Advanced Monetary Economics

Tutorial 5

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Lecture

- start building theoretical model for the interest-rate channel

- answer the question: what is the effect of monetary policy on the economy?

- assumptions:

  1. one representative household

  2. one representative firm

  3. closed economy, no investment, no government consumption

  4. flexible prices

1 Transaction frictions

Suppose there are monetary frictions, which are responsible for households to demand a positive amount of money, even when money is dominated in rate of return by an alternative store of value. The single period utility function $u(c_t, l_t)$ is increasing, concave, twice continuously differentiable and satisfies the Inada conditions. Households are endowed with holdings of money and nominal one-period bonds; the latter earning the nominal interest rate i_{t-1}. The period by period budget constraint of a representative household is given by

$$P_t c_t + M_t + B_t \leq P_t w_t n_t + M_{t-1} + (1 + i_{t-1}) B_{t-1}.$$

The main novel assumption in this exercise refers to the transactions in the goods market. Assume that purchases of consumption goods are costly, as they require transaction services. These services are produced with money holdings and with shopping time s_t, i.e., time spend in the goods market.

1. How does this change the time constraint of the household?

2. Let us introduce a shopping time function, which specifies the time required for shopping as a function of consumption purchases and of money holdings. Intuitively, the shopping time increases with consumption and decreases with (the real value of) money holdings:

$$s_t = H\left(c_t, M_t/P_t\right),$$

where $H_c > 0, H_{cc} > 0, H_m \leq 0, H_{mm} \geq 0$ and $H_{cm} \leq 0$. Provide an interpretation of these functional properties.

3. There is a non-negativity restriction on money, which implies that households can only borrow from the government at the rate $1 + i_t$. Derive the households first order conditions. When does the time constraint bind?

4. Derive the *arbitrage freeness condition* between money and bonds by combining the first order conditions for bonds and money. Can you also derive the money demand of households from this equation as a function of consumption, leisure, and the interest rate?

Solution

1. The time constraint of the household is

$$l_t + n_t + s_t \leq 1.$$

2. The functional properties imply that the (time) costs associated with transactions rises with consumption expenditures, $H_c > 0$, and declines with real balances, $H_m \leq 0$, in a convex way. The last property $H_{cm} \leq 0$ implies that the marginal costs associated with higher consumption expenditures do not rise with higher money holdings.

3. The Lagrangean is

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \{ u(c_t, l_t) + \mu_t (1 - l_t - n_t - H(c_t, m_t)) + \psi_t m_t \\ &\quad + \lambda_t (w_t n_t + (1 + i_{t-1}) b_{t-1}/\pi_t + m_{t-1}/\pi_t - c_t - m_t - b_t) \}.\end{aligned}$$

The first order conditions are

$$\begin{aligned}
u_{ct} &= \lambda_t + \mu_t H_c(c_t, m_t) \\
u_{lt} &= \mu_t \\
\lambda_t w_t &= \mu_t \\
\lambda_t + \mu_t H_m(c_t, m_t) &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} + \psi_t \\
\lambda_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} (1 + i_t) .
\end{aligned}$$

The time constraint is binding if $\lambda_t w_t = \mu_t > 0$, that is, if there is a positive wage rate.

4. To derive the *arbitrage freeness condition* between money and bonds use the first order conditions

$$\begin{aligned}
\lambda_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} + \psi_t - \mu_t H_m(c_t, m_t) \\
\lambda_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} (1 + i_t) .
\end{aligned}$$

and equate them to obtain

$$\begin{aligned}\beta\frac{\lambda_{t+1}}{\pi_{t+1}}(1+i_t) &= \beta\frac{\lambda_{t+1}}{\pi_{t+1}}+\psi_t-\mu_t H_m(c_t,m_t)\\ \beta\frac{\lambda_{t+1}}{\pi_{t+1}}i_t &= \psi_t-\mu_t H_m(c_t,m_t).\end{aligned}$$

