### Heaven's light is our guide"

# Rajshahi University of Engineering & Technology Department of Computer Science & Engineering

**Discrete Mathematics** 

Course No.: CSE 2101

Chapter 7: Relations

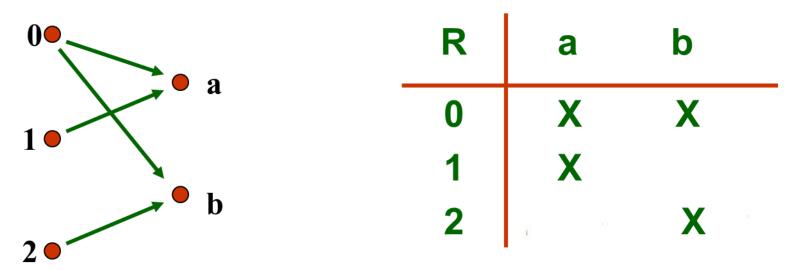
Prepared By: Julia Rahman



#### **DEFINITION 1:**

Let A and B be sets. A *binary relation* from A to B is a subset of A x B.

- ✓ In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.
- ✓ Notation:  $aRb \Leftrightarrow (a, b) \in R$  $aRb \Leftrightarrow (a, b) \notin R$
- ✓ When (a, b) belongs to R, a is said to be related to b by R.
- ✓ Binary relations represent relationships between the elements of two sets.



**EXAMPLE 2:** Let A be the set of all cities, and let B be the set of the 50 states in the United States of America. Define the relation R by specifying that (a, b) belongs to R if city a is in state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R.

#### **4** Functions as relations:

- $\checkmark$  The graph of a function f is the set of ordered pairs (a, b) such that b = f(a)
- ✓ The graph of f is a subset of A \* B  $\rightarrow$  it is a relation from A to B
- ✓ Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph
- ✓ Relations are generalization of functions

#### Definition 2:

A relation on the set A is a relation from A to A.

❖ In other words, a relation on a set A is a subset o f A x A.

**EXAMPLE 4:** Let A be the set  $\{1, 2, 3, 4\}$ . Which ordered pairs are in the relation  $R = \{(a, b) \mid a \text{ divides } b\}$ ?

**Solution:** Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

**EXAMPLE 6:** How many relations are there on a set with n elements? **Solution:** A relation on a set A is a subset of A x A. Because A x A has  $n^2$  elements when A has n elements, and a set with m elements has  $2^m$  subsets, there are  $2^{n^2}$  subsets of A x A. Thus, there are  $2^{n^2}$  relations on a set with n elements. For example, there are  $2^{3^2} = 2^9 = 5$  1 2 relations on the set  $\{a, b, c\}$ .

#### **Properties of Relations**

#### **4** Definition 3:

A relation R on a set A is called *reflexive* if  $(a, a) \in R$  for every element  $a \in A$ .  $\forall a((a, a) \in R)$ 

**EXAMPLE 7:** Consider the following relations on {1, 2, 3, 4}:

$$\begin{split} R_1 &= \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\} \\ R_2 &= \{(1,1), (1,2), (2,1)\} \\ R_3 &= \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\} \\ R_4 &= \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\} \\ R_5 &= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\} \\ R_6 &= \{(3,4)\} \end{split}$$

Which of these relations are reflexive?

**Solution:**  $R_3$  and  $R_5$ : reflexive  $\Leftarrow$  both contain all pairs of the form (a, a): (1,1), (2,2), (3,3) & (4,4).

 $R_1$ ,  $R_2$ ,  $R_4$  and  $R_6$ : not reflexive  $\Leftarrow$  not contain all of these ordered pairs. (3,3) is not in any of these relations.

#### **4** Definition 4:

A relation R on a set A is called *symmetric* if  $(b,a) \in R$  whenever  $(a,b) \in R$ , for all  $a,b \in A$ .

