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A technique for the identification of linear periodic state-space models

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A technique for the identification of linear periodic state-space models

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A non-stochastic state-space identification algorithm for linear time-invariant systems is modified for use with periodic linear systems. Like its predecessor, the modified algorithm forms block Hankel matrices from input–output data and uses the singular value decomposition of these Hankel matrices to compute state vector sequences. The state vector sequences are then used to compute system matrices associated with the periodic linear system by solving an (overdetermined) linear system. The modifications to the original algorithm are as follows. First, the periodic system of period p is viewed as p separate time-invariant period-mapped systems. This technique allows the structure of a periodic Hankel matrix to be deduced, which in turn allows a state vector sub-sequence for the periodic system to be computed. When a complete state vector sequence is computed, it is used directly to construct the periodic state-space models. Second, similarities in the structure of each of the periodic Hankel matrices associated with the p time-invariant period-mapped systems is exploited in such a way that not only reduces the amount of computation necessary, but also improves the accuracy of the computation of the system matrices.

1. Introduction

A paper by Moonen *et al.* (1989) gave a method for the identification of system matrices from a finite number of input–output (I/O) data. This method is completely geometric in its approach; it uses no stochastic techniques whatsoever. The authors reported that if a system is 'sufficiently excited' by its inputs, if the system order is not underestimated, and if the inputs are arranged in an input Hankel matrix and its associated outputs are arranged in an output Hankel matrix, then the concatenation of the output Hankel matrix to the input Hankel matrix adds only n dimensions to the row space of the input Hankel matrix, where n is the dimension of the state vector. Further, these n dimensions represent the space spanned by a state vector sequence. This property is true for two sequential sets of input and output Hankel matrices, i.e., the concatenation of two sequential output Hankel matrices to their associated input Hankel matrices adds only n dimensions to the span of the row space of concatenated Hankel matrices. Moonen *et al.* (1989) showed the consequence of this is that the intersection of the row spaces of the two successive sets of Hankel matrices contains a basis for a state vector sequence associated with the second set of input and output Hankel matrices. Further, they provided an algorithm by which a state vector sequence can be computed via two singular value decompositions. Once a state vector sequence is computed, the state, input, output and throughput matrices associated with the

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standard linear model are computed as a least-squares solution of a linear system. Additionally, the algorithm is constructed in such a way that the right singular basis of the concatenated Hankel matrix is a common factor of each of the terms in the linear system and therefore is discarded in the computation of the system models. This is one of the most important features of the original algorithm; it greatly enhances the computational efficiency and noise insensitivity of the algorithm. Although completely novel in its approach, it is related to other algorithms (Kung 1978, Zeiger and McEwen 1974) which use a singular value decomposition of a block Hankel matrix to identify state-space models. (For further details, see Moonen *et al.* 1989 and Moonen and Vandewalle 1990.)

Several problems arise when this algorithm is applied to periodic systems. First and foremost, the algorithm works only for time-invariant systems. For a periodic linear system with period p, this problem is overcome by creating p separate time-invariant systems which map the states from one period to the next. Such time-invariant systems are referred to as period-mapped systems in this paper. It should be noted, however, that the period-mapped systems are *not themselves identified*. Rather, they provide a theoretical framework from which the structure of Hankel matrices for periodic systems may be deduced. From these periodic Hankel matrices, state vector sequences are computed which are then used to compute the periodic state-space models directly.

Second, the description of a periodic system by p separate time-invariant systems would seemingly require $p + 1$ large singular value decompositions, from which $p + 1$ state vector sequences could be computed, in which case it would be prohibitively expensive to compute so many large singular value decompositions. Furthermore, such an algorithm would require that the right singular bases be used in addition to the left singular bases in the computation of the system matrices, since the right singular bases would not be common factors in the p systems of linear equations used to compute the system matrices. Since the right singular bases are generally the source of noise sensitivity as well as computational load in this type of algorithm, such a technique is ruled out.

Fortunately, there are structural similarities to the $p + 1$ Hankel matrices that are associated with the time-invariant period-mapped systems. These similarities may be exploited in such a way that only one (large) singular value decomposition and $p - 1$ much smaller singular value decompositions are necessary. Furthermore, the right singular basis of the large singular value decomposition is a common factor in all of the p systems of linear equations from which the system matrices are computed; it is not used in the actual system matrix computations. Thus, the algorithm presented herein retains the noise insensitivity and efficiency of its progenitor.

The motivation for developing the algorithms in this paper is due to the renewed interest in the area of periodic linear systems, as various problems of interest in control may be posed within the framework of periodic linear systems. These include multi-rate discrete-time systems (Al Rahmani and Franklin 1990, Georgiou and Khargonekar 1987, Meyer and Burrus 1975), state estimation in the presence of cyclo-stationary white gaussian noise (Gardner 1991), optimal control of periodic systems (Bojanczyk *et al.* 1992, Hench and Laub 1994), and the identification of system models for periodic linear systems (Hench 1993). Furthermore, periodic systems arise naturally when considering the dynamics of mechanical systems with rotational motion, such as electric motors, etc., and thus the identification of system models from I/O data may prove useful.

The paper is organized in the following way. Section 2 is devoted to the decomposition of a periodic time-varying system into a set of time-invariant period-mapped systems which accommodate the results of Moonen *et al.* (1989). Section 3 elaborates how models of system matrices are computed. Section 4 indicates how it is possible to streamline the computation of the state vector sequences and how to improve the accuracy of the solution. An on-line variant of the off-line algorithm discussed in §§ 3 and 4 is presented in § 5. Numerical examples are provided in § 6. Finally, since indices tend to proliferate in the treatment of periodic systems, Table 1 contains the conventions which will be observed throughout the paper.