Rearranging yields

$$\psi_t=\beta\frac{\lambda_{t+1}}{\pi_{t+1}}i_t+\mu_t H_m(c_t,m_t).$$

The money holdings of the household equals zero when the multiplier ψ_t is strictly positive. Conversely, if $\psi_t=0$, households hold money. Then, the right hand side of the condition has to be equal to zero. The condition can thus be interpreted as an arbitrage freeness condition between bonds and money. If $\psi_t=0$, the arbitrage condition determines the money demand of households

$$\begin{aligned}
0 &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} i_t + \mu_t H_m (c_t, m_t) \\
\mu_t H_m (c_t, m_t) &= -\beta \frac{\lambda_{t+1}}{\pi_{t+1}} i_t \\
H_m (c_t, m_t) &= \frac{1}{\mu_t} \left(-\beta \frac{\lambda_{t+1}}{\pi_{t+1}} i_t \right) \\
&= \frac{1}{u_{lt}} \left(-\beta \frac{\lambda_{t+1}}{\pi_{t+1}} i_t \right) \\
&= \frac{i_t}{u_{lt}} \left(-\beta \frac{\lambda_{t+1}}{\pi_{t+1}} \right) \\
&= \frac{i_t}{u_{lt}} \left(-\frac{\lambda_t}{1 + i_t} \right) \\
&= \frac{i_t}{1 + i_t} \frac{u_{lt} H_{ct} - u_{ct}}{u_{lt}}
\end{aligned}$$

This gives an implicit expression for money demand

$$H_m(c_t, m_t) = \frac{i_t}{1+i_t}\frac{u_{lt}H_{ct} - u_{ct}}{u_{lt}} \rightarrow m_t = m^d(i_t, c_t, l_t).$$

2 Money-in-the-utility

Suppose the representative household has the following utility function

$$u(c_t, l_t, M_t/P_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\phi(M_t/P_t)^{1-\sigma_m}}{1-\sigma_m} + \gamma\frac{l_t^{1+\vartheta}}{1+\vartheta},$$

which is separable in all arguments.

1. Maximize the intertemporal welfare of the household subject to the budget constraint given in the previous exercise and to the time constraint $l_t + n_t \leq 1$.

2. Derive the money demand as a function of the interest rate and current consumption and then derive the interest elasticity of money demand $\frac{\partial m_t}{\partial i_t}\frac{i_t}{m_t}$.

3. Linearize the money demand function and provide an interpretation of the determinants of real money holdings. Which values can you reasonably expect for the interest rate elasticity of money demand?

Solution

1. The Lagrange function is

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty}\beta^t\left\{\frac{c_t^{1-\sigma}}{1-\sigma}+\frac{\phi\left(M_t/P_t\right)^{1-\sigma_m}}{1-\sigma_m}+\gamma\frac{l_t^{1+\vartheta}}{1+\vartheta}\right. \\
&\quad +\mu_t\left(1-l_t-n_t\right) \\
&\quad \left.+\lambda_t\left(w_t n_t+\left(1+i_{t-1}\right)b_{t-1}/\pi_t+m_{t-1}/\pi_t-c_t-m_t-b_t\right)\right\}.
\end{aligned}$$

first order conditions are

$$\begin{aligned}
c_t^{-\sigma} &= \lambda_t \\
\gamma l_t^{\vartheta} &= \mu_t \\
\mu_t &= \lambda_t w_t \\
\phi m_t{}^{\sigma_m}+\beta\lambda_{t+1}\pi_{t+1}^{-1} &= \lambda_t \\
\beta\left(1+i_t\right)\lambda_{t+1}\pi_{t+1}^{-1} &= \lambda_t
\end{aligned}$$

Combining the second and third equation gives

$$\gamma l_t^{\vartheta}=\lambda_t w_t$$

Combining this with the first one and the binding time constraint $1 = l_t + n_t$ yields

$$\begin{aligned}\gamma\left(1-n_t\right)^{\vartheta} &= \lambda_t w_t \\ &= c_t^{-\sigma} w_t \\ \gamma\left(1-n_t\right)^{\vartheta} - c_t^{-\sigma} w_t &= 0\end{aligned}$$

Eliminating λ_t in the conditions for real money and bonds with the fist order condition for consumption and putting things together gives

$$\begin{aligned}\gamma\left(1-n_t\right)^{\vartheta} - w_t c_t^{-\sigma} &= 0 \\ \beta\left(1+i_t\right) c_{t+1}^{-\sigma}\pi_{t+1}^{-1} &= c_t^{-\sigma} \\ \phi m_t^{-\sigma_m} + \beta c_{t+1}^{-\sigma}\pi_{t+1}^{-1} &= c_t^{-\sigma}.\end{aligned}$$

