Notes on differential geometry: Riemannian geometry and Vector bundles (unfinished)

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Contents

1 Riemannian manifolds

1.1 Riemannian metric, cohomology

In this article, we will denote an inner product on T_pM by $\langle \cdot, \cdot \rangle_p$.

Definition 1.1 (Riemannian metric)**.** Let M be a smooth manifold. The Riemannian metric $\langle \cdot, \cdot \rangle$ is the smooth assignment

$$p \mapsto \langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}.$$

Definition 1.2 (Riemannian manifold)**.** Let M be a smooth manifold, $\langle \cdot, \cdot \rangle$ a Riemannian metric on M. Then the pair $(M, \langle \cdot, \cdot \rangle)$ is called a Riemannian manifold.

Theorem 1.3 (every manifold has a metric)**.** *Let M be a smooth manifold, then M can be equipped with a Riemannian metric.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M, and $\{\rho_\alpha\}$ a partition of unity subordinate to the open cover $\{U_\alpha\}$. Then $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for each α, and the collection $\{\operatorname{supp} \rho_\alpha\}$ is locally finite. Thus,

$$\sum_\alpha \rho_\alpha \langle \cdot, \cdot \rangle_\alpha$$

defines a smooth Riemannian metric on M. □

Lemma 1.4 (Circle around a puncture induces the zero map)**.** Let Σ be a compact, orientable surface and let $p \in \Sigma$. Let

$$i : C \hookrightarrow \Sigma \setminus \{p\}$$

be the inclusion of a circle C encircling the puncture. Then the induced map

$$i^* : H^1(\Sigma \setminus \{p\}) \to H^1(C)$$

is the zero map.

Proof. The isomorphism

$$H^1(C) \cong H^1(S^1) \cong \mathbb{R}$$

is given by integration. The circle C is the boundary of $\Sigma \setminus D$, where D is an open disk bounded by C. Let ω be any closed 1-form on $\Sigma \setminus \{p\}$. Then

$$\int_C i^*\omega = \int_{\partial(\Sigma \setminus D)} i^*\omega = \int_{\Sigma \setminus D} d\omega = 0,$$

since ω is closed. Since ω was chosen arbitrarily, it follows that

$$i^*[\omega] = 0$$

for any cohomology class. □

Theorem 1.5 (Cohomology of compact orientable surfaces of genus g)**.** *For any compact, orientable surface M_g of genus g,*

$$H^0(M_g) = H^2(M_g) = \mathbb{R}, \quad H^1(M_g) = \mathbb{R}^{2g},$$

and

$$H^k(M_g) = 0, \quad \forall k > 1.$$

Proof. These surfaces are connected, thus $H^0(M_g) = \mathbb{R}$ is immediate. We first prove the case $g = 2$, and then reason by induction.
Choose U, V to be punctured torii. Then $U \cap V$ is homeomorphic to S^1. The Mayer–Vietoris sequence is

$$\begin{tikzcd} H^1(M_2) \arrow[r, "i^*"] & H^1(U) \oplus H^1(V) \arrow[r, "j^*"] & \mathbb{R} \arrow[d, "d^*"] & & \\ & & H^2(M_2) \arrow[r, "i^*"] & H^2(U) \oplus H^2(V) \arrow[r] & 0 \end{tikzcd}$$

From differential topology, we know that $H^k(U) = H^k(V) = 0$ for all $k > 1$ and $H^1(U) = H^1(V) = \mathbb{R}^2$. By the previous lemma, $j^* = 0$. Thus, by exactness,

$$\operatorname{im} j^* = \ker d^* = 0,$$

so d^* is injective and therefore $H^2(M_2) \cong \mathbb{R}$.
The Euler-characteristic type method then implies

$$\dim H^1(M_2) - \dim H^1(U) - \dim H^1(V) - 1 + 1 = 0,$$

i.e. $H^1(M_2) \cong \mathbb{R}^4$.
The induction hypothesis is that $H^1(M_{g-1}) = \mathbb{R}^{2(g-1)}$ and $H^2(M_{g-1}) = \mathbb{R}$. Cover M_{g-1} with

$$U = M_{g-1} \setminus \{p\}, \quad V = D,$$

where D is an open disk. Since D is contractible, $H^k(D) = 0$ for all $k > 0$, and $U \cap V$ is a punctured disk, homotopy equivalent to S^1. The Mayer–Vietoris sequence is

$$\begin{tikzcd} H^1(M_{g-1}) \arrow[r, "i^*"] & H^1(U) \arrow[r, "j^*"] & \mathbb{R} \arrow[d, "d^*"] & & \\ & & \mathbb{R} \arrow[r, "i^*"] & H^2(U) \arrow[r] & 0 \end{tikzcd}$$

