Identification of Nonlinear Systems in Frequency Domain

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Abstract

A frequency domain technique for identifying a class of nonlinear systems is presented. The class of nonlinear systems we consider in this paper consists of a power series nonlinearity sandwiched between two linear systems and the problem we address is one of identifying the transfer functions of the linear systems and the coefficients of the power series nonlinearity from the terminal behavior of the system. The identification procedure is based on a frequency domain model of the nonlinear system. The system is modeled by a set of nonlinear transfer functions in the frequency domain and the relationships between the transfer functions of the linear system, the power series coefficients of the nonlinearity, and the nonlinear transfer functions of the system are developed. The identification procedure is based on a simple algorithm for factoring the nonlinear transfer functions of the system. The experimental data required for applying our identification procedure consists of the amplitude response of the system to multitone sinusoidal input. A minimum phase transfer function in a single variable is used to approximate the experimental data and the repeated application of the factoring algorithm leads to the identification of the system. Examples illustrating the use of the proposed procedure are also presented.

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I. Introduction

Nonlinear system models consisting of a power series nonlinearity sandwiched in between linear systems are often used to model a variety of physical systems such as mixers in communication systems [1, 2], tracking loops in radar systems [3], and imaging process involving film [4]. With reference to Fig. 1, an identification of this nonlinear system consists of experimentally determining the transfer functions of the linear systems and the coefficients of the power series nonlinearity. This identification can be done easily if one has access to the system at points *a, b, c, d;* the transfer functions $H(f)$ and $G(f)$ can be determined experimentally from frequency response data between terminals *a-b* and *c-d* respectively and the power series coefficients can be determined from tests between terminals *b-c*. However, in many physical systems, access is limited to the input-output terminals *a-d* and the identification of the system is to be done based on data obtained between terminals *a-d*. We present here a frequency domain method for identifying a nonlinear system such as in Fig. 1, using data from input-output terminals of the system.

As a first step in the development of the identification procedure, we characterize the system in terms of a set of nonlinear transfer functions. We then develop relationships between $H(f)$, $G(f)$, C_k's and the nonlinear transfer functions. These relationships in frequency domain are utilized to derive a factoring algorithm which allows us to extract essential parameters of $H(f)$ and $G(f)$ along with C_k's from the nonlinear transfer functions. In order to apply the identification algorithm, the nonlinear transfer functions of the system are first obtained by using a multitone signal at terminal *a* and measuring the amplitude of the response at terminal *d*. The multivariable nonlinear transfer function data is converted to a linear (single variable) transfer function through a simple transformation. A least square curve fitting technique is applied to fit a minimum phase linear transfer function to the transformed data. The factoring algorithm is then applied to extract the poles and zeroes of $H(f)$, $G(f)$ and the coefficients C_k's. We present a number of examples illustrating the use of our algorithm.

II. Frequency Domain Representation of Nonlinear Systems

A. Volterra Series Representation

Considerable amount of effort has been made in recent years to develop frequency domain representation of nonlinear systems based on Wiener-Volterra series representation of input-output behavior. Some of the recent works using Volterra series approach include distortion analysis of nonlinear circuits [5, 6], analysis of communication systems [1, 2, 7], and design of a class of nonlinear systems [8]. The identification procedure we present in this paper is based on the Volterra series representation of nonlinear systems and hence we present a brief review of this approach here.

Wiener suggested that the Volterra series can be used to

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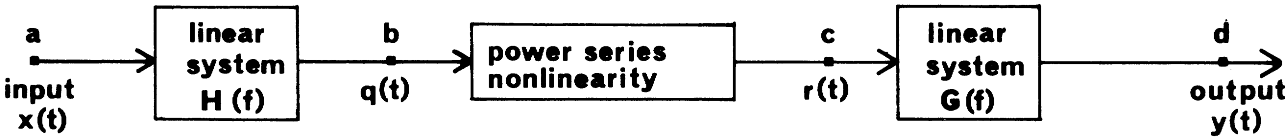

Fig. 1. Model of a nonlinear system $r(t) = \sum_{k=1}^{n} C_k [q(t)]^k$

represent the output $y(t)$ of a nonlinear system in terms of the input $x(t)$ by,

$$y(t) = \sum_{k=1}^{\infty} y_k(t), \tag{1}$$

where,

$$y_k(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h^{(k)}(\tau_1, \tau_2, \cdots, \tau_k) \cdot x(t-\tau_1)x(t-\tau_2)\cdots x(t-\tau_k)\, d\tau_1 \cdots d\tau_k. \tag{2}$$

In (2), $h^{(k)}(\tau_1, \cdots, \tau_k)$ is called the kth order kernel or the nonlinear impulse response of order k.