2. Combining the second and third condition yields

$$\begin{aligned}\phi m_t^{-\sigma_m} + \beta c_{t+1}^{-\sigma}\pi_{t+1}^{-1} &= \beta\left(1+i_t\right) c_{t+1}^{-\sigma}\pi_{t+1}^{-1} \\ &= \beta c_{t+1}^{-\sigma}\pi_{t+1}^{-1} + \beta i_t c_{t+1}^{-\sigma}\pi_{t+1}^{-1} \\ \phi m_t^{-\sigma_m} &= \beta i_t c_{t+1}^{-\sigma}\pi_{t+1}^{-1}.\end{aligned}$$

Using the f.o.c for bonds $c_{t+1}^{-\sigma}\pi_{t+1}^{-1} = \frac{c_t^{-\sigma}}{\beta(1+i_t)}$ again gives

$$\begin{aligned}
\phi m_t^{-\sigma_m} &= \beta i_t c_{t+1}^{-\sigma}\pi_{t+1}^{-1} \\
&= \frac{\beta i_t c_t^{-\sigma}}{\beta\left(1+i_t\right)} \\
&= \frac{i_t c_t^{-\sigma}}{\left(1+i_t\right)}
\end{aligned}$$

Solving for m_t yields the following money demand function

$$\begin{aligned}
m_t^{-\sigma_m} &= \phi^{-1}\left(\frac{i_t}{1+i_t}\right)c_t^{-\sigma} \\
&= \phi^{1/\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1/\sigma_m} c_t^{\sigma/\sigma_m}.
\end{aligned}$$

Take the first derivative of this expression w.r.t. i_t

$$\begin{aligned}
\frac{\partial m_t}{\partial i_t} &= -\frac{1}{\sigma_m}\phi^{1/\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1/\sigma_m-1} c_t^{\sigma/\sigma_m}\left[\frac{1}{1+i_t}-\frac{i_t}{(1+i_t)^2}\right] \\
&= -\frac{1}{\sigma_m}\phi^{1/\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1/\sigma_m-1} c_t^{\sigma/\sigma_m}\frac{1}{(1+i_t)^2}.
\end{aligned}$$

Multiply by $\frac{i_t}{m_t} = \frac{i_t}{\phi^{1/\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1/\sigma_m} c_t^{\sigma/\sigma_m}}$ for the elasticity

$$
\begin{aligned}
\frac{\partial m_t}{\partial i_t}\frac{i_t}{m_t} &= \frac{-i_t \frac{1}{\sigma_m}\phi^{1/\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1/\sigma_m - 1} c_t^{\sigma/\sigma_m}\frac{1}{(1+i_t)^2}}{\phi^{1/\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1/\sigma_m} c_t^{\sigma/\sigma_m}} \\
&= -i_t\frac{1}{\sigma_m}\left(\frac{i_t}{1+i_t}\right)^{-1}\frac{1}{(1+i_t)^2} \\
&= -i_t\frac{1}{\sigma_m}\frac{1+i_t}{i_t}\frac{1}{(1+i_t)^2} \\
&= -\frac{1}{\sigma_m}\frac{1}{(1+i_t)}
\end{aligned}
$$

and rearrange terms for the the corresponding interest rate elasticity

$$
\frac{\partial m_t}{\partial i_t}\frac{i_t}{m_t} = -\frac{1}{\sigma_m}\frac{1}{1+i_t}.
$$

3. Taking logs of the money demand function provides an often more convenient

linear form

$$\ln m_t = -\frac{1}{\sigma_m} \ln \frac{i_t}{1+i_t} + \frac{\sigma}{\sigma_m} \ln c_t - \frac{1}{\sigma_m} \ln \phi.$$

The coefficient $-\frac{1}{\sigma_m}$ is often referred to as the interest rate elasticity, while it is actually the elasticity with respect to $\frac{i_t}{1+i_t}$. But given that values of i_t are usually small, such that $\frac{1}{1+i_t} \approx 1$, this is a reasonable approximation. The linear equation can easily be taken to the data. Typical estimates for the interest rate elasticity are between 0.1 and 0.5.