2. Periodic systems

This paper is concerned with linear discrete-time periodic systems of the form

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}_k \mathbf{x}_k + \mathbf{B}_k \mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}_k \mathbf{x}_k + \mathbf{D}_k \mathbf{u}_k \end{aligned} \tag{1}$$

where the matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$ and $\mathbf{D} \in \mathbb{R}^{l\times m}$ are periodic with period $p \in \mathbb{N}$, i.e., for all k,

$$\begin{aligned} \mathbf{A}_k &= \mathbf{A}_{k+p} \\ \mathbf{B}_k &= \mathbf{B}_{k+p} \\ \mathbf{C}_k &= \mathbf{C}_{k+p} \\ \mathbf{D}_k &= \mathbf{D}_{k+p} \end{aligned}$$

The matrices $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{C}_k$ and $\mathbf{D}_k$ are the unknown systems matrices that are to be constructed from the recorded I/O sequences $\hat{\mathbf{u}}_k$, $\hat{\mathbf{u}}_{k+1}$, ... and $\hat{\mathbf{y}}_k$, $\hat{\mathbf{y}}_{k+1}$, ..., where

$$\begin{aligned} \hat{\mathbf{u}}_k &= \mathbf{u}_k + \mathbf{w}_k \\ \hat{\mathbf{y}}_k &= \mathbf{u}_k + \mathbf{v}_k \end{aligned} \tag{2}$$

The additional inputs $\mathbf{w}_k$ and $\mathbf{v}_k$ are unknown, unmodelled sequences that account for such factors as measurement noise, unknown disturbances, model

Index	Description
i	index related to the number of rows in a Hankel matrix
j	index related to the number of columns in a Hankel matrix
k	'running' (time) index of a linear system
l	number of system outputs
m	number of system inputs
n	number of system states
p	period of a periodic system
r	number between 0 and p, inclusively
κ	fixed number representing an initial (time) value

Table 1. Index conventions.

mismatch, nonlinearities, etc. Since the algorithm of this paper, like its progenitor by Moonen *et al.* (1989), is non-stochastic in its approach, consideration of these terms is deferred to § 3 and § 4, where they will be viewed as residuals of a set of overdetermined equations. This deferral permits greater simplicity in the exposition of the main ideas of this paper. Also, we insist that the system is completely (pointwise) observable; namely, that the matrix pair $[\mathbf{C}_k, \mathbf{A}_k]$ is observable for all time k.

As is the case with most treatments of periodic systems (Bittanti 1991, Flamm 1991, Meyer and Burrus 1975, Park and Verriest 1989), the first task is to find a time-invariant representation of the periodic linear system in (1). This is particularly important since the identification scheme modified by this paper is designed to work with time-invariant systems. This can be done in the following standard way (Meyer and Burrus 1975). Once a time $k = \kappa$ is fixed, the time-invariant system which maps x_κ to $x_{\kappa+p}$ may be written as the period-mapped system

$$\bar{x}_{k+1}[\kappa] = \bar{\mathbf{A}}[\kappa]\bar{x}_k[\kappa] + \bar{\mathbf{B}}[\kappa]\bar{u}_k[\kappa] \tag{3}$$

$$\bar{y}_k[\kappa] = \bar{\mathbf{C}}[\kappa]\bar{x}_k[\kappa] + \bar{\mathbf{D}}[\kappa]\bar{u}_k[\kappa] \tag{4}$$

For example, the system matrices associated with the period-mapped system with $\kappa = 1$ may be written as:

$$\begin{aligned}
\bar{\mathbf{A}}[1] &= \mathbf{A}_p\mathbf{A}_{p-1}\dots\mathbf{A}_1 \\
\bar{\mathbf{B}}[1] &= [\mathbf{A}_p\mathbf{A}_{p-1}\dots\mathbf{A}_2\mathbf{B}_1 \quad \mathbf{A}_p\mathbf{A}_{p-1}\dots\mathbf{A}_3\mathbf{B}_2 \quad \dots \quad \mathbf{A}_p\mathbf{B}_{p-1} \quad \mathbf{B}_p] \\
\bar{\mathbf{C}}[1] &= \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2\mathbf{A}_1 \\ \vdots \\ \mathbf{C}_p\mathbf{A}_{p-1}\dots\mathbf{A}_1 \end{bmatrix} \\
\bar{\mathbf{D}}[1] &= \begin{bmatrix} \mathbf{D}_1 & 0 & \cdots & 0 & 0 \\ \mathbf{C}_2\mathbf{B}_1 & \mathbf{D}_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{C}_{p-1}\mathbf{A}_{p-2}\dots\mathbf{A}_2\mathbf{B}_1 & \mathbf{C}_{p-1}\mathbf{A}_{p-1}\dots\mathbf{A}_3\mathbf{B}_2 & \cdots & \mathbf{D}_{p-1} & 0 \\ \mathbf{C}_p\mathbf{A}_{p-1}\dots\mathbf{A}_2\mathbf{B}_1 & \mathbf{C}_p\mathbf{A}_{p-1}\dots\mathbf{A}_3\mathbf{B}_2 & \cdots & \mathbf{C}_p\mathbf{B}_{p-1} & \mathbf{D}_p \end{bmatrix}
\end{aligned} \tag{5}$$

Note that a state sequence of the period-mapped system is a sub-sequence of the periodic system, i.e.

$$\bar{x}_k[\kappa] = x_{\kappa+pk} \tag{6}$$

and the augmented input and output vectors $\bar{u}_k[\kappa]$ and $\bar{y}_k[\kappa]$ are created by the concatenation of the input and output vectors u_k and $y_k[\kappa]$, respectively:

$$\left.\begin{aligned}
\bar{u}_k[\kappa] &= \begin{bmatrix} u_{\kappa+pk} \\ u_{\kappa+pk+1} \\ \vdots \\ u_{\kappa+p(k+1)-1} \end{bmatrix} \\
\bar{y}_k[\kappa] &= \begin{bmatrix} y_{\kappa+pk} \\ y_{\kappa+pk+1} \\ \vdots \\ y_{\kappa+p(k+1)-1} \end{bmatrix}
\end{aligned}\right\} \tag{7}$$

