# X.12
# CALCULATION OF REFERENCE FRAMES ALONG A SPACE CURVE

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Three-dimensional space curves can represent the path of an object or the boundary of a surface patch. They can also participate in various free-form geometric constructions. For example, the *generalized cylinder* (a cylinder with arbitrary cross-sections along a central, space curve axis) is used in computer graphics to good effect. Establishing reference frames for the cross-sections of a generalized cylinder, or for any other geometric use, is the subject of this Gem.

We restrict the central axis to the familiar three-dimensional cubic curve, which we represent by its polynomial coefficients, the three-dimensional vectors **A**, **B**, **C**, and **D**. A point $P$ on the curve is computed according to its parametric position, $t$:

$$P = \mathbf{A}t^3 + \mathbf{B}t^2 + \mathbf{C}t + \mathbf{D}.$$

When constructing a polygonal generalized cylinder, each cross-section must be aligned properly with its neighbors so that the structure does not twist. This alignment is usually provided by a *reference frame,* a point and three orthogonal vectors that define position and orientation along the central axis of the cylinder (see Fig. 1).

One of the more intuitive reference frames is due to Frenet (see Fig. 2); the frame consists of a unit length tangent, **T**, to the central axis; a principal normal, **N**; and a binormal, **B**. **T** is computed simply as the unit length velocity vector, **V**; **V** is the derivative of the curve

$$\mathbf{V} = 3\mathbf{A}t^2 + 2\mathbf{B}t + \mathbf{C}.$$

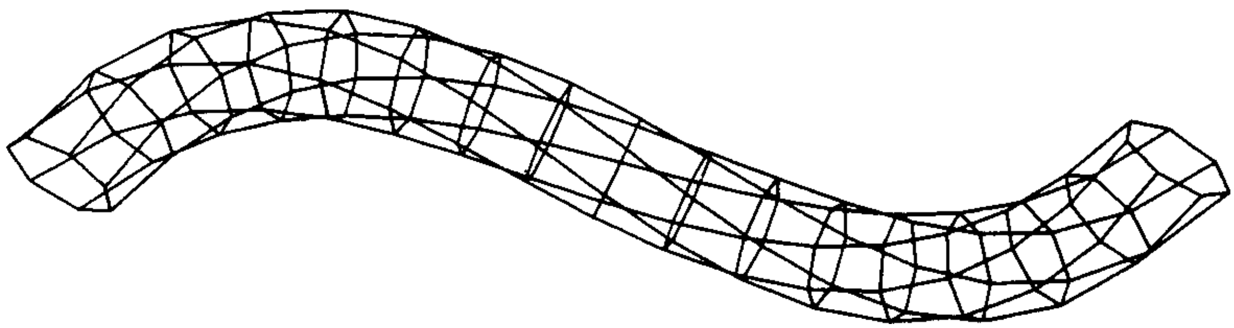

Figure 1. Polygons resulting from twisting reference frames.

The principal normal is often defined to be in the direction of curvature, $\mathbf{K} = \mathbf{V} \times \mathbf{Q} \times \mathbf{V}/|\mathbf{V}|^4$. $\mathbf{Q}$ is the acceleration of the curve, that is, the derivative of velocity, $6\mathbf{A}t + 2\mathbf{B}$. Thus,

$$\mathbf{T} = \mathbf{V}/|\mathbf{V}|, \ \mathbf{N} = \mathbf{K}/|\mathbf{K}|, \text{ and } \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

The Frenet frame is convenient because it can be analytically computed at arbitrary points along the curve. Unfortunately, it is undefined wherever the curvature is degenerate, such as at points of inflection or along straight sections of curve. Worse, the curvature vector can suddenly reverse direction on either side of an inflection point, inflicting a violent twist in a progression of Frenet frames.

This problem was discussed by Shani and Ballard (1984), who proposed an iterative solution to minimize torsion, that is, rotation around the tangent to a curve. This technique was used to compute reference frames for the tree branches in J. Bloomenthal (1985).