Frequency domain representation of the input/output relationship in (1) and (2) can be easily obtained as follows. Let $X(f)$ and $Y(f)$ denote the Fourier transforms of the input and output, respectively. The Fourier transform $H^{(k)}(f_1, f_2, \cdots, f_k)$ of the kth order kernel (nonlinear impulse response of order k) can be written as

$$H^{(k)}(f_1, f_2, \cdots, f_k) = \int_{-\infty}^{\infty} \cdots \int h^{(k)}(\tau_1, \tau_2, \cdots, \tau_k) \prod_{i=1}^{k} \exp \cdot (-j2\pi f_i \tau_i)\, d\tau_i. \tag{3}$$

$H^{(k)}(f_1, f_2, \cdots, f_k)$ is called the nonlinear transfer function of order k. Writing the inverse Fourier transform, we have

$$h^{(k)}(\tau_1, \tau_2, \cdots, \tau_k) = \int_{-\infty}^{\infty} \cdots \int H^{(k)}(f_1, f_2, \cdots, f_k) \prod_{i=1}^{k} \cdot \exp(j2\pi f_i \tau_i)\, df_i. \tag{4}$$

Substituting from (3) into (2), we have

$$y_k(t) = \int_{-\infty}^{\infty} \cdots \int H^{(k)}(f_1, f_2, \cdots, f_k) \prod_{i=1}^{k} \cdot X(f_i) \exp(j2\pi f_i t)\, df_i, \tag{5}$$

and taking the Fourier transform on both sides of (5)

$$Y_k(f) = \int_{-\infty}^{\infty} \cdots \int H^{(k)}(f_1, \cdots, f_k) \cdot \delta(f - f_1 - f_2 - \cdots - f_k) \prod_{i=1}^{k} X(f_i)\, df_i \tag{6}$$

where $\delta(\cdot)$ is the delta function. From (1) we obtain

$$Y(f) = \sum_{k=1}^{\infty} Y_k(f). \tag{7}$$

B. Experimental Method for Determining $H^{(k)}(f_1, f_2, \cdots, f_k)$

The nonlinear transfer functions $H^{(k)}(f_1, f_2, \cdots, f_k)$ can be determined experimentally by applying a multitone signal at the input to the system and measuring the response at certain frequencies. To illustrate this, let us look at how we can determine a third order transfer function $H^{(3)}(f_1, f_2, f_3)$ of a nonlinear system. Suppose the input $x(t)$ consists of three discrete tones at f_1, f_2 and f_3.

$$x(t) = \sum_{k=1}^{3} A_k \cos(2\pi f_k + \theta_k). \tag{8}$$

Taking the Fourier transform of $x(t)$ and substituting in (6) and (7), we can obtain an expression for $Y_3(f)$ which would involve a total of 22 terms, each at a distinct frequency if f_1, f_2, f_3 are chosen to be incommensurable. Now if we look at the output $Y(f)$ at $f = f_1 + f_2 + f_3$, (This term can be generated by the third-order effect or by higher order odd terms. The effects of higher order odd terms in the power series nonlinearity are ignored for the sake of simplicity.)

$$|Y(f_1 + f_2 + f_3)| = (3/2)|A_1 A_2 A_3||H^{(3)}(f_1, f_2, f_3)|. \tag{9}$$

In (9) we look at the absolute values, since the output $Y(f)$ will have a large number of sinusoidal terms and the measurement of phase will be difficult. Equations (8) and (9) suggest a procedure for obtaining $|H^{(3)}(f_1, f_2, f_3)|$, i.e., apply a three tone input as in (8) to the system, measure the amplitude response at $f_1 + f_2 + f_3$ and compute $|H^{(3)}(f_1, f_2, f_3)|$ using (9). Higher order transfer functions can be determined in a similar manner.

C. Nonlinear Transfer Functions of the System Shown in Figure 1

Returning to the system shown in Fig. 1,

$$q(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau, \tag{10}$$

and

$$y(t) = \int_{-\infty}^{\infty} g(\tau)r(t-\tau)\, d\tau, \tag{11}$$

where $h(\cdot)$ and $g(\cdot)$ are impulse response of the linear systems preceding and following the nonlinearity. Also, we have

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$$r(t) = \sum_{k=1}^{n} C_k [q(t)]^k$$

$$= \sum_{k=1}^{n} C_k \int\int_{-\infty}^{\infty} \cdots \int (\prod_{i=1}^{k} h(\tau_i) x(t-\tau_i)\, d\tau_i). \tag{12}$$

Substituting (12) in (11) and writing the output as in (1) and (2),

$$y(t) = \sum_{k=1}^{n} y_k(t),$$

$$y_k(t) = \int_{-\infty}^{\infty} \cdots \int h^{(k)}(\tau_1, \tau_2, \cdots, \tau_k) (\prod_{m=1}^{k} x(t-\tau_m)\, d\tau_m),$$

we obtain the kth order nonlinear impulse response of the nonlinear system shown in Fig. 1 as,

$$h^{(k)}(\tau_1, \tau_2, \cdots, \tau_k) = C_k \int_{-\infty}^{\infty} g(\rho) h(\tau_1 - \rho) \cdots h(\tau_k - \rho)\, d\rho,$$

or the kth order nonlinear transfer function of the system as

$$H^{(k)}(f_1, f_2, \cdots, f_k) = C_k H(f_1) H(f_2) \cdots H(f_k) G(f_1 + f_2 \cdots + f_k) \tag{13}$$

Equation (13) explicitly displays the relationship between the nonlinear transfer function $H^{(k)}$ and the linear transfer functions H, G and the power series coefficients $C_1, C_2, \cdots, C_k$. In the following section, we develop additional relationships which along with (13) form the basis of our identification procedure.