With this definition it is possible to find a state vector sequence for the periodic system. This may be accomplished by forming Hankel matrices for each of the period-mapped systems. Proceeding along the lines of Moonen *et al.* (1989), two Hankel matrices associated with the period-mapped system may be formed in the following way:

$$\mathbf{H}_1[\kappa] = \begin{bmatrix} \bar{u}_0[\kappa] & \bar{u}_1[\kappa] & \cdots & \bar{u}_{j-1}[\kappa] \\ \bar{y}_0[\kappa] & \bar{y}_1[\kappa] & \cdots & \bar{y}_{j-1}[\kappa] \\ \bar{u}_1[\kappa] & \bar{u}_2[\kappa] & \cdots & \bar{u}_j[\kappa] \\ \bar{y}_1[\kappa] & \bar{y}_2[\kappa] & \cdots & \bar{y}_j[\kappa] \\ \vdots & \vdots & & \vdots \\ \bar{u}_{i-1}[\kappa] & \bar{u}_i[\kappa] & \cdots & \bar{u}_{i+j-2}[\kappa] \\ \bar{y}_{i-1}[\kappa] & \bar{y}_i[\kappa] & \cdots & \bar{y}_{i+j-2}[\kappa] \end{bmatrix} \tag{8}$$

$$\mathbf{H}_2[\kappa] = \begin{bmatrix} \bar{u}_i[\kappa] & \bar{u}_{i+1}[\kappa] & \cdots & \bar{u}_{i+j-1}[\kappa] \\ \bar{y}_i[\kappa] & \bar{y}_{i+1}[\kappa] & \cdots & \bar{y}_{i+j-1}[\kappa] \\ \bar{u}_{i+1}[\kappa] & \bar{u}_{i+2}[\kappa] & \cdots & \bar{u}_{i+j}[\kappa] \\ \bar{y}_{i+1}[\kappa] & \bar{y}_{i+2}[\kappa] & \cdots & \bar{y}_{i+j}[\kappa] \\ \vdots & \vdots & & \vdots \\ \bar{u}_{2i-1}[\kappa] & \bar{u}_{2i}[\kappa] & \cdots & \bar{u}_{2i+j-2}[\kappa] \\ \bar{y}_{2i-1}[\kappa] & \bar{y}_{2i}[\kappa] & \cdots & \bar{y}_{2i+j-2}[\kappa] \end{bmatrix} \tag{9}$$

Roughly speaking, the indices i and j are chosen in such a way that they are large enough such that the matrices $\mathbf{H}_1[\kappa]$ and $\mathbf{H}_2[\kappa]$ contain all of the necessary information about the system. In practice, the index j is chosen such that $j \gg \max(mi, li)$, in order to ensure relative noise-insensitivity and computational efficiency of the modified algorithm.

Now, the state sequence $x_{\kappa+ip}, x_{\kappa+(i+1)p}, \ldots$ is determined by means of the following theorem.

Theorem 1: *Let the matrix $\mathscr{X}$ be defined as*

$$\mathscr{X}[\kappa] = [x_{\kappa+ip}, x_{\kappa+(i+1)p}, \ldots, x_{\kappa+(i+j-1)p}]$$

then

$$\mathrm{span}_{\mathrm{row}}(\mathscr{X}[\kappa]) = \mathrm{span}_{\mathrm{row}}(\mathbf{H}_1[\kappa]) \cap \mathrm{span}_{\mathrm{row}}(\mathbf{H}_2[\kappa])$$

Proof: Clearly this sequence is the state-sequence that results from the time-invariant system defined in (4). As a result, Theorem 3 in Moonen *et al.* (1989) completes the proof. □

To determine the state sequence, we apply a similar technique as in Moonen's algorithm. First, a permutation matrix $\mathbf{P}_a$ as well as a modified Hankel matrix $\mathbf{H}_a[\kappa]$ are defined such that

$$\begin{bmatrix} \mathbf{H}_{1a}[\kappa] \\ \mathbf{H}_{2a}[\kappa] \end{bmatrix} = \mathbf{P}_a \begin{bmatrix} \mathbf{H}_1[\kappa] \\ \mathbf{H}_2[\kappa] \end{bmatrix}$$
$$\mathbf{H}_a[\kappa] = \mathbf{P}_a \mathbf{H}[\kappa] \tag{10}$$

where $\mathbf{P}_a$ exchanges the rows of $\mathbf{H}[\kappa]$ so that they are ordered sequentially in time:

$$\mathbf{H}_a[\kappa] = \begin{bmatrix} u_\kappa & u_{\kappa+p} & \cdots \\ y_\kappa & y_{\kappa+p} & \cdots \\ u_{\kappa+1} & u_{\kappa+p+1} & \cdots \\ y_{\kappa+1} & y_{\kappa+p+1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

This reordering is a slight departure from Moonen's algorithm, and indeed is not necessary for the purposes of this section. Nevertheless, it is introduced here as it will be an important element of the algorithm in § 4. Note that such a reordering of the rows does not affect the conditions of Theorem 1.

