$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

$$1-1+1-1+1........=?$$

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4....}}}}$$

Discrete mathematics

The Foundations: Logic and Proofs

$$\exists_{x\in\Re}\exists_{y\in\Re}(x=y)$$

$$\forall_x(\Re\,|\,x)$$

$$\sum_{x=1}^{\infty}\frac{1}{x}=?$$

$$\sum_{x=1}^{\infty}x=?$$

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Chapter Summary

- Propositional Logic
  - Propositional Logic (1.1) - The Language of Propositions
  - Applications (1.2)
  - Propositional Equivalences (1.3)
- Predicate Logic
  - Predicates and Quantifiers (1.4)
  - Nested Quantifiers (1.5)
- Proofs
  - Rules of Inference (1.6)
  - Introduction to Proofs (1.7)
  - Proof Methods and Strategy(1.8)

Propositional Logic

Section 1.1

Section Summary

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth Tables

Propositions

- A **proposition** is a declarative sentence that is either true or false.
- Examples of propositions:
  a) The Moon is made of green cheese.
  b) Dhaka is the capital of Bangladesh.
  c) Toronto is the capital of Canada.
  d) $1 + 0 = 1$
  e) $0 + 0 = 2$
- Examples that are not propositions.
  a) Sit down!
  b) What time is it?
  c) $x + 1 = 2$
  d) $x + y = z$

Propositional Logic

- Constructing Propositions
  - Propositional Variables: p, q, r, s, …
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
  - Compound Propositions; constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$

Negation

– The **negation** of a proposition p is denoted by $\neg p$ and has this truth table:

p	**¬p**
T	F
F	T

- **Example**: If p denotes "The earth is round.", then $\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction

- The **conjunction** of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	p ∧ q
T	T	**T**
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes "I am at home and it is raining."

Disjunction

- The **disjunction** of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	p ∨q
T	T	T
T	F	T
F	T	T
F	F	**F**

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \vee q$ denotes "I am at home or it is raining."

The Connective Or in English

- In English "or" has two distinct meanings.
  - **"Inclusive Or"** - In the sentence **"Students who have taken CS202 or Math120 may take this class**," we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of **disjunction**. For $p \vee q$ to be true, either one or both of p and q must be true.
  - **"Exclusive Or"** - When reading the sentence **"Soup or salad comes with this entrée**," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$ **, one of p and q must be true, but not both**. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a **conditional statement** or **implication** which is read as **"if *p*, then *q*"** and has this truth table:

p	q	p →q
T	T	T
T	F	**F**
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes "If I am at home then it is raining."
- In $p \rightarrow q$, $\boldsymbol{p}$ is the **hypothesis (antecedent or premise)** and $\boldsymbol{q}$ is the **conclusion (or consequence).**

Understanding Implication

p	q	$p \rightarrow q$
T	T	T
T	F	**F**
F	T	T
F	F	T

- These implications are perfectly fine, but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates."
  - "If the moon is made of green cheese then I'm on welfare."
  - "If $1 + 1 = 3$, then your grandma wears combat boots."

Different Ways of Expressing $p \to q$

- if p, then q
- p implies q
- if p, q
- p only if q
- q unless $\neg p$
- q when p
- q if p
- q whenever p
- p is sufficient for q
- q follows from p
- q is necessary for p
- a necessary condition for p is q
- a sufficient condition for q is p

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the **biconditional** proposition $p \leftrightarrow q$, read as **"p if and only if q ."** The **biconditional** $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q \ (\approx (p \rightarrow q) \wedge (q \rightarrow p))$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:

  - **p is necessary and sufficient for q**
  - **if p then q, and conversely**
  - **p iff q**

Truth Tables For Compound Propositions

- Construct a truth table for

$$p \vee q \rightarrow \neg r$$

p	q	r	$\neg$r	p $\vee$ q	p $\vee$ q $\rightarrow$ $\neg$r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are **equivalent** if they always have the **same truth value**.
- **Example**: Show using a truth table that the conditional is equivalent to the **contrapositive**.

Solution:

					Contrapositive of $p \rightarrow q$
p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$$\boxed{p \rightarrow q \equiv \neg q \rightarrow \neg p}$$

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the **converse** nor **inverse** of an implication are not equivalent to the implication.

Solution:

					the **inverse** of $p \to q$	the **converse** of $p \to q$
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

$$p \to q \neq \neg p \to \neg q$$
$$p \to q \neq q \to p$$

Problem

- How many rows are there in a truth table with **n** propositional variables?

  Solution: $\mathbf{2^n}$ [Chapter 6].

- **Note**: that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

- $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
- If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Applications of Propositional Logic

Section 1.2

Applications of Propositional Logic: Summary

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits

Translating English Sentences

- Steps to convert an English sentence to a statement in propositional logic
  - **Identify** atomic propositions
  - **Represent** using propositional variables.
  - **Determine** appropriate logical connectives
- "If I go to Harry's or to the country, I will not go shopping."
  - p: I go to Harry's
  - q: I go to the country.
  - r: I will go shopping.

If p or q then not r.

$$(p \vee q) \rightarrow \neg r$$

Example

Problem: Translate the following sentence into propositional logic:

"**You can access the Internet from campus only if you are a computer science major or you are not a freshman.**"

One Solution: Let

a: "You can access the internet from campus,"

c: "You are a computer science major,"

f: "You are a freshman."

$$a \rightarrow (c \vee \neg f)$$

System Specifications

- System and Software engineers take requirements in English and express them in a precise specification language based on logic.

