big[\sum_{i=k-1}^{n} (-1)^{i-k} C_i^{k-1} r_i^{(i-k+1)} \Big] w \tag{19}$$

for $0 \leq k \in \mathbb{N} \leq n_r$, and where we have

$$C_n^k = \frac{n!}{(n-k)!k!} = \binom{n}{k} \tag{20}$$

For the proof by induction for the formulas (18)-(19), we refer to [14]. For a general proper input-output model, we find the corresponding canonical state space form (21)-(22), which is defined as a function of the input signals u and the states x. To illustrate how these matrices where obtained, we take a closer look to the second order case:

$$x_1 = \left[r_1^{(0)} - r_2^{(1)} \right] w^{(0)} + \left[r_1^{(0)} \right] w^{(1)} \tag{23}$$

$$x_2 = \left[r_2^{(0)} \right] w^{(0)} \tag{24}$$

$$\dot{x}_1 = \Big[- r_0^{(0)} + r_1^{(1)} - r_2^{(2)} \Big] w \tag{25}$$

$$\dot{x}_2 = \Big[- r_1^{(0)} + 2 r_2^{(1)} \Big] w + x_1 \tag{26}$$

Using (23)-(26), we can extract the state space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -r_0 + \dot{r}_1 - \ddot{r}_2 \\ -r_1 + 2\dot{r}_2 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} a_2 & -b_2 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & \frac{-a_0 + \dot{a}_1 - \ddot{a}_2}{a_2} \\ 1 & \frac{-a_1 + 2\dot{a}_2}{a_2} \end{bmatrix} x + \begin{bmatrix} b_0 - \dot{b}_1 + \ddot{b}_2 - (a_0 - \dot{a}_1 + \ddot{a}_2) \frac{b_2}{a_2} \\ b_1 - 2\dot{b}_2 - (a_1 - 2\dot{a}_2) \frac{b_2}{a_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \frac{1}{a_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b_2}{a_2} \end{bmatrix} u$$

Again, we find that the highest order coefficient should not become zero $a_{n_a} \neq 0$. The state space formulas simplify if we assume that the system is strictly proper $b_{n_a} = b_2 = 0$.

$$\dot{x} = \begin{bmatrix} 0 & \frac{-a_0 + \dot{a}_1 - \ddot{a}_2}{a_2} \\ 1 & \frac{-a_1 + 2\dot{a}_2}{a_2} \end{bmatrix} x + \begin{bmatrix} b_0 - \dot{b}_1 \\ b_1 \end{bmatrix} u \tag{27}$$

$$y = \begin{bmatrix} 0 & \frac{1}{a_2} \end{bmatrix} x \tag{28}$$

As in the literature, we find that the resulting state space realization has a rational dynamic dependence on the original input-output coefficients. Obviously, in the LTI case, the a_i and b_i in (5) are constants. Therefore, their derivatives are zero, and we find a known observability canonical form.

This dynamic dependence commonly increases the complexity of the synthesized controller, which is clearly unwanted. In [11], a criterion is given that indicates if the additional complex dynamics can be neglected. In general, we can let an optimization routine choose which basis functions are truly needed, by promoting sparsity. Basically we now have a regressor selection problem.

B. Fitting the LPTV model coefficients with LPV functions

At this point, we can identify an LPTV input-output equation, and transform it into a canonical LPTV state space form. We now want to link the SS model coefficients to the parameter variation $p(t)$. To get an idea of which basis functions to use, we will apply the proposed transformation to a parameter-varying input-output equation, and see what the resulting state space needs. To this end, we apply the chain rule of derivation

$$\tfrac{d}{dt} a_i(p(t)) = \tfrac{d}{dp} a_i(p(t)) \tfrac{dp}{dt} \tag{29}$$

$$\tfrac{d^2}{dt^2} a_i(p(t)) = \tfrac{d^2}{dp^2} a_i(p(t)) \big(\tfrac{dp}{dt}\big)^2 + \tfrac{d}{dp} a_i(p(t)) \tfrac{dp^2}{dt^2} \tag{30}$$