By the previous lemma, $j^* = 0$. Thus $H^1(U) = \ker j^* = \operatorname{im} i^*$, and since i^* is injective,

$$H^1(U) \cong H^1(M_{g-1}) = \mathbb{R}^{2(g-1)}.$$

Using the Euler characteristic type method, we find that $H^2(U) = 0$.
Now cover M_g by $U = M_{g-1} \setminus \{p\}$ and V a punctured torus. Since $U \cap V$ is homeomorphic to a cylinder, the Mayer–Vietoris sequence is

$$\begin{tikzcd} H^1(M_g) \arrow[r, "i^*"] & H^1(U) \oplus H^1(V) \arrow[r, "j^*"] & \mathbb{R} \arrow[d, "d^*"] & & \\ & & H^2(M_g) \arrow[r, "i^*"] & H^2(U) \oplus H^2(V) \arrow[r] & 0 \end{tikzcd}$$

Using the same reasoning as for M_2, we obtain $H^2(M_g) = \mathbb{R}$, and therefore

$$\dim H^1(M_g) = 2g - 2 + 2 - 1 + 1 = 2g,$$

as desired. $\square$

1.2 Connections

While we can properly define directional derivatives on curves and surfaces, we usually can't do that on arbitrary Riemannian manifolds. That is why we introduce connections.
Recall that $\mathfrak{X}(M)$ denotes the Lie algebra of smooth vector fields on M.

Definition 1.6 (Affine connection)**.** The $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is called an affine connection if it is $C^\infty(M)$-linear in the first argument and satisfies the Leibniz rule

$$\nabla_X(fY) = (Xf)Y + f\nabla_X Y.$$

Definition 1.7 (Curvature of a connection)**.** Let ∇ be an affine connection. The curvature of ∇ is

$$R(X,Y) = [\nabla_X Y, \nabla_Y X] - \nabla_{[X,Y]}.$$

Definition 1.8 (Torsion of a connection)**.** The torsion of ∇ is

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Definition 1.9 (Riemannian connection)**.** An affine connection ∇ is said to be Riemannian if it is torsion-free and compatible with the metric:

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

Proposition 1.10 (local operator)**.** *An affine connection is a local operator.*

Proof. Let $s \in \Gamma(TM)$ vanish on an open set U, and let f be a bump function supported in U such that $f(p) = 1$. Then $fs \equiv 0$, hence

$$0 = \nabla_X(fs) = (Xf)s + f\nabla_X s.$$

Evaluating at p yields $\nabla_X s(p) = 0$, and since p was arbitrary, $\nabla_X s = 0$ on U. □

2 Vector bundles

2.1 The geometry of vector bundles

2.1.1 Some definitions

Let M be a Riemannian manifold.

Definition 2.1 (Partition of unity)**.** A *partition of unity* is a collection $\{\rho_\alpha\}$ of smooth maps such that

$$\sum_\alpha \rho_\alpha = 1,$$

and $\{\operatorname{supp}\rho_\alpha\}$ is locally finite, i.e. for any $p \in M$, there exists a neighborhood U intersecting only finitely many of the $\{\operatorname{supp}\rho_\alpha\}$.
Moreover, given an open cover $\{U_\alpha\}$ of M, if $\{\operatorname{supp}\rho_\alpha\} \subset U_\alpha$ for all α, then the partition $\{\rho_\alpha\}$ is said to be *subordinate* to the cover $\{U_\alpha\}$.

By the local finiteness property, on each neighborhood U intersecting finitely many supports, sums of products of partition maps with any quantity are always finite, since in U, only finitely many of the ρ_α are nonzero.

Definition 2.2 (Vector bundle)**.** A smooth surjection $\pi : E \to M$ is called a *vector bundle of rank n* if, given any $p \in M$, the fiber $E_p = \pi^{-1}(\{p\})$ is a vector space of dimension n, and if, given any open neighborhood U of p, there exists a fiber-preserving diffeomorphism

$$\phi : E_U = \pi^{-1}(U) \to U \times \mathbb{R}^n$$

that restricts to an isomorphism on each fiber, i.e.

$$\phi_p : E_p \to \{p\} \times \mathbb{R}^n$$

is a vector space isomorphism.
We say that U is a *trivializing open set*, and that ϕ_U is a *local trivialization* of E. If the diffeomorphism is defined globally, i.e. $\phi : E \to M \times \mathbb{R}^n$, then the bundle is said to be *trivial.*

Any trivial bundle has a smooth frame; thus, since E_U is trivialized by U, it can always be equipped with a frame.

2.1.2 Bundle connections

Let E be a vector bundle over a Riemannian manifold M, and recall that $\Gamma(E)$ denotes the set of sections on E.

Definition 2.3 (connection on a vector bundle)**.** A map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E),$$

written $\nabla_X s$ for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, is called a connection on the vector bundle E if it is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in s, and satisfies the Leibniz rule

$$\nabla_X(fs) = (Xf)s + f\nabla_X s.$$

A section s is said to be flat if $\nabla_X s = 0$ for any vector field X.