III. Identification Procedure

In the previous section we derived the relationship between some structural properties of nonlinear systems and the form of their nonlinear transfer functions. We now present a procedure for identifying the components of the system $G(f)$, $H(f)$ and the C_k's.

The procedure we present is based on the relationships derived in the previous sections and a simple algorithm for factoring multivariable polynomials. We will present a general factoring procedure first and then point out how this factoring leads to the identification of the poles and zeroes of $G(f)$, $H(f)$ and the C_k's. Examples will be presented to illustrate the identification procedure.

Let $H^{(k)}(f_1, f_2, \cdots, f_k)$ be the kth order nonlinear transfer function of a system corresponding to the one shown in Fig. 1. We will assume that $H^{(k)}$ is given to us as a ratio of multivariable polynomials,

$$H^{(k)}(f_1, f_2, \cdots, f_k) = \sum_{i_1=0}^{p_1} \sum_{i_2=0}^{p_2} \cdots \sum_{i_k=0}^{p_k} a_{i_1 i_2 \cdots i_k} (\prod_{j=1}^{k} f_j^{i_j}) / \sum_{i_1=0}^{q_1} \sum_{i_2=0}^{q_2} \cdots \sum_{i_k=0}^{q_k} b_{i_1 i_2 \cdots i_k} \prod_{j=1}^{k} f_j^{i_j}) \tag{14}$$

$$= N^{(k)}(f_1, f_2, \cdots, f_k) / D^{(k)}(f_1, f_2, \cdots, f_k). \tag{15}$$

Our problem now is to find $H(f)$, $G(f)$ and the C_k's from $H^{(k)}$.

Let n be the degree of the numerator polynomial $N^{(k)}$ and let m be the degree of the denominator polynomial $D^{(k)}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_{n_1}$ be the zeroes of $N^{(k)}(f, f, f, \cdots, f)$, a polynomial in a single variable f, and let $\beta_1, \beta_2, \cdots, \beta_{m_1}$ be the zeroes of $D^{(k)}(f, f, \cdots, f)$. The following lemma establishes a simple procedure for factoring a class of multivariable polynomials in terms of the zeroes of a single variable polynomial. This factoring algorithm directly leads to the identification procedure.

Lemma: $H^{(k)}(f_1, f_2, \cdots, f_k)$ is the kth order nonlinear transfer function of the system shown in Fig. 1, if and only if the following relationships hold.

(i) If α_i ($i = 1, 2, \cdots, n_1$) is a zero of multiplicity $r < k$, then $(f_1 + f_2 + \cdots + f_k - k\alpha_i)^r$ can be factored out of $N^{(k)}(f_1, f_2, \cdots, f_k)$.

(ii) If α_i ($i = 1, 2, \cdots, n_1$) is a zero of multiplicity $r \geq k$, then $[(f_1 - \alpha_i)(f_2 - \alpha_i) \cdots (f_k - \alpha_i)]^p$ $[f_1 + f_2 + \cdots + f_i - k\alpha_i]^q$ can be factored from $N^{(k)}(f_1, f_2, \cdots, f_k)$ for some value of p, q satisfying $pk + q = r; p, q \geq 0$.

(iii) A similar relationship exists for $D^{(k)}(f_1, f_2, \cdots, f_k)$ in terms $\beta_1, \beta_2, \cdots, \beta_{m_1}$.

(iv) Let $\alpha_i^{(1)}$, $i = 1, 2, \cdots, s_1$, $\alpha_i^{(1)} \in \{\alpha_i; i = 1, 2, \cdots, n_1\}$ be a zero of $N^{(k)}(f, f, \cdots, f)$ associated with factors of the form $(f_1 - \alpha_i^{(1)})$ of multiplicity $m_i^{(1)}$ of $N^{(k)}(f_1, f_2, \cdots, f_k)$,
$\alpha_i^{(2)}$, $i = 1, 2, \cdots, s_2$, $\alpha_i^{(2)} \in \{\alpha_i; i = 1, 2, \cdots, n_1\}$ be a zero of $N^{(k)}(f, f, \cdots, f)$ associated with factors of the form $(f_1 + f_2 + \cdots + f_k - k\alpha_i^{(2)})$ of multiplicity $m_i^{(2)}$ of $N^{(k)}(f_1, f_2, \cdots, f_k)$,
$\beta_i^{(1)}$, $i = 1, 2, \cdots, s_3$, $\beta_i^{(1)} \in \{\beta_i; i = 1, 2, \cdots, n_2\}$ be a zero of $D^{(k)}(f, f, \cdots, f)$ associated with factors of the form $(f_1 - \beta_i^{(1)})$ of multiplicity $m_i^{(3)}$ of $D^{(k)}(f_1, f_2, \cdots, f_k)$, and
$\beta_i^{(2)}$, $i = 1, 2, \cdots, s_4$, $\beta_i^{(2)} \in \{\beta_i; i = 1, 2, \cdots, n_2\}$ be a zero of $D^{(k)}(f, f, \cdots, f)$ associated with factors of the form $(f_1 + f_2 + f_3 + \cdots + f_k - k\beta_i^{(2)})$ of multiplicity $m_i^{(4)}$ of $D^{(k)}(f_1, f_2, \cdots, f_k)$, then

