Theoretical Computer Science Cheat Sheet

Definitions

$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)\ \forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0\ \forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf_{n\to\infty} a_n$	$\lim_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup_{n\to\infty} a_n$	$\lim_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\dots\pi_n$ on $\{1, 2, \dots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\binom{n}{k} = \binom{n}{n-k}$,

4. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\left[{n \atop 1}\right] = (n-1)!$, **15.** $\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\left[{n \atop n}\right] = 1$, **17.** $\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ **26.** $\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0$ for $n \ne 0$,

34. $\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

- $r_8(n) = 16\sum_{d|n}(-1)^{n+d}d^3$

Gauss Circle Theorem

- The Gauss circle problem is the problem of determining how many integer lattice points there are in a circle centered at the origin and with radius r.
- Since the equation of this circle is given in Cartesian coordinates by $x^2+y^2=r^2$, the question is equivalently asking how many pairs of integers m and n there are such that $m^2+n^2<=r^2$
- If the answer for a given r is denoted by $N(r)$ then
$$N(r) = 1 + 4\sum_{i=0}^{\infty}\left(\left\lfloor\frac{r^2}{4i+1}\right\rfloor - \left\lfloor\frac{r^2}{4i+3}\right\rfloor\right)$$
- A much simpler sum appears if the sum of squares function r2(n) is defined as the number of ways of writing the number n as the sum of two squares. Then
$$N(r) = \sum_{n=0}^{r^2} r_2(n).$$

3. Combinatorics

Notes

- $\sum_{0\le k\le n}\binom{n-k}{k}$ =Fib_{n+1}
- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$
- $\boldsymbol{k\binom{n}{k} = n\binom{n-1}{k-1}}$
- $\boldsymbol{\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}}$
- $\boldsymbol{\sum_{i=0}^{n}\binom{n}{i} = 2^n}$
- $\boldsymbol{\sum_{i\ge 0}\binom{n}{2i} = 2^{n-1}}$
- $\boldsymbol{\sum_{i\ge 0}\binom{n}{2i+1} = 2^{n-1}}$
- $\boldsymbol{\sum_{i=0}^{k}(-1)^i\binom{n}{i} = (-1)^k\binom{n-1}{k}}$
- $\boldsymbol{\sum_{i=0}^{k}\binom{n+i}{i} = \binom{n+k+1}{k}}$
- $\boldsymbol{\sum_{i=0}^{k}\binom{n+i}{n} = \binom{n+k+1}{k}}$
- $\boldsymbol{\sum_{i=0}^{k}\binom{i}{n} = \binom{k+1}{n+1}}$
- $1.\binom{n}{1} + 2.\binom{n}{2} + 3.\binom{n}{3} + \ldots + n.\binom{n}{n} = n.2^{n-1}$
- $1^2.\binom{n}{1} + 2^2.\binom{n}{2} + 3^2.\binom{n}{3} + \ldots + n^2.\binom{n}{n} = (n+n^2).2^{n-2}$
- Vandermonde's Identity: $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$
- Hockey-Stick Identity: $n, r \in \mathbb{N}, n > r, \sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}$
- $\sum_{i=0}^{k}\binom{k}{i}^2 = \binom{2k}{k}$
- $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$
- $\sum_{k=q}^{n}\binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$
- $\boldsymbol{\sum_{i=0}^{n}3^i\binom{n}{i} = 4^i}$
- $\boldsymbol{\sum_{i=0}^{n}\binom{2n}{i} = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}}$
- $\boldsymbol{\sum_{i=1}^{n}\binom{n}{i}\binom{n-1}{i-1} = \binom{2n-1}{n-1}}$
- $\boldsymbol{\sum_{i=0}^{n}\binom{2n}{i}^2 = \frac{1}{2}\left\{\binom{4n}{2n} + \binom{2n}{n}^2\right\}}$
- An integer n ≥ 2 is prime if and only if all the intermediate binomial coefficients $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$ are divisible by n.
- $\binom{n+k}{k}$divides $\frac{lcm(n,n+1,\ldots,n+k)}{n}$
- Kummer's theorem states that for given integers n ≥ m ≥ 0 and a prime number p, the largest power of p dividing $\binom{n}{m}$ is equal to the number of carries when m is added to n − m in base p.
- Number of different binary sequences of length n such that no two 0's are adjacent=Fib_{n+1}
- Combination with repetition: Let's say we choose k elements from an n-element set, the order doesn't matter and each element can be chosen more than once. In that case, the number of different combinations is: $\binom{n+k-1}{k}$
- Number of ways to divide n different persons in n/k equal groups i.e. each having size k is $\binom{n-1}{k-1}$
- The number non-negative solution of the equation $x_1+x_2+x_3+\ldots+x_k=n$ is $\binom{n+k-1}{n}$
- Number of binary sequence of length n and with k '1' is $\binom{n}{k}$
- The number of ordered pairs (a, b) of binary sequences of length n, such that the distance between them is k, can be calculated as follows: $\binom{n}{k}\cdot 2^n$.
The distance between a and b is the number of components that differs in a and b — for example, the distance between (0, 0, 1, 0) and (1, 0, 1, 1) is 2).

