Heaven's light is our guide"

Rajshahi University of Engineering & Technology
Department of Computer Science & Engineering

Discrete Mathematics

Course No. : 305

Chapter 2: Basic Structures: Sets, Functions, Sequences and Sums

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2.1 Sets

Sets

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - ✓ Important for counting.
  - ✓ Programming languages have set operations.
- **Definition:** A ***set*** is an unordered collection of objects.

  Example: the students in this class

  the chairs in this room
- The objects in a set are called the ***elements***, or ***members*** of the set. A set is said to *contain* its elements.
  - ✓ The notation $a \in A$ denotes that a is an element of the set A.
  - ✓ If a is not a member of A, write $a \notin A$

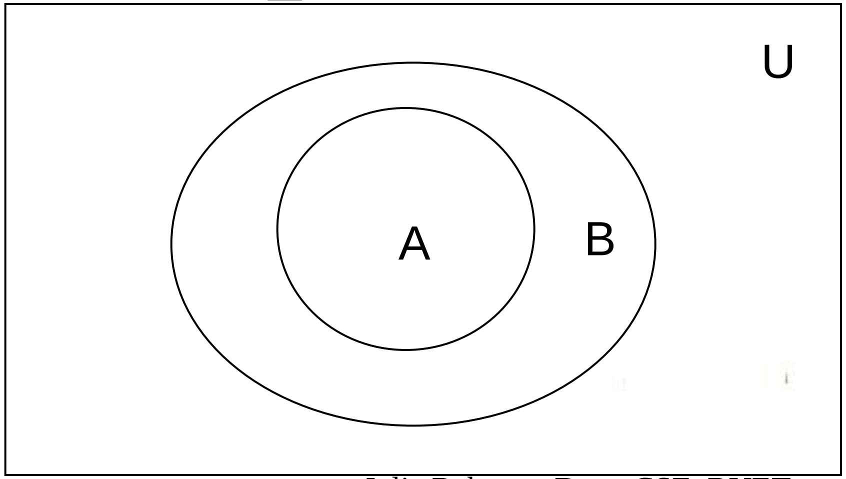
Example 1: The set V of all vowels in English alphabet can be written as {a,e,i,o,u}.

Example 2: The set O of odd positive integers less than 10 can be express by O={1,3,5,7,9}.

Sets

- **Definition 3:**
  - ✓ Two set are ***equal*** if and only if they have the same elements.
  - ✓ If A and B are sets, then A and B are equal if and only if $\forall x\ (x \in A \leftrightarrow x \in B)$
  - ✓ Example {1,3,5}={3,5,1}

- $\varnothing$ empty set (null set).
- **Singleton set:** Set contains one element.
- **Definition 4:**
  - ✓ The set A is said to be a **subset** of B if and only if every element of A is also an element of B.
  - ✓ We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.



Sets

- **Proper Subsets:**
  - ✓ If $A \subseteq B$, but $A \neq B$, then we say A is a ***proper subset*** of B, denoted by $A \subset B$.
  - ✓ If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

- **Definition 5:**
  - ✓ If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is ***finite*.**

    Otherwise it is ***infinite*.**
  - ✓ The ***cardinality*** of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.
  - ✓ **Examples**:
    1. $|\varnothing| = 0$
    2. Let S be the letters of the English alphabet. Then $|S| = 26$
    3. $|\{1,2,3\}| = 3$
    4. $|\{\varnothing\}| = 1$
    5. The set of integers is infinite

Sets

- **Definition 7:**
  - ✓ The set of all subsets of a set A, denoted P**(**A**)**, is called the ***power set*** of A.
  - ✓ **Example**: If A = {a,b} then

    P(A) = {ø, {a},{b},{a,b}}
  - ✓ If a set has n elements, then the cardinality of the power set is 2^n.

Example 13: What is the power set of the set {0, 1, 2}?

Solution: S={0,1,2}. P(S) = {∅,{0}, {1}, {2},{0,1}, {0,2}, {1,2},{0,1,2}}

- **Tuples:**
  - ✓ The ***ordered n-tuple*** $(a_1,a_2,\ldots..,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
  - ✓ Two n-tuples are ***equal*** if and only if their corresponding elements are equal.
  - ✓ 2-tuples are called ***ordered pairs***.
  - ✓ The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
  - ✓ In other words, $(a_1,a_2,\ldots,a_n)=(b_1,b_2,\ldots,b_n)$ if and only if $a_i=b_i$ for $i=1,2,\ldots,n$.