$$H^{(k)}(f_1, f_2, \cdots, f_k) = C_k [\prod_{j=1}^{k} H(f_j)]\, G(\sum_{j=1}^{k} f_j)$$

where

$$H(f) = \prod_{i=1}^{s_1} (f - \alpha_i^{(1)})^{m_i^{(1)}} / \prod_{i=1}^{s_3} (f - \beta_i^{(1)})^{m_i^{(3)}}$$

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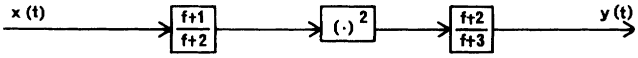

Fig. 2. Nonlinear system corresponding to example 1.

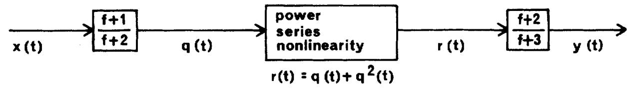

Fig. 3. Nonlinear system given in example 2.

$$G(f) = \prod_{i=1}^{s_2} (f - \alpha_i^{(2)})^{m_i^{(2)}} \Big/ \prod_{i=1}^{s_4} (f - \beta_i^{(2)})^{m_i^{(4)}}. \tag{16}$$

The proof of this lemma follows directly from the form of $H^{(k)}(f_1, f_2, \cdots, f_k)$ for the system shown in Fig. 1. For this system,

$$H^{(k)}(f_1, f_2, \cdots, f_k) = C_k(\prod_{i=1}^{k} H(f_i))\, G(\sum_{i=1}^{k} f_i).$$

From the previous equation, we note that if the numerator of $H(f)$ has a zero at σ_i, with multiplicity q, then the numerator of $H^{(k)}(f, f, \cdots, f)$ will have a zero of multiplicity kv. Also if the numerator of $G(f)$ has a zero at λ_i, with the multiplicity r, then the numerator of $H^{(k)}(f, f, \cdots, f)$ will have a zero of multiplicity r at (λ_i/k). Hence (i) and (ii) follow. A similar argument can be used to prove (iii). Part (iv) follows from the fact that $H^{(k)}(f, f, \cdots, f)$ can not have any poles and zeroes which do not appear in step (i) and (ii).

Example 1: Identify $H(f)$, $G(f)$ and C_2 from the following 2nd order nonlinear transfer function.

$$\begin{aligned} H^{(2)}(f_1, f_2) = {} & (2 + 3f_1 + 3f_2 + f_1^2 + f_2^2 + 4f_1f_2 + f_1^2f_2 + f_2^2f_1)/ \\ & (12 + 10f_1 + 10f_2 + 2f_1^2 + 2f_2^2 \\ & + 7f_1f_2 + f_1^2f_2 + f_2^2f_1). \end{aligned}$$

Substituting $f_1 = f_2 = f$, we get

$$\begin{aligned} H^{(2)}(f, f) &= N^{(2)}(f, f)/D^{(2)}(f, f) \\ &= (2 + 6f + 6f^2 + 2f^3)/(12 + 20f + 11f^2 + 2f^3). \end{aligned}$$

It is easy to see that $N^{(2)}(f, f)$ has a zero at -1 with multiplicity 3. Hence $[(f_1 + 1)(f_2 + 1)]^p\ [f_1 + f_2 + 2]^q$ must be a factor (the only factoring) of $N^{(2)}(f_1, f_2)$ for some value of p, q satisfying $(2p + q) = 3$. It is easy to verify that

$$N^{(2)}(f_1, f_2) = (f_1 + 1)(f_2 + 1)(f_1 + f_2 + 2).$$

Similarly we find that

$$D^{(2)}(f_1, f_2) = (f_1 + 2)(f_2 + 2)(f_1 + f_2 + 3).$$

Applying the last step of the lemma, we find that

$$H(f) = (f + 1)/(f + 2), \text{ and } G(f) = (f + 2)/(f + 3)$$

and the system is shown in Fig. 2.

