Name______________________ Class______________ Date__________

6.1 Adding and Subtracting Polynomials

Resource Locker

Essential Question: How do you add or subtract two polynomials, and what type of expression is the result?

Explore Identifying and Analyzing Monomials and Polynomials

A polynomial function of degree n has the *standard form* $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_2, a_1$, and a_0 are real numbers and $a_n \neq 0$. The expression $a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0$ is called a **polynomial**, and each term of a polynomial is called a **monomial**. A monomial is the product of a number and one or more variables with whole-number exponents. A polynomial is a monomial or a sum of monomials. The *degree of a monomial* is the sum of the exponents of the variables, and the *degree of a polynomial* is the degree of the monomial term with the greatest degree. The *leading coefficient* of a polynomial is the coefficient of the term with the greatest degree.

(A) Identify the monomials: x^3, $y + 3y^2 - 5y^3 + 10$, a^2bc^{12}, 76

Monomials: ______________

Not monomials: ______________

(B) Identify the degree of each monomial.

Monomial	x^3	a^2bc^{12}	76
Degree			

(C) Identify the terms of the polynomial $y + 3y^2 - 5y^3 + 10$. ______________

(D) Identify the coefficient of each term.

Term	y	$3y^2$	$-5y^3$	10
Coefficient				

(E) Identify the degree of each term.

Term	y	$3y^2$	$-5y^3$	10
Degree				

(F) Write the polynomial in standard form. ______________

(G) What is the leading coefficient of the polynomial? ______________

Complete these exercises to review skills you will need for this module.

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

Adding and Subtracting Polynomials

Example 1 Subtract.

$(7a^3 - 4a^2 + 11) - (3a^2 - 2a + 5)$

$7a^3 - 4a^2 + 11 - 3a^2 + 2a - 5$ Multiply by -1.

$7a^3 - 7a^2 + 2a + 6$ Combine like terms.

Add or subtract the polynomials.

1. $(m^5 + 4m^2 + 6) - (3m^5 - 8m^2)$

2. $(k^2 + 3k + 1) + (k^2 - 8)$

Algebraic Expressions

Example 2 Simplify the expression $5x^3 - 10x^2 + x^3 + 10$.

$6x^3 - 10x^2 + 10$ Combine like terms.

Simplify each expression.

3. $6x - 2x^2 - 2x$

4. $(5x)(2x^2) - x^2$

5. $4(2x - 3y) + 2(x + y)$

6. $4(a + b) - 7(a + 2b)$

Multiplying Polynomials

Example 3 Multiply. $(2a - b)(a + ab + b)$

$(2a - b)(a + ab + b) = 2a(a + ab + b) - b(a + ab + b)$

$= 2a \cdot a + 2a \cdot ab + 2a \cdot b - b \cdot a - b \cdot ab - b \cdot b$

$= 2a^2 + 2a^2b + 2ab - ab - ab^2 - b^2$

$= 2a^2 + 2a^2b + ab - ab^2 - b^2$

Multiply the polynomials.

7. $(x^2 - 4)(x + y)$

8. $(3m + 2)(3m^2 - 2m + 1)$

Explain 1 Adding Polynomials

add polynomials, combine like terms.

xample 1 **Add the polynomials.**

) $(4x^2 - x^3 + 2 + 5x^4) + (-x + 6x^2 + 3x^4)$

$$\begin{array}{rrrrr} 5x^4 & -x^3 & +4x^2 & & +2 \\ +3x^4 & & +6x^2 & -x & \\ \hline 8x^4 & -x^3 & +10x^2 & -x & +2 \end{array}$$

Write in standard form.

Align like terms.

Add.

) $(10x - 18x^3 + 6x^4 - 2) + (-7x^4 + 5 + x + 2x^3)$

$(6x^4 - 18x^3 + 10x - 2) + (-7x^4 + 2x^3 + x + 5)$ Write in standard form.

$= (6x^4 - \boxed{}) + (\boxed{} + 2x^3) + (\boxed{} + x) + (-2 + \boxed{})$ Group like terms.

$= \boxed{} - 16x^3 + \boxed{} + 3$ Add.

Reflect

Is the sum of two polynomials always a polynomial? Explain.

our Turn

d the polynomials.

$(17x^4 + 8x^2 - 9x^7 + 4 - 2x^3) + (11x^3 - 8x^2 + 12)$

$(-8x + 3x^{11} + x^6) + (4x^4 - x + 17)$

Ⓐ $(12x^3 + 5x - 8x^2 + 19) - (6x^2 - 9x + 3 - 18x^3)$

Write in standard form.	$12x^3$	$-8x^2$	$+5x$	$+19$
Align like terms and add the opposite.	$+18x^3$	$-6x^2$	$+9x$	-3
Add.	$30x^3$	$-14x^2$	$+14x$	$+16$

Ⓑ $(-4x^2 + 8x^3 + 19 - 5x^5) - (9 + 2x^2 + 10x^5)$

Write in standard form and add the opposite. $(-5x^5 + 8x^3 - 4x^2 + 19) + (-10x^5 - 2x^2 - 9)$

Group like terms $= \left(-5x^5 - \boxed{}\right) + \left(\boxed{}\right) + \left(\boxed{} - 2x^2\right) + \left(\boxed{} - 9\right)$

Add $= \boxed{} + 8x^3 - \boxed{} + 10$

Reflect

5. Is the difference of two polynomials always a polynomial? Explain.

____

____

____

Your Turn

Subtract the polynomials.

6. $(23x^7 - 9x^4 + 1) - (-9x^4 + 6x^2 - 31)$

7. $(7x^3 + 13x - 8x^5 + 20x^2) - (-2x^5 + 9x^2)$

Explain 3 Modeling with Polynomial Addition and Subtraction

Polynomial functions can be used to model real-world quantities. If two polynomial functions model quantities that are two parts of a whole, the functions can be added to find a function that models the quantity as a whole. If the polynomial function for the whole and a polynomial function for a part are given, subtraction can be used to find the polynomial function that models the other part of the whole.

Example 3 **Find the polynomial that models the problem and use it to estimate the quantity.**

(A) The data from the U.S. Census Bureau for 2005–2009 shows that the number of male students enrolled in high school in the United States can be modeled by the function $M(x) = -10.4x^3 + 74.2x^2 - 3.4x + 8320.2$, where x is the number of years after 2005 and $M(x)$ is the number of male students in thousands. The number of female students enrolled in high school in the United States can be modeled by the function $F(x) = -13.8x^3 + 55.3x^2 + 141x + 7880$, where x is the number of years after 2005 and $F(x)$ is the number of female students in thousands. Estimate the total number of students enrolled in high school in the United States in 2009.

In the equation $T(x) = M(x) + F(x)$, $T(x)$ is the total number of students in thousands.

Add the polynomials.

$$\left(-10.4x^3 + 74.2x^2 - 3.4x + 8320.2\right) + \left(-13.8x^3 + 55.3x^2 + 141x + 7880\right)$$

$$= \left(-10.4x^3 - 13.8x^3\right) + \left(74.2x^2 + 55.3x^2\right) + \left(-3.4x + 141x\right) + \left(8320.2 + 7880\right)$$

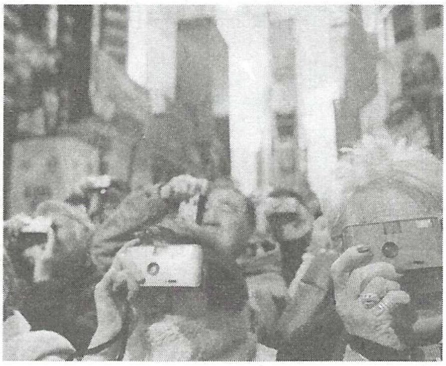
$$= -24.2x^3 + 129.5x^2 + 137.6x + 16{,}200.2$$

The year 2009 is 4 years after 2005, so substitute 4 for x.

$$-24.2(4)^3 + 129.5(4)^2 + 137.6(4) + 16{,}200.2 \approx 17{,}274$$

About 17,274 thousand students were enrolled in high school in the United States in 2009.

(B) The data from the U.S. Census Bureau for 2000–2010 shows that the total number of overseas travelers visiting New York and Florida can be modeled by the function $T(x) = 41.5x^3 - 689.1x^2 + 4323.3x + 2796.6$, where x is the number of years after 2000 and $T(x)$ is the total number of travelers in thousands. The number of overseas travelers visiting New York can be modeled by the function $N(x) = -41.6x^3 + 560.9x^2 - 1632.7x + 6837.4$, where x is the number of years after 2000 and $N(x)$ is the number of travelers in thousands. Estimate the total number of overseas travelers to Florida in 2008.



In the equation $F(x) = T(x)$ ☐ $N(x)$, $F(x)$ is the number of travelers to Florida in thousands.

Subtract the polynomials.

$$\left(41.5x^3 - 689.1x^2 + 4323.3x + 2796.6\right) \;\square\; \left(-41.6x^3 + 560.9x^2 - 1632.7x + 6837.4\right)$$

$$= \left(41.5x^3 - 689.1x^2 + 4323.3x + 2796.6\right) + \left(41.6x^3 - 560.9x^2 + 1632.7x - 6837.4\right)$$

$= (41.3x^3 + \square) + (\square \; 360.9x^2) + (\square + 1632.7x) + (2798.6 \; \square)$

$= \square x^3 - \square x^2 + \square x - \square$

The year 2008 is 8 years after 2000, so substitute $\square$ for x.

$83.1(8)^3 - 1250(8)^2 + 5956(8) - 4040.8 \approx \square$

About $\square$ thousand overseas travelers visited Florida in 2008.

ur Turn

According to the data from the U.S. Census Bureau for 1990–2009, the number of commercially owned automobiles in the United States can be modeled by the function $A(x) = 1.4x^3 - 130.6x^2 + 1831.3x + 128{,}141$, where x is the number of years after 1990 and $A(x)$ is the number of automobiles in thousands. The number of privately-owned automobiles in the United States can be modeled by the function $P(x) = -x^3 + 24.9x^2 - 177.9x + 1709.5$, where x is the number of years after 1990 and $P(x)$ is the number of automobiles in thousands. Estimate the total number of automobiles owned in 2005.

Elaborate

How is the degree of a polynomial related to the degrees of the monomials that comprise the polynomial?

How is polynomial subtraction based on polynomial addition?

How would you find the model for a whole if you have polynomial functions that are models for the two distinct parts that make up that whole?

Essential Question Check-In What is the result of adding or subtracting polynomials?

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. Write the polynomial $-23x^7 + x^9 - 6x^3 + 10 + 2x^2$ in standard form, and then identify the degree and leading coefficient.

Add the polynomials.

2. $(82x^8 + 21x^2 - 6) + (18x + 7x^8 - 42x^2 + 3)$

3. $(15x - 121x^{12} + x^9 - x^7 + 3x^2) + (x^7 - 68x^2 - x^9)$

4. $(16 - x^2) + (-18x^2 + 7x^5 - 10x^4 + 5)$

5. $(x + 1 - 3x^2) + (8x + 21x^2 - 1)$

6. $(64 + x^3 - 8x^2) + (7x + 3 - x^2) + (19x^2 - 7x - 2)$

7. $(x^4 - 7x^3 + 2 - x) + (2x^3 - 3) + (1 - 5x^3 - x^4 + x)$

Subtract the polynomials.

8. $(-2x + 23x^5 + 11) - (5 - 9x^3 + x)$

Find the polynomial that models the problem and use it to estimate the quantity.

14. A rectangle has a length of x and a width of $5x^3 + 4 - x^2$. Find the perimeter of the rectangle when the length is 5 feet.

15. A rectangle has a perimeter of $6x^3 + 9x^2 - 10x + 5$ and a length of x. Find the width of the rectangle when the length is 21 inches.