Proceeding, a singular value decomposition of the matrix $\mathbf{H}_a[\kappa]$ is computed, where

$$\mathbf{H}_a[\kappa] = \begin{bmatrix} \mathbf{U}_{11a}[\kappa] & \mathbf{U}_{12a}[\kappa] \\ \mathbf{U}_{21a}[\kappa] & \mathbf{U}_{22a}[\kappa] \end{bmatrix} \begin{bmatrix} \mathbf{S}_{11a}[\kappa] & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}_a^{\mathrm{T}}[\kappa] \tag{12}$$

and where the submatrices are elements of the following sets:

$$\begin{aligned} \mathbf{U}_{11a}[\kappa] &\in \mathbb{R}^{((m+l)pi)\times(2mpi+n)} \\ \mathbf{U}_{12a}[\kappa] &\in \mathbb{R}^{((m+l)pi)\times(2lpi-n)} \\ \mathbf{U}_{21a}[\kappa] &\in \mathbb{R}^{((m+l)pi)\times(2mpi+n)} \\ \mathbf{U}_{22a}[\kappa] &\in \mathbb{R}^{((m+l)pi)\times(2lpi-n)} \\ \mathbf{S}_{11a}[\kappa] &\in \mathbb{R}^{(2mpi+n)\times(2mpi+n)} \end{aligned}$$

The state vector sequence

$$\mathscr{X}[\kappa] = [\boldsymbol{x}_{\kappa+ip} \quad \boldsymbol{x}_{\kappa+(i+1)p} \quad \cdots \quad \boldsymbol{x}_{\kappa+(i+j-1)p}]$$

may be calculated as

$$\mathscr{X}[\kappa] = \mathbf{U}_{qa}^{\mathrm{T}}[\kappa]\mathbf{U}_{12a}^{\mathrm{T}}[\kappa]\mathbf{H}_{1a}[\kappa] \tag{13}$$

where $\mathbf{U}_{qa}^{\mathrm{T}}[\kappa]$ is an $n \times (2lpi - n)$ matrix which reduces the linearly dependent row vectors of $\mathbf{U}_{12a}^{\mathrm{T}}[\kappa]\mathbf{H}_{1a}[\kappa]$ to n linearly independent row vectors by means of the singular value decomposition of $\mathbf{U}_{12a}^{\mathrm{T}}[\kappa]\mathbf{U}_{11a}[\kappa]\mathbf{S}_{11a}[\kappa]$:

$$\mathbf{U}_{12a}^{\mathrm{T}}[\kappa]\mathbf{U}_{11a}[\kappa]\mathbf{S}_{11a}[\kappa] = [\mathbf{U}_{qa}[\kappa], \mathbf{U}_{qa}^{\perp}[\kappa]] \begin{bmatrix} \mathbf{S}_{qa}[\kappa] & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{qa}^{\mathrm{T}}[\kappa] \\ \mathbf{V}_{qa}^{\perp\mathrm{T}}[\kappa] \end{bmatrix} \tag{14}$$

3. Computation of system matrix models

Once an appropriate state vector sequence is computed, it is possible to determine system matrix models by solving a set of (overdetermined) linear equations. First, the matrices associated with the inputs and output are defined as follows, with the definition for the state sequence matrix repeated for convenience:

$$\begin{aligned} \mathscr{U}[\kappa] &= [\boldsymbol{u}_{\kappa+ip} \quad \boldsymbol{u}_{\kappa+(i+1)p} \quad \cdots \quad \boldsymbol{u}_{\kappa+(i+j-1)p}] \\ \mathscr{Y}[\kappa] &= [\boldsymbol{y}_{\kappa+ip} \quad \boldsymbol{y}_{\kappa+(i+1)p} \quad \cdots \quad \boldsymbol{y}_{\kappa+(i+j-1)p}] \\ \mathscr{X}[\kappa] &= [\boldsymbol{x}_{\kappa+ip} \quad \boldsymbol{x}_{\kappa+(i+1)p} \quad \cdots \quad \boldsymbol{x}_{\kappa+(i+j-1)p}] \end{aligned} \tag{15}$$

Using these definitions, the system matrices may be determined by the system of linear equations

$$\begin{bmatrix} \mathscr{X}[\kappa+1] \\ \mathscr{Y}[\kappa] \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\kappa & \mathbf{B}_\kappa \\ \mathbf{C}_\kappa & \mathbf{D}_\kappa \end{bmatrix} \begin{bmatrix} \mathscr{X}[\kappa] \\ \mathscr{U}[\kappa] \end{bmatrix} \tag{16}$$

It is worth noting that this formulation does not take advantage of the previous decompositions; however, by (12) and (13), the inputs, outputs and states may be written in terms of a singular value decomposition:

$$\begin{aligned} \mathscr{U}[\kappa] &= \mathbf{U}_a[\kappa](pi(m+l)+1 : pi(m+l)+m, :)\mathbf{S}_a[\kappa]\mathbf{V}_a^T[\kappa] \\ \mathscr{Y}[\kappa] &= \mathbf{U}_a[\kappa](pi(m+l)+m+1 : (pi+1)(m+l), :)\mathbf{S}_a[\kappa]\mathbf{V}_a^T[\kappa] \\ \mathscr{X}[\kappa] &= \mathbf{U}_{qa}^T[\kappa]\mathbf{U}_{12a}^T[\kappa]\mathbf{U}_a[\kappa](1 : pi(m+l), :)\mathbf{S}_a[\kappa]\mathbf{V}_a^T[\kappa] \end{aligned} \tag{17}$$

where the column and row indexing above follows the standard syntax found in MATLAB© and in Moonen *et al.* (1989), namely, that $\mathbf{M}(p:q, r:s)$ is the submatrix of $\mathbf{M}$ containing rows p to q and columns r to s. If the colon appears by itself in the first position inside the parentheses, it means all of the rows of the matrix, and all of the columns if it is in the second.

