TARN

DEBORSHEE SEN, DSAP, NUS

April 10, 2015

Contents

1 SMC For TARN Option

We can use the ideas above and the ideas of SMC Samplers as in [2] and to estimate the prices of Target Accrual Redemption Note options.

1.1 Description of the Options

Here we describe a (very simplified) TARN option. Consider a function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = (0, \infty)$. We have a set of monitoring dates $t_1, \dots, t_m$. Let $f = f^+ - f^-$ denote the usual positive and negative parts of f. Consider the gain and loss processes:

$$G_j = \sum_{i=1}^{j} f^+(S_{t_i}),\ L_j = \sum_{i=1}^{j} f^-(S_{t_i}) \tag{1}$$

The following stopping times are defined:

$$\tau^{(G)} = \min\{j : G_j \geq \Gamma_G\},\ \tau^{(L)} = \min\{j : L_j \geq \Gamma_L\} \text{ and } \tau = \min\{\tau^{(G)}, \tau^{(L)}, m\} \tag{2}$$

The payoff of this option is $\sum_{i=1}^{\tau} f(S_{t_i})$ and we want to estimate its price

$$\mathrm{P} = \mathbb{E}\left[\sum_{i=1}^{\tau} f(S_{t_i})\right] \tag{3}$$

1.2 SMC Samplers

SMC samplers were (first ?) presented in [5] and were used in [2] to estimate the price of Asian Options. Suppose that we want to sample from a sequence of target distributions $\{\pi_n\}_{0\leq n\leq p}$ on a common space. The algorithm samples from a sequence of distributions of increasing dimensions. Introduce a seuence of auxiluary measures $\{\tilde{\pi}_n\}_{0\leq n\leq p}$ on spaces of increasng measures, such that they admit the $\{\pi_n\}_{0\leq n\leq p}$ as marginals. The folowing sequence of auxiliary measures is used

$$\tilde{\pi}(x_{0:n}) = \pi_n(x_n) \prod_{j=0}^{n-1} L_j(x_{j+1}, x_j) \tag{4}$$

where $\{L_n\}_{0\leq n\leq p-1}$ are a sequence of Maarkov Kernels that act backward in time and are termed backward Markov kernels.The algorithm samples forward using using kernels $\{K_n\}$. The chice of backward kernels is made as the incremental weights are

$$W_n(x_{n-1:n}) = \frac{\pi_n(x_n) L_{n-1}(x_n, x_{n-1})}{\pi_{n-1}(x_{n-1}) K_n(x_{n-1}, x_n)},\ n \geq 1 \tag{5}$$

which allows for fast computation and avoids a path degeneracy effect. The kernels $K_n(x_{n-1}, \cdot)$ are chosen to be $\pi_n(\cdot)$ invariant (by using a Metropolis-Hastings proposal step) and the backward kernel used is

$$L_{n-1}(x_n, x_{n-1}) = \frac{\pi_n(x_{n-1}) K_n(x_{n-1}, x_n)}{\pi_n(x_n)} \tag{6}$$

and in this case the incremental weights (5) at time n simplify to

$$W_n(x_{n-1}) = \frac{\pi_n(x_{n-1})}{\pi_{n-1}(x_{n-1})} \tag{7}$$

We refer to [2] and [5] for more details.

1.3 Estimating Price of TARN Options

We again assume that the underlying asset evolves as (**??**). We work on a log-scale and consider the Euler-Maruyama discretization (**??**). If τ were deterministic in (3), the authors in [2] have shown how one can use SMC samplers to estimate $\mathbb{E}\left|\sum_{i=1}^{j} f(S_{t_i})\right|$, where we still use $f(\cdot)$ to avoid changing notations. They do it as follows: (everything is conditioned on R_0)

SIR: first sample M particles from the sequence of densitities which are proportional to

$$p(r_1), p(r_{1:2}), \ldots, p(r_{1:t_1-1}), |f(r_{t_1})|^{\kappa_1} p(r_{1:t_1})$$
$$p(r_{1:t_1+1}), p(r_{1:t_1+2}), \ldots, p(r_{1:t_2-1}), \left|\sum_{i=1}^{2} f(r_{t_i})\right|^{\kappa_2} p(r_{1:t_2})$$
$$\vdots$$
$$p(r_{1:t_{m-1}+1}), p(r_{1:t_{m-1}+2}), \ldots, p(r_{1:t_m-1}), |\textstyle\sum_{i=1}^{n} f(r_{t_i})|^{\kappa_n} p(r_{1:t_m})$$

where $0 \leq \kappa_1 < \ldots < \kappa_n < 1$ and the process densities are used as proposals.