Proposition 2.4 (linear combinations)**.** *Let ∇^k be connections, and t_k numbers such that $\sum_{k=1}^n t_k =$ 1. Then*

$$\nabla = \sum_{k=1}^n t_k \nabla^k$$

is a connection.

Proof. Let $f \in C^\infty(M)$, $s, t \in \Gamma(E)$ and $a \in \mathbb{R}$. Then

$$\nabla_{fX} s = \sum t_k \nabla^k_{fX} s = \sum t_k f \nabla_X s = f \sum t_k \nabla_X s = f \nabla_X s.$$

Similarly, $\nabla_X(as + t) = a\nabla_X s + \nabla_X t$. Lastly,

$$\nabla_X(fs) = \sum t_k \nabla^k_X(fs) = \sum t_k[(Xf)s + f\nabla^k_X s] = (Xf)s + f \sum t_k \nabla^k_X s = (Xf)s + f\nabla_X s.$$

□

Definition 2.5 (bundle isomorphism)**.** Let E, F be vector bundles, and $f : E \to F$ a bundle map. Then f is a bundle isomorphism if there exists a bundle map $g : F \to E$ such that

$$f \circ g = 1_F \quad \text{and} \quad g \circ f = 1_E.$$

Lemma 2.6 (connection on trivial bundles)**.** Let E be a trivial bundle over M. Then there exists a connection on E.

Proof. Let $f : E \to M \times \mathbb{R}^n$ be a bundle isomorphism, and let $\{v_1, \ldots, v_n\}$ be a basis for $\mathbb{R}^n$. Then

$$s_i : p \mapsto (p, v_i)$$

is a frame for the product bundle. Define $e_i = \phi^{-1} \circ s_i$. Then for $s \in \Gamma(E)$, $s = \sum a^i e_i$. Let X be a vector field on M and define

$$\nabla_X s = \sum (X a^i) e_i.$$

It is immediate that ∇ is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in s, and satisfies the Leibniz rule. $\square$

Theorem 2.7 (existence of a connection)**.** *Every vector bundle has a connection.*

Proof. Let E be a vector bundle over M, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the trivializing open cover $\{U_\alpha\}$. Define the connection ∇^α on E_{U_α} (exists by Lemma 2.6). Then define

$$\nabla = \sum_\alpha \rho_\alpha \nabla^\alpha.$$

At any $p \in M$, only finitely many terms are nonzero, so the sum is well-defined. By Proposition 1.2.2, ∇ defines a connection globally. $\square$

2.1.3 Riemannian metric on a vector bundle

Definition 2.8 (Riemannian metric on a bundle)**.** Let E be a vector bundle over a smooth manifold M. A Riemannian metric $\langle \cdot, \cdot \rangle$ on E is a smooth assignment

$$p \mapsto \langle \cdot, \cdot \rangle_p : E_p \times E_p \to \mathbb{R}$$

such that for any smooth sections s, t, $\langle s, t \rangle$ is smooth on M.

Lemma 2.9 (metric on a trivial bundle)**.** Every trivial bundle has a metric.

Proof. Let E be trivial with bundle isomorphism $\phi : E \to M \times \mathbb{R}^n$, and consider the Euclidean metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$. For $v, w \in E_p$, define

$$\langle v, w \rangle = \langle \phi_p(v), \phi_p(w) \rangle_{\mathbb{R}^n}.$$

This defines a smooth inner product on E. $\square$

Theorem 2.10 (existence of a metric)**.** *Every vector bundle has a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be a cover of M with trivializing open sets, and $\{\rho_\alpha\}$ a partition of unity subordinate to this cover. By Lemma 1.3.2, each E_{U_α} has a metric $\langle \cdot, \cdot \rangle_\alpha$. Define

$$\langle \cdot, \cdot \rangle = \sum \rho_\alpha \langle \cdot, \cdot \rangle_\alpha.$$

By the same reasoning as for Riemannian metrics on manifolds, this defines a smooth Riemannian metric on E. $\square$

An important remark is that in a vector bundle, the total space E (which we've been referring to as the bundle itself as an abuse of language so far) is a smooth manifold, which means that it can be equipped with a Riemannian metric. This is where we need to be careful : the bundle Riemannian metric needs not coincide with the manifold Riemannian metric, they're not necessarily the same objects at all.
Now, obviously we don't have a direct analogue of Riemannian connections, because with this definition of connections, torsion doesn't make sense if E isn't TM. However, it is still desirable to define compatibility with the metric.

Definition 2.11 (Metric connection)**.** Let E be a Riemannian bundle over a manifold M, ∇ a connection on E, and $s, t \in \Gamma(E)$. Then ∇ is called a *metric connection* if it is compatible with the metric, i.e.

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

As you might have guessed, every vector bundle has a metric connection. In order to prove that, we first need the following results.

