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ADVANCED MICROECONOMICS

Consumer Theory II - Utility Maximization

Winter Term 2025/2026

Outline

Overview

Literature

- Varian (2022): Chapter 4+5 *(for refresher of basic concepts)*
- Autor (2016): Lecture Notes 4

Recap: Preferences

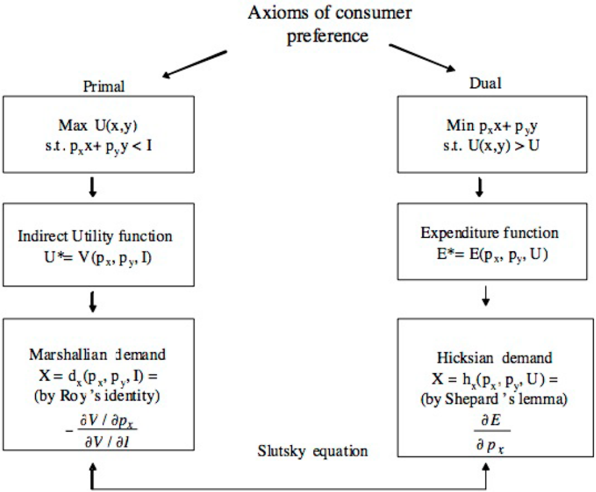
- Preferences are the foundation of decisions taken by consumers.
- Mathematically, preferences can be expressed in form of binary preference relations between goods/alternatives.
- If preferences are rational (i.e., complete and transitive) and continuous, one can always find a utility function that represents the preferences (i.e., keeps the ordering of alternatives).
- Utility functions can be represented as indifference curves.
- Additional properties of preferences (monotonicity and convexity) buy additional (nice) properties of the utility functions and indifference curves.
- For consumer theory usually an ordinal interpretation (ranking of alternatives) of utility functions suffices.

Overview

Optimal Decisions based on Preferences

- **Assumptions**:
  - Consumers have well-defined preferences and we thus have a utility function that represents these preferences.
  - Consumers know which alternatives they can afford (they know their budget, which depends on income and prices).
- Two mathematical approaches to find optimal consumption given budget constraints:
  1. Constrained utility maximization (today)
  2. Expenditure minimization for given utility level (next lecture)
- Both approaches deliver equal results ("duality").

Duality



Source: Autor (2016), Lecture Notes 4, p. 2.

Overview

Overview

Constrained utility maximization

- Three things needed to model a consumer's decision
  1. Utility function: Preferences
  2. Budget constraint: Income (I)
  3. Price vector: Prices of available goods (e.g. p_x price of x and p_y price of y)

- (2) and (3) deliver the **budget constraint**, e.g.

$$I \geq p_x x + p_y y$$

- Consumer problem: Maximize utility for a given budget constraint

The consumer's decision problem

- Typical characteristics of the solution:

Budget exhaustion

Entire budget is spent (more is better) $\rightarrow$ non-satiation

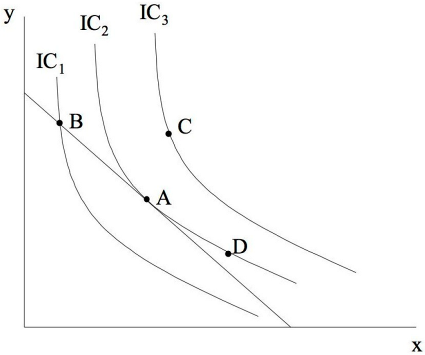
Psychic trade-off = market trade-off (for most solutions)

- Psychic trade-off - Rate at which a person is willing to give up one good for another while maintaining the same level of utility = MRS
- Market trade-off - Rate at which a person can exchange one good for another at the market = price ratio

Overview

Graphical illustration

- Utility maximization reached in **point (A)**

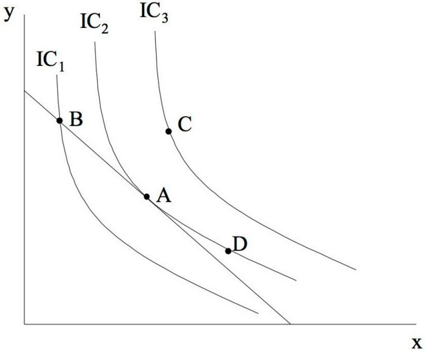


Source: Autor (2016), Lecture Notes 4, p. 3

- Preferences: $A \succ B$, $A \sim D$ $C \succ A$
- Why might we expect someone to choose A?