4. Computational streamlining

The system of equations in (16) and (17) seems to imply that $p+1$ singular value decompositions of $p+1$ different Hankel matrices must be performed. This is unacceptable since such an approach is prohibitively expensive to compute and too sensitive to noise due to the fact that the right singular bases must be used in the solution of the linear system in (16) in the case where $\mathscr{U}[\kappa]$, $\mathscr{Y}[\kappa]$ and $\mathscr{X}[\kappa]$ are written in terms of a singular value decomposition of their associated Hankel matrix. Nevertheless, it is possible to provide a method whereby the right singular bases need not be used to compute the system matrices. To this end, first define the following matrix:

$$\mathbf{H}_3[\kappa] = \begin{bmatrix} \bar{u}_{2i}[\kappa] & \bar{u}_{2i+1}[\kappa] & \cdots & \bar{u}_{2i+j-1}[\kappa] \\ \bar{y}_{2i}[\kappa] & \bar{y}_{2i+1}[\kappa] & \cdots & \bar{y}_{2i+j-1}[\kappa] \end{bmatrix} \tag{18}$$

and append it to $\mathbf{H}_1[\kappa]$ and $\mathbf{H}_2[\kappa]$ and so redefine

$$\hat{\mathbf{H}}[\kappa] = \begin{bmatrix} \mathbf{H}_1[\kappa] \\ \mathbf{H}_2[\kappa] \\ \mathbf{H}_3[\kappa] \end{bmatrix} \tag{19}$$

Next, define the permutation matrix $\mathbf{P}_b$ as well as a modified Hankel matrix $\mathbf{H}_b[\kappa]$ such that the rows of $\mathbf{H}_b[\kappa] = \mathbf{P}_b\hat{\mathbf{H}}[\kappa]$ are ordered in time in the same way as with $\mathbf{H}_a[\kappa]$. Next, define a singular value decomposition of $\mathbf{H}_b[\kappa]$ to be

$$\mathbf{H}_b[\kappa] = \begin{bmatrix} \mathbf{U}_{11b}[\kappa] & \mathbf{U}_{12b}[\kappa] \\ \mathbf{U}_{21b}[\kappa] & \mathbf{U}_{22b}[\kappa] \end{bmatrix} \begin{bmatrix} \mathbf{S}_{11b}[\kappa] & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}_b^T[\kappa] \tag{20}$$

where the submatrices are elements of the following sets:

$$\begin{aligned} \mathbf{U}_{11b}[\kappa] &\in \mathbb{R}^{((m+l)pi)\times(mp(2i+1)+n)} \\ \mathbf{U}_{12b}[\kappa] &\in \mathbb{R}^{((m+l)(i+1)p)\times(lp(2i+1)-n)} \\ \mathbf{U}_{21b}[\kappa] &\in \mathbb{R}^{((m+l)pi)\times(mp(2i+1)+n)} \end{aligned}$$

$$\mathbf{U}_{22\mathrm{b}}[\kappa] \in \mathbb{R}^{((m+l)(i+1)p)\times(lp(2i+1)-n)}$$
$$\mathbf{S}_{11\mathrm{b}}[\kappa] \in \mathbb{R}^{(mp(2i+1)+n)\times(mp(2i+1)+n)}$$

The sequence $\mathscr{X}[\kappa]$ may be computed the same way as in the previous section:

$$\mathscr{X}[\kappa] = \mathbf{U}_{q\mathrm{b}}^{\mathrm{T}}[\kappa]\mathbf{U}_{12\mathrm{b}}^{\mathrm{T}}[\kappa]\mathbf{U}_{\mathrm{b}}[\kappa](1 : pi(m+l), :)\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa] \tag{21}$$

Further, the time-invariance and the block Hankel structure of the matrix $\mathbf{H}_{\mathrm{b}}[\kappa]$ ensure that

$$\mathscr{X}[\kappa + p] = \mathbf{U}_{q\mathrm{b}}^{\mathrm{T}}[\kappa]\mathbf{U}_{12\mathrm{b}}^{\mathrm{T}}[\kappa]\mathbf{U}_{\mathrm{b}}(p(m+l)+1 : p(i+1)(m+l), :)\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa] \tag{22}$$

The computation of $\mathscr{X}[\kappa + r]$ for $1 \leq r \leq p-1$ in such a way that its last term is $\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]$ presents more of a challenge. First, define the matrices $\mathbf{P}_1, \ldots, \mathbf{P}_{p-1}$ to be

$$\mathbf{P}_{\mathrm{r}} = [0_{2pi(m+l)\times r(m+l)} \quad \mathbf{I}_{2pi(m+l)\times 2pi(m+l)} \quad 0_{2pi(m+l)\times(p-r)(m+l)}]$$

By the structure of the Hankel matrices $\mathbf{H}[\kappa + r]$, it is evident that

$$\mathbf{H}_{\mathrm{a}}[\kappa + r] = \mathbf{P}_{\mathrm{r}}\mathbf{H}_{\mathrm{b}}[\kappa] \tag{23}$$

Further, the singular value decomposition of $\mathbf{H}_{\mathrm{a}}[\kappa + r]$ can be related to $\mathbf{H}_{\mathrm{b}}[\kappa]$ by the following theorem.