Lemma 2.12 (Existence on a trivial bundle)**.** On a trivial vector bundle E, the connection induced by the trivialization is compatible with the metric.

Proof. Let $\{v_i\}$ be the standard basis for $\mathbb{R}^n$, and $\{e_i\} = \phi^{-1}(p, v_i)$ be a frame for E, then $\{e_i\}$ is flat. Take sections s, t, then $s = \sum a^i e_i$ and $t = \sum b^j e_j$, and thus, given $X \in \mathfrak{X}(M)$,

$$X\langle s, t\rangle = X\left(\sum a^i b^i\right) = \sum (Xa^i)b^i + \sum a^i X b^i = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

$\square$

Proposition 2.13 (Linear combinations)**.** *Let ∇^i be metric connections on a Riemannian bundle E, and let t^i be numbers such that $\sum t^i = 1$. Then $\nabla = \sum t^i \nabla^i$ is compatible with the metric.*

Proof. Take two sections, s and t, and a vector field X on the manifold M. Then,

$$\begin{aligned}
\langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle &= \left\langle \sum t^i \nabla^i_X s, t\right\rangle + \left\langle s, \sum t^i \nabla^i_X t\right\rangle \\
&= \sum t^i \left(\langle \nabla^i_X s, t\rangle + \langle s, \nabla^i_X t\rangle\right) \\
&= \sum t^i X\langle s, t\rangle = X\langle s, t\rangle.
\end{aligned}$$

$\square$

Theorem 2.14 (Existence of a metric connection)**.** *Every Riemannian bundle has a metric connection.*

Proof. Let $\{U_\alpha\}$ be a cover of M of open sets that trivialize E, the Riemannian bundle over M, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. On E_{U_α}, define a metric connection ∇^α using the trivialization. Set

$$\nabla = \sum \rho_\alpha \nabla^\alpha.$$

Using local finiteness and the previous proposition, this defines a metric connection on E. $\square$

Lastly, we define the curvature of a bundle connection.

Definition 2.15 (Curvature)**.** The curvature of a connection ∇ on a vector bundle E is the tensor

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

defined by

$$R(X, Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s.$$

3 Geodesics

3.1 Covariant derivative

To motivate this part, we first need to define vector fields along curves.

Definition 3.1 (Vector field along a curve)**.** Let M be a Riemannian manifold, and $c : [a, b] \to M$ a curve. Then, a vector field X along the curve $c(t)$ is a map

$$X : [a, b] \to \bigsqcup_{t \in [a,b]} T_{c(t)}M,$$

such that $X(t) \in T_{c(t)}M$.

Such a vector field can be induced by a vector field $\tilde{X} \in \mathfrak{X}(M)$, by setting $X(t) = \tilde{X}_{c(t)}$.
It turns out that in $\mathbb{R}^n$, we can differentiate these vector fields along curves. Thus, an obvious question would be whether this can be generalized to any Riemannian manifold, and the answer is yes. In fact, an affine connection naturally induces a unique map, called a covariant derivative, allowing us to do that.
From now on, $\Gamma(TM|_{c(t)})$ denotes the space of vector fields along a curve $c : [a, b] \to M$.

Definition 3.2 (Covariant derivative)**.** Let ∇ be an affine connection on M, then the covariant derivative associated to ∇ is an $\mathbb{R}$-linear map

$$\frac{D}{dt} : \Gamma(TM|_{c(t)}) \to \Gamma(TM|_{c(t)})$$

satisfying the Leibniz rule, and such that if $\tilde{X}$ induces X, then

$$\frac{DX}{dt} = \nabla_{c'(t)}\tilde{X}.$$

Theorem 3.3 (Existence and uniqueness)**.** *Given any Riemannian manifold M, and any affine connection ∇ on M, there exists a unique covariant derivative associated to ∇.*

Proof. Let U be a trivializing open subset of M, and let $\{e_1, \ldots, e_n\}$ be a frame for TU. Then,

$$X(t) = \sum a^i(t) e_{i,c(t)}$$

for some coefficients a^i, $i = 1, \ldots, n$.
Suppose existence, then

$$\frac{DX}{dt} = \sum \frac{D}{dt}(a^i e_i) = \sum \left(\frac{da^i}{dt} e_i + a^i \nabla_{c'(t)} e_i \right).$$

Assume that another covariant derivative $\frac{\tilde{D}}{dt}$ exists, then it satisfies the same properties as $\frac{D}{dt}$, and thus

$$\frac{\tilde{D}X}{dt} = \sum \left(\frac{da^i}{dt} e_i + a^i \nabla_{c'(t)} e_i \right) = \frac{DX}{dt},$$

proving uniqueness.

