

Notes on differential geometry: Riemannian geometry and Vector bundles (unfinished)

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Contents

| | | |
|----------|--------------------------------------|----------|
| 1 | Riemannian manifolds | 2 |
| 1.1 | Riemannian metric, cohomology | 2 |
| 1.2 | Connections | 4 |
| 2 | Vector bundles | 4 |
| 2.1 | The geometry of vector bundles | 4 |
| 2.1.1 | Some definitions | 4 |
| 2.1.2 | Bundle connections | 5 |
| 2.1.3 | Riemannian metric on a vector bundle | 6 |

1 Riemannian manifolds

1.1 Riemannian metric, cohomology

In this article, we will denote an inner product on $T_p M$ by $\langle \cdot, \cdot \rangle_p$.

Definition 1.1 (Riemannian metric). Let M be a smooth manifold. The Riemannian metric $\langle \cdot, \cdot \rangle$ is the smooth assignment

$$p \mapsto \langle \cdot, \cdot \rangle_p : T_p M \times T_p M \rightarrow \mathbb{R}.$$

Definition 1.2 (Riemannian manifold). Let M be a smooth manifold, $\langle \cdot, \cdot \rangle$ a Riemannian metric on M . Then the pair $(M, \langle \cdot, \cdot \rangle)$ is called a Riemannian manifold.

Theorem 1.3 (every manifold has a metric). *Let M be a smooth manifold, then M can be equipped with a Riemannian metric.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M , and $\{\rho_\alpha\}$ a partition of unity subordinate to the open cover $\{U_\alpha\}$. Then $\text{supp } \rho_\alpha \subset U_\alpha$ for each α , and the collection $\{\text{supp } \rho_\alpha\}$ is locally finite. Thus,

$$\sum_\alpha \rho_\alpha \langle \cdot, \cdot \rangle_\alpha$$

defines a smooth Riemannian metric on M . \square

Lemma 1.4 (Circle around a puncture induces the zero map). Let Σ be a compact, orientable surface and let $p \in \Sigma$. Let

$$i : C \hookrightarrow \Sigma \setminus \{p\}$$

be the inclusion of a circle C encircling the puncture. Then the induced map

$$i^* : H^1(\Sigma \setminus \{p\}) \rightarrow H^1(C)$$

is the zero map.

Proof. The isomorphism

$$H^1(C) \cong H^1(S^1) \cong \mathbb{R}$$

is given by integration. The circle C is the boundary of $\Sigma \setminus D$, where D is an open disk bounded by C . Let ω be any closed 1-form on $\Sigma \setminus \{p\}$. Then

$$\int_C i^* \omega = \int_{\partial(\Sigma \setminus D)} i^* \omega = \int_{\Sigma \setminus D} d\omega = 0,$$

since ω is closed. Since ω was chosen arbitrarily, it follows that

$$i^*[\omega] = 0$$

for any cohomology class. \square

Theorem 1.5 (Cohomology of compact orientable surfaces of genus g). *For any compact, orientable surface M_g of genus g ,*

$$H^0(M_g) = H^2(M_g) = \mathbb{R}, \quad H^1(M_g) = \mathbb{R}^{2g},$$

and

$$H^k(M_g) = 0, \quad \forall k > 1.$$

Proof. These surfaces are connected, thus $H^0(M_g) = \mathbb{R}$ is immediate. We first prove the case $g = 2$, and then reason by induction.

Choose U, V to be punctured torii. Then $U \cap V$ is homeomorphic to S^1 . The Mayer–Vietoris sequence is

$$\begin{array}{ccccccc} H^1(M_2) & \xrightarrow{i^*} & H^1(U) \oplus H^1(V) & \xrightarrow{j^*} & \mathbb{R} & & \\ & & & & \downarrow d^* & & \\ & & & & H^2(M_2) & \xrightarrow{i^*} & H^2(U) \oplus H^2(V) \longrightarrow 0 \end{array}$$

From differential topology, we know that $H^k(U) = H^k(V) = 0$ for all $k > 1$ and $H^1(U) = H^1(V) = \mathbb{R}^2$. By the previous lemma, $j^* = 0$. Thus, by exactness,

$$\text{im } j^* = \ker d^* = 0,$$

so d^* is injective and therefore $H^2(M_2) \cong \mathbb{R}$.

The Euler-characteristic type method then implies

$$\dim H^1(M_2) - \dim H^1(U) - \dim H^1(V) - 1 + 1 = 0,$$

i.e. $H^1(M_2) \cong \mathbb{R}^4$.

