# CFD in Python Lecture -1

Tuesday, July 9, 2024 12:11 PM

## Forward differencing Method

$$f'(x_0) = \boxed{\frac{f(x_0+h) - f(x_0)}{h}} - O(h)$$

$h \rightarrow$ grid spacing

$\boxed{h < 1}$

First order estimate

Approximation value

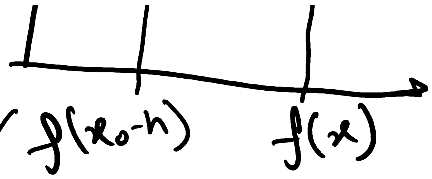
Truncation error

leading term is the order of $h$

$\left(\frac{h}{2!} f''(x_0) - \cdots\right)$

## Backward differencing Method

$$f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} - O(h)$$

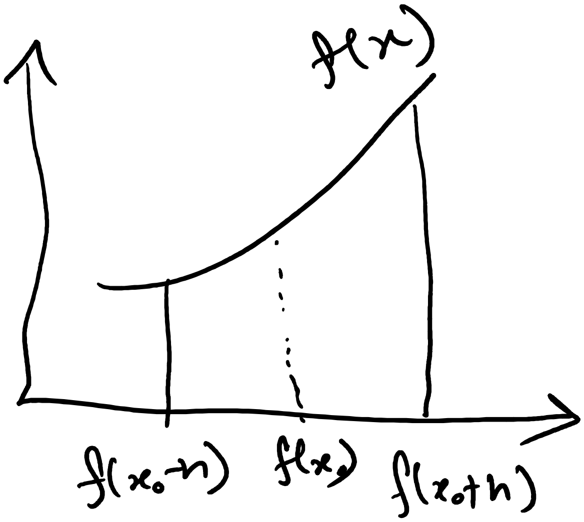

For backward differencing method


forward differencing method

# Central Differencing Method

$$f'(x_0) = \frac{f(x_0-h) - f(x_0+h)}{2h} - O(h^2)$$

Truncation error is in the order of $h^2$




Central differencing method

Solve: $\boxed{-4x^2+7x^2-3x+9}$ → Polynomial to solve using forward, ~~backward~~

Given, $x_0 = 0$

using

Given, $x_0 = 0$

$\boxed{h = 0.25}$ → Grid spacing

forward, ~~backwar~~ backward and central differencing method

# Second Order Derivative

$$\boxed{f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - O(h^2)}$$

→ Second derivative with second order accuracy

→ Central differencing

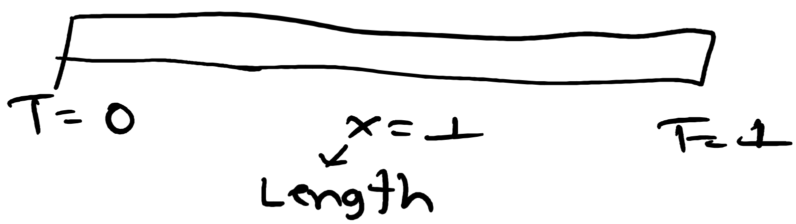
Forward differencing

→ $$f''(x_0) = \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2}$$

Backward differencing

→ $$f'(x_0) = \frac{f(x_0) - 2f(x_0-h) + f(x_0-2h)}{h^2}$$

# One dimensional Problem

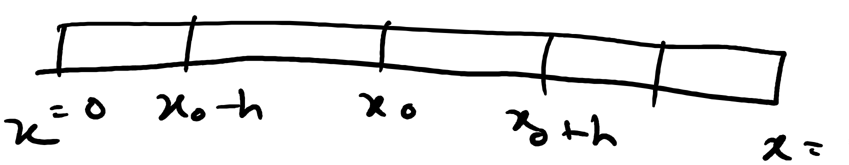



1-D diffusion equation,

$$\frac{d^2T}{dx^2} = 0$$

Initially $T(t=0, x<1) = 0$ and
$T(t=0, x=1) = 1$

Dividing the geometry in computational mesh




Converging solution,

criteria

$$\sum_{x=x_0} |T_{new} - T_{old}| < e$$

Tolerance number

$$\sum |T_{new} - T_{old}| < e$$

$$\sum_{x} \left| T_{new} - T_{old} \right| < e$$

$$\sum_{x} \left| \frac{T_{new} - T_{old}}{T_{old}} \right| < e$$

Exercise: 1D convection diffusion equation

$$u \frac{dT}{dx} + \frac{d^2T}{dx^2} = 0$$

Now, $\frac{d^2T}{dx^2} = T''(x)$

$$T''(x) = \frac{T(x_0+h) - 2T(x_0) + T(x_0-h)}{h^2}$$

[using central differencing method]