With this construction, existence is guaranteed. In fact, $\frac{D}{dt}$ is obviously $\mathbb{R}$-linear, it satisfies the Leibniz rule since given $f \in C^\infty(U)$,

$$\begin{aligned}\frac{D(fX)}{dt} &= \sum\left(\frac{d(fa^i)}{dt}e_i + fa^i\nabla_{c'(t)}e_i\right)\\ &= \sum\left(\left(\frac{df}{dt}a^i + f\frac{da^i}{dt}\right)e_i + a^i\nabla_{c'(t)}e_i\right)\\ &= \frac{df}{dt}\sum a^i e_i + f\sum\left(\frac{da^i}{dt}e_i + a^i\nabla_{c'(t)}e_i\right) = \frac{df}{dt}X + f\frac{DX}{dt}.\end{aligned}$$

Lastly, $\tilde{X} = \sum \tilde{a}^i e_{i,p}$, thus

$$\begin{aligned}\nabla_{c'(t)}\tilde{X} &= \sum \nabla_{c'(t)}(\tilde{a}^i e_{i,p})\\ &= \sum\left(\frac{d(\tilde{a}^i \circ c(t))}{dt}e_{i,c(t)} + \tilde{a}^i \circ c(t)\nabla_{c'(t)}e_i\right).\end{aligned}$$

But since $\tilde{X}_{c(t)} = X(t)$, unicity of basis expansion guarantees that $a^i = \tilde{a}^i \circ c$, therefore

$$\nabla_{c'(t)}\tilde{X} = \sum \frac{da^i}{dt}e_{i,c(t)} + a^i\nabla_{c'(t)}e_i = \frac{DX}{dt}.$$

Lastly, the proof that it is a well defined map is easy. Take another frame, then use unicity to deduce that the covariant derivative is independent of the frame. □

If ∇ is compatible with the metric, it turns out that covariant derivatives interact nicely with the metric.

Theorem 3.4 (Compatibility with the metric)**.** *Suppose ∇ is compatible with the metric on M, then for vector fields X, Y along a curve $c(t)$,*

$$\frac{d}{dt}\langle X, Y\rangle = \left\langle \frac{DX}{dt}, Y\right\rangle + \left\langle X, \frac{DY}{dt}\right\rangle.$$

Proof. Take a framed open set U with orthonormal frame $\{e_1, \dots, e_n\}$, then on U we write $X = \sum a^i e_i$ and $Y = \sum b^j e_j$. Thus,

$$\frac{d}{dt}\langle X, Y\rangle = \frac{d}{dt}\left(\sum a^i b^i\right) = \sum\left(\frac{da^i}{dt}b^i + a^i\frac{db^i}{dt}\right).$$

By definition of the covariant derivative,

$$\frac{DX}{dt} = \sum\left(\frac{da^i}{dt}e_i + a^i\nabla_{c'(t)}e_i\right), \quad \frac{DY}{dt} = \sum\left(\frac{db^j}{dt}e_j + e^j\nabla_{c'(t)}e_j\right).$$

Thus,

$$\begin{aligned}\langle\frac{DX}{dt}, Y\rangle + \langle X, \frac{DY}{dt}\rangle &= \sum\frac{da^i}{dt}b^i + \sum a^i b^j\langle\nabla_{c'(t)}e_i, e_j\rangle\\ &+ \sum a^i b^j\langle e_i, \nabla_{c'(t)}e_j\rangle + \sum a^i\frac{db^j}{dt}.\end{aligned}$$

But

$$\sum a^i b^j\langle\nabla_{c'(t)}e_i, e_j\rangle + \sum a^i b^j\langle e_i, \nabla_{c'(t)}e_j\rangle = \sum a^i b^j c'(t)\langle e_i, e_j\rangle,$$

and this last sum is exactly

$$\sum a^i b^j c'(t)\delta^i_j = 0.$$

□

A natural question is to determine how the covariant derivative behaves under pushforwards. Suppose $f : M \to N$ is a smooth map of manifolds. If X is a vector field along the curve $c(t)$, then $(f_*X)(t) = f_{*,c(t)}(X(t))$ is a vector field along the curve $f \circ c$ in N. We consider the case where f is a diffeomorphism.

Definition 3.5 (Connection-preserving diffeomorphism)**.** Let (M, ∇^M) and (N, ∇^N) be Riemannian manifolds, and $f : M \to N$ a diffeomorphism. Then f is connection-preserving if given any $X, Y \in \mathfrak{X}(M)$,

$$f_*(\nabla_X^M Y) = \nabla_{f_*X}^N f_*Y.$$

Proposition 3.6 (Pushforward of covariant derivative)**.** *Let $f : M \to N$ be connection-preserving, $\frac{D^M}{dt}$ and $\frac{D^N}{dt}$ be the covariant derivatives associated to ∇^M and ∇^N respectively, and let $c(t)$ be a curve in M. If X is a vector field along c, then*