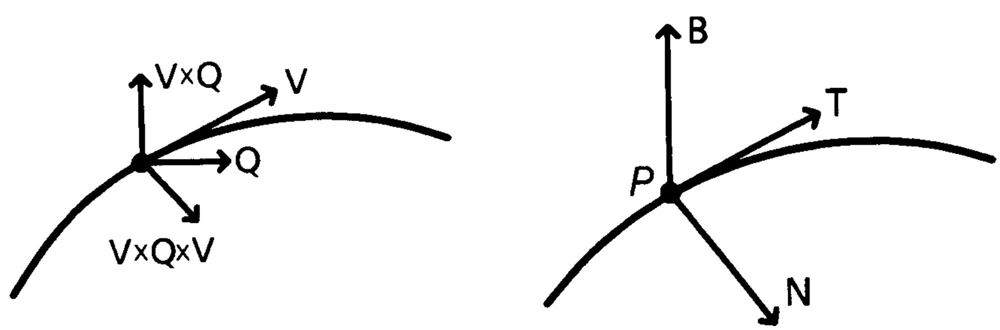


Figure 2. Curvature (left) and a Frenet frame (right).

Papers by Klok (1986) and M. Bloomenthal (1988) discuss rotation minimizing frames in some detail. They both observe that a rotation minimizing frame does not necessarily produce the intuitively desired result; in the case of a helical curve, for example, the Frenet frame appears more desirable.

The idea behind rotation minimizing frames is to define an initial reference frame at the beginning of the curve and then propagate the frame along the curve using small, local rotations. This method is immune to degeneracies in the curvature vector; it does not, unfortunately, permit analytical computation of a reference frame.

The first frame usually can be computed using curvature, as illustrated in Fig. 2. If the curvature is degenerate, then **N** can be any unit length vector perpendicular to **T**. Given the initial frame, subsequent frames are generated, *in order*, by computing $P$ and **T** at the new location on the curve. The old reference frame is then rotated such that the old **T** aligns itself with the new **T**. The rotation creates a new **N** and **B**, which, with the new $P$ and **T**, define a new reference frame. The axis of rotation is given by $\mathbf{T0} \times \mathbf{T1}$ and $\alpha = \cos^{-1}((\mathbf{T0} \cdot \mathbf{T1})/(|\mathbf{T0}|\ |\mathbf{T1}|))$. In Fig. 3, $\{P0, \mathbf{T0}, \mathbf{N0}, \mathbf{B0}\}$ becomes $\{P1, \mathbf{T1}, \mathbf{N1}, \mathbf{B1}\}$.

As the curve becomes relatively straight, the difference between **T0** and **T1** becomes small. If **T0** = **T1**, their cross-product is undefined and no axis is available to perform the rotation: this is not a problem, because the amount of rotation is zero.

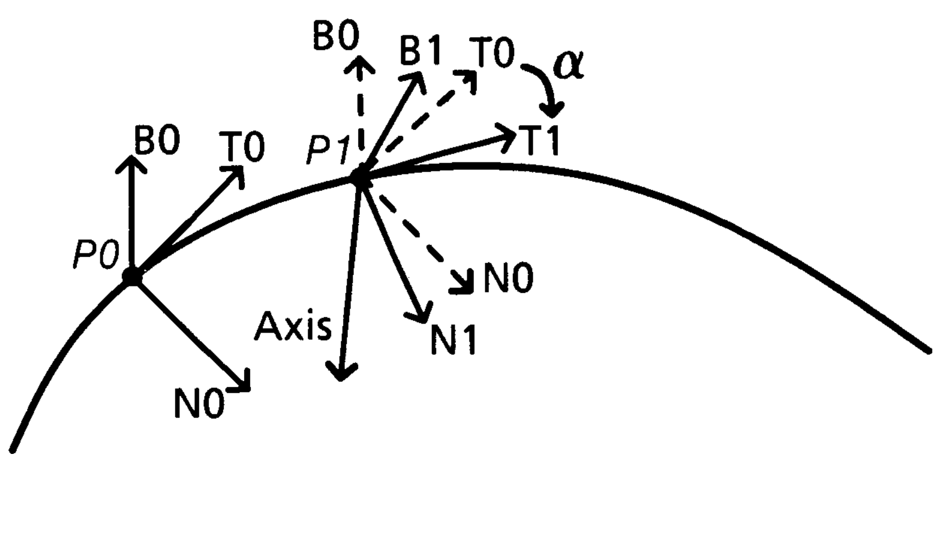


Figure 3. Computing a reference frame from the previous frame.

Although the tangent is needed to compute the reference frame, only the point $P$, normal $\mathbf{N}$, and binormal $\mathbf{B}$ are needed to transform the cross-section into the plane defined by $\mathbf{N}$ and $\mathbf{B}$.

