① $$\mathcal{L}(sP) = \left\langle \left(\frac{\ell_i}{\ell_{i+2}}\right)^{u} \left(\frac{\ell_{i+1}}{\ell_i}\right)^{v-u} \;\middle|\; \begin{array}{l} 0 \le v \le q \\ v+1 \le 3u \\ i=1,2,3 \end{array} \quad (q+1)u - v \le s \right\rangle$$

$P = P_1 + P_2 + P_3$ place of deg 3

ℓ_i tangent at P_i.

$P_{i+1} = P_i^{Fr}$ $\quad \ell_{i+1} = \ell_i^{Fr}$

$$\mathcal{L}(sP)/\mathcal{L}((s-1)P) = \left\langle \left(\frac{\ell_i}{\ell_{i+2}}\right)^{u} \left(\frac{\ell_{i+1}}{\ell_i}\right)^{v-u} \;\middle|\; \begin{array}{l} i=1,2,3 \\ s=(q+1)u-v \end{array} \right\rangle$$

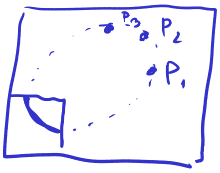
② $C_{\mathcal{L}}(D, sP)$ $\quad D = Q_1 + \ldots + Q_{q^3}$ affine rat. pl.

? $C_{\mathcal{L}}(Q_\infty + D, sP)$ (mistake in our GAP package!)

? If f taking values in $\mathbb{F}_q$ in $Q_1, \ldots, Q_{q^3} \overset{?}{\Rightarrow} f(Q_\infty) \in \mathbb{F}_q$?

(probably yes, if "degree" of f is not very high)

$AG(2, q^6)$

③ 

$AG(2, q^2)$

$Aut(\mathcal{H}_q) \cong PGU(3,q)$

the stabilizer of $P = P_1 + P_2 + P_3$ has the structure

$\langle \alpha \rangle$ cyclic subgroup of order $q^2 - q + 1$, $P_i^\alpha = P_i$.

$\langle \gamma \rangle$ —— " —— 3, $P_i^\gamma = P_{i+1}$

Prop: The basis above consists of eigenvectors of α.

Prop: We can give the known generators of $C_{\mathcal{L}}(D(+Q_\infty), sP)|_{\mathbb{F}_q}$ in terms of the basis above for $s = 2g$ and $s = 2g+1$.

④ Conjecture: $$\dim C_{\mathcal{L}}(D, sP)|_{\mathbb{F}_q} = \begin{cases} 1 & \text{for } s \le 2g-1 \\ 7 & \text{for } s = 2g \quad q > 2 \\ 10 & \text{for } s = 2g+1 \text{ and } q > 3 \end{cases}$$

We can prove:

- $\dim \ge \ldots$, we can give the generators explicitly
- $s \le \frac{2g}{3}$

⑤ $$R(X,Y) = X \prod_{\substack{c \in \mathbb{F}_{q^2} \\ c^q + c \ne 0}} (Y - c)$$ nr of poss for $c = q^2 - q$

$\deg R = q^2 - q + 1$

⊛ $R(x,y) \ne 0$ in the function field, since $X^{q+1} - Y - Y^q \nmid R(X,Y)$.

$$\mathbb{F}_{q^2}(\mathcal{H}_q) = \langle x, y \mid x^{q+1} - y - y^q = 0 \rangle$$

⊛ $R(Q_i) = 0$ for all $i = 1, \ldots, q^3$

$\mathrm{Div}(R) = Q_1 + \ldots + Q_{q^3} - q^3 P_\infty$

$$x^q R(x,y) = x^{q+1} \prod_{c^q+c\neq 0} (y-c) = x^{q+1} \frac{\prod_{c\in\mathbb{F}_{q^2}}(y-c)}{\prod_{c^q+c=0}(y-c)} = (y+y^q) \frac{y^{q^2}-y}{y^q+y} = y^{q^2}-y$$

$$\begin{aligned} x^{q^2}-x &= x\left((x^{q+1})^{q-1}-1\right) \\ &= x \prod_{\alpha\in\mathbb{F}_q^*} (x^{q+1}-\alpha) = x \prod_{\alpha\in\mathbb{F}_q^*} (y+y^q-\alpha) = x \prod_{\alpha\in\mathbb{F}_q^*} \cdot \prod_{c^q+c=\alpha} (y-c) \\ &= x \prod_{c^q+c\neq 0} (y-c) = R(x,y) \end{aligned}$$

Prop: In the function field, we have $x^q R(x,y) = y^{q^2}-y$ and $R(x,y) = x^{q^2}-x$.

(6) Trivially : $f(Q_i) \in \mathbb{F}_q$ for $i = 1,\dots, q^3+1$

$\iff$ $f^q - f = A\,(X^{q+1}-Y-Y^q) + B\cdot R(X,Y)$

for some polynomials $A(X,Y), B(X,Y)$