Graphical illustration

- Utility maximization reached in **point (A)**

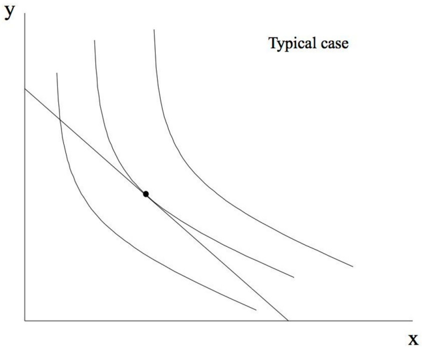


Source: Autor (2016), Lecture Notes 4, p. 3

- Why?
    - Slope of budget set is $-\frac{p_x}{p_y}$
    - Slope of indifference curves is $MRS = \frac{MU_x}{MU_y}$
    - In (A) $\rightarrow$ market trade-off $=$ psychic trade-off (MRS)

Interior solutions

The typical case

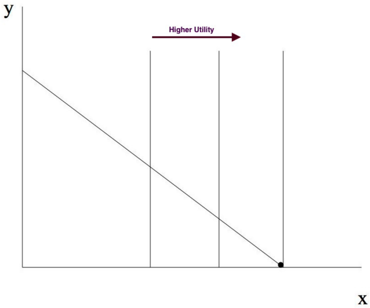


Source: Autor (2016), Lecture Notes 4, p. 4

- The consumer chooses a combination of x and y (because she gains positive utility from the consumption of both goods)

Corner solutions - Example 1

- Vertical indifference curves - Consumer is indifferent to the consumption of one good (here y), utility increases only with consumption of x, consumer purchases x exclusively

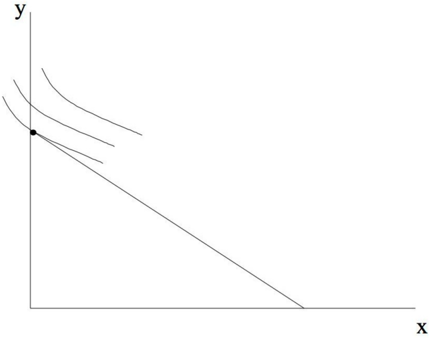


Source: Autor (2016), Lecture Notes 4, p. 5

- Consumers with these preferences would consume negative amounts of y to gain more x if they could

Corner solutions - Example 2

- Very strong preferences for good y (relative to x)
  - For example: Steak for a vegetarian.

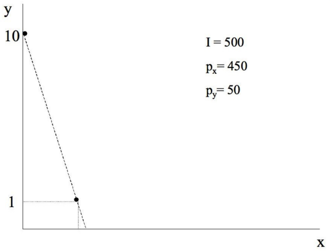


Source: Autor (2016), Lecture Notes 4, p. 6

- Consumers with these preferences would consume negative amounts of x to gain more y if they could.

Corner solution - Example 3

- Indivisibility of x and y (integer constraint) $\rightarrow$ only two possible bundles of goods: (0,10) (=corner solution) or (1,1)
  - Example: Car and parking permits - If I have an own car (x) the city only issues me one permit (y), if I don't have a car, I am allowed to buy 10 (for visitors, rentals).



Source: Autor (2016), Lecture Notes 4, p. 7.

- Normally, we just ignore this problem and assume that goods are divisible

Overview

Mathematical Optimization

Lagrange Approach

An approach for constrained optimization problem by transforming it into an unconstrained problem (using a Lagrange multiplier λ)

Goal:

$$\max_{x,y} U(x,y) \text{ s.t. (such that) } p_x x + p_y y \leq I$$

Lagrange function:

$$\mathcal{L} = U(x,y) + \lambda(I - p_x x - p_y y)$$

Lagrange Approach

- First order conditions (FOCs)
  1. $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial U}{\partial x} - \lambda p_x = MU_x - \lambda p_x = 0$
  2. $\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial U}{\partial y} - \lambda p_y = MU_y - \lambda p_y = 0$
  3. $\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0$
- Rearranging (1) and (2) gives

$$\lambda = \frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y} \qquad (MRS = \text{price ratio})$$

- (3) tells us that entire budget is spent in optimum (budget exhaustion)

How to interpret the Lagrange Multiplier (λ)?