16. Cho is making a rectangular garden, where the length is x feet and the width is $4x - 1$ feet. He wants to add garden stones around the perimeter of the garden once he is done. If the garden is 4 feet long, how many feet will Cho need to cover with garden stones?

17. Employment The data from the U.S. Census Bureau for 1980–2010 shows that the median weekly earnings of full-time male employees who have at least a bachelor's degree can be modeled by the function $M(x) = 0.009x^3 - 0.29x^2 + 30.7x + 439.6$, where x is the number of years after 1980 and $M(x)$ is the median weekly earnings in dollars. The median weekly earnings of all full-time employees who have at least a bachelor's degree can be modeled by the function $T(x) = 0.012x^3 - 0.46x^2 + 56.1x + 732.3$, where x is the number of years after 1980 and $T(x)$ is the median weekly earnings in dollars. Estimate the median weekly earnings of a full-time female employee with at least a bachelor's degree in 2010.

20. Geography The data from the U.S. Census Bureau for 1982–2003 shows that the surface area of the United States that is covered by rural land can be modeled by the function $R(x) = 0.003x^3 - 0.086x^2 - 1.2x + 1417.4$, where x is the number of years after 1982 and $R(x)$ is the surface area in millions of acres. The total surface area of the United States can be modeled by the function $T(x) = 0.0023x^3 + 0.034x^2 - 5.9x + 1839.4$, where x is the number of years after 1982 and $T(x)$ is the surface area in millions of acres. Estimate the surface area of the United States that is not covered by rural land in 2001.

21. Determine which polynomials are monomials. Choose all that apply.

a. $4x^3y$

b. $12 - x^2 + 5x$

c. $152 + x$

d. 783

e. x

f. $19x^{-2}$

g. $4x^4x^2$

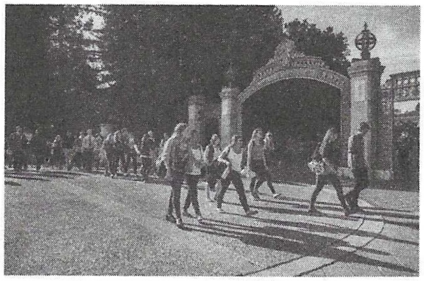
H.O.T. Focus on Higher Order Thinking

22. Explain the Error Colin simplified $\left(16x + 8x^2y - 7xy^2 + 9y - 2xy\right) - \left(-9xy + 8xy^2 + 10x^2y + x - 7y\right)$. His work is shown below. Find and correct Colin's mistake.

$$\begin{aligned}
&\left(16x + 8x^2y - 7xy^2 + 9y - 2xy\right) - \left(-9xy + 8xy^2 + 10x^2y + x - 7y\right) \\
&= \left(16x + 8x^2y - 7xy^2 + 9y - 2xy\right) + \left(9xy - 8xy^2 - 10x^2y - x + 7y\right) \\
&= \left(16x - x\right) + \left(8x^2y - 7xy^2 - 8xy^2 - 10x^2y\right) + \left(9y + 7y\right) + \left(-2xy + 9xy\right) \\
&= 15x - 17x^2y^2 + 16y + 7xy
\end{aligned}$$

18. **Business** From data gathered in the period 2008–2012, the yearly value of U.S. exports can be modeled by the function $E(x) = -228x^3 + 2552.8x^2 - 6098.5x + 11{,}425.8$, where x is the number of years after 2008 and $E(x)$ is the value of exports in billions of dollars. The yearly value of U.S. imports can be modeled by the function $I(x) = -400.4x^3 + 3954.4x^2 - 11{,}128.8x + 17{,}749.6$, where x is the number of years after 2008 and $I(x)$ is the value of imports in billions of dollars. Estimate the total value the United States imported and exported in 2012.

19. **Education** From data gathered in the period 1970–2010, the number of full-time students enrolled in a degree-granting institution can be modeled by the function $F(x) = 8.7x^3 - 213.3x^2 + 2015.5x + 3874.9$, where x is the number of years after 1970 and $F(x)$ is the number of students in thousands. The number of part-time students enrolled in a degree-granting institution can be modeled by the function $P(x) = 12x^3 - 285.3x^2 + 2217x + 1230$, where x is the number of years after 1970 and $P(x)$ is the number of students in thousands. Estimate the total number of students enrolled in a degree-granting institution in 2000.



23. Critical Reasoning Janice is building a fence around a portion of her rectangular yard. The length of yard she will enclose is x, and the width is $2x^2 - 98x + 5$, where the measurements are in feet. If the length of the enclosed yard is 50 feet and the cost of fencing is $13 per foot, how much will Janice need to spend on fencing?

24. Multi-Step Find a polynomial expression for the perimeter of a trapezoid with legs of length x and bases of lengths $0.1x^3 + 2x$ and $x^2 + 3x - 10$ where each is measured in inches.

a. Find the perimeter of the trapezoid if the length of one leg is 6 inches.

b. If the leg length is increased by 5 inches, will the perimeter also increase? By how much?

25. Communicate Mathematical Ideas Present a formal argument for why the set of polynomials is closed under addition and subtraction. Use the polynomials $ax^m + bx^m$ and $ax^m - bx^m$, for real numbers a and b and whole number m, to justify your reasoning.

Lesson Performance Task

The table shows the average monthly maximum and minimum temperatures for Death Valley throughout one year.

Month	Maximum Temperature	Minimum Temperature
January	67	40
February	73	46
March	82	55
April	91	62
May	101	73
June	110	81
July	116	88
August	115	86
September	107	76
October	93	62
November	77	48
December	65	38

Use a graphing calculator to find a good fourth-degree polynomial regression model for both the maximum and minimum temperatures. Then find a function that models the range in monthly temperatures and use the model to estimate the range during September. How does the range predicted by your model compare with the range shown in the table?

Name ____________________ Class ____________ Date ________

6.2 Multiplying Polynomials

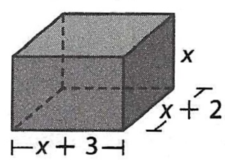
Essential Question: How do you multiply polynomials, and what type of expression is the result?

Resource Locker

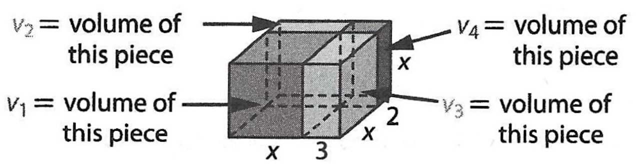
Explore Analyzing a Visual Model for Polynomial Multiplication

The volume of a rectangular prism is the product of the length, width, and height of that prism. If the dimensions are all known, then the volume is a simple calculation. What if some of the dimensions are given as *binomials*? A **binomial** is a polynomial with two terms. How would you find the volume of a rectangular prism that is $x + 3$ units long, $x + 2$ units wide, and x units high? The images below show two methods for finding the solution.

$$V = \text{length} \times \text{width} \times \text{height}$$
$$= (x + 3)(x + 2)x$$



$$V = V_1 + V_2 + V_3 + V_4$$



(A) The first model shows the rectangular prism, and its volume is calculated directly as the product of two binomials and a monomial.

(B) The second image divides the rectangular prism into ________ smaller prisms, the dimensions of which are each ________.

(C) The volume of a cube (V_1) where all sides have a length of x, is ________.

(D) The volume of a rectangular prism (V_2) with dimensions x by x by 2 is ________.

(E) The volume of a rectangular prism (V_3) with dimensions x by x by 3 is ________.

(F) The volume of a rectangular prism (V_4) with dimensions x by 3 by 2 is ________.

(G) So the volume of the rectangular prism is the sum of the volumes of the four smaller regions.

$$V_1 + V_2 + V_3 + V_4 = \boxed{} + \boxed{} + \boxed{} + \boxed{}$$
$$= \boxed{}$$

(B) $(3x - 4)(2 + x - 7x^2)$

Find the product by multiplying vertically.

$-7x^2 + \boxed{} + 2$ $3x - 4$	Write each polynomial in standard form.
$\boxed{} - 4x - 8$	Multipy -4 and $(-7x^2 + x + 2)$.
$\boxed{} + 3x^2 + 6x$	Multipy $\boxed{}$ and $(-7x^2 + x + 2)$.
$-21x^3 + \boxed{} + 2x - 8$	Combine like terms.

Therefore, $(3x - 4)(2 + x - 7x^2) =$ ______________________.

Your Turn

3. $(3 + 2x)(4 - 7x + 5x^2)$

4. $(x - 6)(3 - 8x - 4x^2)$

Explain 2 Modeling with Polynomial Multiplication

Many real-world situations can be modeled with polynomial functions. Sometimes, a situation will arise in which a model is needed that combines two quantities modeled by polynomial functions. In this case, the desired model would be the product of the two known models.

Example 2 **Find the polynomial function modeling the desired relationship.**

(A) Mr. Silva manages a manufacturing plant. From 1990 through 2005, the number of units produced (in thousands) can be modeled by $N(x) = 0.02x^2 + 0.2x + 3$, where x is the number of years since 1990. The average cost per unit (in dollars) can be modeled by $C(x) = -0.002x^2 - 0.1x + 2$, where x is the number of years since 1990. Write a polynomial $T(x)$ that can be used to model Mr. Silva's total manufacturing cost for those years.

The total manufacturing cost is the product of the number of units made and the cost per unit.

$T(x) = N(x) \cdot C(x)$

Your Turn

5. Brent runs a small toy store specializing in wooden toys. From 2000 through 2012, the number of toys Brent made can be modeled by $N(x) = 0.7x^2 - 2x + 23$, and the average cost to make each toy can be modeled by $C(x) = -0.004x^2 - 0.08x + 25$, where x is the number of years since 2000. Write a polynomial that can be used to model Brent's total cost for making the toys, $T(x)$, for those years.

Explain 3 Verifying Polynomial Identities

You have already seen certain special polynomial relationships. For example, a difference of two squares can be easily factored: $x^2 - a^2 = (x + a)(x - a)$. This equation is an example of a **polynomial identity**, a mathematical relationship equating one polynomial quantity to another. Another example of a polynomial identity is

$$(x + a)^2 - (x - a)^2 = 4ax.$$

The identity can be verified by simplifying one side of the equation to match the other.

Example 3 **Verify the given polynomial identity.**

(A) $(x + a)^2 - (x - a)^2 = 4ax$

The right side of the identity is already fully simplified. Simplify the left-hand side.

$$(x + a)^2 - (x - a)^2 = 4ax$$

$$x^2 + 2ax + a^2 - (x^2 - 2ax + a^2) = 4ax \qquad \text{Square each binomial.}$$

$$x^2 + 2ax + a^2 - x^2 + 2ax - a^2 = 4ax \qquad \text{Distribute the negative.}$$

$$\cancel{x^2 - x^2} + 2ax + 2ax + \cancel{a^2 - a^2} = 4ax \qquad \text{Rearrange terms.}$$

$$4ax = 4ax \qquad \text{Simplify.}$$

Therefore, $(x + a)^2 - (x - a)^2 = 4ax$ is a true statement.

(B) The identity $(x + y)^2 = x^2 + 2xy + y^2$ can be used for mental-math calculations to quickly square numbers.

Find the square of 27.

Find two numbers whose sum is equal to 27.

Let $x = \boxed{}$ and $y = 7$

Evaluate

$$\left(20 + \boxed{}\right)^2 = 20^2 + \boxed{} + 7^2$$

$$27^2 = 400 + \boxed{} + 49$$

$$27^2 = \boxed{}$$

Verify by using a calculator to find 27^2.