A relation R on a set A such that for all  $a,b \in A$ , if  $(a,b) \in R$  and  $(b,a) \in R$ , then a=b is called *antisymmetric*.

$$\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$$
  
 $\forall a \forall b(((a,b) \in R \land (b,a) \in R) \rightarrow (a=b))$ 

**EXAMPLE 10:** Consider the following relations on {1, 2, 3, 4}:

```
\begin{split} R_1 &= \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\} \\ R_2 &= \{(1,1), (1,2), (2,1)\} \\ R_3 &= \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,3), (4,4)\} \\ R_4 &= \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\} \\ R_5 &= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\} \\ R_6 &= \{(3,4)\} \end{split}
```

Which of these relations are symmetric and which are anti symmetric?

#### **Solution:**

 $R_2 \& R_3$ : symmetric  $\Leftarrow$  each case (b, a) belongs to the relation whenever (a, b) does.

For  $R_2$ : only thing to check that both (1,2) & (2,1) belong to the relation For  $R_3$ : it is necessary to check that both (1,2) & (2,1) belong to the relation. None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.

 $R_4$ ,  $R_5$  and  $R_6$ : antisymmetric  $\Leftarrow$  for each of these relations there is no pair of elements a and b with  $a \neq b$  such that both (a, b) and (b, a) belong to the relation. None of the other relations is antisymmetric.: find a pair (a, b) with  $a \neq b$  so that (a, b) and (b, a) are both in the relation.

**EXAMPLE 12:** Is the "divides" relation on the set of positive integers symmetric? Is it anti symmetric?

**Solution:** This relation is not symmetric because  $1 \mid 2$ , but  $2 \mid 1$ . It is anti symmetric, for if a and b are positive integers with a  $\mid b$  and b  $\mid a$ , then a = b.

**Definition 5:** 

A relation R on a set A is called *transitive* if whenever  $(a,b) \in R$  and  $(b,c) \in R$ , then  $(a,c) \in R$ , for all  $a,b,c \in A$ .

$$\forall a \forall b \forall c(((a,b) \in R \land (b,c) \in R) \rightarrow (a,c) \in R)$$

**EXAMPLE 15:** Is the "divides" relation on the set of positive integers transitive? **Solution:** Suppose that a divides b and b divides c. Then there are positive integers k and l such that b = ak and c = bl. Hence, c = a(kl), so a divides c. It follows that this relation is transitive.

**EXAMPLE 13:** Consider the following relations on {1, 2, 3, 4}:

```
\begin{split} R_1 &= \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\} \\ R_2 &= \{(1,1), (1,2), (2,1)\} \\ R_3 &= \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)\} \\ R_4 &= \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\} \\ R_5 &= \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\} \\ R_6 &= \{(3,4)\} \end{split}
```

Which of these relations are transitive?

#### **Solution:**

- $R_4$ ,  $R_5$  &  $R_6$ : transitive  $\leftarrow$  verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation
- $R_4$  transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1), (4,1) and (4,2) belong to  $R_4$ . Same reasoning for  $R_5$  and  $R_6$ .
- $R_1$ : not transitive  $\Leftarrow$  (3,4) and (4,1) belong to  $R_1$ , but (3,1) does not.
- $R_2$ : not transitive  $\Leftarrow$  (2,1) and (1,2) belong to  $R_2$ , but (2,2) does not.
- $R_3$ : not transitive  $\Leftarrow$  (4,1) and (1,2) belong to  $R_3$ , but (4,2) does not.