Given that,

$$\frac{d^2T}{dx^2} = T''(x) = 0$$

Therefore,

$$0 = T(x_0+h) - 2T(x_0) + T(x_0-h)$$

$$\therefore T(x_0) = \frac{1}{2}\left(T(x_0+h) + T(x_0-h)\right)$$

$$\Rightarrow T(x_0) = \frac{1}{2}\left(T(x_0+h) + T(x_0-h)\right)$$

Using Forward differencing scheme,

$$T''(x) = \frac{T(x_0+2h) - 2T(x_0+h) + T(x_0)}{h^2}$$

$$T(x_0) - 2T(x_0+h) + T(x_0+2h) = 0$$

$$\Rightarrow T(x_0) = 2T(x_0+h) - T(x_0+2h)$$

Using Backward differencing scheme,

$$T''(x) = \frac{T(x_0) - 2T(x_0-h) + T(x_0-2h)}{h^2}$$

$$\Rightarrow T(x_0) - 2T(x_0-h) + T(x_0-2h) = 0$$

$$\Rightarrow T(x_0) = 2T(x_0-h) - T(x_0-2h)$$

For exercise problem,

$$u\frac{dT}{dx} + \frac{d^2T}{dx^2} = 0$$

taking $u$ as constant,

using central differencing method,

using central difference

$$\Rightarrow U \times \frac{T(x_0+h) - T(x_0-h)}{2h} + \frac{T(x_0+h) - 2T(x_0) + T(x_0-h)}{h^2} = 0$$

$$\Rightarrow \frac{Uh\left(T(x_0+h) - T(x_0-h)\right) + 2T(x_0+h) - 4T(x_0) + 2T(x_0-h)}{2h^2} = 0$$

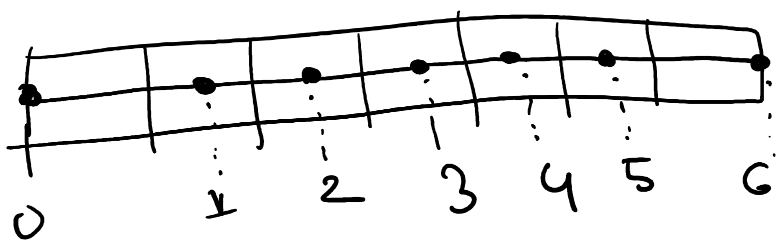
$$\Rightarrow Uh\left(T(x_0+h) - T(x_0-h)\right) + 2T(x_0+h) - 4T(x_0) + 2T(x_0-h) = 0$$

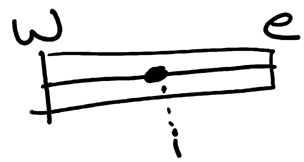
$$\Rightarrow T(x_0+h)\left\{Uh+2\right\} + T(x_0-h)\left\{2-Uh\right\} = 4T(x_0)$$

$$\Rightarrow T(x_0) = \frac{1}{4}\left[(Uh+2)\,T(x_0+h) + (2-Uh)\,T(x_0-h)\right]$$

Volume

# Generalized Finite Volume Method



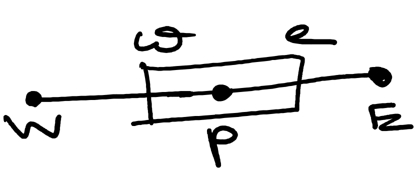




$$\left.\frac{dT}{dx}\right|_e - \left.\frac{dT}{dx}\right|_w = 0$$

Heat conduction governing equation

Generalized $\rightarrow$ $\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$ [At first we consider $k = const$]

$k \rightarrow$ Thermal conductivity




Now employing Gauss divergence theorem.