SMC samplers: Given the samples $\{R_{1:N}^{(l)}\}_{l=1}^{M}$ from SIR, use SMC samplers (as in section 3.2) to sample from the sequence of densities

$$\tilde{\pi}_j(r_{1:N}) \propto \left|\sum_{i=1}^{m} f(r_{t_i})\right|^{\tilde{\kappa}_j} p(r_{1:N}) \tag{8}$$

for $j = 1, 2, \ldots, p$, where $\kappa_n = \tilde{\kappa}_1 < \tilde{\kappa}_2 < \ldots < \tilde{\kappa}_m = 1$. In the notation of section 3.2, $x = r_{1:N}$. We use a random walk Metropolis-Hastings proposal as the kernel $K_n(x_{n-1}, x_n)$.

We use their idea and the idea of what we called non-blind SMC to estimate (3) as follows:

$$\begin{aligned}
\mathrm{P} &= \mathbb{E}\left[\sum_{i=1}^{\tau} f(S_{t_i})\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{\tau} \left(f^+(R_{t_i}) - f^-(R_{T_i})\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\sum_{i=1}^{\tau} \left(f^+(R_{t_i}) - f^-(R_{T_i})\right)\middle|\tau\right)\right] \\
&= \sum_{j=1}^{m}\left[\mathbb{E}\left(\sum_{i=1}^{j} f^+(R_{t_n})\middle|\tau = j\right)\mathbb{P}(\tau = j) - \mathbb{E}\left(\sum_{i=1}^{j} f^-(R_{t_n})\middle|\tau = j\right)\mathbb{P}(\tau = j)\right]
\end{aligned} \tag{9}$$

The terms $\mathbb{P}(\tau = j)$ can be estimated as follows:

For $1 \leq j \leq m-1$,

$$\begin{aligned}
\mathbb{P}(\tau = j) = \quad & \mathbb{P}(G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, G_j \geq \Gamma_G) \\
&+ \mathbb{P}(G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, L_j \geq \Gamma_L) \\
&- \mathbb{P}(G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, G_j > \Gamma_L, L_j > \Gamma_L)
\end{aligned} \tag{10}$$

For $j = m$,

$$\begin{aligned}
\mathbb{P}(\tau = m) = \quad & \mathbb{P}(G_{1:m} < \Gamma_G, L_{1:m} < \Gamma_M) \\
&+ \mathbb{P}(G_{1:m-1} < \Gamma_G, L_{1:m-1} < \Gamma_L, G_j \geq \Gamma_G) \\
&+ \mathbb{P}(G_{1:m-1} < \Gamma_G, L_{1:m-1} < \Gamma_L, L_m \geq \Gamma_L) \\
&- \mathbb{P}(G_{1:m-1} < \Gamma_G, L_{1:m-1} < \Gamma_L, G_m > \Gamma_L, L_m > \Gamma_L)
\end{aligned} \tag{11}$$

These simply follows from the fact that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$ for events A and B. The terms in (10) and (11) can be written as:

$$\begin{aligned}
\mathbb{P}(G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, G_j \geq \Gamma_G) &= \mathbb{E}\left[\mathbb{1}\{G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, G_j \geq \Gamma_G\}\right] \\
&= \mathbb{E}\left[\left(\prod_{i=1}^{j-1} \mathbb{1}\{G_i < \Gamma_G, L_i < \Gamma_L\}\right) \mathbb{1}\{G_j \geq \Gamma_G\}\right],
\end{aligned}$$