We can see that the new required basis functions are a mix of derivatives of the parameter-varying coefficients $a_i(p(t))$ and $b_i(p(t))$ to $p(t)$, and the time derivatives of the scheduling parameter itself. If either of these components becomes small, the term is negligable. Suppose the coefficients of the differential equation have a polynomial dependency on $p(t)$. Then the higher derivatives $\frac{d^i}{dp^i}$ are zero, and we find a possibly big, but finite set of new basis functions. Generally, this hold for all smooth functions $f(p(t))$, over the finite interval $[p_-, p_+]$ because they can be approximated by a polynomial basis in p.

Note however, that we have a rational form if a_n is not constant. In this case, in theory we will need an infinite number of basis functions. In practice we will then have to approximate the parameter variation, with a truncated sum, but then the fitting step will not be exact.

V. Simulation Example: A Bandpass Filter

To illustrate the proposed conversion from LPTV input-output to LPTV state space, we simulate a time-varying fourth order Chebyshev bandpass filter.

$$\dot{x} = \begin{bmatrix} 0 & 0 & \dots & 0 & \frac{\sum_{i=0}^{n} (-1)^{i-1} C_i^0 a_i^{(i)}}{a_n} \\ 1 & 0 & \dots & 0 & \frac{\sum_{i=1}^{n} (-1)^{i-2} C_i^1 a_i^{(i-1)}}{a_n} \\ & \ddots & & & \vdots \\ 0 & 0 & \dots & 1 & \frac{\sum_{i=n-1}^{n} (-1)^{i-n} C_i^{n-1} a_i^{(i-n+1)}}{a_n} \end{bmatrix} x + \begin{bmatrix} \frac{b_n \sum_{i=0}^{n} (-1)^{i-1} C_i^0 a_i^{(i)} - a_n \sum_{i=0}^{n} (-1)^{i-1} C_i^0 b_i^{(i)}}{a_n} \\ \frac{b_n \sum_{i=1}^{n} (-1)^{i-2} C_i^1 a_i^{(i-1)} - a_n \sum_{i=1}^{n} (-1)^{i-2} C_i^1 b_i^{(i-1)}}{a_n} \\ \vdots \\ \frac{b_n \left(-a_{n-1} + C_n^{n-1} a_n^{(1)}\right) + a_n \left(b_{n-1} - C_n^{n-1} b_n^{(1)}\right)}{a_n} \end{bmatrix} u \tag{21}$$

$$y = \begin{bmatrix} 0 & 0 & \dots & \frac{1}{a_n} \end{bmatrix} x + \begin{bmatrix} \frac{b_n}{a_n} \end{bmatrix} u \tag{22}$$

$$
\begin{aligned}
y^{(4)} &+ 0.129y^{(3)} + (0.5083 + 2p + 2p^2)y^{(2)} \\
&+ (0.031 + 0.129p + 0.129p^2)y^{(1)} \\
&+ (0.0576 + 0.48p + 1.48p^2 + 2p^3 + p^4)y \\
&= 0.02u^{(2)}
\end{aligned} \tag{31}
$$

with $0 < p(t) < 1$. Figure 1 shows some frozen transfer functions in this range. We simulate $N = 2000$ equidistant time samples at a sample frequency $f_s = 0.6366$ Hz. The bandpass filter moves (periodically) from around 0.08 Hz to 0.25 Hz, which is a pretty big variation with respect to f_s.

Using the state definitions (18)-(19), we obtain a fourth order LPTV state space description (32)-(33). Figure 2 shows the resulting steady state output spectrum. A random phase multisine [13], shown in Figure 2a, was applied at the input. Figure 2b illustrates the effects of the small, slow parameter variation. We can still recognize the shape of the Chebyshev filter, but extra spectral lines appear around the excited frequencies, forming "skirts". Finally, we take the time variation equal to $p(t) = 0.5 + 0.5\sin(2\pi 2 f_0 t)$, with the base frequency $f_0 = f_s/N$. The resulting output spectrum is shown in Figure 2c. The skirts start overlapping, and the dependence on the time variation can no longer be identified with the naked eye. The rate of change is slow, but the amplitude is rather large, as illustrated in Figure 1. Clearly, not only the speed, but also the amplitude of the time variation is important.