If $(C_x, C_y)$ is on the cross-section (Fig. 4), $(P_x + C_x\mathbf{N_x} + C_y\mathbf{B_x}, P_y + C_x\mathbf{N_y} + C_y\mathbf{B_y}, P_z + C_x\mathbf{N_z} + C_y\mathbf{B_z})$ is a three-dimensional point properly positioned on the surface of the generalized cylinder. This is conveniently expressed in matrix form:

$$P_{surface} = [C_x,\ C_y,\ 1][\mathbf{M}], \text{ where } \mathbf{M} = \begin{bmatrix} \mathbf{N_x} & \mathbf{N_y} & \mathbf{N_z} \\ \mathbf{B_x} & \mathbf{B_y} & \mathbf{B_z} \\ P_x & P_y & P_z \end{bmatrix}.$$

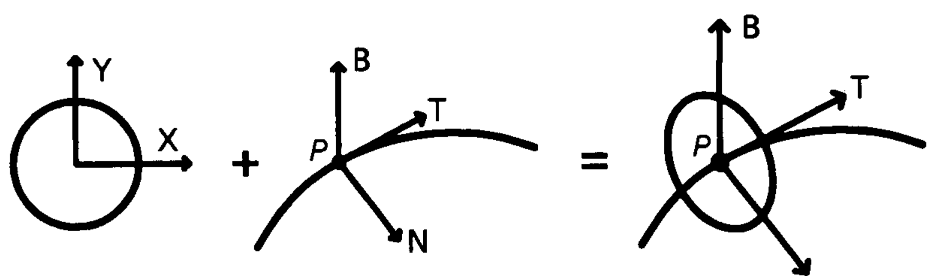


Figure 4. Positioning and orienting a cross-section.

Note that the results depend on the distance between successive reference frames. Reference frames a small distance apart will, naturally, follow the path of the curve more closely. With large distances it is possible to miss turns of the curve; such an error is then propagated along the curve. Implementors may find it advantageous to create several intermediate reference frames in order to establish one at a desired location. Also, a cross-section at the beginning of a closed curve will not necessarily align with the cross-section at the end of the curve.

Figure 5 was created with the technique described here; note that the cross-sections change as they progress along the curve. Also, there are more cross-sections where the curvature is relatively high. The number of cross-sections can also depend on the change in cross-sections (whether radius or shape).

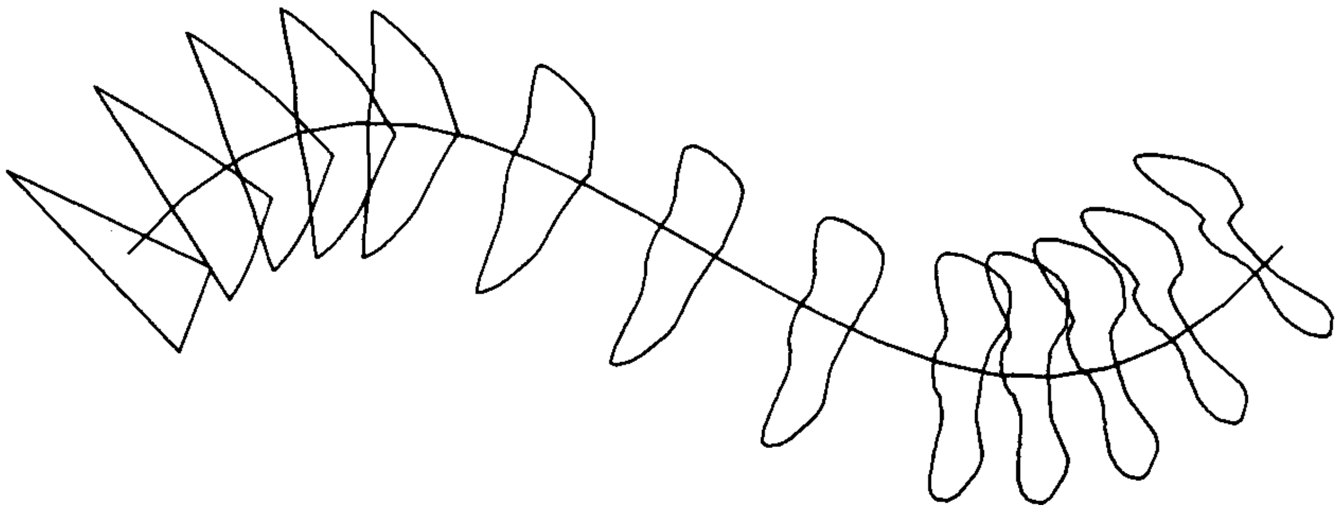

Figure 5. A generalized cylinder with changing cross-sections.