Sets

- **Cartesian Products:**
  - ✓ The ***Cartesian Product*** of two sets *A* and *B*, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \land b \in B\}$$

  - ✓ Note that A×B ≠ B×A in general. A×B = B×A if and only if A=∅, B = ∅, or A=B.

Example: Let A={1,2} and B={a,b,c}. Then A×B= {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}.

- **Definition:**
  - ✓ A subset R of the Cartesian product of A×B is a ***relation*** from the set A to the set B.
  - ✓ For example, R= {(1,a), (1,c),(2,a),(2,b),(2,c)} is a relation from A={1,2} to B={a,b,c}.

Sets

- **Definition 10:**

The ***Cartesian product*** of the sets A_1, A_2, …,A_n, denoted by $A_1 \times A_2 \ldots \times A_n$ is the set of ordered n-tuples (a_1,a_2,…,a_n), where a_i belongs to A_i for i=1,2,…,n.

$$A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example 18:

What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution:

$A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),$
$(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)\}$

2.2 Set operations

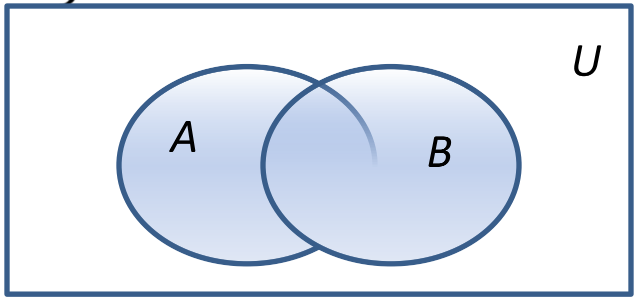
Set Operations

- **Definition 1:**

  Let A and B be two sets. The ***union*** of the sets A and B, denoted by A∪B, is the set that contains those elements that are either in A or in B.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

Example 1: What is {1,2,3} ∪ {3, 4, 5}?
Solution: {1,2,3,4,5}

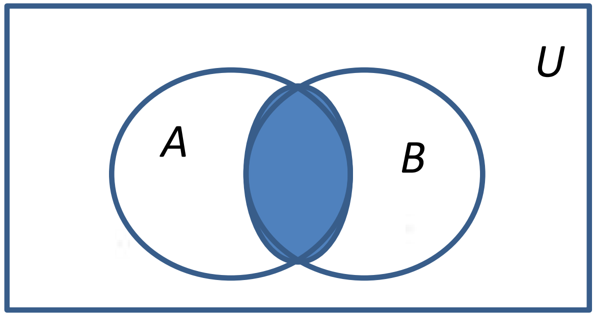


Venn Diagram for $A \cup B$

- **Definition 2:**

  Let A and B be two sets. The ***intersection*** of the sets A and B, denoted by A∩B, is the set that contains those elements that are in both A and B.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$



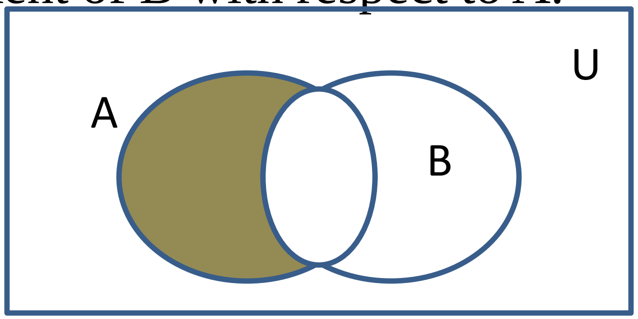
- **Definition 3:** If the intersection is empty, then *A* and *B* are said to be ***disjoint***.

Example 3: What is {1,2,3} ∩ {3,4,5}?
Solution: {3}
Example 5: What is {1,2,3} ∩ {4,5,6}?
Solution: ∅

Set Operations

- $|A \cup B| = |A| + |B| - |A \cap B|$
  The generalization of the above result is called the ***principle of inclusion-exclusion.***
- **Definition 10:**
  Let A and B be two sets. The ***difference*** of the sets A and B, denoted by A-B, is the set that contains those elements that are in A but not in B. The difference of the sets A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



Venn Diagram for $A - B$

Example 6: {1,3,5} -{1,2,3}={5}.
{1,2,3} -{1,3,5}={2}.