With the LPTV IO identification method from [8], we find the exact Fourier coefficients as described by the differential equation (10). In the time domain, we find the given Chebyshev bandpass filter equation (31), up to a constant.

Let us transform this input-output equation into the proposed canonical state space form, and simulate the steady state response in the frequency domain. Figure 2c depicts the difference • between the predicted IO and SS output. The errors are at the level of the numerical precision of Matlab. We can thus conclude that the transformation is exact.

In a final step, we now want to identify the relation between the parameter variation $p(t)$ and the coefficients of the linear periodic time-varying state space equations (32)-(33). How do we choose the different basis functions, and how can we distinguish them from one another? As explained in Section IV-B, in the Chebyshev filter example, we can compute the exact parameter varying state space representation (34)-(35) as well, by applying the chain rule. This is only possible if we have a parameter-varying input-output description available.

The state space description (34) gives valuable insight

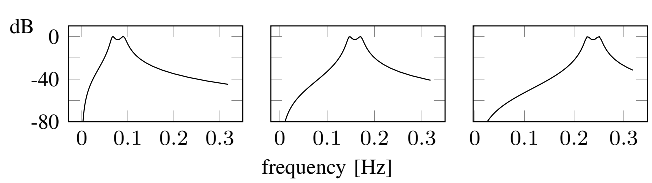
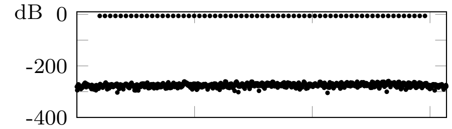
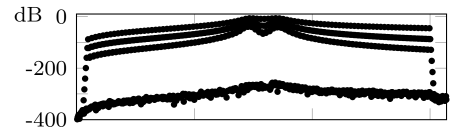


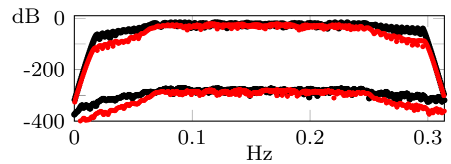
Fig. 1. Frozen transfer functions (in dB) for the fourth order Chebyshev filter. Note that the time variation is slow, but big with respect to f_s.



(a) input spectrum



(b) output spectrum for a small time variation



(c) output spectrum for $p(t) = 0.5 + 0.5\sin(2\pi 2 f_0 t)$

Fig. 2. Spectra • (in dB) for the time-varying Chebyshev filter, over the entire measurement record. (a) The input is a random phase multisine, which has a flat amplitude spectrum. (b) If we keep the time variation small, the harmonics can be distinguished from one another. If the time variation is fast or big, the skirts start to overlap (c). The red dots • represent the difference between the original IO model and the corresponding SS representation. The errors are at the same level as the numerical accuracy of Matlab.

in the LPV fitting problem. In the filter example, the time variation is periodic with a frequency $f_p = 2f_0$. Both $p(t) = \sin(2\pi f_p t)$ and $p^{(2)}(t) = -(2\pi f_p)^2 \sin(2\pi f_p t)$ create harmonics at the same frequencies. If we only apply a single sine as the scheduling parameter, we cannot discern these two components. The scheduling has to be rich enough to identify the parameter dependency. We thus find that experiment design becomes important for the input and scheduling (periodic and synchronized), but also for the frequency content of the scheduling. As with the input, the frequency band of interest should be excited. To this end, we simulated the time-varying Chebyshev filter again, but changed the scheduling signal to a multitone:

