(3) Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors satisfying $4x_1 - 3x_2 + x_3 = 0$ is – (i) the null-space of T. (ii) the range of T.

Nimay pointed this out correctl in class,

(i) Given the null space, find the vectors not in null space but choose them orthogonally. Why?

But still, for the sake of mathematical rigour, you are reqd. to do what I was trying to explain.

$4x_1 - 3x_2 + x_3 = 0$: plane in $\mathbb{R}^3$: dim = 2

$B_V = \{(3,4,0), (0,1,3)\}$

Extend this basis to $\mathbb{R}^3$, consider the normal to the plane $(4,-3,1)$

$B_{\mathbb{R}^3} = \{(3,4,0), (0,1,3), (4,-3,1)\}$

Now if I know values that Av takes for $v \in B_{\mathbb{R}^3}$ then I can uniquely determine A

say $\left.\begin{array}{l} Av_1 = w_1 \\ Av_2 = w_2 \\ Av_3 = w_3 \end{array}\right\} \Rightarrow [Av_1 \; Av_2 \; Av_3] = [w_1 \; w_2 \; w_3]$

$= A[v_1 \; v_2 \; v_3]$

$$\boxed{A = [v_1 \; v_2 \; v_3]^{-1}[w_1 \; w_2 \; w_3]}$$

Since $\begin{pmatrix} 3 \\ \cdot \\ \cdot \end{pmatrix}, \begin{pmatrix} 0 \\ \cdot \\ \cdot \end{pmatrix}$ is null space

Since $\begin{pmatrix}3\\4\\0\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}$ in null space

$$A\begin{pmatrix}3\\4\\0\end{pmatrix} = A\begin{pmatrix}0\\1\\3\end{pmatrix} = 0$$

give $A\begin{pmatrix}4\\-3\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \neq 0$

$$A\begin{bmatrix}3 & 0 & 4\\4 & 1 & -3\\0 & 3 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$A = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}5/39 & 2/13 & -4/39\\-4/39 & 1/26 & 25/39\\2/13 & -3/26 & 1/26\end{bmatrix}$$

$$A = \begin{bmatrix}2/13 & -3/26 & 1/26\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

which is actually what Nimay said.

(ii) If given the range space, simply find a basis vector set of the given space and use them as the columns of your transformation matrix.

Consider $B_{\mathbb{R}^3} = (e_1, e_2, e_3)$

Choose $\quad Ae_1 = \begin{pmatrix}3\\4\end{pmatrix}$ $\quad$ Choose Ae_1, Ae_2, Ae_3 but choose ... $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$

careful, choose them s.t they are l.I or else 0. Why?

$$Ae_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

$$Ae_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Again exactly what was pointed out.

(4) Let $\mathcal{P}[x]$ denote the space of all real polynomials in one variable. Let

$$V = \{p(x) \in \mathcal{P}[x] \ : p(0) = 0\}.$$

Prove that taking the derivative defines a one-to-one linear transformation from $D : V \longrightarrow \mathcal{P}$ and $D^{-1}(p)(x) = \int_0^x p(t)\,dt$.

$D : V \to P$

$p(x) \in V$ $\quad D(p(x)) = 0 \Rightarrow p(x) = \text{constant } \forall x \in \mathbb{R}$

$p(x) = p(0) = 0 \Rightarrow N(D) = \{0\}$

$\therefore$ injective

Fundamental Thm of Calculus: MA 109/11 में पढ़ा

Fundamental Thm of Calculus ... था!

Let f be a continuous real valued function on a closed interval $[a,b]$. Let F be the f^n defined $\forall x \in [a,b]$ by

$$F(x) = \int_a^x f(t)\,dt$$

Then $F'(x) = f(x)$ $\forall x \in (a,b)$

1 $D^{-1}(p)(x) = \int_0^x p(t)\,dt\,.$

consider $F(x) = \int_0^x p(t)\,dt$

$\Rightarrow F'(x) = p(x)$

$D(F(x)) = p(x)$

$F(x) = D^{-1}(p(x)) = \int_0^x p(t)\,dt$