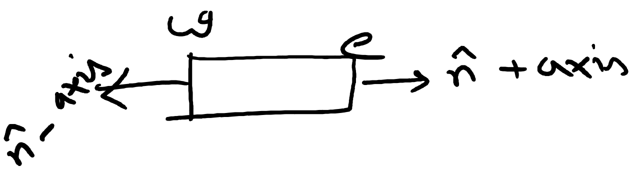
$$\int \frac{d}{dx}\left(k\frac{dT}{dx}\right)dV = 0$$

$$\int \frac{d}{dx}\left(k\frac{dT}{dx}\right)dV = 0$$

$$\int \hat{n}\cdot\left(k\frac{dT}{dx}\right)dA = 0 \quad [\hat{n} \rightarrow \text{Normal vector}]$$

Let,
$\hat{n} \rightarrow +x$ axis at E
$\hat{n} \rightarrow -x$ axis at W

Now,
$$kA\frac{dT}{dx}\Big|_e - kA\frac{dT}{dx}\Big|_w = 0$$




$$k_eA_e\left(\frac{T_E - T_P}{h_{PE}}\right) - k_wA_w\left(\frac{T_P - T_W}{h_{WP}}\right) = 0$$

$k_e \rightarrow$ Thermal conductivity at 'e' face
$k_w \rightarrow$ " " at 'w' face.
$T_E \rightarrow$ Temperature at E
$T_W \rightarrow$ Temperature at W
$T_P \rightarrow$ Temperature at P
$h_{PE} \rightarrow$ spacing between P and E

$h_{pE} \rightarrow$ Spacing between P and E

$h_{pw} \rightarrow$ Spacing between P and w

Now, $a_e = \frac{k_e A_e}{h_{pE}}$ | $a_p = \frac{k_e A_e}{h_{pE}} + \frac{k_w A_w}{h_{pw}}$

$a_w = \frac{k_w A_w}{h_{pw}}$

Therefore,

$$\boxed{a_e T_E + a_w T_w = a_p T_p}$$

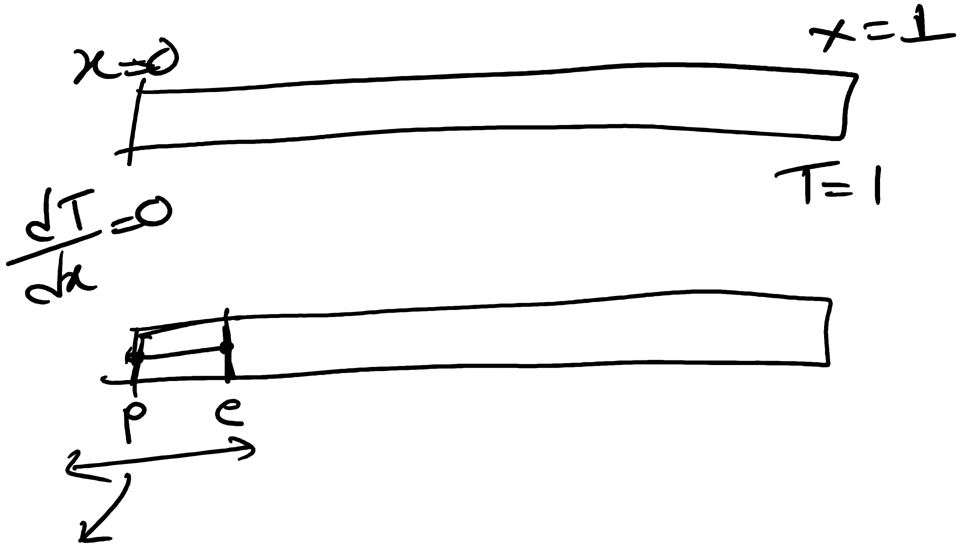
General equation

# Dirichlet Boundary Condition

when the value is specied at the boundary. Then it is called Dirichlet Boundary condition