- λ captures the additional utility that a consumer receives from one additional unit of income (which is the same independent of the good it is spent on at the optimal level of consumption)

$$\lambda = \frac{MU_x}{p_x} = \frac{MU_y}{p_y} = \frac{MU_n}{p_n}$$

$\rightarrow$ λ is also called the **shadow price** of the budget constraint

$$\lambda = \frac{dU(x^*, y^*)}{dI}$$

  - where $x = x^*$ and $y = y^*$ are the amounts of x and y consumed at the optimum

- Remember: this increase in utility cannot be interpreted cardinally, just ordinally!

Mathematical derivation of the shadow price

- Proof: $\lambda = \frac{dU(x^*,y^*)}{dI}$.
- Change in utility at optimal level $(x = x^*, y = y^*)$ with change in income:

$$
\begin{aligned}
\left.\frac{d\mathcal{L}}{dI}\right|_{x=x^*,y=y^*} &= \left.\frac{\partial \mathcal{L}}{\partial x}\frac{\partial x}{\partial I}\right|_{x^*,y^*} + \left.\frac{\partial \mathcal{L}}{\partial y}\frac{\partial y}{\partial I}\right|_{x^*,y^*} + \frac{\partial \mathcal{L}}{\partial I} \\
&= (MU_x - \lambda p_x)|_{x^*,y^*}\frac{\partial x^*}{\partial I} + (MU_y - \lambda p_y)|_{x^*,y^*}\frac{\partial y^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial I}
\end{aligned}
$$

- Remember FOCs: $MU_x - \lambda p_x = 0$ and $MU_y - \lambda p_y = 0$

$$
\begin{aligned}
\left.\frac{d\mathcal{L}}{dI}\right|_{x=x^*,y=y^*} &= \frac{\partial \mathcal{L}}{\partial I} \\
&= \lambda
\end{aligned}
$$

Envelope Theorem

- Knowing the Envelope Theorem would have spared us from doing the calculations on the last slide.

Envelope Theorem in Utility Maximization

In the optimum (small) changes in income will not result in changes in the optimal values of x and y, but will only lead to changes in utility by relaxing the budget constraint

- The theorem applies to situations more generally where multiple values are chosen to optimize a function and we see (small) changes in one variable at the optimum.

Envelope Theorem in General

At the optimum in case of small changes only the **direct effect** of the change is relevant, **indirect effects** resulting from re-optimization of other variables can be ignored.

Overview

Non-negativity constraint

- So far we have only looked at interior solutions, where consumers buy non-zero amount of both goods.
- Remember: Corner solutions can exist if consumers have strong preferences for one of the two goods.
- Consumers with these preferences would consume negative amounts of one good to gain even more of the other good if they could but this is not possible

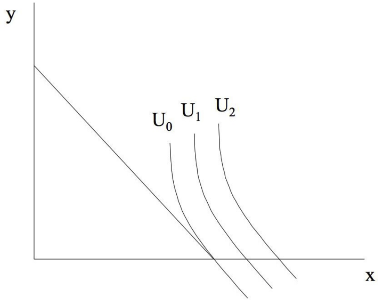
$\rightarrow$ Additional constraints:

Non-negativity constraint

$x \geq 0$ and $y \geq 0$

Corner Solutions

- The non-negativity constraints are irrelevant for interior solutions as they do not bind. They can therefore be ignored
- But what if we had a corner solution (e.g. demand for steak for a vegetarian)?



- Non-negativity constraints would be binding and would have to be taken into account in the optimization
- In most cases, we won't get MRS = price ratio because it would give us a negative consumption level for one good.

Kuhn-Tucker Approach

- Kuhn-Tucker approach allows for incorporating non-negativity constraints
- The optimization approach following Kuhn and Tucker (1951) allows for explicitly incorporating **non-negativity constraints** in an optimization problem.
    - For simplicity, we will assume only the constraint on y is relevant. (It works analogously for x).
- Optimization problem:

$$\max_{x,y\geq 0} U(x,y) \text{ s.t.} p_x x + p_y y \leq I$$

- Lagrange function:

$$\mathcal{L} = U(x,y) + \lambda(I - p_x x - p_y y) + \mu(y - 0)$$

Kuhn-Tucker Conditions for a maximum

First Order Conditions

1. $\frac{\partial \mathcal{L}}{\partial x} = MU_x - \lambda p_x = 0$
2. $\frac{\partial \mathcal{L}}{\partial y} = MU_y - \lambda p_y + \mu = 0$
3. $\mu y = 0$
   - Complementary slackness - Either the constraint is slack ($\mu = 0$) or the constraint is binding ($y = 0$)
4. $\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0$