Theorem 2: *Let the singular value decompositions of* $\mathbf{H}_{\mathrm{b}}[\kappa]$ *and* $\mathbf{H}_{\mathrm{a}}[\kappa + r]$ *be defined as*

$$\mathbf{H}_{\mathrm{b}}[\kappa] = \mathbf{U}_{\mathrm{b}}[\kappa]\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]$$

and

$$\mathbf{H}_{\mathrm{a}}[\kappa + r] = \mathbf{U}_{\mathrm{a}}[\kappa + r]\mathbf{S}_{\mathrm{a}}[\kappa + r]\mathbf{V}_{\mathrm{a}}^{\mathrm{T}}[\kappa + r]$$

with $j \gg i$. *Further, let the singular value decomposition of the product*

$$\mathbf{P}_{\mathrm{r}}\mathbf{U}_{\mathrm{b}}[\kappa]\mathbf{S}_{\mathrm{b}}[\kappa](:, 1 : mp(2i+1)+n) = \tilde{\mathbf{U}}[\kappa, r]\tilde{\mathbf{S}}[\kappa, r]\tilde{\mathbf{W}}^{\mathrm{T}}[\kappa, r] \tag{24}$$

Then, the terms of the singular value decomposition of $\mathbf{H}_{\mathrm{a}}[\kappa + r]$ *may be written as*

$$\begin{aligned} \mathbf{U}_{\mathrm{a}}[\kappa + r] &= \tilde{\mathbf{U}}[\kappa, r] \\ \mathbf{S}_{\mathrm{a}}[\kappa + r] &= [\tilde{\mathbf{S}}[\kappa, r] \| 0] \\ \mathbf{V}_{\mathrm{a}}^{\mathrm{T}}[\kappa + r] &= \tilde{\mathbf{V}}^{\mathrm{T}}[\kappa, r]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa] \end{aligned} \tag{25}$$

where

$$\tilde{\mathbf{V}}[\kappa, r] = \begin{bmatrix} \tilde{\mathbf{W}}[\kappa, r] & 0 \\ 0 & \mathbf{I} \end{bmatrix} \tag{26}$$

Proof: Since the permutation matrix $\mathbf{P}_{\mathrm{r}}$ does not affect the basis for the right null space of $\mathbf{H}_{\mathrm{b}}[\kappa]$, the basis for the right null space of $\mathbf{H}_{\mathrm{a}}[\kappa + r]$ can be taken to be the same as $\mathbf{H}_{\mathrm{b}}[\kappa]$. The remainder of the proof is complete by direct substitution. □

Now it is possible to replace the previous definitions of $\mathscr{U}[\kappa + r]$, $\mathscr{Y}[\kappa + r]$

and $\mathscr{X}[\kappa + r]$ in such a way that they all have the term $\mathscr{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]$ as their last term:

$$\left.\begin{array}{ll}
\mathscr{U}[\kappa + r] = \mathbf{U}_{\mathrm{b}}[\kappa](_{pi(m+l)+(m+l)r+1:pi(m+i)+(m+l)r+m,:})\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]; & 0 \leqslant r \leqslant p \\
\mathscr{Y}[\kappa + r] = \mathbf{U}_{\mathrm{b}}[\kappa](_{pi(m+l)+(m+l)r+m+1:pi(m+l)+(m+l)(r+1),:})\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]; & 0 \leqslant r \leqslant p \\
\mathscr{X}[\kappa + r] = \mathbf{U}_{q\mathrm{b}}^{\mathrm{T}}[\kappa]\mathbf{U}_{12}^{\mathrm{T}}[\kappa]\mathbf{U}_{\mathrm{b}}[\kappa](_{1:pi(m+l),:})\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]; & r = 0 \\
\mathscr{X}[\kappa + r] = \tilde{\mathbf{U}}_{q\mathrm{b}}^{\mathrm{T}}[\kappa, r]\tilde{\mathbf{U}}_{12}^{\mathrm{T}}[\kappa, r]\tilde{\mathbf{U}}_{\mathrm{b}}[\kappa, r](_{1:pi(m+l),:})\tilde{\mathbf{S}}_{\mathrm{b}}[\kappa, r]\tilde{\mathbf{V}}^{\mathrm{T}}[\kappa, r]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa] & \\
 & 1 \leqslant r \leqslant p - 1 \\
\mathscr{X}[\kappa + r] = \mathbf{U}_{q\mathrm{b}}^{\mathrm{T}}[\kappa]\mathbf{U}_{12}^{\mathrm{T}}[\kappa]\mathbf{U}_{\mathrm{b}}[\kappa](_{p(m+l)+1:p(i+1)(m+l),:})\mathbf{S}_{\mathrm{b}}[\kappa]\mathbf{V}_{\mathrm{b}}^{\mathrm{T}}[\kappa]; & r = p
\end{array}\right\} \quad (27)$$

Note that the term $\tilde{\mathbf{U}}_{q\mathrm{b}}$ is defined analogously to $\mathbf{U}_{q\mathrm{a}}$ in (14). Its purpose is to reduce the linear dependent rows in

$$\tilde{\mathbf{U}}_{12\mathrm{b}}^{\mathrm{T}}[\kappa, r]\mathbf{H}_{\mathrm{a}}[\kappa + r]$$

and is defined by means of the singular value decomposition on the product $\tilde{\mathbf{U}}_{12}^{\mathrm{T}}[\kappa, r]\tilde{\mathbf{U}}_{11}[\kappa, r]\tilde{\mathbf{S}}[\kappa, r]$.

At this point a few remarks are in order.

Remark 1: With the modified definitions of $\mathscr{X}[\kappa + r]$ for $0 < r < p$ in (27), it is possible to discard the right singular basis $\mathbf{V}_{\mathrm{b}}[\kappa]$ in the computation of the system matrices in (16). This not only increases the accuracy of the modified algorithm, but also increases its computational efficiency, since the singular value decomposition of $\mathbf{H}_{\mathrm{a}}[\kappa + r]$ for $0 < r < p$ requires much less computation than the singular value decomposition of $\mathbf{H}_{\mathrm{b}}[\kappa]$. □

Remark 2: In the previous sections it was assumed that the I/O measurements $\hat{u}$ and $\hat{y}$ were noiseless measurements of the inputs and outputs of a perfectly linear periodic system, i.e., $\hat{u} = u$ and $\hat{y} = y$. In practice, the measurements will be corrupted by a number of factors, causing the various Hankel matrices in the algorithm to be of full row rank. Fortunately, the singular value decomposition provides a natural way of determining the numerical rank of the Hankel matrices. Singular values which have been determined to have magnitude less than an appropriate threshold are set to zero. □