# Neumann boundary condition

When gradient is specied at the boundary then it is called boundary then it is called Neumann boundary condition




$$\int \frac{d}{dx}\left(k\frac{dT}{dx}\right)dV = 0$$

$$\hookrightarrow \int \hat{n}\left(k\frac{dT}{dx}\right)dA = 0$$

$$\Rightarrow kA\frac{dT}{dx}\Big|_e - kA\frac{dT}{dx}\Big|_p = 0$$

Given that $\frac{dT}{dx}\Big|_p = 0$

Then,

$$k_eA_e\left(\frac{T_E - T_P}{h_{PE}}\right) = 0$$

here, $a_W = 0$

and so,

$$a_E = \frac{k_eA_e}{h_{PE}}$$

$$a_E = \frac{k_e A_e}{h_{PE}}$$

and $\boxed{a_P T_P = a_E T_E}$

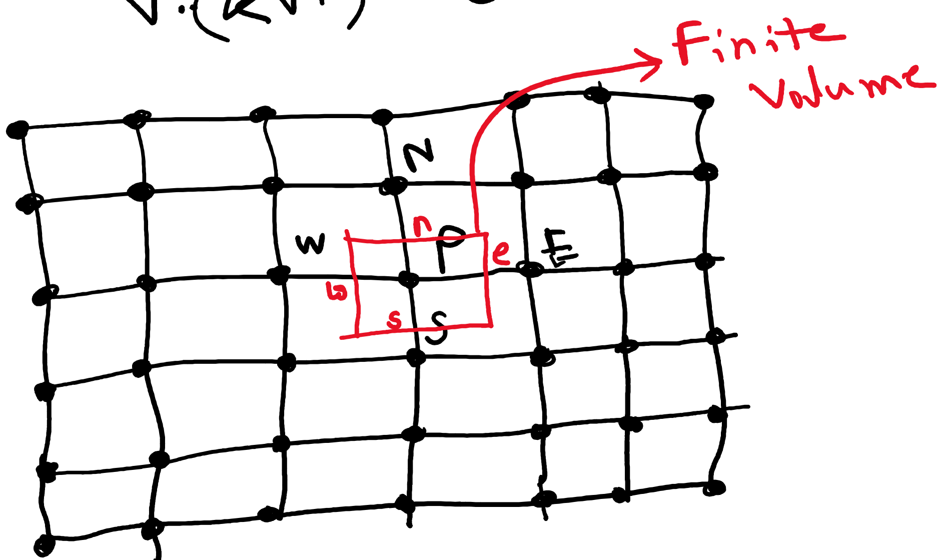
# 2D Heat Diffusion

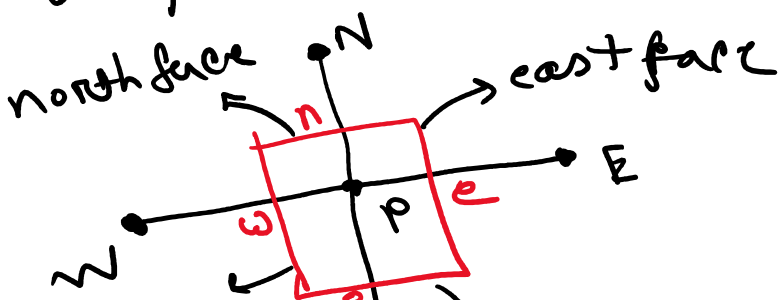
Governing Equation,

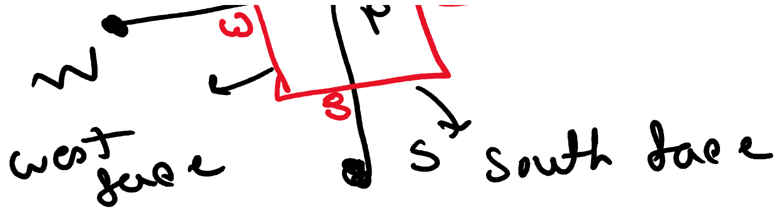
For 1D $\rightarrow$ $\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$

For 2D $\rightarrow$ $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = 0$

$$\nabla\cdot(k\nabla T) = 0$$






Finite Volume,

$$\int \left[ \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \right] dv = 0$$

using Gauss divergence theorem,

$$\int \hat{n} \cdot \left(k\frac{\partial T}{\partial x} + k\frac{\partial T}{\partial y}\right) dA = 0$$