#### **Combining Relations**

- ✓ Two relations can be combined in any way two sets can be combined.
- **Example:** Let  $A = \{1, 2, 3\}$  and  $B = \{1, 2, 3, 4, \}$ . The relations  $R_1 = \{(1,1), (2,2), (3,3)\}$  and  $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$  can be combined to obtain:  $R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$   $R_1 \cap R_2 = \{(1,1)\}$   $R_1 R_2 = \{(2,2), (3,3)\}$   $R_2 R_1 = \{(1,2), (1,3), (1,4)\}$

#### **Definition 6:**

Let R be relation from A to B, S be relation from B to C. The *composite* of R & S is the relation consisting of ordered pairs (a,c), where  $a \in A$ ,  $c \in C$ , and there exists an element  $b \in B$  such that  $(a,b) \in R$  and  $(b,c) \in S$ . (denoted by  $S \circ R$ )

- **Example 20:** What is the composite of relations R & S, where R is the relation from  $\{1, 2, 3\}$  to  $\{1, 2, 3, 4\}$  with  $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$  and S is the relation from  $\{1,2,3,4\}$  to  $\{0,1,2\}$  with  $S = \{(1,0), (2,0), (3,1), (3,2), (4,1)\}$ ?
- **Solution:** S∘R is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs (2, 3) in R and (3, 1) in S

### 5.3 Permutations and Combinations

S produce the ordered pair (2, 1) in S $\circ$ R. Computing all the ordered pairs in the composite, we find

S o R = 
$$\{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$$
.

#### Definition 7:

Let R be a relation on the set A. The *powers*  $R^n$ , n=1,2,3,..., are defined recursively by  $R^1=R$ , and  $R^{n+1}=R^n \circ R$ .

The definition shows that  $R^2 = R \circ R$ ,  $R^3 = R^2 \circ R = (R \circ R)$  o R, and so on.

**Example 22:** Let  $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$ . Find the powers  $R^n$ , n = 2, 3 4,...

**Solution:** Because  $R^2 = R$  o R, we find that  $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$ . Furthermore, because  $R^3 = R^2$  o R,  $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$ . Additional computation shows that  $R^4$  is the same as  $R^3$ , so  $R^4 = (1, 1), (2, 1), (3, 1), (4, 1)\}$ . It also follows that  $R^n = R^3$  for n = 5, 6, 7, . . . . The reader should verify this.

**Theorem 1:** The relation R on a set A is transitive if and only if  $R^n \subseteq R$  for n=1,2,3,...

- ✓ Relationship among elements of more than 2 sets often arise: n-ary relations.
- ✓ Used to represent computer databases.

#### **Definition 1:**

Let  $A_1, A_2, ..., A_n$  be sets. An *n-ary relations* on these sets is a subset of  $A_1 \times A_2 \times ... \times A_n$ . The sets  $A_1, A_2, ..., A_n$  are called the *domains* of the relation, and n is called its *degree*.

**Example 1:** Let R be the relation on N \* N \* N consisting of triples (a, b, c) where a, b, and c are integers with a<b<c. Then  $(1,2,3) \in R$ , but  $(2,4,3) \notin R$ . The degree of this relation is 3. Its domains are equal to the set of integers.

#### **Lange 19** Databases and Relations:

- ✓ **Relational database model** has been developed for information processing
- ✓ A database consists of *records*, which are n-tuples made up of *fields*
- ✓ The fields contains information such as:

  Name, Student #, Major, Grade point average of the student
- ✓ The relational database model represents a database of records or n-ary relation
- ✓ The relation is R(Student-Name, Id-number, Major, GPA)
- ✓ Relations used to represent databases are called *tables*, each column of the table corresponds to an attribute of the database.

  Julia Rahman, Dept. CSE, RUET

13

- ✓ A domain of an n -ary relation is called a *primary key* when the value of the n -tuple from this domain determines the n -tuple. That is, a domain is a primary key when no two n -tuples in the relation have the same value from this domain.
- ✓ Records are often added to or deleted from databases. Because of this, the property that a domain is a primary key is time-dependent.
- ✓ Consequently, a primary key should be chosen that remains one whenever the database is changed.
- ✓ The current collection of n -tuples in a relation is called the *extension* of the relation.
- ✓ The more permanent part of a database, including the name and attributes of the database, is called its *intension*.
- ✓ Combinations of domains can also uniquely identify n -tuples in an n-ary relation.
- ✓ When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a *composite key*.