$$f_*\left(\frac{D^M X}{dt}\right) = \frac{D^N f_* X}{dt}.$$

Proof. Fix t, and take a framed neighborhood of $c(t)$ with frame $\{e_1, \ldots, e_n\}$. Define $\bar{e}_i = f_*e_i$, then $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a frame for the open set $f(U)$. If

$$X = \sum a^i e_{i,c(t)},$$

then

$$f_*X = \sum a^i \bar{e}_{i,f\circ c(t)}.$$

Since the points at which we take the frame are now quite clear, we will omit them from the subscripts. Then, pushing forward the covariant derivative of X, we obtain

$$f_* \frac{DX}{dt} = \sum \left(\frac{da^i}{dt} \bar{e}_i + a^i \nabla^N_{f_*c'(t)} \bar{e}_i \right),$$

and since $f_*c' = (f \circ c)'$, applying the covariant derivative $\frac{D^N}{dt}$ to f_*X completes the proof. $\square$

3.2 Christoffel symbols, geodesic curves

Geodesics have the nice property of being solutions to systems of differential equations, with given initial conditions. Thus, their existence and uniqueness is surprisingly guaranteed not by differential geometry, but by the Cauchy-Lipschitz theorem, a famous result from the theory of ordinary differential equations.
However, proving this property is no trivial task, and the first step is defining the Christoffel symbols for an arbitrary Riemannian manifold M.

Definition 3.7 (Christoffel symbols)**.** Consider an arbitrary Riemannian manifold M, equipped with an affine connection ∇. Choose a chart $(U, x^1, \ldots, x^n)$ around a point p, and setting $\frac{\partial}{\partial x^i} = \partial_i$, let $\{\partial_1, \ldots, \partial_n\}$ be the frame for T_pM. Then,

$$\nabla_{\partial_i} \partial_j = \sum \Gamma_{ij}^k \partial_k$$

for some numbers Γ_{ij}^k. These numbers are called the Christoffel symbols of the connection ∇.

We can now explore some results about geodesics. A geodesic curve is, in simple terms, the shortest path from a point to another. In $\mathbb{R}^n$, this path is a straight line, so geodesics can be thought of as the generalization of straight lines to arbitrary manifolds.
From now on, M is always a Riemannian manifold, ∇ an affine connection (unless stated otherwise) on M, and $c : I \to M$ a curve defined on some interval I. Moreover, the tangent vector to $c(t)$ is $T(t) = c'(t)$.

Definition 3.8 (Geodesic curve)**.** A curve $c(t)$ in M is a geodesic curve if

$$\frac{DT}{dt} = 0.$$

This definition, while simple, lacks any sort of comparability. In fact, computing the covariant derivative requires computing an affine connection, which often isn't an easy task. Fortunately, some simple results fix this problems, such as the following proposition.
Define a norm on vectors by setting $\|X_p\| = \sqrt{\langle X_p, X_p\rangle}$, then the speed of a curve $c(t)$ is $\|c'(t)\|$.

Proposition 3.9 (Geodesic has constant speed)**.** *A curve $c(t)$ is geodesic if and only if it has constant speed.*

Proof. Define $f(t) = \langle T, T\rangle$, then the speed is constant if and only if $f'(t) = 0$, where

$$f'(t) = 2\left\langle \frac{DT}{dt}, T \right\rangle.$$

If $c(t)$ is geodesic, then $\frac{DT}{dt} = 0$ and thus $f' = 0$, and conversely if the speed is constant, then $f' = 0$, implying that $\frac{DT}{dt} = 0$ (if $T = 0$, then the curve is a constant, and thus trivially a geodesic). □

This proposition provides a simple way of verifying whether or not a curve is geodesic. As an example, the great circles on the sphere S^2 are geodesics.
In fact, take a surface $M \subset \mathbb{R}^3$, and give it its Riemannian connection ∇, then if N is the normal vector field of M it can be shown (by showing it satisfies the properties of a Riemannian metric, then use unicity) that

$$\nabla_X Y = (D_X Y)_{\tan},$$

where $D_X Y$ is the directional derivative and

$$X_{\tan} = X - \langle X, N\rangle N$$

is the tangent component of a vector field. An easy corollary is that

$$\frac{DX}{dt} = \left(\frac{dX}{dt}\right)_{\tan}.$$

Exemple 3.10 (Unit sphere)**.** *Let $c(t)$ be a great circle in S^2, parametrized by arc-length, then*

$$\|c'(t)\|^2 = \langle c', c'\rangle = 1.$$

Differentiate on both sides with respect to t, then it follows that

$$2\langle c'', c'\rangle = 0,$$

i.e. c'' is orthogonal to c'. In particular, since $c' = T$, c'' is a normal vector field of $c(t)$, and therefore

$$\frac{DT}{dt} = \left(\frac{dT}{dt}\right)_{tan} = c''_{tan} = 0.$$

Thus, $c(t)$ is a geodesic curve.