Example 2: Identify the nonlinear system which has the following first (linear) and second order transfer functions.

$$H^{(1)}(f_1) = (f_1 + 1)/(f_1 + f_3)$$

$$\begin{aligned} H^{(2)}(f_1, f_2) = {} & (2 + 3f_1 + 3f_2 + f_1^2 + f_2^2 + 4f_1f_2 + f_1^2f_2 + f_2^2f_1)/ \\ & (12 + 10f_1 + 10f_2 + 2f_1^2 + 2f_2^2 \\ & + 7f_1f_2 + f_1^2f_2 + f_2^2f_1). \end{aligned}$$

From the previous example, from $H^{(2)}$ we get $H = (f + 1)/(f + 2)$, $G = (f + 2)/(f + 3)$, $C_2 = 1$, and from (13), $H^{(1)}(f_1) = H(f_1)G(f_1)C_1$ or $C_1 = 1$; the system is shown in Fig. 3.

IV. System Identification from Experimental Data

In the previous section we presented a method for identifying the elements of a nonlinear system if the nonlinear transfer functions of the system are given to us in closed form. However, in practical situations these functions have to be determined experimentally at some discrete values of frequencies and the identification has to be carried out based on the limited amount of data. To illustrate some of the difficulties in the experimental procedure and to point out how to circumvent these difficulties let us consider the problem of identifying a nonlinear system with a power series nonlinearity with three terms.

A. Data

The data needed for identifying a third order nonlinear system can be obtained by injecting three tones at frequencies f_1, f_2, and f_3 (these frequencies in general have to be incommensurable) at point a in Fig. 1. We measure the amplitude of the response of the system at various frequencies, i.e., we measure $|Y(f)|$ for values of f which are algebraic combinations of f_1, f_2, and f_3. From Section II, we know that

$$\begin{aligned} |Y(f_1 + f_2 + f_3)| &= (3/2)\ |A_1A_2A_3\, H^{(3)}(f_1, f_2, f_3)| \\ |Y(f_1 + f_2)| &= |H^{(2)}(f_1, f_2)A_1A_2|. \end{aligned} \tag{17}$$

These equations enable us to obtain $|H^{(3)}(f_1, f_2, f_3)|$ and $|H^{(2)}(f_1, f_2)|$, $|H^{(2)}(f_1, f_3)|$ and $|H^{(2)}(f_2, f_3)|$. However if we look at $|Y(f_1)|$ for example,

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$$|Y(f_1)| = |C_1 A_1 H^{(1)}(f_1) + (3/4)C_3 A_1^3 H^{(3)}(f_1, f_1, -f_1)$$
$$+ (3/2)C_3 A_1 A_2^2 H^{(3)}(f_1, f_2, -f_2)$$
$$+ (3/2)C_3 A_1 A_3^2 H^{(3)}(f_1, f_3, -f_3)| \qquad (18)$$

i.e., the third order nonlinearity gives rise to a term at frequency f_1, which mixes with the response due to the linear portion and we cannot separate these terms. However, if we make the following sets of assumptions,

$$|C_1| \gg |C_3| \gg |C_5| > \cdots,$$

and

$$|C_2| \gg |C_4| \gg |C_6| > \cdots, \qquad (19)$$

then we can ignore the effects of higher order terms on lower order responses. For example we can approximate

$$|Y(f_1)| \approx |C_1 A_1 H^{(1)}(f_1)| \qquad (20)$$

Using (17) and (20) we can obtain $H^{(1)}$ at three distinct frequencies, $H^{(2)}$ at nine distinct frequencies and $H^{(3)}$ at 19 distinct frequencies. By changing the values of f_1, f_2, and f_3 and repeating the experiment we obtain the values of $H^{(1)}, H^{(2)}$, and $H^{(3)}$ over a wide range of frequencies. In repeating the experiment f_1, f_2, and f_3 should be nearly but not exactly equal.

B. Curve Fitting, Factoring and Identification

Before applying the factoring algorithm, the data $|H^{(1)}(f_1)|$, $|H^{(2)}(f_1, f_2)|$, $|H^{(3)}(f_1, f_2, f_3)|$ have to be expressed in a closed form as in (14). This can be done using a numerical curve fitting technique. However, to apply the factoring algorithm, we only need to express the data $|H^{(1)}(f)|$, $|H^{(2)}(f, f)|$ and $|H^{(3)}(f, f, f)|$ as a ratio of polynomials in a single variable f. $|H^{(k)}(f, f, f, \cdots, f)|$ is obtained from $|H^{(k)}(f_1, f_2, \cdots, f_k)|$ by letting $f_1 \to f_2 \to \cdots \to f_k \to f$.