Now considering the four faces.

$$kA\frac{dT}{dx}\Big|_e - kA\frac{dT}{dx}\Big|_w + kA\frac{dT}{dy}\Big|_n - kA\frac{dT}{dy}\Big|_s = 0$$

$$k_eA_e\left(\frac{T_E - T_P}{h_{PE}}\right) - k_wA_w\left(\frac{T_P - T_W}{h_{WP}}\right) + k_nA_n\left(\frac{T_N - T_P}{h_{PN}}\right) - k_sA_s\left(\frac{T_P - T_S}{h_{PS}}\right) = 0$$

Now we can write,

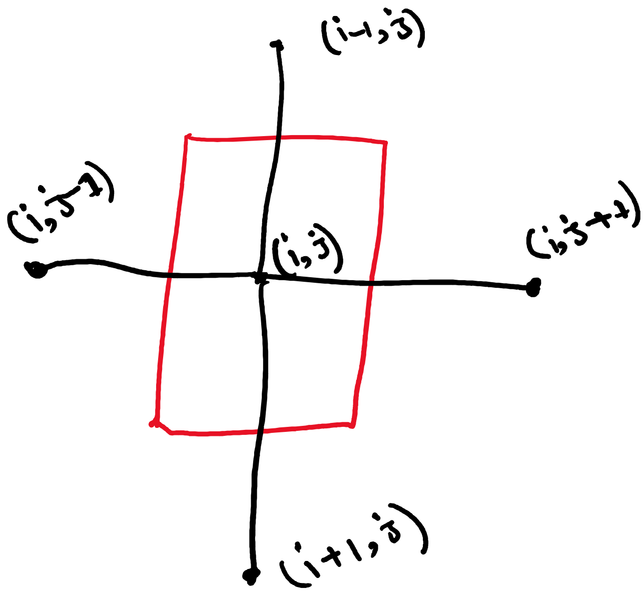
$$\boxed{a_PT_P = a_ET_E + a_WT_W + a_NT_N + a_ST_S}$$

$$\boxed{a_p T_p = a_E T_E + a_W T_W + a_N T_N + a_S T_S}$$

$$a_E = \frac{k_e A_e}{h_{pE}} \qquad a_N = \frac{A_n K_n}{h_{pN}}$$

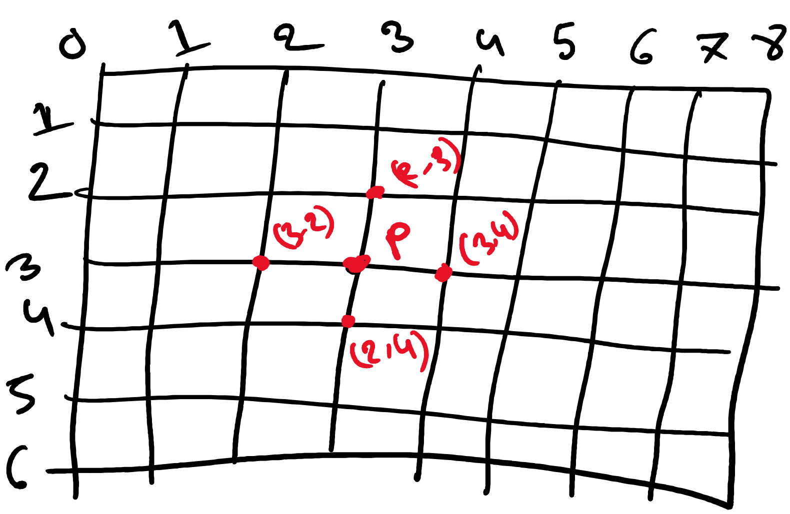
$$a_W = \frac{k_w A_w}{h_{wp}} \qquad a_S = \frac{A_s K_s}{h_{ps}}$$