3 Shopping time

When households want to go shopping, they need to find the place where to buy the goods. This time cannot used for leisure. Thus, the period utility function might look like

$$v\left(c_t, s_t\right) = u^c(c_t) - u^s\left(s_t\right)$$

where s_t is the *shopping time*. Assume the functional form for $s_t = \left(\frac{P_t c_t}{M_t}\right)^{\varphi}, \ \ \varphi \in (0,1)$. Furthermore, assume that $u^c(c_t) = \ln(c_t)$ and $u^s\left(s_t\right) = \ln s_t$. How do these assumption motivate the money-in-the-utility function?

Solution

The shopping time increases the more goods $P_t c_t$ are bought and decreases in the transaction costs of nominal money holdings M_t. Using the functional form in v gives

$$v = u^c(c_t) - u^s\left(\left(\frac{P_t c_t}{M_t}\right)^{\varphi}\right).$$

Using that v is logarithmic in both arguments gives

$$\begin{aligned} v &= \ln c_t - \ln\left(\frac{P_t c_t}{M_t}\right)^{\varphi} = \ln c_t - \varphi\left[\ln c_t - \ln\left(M_t/P_t\right)\right] \\ &= (1-\varphi)\ln c_t + \varphi \ln\left(M_t/P_t\right) \end{aligned}$$

and with $u = v/(1-\varphi)$ and $\gamma = \varphi/\left(1-\varphi\right) > 0$ the normalized utility function u is

$$u = \ln c_t + \gamma \ln \left(M_t/P_t\right).$$

This expressions shows that the real value of money delivers utility. As such the approach of "Money-in-the-utility-function" (MIU) can be seen as a short cut for money providing transaction services.

4 Cash-in-advance constraint

An alternative approach for motivating money stresses the role of money as a medium of exchange. In particular, assume that households have to use money to buy goods. Such a restriction is called a *cash-in-advance (CIA) constraint*. Suppose that it has the following form:

$$P_t c_t \leq M_t.$$

According to this definition, households enter the asset market before they trade in the goods market, thus their cash holding equal M_t when they enter the goods market.

1. Look at the CIA constraint. What is the likely impact of money injections by the central bank?

2. Assume that the households' budget constraint is

$$P_t c_t + M_t + B_t \leq P_t w_t n_t + M_{t-1} + (1 + i_{t-1}) B_{t-1} - P_t \tau_t$$

   and that the real wage is given and equal to $w_t = 1$. τ_t are lump-sum taxes. Maximize the representative households' utility $u(c_t, l_t) = u(c_t, 1 - n_t)$ over its lifetime subject to the budget and cash-in-advance constraint. When is the CIA constraint binding? Compare the consumption-leisure optimality condition to the one for an unrestricted goods market. How can the central bank eliminate the distortion in the goods market?

3. The resource constraint is $y_t = c_t$ and the production function is $y_t = n_t$. The central bank issues money and transfers seigniorage $P_t \tau_t^m = M_t - M_{t-1}$ to the government. Derive the consolidated budget constraint of the public sector and provide an interpretation.

4. Define the competitive equilibrium. Suppose the central bank controls the interest rate. How can it achieve that the CIA constraint does not bind? Is this optimal?

5. Why do we need the assumption that $i_t \geq 0$. What would a clever household otherwise do?

Solution

1. Money injections relax the cash constraint and tend to increase households' purchases of consumption goods.

2. The Lagrangean is

$$\begin{aligned}\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t \{u(c_t, 1-n_t) + \varphi_t (m_t - c_t) \\ &\quad + \lambda_t (w_t n_t + (1+i_{t-1}) b_{t-1}/\pi_t + m_{t-1}/\pi_t - c_t - m_t - b_t - \tau_t)\}.\end{aligned}$$