The curve fitting technique which we propose can be formulated and developed as follows. We are given the amplitude $|F(j\omega)|$ of a minimum phase transfer function $F(f), f = j\omega$ at L values of $\omega = \omega_k$, $k = 1, 2, \cdots, L$. From this data we wish to determine $F(f)$ in the form

$$F(f) = (c_0 + c_1 f + c_2 f^2 + \cdots + c_M f^M)/$$
$$(1 + d_1 f + d_2 f^2 + \cdots + d_N f^N) \qquad (21)$$

such that the error between the function $|F(j\omega)|$ and the L measured values of $|F(j\omega)|$ is minimized. Applying the least-squared- error criterion to the function in (21) directly in order to find c_i's and d_i's will result in highly nonlinear equations, which are extremely difficult to solve even with some iterative procedure. (A variety of methods has been proposed in the literature for iteratively solving these nonlinear equations. Most of these methods require both the amplitude and phase response data; and convergence is a problem when the data spans several decades of frequency.) Therefore we start from the magnitude-squared function

$$|F(j\omega)|^2 = G(\omega^2) = (a_0 + a_1\omega^2 + a_2\omega^4 + \cdots + a_M\omega^{2M})$$
$$/(1 + b_1\omega^2 + b_2\omega^4 + \cdots + b_N\omega^{2N})$$
$$= P(\omega^2)/Q(\omega^2) \qquad (22)$$

and try to find a_i's and b_i's such that

$$E = \sum_{i=1}^{L} [w_k(G_k - D_k^2)]^2 \qquad (23)$$

is minimized. In (23), G_k denotes $G(\omega_k^2)$, D_k is the measured value of the amplitude response at ω_k, and w_k is an arbitrary weighting function. From the magnitude-squared function in (22) a minimum-phase transfer can then be obtained [11].

We propose the following iterative procedure for determining the a's and b's. Let $E_t^{M,N}$ denote the sum of the weighted squared errors after t iterations, and let P_{k_t}, Q_{k_t}, a_{j_t}, b_{j_t} refer to the values of P_k, Q_k, a_j, and b_j at the end of t iterations. With the weight set equal to $Q_{k_t}/Q_{k_{t-1}}$, $E_t^{M,N}$ is defined as

$$E_t^{M,N} = \sum_{k=1}^{L} [(Q_{k_t}/Q_{k_{t-1}})(G_{k_t} - D_k^2)]^2$$
$$= \sum_{k=1}^{L} (P_{k_t} - D_k^2 Q_{k_t})^2 / Q_{k_{t-1}}^2 \qquad (24)$$

where M, N are used to reflect the fact that the error is a function of M and N, the degrees of numerator and denominator polynomials in (21) and (22). Here the squared error is weighted more evenly at all frequencies to overcome the problem of poor fit at low frequencies with Levy's method [9]. Setting the partials of $E_t^{M,N}$ to 0 with respect to a_{j_t} and b_{j_t}, we obtain,

$$\sum_{j=0}^{M} [\sum_{k=1}^{L} (\omega_k^{2(i+j)})/(Q_{k_{t-1}}^2)]\, a_{j_t}$$
$$- \sum_{j=1}^{N} [\sum_{k=1}^{L} (D_k \omega_k^{2(i+j)})/(Q_{k_{t-1}}^2)]\, b_{j_t} = \sum_{k=1}^{L} (D_k \omega_k^{2i})/(Q_{k_{t-1}}^2)$$
$$i = 0, 1, \cdots, M, \qquad (25)$$

$$\sum_{j=0}^{M} [\sum_{k=1}^{L} (D_k^2 \omega_k^{2(i+j)})/(Q_{k_{t-1}}^2)]\, a_{j_t}$$
$$- \sum_{j=1}^{N} [\sum_{k=1}^{L} (D_k^4 \omega_k^{2(i+j)})/(Q_{k_{t-1}}^2)]\, b_{j_t}$$
$$= \sum_{k=1}^{L} (D_k^4 \omega_k^{2i})/(Q_{k_{t-1}}^2)$$
$$i = 1, 2, \cdots, N. \qquad (26)$$

Observe that (26) and (27) represent a set of *linear equations*

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TABLE I

Amplitude Response Data of a Nonlinear System

Logarithm of Frequency Log (γ_k) $\gamma_k=(\omega_k)/(2\tau)$	First Order (Linear) Amplitude Response Log $\lvert H^{(1)}(\gamma_k)\rvert$	Second Order Amplitude Response Log $\lvert H^{(2)}(\gamma_k,\gamma_k)\rvert$	Third Order Amplitude Response Log $\lvert H^{(3)}(\gamma_k,\gamma_k,\gamma_k)\rvert$
-2.3010	-3.4842	-5.3865	-7.4137
-2.1505	-3.3329	-5.2344	-7.2607
-2.0001	-3.1808	-5.0806	-7.1053
-1.8496	-3.0271	-4.9236	-6.9451
-1.6992	-2.8701	-4.7603	-6.7756
-1.5487	-2.7073	-4.5853	-6.5886
-1.3983	-2.5341	-4.3897	-6.3710
-1.2478	-2.3445	-4.1616	-6.1051
-1.0974	-2.1324	-3.8895	-5.7742
-0.9469	-1.8961	-3.5700	-5.3739
-0.7965	-1.6392	-3.2119	-4.9183
-0.6460	-1.3702	-2.8335	-4.4344
-0.4956	-1.0993	-2.4563	-3.9519
-0.3451	-0.8382	-2.1013	-3.4951
-0.1947	-0.6000	-1.7864	-3.0832
-0.0442	-0.3989	-1.5255	-2.7334
0.1061	-0.2455	-1.3265	-2.4602
0.2566	-0.1407	-1.1891	-2.2683
0.4070	-0.0764	-1.1035	-2.1474
0.5575	-0.0400	-1.0544	-2.0777
0.7079	-0.0205	-1.0280	-2.0400
0.8584	-0.0103	-1.0142	-2.0203
1.0088	-0.0052	-1.0071	-2.0102
1.1593	-0.0026	-1.0035	-2.0051
1.3097	-0.0013	-1.0018	-2.0025