Possible solutions

Three cases (for equation 3 to hold):

1. $y \neq 0, \mu = 0$
2. $y = 0, \mu = 0$
3. $y = 0, \mu \neq 0$ (as $\mu \geq 0$, here $\mu > 0$)

Kuhn-Tucker Approach

1. Case: $y \neq 0, \mu = 0$

- FOC (2) gives $MU_y - \lambda p_y = 0$
- ... and thus $\frac{MU_y}{p_y} = \lambda$
- With FOC (1), we get

$$\frac{MU_x}{p_x} = \lambda = \frac{MU_y}{p_y}.$$

- or

$$\frac{p_x}{p_y} = \frac{MU_x}{MU_y}.$$

$\rightarrow$ i.e. if the non-negativity constraint is not binding, we get the usual interior solution

Kuhn-Tucker Approach

2. Case: $y = 0, \mu = 0$

- FOC (2) $MU_y - \lambda p_y = 0$
- and thus $\frac{MU_y}{p_y} = \lambda$.
- With FOC 1, we get

$$\frac{MU_x}{p_x} = \lambda = \frac{MU_y}{p_y}.$$

- or

$$\frac{p_x}{p_y} = \frac{MU_x}{MU_y}.$$

→ Since $y = 0$ we have a corner solution, although $\mu = 0$ and thus the non-negativity constraint is slack.

- I.e. in this case, the corner solution is equal to the optimum without non-negativity constraints (market trade-off = psychic trade-off)

Kuhn-Tucker Approach

3. Case: $y = 0, \mu > 0$

- FOC (2) gives $MU_y - \lambda p_y + \mu = 0$
- Since $\mu > 0$, we know $MU_y - \lambda p_y < 0$
- and thus
$$\frac{MU_y}{p_y} < \lambda$$
- With FOC (1) this gives
$$\lambda = \frac{MU_x}{p_x} > \frac{MU_y}{p_y}$$
- Marginal utility of x relative to p_x is higher than marginal utility of y compared to p_y
- Interpretation: At a given price ratio, consumer would like to consume even more x and less y, but can't as the non-negativity constraint is binding.

Overview

Indirect utility function

- The indirect utility function ($V()$) is the value of the maximized utility
- It measures the maximal utility that a consumer can achieve as a function of income and price levels for a given utility function (value function)
- The indirect utility function results from plugging $x^*(p, I)$ and $y^*(p, I)$ into the (direct) utility function
- Distinction
  - (Direct) utility function $U(x, y)$.
  - Indirect utility function $V(p, I) = U(x^*(p, I), y^*(p, I))$.

Indirect utility function

- Why? $V(p, I)$ is useful because it spares us to re-calculate maximal utility when prices or income change.
  - If we have derived the indirect utility function once, we can calculate the maximum utility for any combination of income and prices.
  - Helpful when working with individual demand functions (i.e. quantity of consumption as a function of price and income)

Indirect utility function - Formal

Derivation

- For every combination of
  - Utility function ($U(x_1, x_2, ..., x_n)$)
  - Income (I)
  - Price vector ($p_1, p_2, ..., p_n$)

  constrained maximization results in optimally demanded amounts of the goods

$$x_1^*(p_1, p_2, ..., p_n, I), \; ..., x_n^*(p_1, p_2, ..., p_n, I).$$

- Plugging these into $U()$ results in the indirect utility function

$$V(p_1, ..., p_n, I) = U(x_1^*(p_1, p_2, ..., p_n, I), ..., x_n^*(p_1, p_2, ..., p_n, I)).$$

Overview

Example

- **Example** $U(x,y) = x^{\frac{1}{2}} y^{\frac{1}{2}}$
- This utility function satisfies the preference axioms
  - completeness (defined for all possible values of $x, y > 0$)
  - transitivity
  - continuity
  - non-satiation

$$\begin{aligned}\frac{\partial U}{\partial x} = MU_x &= \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}} > 0 \;\; \forall x > 0 \\ \frac{\partial U}{\partial y} = MU_y &= \frac{1}{2} x^{\frac{1}{2}} y^{-\frac{1}{2}} > 0 \;\; \forall y > 0\end{aligned}$$

