Notes on differential geometry: Riemannian geometry and Vector bundles (unfinished)

Sacha Sallustrau-Dendura

January 2026

Contents

1 Riemannian manifolds

1.1 Riemannian metric, cohomology

In this article, we will denote an inner product on T_pM by $\langle \cdot, \cdot \rangle_p$.

Definition 1.1 (Riemannian metric)**.** Let M be a smooth manifold. The Riemannian metric $\langle \cdot, \cdot \rangle$ is the smooth assignment

$$p \mapsto \langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}.$$

Definition 1.2 (Riemannian manifold)**.** Let M be a smooth manifold, $\langle \cdot, \cdot \rangle$ a Riemannian metric on M. Then the pair $(M, \langle \cdot, \cdot \rangle)$ is called a Riemannian manifold.

Theorem 1.3 (every manifold has a metric)**.** *Let M be a smooth manifold, then M can be equipped with a Riemannian metric.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M, and $\{\rho_\alpha\}$ a partition of unity subordinate to the open cover $\{U_\alpha\}$. Then $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for each α, and the collection $\{\operatorname{supp} \rho_\alpha\}$ is locally finite. Thus,

$$\sum_\alpha \rho_\alpha \langle \cdot, \cdot \rangle_\alpha$$

defines a smooth Riemannian metric on M. □

Lemma 1.4 (Circle around a puncture induces the zero map)**.** Let Σ be a compact, orientable surface and let $p \in \Sigma$. Let

$$i : C \hookrightarrow \Sigma \setminus \{p\}$$

be the inclusion of a circle C encircling the puncture. Then the induced map

$$i^* : H^1(\Sigma \setminus \{p\}) \to H^1(C)$$

is the zero map.

Proof. The isomorphism

$$H^1(C) \cong H^1(S^1) \cong \mathbb{R}$$

is given by integration. The circle C is the boundary of $\Sigma \setminus D$, where D is an open disk bounded by C. Let ω be any closed 1-form on $\Sigma \setminus \{p\}$. Then

$$\int_C i^*\omega = \int_{\partial(\Sigma \setminus D)} i^*\omega = \int_{\Sigma \setminus D} d\omega = 0,$$

since ω is closed. Since ω was chosen arbitrarily, it follows that

$$i^*[\omega] = 0$$

for any cohomology class. □

Theorem 1.5 (Cohomology of compact orientable surfaces of genus g)**.** *For any compact, orientable surface M_g of genus g,*

$$H^0(M_g) = H^2(M_g) = \mathbb{R}, \quad H^1(M_g) = \mathbb{R}^{2g},$$

and

$$H^k(M_g) = 0, \quad \forall k > 1.$$

Proof. These surfaces are connected, thus $H^0(M_g) = \mathbb{R}$ is immediate. We first prove the case $g = 2$, and then reason by induction.

Choose U, V to be punctured torii. Then $U \cap V$ is homeomorphic to S^1. The Mayer–Vietoris sequence is

$$\begin{tikzcd} H^1(M_2) \arrow[r, "i^*"] & H^1(U) \oplus H^1(V) \arrow[r, "j^*"] & \mathbb{R} \arrow[d, "d^*"] & & \\ & & H^2(M_2) \arrow[r, "i^*"] & H^2(U) \oplus H^2(V) \arrow[r] & 0 \end{tikzcd}$$

From differential topology, we know that $H^k(U) = H^k(V) = 0$ for all $k > 1$ and $H^1(U) = H^1(V) = \mathbb{R}^2$. By the previous lemma, $j^* = 0$. Thus, by exactness,

$$\operatorname{im} j^* = \ker d^* = 0,$$

so d^* is injective and therefore $H^2(M_2) \cong \mathbb{R}$.

The Euler-characteristic type method then implies

$$\dim H^1(M_2) - \dim H^1(U) - \dim H^1(V) - 1 + 1 = 0,$$

i.e. $H^1(M_2) \cong \mathbb{R}^4$.

The induction hypothesis is that $H^1(M_{g-1}) = \mathbb{R}^{2(g-1)}$ and $H^2(M_{g-1}) = \mathbb{R}$. Cover M_{g-1} with

$$U = M_{g-1} \setminus \{p\}, \quad V = D,$$

where D is an open disk. Since D is contractible, $H^k(D) = 0$ for all $k > 0$, and $U \cap V$ is a punctured disk, homotopy equivalent to S^1. The Mayer–Vietoris sequence is

$$\begin{tikzcd} H^1(M_{g-1}) \arrow[r, "i^*"] & H^1(U) \arrow[r, "j^*"] & \mathbb{R} \arrow[d, "d^*"] & & \\ & & \mathbb{R} \arrow[r, "i^*"] & H^2(U) \arrow[r] & 0 \end{tikzcd}$$

By the previous lemma, $j^* = 0$. Thus $H^1(U) = \ker j^* = \operatorname{im} i^*$, and since i^* is injective,

$$H^1(U) \cong H^1(M_{g-1}) = \mathbb{R}^{2(g-1)}.$$

Using the Euler characteristic type method, we find that $H^2(U) = 0$.