in a_{j_t}'s and b_{j_t}'s which can be solved easily. The iterations are continued until the solution converges. This modified iterative solution is similar to the one used by Sanathanan and Koerner [10] for determining the transfer function from the real and imaginary part of the system response. However, our method is developed for determining the transfer function from amplitude response alone.

To illustrate the application of the approximation algorithm, consider the problem of fitting $H^{(1)}(f), H^{(2)}(f,f)$, $H^{(3)}(f,f,f)$ to the data displayed in Table I. The iterative algorithm was applied to the data with $M = 2, N = 2$, for $H^{(1)}$; $M = 3, N = 3$ for $H^{(2)}$. $H^{(3)}(f,f,f)$ need not be factored for system identification and hence was not approximated. The iterative algorithm converged in three steps and yielded the following fits.

$$|H^{(1)}(j\omega)|^2 = (\omega^4 + .27980\omega^2)/(\omega^4 + 100.01\omega^2 + 2310.55)$$

$$|H^{(2)}(j\omega, j\omega)|^2 = (.01)\omega^2(\omega^4 + .50454\omega^2 + .06218)/\omega^6 + 137.03\omega^4 + 5247.47\omega^2 + 36927.62.$$

Using the minimum phase assumption, we obtain [11]

$$H^{(1)}(j\omega) = \pm j\omega(j\omega + .53)/(j\omega + 6.0185)(j\omega + 7.986)$$

$$H^{(2)}(j\omega, j\omega) = \pm(0.1)(j\omega)(j\omega + .539)(j\omega + .463)/(j\omega + 3.004)(j\omega + 7.869)(j\omega + 8.128),$$

or letting f stand for $j\omega$, we have

$$H^{(2)}(f,f) = \pm(0.1)f(f + .539)(f + .463)/(f + 3.004)(f + 7.869)(f + 8.128).$$

The ± is necessary since we cannot uniquely resolve the sign with magnitude information and minimum phase assumption. We now need to apply the factoring algorithm to the following transfer functions.

$$H^{(1)}(f) = \pm f(f + .53)/(f + 6.0185)(f + 7.986)$$

$$H^{(2)}(f,f) = \pm(0.1)(f + .5398)(f + .463)f/\cdot(f + 7.869)(f + 8.128)(f + 3.004).$$

To do the identification, we need to verify if a pole or zero of $H^{(1)}(f)$ appears scaled or with multiplicity in $H^{(2)}(f,f)$. Starting with the f term in the numerator in $H^{(1)}(f)$, it does not appear with multiplicity 2 in the numerator of $H^{(2)}(f)$.

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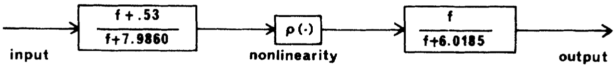

Fig. 4. Nonlinear system identified from data in Table I.
$\rho(\cdot) = C_1 + C_2(\cdot)^2 + C_3(\cdot)^3$,
$|C_1| = 1$, $|C_2| = 0.1$, $|C_3| = 0.01$

Hence the zero at the origin goes with $G(f)$. The zero at $-.53$ in $H^{(1)}$ and the zeroes at $-.538$ and $-.463$ in $H^{(2)}$ are close enough that we can say that $(f+.53)$ occurs with multiplicity 2 in $H^{(2)}(f,f)$; i.e., $(f+.53)$ is a zero of $H(f)$. A similar reasoning identifies $(f+7.986)$ as a pole of $H(f)$, (i.e., we treat $(f+7.869)$, $(f+8.128)$ and $(f+7.986)$ as the same factors). A comparison of $(f+6.0185)$ and $(f+3.004)$ indicates that $(2f+6.008)$ is a pole of $H^{(2)}(f,f)$ and hence $(f+6.0185)$ is a pole of $G(f)$. Hence the system consists of

$$H(f) = (f+.53)/(f+7.986)$$

$$G(f) = (f)/(f+6.0185),$$

and $|C_1| = 1.0$, $|C_2| = 0.1$.