Example: Express in propositional logic:

"The automated reply cannot be sent when the file system is full"

Solution: One possible solution: Let p denote "The automated reply can be sent" and q denote "The file system is full."

$$q \rightarrow \neg p$$

Consistent System Specifications

Definition: A list of propositions is **consistent** if it is possible to assign truth values to the **proposition variables** so that each proposition is true.

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."

Solution: Let

p: "The diagnostic message is stored in the buffer."

q: "The diagnostic message is retransmitted"

The specification can be written as:

- $p \vee q$
- $\neg p$,
- $p \rightarrow q$.

p	q	$p \vee q$	$\neg p$	$p \rightarrow q$
T	T	T	F	T
T	F	T	F	F
F	T	T	T	T
F	F	F	T	T

When p is false and q is true all three statements are true. **So the specification is consistent**.

Consistent System Specifications

Exercise: Are these specifications consistent?

- "The diagnostic message is stored in the buffer or it is retransmitted."
- "The diagnostic message is not stored in the buffer."
- "If the diagnostic message is stored in the buffer, then it is retransmitted."
- "The diagnostic message is not retransmitted is added."

Solution: Let

p: "The diagnostic message is stored in the buffer."

q: "The diagnostic message is retransmitted"

The specification can be written as:

- ❑ $p \lor q$
- ❑ $\neg p$,
- ❑ $p \to q$.
- ❑ $\neg q$

p	q	$p \lor q$	$\neg p$	$p \to q$	$\neg q$
T	T	T	F	T	F
T	F	T	F	F	T
F	T	T	T	T	F
F	F	F	T	T	T

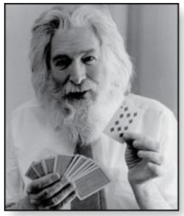
When p is false and q is true all three statements are true. **So the specification is not consistent**.

Boolean Searches

- Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages.
- Because these searches employ techniques from propositional logic, they are called **Boolean searches**.

Logic Puzzles

- An island has two kinds of inhabitants, **knights**, who always tell the truth, and **knaves**, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."



Raymond Smullyan (Born 1919)

Example: What are the types of A and B?

Solution: Let

p: A is a knight
q: B is a knight,
¬p: A is a knave
¬q: B is a knave.

- If A is a knight, then *p* is true.
  - **1st Line:** Since knights tell the truth, **q: B is a knight** must also be true.
  - **2nd Line**: Since knights tell the truth, $(p \land \neg q) \lor (\neg p \land q)$: **the two of us are of opposite types** would have to be true, but it is not.
  - So, A is not a knight and therefore ¬p must be true.
- If A is a knave,
  - **1st Line:** Since knaves always lie, ¬q or B must not be a knight.
  - **2nd Line:** Since knaves always lie, $(p \land \neg q) \lor (\neg p \land q)$: **the two of us are of opposite types** would have to be **false**.
  - both $\neg p$ and $\neg q$ hold.

Logic Puzzles

- An island has two kinds of inhabitants, **knights**, who always tell the truth, and **knaves**, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

Example: What are the types of A and B?

Solution:

A	B	Truth values of statements	possible?
Knight	Knave	Knight says "B is a knight." (F) Knave says "The two of us are of opposite types."(F)	No
Knight	Knight	Knight says "B is a knight." (T) Knight says "The two of us are of opposite types."(F)	No
Knave	Knight	Knave says "B is a knight." (F) Knight says "The two of us are of opposite types."(F)	No
Knave	Knave	Knave says "B is a knight." (T) Knave says "The two of us are of opposite types."(T)	Yes

Logic Puzzles

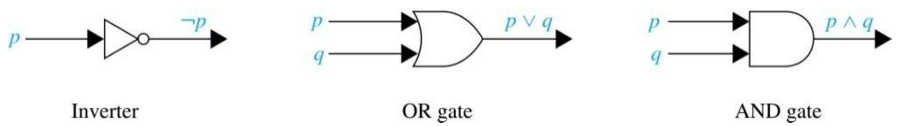
- A father tells his two children, a boy and a girl, to play in their backyard without getting dirty. However, while playing, both children get mud on their foreheads. When the children stop playing,
  - the father says "At least one of you has a muddy forehead,"
  - and then asks the children to answer "Yes" or "No" to the question: "Do you know whether you have a muddy forehead?"
- **Example:** The father asks this question twice. What will the children answer each time this question is asked, assuming that a child can see whether his or her sibling has a muddy forehead, but cannot see his or her own forehead? Assume that both children are honest and that the children answer each question simultaneously.

Solution:

Homework

Logic Circuits
(Studied in depth in Chapter 12)

- Electronic circuits; each input/output signal can be viewed as a 0 or 1.
  - 0 represents **False**
  - 1 represents **True**
- Complicated circuits are constructed from three basic circuits called gates.



  - The inverter (**NOT gate**)takes an input bit and produces the negation of that bit.
  - The **OR gate** takes two input bits and produces the value equivalent to the disjunction of the two bits.
  - The **AND gate** takes two input bits and produces the value equivalent to the conjunction of the two bits.
- More complicated digital circuits can be constructed by combining these basic circuits to produce the desired output given the input signals by building a circuit for each piece of the output expression and then combining them. For example: