MA106 Tutorial 1

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Q20

Method 1

Multiplying (I+N) and the equation given simplifies to

$$(I+N)(I-N+N^2\dots(-1)^{n-1}N^{n-1}) = I-(-1)^nN^n)$$

How? **Hint**: Think of $(1+x)(1-x+x^2...)$. What if n is odd? What if even?
For n=3, $(1+x)(1-x+x^2) = (1+x^3)$. For n=4, $(1+x)(1-x+x^2-x^3) = 1-x^4$. Do you see it?

Okay, so how do you prove it formally? Again, **Prove By Induction!!**. If you did not understand PBI, Check here [Help]. Now attempt proving the claim for this problem.

Another thing to notice is that (I+N) and the large equation infact commute, because I and N actually commute. Now that we have the product, to prove that this product is equal to I, we need to show $N^n = 0$, is it actually? Try checking it manually for matrices of such type for n = 2,3.

Let $N = [n_{ij}]$. where $n_{ij} = 0$ for all $i \geq j$ and for remaining positions, not all $n_{ij} = 0$. Now consider the product $A = N^2 = NN$. Let $A = [a_{ij}]$. Now,

$$a_{ij} = \sum_{n=1}^{k} n_{ik}n_{kj}$$

Case 1: $i \geq j$
$n_{ik} = 0$ when $k \leq i$ and $n_{kj} = 0$ when $k \geq j$. Hence the product $n_{ik}n_{kj}$ is 0 for all values of k, as for any value of k, atleast one among n_{ik}, n_{jk} is 0.
Case 2: $i < j$
Same as previous, we see $n_{ik}n_{kj} = 0$ for $k \leq i$ and $k \geq j$. But since $i < j$. What about the values between i, j. For the values between i and j, the product may be non-zero. Hence we conclude that a_{ij} is non zero if $i < j$ and there exists any number between i and j.
If i=1, j=2 , $a_{ij} = 0$, since there is no number between 1 and 2. But for i=1, j=3 $a_{13} = n_{12}n_{23}$ which need not be zero. Hence in N^2, we find n-1 additional zeros that N. Additional zeros are at positions $a_{12}, a_{23}, \dots, a_{(n-1)n}$.

Similarly, when multiply N^2 by another N, we would notice $n-2$ additional zeros, and this continues till all positions conquered by zeros in N^n.

For the second part of the question, this N is a different N. Given a nilpotent matrix N, lets say with index of nilpotency **p**. That means $p \geq 1$ and $N^k = 0$ for all $k \geq p$. The following holds. Don't be surprised, we already saw this above!

$$(I+N)(I-N+N^2\dots(-1)^{p-1}N^{p-1}) = I-(-1)^pN^p)$$

Now replace N with -N in above equation, and also put $N^p = 0$. We observe that (I-N) indeed has an inverse, by definition!

Method 2(Using Induction)

Let e_j be the unit vector along j^{th} axis in $\mathbb{R}^n$ space. Ne_j is the j^{th} column of N, which can be rewritten as $Ne_j = \sum_{i=1}^{j-1} a_{ij} e_i$. Verify yourself!

Proof By Induction:
Proposition: $N^k e_j = 0$ for $j \leq k$, $1 \leq k \leq n$.

Base Case: k=1, $Ne_1 = C_1$(first column of matrix N), which was all 0, therefore $Ne_1 = 0$. Hence, base case satisfies the Proposition.

Induction Step:
Induction Hypothesis: Assume proposition holds for $k \leq k_0$. Now $A^{k+1}e_i = AA^k e_i = 0$ for $i \leq k$. For $k = k_0 + 1$, $A^{k_0+1}e_{k_0+1} = A^{k_0}(Ae_{k_0+1}) = A^{k_0} \sum_{i=1}^{k_0} a_{i,k_0+1} e_i = \sum_{i=1}^{k_0} a_{i,k_0+1} A^{k_0} e_i = 0$. Therefore, proposition holds for $k = k_0 + 1$.

Hence Proved :)

PS: Revisit this problem, when you learn of eigenvalues. For any upper triangular matrix, the eigenvalues are the diagonal entries. Why?. And for any $n \times n$ matrix A, its characteristic equation is given by $(A - \lambda_1)(A - \lambda_2) \ldots (A - \lambda_n) = 0$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A.(Recall **Cayley-Hamilton Theorem**) For this problem, N has got all diagonal entries zero, hence its characteristic equation simplifies to $N^n = 0$. **Kaboom**!!!