Set Operations

- **Definition 5:**

  Let U be the universal set. The complement of the set A, denoted by A, the ***complement of A with respect to U***. In other words, the complement of A is U-A.

$$\bar{A} = \{x \in U \mid x \notin A\}$$

Example : If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$?

Solution: $\{x \mid x \leq 70\}$

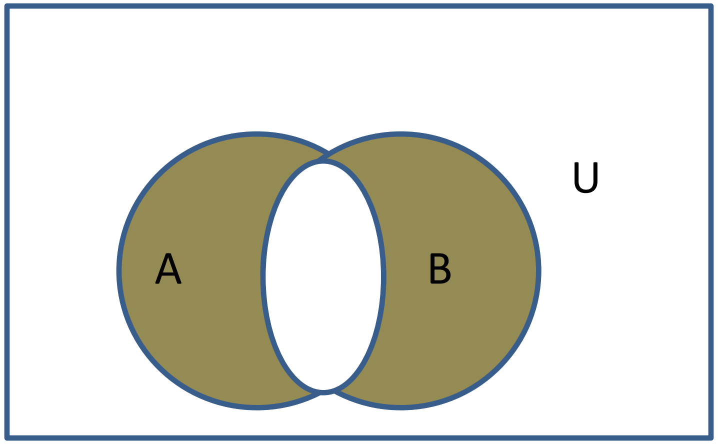
Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$
$A = \{1,2,3,4,5\}$,
$B = \{4,5,6,7,8\}$

1. $A \cup B$ **Solution:** $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$ **Solution:** $\{4,5\}$
3. $\bar{A}$ **Solution:** $\{0,6,7,8,9,10\}$
4. B^c **Solution:** $\{0,1,2,3,9,10\}$
5. $A - B$ **Solution:** $\{1,2,3\}$
6. $B - A$ **Solution:** $\{6,7,8\}$

Set Operations

- **Definition 10:**

  The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set $(A - B) \cup (B - A)$



Example:

What is the output ,U = {0,1,2,3,4,5,6,7,8,9,10} A = {1,2,3,4,5} B ={4,5,6,7,8}

Solution:

{1,2,3,6,7,8}

Set Operations

Identity	Name
$A \cup \emptyset = A$ and $A \cap U = A$	Identity laws
$A \cup U = U$ and $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ and $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ and $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ and $A \cap \overline{A} = \emptyset$	Complement laws

Set Operations

- **Proving Set Identities:**

  Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and 2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

Set Operations

These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$$
\begin{array}{ll}
x \in \overline{A} \cup \overline{B} & \text{by assumption} \\
(x \in \overline{A}) \vee (x \in \overline{B}) & \text{defn. of union} \\
(x \notin A) \vee (x \notin B) & \text{defn. of complement} \\
\neg(x \in A) \vee \neg(x \in B) & \text{defn. of negation} \\
\neg((x \in A) \wedge (x \in B)) & \text{by 1st De Morgan Law for Prop Logic} \\
\neg(x \in A \cap B) & \text{defn. of intersection} \\
x \in \overline{A \cap B} & \text{defn. of complement}
\end{array}
$$

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x \mid x \notin A \cap B\} & \text{by defn. of complement} \\
& = & \{x \mid \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
& = & \{x \mid \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
& = & \{x \mid \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
& & & \text{for Prop Logic} \\
& = & \{x \mid x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
& = & \{x \mid x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
& = & \{x \mid x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
& = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Set Operations

Membership Table:

Example : Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Set Operations

- **Generalized Unions and Intersections:**
  - Let A1, A2 ,…., An be an indexed collection of sets.

    We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

  - For i = 1,2,…, let Ai = {i, i + 1, i + 2, ….}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

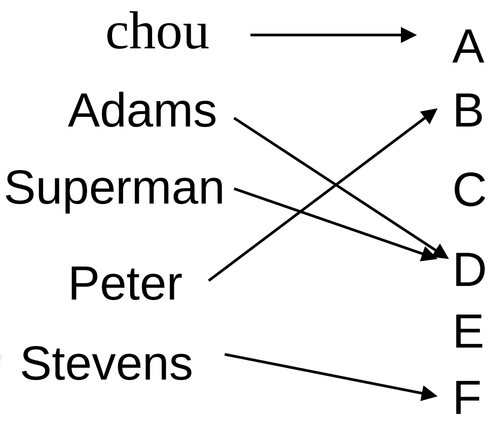
$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