The induction hypothesis is that $H^1(M_{g-1}) = \mathbb{R}^{2(g-1)}$ and $H^2(M_{g-1}) = \mathbb{R}$. Cover M_{g-1} with

$$U = M_{g-1} \setminus \{p\}, \quad V = D,$$

where D is an open disk. Since D is contractible, $H^k(D) = 0$ for all $k > 0$, and $U \cap V$ is a punctured disk, homotopy equivalent to S^1 . The Mayer–Vietoris sequence is

$$\begin{array}{ccccccc} H^1(M_{g-1}) & \xrightarrow{i^*} & H^1(U) & \xrightarrow{j^*} & \mathbb{R} & & \\ & & & & \downarrow d^* & & \\ & & & & \mathbb{R} & \xrightarrow{i^*} & H^2(U) \longrightarrow 0 \end{array}$$

By the previous lemma, $j^* = 0$. Thus $H^1(U) = \ker j^* = \text{im } i^*$, and since i^* is injective,

$$H^1(U) \cong H^1(M_{g-1}) = \mathbb{R}^{2(g-1)}.$$

Using the Euler characteristic type method, we find that $H^2(U) = 0$.

Now cover M_g by $U = M_{g-1} \setminus \{p\}$ and V a punctured torus. Since $U \cap V$ is homeomorphic to a cylinder, the Mayer–Vietoris sequence is

$$\begin{array}{ccccccc} H^1(M_g) & \xrightarrow{i^*} & H^1(U) \oplus H^1(V) & \xrightarrow{j^*} & \mathbb{R} & & \\ & & & & \downarrow d^* & & \\ & & & & H^2(M_g) & \xrightarrow{i^*} & H^2(U) \oplus H^2(V) \longrightarrow 0 \end{array}$$

Using the same reasoning as for M_2 , we obtain $H^2(M_g) = \mathbb{R}$, and therefore

$$\dim H^1(M_g) = 2g - 2 + 2 - 1 + 1 = 2g,$$

as desired. \square

1.2 Connections

While we can properly define directional derivatives on curves and surfaces, we usually can't do that on arbitrary Riemannian manifolds. That is why we introduce connections.

Recall that $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M .

Definition 1.6 (Affine connection). The \mathbb{R} -bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$$

is called an affine connection if it is $C^\infty(M)$ -linear in the first argument and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

Definition 1.7 (Curvature of a connection). Let ∇ be an affine connection. The curvature of ∇ is

$$R(X, Y) = [\nabla_X Y, \nabla_Y X] - \nabla_{[X, Y]}.$$

Definition 1.8 (Torsion of a connection). The torsion of ∇ is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Definition 1.9 (Riemannian connection). An affine connection ∇ is said to be Riemannian if it is torsion-free and compatible with the metric:

$$Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle.$$

Proposition 1.10 (local operator). *An affine connection is a local operator.*

Proof. Let $s \in \Gamma(TM)$ vanish on an open set U , and let f be a bump function supported in U such that $f(p) = 1$. Then $fs \equiv 0$, hence

$$0 = \nabla_X(fs) = (Xf)s + f\nabla_X s.$$

Evaluating at p yields $\nabla_X s(p) = 0$, and since p was arbitrary, $\nabla_X s = 0$ on U . \square

2 Vector bundles

2.1 The geometry of vector bundles

2.1.1 Some definitions

Let M be a Riemannian manifold.

Definition 2.1 (Partition of unity). A *partition of unity* is a collection $\{\rho_\alpha\}$ of smooth maps such that

$$\sum_\alpha \rho_\alpha = 1,$$

and $\{\text{supp } \rho_\alpha\}$ is locally finite, i.e. for any $p \in M$, there exists a neighborhood U intersecting only finitely many of the $\{\text{supp } \rho_\alpha\}$.

Moreover, given an open cover $\{U_\alpha\}$ of M , if $\{\text{supp } \rho_\alpha\} \subset U_\alpha$ for all α , then the partition $\{\rho_\alpha\}$ is said to be *subordinate* to the cover $\{U_\alpha\}$.

By the local finiteness property, on each neighborhood U intersecting finitely many supports, sums of products of partition maps with any quantity are always finite, since in U , only finitely many of the ρ_α are nonzero.

Definition 2.2 (Vector bundle). A smooth surjection $\pi : E \rightarrow M$ is called a *vector bundle of rank n* if, given any $p \in M$, the fiber $E_p = \pi^{-1}(\{p\})$ is a vector space of dimension n , and if, given any open neighborhood U of p , there exists a fiber-preserving diffeomorphism

$$\phi : E_U = \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$$

that restricts to an isomorphism on each fiber, i.e.

$$\phi_p : E_p \rightarrow \{p\} \times \mathbb{R}^n$$

is a vector space isomorphism.

We say that U is a *trivializing open set*, and that ϕ_U is a *local trivialization* of E . If the diffeomorphism is defined globally, i.e. $\phi : E \rightarrow M \times \mathbb{R}^n$, then the bundle is said to be *trivial*.