Now cover M_g by $U = M_{g-1} \setminus \{p\}$ and V a punctured torus. Since $U \cap V$ is homeomorphic to a cylinder, the Mayer–Vietoris sequence is

$$\begin{tikzcd} H^1(M_g) \arrow[r, "i^*"] & H^1(U) \oplus H^1(V) \arrow[r, "j^*"] & \mathbb{R} \arrow[d, "d^*"] & & \\ & & H^2(M_g) \arrow[r, "i^*"] & H^2(U) \oplus H^2(V) \arrow[r] & 0 \end{tikzcd}$$

Using the same reasoning as for M_2, we obtain $H^2(M_g) = \mathbb{R}$, and therefore

$$\dim H^1(M_g) = 2g - 2 + 2 - 1 + 1 = 2g,$$

as desired. □

1.2 Connections

While we can properly define directional derivatives on curves and surfaces, we usually can't do that on arbitrary Riemannian manifolds. That is why we introduce connections.

Recall that $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M.

Definition 1.6 (Affine connection)**.** The $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is called an affine connection if it is $C^\infty(M)$-linear in the first argument and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

Definition 1.7 (Curvature of a connection)**.** Let ∇ be an affine connection. The curvature of ∇ is

$$R(X, Y) = [\nabla_X Y, \nabla_Y X] - \nabla_{[X,Y]}.$$

Definition 1.8 (Torsion of a connection)**.** The torsion of ∇ is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Definition 1.9 (Riemannian connection)**.** An affine connection ∇ is said to be Riemannian if it is torsion-free and compatible with the metric:

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

Proposition 1.10 (local operator)**.** *An affine connection is a local operator.*

Proof. Let $s \in \Gamma(TM)$ vanish on an open set U, and let f be a bump function supported in U such that $f(p) = 1$. Then $fs \equiv 0$, hence

$$0 = \nabla_X(fs) = (Xf)s + f\nabla_X s.$$

Evaluating at p yields $\nabla_X s(p) = 0$, and since p was arbitrary, $\nabla_X s = 0$ on U. $\square$

2 Vector bundles

2.1 The geometry of vector bundles

It just occurred to me that in two of my previous posts, I used partitions of unity without defining them beforehand. In this first part, we define these important concepts.

2.1.1 Some definitions

Let M be a Riemannian manifold.

Definition 2.1 (Partition of unity)**.** A *partition of unity* is a collection $\{\rho_\alpha\}$ of smooth maps such that

$$\sum_\alpha \rho_\alpha = 1,$$

and $\{\operatorname{supp} \rho_\alpha\}$ is locally finite, i.e. for any $p \in M$, there exists a neighborhood U intersecting only finitely many of the $\{\operatorname{supp} \rho_\alpha\}$.

Moreover, given an open cover $\{U_\alpha\}$ of M, if $\{\operatorname{supp} \rho_\alpha\} \subset U_\alpha$ for all α, then the partition $\{\rho_\alpha\}$ is said to be *subordinate* to the cover $\{U_\alpha\}$.

By the local finiteness property, on each neighborhood U intersecting finitely many supports, sums of products of partition maps with any quantity are always finite, since in U, only finitely many of the ρ_α are nonzero.

Definition 2.2 (Vector bundle)**.** A smooth surjection $\pi : E \to M$ is called a *vector bundle of rank* n if, given any $p \in M$, the fiber $E_p = \pi^{-1}(\{p\})$ is a vector space of dimension n, and if, given any open neighborhood U of p, there exists a fiber-preserving diffeomorphism

$$\phi : E_U = \pi^{-1}(U) \to U \times \mathbb{R}^n$$

that restricts to an isomorphism on each fiber, i.e.

$$\phi_p : E_p \to \{p\} \times \mathbb{R}^n$$

is a vector space isomorphism.

We say that U is a *trivializing open set*, and that ϕ_U is a *local trivialization* of E. If the diffeomorphism is defined globally, i.e. $\phi : E \to M \times \mathbb{R}^n$, then the bundle is said to be *trivial.*

Any trivial bundle has a smooth frame; thus, since E_U is trivialized by U, it can always be equipped with a frame.