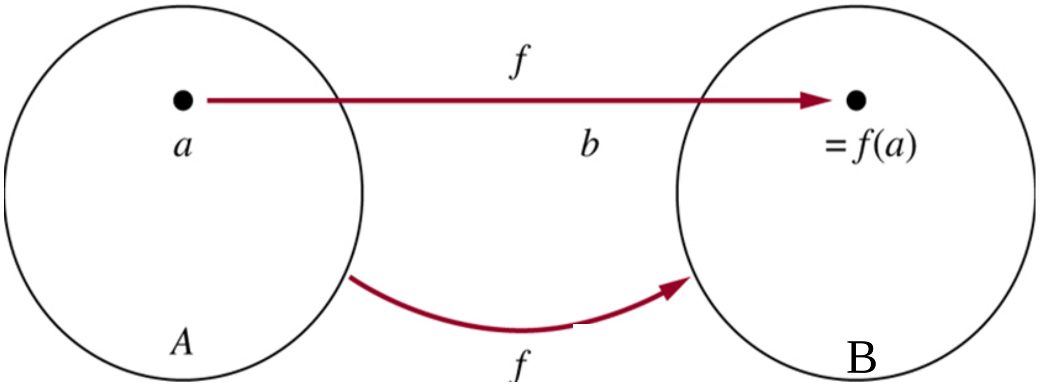
2.3 Functions

Functions

- **Definitions 1:**
  - Let A and B be nonempty sets. A ***function*** from A to B is an assignment of exactly one element of B to each element of A. We write f(a)=b if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f:A \rightarrow B$.
  - Functions are sometimes called ***mappings*** or ***transformations***.
- **Definitions 2:**

  If f is a function from A to B, we say that A is the ***domain*** of f and B is the ***codomain*** of f. If f(a)=b, we say that b is the ***image*** of a, and a is a ***preimage*** of b. The ***range*** of f is the set of all images of A. Also, if f is a function from A to B, we say that f ***maps*** A to B.





Functions

- **Example 3:** Let f be a function that assigns the last two bits of a bit string of length 2 or greater to that string. Foe example, f(100001)=01. Then the domain of f the set of all bit strings of length 2 or greater, and both the codomain and range are the set {00,01,10,11}.

- **Example 4:** Let f:Z→Z assign the square of an integer to this integer. Then $f(x) = x^2$ where the domain of f is the set of integers, we take the codmain of f to be the domain of f is the set of integers, and the range of f is the set of integers that are perfect squares, namely, {0,1,4,9,…}.

- **Real-valued functions:**

  Let f1 and f2 be functions from A to R. Then f1+f2 and f1f2 are the functions from A to R defined by

  $$(f1+f2)(x)=f1(x)+f2(x)$$
  $$(f1f2)(x)=f1(x)f2(x)$$

- **Example 6:** Let f1 and f2 be functions from R to R such that $f1(X)= X^2$ and $f2(X)=X - X^2$. f1+f2=? And f1f2=?

  Solution: (f1+f2)(X)= f1(X)+f2(X) = X and $(f1f2)(X) = X^3 – X^4$

Functions

- **Definitions 4:**

  Let f be a function from A to B and S be a subset of A. The ***image*** of S under the function f is the subset of B that consists of the images of elements of S. We denote the image of S by f(S).

Example 7: Let A={a,b,c,d,e} and B={1,2,3,4} with f(a)=2, f(b)=1, f(c)=4, f(d)=1, and f(e)=1. The image of the set S={b,c,d} is the set f(S)={1,4}.

- **Definitions 5:**
  - ✓ A function f is said to be ***one-to-one,*** or ***injective***, if and only if f(a)=f(b) implies a=b for all a and b in the domain of f. A function is said to be an ***injection*** if it is one-to-one.
  - ✓ Note that f is one-to-one if and only if f(a)≠f(b) whenever s a ≠ b.

Example 8: Determine whether the function f from {a,b,c,d} to {1,2,3,4,5} with f(a)=4, f(b)=5, f(c)=1, and f(d)=3 is one-to-one.

Solution: Function f is one to one because f takes on different values at the four elements of its domain.

Functions

Example 9: Determine whether the function $f(x)=X^2$ from the set of integers to the set of integers is one-to-one.
Solution: The function $f(x)=X^2$ is not one to one because for instance, $f(1)=1$ and $f(-1)= 1$, but $1\neq -1$.