Theorem 3.11 (System of differential equations)**.** *A curve $c(t)$ is geodesic if and only if on any chart $(U, x^1, \ldots, x^n)$, its components $y^i = x^i \circ c$ satisfy the differential equations*

$$y''^k + \sum_{i,j} \Gamma^k_{ij} y'^i y'^j = 0.$$

Proof. The tangent vector is

$$T(t) = c'(t) = \sum_j y'^j \partial_j,$$

and therefore

$$\frac{DT}{dt} = \sum_j \left(y''^j \partial_j + y'^j \nabla_{c'(t)} \partial_j \right).$$

But notice that

$$\nabla_{c'(t)} \partial_j = \nabla_{\sum_i y'^i \partial_i} \partial_j = \sum_i y'^i \nabla_{\partial_i} \partial_j = \sum_{i,k} y'^i \Gamma^k_{ij} \partial_k,$$

thus

$$\frac{DT}{dt} = \sum_j y''^j \partial_j + \sum_{i,j,k} y'^k y'^i \Gamma^k_{ij} \partial_k = \sum_k \left(y''^k + \sum_{i,j} y'^i y'^j \Gamma^k_{ij} \right) \partial_k.$$

Thus,

$$\frac{DT}{dt} = 0 \iff y''^k + \sum_{i,j} y'^i y'^j \Gamma^k_{ij} = 0, \quad \forall k,$$

as desired. □

Using the Cauchy-Lipschitz theorem (also named Picard-Lindelöf theorem), it follows that given any $p \in M$ and any $X_p \in T_pM$, there exists a unique geodesic curve $c(t)$ such that $c(0) = p$ and $c'(0) = X_p$.

Proposition 3.12 (Geodesics under connection-preserving diffeomorphisms)**.** *Let $f : M \to N$ be a connection-preserving diffeomorphism, and $c(t)$ a geodesic curve in M. Then, $f \circ c$ is a geodesic curve in N.*

Proof. The tangent vector field of $f \circ c$ is

$$\bar{T}(t) = (f \circ c)'(t) = f_* c'(t) = f_* T.$$

Thus, by Proposition 2.1.6,

$$\frac{\bar{D}\bar{T}}{dt} = \frac{\bar{D} f_* T}{dt} = f_* \left(\frac{DT}{dt} \right).$$

But $\frac{DT}{dt} = 0$ since $c(t)$ is geodesic, thus it follows that

$$\frac{\bar{D}\bar{T}}{dt} = 0.$$

Therefore, $f \circ c$ is a geodesic curve in N, as claimed. □

4 Lie groups

4.1 Exponential map

We define the exponential map on a Lie algebra $\mathfrak{g}$ of a Lie group G. To do that, we need some results on integral curves and flows. I will be following Tu's *Differential Geometry* book closely.

Definition 4.1 (Integral curve)**.** An integral curve $\phi_t : (a, b) \to G$ of a (left-invariant) vector field X is a curve satisfying

$$\frac{d\phi_t}{dt} = X_{\phi_t}.$$

If the curve starts at $g \in G$, then unicity is guaranteed by the Cauchy-Lipschitz theorem.

Definition 4.2 (Flow)**.** A flow on a manifold M is a map $(t, p) \mapsto \phi_t(p)$ satisfying

$$\phi_0(p) = p, \quad \forall p \in M,$$

and for all $s, t \in \mathbb{R}$,

$$\phi_s \circ \phi_t(p) = \phi_{s+t}(p).$$

Recall that on the Lie group G, l_g denotes left multiplication by g.

Proposition 4.3 (Left translates)**.** *Let ϕ_t be an integral curve on G, $g \in G$, and X a left-invariant field. Then $g\phi_t(p)$ is the integral curve starting at gp.*

Proof. We differentiate $g\phi_t(p)$:

$$\frac{d(g\phi_t(p))}{dt} = (l_g)_* \phi_t'(p) = X_{g\phi_t(p)},$$

since ϕ_t is an integral curve, and X is left-invariant. At $t = 0$, $g\phi_0(p) = gp$. Using unicity of integral curves completes the proof. □

Corollary 4.4 (Commutativity)**.** *Consider a flow ϕ_t on G. Then, for any left-invariant vector field X on G and any $g \in G$,*

$$l_g \circ \phi_t = \phi_t \circ l_g.$$

Proof. Let $p \in G$, then obviously $g\phi_t(p) = l_g \circ \phi_t(p)$, and $\phi_t(gp) = \phi_t \circ l_g(p)$. Since p was arbitrary, the result holds on all G. □