$27^2 = \boxed{}$

Your Turn

8. The identity $(x + y)(x - y) = x^2 - y^2$ can be used for mental-math calculations to quickly multiply two numbers in specific situations.

Find the product of 37 and 43. (Hint: What values should you choose for x and y so the equation calculates the product of 37 and 43?)

9. The identity $(x - y)^2 = x^2 - 2xy + y^2$ can also be used for mental-math calculations to quickly square numbers.

Find the square of 18. (Hint: What values should you choose for x and y so the equation calculates the square of 18?)

4. $(x^2 + 9x + 7)(3x^2 + 9x + 5)$

5. $(2x + 5y)(3x^2 - 4xy + 2y^2)$

6. $(x^3 + x^2 + 1)(x^2 - x - 5)$

7. $(4x^2 + 3x + 2)(3x^2 + 2x - 1)$

11. **Physics** An object thrown in the air has a velocity after t seconds that can be described by $v(t) = -9.8t + 24$ (in meters/second) and a height $h(t) = -4.9t^2 + 24t + 60$ (in meters). The object has mass $m = 2$ kilograms. The kinetic energy of the object is given by $K = \frac{1}{2}mv^2$, and the potential energy is given by $U = 9.8mh$. Find an expression for the total kinetic and potential energy $K + U$ as a function of time. What does this expression tell you about the energy of the falling object?

Verify the given polynomial identity.

12. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$

13. $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

14. $x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$

23. Determine how many terms there will be after performing the polynomial multiplication.

a. $(5x)(3x)$	☐ 1	☐ 2	☐ 3	☐ 4
b. $(3x)(2x + 1)$	☐ 1	☐ 2	☐ 3	☐ 4
c. $(x + 1)(x - 1)$	☐ 1	☐ 2	☐ 3	☐ 4
d. $(x + 2)(3x^2 - 2x + 1)$	☐ 1	☐ 2	☐ 3	☐ 4

H.O.T. Focus on Higher Order Thinking

24. Multi-Step Given the polynomial identity: $x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$

a. Verify directly by expanding the right hand side.

b. Use another polynomial identity to verify this identity. $\left(\text{Note that } a^6 = (a^2)^3 = (a^3)^2\right)$

25. Communicate Mathematical Ideas Explain why the set of polynomials is closed under multiplication.

26. Critical Thinking Explain why every other term of the polynomial product $(x - y)^5$ written in standard form is subtracted when $(x - y)$ is raised to the fifth power.

Lesson Performance Task

The table presents data about oil wells in the state of Oklahoma from 1992 through 2008.

Year	Number of Wells	Average Daily Oil Production per Well (Barrels)
2008	83,443	2.178
2007	82,832	2.053
2006	82,284	2.108
2005	82,551	2.006
2004	83,222	2.10
2003	83,415	2.12
2002	83,730	2.16
2001	84,160	2.24
2000	84,432	2.24
1999	85,043	2.29
1998	85,691	2.49
1997	86,765	2.62
1996	88,144	2.66
1995	90,557	2.65
1994	91,289	2.73
1993	92,377	2.87
1992	93,192	2.99

a. Given the data in this table, use regression to find models for the number of producing wells (cubic regression) and average daily well output (quadratic regression) in terms of t years since 1992.

b. Find a function modeling the total daily oil output for the state of Oklahoma.

15. $(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2$

Find the square of the number or the product of the numbers using one or more of these identities.

$(x + y)^2 = x^2 + 2xy + y^2$, $(x + y)(x - y) = x^2 - y^2$, or $(x - y)^2 = x^2 - 2xy + y^2$.

16. 43^2

17. 32^2

18. 89^2

19. 47^2

20. $54 \cdot 38$

21. $58 \cdot 68$

22. **Explain the Error** A polynomial identity for the difference of two cubes is $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. A student uses the identity to factor $64 - 27x^6$. Identify the error the student made, and then correct it.

Each term of $64 - 27x^6$ is a perfect cube. Let $a = 4$ and $b = 3x^2$. Then:

$$\begin{aligned} 64 - 27x^6 &= 4^3 - (3x^2)^3 \\ &= (4 - 3x^2)(4^2 + 4(-3x^2) + (-3x^2)^2) \\ &= (4 - 3x^2)(16 - 12x^2 + 9x^4) \end{aligned}$$

Write a polynomial function to represent the new value.

8. The volume of a stock, or number of shares traded, is modeled over time during a given day by $S(x) = x^5 - 3x^4 + 10x^2 - 6x + 30$. The cost per share of that stock during that day is modeled by $C(x) = 0.004x^4 - 0.02x^2 + 0.3x + 4$. Write a polynomial function $V(x)$ to model the changing value during that day of the trades made of shares of that stock.

9. A businessman models the number of items (in thousands) that his company sold from 1998 through 2004 as $N(x) = -0.1x^3 + x^2 - 3x + 4$ and the average price per item (in dollars) as $P(x) = 0.2x + 5$, where x represents the number of years since 1998. Write a polynomial $R(x)$ that can be used to model the total revenue for this company.

10. Biology A biologist has found that the number of branches on a certain rare tree can be modeled by the polynomial $b(y) = 4y^2 + y$ where y is the number of years after the tree reaches a height of 6 feet. The number of leaves on each branch can be modeled by the polynomial $l(y) = 2y^3 + 3y^2 + y$. Write a polynomial describing the total number of leaves on the tree.

Elaborate

10. What property is employed in the process of polynomial multiplication?

11. How can you use unit analysis to justify multiplying two polynomial models of real-world quantities?

12. Give an example of a polynomial identity and how it's useful.

13. **Essential Question Check-In** When multiplying polynomials, what type of expression is the product?

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. The dimensions for a rectangular prism are $x + 5$ for the length, $x + 1$ for the width, and x for the height. What is the volume of the prism?

Perform the following polynomial multiplications.

2. $(3x - 2)(2x^2 + 3x - 1)$

3. $(x^3 + 3x^2 + 1)(3x^2 + 6x - 2)$

Ⓑ $(a+b)(a^2-ab+b^2)=a^3+b^3$

The right side of the identity is already fully simplified. Simplify the left-hand side.

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$

$a(a^2)+a\left(\boxed{}\right)+a(b^2)+b(a^2)+\boxed{}(-ab)+b(b^2)=a^3+b^3$ Distribute a and b.

$a^3-a^2b+ab^2+\boxed{}-ab^2+\boxed{}=a^3+b^3$ ________

$a^3-\boxed{}+a^2b+ab^2-\boxed{}+b^3=a^3+b^3$ Rearrange terms.

$a^3\boxed{}b^3=a^3+b^3$ Combine like terms.

Therefore, $(a+b)(a^2-ab+b^2)=a^3+b^3$ is a ______ statement.

Your Turn

6. Show that $a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$.

7. Show that $(a-b)(a^2+ab+b^2)=a^3-b^3$.

Explain 4 Using Polynomial Identities

The most obvious use for polynomial identities is simplifying algebraic expressions, but polynomial identities often turn out to have nonintuitive uses as well.

Example 4 **For each situation, find the solution using the given polynomial identity.**

Ⓐ The polynomial identity $(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$ can be used to identify Pythagorean triples. Generate a Pythagorean triple using $x=4$ and $y=3$.

Substitute the given values into the identity.

$$(4^2+3^2)^2=(4^2-3^2)^2+(2\cdot4\cdot3)^2$$
$$(16+9)^2=(16-9)^2+(24)^2$$
$$(25)^2=(7)^2+(24)^2$$
$$625=49+576$$
$$625=625$$

Therefore, 7, 24, 25 is a Pythagorean triple.

Multiply the two polynomials.

$$\begin{array}{rrrrr}
 & & 0.02x^2 & + 0.2x & + 3 \\
 & \times & -0.002x^2 & - 0.1x & + 2 \\
\hline
 & & 0.04x^2 & + 0.4x & + 6 \\
 & -0.002x^3 & - 0.02x^2 & - 0.3x & \\
-0.00004x^4 & - 0.0004x^3 & - 0.006x^2 & & \\
\hline
-0.00004x^4 & - 0.0024x^3 & + 0.014x^2 & + 0.1x & + 6
\end{array}$$

Therefore, the total manufacturing cost can be modeled by the following polynomial, where x is the number of years since 1990.

$T(x) = -0.00004x^4 - 0.0024x^3 + 0.014x^2 + 0.1x + 6$

(B) Ms. Liao runs a small dress company. From 1995 through 2005, the number of dresses she made can be modeled by $N(x) = 0.3x^2 - 1.6x + 14$, and the average cost to make each dress can be modeled by $C(x) = -0.001x^2 - 0.06x + 8.3$, where x is the number of years since 1995. Write a polynomial that can be used to model Ms. Liao's total dressmaking costs, $T(x)$, for those years.

The total dressmaking cost is the product of the number of dresses made and the cost per dress.

$T(x) = N(x) \cdot C(x)$

Multiply the two polynomials.

$$\begin{array}{rrrrr}
 & & 0.3x^2 & - 1.6x & + 14 \\
 & \times & -0.001x^2 & \boxed{} & + 8.3 \\
\hline
 & & 2.49x^2 & - 13.28x & \boxed{} \\
 & -0.018x^3 & \boxed{} & - 0.84x & \\
-0.0003x^{\boxed{}} & + 0.0016x^3 & - 0.014x^2 & & \\
\hline
-0.0003x^{\boxed{}} & - 0.0164x^3 & + 2.572x^2 & \boxed{} & + 116.2
\end{array}$$

Therefore, the total dressmaking cost can be modeled by the following polynomial, where x is the number of years since 1995.

$T(x) =$ ______________________________

Reflect

1. If all three dimensions were binomials, how many regions would the rectangular prism be divided into?

2. **Discussion** Can this method be applied to finding the volume of other simple solids? Are there solids that this process would be difficult to apply to? Are there any solids that this method cannot be applied to?

Explain 1 Multiplying Polynomials

Multiplying polynomials involves using the product rule for exponents and the distributive property. The product of two monomials is the product of the coefficients and the sum of the exponents of each variable.

$$5x \cdot 6x^3 = 30x^{1+3} \qquad -2x^2y^4z \cdot 5y^2z = -10x^2y^{4+2}z^{1+1}$$
$$= 30x^4 \qquad = -10x^2y^6z^2$$

When multiplying two binomials, the distributive property is used. Each term of one polynomial must be multiplied by each term of the other.

$$\begin{aligned}(2+3x)(1+x) &= 2(1+x) + 3x(x+1)\\ &= 2(1) + 2(x) + 3x(x) + 3x(1)\\ &= 2 + 2x + 3x^{1+1} + 3x\\ &= 2 + 5x + 3x^2\end{aligned}$$

The polynomial $2 + 5x + 3x^2$ is called a **trinomial** because it has three terms.

Example 1 **Perform the following polynomial multiplications.**

(A) $(x+2)(1-4x+2x^2)$

Find the product by multiplying horizontally.

$(x+2)(2x^2-4x+1)$	Write the polynomials in standard form.
$x(2x^2) + x(-4x) + x(1) + 2(2x^2) + 2(-4x) + 2(1)$	Distribute the x and the 2.
$2x^3 - 4x^2 + x + 4x^2 - 8x + 2$	Simplify.
$2x^3 - 7x + 2$	Combine like terms.

Therefore, $(x+2)(2x^2-4x+1) = 2x^3 - 7x + 2$.

Name________________ Class__________ Date______

6.3 The Binomial Theorem

Essential Question: How is the Binomial Theorem useful?