Example 10: Determine whether the function $f(x)=x+1$ from the set of real numbers to itself is one-to-one.
Solution: The function $f(x)= x+1$ is a one to one.to demonstrate this, note that $x+1 \neq y+1$ when $x\neq -y$.

- **Definitions 6:**

  A function f which domain and codomain are the set of real numbers is called ***increasing*** if $f(x)\leq f(y)$, and ***strictly increasing*** if $f(x)<f(y)$, whenever $x<y$, and x and y are in the domain of f. Similarly, called ***decreasing*** if $f(x)\geq f(y)$, and ***strictly increasing*** if $f(x)>f(y)$, whenever $x<y$, and x and y are in the domain of f.

Functions

- **Definitions 7:**

  A function f from A to B is called ***onto***, or ***surjective*** if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$. A function f is called a ***surjection*** if it is onto.

Example 11: Determine whether the function f from {a,b,c,d} to {1,2,3} with f(a)=3, f(b)=2, f(c)=1, and f(d)=3 is onto.

Solution: Because all three elements of the codomain are images of elements of domain, so it is onto.

Example 12: Determine whether the function $f(x)= X^2$ from the set of integers to the set of integers is onto.

Solution: The function $f(x)=X^2$ is not onto because there is no integer x with $X^2 = -1$ for instance.

Example 13: Determine whether the function f(x)=x+1 from the set of real numbers to itself is onto.

Solution: The function f(x)= x+1 is a onto because for every integer y there is an integer x, such that f(x) = y.

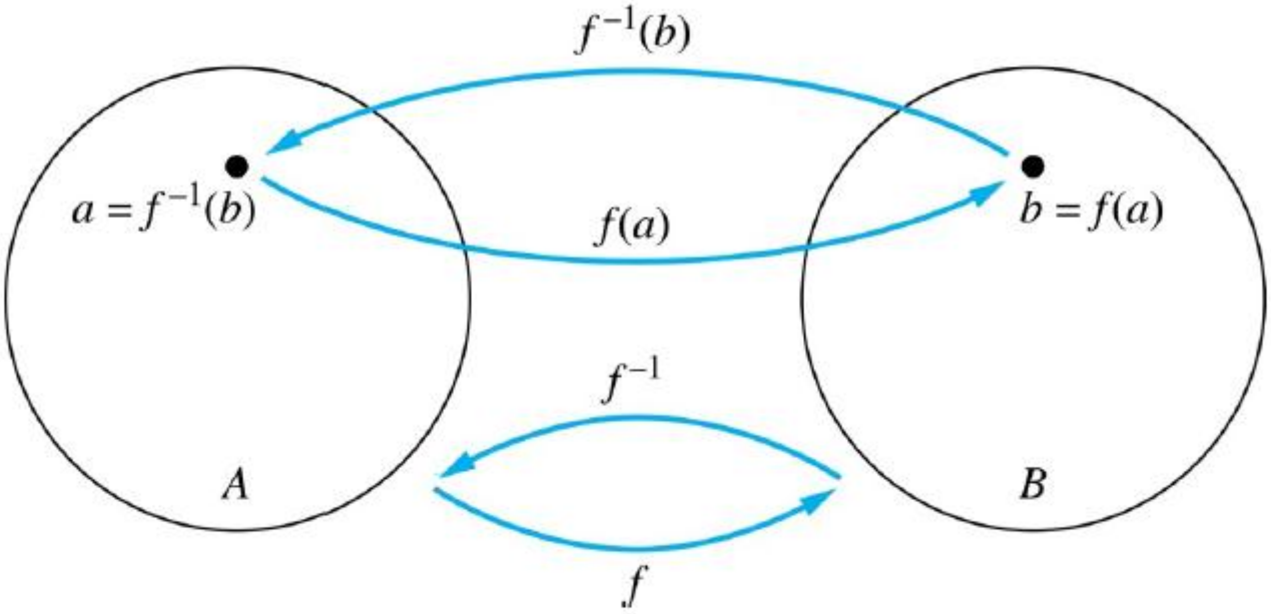
Functions

- **Definitions 8:**
  The function f is a ***one-to-one correspondence***, or ***bijection***, if it is both one-to-one and onto.