$$p(t) = 0.5 + 0.3\sin(2\pi 2 f_0 t) + 0.2\sin(2\pi 3 f_0 t)$$

If we then use the basis functions

$$[1,\ p,\ p^2,\ p^3,\ p^4,\ p^{(1)},\ p\,p^{(1)},\ (p^{(1)})^2,\ p^{(2)},\ p\,p^{(2)}]$$

by a simple least squares regression on the estimated time-varying state space model, we find the exact coefficients shown in (34). The matrix coefficient functions can also be approximated with a purely polynomial basis, but then the performance of the identified state space model will obviously decrease. It is well-known that fast-varying dynamics commonly dominate the system performance. We can see that the impact of the derivatives will increase as the parameter variation becomes faster.

Once we have an affine LPV state space model, we can compute the output spectrum time-efficiently, by exploiting the sparsity and band dominant structure [3], [4]. The model can also be transformed to the time domain, but there we need a computationally expensive ODE solver to simulate

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & \frac{-a_0+a_1^{(1)}p^{(1)}-\left[a_2^{(2)}(p^{(1)})^2+a_2^{(1)}p^{(2)}\right]}{a_4} \\ 1 & 0 & 0 & \frac{-a_1+2a_2^{(1)}p^{(1)}}{a_4} \\ 0 & 1 & 0 & \frac{-a_2}{a_4} \\ 0 & 0 & 1 & \frac{-a_3}{a_4} \end{bmatrix} x + \begin{bmatrix} b_2^{(2)}(p^{(1)})^2+b_2^{(1)}p^{(2)} \\ -2b_2^{(1)}p^{(1)} \\ b_2 \\ 0 \end{bmatrix} u \tag{32}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{a_4} \end{bmatrix} x \tag{33}$$

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & -\left(0.0576+0.48p+1.48p^2+2p^3+p^4\right)+\left(0.129+0.285p\right)p^{(1)}-4(p^{(1)})^2-\left(2+4p\right)p^{(2)}\Big) \\ 1 & 0 & 0 & -\left(0.031+0.129p+0.129p^2\right)+2\left(2+4p\right)p^{(1)} \\ 0 & 1 & 0 & -\left(0.5083+2p+2p^2\right) \\ 0 & 0 & 1 & -0.129 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} u \tag{34}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x \tag{35}$$

the dynamic behavior. Naturally, it is advised to use the model only in the frequency range used in the identification experiment, for both the input and scheduling.

VI. Conclusion

The main goal of this paper was to eliminate the possible problems of initialization of the recently developed frequency domain LPV SS identification technique [3]. The linear periodic time-varying input-output identification technique in [8] performs very well in this aspect. There, the initialization is easy, by means of a (generalized total) least squares solution, and it also works for fast time variations. The big remaining problem is the selection of the basis functions for the affine parameter-varying state space model, but this has always been troublesome. At least now we have an educated guess: monomials $p^i p^{(j)}$ consisting of powers in $p(t)$, and the time derivatives of the parameter variation $p(t)$. By studying the transformation of a LPV IO equation into a LPV state space form, we have also shown that experiment design is important not only for the input, but for the scheduling as well. To fit the identified IO model perfectly, the scheduling has to be rich in frequency content as well.

The advantages of the proposed algorithm remain. In the frequency domain, we can estimate a non-parametric noise model. Also, the equations will remain structured and sparse, so we can still evaluate the model time-efficiently.

Summarized, the main contribution of this paper is a step towards an improved initialization procedure for the estimation of an affine LPV state space model. The complete SS identification method comprises four steps:

1) A linear periodic time-varying input-output model is estimated using a known algorithm [8].
2) The LPTV IO model is transformed into a canonical state space form.
3) This periodic LTV state space representation is approximated with an affine parameter-varying SS model.
4) The LPV SS model is optimized using the Gauss-Newton method described in [3].

As an additional result, we have established a closed-form expression for the transformation of an arbitrary time-varying input-output differential equation into a state space model. We have shown on a physically relevant example that the presented approach is able to identify a parameter-varying state space model from periodic input-output data.

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