#### TABLE A: Students

Students Names	ID#	Major	GPA
Smith	3214	Mathematics	3.9
Stevens	1412	Computer Science	4.0
Rao	6633	Physics	3.5
Adams	1320	Biology	3.0
Lee	1030	Computer Science	3.7

#### **Operations on n-ary Relations**

**4** There are varieties of operations that are applied on n-ary relations in order to create new relations that answer eventual queries of a database.

#### Definition 2:

Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator SC maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

**Example 7:** To find the records of computer science majors in the n-ary relation R shown in Table 1, we use the operator  $S_{C_1}$  where  $C_1$  is the condition Major = "Computer Science." The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49). Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator  $S_{C_2}$ , where  $C_2$  is the condition GPA >3.5. The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics, 3.90). Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator  $S_{C_3}$ where  $C_3$  is the condition (Major = "Computer Science" /\ GPA > 3.5). The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88). 16 Julia Rahman, Dept. CSE, RUET

#### Definition 3:

The projection  $P_{i_1 i_2 \dots i_m}$  where  $i_1 < i_2 < \dots < i_m$ , maps the n-tuple  $(a_1, a_2, \dots, a_n)$  to the m-tuple  $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ , where m<=n.

- ❖ In other words, the projection deletes n-m of the components of n-tuple, leaving the  $i_1$ th,  $i_2$ th, ..., and  $i_m$ th components.
- **EXAMPLE 8:** What results when the projection  $P_{1,3}$  is applied to the 4-tuples (2, 3, 0, 4), (Jane Doe, 23411101, Geography, 3.14), and  $(a_1, a_2, a_3, a_4)$ ? **Solution:** The projection  $P_{1,3}$  sends these 4-tuples to (2, 0), (Jane Doe,

Geography), and  $(a_1, a_3)$ , respectively.

**Example 9:** What relation results when the projection  $P_{1,4}$  is applied to the relation in Table A?

**Solution:** When the projection P<sub>1,4</sub> is used, the second and third columns of the table are deleted, and pairs representing student names and GPA are obtained. Table B displays the results of this projection.

Students Names	GPA
Smith	3.9
Stevens	4.0
Rao	3.5
Adams	3.0
Lee	3.7

TABLE B: GPAs

#### **Definition 4:**

Let R be a relation of degree m and S a relation of degree n. The  $join J_p(R,S)$ , where  $p \le m$  and  $p \le n$ , is a relation of degree m+n-p that consists of all (m+n-p)-tuples  $(a_1,a_2,...,a_{m-p},c_1,c_2,...,c_p,b_1,b_2,...,b_{n-p})$ , where the m-tuple  $(a_1,a_2,...,a_{m-p},c_1,c_2,...,c_p,b_1,b_2,...,c_p,b_1,b_2,...,b_{n-p})$  belongs to S.

**EXAMPLE 11:** What relation results when the operator  $J_2$  is used to combine the relation displayed in tables C and D?

Professor	Dpt	Course #
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

Dpt	Course #	Room	Time
Computer Science	518	N521	2:00 PM
Mathematics	575	N502	3:00 PM
Mathematics	611	N521	4:00 PM
Physics	544	B505	4:00 PM
Psychology	501	A100	3:00 PM
Psychology	617	A110	11:00 AM
Zoology	335	A100	9:00 AM
Zoology	412	A100	8:00 AM

TABLE C: Teaching Assignments TABLE D: Class Schedule

**Solution:** The join J<sub>2</sub> produces the relation shown in Table E

Professor	Dpt	Course #	Room	Time
Cruz	Zoology	335	A100	9:00 AM
Cruz	Zoology	412	A100	8:00 AM
Farber	Psychology	501	A100	3:00 PM
Farber	Psychology	617	A110	11:00 AM
Grammer	Physics	544	B505	4:00 PM
Rosen	Computer Science	518	N521	2:00 PM
Rosen	Mathematics	575	N502	3:00 PM

Table E: Teaching Schedule