## Discretized Governing Equation

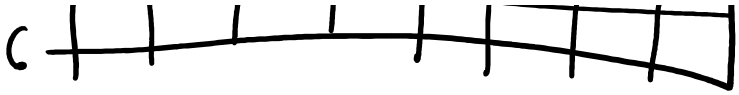



We have,

$$a_p T_p = a_E T_E + a_W T_W + a_N T_N + a_S T_S$$



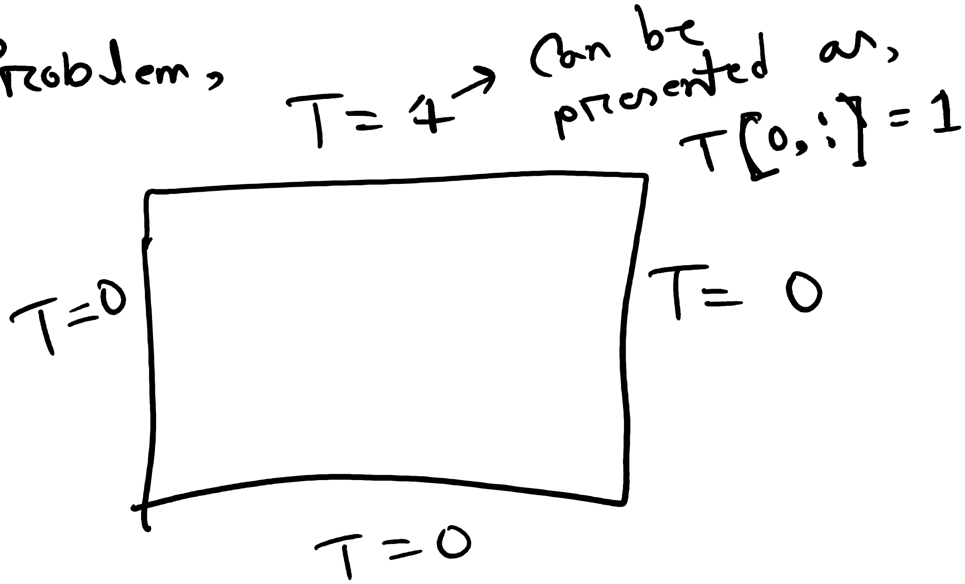

For $i = 3$
$j = 3$



Now we can write,

$$a_P T_{3,3} = a_E T_{3,4} + a_W T_{3,2} + a_N T_{2,3} + a_S T_{2,4}$$

Problem,




# Source Term

Governing equation.

$$\int_V \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + S\right] dV = 0$$

$$\underbrace{\int \hat{n}\left(k\frac{\partial T}{\partial x} + k\frac{\partial T}{\partial y}\right) dA}_{\text{Same as previous one}} + \int S\, dV = 0$$

← Same as p...
one

$$\int s\,dv = \bar{S}\Delta V$$

$\bar{S}$ → Average value of source at control volume

→ constant based source term
→ Not constant

Source term linearization.

$$\bar{S}\nabla V = S_u + \underline{S_p T_p}$$

$S_u$ ↙ constant value; $S_p T_p$ ↓ Non constant

Now we can write,

$$kA\frac{dT}{dx}\Big|_e - kA\frac{dT}{dx}\Big|_w + kA\frac{dT}{dy}\Big|_n - kA\frac{dT}{dy}\Big|_s + S_u + S_p T_p = 0$$

$$\Rightarrow k_e A_e\left(\frac{T_E - T_P}{h_{PE}}\right) - k_w A_w\left(\frac{T_P - T_W}{h_{PW}}\right) + k_n A_n\left(\frac{T_N - T_P}{h_{PN}}\right) - k_s A_s\left(\frac{T_P - T_S}{h_{PS}}\right) + S_u + S_p T_p = 0$$

$$\Rightarrow a_e = \left(\frac{k_e A_e}{h_{PE}}\right) \quad a_w = \left(\frac{k_w A_w}{h_{PW}}\right)$$

$$\Rightarrow a_e = \left(\frac{\quad}{h_{PE}}\right) \quad a_w = \left(\frac{\quad}{h_{PW}}\right)$$

$$a_n = \left(\frac{k_n A_n}{h_{PN}}\right) \quad a_{SP} = \left(\frac{k_s A_s}{h_{PS}}\right)$$

$$a_p = a_e + a_w + a_n + a_{sp} - S_p$$

$$\boxed{a_p T_p = a_e T_E + a_w T_W + a_n T_N + a_s T_s + S_u}$$

Final Governing equation

$$\boxed{S = h q \times A} \rightarrow$$

$q \rightarrow$ Heat source per unit volume

$A \rightarrow$ Area

$h \rightarrow$ spacing

We are considering here constant source term.