Resource Locker

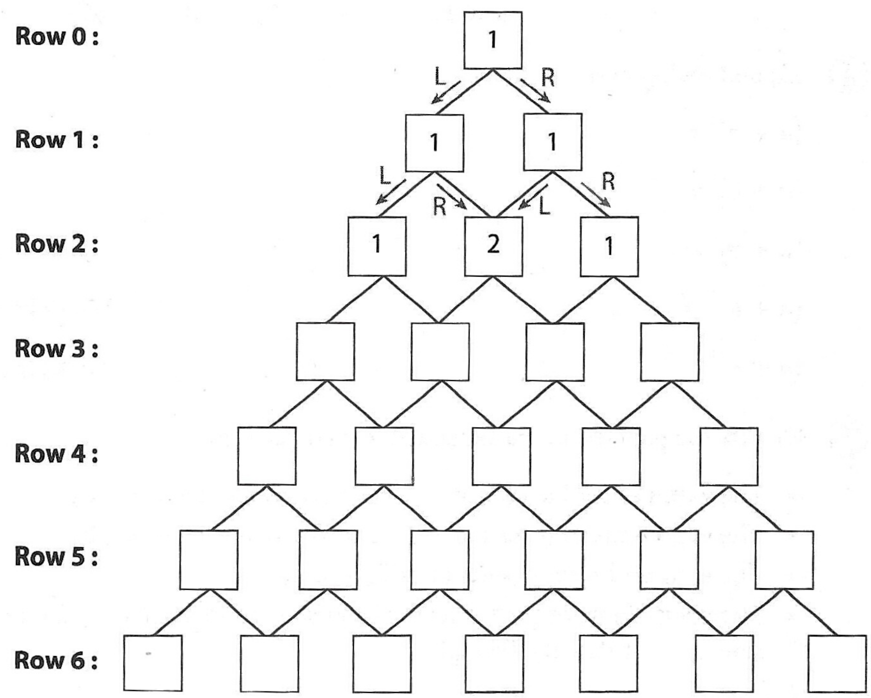
Explore 1 Generating Pascal's Triangle

Pascal's Triangle is a famous number pattern named after the French mathematician Blaise Pascal (1623–1662). You can use Pascal's Triangle to help you expand a power of a binomial of the form $(a + b)^n$.

Use the tree diagram shown to generate Pascal's Triangle. Notice that from each node in the diagram to the nodes immediately below it there are two paths, a left path (L) and a right path (R). You can describe a path from the single node in row 0 to any other node in the diagram using a string of Ls and Rs.

First, notice that there is only one possible path to each node in row 1, which is why a 1 appears in those nodes. In row 2, there is only one possible path, LL, to the first node and only one possible path, RR, to the last node, but there are two possible paths, LR and RL, to the center node.

(A) Complete only rows 3 and 4 of Pascal's Triangle. (You will complete rows 5 and 6 in Step C.) In each node, write the number of possible paths from the top down to that node.



(B) Look for patterns in the tree diagram.

What is the value in the first and last node in each row? ______

For every other node, the value in the node is the ______ of the two values above it.

(C) Using the patterns in Step B, go back to Pascal's Triangle in Step A and complete rows 5 and 6.

Explain 1 Expanding Powers of Binomials Using the Binomial Theorem

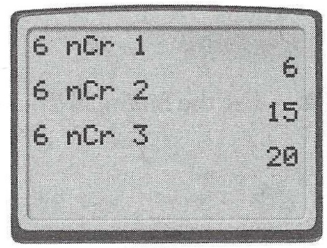
The **Binomial Theorem** states the connection between the terms of the expanded form of $(a+b)^n$ and Pascal's Triangle.

Binomial Theorem

For any whole number n, the binomial expansion of $(a+b)^n$ is given by
$(a+b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_{n-1}a^1b^{n-1} + {}_nC_na^0b^n$
where ${}_nC_r$ is the value in position r (where r starts at 0) of the nth row of Pascal's Triangle.

Since it can be cumbersome to look up numbers from Pascal's Triangle each time you want to expand a power of a binomial, you can use a calculator instead. To do so, enter the value of n, press MATH, go to the **PRB** menu, select **3:nCr**, and then enter the value of r. The calculator screen shows the values for ${}_6C_1$, ${}_6C_2$, and ${}_6C_3$.

Example 1 **Use the Binomial Theorem to expand each power of a binomial.**



(A) $(x-2)^3$

Step 1 Identify the values in row 3 of Pascal's Triangle.

1, 3, 3, and 1

Step 2 Expand the power as described by the Binomial Theorem, using the values from Pascal's Triangle as coefficients.

$$(x-2)^3 = 1x^3(-2)^0 + 3x^2(-2)^1 + 3x^1(-2)^2 + 1x^0(-2)^3$$

Step 3 Simplify.

$$(x-2)^3 = x^3 - 6x^2 + 12x - 8$$

(B) $(x+y)^7$

Step 1 Use a calculator to determine the values of ${}_7C_0$, ${}_7C_1$, ${}_7C_2$, ${}_7C_3$, ${}_7C_4$, ${}_7C_5$, ${}_7C_6$, and ${}_7C_7$.

____

Step 2 Expand the power as described by the Binomial Theorem, using the values of ${}_7C_0$, ${}_7C_1$, ${}_7C_2$, ${}_7C_3$, ${}_7C_4$, ${}_7C_5$, ${}_7C_6$, and ${}_7C_7$ as coefficients.

$$(x+y)^7 = \square x^{\square}y^{\square} + \square x^{\square}y^{\square} + \square x^{\square}y^{\square} + \square x^{\square}y^{\square}$$
$$+ \square x^{\square}y^{\square} + \square x^{\square}y^{\square} + \square x^{\square}y^{\square} + \square x^{\square}y^{\square}$$

Step 3 Simplify.

$$(x+y)^7 = x^{\square} + \square x^{\square}y + \square x^{\square}y^{\square} + \square x^{\square}y^{\square} + \square x^{\square}y^{\square}$$
$$+ \square x^{\square}y^{\square} + \square xy^{\square} + y^{\square}$$

Example 2 **One in 5 boats traveling down a river bypass a harbor at the mouth of the river and head out to sea. Currently, 4 boats are traveling down the river and approaching the mouth of the river.**

(A) What is the probability that exactly 2 of the 4 boats head out to sea?

The probability that a boat will head out to sea is $\frac{1}{5}$, or 0.2.

Substitute 4 for n, 2 for r, 0.2 for p, and 0.8 for q.

$$P(2) = {}_4C_2(0.2)^2(0.8)^{4-2}$$
$$= 6(0.2)^2(0.8)^2$$
$$= 6(0.04)(0.64)$$
$$= 0.1536$$

So, the probability that exactly 2 of the 4 boats will head out to sea is 0.1536, or 15.36%.

(B) What is the probability that at least 2 of the 4 boats will head out to sea?

To find the probability that at least 2 of the 4 boats will head out to sea, find the probability that 2, ____, or ____ boats will head out to sea and add the probabilities.

From Part A, you know that $P(2) = 0.1536$.

$$P(3) = {}_4C_{\square}(0.2)^{\square}(0.8)^{\square}$$
$$= 4(\square)(\square)$$
$$= \square$$

$$P(4) = {}_4C_{\square}(0.2)^{\square}(0.8)^{\square}$$
$$= 1(\square)(\square)$$
$$= \square$$

$$P(\text{at least } 2) = P(2 \text{ or } 3 \text{ or } 4)$$
$$= P(2) + P(3) + P(4)$$
$$= 0.1536 + \square + \square$$
$$= \square$$

So, the probability that at least 2 of the 4 boats will head out to sea is 0.1808, or 18.08%.

Evaluate: Homework and Practice

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

1. The path LLRRLLR leads to which node in which row of Pascal's Triangle? What is the value of that node?

2. Without expanding the power, determine the middle term of $(a + b)^8$. Explain how you found your answer.

Use the Binomial Theorem to expand each power of a binomial.

3. $(x + 6)^3$

4. $(x - 5)^4$

5. $(x + 3)^6$

6. $(2x - 1)^3$

7. $(3x + 4)^5$

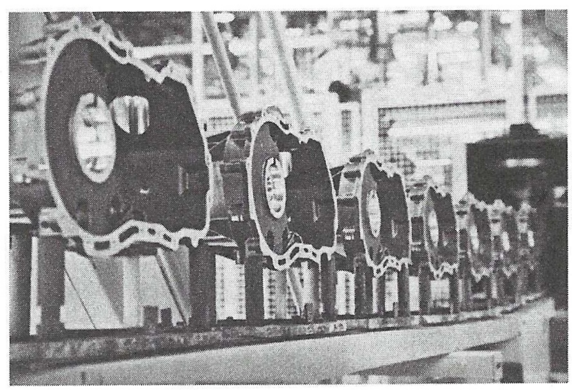
8. $(2x - 3)^7$

Ellen takes a multiple-choice quiz that has 5 questions, with 4 answer choices for each question.

19. What is the probability that she will get exactly 2 answers correct by guessing?

20. What is the probability that Ellen will get at least 3 answers correct by guessing?

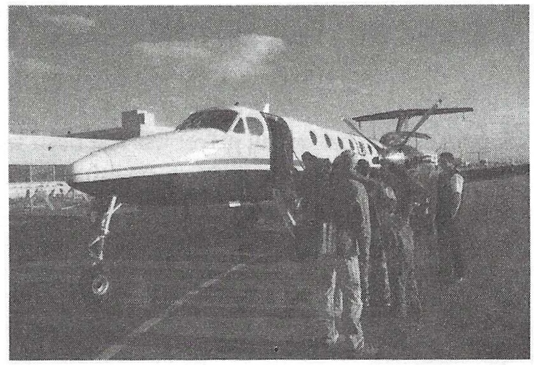
Manufacturing **A machine that makes a part used in cars has a 98% probability of producing the part within acceptable tolerance levels. The machine makes 25 parts per hour.**



21. What is the probability that the machine will make exactly 20 acceptable parts in an hour?

22. What is the probability that the machine makes 23 or fewer acceptable parts?

26. **Represent Real-World Situations** A small airline overbooks flights on the assumption that some passengers will not show up. The probability that a passenger shows up is 0.8. What number of tickets can the airline sell for a 20-seat flight and still have a probability of seating everyone that is at least 90%? Explain your reasoning.



Reflect

1. Using strings of Ls and Rs, write the paths that lead to the second node in row 3 of Pascal's Triangle. How are the paths alike, and how are they different?

2. The path LLRLR leads to which node in which row of Pascal's Triangle? What is the value of that node?

Explore 2 Relating Pascal's Triangle to Powers of Binomials

As shown, the value in position r of row n of Pascal's Triangle is written as ${}_nC_r$, where the position numbers in each row start with 0. In this Explore, you will see how the values in Pascal's Triangle are related to powers of a binomial.

Row 0 : $\longrightarrow$ ${}_0C_0$

Row 1 : $\longrightarrow$ ${}_1C_0$ ${}_1C_1$

Row 2 : $\longrightarrow$ ${}_2C_0$ ${}_2C_1$ ${}_2C_2$

Row 3 : $\longrightarrow$ ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$

(A) Expand each power.

$(a+b)^0 = \boxed{}$

$(a+b)^1 = \boxed{}$

$(a+b)^2 = \boxed{}$ Square of a binomial

$(a+b)^3 = \boxed{}$ Multiply $(a+b)^2$ by $(a+b)$.

$(a+b)^4 = \boxed{}$ Multiply $(a+b)^3$ by $(a+b)$.

(B) Identify the patterns in the expanded form of $(a+b)^n$.

- The exponents of a start at ____ and [increase/decrease] by ____ each term.
- The exponents of b start at ____ and [increase/decrease] by ____ each term.
- The sum of the exponents in each term is ____ .
- The coefficients of the terms in the expanded form of $(a+b)^n$ are the values in row ____ of Pascal's Triangle.