Catalan numbers

- ✓ $C_n = \frac{1}{n+1}\binom{2n}{n}$
- ✓ $C_0 = 1, C_1=1$ and $C_n = \sum_{k=0}^{n-1}C_k C_{n-1-k}$
- ✓ 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786
- ✓ Number of correct bracket sequence consisting of n opening and n closing brackets.
- ✓ The number of ways to completely parenthesize n+1 factors.
- ✓ The number of triangulations of a convex polygon with +2 sides (i.e. the number of partitions of polygon into disjoint triangles by using the diagonals).
- ✓ The number of ways to connect the 2n points on a circle to form n disjoint i.e. non-intersecting chords.
- ✓ The number of monotonic lattice paths from point (0,0) to point (n,n) in a square lattice of size n×n, which do not pass above the main diagonal (i.e. connecting (0,0) to (n,n)).
- ✓ Number of permutations of length n that can be stack sorted (i.e. it can be shown that the rearrangement is stack sorted if and only if there is no such index i<j<k, such that $a_k<a_i<a_j$).
- ✓ The number of non-crossing partitions of a set of n elements.
- ✓ The number of rooted full binary trees with n+1 leaves (vertices are not numbered). A rooted binary tree is full if every vertex has either two children or no children.
- ✓ The number of Dyck words of length 2n. A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's For example, the following are the Dyck words of length 6: XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY.
- ✓ The number of different ways a convex polygon with n + 2 sides can be cut into triangles by connecting vertices with straight lines (a form of Polygon triangulation)
- ✓ Number of permutations of {1, …, n} that avoid the pattern 123 (or any of the other patterns of length 3); that is, the number of permutations with no three-term increasing subsequence. For n = 3, these permutations are 132, 213, 231, 312 and 321. For n = 4, they are 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312 and 4321
- ✓ Number of ways to tile a stairstep shape of height n with n rectangles.
- ✓ $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$
- ✓ The number of expressions containing *n* pairs of parentheses, which are correctly matched and which contain *k* distinct nestings. For instance, *N*(4, 2) = 6 as with four pairs of parentheses six sequences can be created which each contain two times the sub-pattern '()':

  ()((())) (())(()) (()(())) ((()())) ((())()) ((()))()

- ✓ The number of paths from (0, 0) to (2*n*, 0), with steps only northeast and southeast, not straying below the *x*-axis, with *k* peaks. And sum of all number of peaks is Catalan number.

Stirling numbers of the first kind

- ✓ The Stirling numbers of the first kind count permutations according to their number of cycles (counting fixed points as cycles of length one).
- ✓ S(n,k) counts the number of permutations of n elements with k disjoint cycles.
- ✓ $S(n,k) = (n-1) * S(n-1,k) + S(n-1,k-1),$
  $where\ S(0,0) = 1, S(n,0) = S(0,n) = 0$
- ✓ $\sum_{k=0}^{n}S(n,k) = n!$