Example 14: Determine whether the function f from {a,b,c,d} to {1,2,3,4} with f(a)=4, f(b)=2, f(c)=1, and f(d)=3 is a bijection.
Solution: Function f is one to one and onto. Function f is one to one because f takes on different values at the four elements of its domain. Three elements of the codomain are images of elements of domain, so it is onto. Hence it is bijection.
Example 15: Let A be a set. The identity function on A is the function $i_A: A \to A$, where $i_A(x)=x$ for all $x \to A$. The function i_A is a bijection.

Functions

- **Definitions 9:**
  - Let f be a one-to-one correspondence from the set A to the set B. The ***inverse function*** of f is the function that assigns to an element b belonging to B the unique element a in A such that f(a)=b. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b)=a$ when f(a)=b.
  - A one-to-one correspondence is called ***invertible*** because we can define an inverse of the function. A function is ***not invertible*** if it is not invertible.
  - If f is not a bijection then the inverse does not exist.



Functions

Example 16: Let f be the function from {a,b,c} to {1,2,3} with f(a)=3, f(b)=2, and f(c)=1. Is f invertible, and if f is invertible, what is its inverse?

Solution: Function f is invertible because it is one to one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$ and $f^{-1}(3) = b$.

Example 17: Let $f:Z \rightarrow Z$ be such that $f(x)=x+1$. Is f invertible, and if f is invertible, what is its inverse?

Solution: The function f has an inverse because it is one to one correspondence. To reverse the correspondence, suppose that y is the image of x, so that $y=x+1$. then $x=y-1$. This means that $y-1$ is the unique element of Z that is sent to y by f. Consequently, $f^{-1}(y) = y-1$.

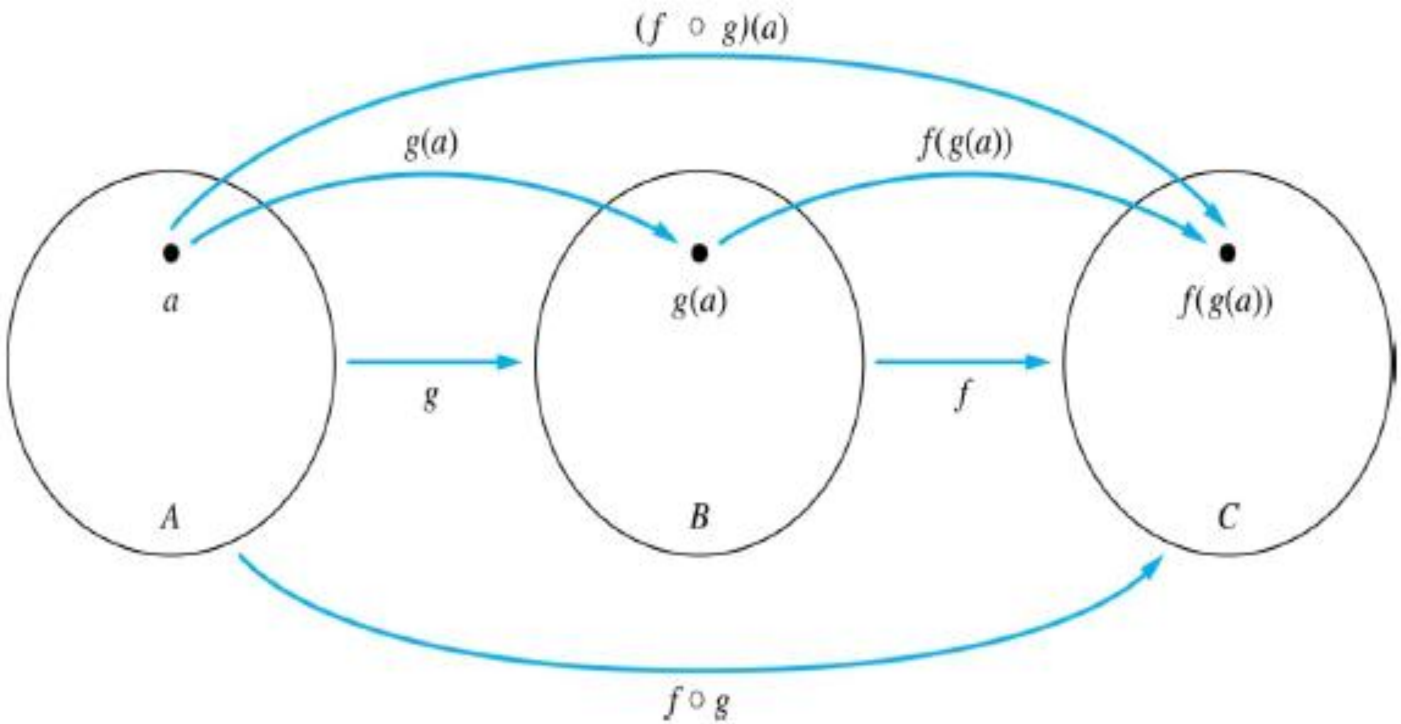
Example 18: Let f be the function from R to R with $f(x)= X^2$. Is f invertible?