Remark 3: In this work, various schemes to streamline the computation were investigated, but are not reported. The variations included different definitions of $\mathbf{H}_{\mathrm{b}}[\kappa]$ and $\mathbf{P}_{\mathrm{r}}$. For example, it would be possible to define $\mathbf{H}_{\mathrm{b}}[\kappa]$ with less rows. While this would produce a more elegant equation for $\mathscr{X}[\kappa + r]$ in (27), numerical experiments like those in § 6 suggest that the alternative schemes would produce state models with substantial asymptotic bias. This indicates the importance that the Hankel matrix $\mathbf{H}_{\mathrm{b}}[\kappa]$ be formed from an underlying time-invariant system, as it is in this paper. This has another underlying advantage; it makes it possible to apply the results of Moonen and Vandewalle (1990), which use a quotient singular value decomposition to identify periodic state-space models. This, however, is beyond the scope of this paper. □

5. On-line estimation

The off-line algorithm of the previous section may be easily converted to an on-line algorithm. In other words, the estimation of state-space models may be updated after each new period, i.e., after p time-steps, to reflect the new information contained in the most recent I/O data measurement. This may be accomplished by two standard techniques: either a sliding window, or an exponentially weighted 'forgetting-factor'. Both of these techniques are analogous to those in Moonen *et al.* (1989), but are repeated here for completeness. The former technique is rather straightforward. It requires that the first column of $\mathbf{H}_b[\kappa]$ is deleted as each new column is added to the matrix $\mathbf{H}_b[\kappa]$. From the updated (and downdated) matrix $\mathbf{H}_b[\kappa]$ new models of system matrices are computed using the method of § 4.

The latter technique requires that each of the columns of the matrix $\mathbf{H}_b[\kappa]$ be weighted by a forgetting factor $\alpha \leq 1$ as each new column is added. Thus, a column of data which corresponds to data np time-steps earlier is weighted by α^n, which reduces the influence of the older data on the estimation of the state space models. This method is perhaps preferable to the sliding window approach, since the old data comprising the columns of $\mathbf{H}_b[\kappa]$ need not be used for its update, thus reducing the overall computational load and storage necessary for the update of system models.

The 'forgetting-factor' on-line algorithm may be implemented as follows. First, define the Hankel matrix $\mathbf{H}_b^{(n)}[\kappa]$ as the Hankel matrix $\mathbf{H}_b[\kappa]$ formed with n columns of data. This matrix will never be explicitly formed. Rather, the matrices $\mathbf{U}_b^{(n)}[\kappa]$ and $\mathbf{S}_b^{(n)}[\kappa]$ from its singular value decomposition, defined analogously to those in (20), will be constructed as each new column of data is available. The first step is to initialize $\mathbf{U}_b^{(0)}[\kappa]$ and $\mathbf{S}_b^{(0)}[\kappa]$ as

$$\mathbf{U}_b^{(0)}[\kappa] = \mathbf{I}_{p(2i+1)(m+l)\times p(2i+1)(m+l)}$$

$$\mathbf{S}_b^{(0)}[\kappa] = 0_{p(2i+1)(m+l)\times p(2i+1)(m+l)}$$

where p, m, l and i are defined as if matrix $\mathbf{H}_b^{(0)}[\kappa]$ were explicitly formed. Further, the vector forming a new column of data to be appended to the unformed matrix $\mathbf{H}_b^{(n-1)}[\kappa]$ is defined as $\boldsymbol{h}^{(n)}[\kappa]$. With these definitions it is possible to update the factors $\mathbf{U}_b^{(n)}[\kappa]$ and $\mathbf{S}_b^{(n)}[\kappa]$. These are derived from the singular value decomposition of the matrix

$$[\alpha \mathbf{U}_b^{(n-1)}[\kappa]\mathbf{S}_b^{(n-1)}[\kappa] | \boldsymbol{h}^{(n)}[\kappa]]$$

Once these factors are updated, new state-space models may be computed using the method of § 4, since only the left singular basis and singular values of the matrix $\mathbf{H}_b^{(n)}[\kappa]$ are needed to compute state-space models $\mathbf{A}_r^{(n)}$, $\mathbf{B}_r^{(n)}$, $\mathbf{C}_r^{(n)}$ and $\mathbf{D}_r^{(n)}$ analogous to those in (27).

6. Numerical examples

A number of numerical examples are provided to illustrate the methods of the previous sections.

Example 1: In this example, a system is constructed as described by (1) with period $p = 3$ and with system matrices

$$\left[\begin{array}{c|c} \mathbf{A}_1 & \mathbf{B}_1 \\ \hline \mathbf{C}_1 & \mathbf{D}_1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} \mathbf{A}_2 & \mathbf{B}_2 \\ \hline \mathbf{C}_2 & \mathbf{D}_2 \end{array}\right] = \left[\begin{array}{cc|c} \frac{1}{5} & 1 & 0 \\ 0 & \frac{2}{5} & 1 \\ \hline 2 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} \mathbf{A}_3 & \mathbf{B}_3 \\ \hline \mathbf{C}_3 & \mathbf{D}_3 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 1 & 1 \\ 0 & 1 & 2 \\ \hline 1 & 1 & 0 \end{array}\right]$$

