THE PIGEONHOLE PRINCIPLE

If k is a positive integer and k + 1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof: The pigeonhole principle will be proven using a proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least k + 1 objects.

Example 1: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example 2: In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

Example 3: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

Generalized pigeonhole principle

Theorem 2: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N / k \rceil$ objects.

Proof: We will use a proof by contradiction. Suppose that none of the boxes contains more than $\lceil N / k \rceil - 1$ objects. Then, the total number of objects is at most

$$k \left(\left\lceil \frac{N}{k} \right\rceil - 1 \right) < k \left(\left(\frac{N}{k} + 1 \right) - 1 \right) = N$$

where the inequality $\lceil N / k \rceil < (N / k) + 1$ has been used. This is a contradiction because there are a total of N objects.