and similarly,

$$\begin{aligned}
\mathbb{P}(G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, L_j \geq \Gamma_L) &= \mathbb{E}\left[\left(\prod_{i=1}^{j-1} \mathbb{1}\{G_i < \Gamma_G, L_i < \Gamma_L\}\right) \mathbb{1}\{L_j \geq \Gamma_L\}\right], \\
\mathbb{P}(G_{1:j-1} < \Gamma_G, L_{1:j-1} < \Gamma_L, G_j \geq \Gamma_G, L_j \geq \Gamma_L) &= \mathbb{E}\left[\left(\prod_{i=1}^{j-1} \mathbb{1}\{G_i < \Gamma_G, L_i < \Gamma_L\}\right) \mathbb{1}\{G_j \geq \Gamma_G, L_j \geq \Gamma_L\}\right], \\
\text{and } \mathbb{P}(G_{1:m} < \Gamma_G, L_{1:m} < \Gamma_L) &= \mathbb{E}\left[\prod_{i=1}^{m} \mathbb{1}\{G_i < \Gamma_G, L_i < \Gamma_L\}\right]
\end{aligned}$$

for $j = 1, 2, \ldots, m$. We can therefore sraightforwardly use the ideas of 'blind' SMC to estimate the terms above. Can we also use 'non blind' SMC?

1.3.1 Simple Weighing Functions

We try some simple weighing functions h_n's. We let $L = 99$ and $U = 101$. The number of repetitions was set to be 100. In this case, we choose two types of h_n's:

- $h_n(x) = h_{gaussian}(x) = e^{-c[x-\log(M)]^2}\ \forall n \notin \{t_1, t_2, \ldots, t_m\}$.
- $h_n(x) = h_{quadratic}(x) = c[x - \log(M)]^2 + 1\ \forall n \notin \{t_1, t_2, \ldots, t_m\}$.

where $M = \sqrt{L \times U}$ (the geometric mean of L and U. We choose four different values of c, namely $1, 0.1, 0.01$ and 0.001. Here are some results:

– $m = 2, k = 90$:

Method	Mean	Relative standard deviation
Blind SMC	6.20×10^{-3}	4.85×10^{-2}
Gaussian with $c = 1$	6.16×10^{-3}	4.58×10^{-2}
Gaussian with $c = 0.1$	6.15×10^{-3}	4.59×10^{-2}
Gaussian with $c = 0.01$	6.10×10^{-3}	4.67×10^{-2}
Gaussian with $c = 0.001$	6.19×10^{-3}	4.44×10^{-2}
Qudratic with $c = 1$	6.18×10^{-3}	4.75×10^{-2}
Qudratic with $c = 0.1$	6.15×10^{-3}	4.64×10^{-2}
Qudratic with $c = 0.01$	6.17×10^{-3}	4.06×10^{-2}
Qudratic with $c = 0.001$	6.11×10^{-3}	4.58×10^{-2}

– $m = 5, k = 90$:

Method	Mean	Relative standard deviation
Blind SMC	2.42×10^{-5}	6.40×10^{-2}
Gaussian with $c = 1$	2.40×10^{-5}	6.63×10^{-2}
Gaussian with $c = 0.1$	2.39×10^{-5}	5.38×10^{-2}
Gaussian with $c = 0.01$	2.40×10^{-5}	5.55×10^{-2}
Gaussian with $c = 0.001$	2.41×10^{-5}	6.19×10^{-2}
Qudratic with $c = 1$	2.39×10^{-5}	6.32×10^{-2}
Qudratic with $c = 0.1$	2.41×10^{-5}	6.23×10^{-2}
Qudratic with $c = 0.01$	2.38×10^{-5}	5.67×10^{-2}
Qudratic with $c = 0.001$	2.43×10^{-5}	5.35×10^{-2}

References

[1] Del Moral, P. and Shevchenko, P.V., *Valuation of Barrier Options Using Sequential Monte Carlo*, arXiv, (2014)

[2] Jasra, A. and Del Moral, P., *Sequential Monte Carlo for Option Pricing*, Stoch. Anal. Appl., (2011)

[3] Del Moral, P. and Murray, M.M., *Sequential Monte Carlo With Highly Informative Observations*, arXiv, (2014)

[4] Glasserman, P. and Staum, J., *Conditioning on One-Step Survival for Barrier Option Simulations*, Operations Research, Vol. 49, No. 6, 923-937 (2001)

[5] Del Moral, P., Doucet, A. and Jasra, A., *Sequential Monte Carlo Samplers*, J. R. Statist. Soc. B (2006)

[6] Doucet, A. and Johansen, A.M., *A Tutorial on Particle Filtering and Smothing: Fifteen Years Later*, Oxford Handbook of Nonlinear Filtering (2011)

[7] Luo, X. and Shevchenko, P.V. *Pricing TARN Using a Finite Difference Method*, arXiv, (2014)