Identities Cont.	Trees
38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix}\binom{k}{m} = \sum_{k=0}^{n} \begin{bmatrix} k \\ m \end{bmatrix} n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\begin{bmatrix} k \\ m \end{bmatrix}$, **39.** $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$, **40.** $\left\{ {n \atop m} \right\} = \sum_k \binom{n}{k} \left\{ {k+1 \atop m+1} \right\} (-1)^{n-k}$, **41.** $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k}$, **42.** $\left\{ {m+n+1 \atop m} \right\} = \sum_{k=0}^{m} k \left\{ {n+k \atop k} \right\}$, **43.** $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^{m} k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}$, **44.** $\binom{n}{m} = \sum_k \left\{ {n+1 \atop k+1} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k}$, **45.** $(n-m)!\binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ {k \atop m} \right\} (-1)^{m-k}$, for $n \geq m$, **46.** $\left\{ {n \atop n-m} \right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix}$, **47.** $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k} \left\{ {m+k \atop k} \right\}$, **48.** $\left\{ {n \atop \ell+m} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ {k \atop \ell} \right\} \left\{ {n-k \atop m} \right\} \binom{n}{k}$, **49.** $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}$.	Every tree with n vertices has $n-1$ edges. Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$: $$\sum_{i=1}^{n} 2^{-d_i} \leq 1,$$ and equality holds only if every internal node has 2 sons.

Stirling numbers of the second kind

- ✓ Stirling number of the second kind is the number of ways to partition a set of n objects into k non-empty subsets.
- ✓ $S(n,k) = k * S(n-1,k) + S(n-1,k-1)$,
  $where\ S(0,0) = 1, S(n,0) = S(0,n) = 0$
- ✓ $S(n,2)=2^{n-1}-1$
- ✓ $S(n,k)*k!$ = number of ways to color n nodes using colors from 1 to k such that each color is used at least once.

Bell number

- ✓ Counts the number of partitions of a set.
- ✓ $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} * B_k$
- ✓ $B_n = \sum_{k=0}^{n} S(n,k)$,where S(n,k) is stirling number of second kind.
- ✓ The number of multiplicative partitions of a squarefree number with i prime factors is the i-th Bell number, B_i.
- ✓ If a deck of n cards is shuffled by repeatedly removing the top card and reinserting it anywhere in the deck (including its original position at the top of the deck), with exactly n repetitions of this operation, then there are n^n different shuffles that can be performed. Of these, the number that return the deck to its original sorted order is exactly B_n. Thus, the probability that the deck is in its original order after shuffling it in this way is B_n/n^n.

Lucas Theorem

- ✓ If p is prime the $\binom{p^a}{k} \equiv 0\ (mod\ p)$
- ✓ For non-negative integers m and n and a prime p, the following congruence relation holds:

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i}\ \ (mod\ p),$$

where,
$m = m_k p^k + m_{k-1} p^{k-1} + \ldots + m_1 p + m_0$,
and
$n = n_k p^k + n_{k-1} p^{k-1} + \ldots + n_1 p + n_0$
are the base p expansions of m and n respectively. This uses the convention that $\binom{m}{n} = 0, when\ m < n$.

Derangement

- ✓ A derangement is a permutation of the elements of a set, such that no element appears in its original position.
- ✓ $d(n) = (n-1) * (d(n-1) + d(n-2))$,
  $where\ d(0) = 1, d(1) = 0$
- ✓ $d(n) = \left\lfloor \frac{n!}{e} \right\rceil, n \geq 1$

4. Burnside Lemma

The task is to count the number of different necklaces from n beads, each of which can be painted in one of the k colors. When comparing two necklaces, they can be rotated, but not reversed (i.e. a cyclic shift is permitted).
Solution:

Geometry

Projective coordinates: triples (x,y,z), not all x, y and z zero.

$$(x,y,z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x,y)	$(x,y,1)$
$y = mx + b$	$(m,-1,b)$
$x = c$	$(1,0,-c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2},$$

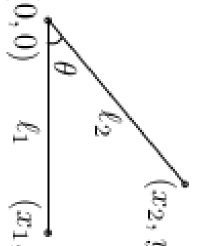
$$\left[|x_1-x_0|^p + |y_1-y_0|^p\right]^{1/p},$$

$$\lim_{p\to\infty} \left[|x_1-x_0|^p + |y_1-y_0|^p\right]^{1/p}.$$

Area of triangle (x_0,y_0), (x_1,y_1) and (x_2,y_2):

$$\frac{1}{2}\,\text{abs}\begin{vmatrix} x_1-x_0 & y_1-y_0 \\ x_2-x_0 & y_2-y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1,y_1)\cdot(x_2,y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0,y_0) and (x_1,y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \frac{4}{3}\pi r^3.$$