: $MU_x = \frac{\partial U}{\partial x} = U_x = \frac{1}{4x} > 0 \;\; \forall x > 0$,
$MU_y = U_y = \frac{3}{4y} > 0 \;\; \forall y > 0$

- ... with decreasing marginal utility $\frac{\partial^2 U}{\partial x^2} = -\frac{1}{4} x^{-\frac{3}{2}} y^{\frac{1}{2}} < 0$ and $\frac{\partial^2 U}{\partial y^2} = -\frac{1}{4} x^{\frac{1}{2}} y^{-\frac{3}{2}} < 0$
- convex (**diminishing MRS**)

Example
Diminishing MRS

- Moving along an indifference curve for this utility function:$\bar{U} = x^{\frac{1}{2}} y^{\frac{1}{2}}$.
- Total derivative: $0 = \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}} dx + \frac{1}{2} x^{\frac{1}{2}} y^{-\frac{1}{2}} dy$
- and thus$-\frac{dy}{dx} = \frac{y}{x} = MRS$
- MRS decreases in x (1st derivative w.r.t. x: $-\frac{y}{x^2} < 0$) and increases in y (1st derivative w.r.t. y: $\frac{1}{x} > 0$).

$\rightarrow$ The more y (and less x) a consumer has, the more y she is willing to give up for one unit of x.

Example

- Further assume $I = 12$ und $p_x = 1, p_y = 2$.
- The **optimization problem** is thus:

$$\max_{x,y>0} x^{\frac{1}{2}} y^{\frac{1}{2}} \quad \text{s.t.} \quad x + 2y \leq 12$$

- Using the Lagrange approach, we get the **objective function**:

$$\mathcal{L} = x^{\frac{1}{2}} y^{\frac{1}{2}} + \lambda(12 - x - 2y)$$

Example

- And the **FOCs**
  1. $\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} - \lambda = 0$
  2. $\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}} - 2\lambda = 0$
  3. $\frac{\partial \mathcal{L}}{\partial \lambda} = 12 - x - 2y = 0$
- **1. and 2.** lead to optimality condition

$$\frac{\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}} = \frac{y}{x} \text{ (MRS)} = \frac{1}{2} \text{ (price ratio)}$$

- And $\lambda = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} = \frac{1}{4}x^{\frac{1}{2}}y^{-\frac{1}{2}}$ (shadow price)
- Using optimality condition and budget constraint, we can calculate $x^* = 6$ and $y^* = 3$

Example of an indirect utility function

- More generally:

$$\frac{y}{x} = \frac{p_x}{p_y} \text{ (optimality condition)}$$

- Plugging these into the budget constraint yields

$$y^* = \frac{I}{2p_y}$$

and

$$x^* = \frac{I}{2p_x}.$$

- Thus, the **indirect utility function** is

$$V(p_x, p_y, I) = \left(\frac{I}{2p_x}\right)^{\frac{1}{2}} \left(\frac{I}{2p_y}\right)^{\frac{1}{2}} = \frac{I}{2\sqrt{p_x p_y}}.$$

Overview

Summary

1. Rational choice is the standard approach to model individual decision making, one way to implement this is constrained utility maximization.
2. For the (often relevant) interior solutions maximum utility is reached at the condition psychic trade-off = market trade-off.
3. Graphically this is the point of tangency between an indifference curve and the budget line, i.e. slope of indifference curve (MRS)=slope of budget line (price ratio).
4. This can be shown mathematically via the Lagrange approach.
5. Often, we just ignore non-negativity constraints on goods. But they can be taken into account by using the Kuhn-Tucker approach to allow for corner solutions
6. Using the solutions to the maximization problem, one can derive the indirect utility function

Main Keywords

- Constrained Utility Maximization
- Budget Constraint
- Budget Exhaustion (Non-satiation)
- Market Trade-off = Psychic Trade-off
- Interior Solutions
- Corner Solutions
- Lagrange Approach
- Lagrange Multiplier (λ) / Shadow Price
- Marginal Utilities (MU_x, MU_y)
- First Order Conditions (FOCs)
- Envelope Theorem
- Non-Negativity Constraint
- Kuhn-Tucker Approach
- Complementary Slackness
- Indirect Utility Function $V(p, I)$

Questions?