# 1D Convection [with Peclet number]

$$a_p T_p = a_e T_e + a_w T_w$$

$a_p \phi_p = a_E \phi_E$

$$a_E = \frac{\gamma}{h} + \frac{\rho u}{2} = a_W$$

[Taking $\gamma$, $h$, $\rho$ and $u$ as constant]

$\gamma \rightarrow$ diffusivity

$h \rightarrow$ grid spacing

$\rho \rightarrow$ density

$u \rightarrow$ velocity

Peclet number needs to be remain below 2

$$P_e = \frac{\rho * u * h}{\gamma}$$

# Upwind Scheme

Steady equation,

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{\partial \phi}{dx}\right)$$

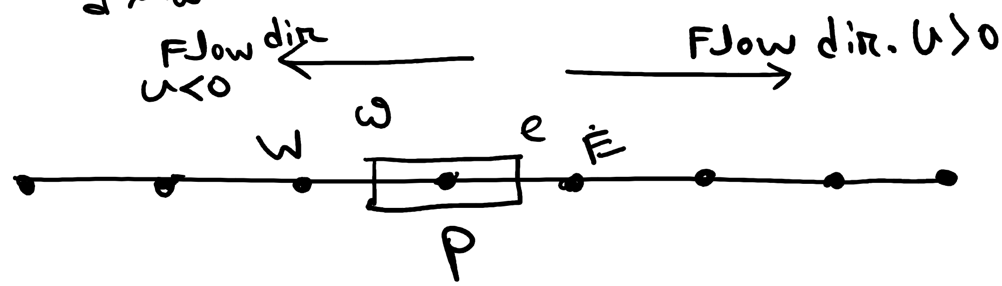
$dx$ $d'$

using gauss divergence theorem

$$\Gamma \left.\frac{d\phi}{dx}\right|_e - \Gamma \left.\frac{d\phi}{dx}\right|_w = (\rho u \phi)_e - (\rho u \phi)_w$$

For pure conduction → Central differential scheme

$$\left.\frac{d\phi}{dx}\right|_w = \frac{\phi_P - \phi_W}{h} \qquad \left.\frac{d\phi}{dx}\right|_e = \frac{\phi_E - \phi_P}{h}$$




The upwind scheme says,

$$\phi_e = \begin{cases} \phi_P, & u > 0 \text{ [Flow is from P to E]} \\ \phi_E, & u < 0 \text{ [Flow is from E to P]} \end{cases}$$

Again,

$$a_w = \begin{cases} a_W, & u > 0 \\ a_P, & u < 0 \end{cases}$$

We have equation,

We have equation,

$$(\rho u \phi A)_e - (\rho u \phi A)_w = \Gamma A \frac{d\phi}{dx}\Big|_e - \Gamma A \frac{d\phi}{dx}\Big|_w$$

taking A as constant

And $\boxed{F = \rho u}$ Strength of convection

Case-1 $\rightarrow u > 0$

$$F_e \phi_P - F_w \phi_W = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

$$\Rightarrow \phi_P = \frac{(D_w + F_w)\phi_W + D_e \phi_E}{D_w + D_e + F_e}$$

Case-2 $\rightarrow u < 0$

$$F_e \phi_E - F_w \phi_P = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$$

$$\Rightarrow D_e \phi_P + D_w \phi_P - F_w \phi_P = D_e \phi_E + D_w \phi_W - F_e \phi_E$$

$$\Rightarrow \phi_P = \frac{(D_e - F_e)\phi_E + D_w \phi_W}{D_e + D_w - F_w}$$

/ 7 P $D_e + D_w - F_w$

For upwind scheme general equation can be written as,

$a_W = D_w + \max(F_w, 0)$

$a_E = D_e + \max(0, -F_e)$