Reflect

3. How many terms are in the expanded form of $(a+b)^n$?

4. Without expanding the power, determine the middle term of $(a+b)^6$. Explain how you found your answer.

5. Without expanding the power, determine the first term of $(a+b)^{15}$. Explain how you found your answer.

Reflect

6. What happens to the signs of the terms in the expanded form of $(x - 2)^3$? Why does this happen?

7. If the number 11 is written as the binomial $(10 + 1)$, how can you use the Binomial Theorem to find 11^2, 11^3, and 11^4? What is the pattern in the digits?

Your Turn

8. Use the Binomial Theorem to expand $(x - y)^4$.

Explain 2 Solving a Real-World Problem Using Binomial Probabilities

Recall that the probability of an event A is written as $P(A)$ and is expressed as a number between 0 and 1, where 0 represents impossibility and 1 represents certainty.

When dealing with probabilities, you will find these two rules helpful.

1. **Addition Rule for Mutually Exclusive Events:** If events A and B are *mutually exclusive* (that is, they cannot occur together), then $P(A \text{ or } B) = P(A) + P(B)$. For example, when rolling a die, getting a 1 and getting a 2 are mutually exclusive events, so $P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

2. **Complement Rule:** The *complement* of event A consists of all of the possible outcomes that are not part of A, and the probability that A does not occur is $P(\text{not } A) = 1 - P(A)$. For example, when rolling a die, the probability of not getting a 2 is $P(\text{not } 2) = 1 - P(2) = 1 - \frac{1}{6} = \frac{5}{6}$.

A **binomial experiment** involves many trials where each trial has only two possible outcomes: success or failure. If the probability of success in each trial is p and the probability of failure in each trial is $q = 1 - p$, the **binomial probability** of exactly r successes in n trials is given by $P(r) = {}_nC_r p^r q^{n-r}$. Since ${}_nC_r = {}_nC_{n-r}$, you can rewrite $P(r)$ as $P(r) = {}_nC_{n-r} p^r q^{n-r}$, which represents the $(n - r)$th term in the expanded form of $(p + q)^n$.

Reflect

9. In words, state the complement of the event that at least 2 of the 4 boats will head out to sea. Then find the probability of the complement.

Your Turn

10. Students are assigned randomly to 1 of 3 guidance counselors at a school. What is the probability that Ms. Banks, one of the school's guidance counselors, will get exactly 2 of the next 3 students assigned?

Elaborate

11. How do the numbers in one row of Pascal's Triangle relate to the numbers in the previous row?

12. How does Pascal's Triangle relate to the power of a binomial?

13. The expanded form of $(p + q)^3$ is $p^3 + 3p^2q + 3pq^2 + q^3$. In terms of a binomial experiment with a probability p of success and a probability q of failure on each trial, what do each of the terms p^3, $3p^2q$, $3pq^2$, and q^3 represent?

14. **Essential Question Check-In** The Binomial Theorem says that the expanded form of $(a + b)^n$ is a sum of terms of the form ${}_nC_r a^{n-r} b^r$ for what values of r?

9. $(x + 2y)^5$

10. $(3x - y)^4$

11. $(5x + y)^4$

12. $(x - 6y)^5$

13. $(5x - 4y)^3$

14. $(4x + 3y)^6$

Use the Binomial Theorem to find the specified term of the given power of a binomial. (Remember that r starts at 0 in the Binomial Theorem, so finding, say, the second term means that $r = 1$.)

15. Find the fourth term in the expanded form of $(x - 1)^6$.

16. Find the second term in the expanded form of $(2x + 1)^4$.

17. Find the third term in the expanded form of $(3x - 2y)^5$.

18. Find the fifth term in the expanded form of $(6x + 8y)^7$.

23. Match each term of an expanded power of a binomial on the right with the corresponding description of the term on the left. (Remember that r starts at 0 in the Binomial Theorem, so finding, say, the second term means that $r = 1$.)

A. Fifth term in the expanded form of $(x + 2)^6$ ______ $640x^2$

B. Fourth term in the expanded form of $(x + 4)^5$ ______ $48x^2$

C. Third term in the expanded form of $(x + 8)^4$ ______ $240x^2$

D. Second term in the expanded form of $(x + 16)^3$ ______ $384x^2$

H.O.T. Focus on Higher Order Thinking

24. **Construct Arguments** Identify the symmetry in the rows of Pascal's Triangle and give an argument based on strings of Ls and Rs to explain why the symmetry exists.

25. **Communicate Mathematical Ideas** Explain why the numbers from Pascal's Triangle show up in the Binomial Theorem.

Lesson Performance Task

Suppose that a basketball player has just been fouled while attempting a 3-point shot and is awarded three free throws. Given that the player is 85% successful at making free throws, calculate the probability that the player successfully makes zero, one, two, or all three of the free throws. Which situation is most likely to occur?

Name ____________________ Class ____________ Date ________

6.4 Factoring Polynomials

Essential Question: What are some ways to factor a polynomial, and how is factoring useful?

Resource Locker

Explore Analyzing a Visual Model for Polynomial Factorization

Factoring a polynomial of degree n involves finding factors of a lesser degree that can be multiplied together to produce the polynomial. When a polynomial has degree 3, for example, you can think of it as the volume of a rectangular prism whose dimensions you need to determine.

(A) The volumes of the parts of the rectangular prism are as follows:

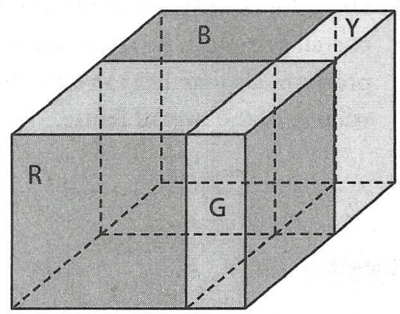
Red(R): $V = x^3$

Green(G): $V = 2x^2$

Yellow(Y): $V = 8x$

Blue(B): $V = 4x^2$

Total volume: $V = x^3 + 6x^2 + 8x$



(B) The volume of the red piece is found by cubing the length of one edge. What is the height of this piece?

____

(C) The volume of a rectangular prism is $V = lwh$, where l is the length, w is the width, and h is the height of the prism. Notice that the green prism shares two dimensions with the cube. What are these dimensions?

____

(D) What is the length of the third edge of the green prism?

____

____

(E) You showed that the width of the cube is ______ and the width of the green prism is ______. What is the width of the entire prism?

____

____

(B) $2x^3 - 20x$

$\_\_\_^3 - \_\_\_x$ — Write out the polynomial.

$\_\_\_(x^2 - 10)$ — Factor out the greatest common monomial.

Reflect

2. Why wasn't the factor $x^2 - 10$ further factored?

---

3. Consider what happens when you factor $x^2 - 10$ over the real numbers and not merely the integers. Find a such that $x^2 - 10 = (x - a)(x + a)$.

Your Turn

4. $3x^3 + 7x^2 + 4x$

Explain 2 Recognizing Special Factoring Patterns

Remember the factoring patterns you already know:

Difference of two squares: $a^2 - b^2 = (a + b)(a - b)$

Perfect square trinomials: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$

There are two other factoring patterns that will prove useful:

Sum of two cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of two cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Notice that in each of the new factoring patterns, the quadratic factor is irreducible over the integers.

Example 2 **Factor the polynomial using a factoring pattern.**

(A) $27x^3 + 64$

$27x^3 + 64$	Write out the polynomial.
$27x^3 = (3x)^3$	Check if $27x^3$ is a perfect cube.
$64 = (4)^3$	Check if 64 is a perfect cube.
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Use the sum of two cubes formula to factor.
$(3x)^3 + 4^3 = (3x + 4)((3x)^2 - (3x)(4) + 4^2)$	
$27x^3 + 64 = (3x + 4)(9x^2 - 12x + 16)$	

Explain 3 Factoring by Grouping

Another technique for factoring a polynomial is grouping. If the polynomial has pairs of terms with common factors, factor by grouping terms with common factors and then factoring out the common factor from each group. Then look for a common factor of the groups in order to complete the factorization of the polynomial.

Example 3 **Factor the polynomial by grouping.**

(A) $x^3 - x^2 + x - 1$

Write out the polynomial.	$x^3 - x^2 + x - 1$
Group by common factor.	$(x^3 - x^2) + (x - 1)$
Factor.	$x^2(x - 1) + 1(x - 1)$
Regroup.	$(x^2 + 1)(x - 1)$

(B) $x^4 + x^3 + x + 1$

Write out the polynomial.	$x^4 + x^3 + x + 1$
Group by common factor.	$(\_\_\_ + \_\_\_) + (x + 1)$
Factor.	$\_\_\_(x + 1) + \_\_\_(x + 1)$
Regroup.	$(\_\_\_ + \_\_\_)(x + 1)$
Apply sum of two cubes to the first term.	$(\_\_\_ - \_\_\_ + 1)(x + 1)(x + 1)$
Substitute this into the expression and simplify.	$(\_\_\_)^2(\_\_\_^2 - \_\_\_ + 1)$

Your Turn

7. $x^3 + 3x^2 + 3x + 2$

Explain 4 Solving a Real-World Problem by Factoring a Polynomial

Remember that the zero-product property is used in solving factorable quadratic equations. It can also be used in solving factorable polynomial equations.

Your Turn

8. **Engineering** A small bank vault is being designed in the shape of a rectangular prism. The vault's sides and top should all be 3 feet thick. The outer length of the vault should be twice the outer width. The outer height should be the same as the outer width.

What should the outer dimensions of the vault be if it is to have 972 cubic feet of space?

Elaborate

9. Describe how the method of grouping incorporates the method of factoring out the greatest common monomial.

10. How do you decide if an equation fits in the sum of two cubes pattern?

11. How can factoring be used to solve a polynomial equation of the form $p(x) = a$, where a is a nonzero constant?

12. **Essential Question Check-In** What are two ways to factor a polynomial?

Factor the polynomial by grouping.

13. $x^3 + 8x^2 + 6x + 48$

14. $x^3 + 4x^2 - x - 4$

15. $8x^4 + 8x^3 + 27x + 27$

16. $27x^4 + 54x^3 - 64x - 128$

17. $x^3 + 2x^2 + 3x + 6$

18. $4x^4 - 4x^3 - x + 1$

Write and solve a polynomial equation for the situation described.

19. Engineering A rectangular two-story horse barn is being designed for a farm. The upper floor will be used for storing hay, and the lower floor will have horse stalls that extend 5 feet from both of the longer walls. The barn's length is twice the barn's width, and the lower floor's ceiling height is 6 feet less than the barn's width. What should the dimensions of the lower floor be if the space not used for stalls is to have a volume of 1920 cubic feet?

20. Arts A piece of rectangular crafting supply is being cut for a new sculpture. You want its length to be 4 times its height and its width to be 2 times its height. If you want the wood to have a volume of 64 cubic centimeters, what will its length, width, and height be?

26. **Explain the Error** Jim was trying to factor a polynomial and produced the following result:

$3x^3 + x^2 + 3x + 1$	Write out the polynomial.
$3x^2(x + 1) + 3(x + 1)$	Group by common factor.
$3(x^2 + 1)(x + 1)$	Regroup.

Explain Jim's error.

27. Factoring can also be done over the complex numbers. This allows you to find all the roots of an equation, not just the real ones.

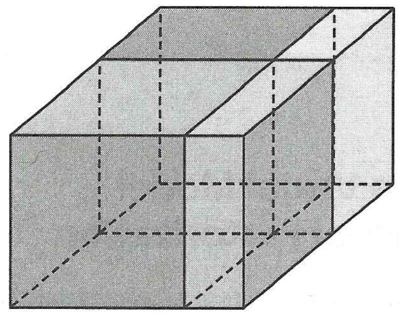
Complete the steps showing how to use a special factor identity to factor $x^2 + 4$ over the complex numbers.