Any trivial bundle has a smooth frame; thus, since E_U is trivialized by U , it can always be equipped with a frame.

2.1.2 Bundle connections

Let E be a vector bundle over a Riemannian manifold M , and recall that $\Gamma(E)$ denotes the set of sections on E .

Definition 2.3 (connection on a vector bundle). A map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E),$$

written $\nabla_X s$ for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, is called a connection on the vector bundle E if it is $C^\infty(M)$ -linear in X , \mathbb{R} -linear in s , and satisfies the Leibniz rule

$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

A section s is said to be flat if $\nabla_X s = 0$ for any vector field X .

Proposition 2.4 (linear combinations). Let ∇^k be connections, and t_k numbers such that $\sum_{k=1}^n t_k = 1$. Then

$$\nabla = \sum_{k=1}^n t_k \nabla^k$$

is a connection.

Proof. Let $f \in C^\infty(M)$, $s, t \in \Gamma(E)$ and $a \in \mathbb{R}$. Then

$$\nabla_{fX}s = \sum t_k \nabla_{fX}^k s = \sum t_k f \nabla_X s = f \sum t_k \nabla_X s = f \nabla_X s.$$

Similarly, $\nabla_X(as + t) = a\nabla_X s + \nabla_X t$. Lastly,

$$\nabla_X(fs) = \sum t_k \nabla_X^k(fs) = \sum t_k [(Xf)s + f \nabla_X^k s] = (Xf)s + f \sum t_k \nabla_X^k s = (Xf)s + f \nabla_X s.$$

□

Definition 2.5 (bundle isomorphism). Let E, F be vector bundles, and $f : E \rightarrow F$ a bundle map. Then f is a bundle isomorphism if there exists a bundle map $g : F \rightarrow E$ such that

$$f \circ g = 1_F \quad \text{and} \quad g \circ f = 1_E.$$

Lemma 2.6 (connection on trivial bundles). Let E be a trivial bundle over M . Then there exists a connection on E .

Proof. Let $f : E \rightarrow M \times \mathbb{R}^n$ be a bundle isomorphism, and let $\{v_1, \dots, v_n\}$ be a basis for \mathbb{R}^n . Then

$$s_i : p \mapsto (p, v_i)$$

is a frame for the product bundle. Define $e_i = \phi^{-1} \circ s_i$. Then for $s \in \Gamma(E)$, $s = \sum a^i e_i$. Let X be a vector field on M and define

$$\nabla_X s = \sum (X a^i) e_i.$$

It is immediate that ∇ is $C^\infty(M)$ -linear in X , \mathbb{R} -linear in s , and satisfies the Leibniz rule. \square

Theorem 2.7 (existence of a connection). *Every vector bundle has a connection.*

Proof. Let E be a vector bundle over M , and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the trivializing open cover $\{U_\alpha\}$. Define the connection ∇^α on E_{U_α} (exists by Lemma 1.2.4). Then define

$$\nabla = \sum_\alpha \rho_\alpha \nabla^\alpha.$$

At any $p \in M$, only finitely many terms are nonzero, so the sum is well-defined. By Proposition 1.2.2, ∇ defines a connection globally. \square

2.1.3 Riemannian metric on a vector bundle

Definition 2.8 (Riemannian metric on a bundle). Let E be a vector bundle over a smooth manifold M . A Riemannian metric $\langle \cdot, \cdot \rangle$ on E is a smooth assignment

$$p \mapsto \langle \cdot, \cdot \rangle_p : E_p \times E_p \rightarrow \mathbb{R}$$

such that for any smooth sections s, t , $\langle s, t \rangle$ is smooth on M .

Lemma 2.9 (metric on a trivial bundle). Every trivial bundle has a metric.

Proof. Let E be trivial with bundle isomorphism $\phi : E \rightarrow M \times \mathbb{R}^n$, and consider the Euclidean metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. For $v, w \in E_p$, define

$$\langle v, w \rangle = \langle \phi_p(v), \phi_p(w) \rangle_{\mathbb{R}^n}.$$

This defines a smooth inner product on E . \square

Theorem 2.10 (existence of a metric). *Every vector bundle has a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be a cover of M with trivializing open sets, and $\{\rho_\alpha\}$ a partition of unity subordinate to this cover. By Lemma 1.3.2, each E_{U_α} has a metric $\langle \cdot, \cdot \rangle_\alpha$. Define

$$\langle \cdot, \cdot \rangle = \sum \rho_\alpha \langle \cdot, \cdot \rangle_\alpha.$$

By the same reasoning as for Riemannian metrics on manifolds, this defines a smooth Riemannian metric on E . \square