(3) A **hyperplane** in $\mathbb{R}^n$ is defined to be the set $u + W$ where $u \in \mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$ having dimension $n-1$. Prove that a hyperplane in $\mathbb{R}^n$ is the set of solutions of a single linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where $a_1, \ldots, a_n, b \in \mathbb{R}$.

Problem: Given an affine subspace (hyperplane) $u+W$, find a linear equation, solution space of which is the hyperplane given.

(1) Consider the problem: Find a linear eqn i.e., $a_1, a_2 \cdots a_n$ s.t. Soln space of $a_1x_1 + a_2x_2 + \cdots a_nx_n = 0$ is W.

$a^Tx = 0 \quad a = (a_1, a_2, a_3 \cdots, a_n)^T$

This means that all vectors in W satisfy the eqn. So it is enough if all vectors of any basis set satisfies the eqn. Yes?

dim $= n-1$

Consider a basis $B = \{w_1, w_2, \cdots w_{n-1}\}$ of W.

So, $a^Tw_1 = a^Tw_2 = a^Tw_3 \cdots = a^Tw_n = 0$

$$a^T[w_1 \quad w_2 \cdots\cdots w_{n-1}] = 0$$

$$A = \underbrace{[w_1 \quad w_2 \cdots \quad w_{n-1}]}_{(n \times n-1)}$$

$a^TA = 0$

$\Rightarrow A^Ta = 0$

A^T is $(n-1 \times n)$ matrix made up of $(n-1)$ linearly independent rows.

$\therefore$ $\text{Rank}(A^T) = n-1$

By Rank-Nullity Theorem: $\text{Nullity}(A^T) = 1$

$\therefore$ There exists a non-zero value v s.t $A^T v = 0$

Choose that v as our a in $\underline{a^T x = 0}$

(2) Now we consider the translation vector u (in $u + W$)

Calculate $a^T u$ (call it b)

and now the eqn you need is finally

$$\underline{a^T x = b = a^T u}$$

(5) Let $P_n[x]$ denote the vector space consisting of the zero polynomial and all real polynomials of degree $\leq n$, where n is fixed. Let S be a subset of all polynomials $p(x)$ in $P_n[x]$ satisfying the following conditions. Check whether S is a subspace; if so, find the dimension of S. (i) $p(0) = 0$; (ii) p is an odd function; (iii) $p(0) = p''(0) = 0$.

(i) $S = \{ p(x) \in P_n[x] \mid p(0) = 0 \}$

a) $p(x) = 0$ identically $\in S$ since it satisfies $p(0) = 0$.

$\therefore 0 \in S$

b) Consider $P_1(x), P_2(x) \in S \Rightarrow P_1(0) = P_2(0) = 0$

Let $p(x) = (\alpha_1 P_1 + \alpha_2 P_2)(x) = \alpha_1 P_1(x) + \alpha_2 P_2(x)$

$p(0) = \alpha_1 p_1(0) + \alpha_2 p_2(0) = \alpha_1 p_1(0) + \alpha_2 p_2(0) = 0$

$\therefore p(x) \in S$

$\therefore$ S is a subspace

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$

$p(0) = 0 \Rightarrow a_0 = 0$

$p(x) \in S \Rightarrow p(x) = a_1 x + a_2 x^2 + \cdots a_n x^n$

$\therefore$ Basis $= \{x, x^2, \cdots x^n\}$

Dim $= |\text{Basis}| = n$

(ii) also a subspace

Dim = largest odd no. $\leq n$

(iii) also a subspace

Dim $= n-1 \quad (n \geq 2)$

Try out yourself