$x^2 + 4$	Write out the polynomial.
$x^2 - (-4)$	____________________
$(x + \_\_\_\_)\ (x - \_\_\_\_)$	Factor.
$(x + 2i)\ (\_\_\_\_)$	Simplify.

28. Find all the imaginary roots of the equation $x^4 - 16 = 0$.

29. Factor $x^3 + x^2 + x + 1$ over the complex numbers.

(F) You determined that the length of the green piece is x. Use the volume of the yellow piece and the information you have derived to find the length of the prism.



(G) Since the dimensions of the overall prism are x, $x + 2$, and $x + 4$, the volume of the overall prism can be rewritten in factored form as $V = (x)(x + 2)(x + 4)$. Multiply these polynomials together to verify that this is equal to the original given expression for the volume of the overall figure.

Reflect

1. **Discussion** What is one way to double the volume of the prism?

Explain 1 Factoring Out the Greatest Common Monomial First

Most polynomials cannot be *factored over the integers*, which means to find factors that use only integer coefficients. But when a polynomial can be factored, each factor has a degree less than the polynomial's degree. While the goal is to write the polynomial as a product of linear factors, this is not always possible. When a factor of degree 2 or greater cannot be factored further, it is called an **irreducible factor**.

Example 1 **Factor each polynomial over the integers.**

(A) $6x^3 + 15x^2 + 6x$

$6x^3 + 15x^2 + 6x$	Write out the polynomial.
$x(6x^2 + 15x + 6)$	Factor out a common monomial, an x.
$3x(2x^2 + 5x + 2)$	Factor out a common monomial, a 3.
$3x(2x + 1)(x + 2)$	Factor into simplest terms.

Note: The second and third steps can be combined into one step by factoring out the greatest common monomial.

(B) $8x^3 - 27$

$8\_\_^3 - 27$	Write out the polynomial.
$8x^3 = (\_\_x)^3$	Check if $8x^3$ is a perfect cube.
$27 = (\_\_)^3$	Check if 27 is a perfect cube.
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	Use the difference of two cubes formula to factor.
$8x^3 - 27 = (\_\_x - \_\_)(\_\_x^2 + \_\_x + \_\_)$	

Reflect

5. The equation $8x^3 - 27 = 0$ has three roots. How many of them are real, what are they, and how many are nonreal?

Your Turn

6. $40x^4 + 5x$

Example 4 **Write and solve a polynomial equation for the situation described.**

(A) A water park is designing a new pool in the shape of a rectangular prism. The sides and bottom of the pool are made of material 5 feet thick. The interior length must be twice the interior height (depth), and the interior width must be three times the interior height. The volume of water that the pool holds must be 6000 cubic feet. What are the exterior dimensions of the pool?

Let x represent the exterior height of the pool, that is, including its bottom. Let h, l, and w represent the interior dimensions. Then the dimensions of the interior of the pool are the following:

$h = x - 5$

$l = 2x - 10$

$w = 3x - 15$

The formula for the volume of a rectangular prism is $V = lwh$. Plug the values into the volume equation.

$V = (2x - 10)(3x - 15)(x - 5)$

$V = (6x^2 - 60x + 150)(x - 5)$

$V = 6x^3 - 90x^2 + 450x - 750$

Now solve for $V = 6000$.

$6000 = 6x^3 - 90x^2 + 450x - 750$

$0 = 6x^3 - 90x^2 + 450x - 6750$

Factor the resulting new polynomial.

$6x^3 - 90x^2 + 450x - 6750$

$= 6x^2(x - 15) + 450(x - 15)$

$= (6x^2 + 450)(x - 15)$

The only real root is $x = 15$. This is the exterior height of the pool.

The interior height of the pool will be 10 feet, the interior length 20 feet, and the interior width 30 feet. Because each side wall of the pool is 5 feet thick, the exterior length is 30 feet and the exterior width is 40 feet.

(B) **Engineering** To build a hefty wooden feeding trough for a zoo, its sides and bottom should be 2 feet thick, and its outer length should be twice its outer width and height.

What should the outer dimensions of the trough be if it is to hold 288 cubic feet of water?

Volume = Interior Length(feet) · Interior Width(feet) · Interior Height(feet)

$288 = (\_\_\_\_ - 4)(\_\_\_\_ - 4)(\_\_\_\_ - 2)$

$288 = \_\_\_\_x^3 - \_\_\_\_x^2 + \_\_\_\_x - \_\_\_\_$

$0 = \_\_\_\_x^3 - \_\_\_\_x^2 + \_\_\_\_x - \_\_\_\_$

$0 = \_\_\_\_(x - \_\_\_\_) + \_\_\_\_(x - \_\_\_\_)$

$0 = \_\_\_\_(x^2 + \_\_\_\_)(x - \_\_\_\_)$

The only real solution is $x =$ ____. The trough is ____ feet long, ____ feet wide, and ____ feet high.

Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

Factor the polynomial, or identify it as irreducible.

1. $x^3 + x^2 - 12x$

2. $x^3 + 5$

3. $x^3 - 125$

4. $x^3 + 5x^2 + 6x$

5. $8x^3 + 125$

6. $2x^3 + 6x$

7. $216x^3 + 64$

8. $8x^3 - 64$

9. $10x^3 - 80$

10. $2x^4 + 7x^3 + 5x^2$

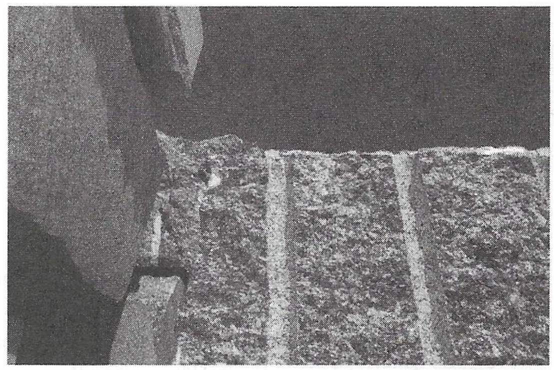
11. $x^3 + 10x^2 + 16x$

12. $x^3 + 9769$

21. Engineering A new rectangular holding tank is being built. The tank's sides and bottom should be 1 foot thick. Its outer length should be twice its outer width and height.

What should the outer dimensions of the tank be if it is to have a volume of 36 cubic feet?

22. Construction A piece of granite is being cut for a building foundation. You want its length to be 8 times its height and its width to be 3 times its height. If you want the granite to have a volume of 648 cubic yards, what will its length, width, and height be?



23. State which, if any, special factoring pattern each of the following polynomial expressions follows:

a. $x^2 - 4$

b. $3x^3 + 5$

c. $4x^2 + 25$

d. $27x^3 + 1000$

e. $64x^3 - x^2 + 1$

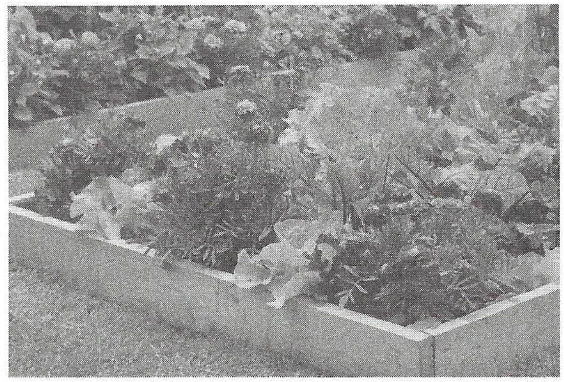
H.O.T. Focus on Higher Order Thinking

24. Communicate Mathematical Ideas What is the relationship between the degree of a polynomial and the degree of its factors?

25. Critical Thinking Why is there no sum-of-two-squares factoring pattern?

Lesson Performance Task

Sabrina is building a rectangular raised flower bed. The boards on the two shorter sides are 6 inches thick, and the boards on the two longer sides are 4 inches thick. Sabrina wants the outer length of her bed to be 4 times its height and the outer width to be 2 times its height. She also wants the boards to rise 4 inches above the level of the soil in the bed. What should the outer dimensions of the bed be if she wants it to hold 3136 cubic inches of soil?



Name ______________________ Class ______________ Date ________

6.5 Dividing Polynomials

Essential Question: What are some ways to divide polynomials, and how do you know when the divisor is a factor of the dividend?

Resource Locker

Explore Evaluating a Polynomial Function Using Synthetic Substitution

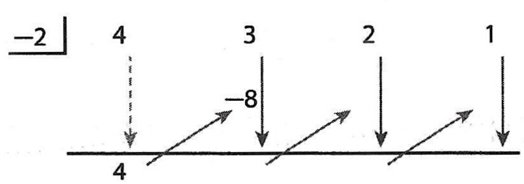
Polynomials can be written in something called nested form. A polynomial in nested form is written in such a way that evaluating it involves an alternating sequence of additions and multiplications. For instance, the nested form of $p(x) = 4x^3 + 3x^2 + 2x + 1$ is $p(x) = x\big(x(4x + 3) + 2\big) + 1$, which you evaluate by starting with 4, multiplying by the value of x, adding 3, multiplying by x, adding 2, multiplying by x, and adding 1.

(A) Given $p(x) = 4x^3 + 3x^2 + 2x + 1$, find $p(-2)$.

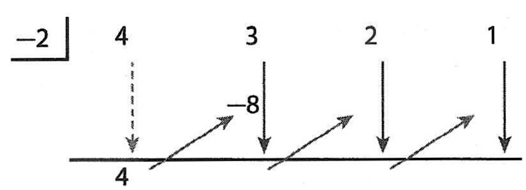
You can set up an array of numbers that captures the sequence of multiplications and additions needed to find $p(a)$. Using this array to find $p(a)$ is called **synthetic substitution**.

Given $p(x) = 4x^3 + 3x^2 + 2x + 1$, find $p(-2)$ by using synthetic substitution. The dashed arrow indicates bringing down, the diagonal arrows represent multiplication by -2, and the solid down arrows indicate adding.

The first two steps are to bring down the leading number, 4, then multiply by the value you are evaluating at, -2.



(B) Add 3 and -8.



Find the value you need to multiply the divisor by so that the first term matches with the first term of the dividend. In this case, in order to get $4x^2$, we must multiply x^2 by $4x$. This will be the first term of the quotient.

$$\begin{array}{r l} & \quad 4x \\ x^2+3x+1 & \big)\overline{\,4x^3+2x^2+3x+5} \end{array}$$

Next, multiply the divisor through by the term of the quotient you just found and subtract that value from the dividend. $(x^2+3x+1)(4x) = 4x^3+12x^2+4x$, so subtract $4x^3+12x^2+4x$ from $4x^3+2x^2+3x+5$.

$$\begin{array}{r l} & \quad 4x \\ x^2+3x+1 & \big)\overline{\,4x^3+2x^2+3x+5} \\ & \underline{-\left(4x^3+12x^2+4x\right)} \\ & \qquad -10x^2-x+5 \end{array}$$

Taking this difference as the new dividend, continue in this fashion until the largest term of the remaining dividend is of lower degree than the divisor.

$$\begin{array}{r l} & \quad 4x-10 \\ x^2+3x+1 & \big)\overline{\,4x^3+2x^2+3x+5} \\ & \underline{-\left(4x^3+12x^2+4x\right)} \\ & \qquad -10x^2-x+5 \\ & \underline{-\left(-10x^2-30x-10\right)} \\ & \qquad\qquad\qquad 29x+15 \end{array}$$

Since $29x+5$ is of lower degree than x^2+3x+1, stop. $29x+15$ is the remainder.