Suppose $X_e \in \mathfrak{g}$ generates the left-invariant vector fields X. Let $\phi_t(e)$ be the integral curve of X starting at e, and denote it by $c_X(t)$. Now, let $s \in \mathbb{R}$. Then

$$\frac{dc_X(st)}{dt} = sc_X'(st) = sX_{c_X(st)}$$

by the integral curve property of c_X. This proves that $c_X(st)$ is an integral curve of sX, and therefore, by unicity, it follows that

$$c_X(st) = c_{sX}(t).$$

Definition 4.5 (Exponential map)**.** The exponential map of a Lie group G with Lie algebra $\mathfrak{g}$ is the map

$$\exp : \mathfrak{g} \to G$$

defined by

$$e^{X_e} = c_X(1).$$

Proposition 4.6. *$e^{tX_e} = c_X(t)$.*

Proof. $e^{tX_e} = c_{tX}(1) = c_X(t)$ by the discussion above. $\square$

Proposition 4.7. *For $g \in G$, the unique integral curve of X starting at g is ge^{tX_e}.*

Proof. The integral curve is

$$\phi_t(g) = \phi_t(ge) = g\phi_t(e) \quad \text{by Proposition 4.3},$$

and $g\phi_t(e) = gc_X(t) = ge^{tX_e}$ by Proposition 4.6. $\square$

Proposition 4.8 (Smoothness)**.** *The exponential map is smooth.*

Proof. A result from ODE theory states that the flow of a smooth vector field is itself smooth. On $G \times \mathfrak{g}$, define the vector field

$$V_{(g,X_e)} = (X_g, 0) = (l_{g*}X_e, 0),$$

then its integral curve starting at (g, X_e) is (ge^{tX_e}, X_e). Thus, the flow of V is the map

$$\phi : \mathbb{R} \times (G \times \mathfrak{g}) \to G \times \mathfrak{g}, \quad (t, (g, X_e)) \mapsto (ge^{tX_e}, X_e).$$

Let $\pi : G \times \mathfrak{g} \to G$ be the projection. Then

$$e^{X_e} = \pi \circ \phi(1, (e, X_e)).$$

Hence, e is a composition of smooth maps, and therefore smooth. $\square$

Proposition 4.9 (Identity)**.** *The pushforward at 0, $e_{*,0} : T_0\mathfrak{g} = \mathfrak{g} \to \mathfrak{g}$, is the identity.*

Proof. On $\mathfrak{g}$, define the smooth curve $c(t) = tX_e$. Then

$$e_{*,0}X_e = \frac{d}{dt}\Big|_{t=0} e \circ c(t) = \frac{d}{dt}\Big|_{t=0} e^{tX_e} = X_e.$$

$\square$

Using Propositions 4.8 and 4.9, the exponential map is a smooth map whose pushforward is an automorphism of $\mathfrak{g}$. By the inverse function theorem, the exponential map is a local diffeomorphism. In particular, there exists an open neighborhood U of $0 \in \mathfrak{g}$ such that e^U is an open set of G. This fact is used to prove the following theorem.

Theorem 4.10 (Generators of connected Lie groups)**.** *Every connected Lie group is generated by $e^{\mathfrak{g}}$, that is, for all $g \in G$, $g = \prod_{k=1}^n e^{A_k}$ for $A_k \in \mathfrak{g}$.*

Proof. Let H be the subgroup of G generated by $e^{\mathfrak{g}}$, and assume for contradiction that $G \setminus H \neq \emptyset$. Since H is a subgroup, it is closed. Moreover, define U to be an open neighborhood of 0 as above. Then $V = e^U \subset e^{\mathfrak{g}} \subset H$ is an open neighborhood of e contained in H. This implies that H is open, since given any $h \in H$, the subgroup property guarantees that $hV \subset H$, and therefore

$$H = \bigcup_{h \in H} hV.$$

This yields a contradiction, since H is a proper, nontrivial clopen subset of G, which was assumed to be connected. $\square$

Recall that if $f : G \to H$ is a group homomorphism, then $f \circ l_g = l_{f(g)} \circ f$ for all $g \in G$. In the next theorem, suppose that G, H are Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ respectively.

Theorem 4.11 (Naturality of the exponential)**.** *If $f : G \to H$ is a Lie group homomorphism, then the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{f_*} & \mathfrak{h} \\ \downarrow e & & \downarrow e \\ G & \xrightarrow{f} & H \end{array}$$

Proof. Suppose $X_e \in \mathfrak{g}$ generates the left-invariant vector field X, and let c be the integral curve of X starting at e. Then

$$c'(t) = X_{c(t)} = (l_{c(t)})_* X_e.$$

By differentiating the homomorphism formula $f_* \circ l_{g*} = l_{f(g)*} \circ f_*$, we have

$$(f \circ c)'(t) = f_* \circ l_{c(t)*} X_e = l_{f \circ c(t)*} \circ f_* X_e,$$

therefore $f \circ c$ is the integral curve of the left-invariant vector field $f_* X$. Then

$$e^{f_* X_e} = c_{f_* X}(1) = f \circ c(1),$$

and by unicity of integral curves, $f \circ c(1) = f(e^{X_e})$, proving commutativity. $\square$