The first order conditions are

$$\begin{aligned} u_{ct} &= \lambda_t + \varphi_t \\ u_{lt} &= \lambda_t w_t \\ \lambda_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} (1+i_t) \\ \lambda_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} + \varphi_t \end{aligned}$$

and the complementary slackness constraints for the budget conditions and for the CIA constraint. The latter are

$$\varphi_t (m_t - c_t) = 0, \quad \varphi_t \geq 0, \quad m_t - c_t \geq 0.$$

Thus, the CIA constraint is binding when $\varphi_t > 0$. To see when this is the case, use the first order conditions

$$
\begin{aligned}
\frac{\lambda_t}{(1+i_t)} &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}} \\
\lambda_t - \varphi_t &= \beta \frac{\lambda_{t+1}}{\pi_{t+1}}
\end{aligned}
$$

equate them

$$
\begin{aligned}
\frac{\lambda_t}{(1+i_t)} &= \lambda_t - \varphi_t \\
\varphi_t &= \lambda_t - \frac{\lambda_t}{(1+i_t)} \\
&= \frac{(1+i_t)\,\lambda_t - \lambda_t}{(1+i_t)} \\
&= \frac{i_t \lambda_t}{(1+i_t)}
\end{aligned}
$$

Use the f.o.c. for leisure

$$u_{lt} = \lambda_t w_t$$

and replace λ_t to derive

$$\varphi_t = \frac{i_t}{(1+i_t)} \frac{u_{lt}}{w_t}.$$

Use this expression in the f.o.c. for consumption

$$\begin{aligned} u_{ct} &= \lambda_t + \varphi_t \\ &= \lambda_t + \frac{i_t}{(1+i_t)} \frac{u_{lt}}{w_t} \\ &= \frac{u_{lt}}{w_t} + \frac{i_t}{(1+i_t)} \frac{u_{lt}}{w_t} \end{aligned}$$

to obtain

$$u_{ct} = \frac{u_{lt}}{w_t} \left(1 + \frac{i_t}{1+i_t} \right).$$

This condition differs from the corresponding condition for the case of an unrestricted goods market, which reads $u_{ct} = \frac{u_{lt}}{w_t}$. What makes this expression different is the appearance of the nominal interest rate, which enters the right hand side of the equation. This feature is due to the property that consumption c_t is a "cash good", whereas leisure l_t is a "credit good". This distortion vanishes if $i_t = 0$.

3. Inserting the resource constraint, the production function, and the budget constraint of the central bank into the household budget constraint yields

$$P_t y_t + P_t \tau_t^m + B_t = P_t w_t y_t + (1 + i_{t-1}) B_{t-1} - P_t \tau_t.$$

Cancelling terms and using that the real wage is $w_t = 1$ gives the consolidated budget constraint of the public sector.

$$P_t \tau_t^m + P_t \tau_t + B_t = (1 + i_{t-1}) B_{t-1}.$$

The government has to pay back outstanding debt including interest. It has two types of taxes available. Lump-sum taxes and seigniorage, and it can issue new bonds.

4. A competitive equilibrium is a set of sequences $\{c_t, n_t, \pi_t, w_t, b_t, m_t, i_t\}$ satisfying the first order conditions of the households and the cash-in-advance constraint

$$\begin{aligned} \varphi_t &> 0 : c_t = m_t \\ \varphi_t &= 0 : c_t \le m_t, \end{aligned}$$

the first order conditions of the firms, $w_t = 1$, and a monetary policy which can either be an interest rate policy (setting i_t) or a money growth rule (setting m_t/m_{t-1}). From the first order conditions $\varphi_t = \frac{i_t}{(1+i_t)}\frac{u_{lt}}{w_t}$, one can see that the multiplier φ_t equals zero if the central bank sets the nominal interest rate equal to its lower bound, namely, zero. In this case, the CIA constraint is not binding. Households are then indifferent between investments in money and bonds, such that money holdings are unrelated to the exact amount of consumption purchases $m_t \geq c_t$. It is not very difficult to guess which kind of monetary policy regime is optimal in this set-up. Since the cash-in-advance constraint is the single distortion in this model, a welfare maximizing policy regime should aim at minimizing this distortion. This can be done by guaranteeing a zero nominal interest rate on bonds. This result is known as the Friedman-rule referring to Friedman (1969).

5. The nominal interest rate on bonds has to satisfy $i_t \geq 0$, i.e., the lower bound on interest rates. This rules out unbounded profits that would be possible when households borrow in bonds and invest in money.