Write the final answer.

$$4x^3+2x^2+3x+5 = \left(x^2+3x+1\right)(4x-10)+29x+15$$

Check.

$$\begin{aligned} 4x^3+2x^2+3x+5 &= \left(x^2+3x+1\right)(4x-10)+29x+15 \\ &= 4x^3+12x^2+4x-10x^2-30x-10+29x+15 \\ &= 4x^3+2x^2+3x+5 \end{aligned}$$

B $\left(6x^4+5x^3+2x+8\right) \div \left(x^2+2x-5\right)$

Write the dividend in standard form, including terms with a coefficient of 0.

Write the division in the same way as you would when dividing numbers.

$$x^2+2x-5\,\big)\overline{\,6x^4+5x^3+0x^2+2x+8}$$

4. $\left(9x^4 + x^3 + 11x^2 - 4\right) \div \left(x^2 + 16\right)$

Explain 2 Dividing $p(x)$ by $x - a$ Using Synthetic Division

Compare long division with synthetic substitution. There are two important things to notice. The first is that $p(a)$ is equal to the remainder when $p(x)$ is divided by $x - a$. The second is that the numbers to the left of $p(a)$ in the bottom row of the synthetic substitution array give the coefficients of the quotient. For this reason, synthetic substitution is also called **synthetic division**.

Long Division	Synthetic Substitution
$\begin{array}{r} 3x^2 + 10x + 20 \\ x - 2 \overline{) 3x^3 + 4x^2 + 0x + 10} \\ \underline{-\left(3x^3 - 6x^2\right)} \\ 10x^2 + 0x \\ \underline{-\left(10x^2 - 20x\right)} \\ 20x + 10 \\ \underline{-20x - 40} \\ 50 \end{array}$	$\begin{array}{r\|rrrr} 2 & 3 & 4 & 0 & 10 \\ & & 6 & 20 & 40 \\ \hline & 3 & 10 & 20 & \boxed{50} \end{array}$

Example 2 **Given a polynomial $p(x)$, use synthetic division to divide by $x - a$ and obtain the quotient and the (nonzero) remainder. Write the result in the form $p(x) = (x - a)(quotient) + p(a)$, then carry out the multiplication and addition as a check.**

Ⓐ $\left(7x^3 - 6x + 9\right) \div \left(x + 5\right)$

By inspection, $a = -5$. Write the coefficients and a in the synthetic division format.

$$\begin{array}{r|rrrr} -5 & 7 & 0 & -6 & 9 \\ & & & & \\ \hline & & & & \end{array}$$

Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{r|rrrr} -5 & 7 & 0 & -6 & 9 \\ & & -35 & 175 & -845 \\ \hline & 7 & -35 & 169 & \boxed{-836} \end{array}$$

Write the result, using the non-remainder entries of the bottom row as the coefficients.

$$\left(7x^3 - 6x + 9\right) = \left(x + 5\right)\left(7x^2 - 35x + 169\right) - 836$$

Check.

$$\begin{aligned} \left(7x^3 - 6x + 9\right) &= \left(x + 5\right)\left(7x^2 - 35x + 169\right) - 836 \\ &= 7x^3 - 35x^2 - 35x^2 - 175x + 169x + 845 - 836 \\ &= 7x^3 - 6x + 9 \end{aligned}$$

Explain 3 Using the Remainder Theorem and Factor Theorem

When $p(x)$ is divided by $x - a$, the result can be written in the form $p(x) = (x - a)q(x) + r$ where $q(x)$ is the quotient and r is a number. Substituting a for x in this equation gives $p(a) = (a - a)q(a) + r$. Since $a - a = 0$, this simplifies to $p(a) = r$. This is known as the **Remainder Theorem**.

If the remainder $p(a)$ in $p(x) = (x - a)q(x) + p(a)$ is 0, then $p(x) = (x - a)q(x)$, which tells you that $x - a$ is a factor of $p(x)$. Conversely, if $x - a$ is a factor of $p(x)$, then you can write $p(x)$ as $p(x) = (x - a)q(x)$, and when you divide $p(x)$ by $x - a$, you get the quotient $q(x)$ with a remainder of 0. These facts are known as the **Factor Theorem**.

Example 3 **Determine whether the given binomial is a factor of the polynomial $p(x)$. If so, find the remaining factors of $p(x)$.**

(A) $p(x) = x^3 + 3x^2 - 4x - 12$; $(x + 3)$

Use synthetic division.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -4 & -12 \\ & & -3 & 0 & 12 \\ \hline & 1 & 0 & -4 & \boxed{0} \end{array}$$

Since the remainder is 0, $x + 3$ is a factor.

Write $q(x)$ and then factor it.

$q(x) = x^2 - 4 = (x + 2)(x - 2)$

So, $p(x) = x^3 + 3x^2 - 4x - 12 = (x + 2)(x - 2)(x + 3)$.

(B) $p(x) = x^4 - 4x^3 - 6x^2 + 4x + 5$; $(x + 1)$

Use synthetic division.

$$\begin{array}{r|rrrrr} -1 & 1 & -4 & -6 & 4 & 5 \\ & & & & & \\ \hline & 1 & & & & \boxed{} \end{array}$$

Since the remainder is ____, $(x + 1)$ ____ a factor. Write $q(x)$.

$q(x) =$ ______

Now factor $q(x)$ by grouping.

$q(x) =$ ______

$=$ ______

$=$ ______

$=$ ______

So, $p(x) = x^4 - 4x^3 - 6x^2 + 4x + 5 =$ ______.

Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

Given $p(x)$, find $p(-3)$ by using synthetic substitution.

1. $p(x) = 8x^3 + 7x^2 + 2x + 4$

2. $p(x) = x^3 + 6x^2 + 7x - 25$

3. $p(x) = 2x^3 + 5x^2 - 3x$

4. $p(x) = -x^4 + 5x^3 - 8x + 45$

Given a polynomial divisor and dividend, use long division to find the quotient and remainder. Write the result in the form *dividend* = (*divisor*)(*quotient*) + *remainder*. You may wish to carry out a check.

5. $(18x^3 - 3x^2 + x - 1) \div (x^2 - 4)$

6. $(6x^4 + x^3 - 9x + 13) \div (x^2 + 8)$

Determine whether the given binomial is a factor of the polynomial $p(x)$. If so, find the remaining factors of $p(x)$.

12. $p(x) = x^3 + 2x^2 - x - 2;\ (x + 2)$

13. $p(x) = 2x^4 + 6x^3 - 5x - 10;\ (x + 2)$

14. $p(x) = x^3 - 22x^2 + 157x - 360;\ (x - 8)$

15. $p(x) = 4x^3 - 12x^2 + 2x - 5;\ (x - 3)$

16. The volume of a rectangular prism whose dimensions are binomials with integer coefficients is modeled by the function $V(x) = x^3 - 8x^2 + 19x - 12$. Given that $x - 1$ and $x - 3$ are two of the dimensions, find the missing dimension of the prism.

H.O.T. Focus on Higher Order Thinking

21. Multi-Step Use synthetic division to divide $p(x) = 3x^3 - 11x^2 - 56x - 50$ by $(3x + 4)$. Then check the solution.

22. Critical Thinking The polynomial $ax^3 + bx^2 + cx + d$ is factored as $3(x - 2)(x + 3)(x - 4)$. What are the values of a and d? Explain.

23. Analyze Relationships Investigate whether the set of whole numbers, the set of integers, and the set of rational numbers are closed under each of the four basic operations. Then consider whether the set of polynomials in one variable is closed under the four basic operations, and determine whether polynomials are like whole numbers, integers, or rational numbers with respect to closure. Use the table to organize.

	Whole Numbers	Integers	Rational Numbers	Polynomials
Addition				
Subtraction				
Multiplication				
Division (by nonzero)				

STUDY GUIDE REVIEW

Polynomials

MODULE 6

Essential Question: How can you use polynomials to solve real-world problems?

Key Vocabulary

binomial *(binomio)*
monomial *(monomio)*
polynomial *(polinomio)*
synthetic division *(división sintética)*
trinomial *(trinomio)*

KEY EXAMPLE (Lesson 6.1)

Subtract: $(5x^4 - x^3 + 2x + 1) - (2x^3 + 3x^2 - 4x - 7)$

$$\begin{array}{rl} 5x^4 - x^3 + 0x^2 + 2x + 1 & \text{Write in standard form.} \\ +\quad -2x^3 - 3x^2 + 4x + 7 & \text{Align like terms and add the opposite.} \\ \hline 5x^4 - 3x^3 - 3x^2 + 6x + 8 & \text{Add.} \end{array}$$

Therefore, $(5x^4 - x^3 + 2x + 1) - (2x^3 + 3x^2 - 4x - 7) = 5x^4 - 3x^3 - 3x^2 + 6x + 8$.

KEY EXAMPLE (Lesson 6.2)

Multiply: $(3x - 2)(2x^2 - 5x + 1)$

$(3x - 2)(2x^2 - 5x + 1)$ Write in standard form.

$3x(2x^2) + 3x(-5x) + 3x(1) + (-2)(2x^2) + (-2)(-5x) + (-2)(1)$ Distribute the $3x$ and the -2.

$6x^3 - 15x^2 + 3x - 4x^2 + 10x - 2$ Simplify.

$6x^3 - 19x^2 + 13x - 2$ Combine like terms.

Therefore, $(3x - 2)(2x^2 - 5x + 1) = 6x^3 - 19x^2 + 13x - 2$.

KEY EXAMPLE (Lesson 6.5)

Divide: $(x^3 + 10x^2 + 13x + 36) \div (x + 9)$

$$\begin{array}{r|l} & x^2 + x + 4 \\ \hline x + 9 & x^3 + 10x^2 + 13x + 36 \\ & -(x^3 + 9x^2) \\ \hline & \quad x^2 + 13x \\ & \quad -(x^2 + 9x) \\ \hline & \qquad 4x + 36 \\ & \qquad -(4x + 36) \\ \hline & \qquad\qquad 0 \end{array}$$

In order to get x^3, multiply by x^2.

Multiply the divisor through by x^2, then subtract.

In order to get x^2, multiply by x.

Multiply the divisor through by x, then subtract.

In order to get $4x$, multiply by 4.

Multiply the divisor through by 4, then subtract.

Therefore, $(x^3 + 10x^2 + 13x + 36) \div (x + 9) = x^2 + x + 4$.

Ready to Go On?

6.1–6.5 Polynomials

Personal Math Trainer

- Online Homework
- Hints and Help
- Extra Practice

Factor the polynomial. *(Lesson 6.4)*

1. $3x^2 + 4x - 4$

2. $2x^3 + 4x^2 - 30x$

3. $9x^2 - 25$

4. $4x^2 - 16x + 16$

Complete the polynomial operation. *(Lesson 6.1, 6.2, 6.3, 6.5)*

5. $(8x^3 - 2x^2 - 4x + 8) + (5x^2 + 6x - 4)$

6. $(-4x^2 - 2x + 8) - (x^2 + 8x - 5)$

7. $5x(x + 2)(3x - 7)$

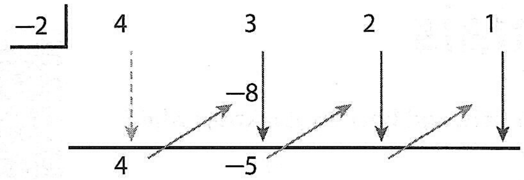
8. $(3x^3 + 12x^2 + 11x - 2) \div (x + 2)$

9. $(x + y)^6$

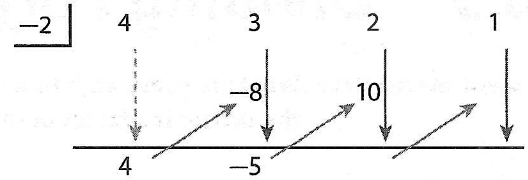
ESSENTIAL QUESTION

10. Write a real-world situation that would require adding polynomials. *(Lesson 6.1)*

(C) Multiply the previous answer by -2.