2.1.2 Bundle connections

Let E be a vector bundle over a Riemannian manifold M, and recall that $\Gamma(E)$ denotes the set of sections on E.

Definition 2.3 (connection on a vector bundle)**.** A map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

written $\nabla_X s$ for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, is called a connection on the vector bundle E if it is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in s, and satisfies the Leibniz rule

$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

A section s is said to be flat if $\nabla_X s = 0$ for any vector field X.

Proposition 2.4 (linear combinations)**.** *Let ∇^k be connections, and t_k numbers such that $\sum_{k=1}^n t_k = 1$. Then*

$$\nabla = \sum_{k=1}^n t_k \nabla^k$$

is a connection.

Proof. Let $f \in C^\infty(M)$, $s, t \in \Gamma(E)$ and $a \in \mathbb{R}$. Then

$$\nabla_{fX} s = \sum t_k \nabla^k_{fX} s = \sum t_k f \nabla_X s = f \sum t_k \nabla_X s = f\nabla_X s.$$

Similarly, $\nabla_X(as + t) = a\nabla_X s + \nabla_X t$. Lastly,

$$\nabla_X(fs) = \sum t_k \nabla^k_X(fs) = \sum t_k[(Xf)s + f\nabla^k_X s] = (Xf)s + f\sum t_k \nabla^k_X s = (Xf)s + f\nabla_X s.$$

□

Definition 2.5 (bundle isomorphism)**.** Let E, F be vector bundles, and $f : E \to F$ a bundle map. Then f is a bundle isomorphism if there exists a bundle map $g : F \to E$ such that

$$f \circ g = 1_F \quad \text{and} \quad g \circ f = 1_E.$$

Lemma 2.6 (connection on trivial bundles)**.** Let E be a trivial bundle over M. Then there exists a connection on E.

Proof. Let $f : E \to M \times \mathbb{R}^n$ be a bundle isomorphism, and let $\{v_1, \dots, v_n\}$ be a basis for $\mathbb{R}^n$. Then

$$s_i : p \mapsto (p, v_i)$$

is a frame for the product bundle. Define $e_i = \phi^{-1} \circ s_i$. Then for $s \in \Gamma(E)$, $s = \sum a^i e_i$. Let X be a vector field on M and define

$$\nabla_X s = \sum (X a^i) e_i.$$

It is immediate that ∇ is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in s, and satisfies the Leibniz rule. $\square$

Theorem 2.7 (existence of a connection)**.** *Every vector bundle has a connection.*

Proof. Let E be a vector bundle over M, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the trivializing open cover $\{U_\alpha\}$. Define the connection ∇^α on E_{U_α} (exists by Lemma 1.2.4). Then define

$$\nabla = \sum_\alpha \rho_\alpha \nabla^\alpha.$$

At any $p \in M$, only finitely many terms are nonzero, so the sum is well-defined. By Proposition 1.2.2, ∇ defines a connection globally. $\square$

2.1.3 Riemannian metric on a vector bundle

Definition 2.8 (Riemannian metric on a bundle)**.** Let E be a vector bundle over a smooth manifold M. A Riemannian metric $\langle \cdot, \cdot \rangle$ on E is a smooth assignment

$$p \mapsto \langle \cdot, \cdot \rangle_p : E_p \times E_p \to \mathbb{R}$$

such that for any smooth sections s, t, $\langle s, t \rangle$ is smooth on M.

Lemma 2.9 (metric on a trivial bundle)**.** Every trivial bundle has a metric.

Proof. Let E be trivial with bundle isomorphism $\phi : E \to M \times \mathbb{R}^n$, and consider the Euclidean metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. For $v, w \in E_p$, define

$$\langle v, w \rangle = \langle \phi_p(v), \phi_p(w) \rangle_{\mathbb{R}^n}.$$

This defines a smooth inner product on E. $\square$

Theorem 2.10 (existence of a metric)**.** *Every vector bundle has a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be a cover of M with trivializing open sets, and $\{\rho_\alpha\}$ a partition of unity subordinate to this cover. By Lemma 1.3.2, each E_{U_α} has a metric $\langle \cdot, \cdot \rangle_\alpha$. Define

$$\langle \cdot, \cdot \rangle = \sum \rho_\alpha \langle \cdot, \cdot \rangle_\alpha.$$

By the same reasoning as for Riemannian metrics on manifolds, this defines a smooth Riemannian metric on E. $\square$