There is no need to do curve fitting and factoring on the $|H^{(3)}(f,f,f)|$ data. Since $H(f)$ and $G(f)$ are already determined, and since we have assumed the system to have the structure shown in Fig. 1, we now need to find C_3 only. Using $H(f)$ and $G(f)$ which are already determined, we find $|C_3| \approx .01$. Hence the system which corresponds to the data in Table I is shown in Fig. 4. The actual system from which the data was obtained had the following transfer functions:

$$H(f) = (f+0.5)/(f+8.0),\ G(f) = (f)/(f+6.0);$$

$$C_1 = 1.0, C_2 = 0.1, C_3 = 0.01.$$

In many physical systems, the nonlinearity can be approximated by as few as five terms. Also the assumption that $|C_1| \gg |C_3| \gg |C_5|$ and $|C_2| \gg |C_4|$ will hold for most systems. Further, in balanced systems only the odd or even terms alone are necessary. Hence the primary data needed for identification consists of $|H^{(2)}(f,f)|$ and $|H^{(3)}(f,f,f)|$. If $|H^{(2)}(f,f)|$ indicates the presence of second order nonlinearity, the factoring should be applied to $|H^{(2)}(f,f)|$ and estimates of $H(f)$ and $G(f)$ should be obtained from $|H^{(2)}(f,f)|$ rather than from higher order transfer functions. This is due to the fact that the curve fitting and factoring on higher order transfer function data will not be very accurate; also a large volume of data will be necessary to do an adequate curve fitting job. However, once $H(f)$ and $G(f)$ are determined, then only a small amount of data will be necessary to extract the magnitudes of the higher order power series coefficients.

The sign of the power series coefficients can be determined from a limited amount of phase information. For example, to determine the sign of C_3 we need to isolate the $f_1+f_2+f_3$ component in the output of a three tone test. This can be done by passing the output of the system through a narrow band pass filter centered at $f_1+f_2+f_3$ and measuring the phase of the filter output with respect to the input signal. Also, prior knowledge about the nature of the nonlinearity might allow the determination of the signs of the power series coefficients without the need for experimental data. For example, if the nonlinearity is of e^{-x} type, then we know that the C's of this nonlinearity will have alternating signs.

V. Conclusions

The problem of identifying a class of nonlinear systems consisting of a power series nonlinearity sandwiched in between two linear systems has been considered. A frequency domain technique for determining the transfer functions of the linear portions and the coefficients of the power series nonlinearity was presented. Some a priori knowledge about the nature of the nonlinearity and the assumption of minimum phase transfer functions for the linear portions is necessary to arrive at a unique identification from experimental amplitude response data. Examples were presented to illustrate the use of the identification procedure.

The main limitation of the procedure is that the nonlinearity should be mild so that only a few terms in a power series expansion are necessary. Also the assumption that the magnitude of higher order coefficients be smaller compared to the lower order coefficients is necessary so that the various nonlinear transfer functions can be determined independently of each other experimentally.

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References

[1] J.J. Bussgang, L. Ehrman, and J.W. Graham, "Analysis of nonlinear systems with multiple inputs," *Proc. IEEE*, vol. 62, pp. 1088-1119, August 1974.
[2] J.W. Graham and L. Ehrman, "Communication receivers interference modeling: Experimental validation of the nonlinear transfer function approach to receiver modeling," *1972 Proc. IEEE Conf. on Communications*, June 1972.
[3] C.T. Leondes, "Volterra series synthesis of nonlinear stochastic tracking systems," *IEEE Trans. Aerospace and Electronic Systems*, vol. AES-11, pp. 245-266, March 1975.
[4] J.W. Goodman, *Introduction to Fourier Optics.* New York: McGraw-Hill, 1968, chapter 6.
[5] S. Narayanan, "Intermodulation distortion of cascaded transistors," *IEEE J. Solid State Circuits*, vol. SC-4, pp. 97-106, June 1969.
[6] S. Narayanan, "Application of Volterra series to intermodulation distortion analysis of transistor feedback amplifiers,"

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IEEE Trans. Circuit Theory, vol. CT-17, pp. 518-527, November 1970.
[7] E. Bedrosian and S.O. Rice, "The output properties of Volterra systems (nonlinear systems with memory driven by harmonic and Gaussian inputs)," *Proc. IEEE,* vol. 59, pp. 1688-1707, December 1971.
[8] W.W. Smith, Jr., and W.J. Rugh, "On the structure of a class of nonlinear systems," *IEEE Trans. Automat. Contr.,* vol. AC-19, pp. 701-705, December 1974.
[9] E.C. Levy, "Complex curve fitting," *IRE Trans. Automat. Contr.,* vol. AC-4, pp. 37-43, 1959.
[10] C.K. Sanathanan and J. Koerner, "Transfer function synthesis as a ratio of two complex polynomials," *IEEE Trans. Automat. Contr.,* vol. AC-8, pp. 56-58, 1963.
[11] S. Seshu and N. Balabanian, *Linear Network Analysis,* Ch. 7. New York: Wiley, 1959.

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