(D) Continue this sequence of steps until you reach the last addition.



(E) $p(-2) = \boxed{}$

Reflect

1. Discussion After the final addition, what does this sum correspond to?

____

____

Explain 1 Dividing Polynomials Using Long Division

Recall that arithmetic long division proceeds as follows.

$$\begin{array}{rl} \text{Divisor} \quad\quad 23 & \leftarrow \text{Quotient} \\ 12\overline{)\,277} & \leftarrow \text{Dividend} \\ \underline{24} & \\ 37 & \\ \underline{36} & \\ 1 & \leftarrow \text{Remainder} \end{array}$$

Notice that the long division leads to the result $\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor}$. Using the numbers from above, the arithmetic long division leads to $\frac{277}{12} = 23 + \frac{1}{12}$. Multiplying through by the divisor yields the result $dividend = (divisor)(quotient) + remainder$. (This can be used as a means of checking your work.)

Example 1 **Given a polynomial divisor and dividend, use long division to find the quotient and remainder. Write the result in the form *dividend = (divisor)(quotient) + remainder* and then carry out the multiplication and addition as a check.**

(A) $\left(4x^3 + 2x^2 + 3x + 5\right) \div \left(x^2 + 3x + 1\right)$

Begin by writing the dividend in standard form, including terms with a coefficient of 0 (if any).

$4x^3 + 2x^2 + 3x + 5$

Write division in the same way as you would when dividing numbers.

$x^2 + 3x + 1\overline{)\,4x^3 + 2x^2 + 3x + 5}$

Divide.

$$
\begin{array}{r}
6x^2 - \boxed{} + \boxed{} \\
x^2 + 2x - 5 \overline{\big) 6x^4 + 5x^3 + 0x^2 + 2x + 8} \\
\underline{-\left(6x^4 + 12x^3 - 30x^2\right)} \\
-7x^3 + 30x^2 + 2x \\
\underline{-\left(-7x^3 \boxed{}\right)} \\
\boxed{} + 8 \\
\underline{-\left(\boxed{}\right)} \\
\boxed{}
\end{array}
$$

Write the final answer.

$6x^4 + 5x^3 + 2x + 8 = \boxed{}$

Check.

Reflect

2. How do you include the terms with coefficients of 0?

____

____

Your Turn

Use long division to find the quotient and remainder. Write the result in the form *dividend* $=$ (*divisor*)(*quotient*) $+$ *remainder* and then carry out a check.

3. $\left(15x^3 + 8x - 12\right) \div \left(3x^2 + 6x + 1\right)$

Ⓑ $\left(4x^4 - 3x^2 + 7x + 2\right) \div \left(x - \frac{1}{2}\right)$

Find a. Then write the coefficients and a in the synthetic division format.

Find a = $\boxed{}$

$$\begin{array}{c|ccccc} \underline{} & 4 & 0 & -3 & 7 & 2 \\ & & & & & \\ \hline & & & & & \end{array}$$

Bring down the first coefficient. Then multiply and add for each column.

$$\begin{array}{c|ccccc} \underline{} & 4 & 0 & -3 & 7 & 2 \\ & & & & & \\ \hline & 4 & & & & \end{array}$$

Write the result.

$\left(4x^4 - 3x^2 + 7x + 2\right) = \boxed{}$

Check.

Reflect

5. Can you use synthetic division to divide a polynomial by $x^2 + 3$? Explain.

Your Turn

Given a polynomial $p(x)$, use synthetic division to divide by $x - a$ and obtain the quotient and the (nonzero) remainder. Write the result in the form $p(x) = (x - a)(\textit{quotient}) + p(a)$. You may wish to perform a check.

6. $\left(2x^3 + 5x^2 - x + 7\right) \div (x - 2)$

7. $\left(6x^4 - 25x^3 - 3x + 5\right) \div \left(x + \frac{1}{3}\right)$

Your Turn

Determine whether the given binomial is a factor of the polynomial $p(x)$. If it is, find the remaining factors of $p(x)$.

8. $p(x) = 2x^4 + 8x^3 + 2x + 8;\ (x + 4)$

9. $p(x) = 3x^3 - 2x + 5;\ (x - 1)$

Elaborate

10. Compare long division and synthetic division of polynomials.

11. How does knowing one linear factor of a polynomial help find the other factors?

12. What conditions must be met in order to use synthetic division?

13. **Essential Question Check-In** How do you know when the divisor is a factor of the dividend?

7. $\left(x^4 + 6x - 2.5\right) \div \left(x^2 + 3x + 0.5\right)$

8. $\left(x^3 + 250x^2 + 100x\right) \div \left(\frac{1}{2}x^2 + 25x + 9\right)$

Given a polynomial $p(x)$, use synthetic division to divide by $x - a$ and obtain the quotient and the (nonzero) remainder. Write the result in the form $p(x) = (x - a)\left(\text{quotient}\right) + p(a)$. You may wish to carry out a check.

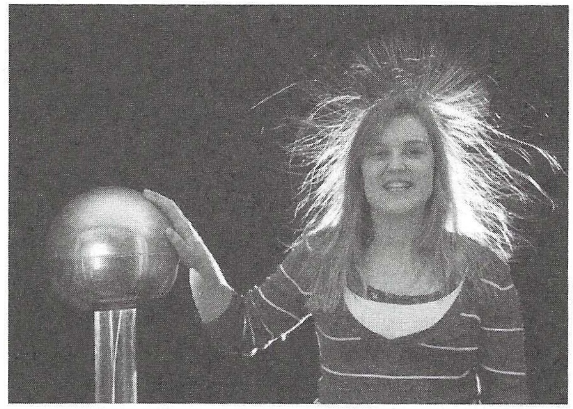
9. $\left(7x^3 - 4x^2 - 400x - 100\right) \div (x - 8)$

10. $\left(8x^4 - 28.5x^2 - 9x + 10\right) \div (x + 0.25)$

11. $\left(2.5x^3 + 6x^2 - 5.5x - 10\right) \div (x + 1)$

17. Given that the height of a rectangular prism is $x + 2$ and the volume is $x^3 - x^2 - 6x$, write an expression that represents the area of the base of the prism.

18. Physics A Van de Graaff generator is a machine that produces very high voltages by using small, safe levels of electric current. One machine has a current that can be modeled by $I(t) = t + 2$, where $t > 0$ represents time in seconds. The power of the system can be modeled by $P(t) = 0.5t^3 + 6t^2 + 10t$. Write an expression that represents the voltage of the system. Recall that $V = \frac{P}{I}$.



19. Geometry The volume of a hexagonal pyramid is modeled by the function $V(x) = \frac{1}{3}x^3 + \frac{4}{3}x^2 + \frac{2}{3}x - \frac{1}{3}$. Given the height $x + 1$, use polynomial division to find an expression for the area of the base.
(Hint: For a pyramid, $V = \frac{1}{3}Bh$.)

20. Explain the Error Two students used synthetic division to divide $3x^3 - 2x - 8$ by $x - 2$. Determine which solution is correct. Find the error in the other solution.

A.	B.
$\begin{array}{r\|rrrr} \underline{2} & 3 & 0 & -2 & -8 \\ & & 6 & 12 & 20 \\ \hline & 3 & 6 & 10 & 12 \end{array}$	$\begin{array}{r\|rrrr} \underline{-2} & 3 & 0 & -2 & -8 \\ & & -6 & 12 & -20 \\ \hline & 3 & -6 & 10 & -28 \end{array}$

Lesson Performance Task

The table gives the attendance data for all divisions of NCAA Women's Basketball.

NCAA Women's Basketball Attendance			
Season	**Years since 2006–2007**	**Number of teams in all 3 divisions**	**Attendance (in thousands) for all 3 divisions**
2006–2007	0	1003	10,878.3
2007–2008	1	1013	11,120.8
2008–2009	2	1032	11,160.3
2009–2010	3	1037	11,134.7
2010–2011	4	1048	11,160.0
2011–2012	5	1055	11,201.8

Enter the data from the second, third, and fourth columns of the table and perform linear regression on the data pairs (t, T) and cubic regression on the data pairs (t, A) where $t =$ years since the 2006–2007 season, $T =$ number of teams, and $A =$ attendance (in thousands).

Then create a model for the average attendance per team: $A_{avg}(t) = \frac{A(t)}{T(t)}$. Carry out the division to write $A_{avg}(t)$ in the form $\textit{quadratic quotient} + \frac{\textit{remainder}}{T(t)}$.

Use an online computer algebra system to carry out the division of $A(t)$ by $T(t)$.

EXERCISES

Simplify. *(Lessons 6.1, 6.2, 6.5)*

1. $(9x^2 + 2x + 12) + (7x^2 + 10x - 13)$

2. $(6x^6 - 4x^5) - (10x^5 - 15x^4 + 8)$

3. $(x - 3)(4x^2 - 2x + 3)$

4. $(9x^4 + 27x^3 + 23x^2 + 10x) \div (x^2 + 2x)$

5. Mr. Alonzo runs a car repair garage. The average income from repairing a car can be modeled by $C(x) = 45x + 150$. If, for one year, the number of cars repaired can be modeled by $N(x) = 9x^2 + 7x + 6$, write a polynomial that can be used to model Mr. Alonzo's business income for that year. Explain. *(Lesson 6.2)*

MODULE PERFORMANCE TASK

What's the Temperature?

A meteorologist studying the temperature patterns for Redding, California, found the average of the daily minimum and maximum temperatures for each month, but the August temperatures are missing.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Average Max Temperature (°F)	55.3	61.3	62.5	69.6	80.5	90.4	98.3	?	89.3	77.6	62.1	54.7
Average Min Temperature (°F)	35.7	40	41.7	46	52.3	61.8	64.7	?	58.8	49.2	41.4	35.2

How can she find the averages for August? She began by fitting the polynomial function shown below to the data for the average maximum temperature, where x is the month, with $x = 1$ corresponding to January, and the temperature is in degrees Fahrenheit.

$$T_{max}(x) = 0.0095x^5 - 0.2719x^4 + 2.5477x^3 - 9.1882x^2 + 17.272x + 45.468$$

She also thinks that a vertical compression of this function will create a function that fits the average minimum temperature data for Redding.

Use this information to find the average high and low temperature for August. Use graphs, numbers, words, or algebra to explain how you reached your conclusion.

MODULE 6
MIXED REVIEW

Assessment Readiness

1. Look at each polynomial division problem. Can the polynomials be divided without a remainder?
Select Yes or No for A–C.

A. $(3x^3 - 5x^2 + 10x + 4) \div (3x + 1)$ ○ Yes ○ No
B. $(2x^2 - 5x - 1) \div (x - 3)$ ○ Yes ○ No
C. $(x^3 - 4x^2 + 2x - 3) \div (x + 2)$ ○ Yes ○ No

2. Consider the polynomial $x^3 - x^2 - 6x$.
Select True or False for each statement.

A. $6x$ can be factored out of every term. ○ True ○ False
B. The completely factored polynomial is $x(x + 2)(x - 3)$. ○ True ○ False
C. $f(x) = x^3 - x^2 - 6x$ has a global minimum. ○ True ○ False

3. Alana completed a problem where she had to find the sum of the polynomials $(3x^2 + 8x - 4)$ and $(-8x^3 - 3x + 4)$. Her answer is 0. Describe and correct her mistake. When graphed, how many times does the sum change directions?

4. A rectangular plot of land has a length of $(2x^2 + 5x - 20)$ and a width of $(3x + 4)$. What polynomial represents the area of the plot of land? Explain how you got your answer.