Solution: f is not invertible. Because $f(2)=f(-2)=4$, f is not one to one function.

Functions

- **Definitions 10:**

  Let g be a function from the set A to the set B and f be a function from the set B to the set C. The ***compositition*** of the function f and g, denoted by f•g, is defined by (f•g)(a)=f(g(a)).



Functions

Example 20: Let g be the function from the set {a,b,c} to itself such that g(a)=b, g(b)=c, and g(c)= a. Let f be the be the function from the set {a,b,c} to {1,2,3} such that f(a)=3, f(b)=2, and f(c)= 1. What is the composition of f and g, and what is the composition of g and f?

Solution: Then $(f \bullet g)(a)=2$, $(f \bullet g)(b)=1$, and $(f \bullet g)(c)=3$. But $g \bullet f$ is not defined. Because the range of f is not a subset of the domain of g.

Example 21: Let f and g be the functions from the set of integers to the set of integers defined by $f(x)=2x+3$ and $g(x)=3x+2$. What is the composition of f and g, and what is the composition of g and f?

Solution: $(f \bullet g)(x)=f(g(x))=f(3x+2)=2(3x+2)+3=6x+7$.

And

$(g \bullet f)(x)=g(f(x))=g(2x+3)=3(2x+3)+2=6x+11$

Example 18: Let f be the function from R to R with $f(x)= X^2$. Is f invertible?

Solution: f is not invertible. Because $f(2)=f(-2)=4$, f is not one to one function.

2.4 Sequences and Summations

Sequences and Summations

- **Definitions 1:**

  A ***sequence*** is a function from the set of integers (usually either the set {0,1,2,…} or the set {1,2,3,…}) to a set S. We use the notation an to denote the images of the integer n. We call an a ***term*** of the sequence.

Example: a_n =1/n for n=1,2,…

(1, 1/2, 1/3,1/4,…)

- **Definitions 2:**

  A ***geometric progression*** is a sequence of the form a, ar, ar^2, …,ar^n,… where the ***initial term*** a and the ***common ratio*** r are real numbers.

Example: 2,10,50,250,1250,…

- **Definitions 3:**

  An ***arithmetic progression*** is a sequence of the form a, a+d, a+2d,…,a+nd,… where the ***initial*** term a and the ***common difference*** d are real numbers.

Example: -1,3,7,11,…

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- **Definitions :**

  The ***string*** is a finite sequence of bits denoted by $a_1, a_2 \ldots\ldots a_n$. The length of the string S is the number of terms in this string. The ***empty*** string, denoted by λ, is the string that has no term.

- **Summations :**

  We use the notation to denote

  $$\sum_{j=m}^{n} a_j, \quad \sum_{j=m}^{n} a_j, \text{ or } \sum_{m \le j \le n} a_j$$

  to repressent $a_m + a_{m+1} + \ldots + a_n$.

  Here the variable j is called the index of summation.

  m is the lower limit, and n us the upperlimit.

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Example:

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$$\sum_{k=4}^{8} (-1)^k = 1$$

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6$$

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- **Some useful summation formula:**

$$\sum_{k=0}^{n} ar^{k} = \frac{ar^{n+1} - a}{r-1}, \quad r \neq 1$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^{n} k^{2} = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^{3} = \frac{n^{2}(n+1)^{2}}{4}$$

$$\sum_{k=0}^{\infty} x^{k} = \frac{1}{1-x}, \quad |x| < 1$$

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^{2}}, \quad |x| < 1$$

Example :

$$\sum_{k=50}^{100} k^{2} = \sum_{k=1}^{100} k^{2} - \sum_{k=1}^{49} k^{2} = \frac{100 \bullet 101 \bullet 201}{6} - \frac{49 \bullet 50 \bullet 99}{6}$$

$$= 297925$$