

Tutorial 1

Q20

Multiplying $(I+N)$ and the equation given simplifies to

$$(I + N)(I - N + N^2 \dots (-1)^{n-1} N^{n-1}) = I - (-1)^n N^n$$

How? Hint: Think of $(1+x)(1-x+x^2\dots)$. What if n is odd? What if even?

For $n=3$, $(1+x)(1-x+x^2) = (1+x^3)$. For $n=4$, $(1+x)(1-x+x^2-x^3) = 1-x^4$. Do you see it?

Okay, so how do you prove it formally? Again, **Prove By Induction!!**. If you did not understand PBI, Check here [\[Help\]](#). Now attempt proving the claim for this problem.

Another thing to notice is that $(I+N)$ and the large equation infact commute, because I and N actually commute. Now that we have the product, to prove that this product is equal to I , we need to show $N^n = 0$, is it actually? Try checking it manually for matrices of such type for $n = 2, 3$.

Let $N = [n_{ij}]$, where $n_{ij} = 0$ for all $i \geq j$ and for remaining positions, not all $n_{ij} = 0$. Now consider the product $A = N^2 = NN$. Let $A = [a_{ij}]$. Now,

$$a_{ij} = \sum_{k=1}^k n_{ik} n_{kj}$$

Case 1: $i \geq j$

$n_{ik} = 0$ when $k \leq i$ and $n_{kj} = 0$ when $k \geq j$. Hence the product $n_{ik} n_{kj}$ is 0 for all values of k , as for any value of k , atleast one among n_{ik}, n_{kj} is 0.

Case 2: $i < j$

Same as previous, we see $n_{ik} n_{kj} = 0$ for $k \leq i$ and $k \geq j$. But since $i < j$. What about the values between i, j . For the values between i and j , the product may be non-zero. Hence we conclude that a_{ij} is non zero if $i < j$ and there exists any number between i and j .

If $i=1, j=2$, $a_{ij} = 0$, since there is no number between 1 and 2. But for $i=1, j=3$ $a_{13} = n_{12} n_{23}$ which need not be zero. Hence in N^2 , we find $n-1$ additional zeros that N . Additional zeros are at positions $a_{12}, a_{23}, \dots, a_{(n-1)n}$.

Similarly, when multiply N^2 by another N , we would notice $n-2$ additional zeros, and this continues till all positions conquered by zeros in N^n .

For the second part of the question, this N is a different N . Given a nilpotent matrix N , lets say with index of nilpotency p . That means $p \geq 1$ and $N^k = 0$ for all $k \geq p$. The following holds. Don't be surprised, we already saw this above!

$$(I + N)(I - N + N^2 \dots (-1)^{p-1} N^{p-1}) = I - (-1)^p N^p$$

Now replace N with $-N$ in above equation, and also put $N^p = 0$. We observe that $(I-N)$ indeed has an inverse, by definition!

PS: Revisit this problem, when you learn of eigenvalues. For any upper triangular matrix, the eigenvalues are the diagonal entries. Why?. And for any $n \times n$ matrix A , its characteristic equation is given by $(A - \lambda_1)(A - \lambda_2) \dots (A - \lambda_n) = 0$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A . (Recall **Cayley-Hamilton Theorem**) For this problem, N has got all diagonal entries zero, hence its characteristic equation simplifies to $N^n = 0$. **Kaboom!!!**