For illustrative purposes, the input $\boldsymbol{u}$ was modelled as a zero-mean normally distributed random vector with variance $\sigma_u = 1$. Also, the measurements $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{y}}$ were corrupted by zero-mean normally distributed random vectors $\boldsymbol{w}_k$ and $\boldsymbol{v}_k$ with $\sigma = 1 \times 10^{-2}$. With $j = 1000$ and $i = 4$, the following models of the system matrices were produced:

$$\left[\begin{array}{c|c} \hat{\mathbf{A}}_1 & \hat{\mathbf{B}}_1 \\ \hline \hat{\mathbf{C}}_1 & \hat{\mathbf{D}}_1 \end{array}\right] = \left[\begin{array}{cc|c} 2{\cdot}2929 & -0{\cdot}5419 & 0{\cdot}4014 \\ 0{\cdot}0005 & 1{\cdot}0196 & 0{\cdot}2478 \\ \hline -2{\cdot}7590 & 0{\cdot}5769 & -0{\cdot}0017 \end{array}\right]$$

$$\left[\begin{array}{c|c} \hat{\mathbf{A}}_2 & \hat{\mathbf{B}}_2 \\ \hline \hat{\mathbf{C}}_2 & \hat{\mathbf{D}}_2 \end{array}\right] = \left[\begin{array}{cc|c} 0{\cdot}1780 & -2{\cdot}0724 & 0{\cdot}1547 \\ -0{\cdot}0036 & -0{\cdot}6509 & -0{\cdot}6277 \\ \hline -2{\cdot}5218 & 4{\cdot}0787 & 0{\cdot}0008 \end{array}\right]$$

$$\left[\begin{array}{c|c} \hat{\mathbf{A}}_3 & \hat{\mathbf{B}}_3 \\ \hline \hat{\mathbf{C}}_3 & \hat{\mathbf{D}}_3 \end{array}\right] = \left[\begin{array}{cc|c} 2{\cdot}3637 & 1{\cdot}0136 & -0{\cdot}1724 \\ 0{\cdot}0077 & -0{\cdot}7007 & 0{\cdot}9046 \\ \hline -2{\cdot}3677 & -2{\cdot}1765 & -0{\cdot}0007 \end{array}\right]$$

The dynamics of any regular periodic linear system may be adequately described by the eigenvalues of its period map $\Pi = \mathbf{A}_p \ldots \mathbf{A}_2\mathbf{A}_1$. The eigenvalues of the period map of the underlying system in this example are

$$\Lambda(\Pi) = \left\{\frac{4}{5}, \frac{3}{5}\right\}$$

The eigenvalues of the period map $\hat{\Pi} = \hat{\mathbf{A}}_3\hat{\mathbf{A}}_2\hat{\mathbf{A}}_1$ of the identified system are

$$\Lambda(\hat{\Pi}) = \{0{\cdot}8000, 0{\cdot}5999\}$$

The relative mean-square error ε in the identified eigenvalues of the period map is $\varepsilon = 1{\cdot}186 \times 10^{-4}$. Note that the estimates of the throughput terms are nearly zero, as expected. In fact, the largest throughput term, defined as

$$\mathbf{D}_{\max} = \max(|\mathbf{D}_1|, |\mathbf{D}_2|, |\mathbf{D}_3|)$$

is $\mathbf{D}_{\max} = 1{\cdot}670 \times 10^{-3}$. This experiment was repeated for different values of σ, yielding the values in Table 2.

Example 2: In this example, a periodic system was constructed with two states as above, with the exception that the poles vary sinusoidally in time, albeit slowly relative to the system dynamics. The input $\boldsymbol{u}$ was again a zero-mean normally distributed random vector with variance $\sigma_u = 1$. The input and output

σ	λ_1	λ_2	ε	$\mathbf{D}_{max}$
1×10^{-8}	0·8000	0·6000	$1{\cdot}609 \times 10^{-10}$	$8{\cdot}312 \times 10^{-10}$
1×10^{-4}	0·8000	0·6000	$2{\cdot}442 \times 10^{-6}$	$2{\cdot}951 \times 10^{-5}$
1×10^{-2}	0·8001	0·5999	$1{\cdot}186 \times 10^{-4}$	$1{\cdot}670 \times 10^{-3}$
1×10^{-1}	0·8052	0·5903	$1{\cdot}010 \times 10^{-2}$	$1{\cdot}450 \times 10^{-2}$
1	0·8971	0·2987	$3{\cdot}166 \times 10^{-1}$	$7{\cdot}715 \times 10^{-2}$

Table 2. Identified system parameters.

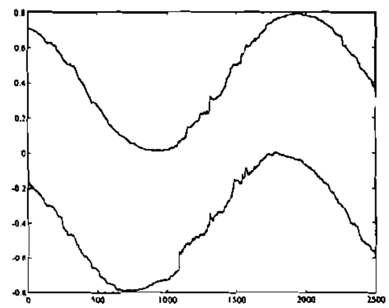

Figure 1. Slowly time-varying periodic system poles.

measurements were disturbed with zero-mean normally distributed random noise with $\sigma = 5 \times 10^{-2}$. The forgetting factor was $\alpha = 0{\cdot}995$. With 3000 I/O samples and $i = 4$, the poles of the periodic system identified by the on-line algorithm are plotted in the Figure.

7. Concluding remarks

This paper has extended the results of Moonen *et al.* (1989) so that they are applicable to the identification of system matrices of periodic linear systems. In that paper, the authors used selected, weighted left singular vectors from the singular value decomposition of a Hankel matrix formed from I/O data in order to provide a basis for a state sequence. This basis was then used to compute state-space models via a least-squares solution of an overdetermined linear system. One of the main features of that algorithm is that the right singular vectors are not used in the computation of the state-space models, reducing the noise-sensitivity and computational intensity of the algorithm.

The extension of this paper requires the representation of a periodic system of period p as a set of p period-mapped time-invariant systems. By recognizing structural relationships between the various Hankel matrices used by this algorithm, it is possible to reduce dramatically the amount of computation necessary by requiring only one large singular value decomposition. This technique also eliminates the need to use the right singular basis to produce models of the system matrices, thereby retaining one of the important features of the time-invariant algorithm.

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