Econometrics I
Problem Set 1

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1. **Deriving the univariate OLS estimator**

   (a) Draw $\mathbb{E}[y_i|x] = \beta_1 + \beta_2 x_i$ on a plane and explain what β_1 and β_2 are.

   (b) Explain the difference between population and sample regression.

   (c) Show that the slope of the OLS estimate is

   $$\hat{\beta}_2 = \frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2} \tag{OLS-sums}$$

   (d) Show that the slope of the OLS estimate can also be written as

   $$\hat{\beta}_2 = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\text{cov}(x, y)}{\text{var}(x)}. \tag{OLS-cov}$$

   Use the covariance formula

   $$\text{cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}).$$

   and the fact that $\bar{x} = \frac{1}{n}\sum_i x$.

   (e) What are the meanings of $\text{var}(x)$ and $\text{cov}(x, y)$?

   (f) What is the only requirement for the OLS method to yield a result for $\hat{\beta}_2$?

2. **Deriving the multivariate OLS estimator**

   (a) Consider now the multivariate regression

   $$y_i = \hat{\beta}_1 + \hat{\beta}_2 x_{2,i} + \ldots + \hat{\beta}_k x_{k,i} + \hat{u}_i \qquad \text{for all } i = 1, ..., n$$

   with k different regressors (the first of which is just the constant). Show that it can be expressed in matrix form as

   $$\underset{(n\times 1)}{y} = \underset{(n\times k)}{X}\,\underset{(k\times 1)}{\hat{\beta}} + \underset{(n\times 1)}{\hat{u}} \tag{m-form}$$

(b) Show that the sum of squared residuals can be expressed as

$$\begin{aligned} SSR(\hat{\beta}, y, X) &= \sum_{i=1}^{n} \hat{u}_i^2 = \hat{u}'\hat{u} = (y - X\hat{\beta})'(y - X\hat{\beta}) \\ &= y'y - 2y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \end{aligned} \tag{SSR}$$

(c) Show that the derivative of (SSR) takes the form

$$\begin{aligned} \frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} &= -2\frac{\partial y'X\hat{\beta}}{\partial \hat{\beta}} + \frac{\partial \hat{\beta}'X'X\hat{\beta}}{\partial \hat{\beta}} \\ &= -2X'y + 2X'X\hat{\beta} = \underset{k\times 1}{0} \end{aligned}$$

and can be solved for the OLS estimator

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = (X'X)^{-1}X'y. \tag{$\hat{\beta}$-OLS}$$

Make use of the differentiation rules stated in the appendix.

(d) What condition is required for the estimates vector ($\hat{\beta}$-OLS) to exist?

Matrix math appendix

Basic Properties

- In general, $\mathbf{AB} \neq \mathbf{BA}$.
- Let $\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$, then $\mathbf{A}' = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$

  and $\mathbf{A}'\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 + a_3^2 & a_1b_1 + a_2b_2 + a_3b_3 \\ a_1b_1 + a_2b_2 + a_3b_3 & b_1^2 + b_2^2 + b_3^2 \end{bmatrix}$
- Let vector $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and vector $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then $a' \cdot b = a_1b_1 + a_2b_2 + a_3b_3$.
- Inverses: $\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}_n$, where $\mathbf{A}$ is an $n \times n$ square matrix.
- Distributivity:

$$\begin{aligned} \mathbf{A}(\mathbf{B} + \mathbf{C}) &= \mathbf{AB} + \mathbf{AC} \\ (\mathbf{A} + \mathbf{B})\mathbf{C} &= \mathbf{AC} + \mathbf{BC} \end{aligned}$$

- Associativity: $\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- Exponents: $\mathbf{A}^k = \underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{k,\text{times}}$
- Transposition:

$$\begin{aligned}(\mathbf{A}+\mathbf{B})' &= \mathbf{A}'+\mathbf{B}' \\ (\mathbf{AB})' &= \mathbf{B}'\mathbf{A}'\end{aligned}$$

Some Differentiation Rules

1. When we differentiate a scalar or a (scalar) function y with respect to a vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}'$, we have that:

$$\frac{\partial y}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \cdots & \frac{\partial y}{\partial x_n} \end{bmatrix}'.$$

   That is, the result of the derivative is a vector containing the derivatives with respect to each element of $\mathbf{x}$.

2. Let $a' = [a_1 \ a_2 \ \dots \ a_n]$ and let $z = [z_1 \ z_2 \ \dots \ z_n]'$, then $a'z = \sum_{i=1}^n a_i z_i$ and

$$\frac{\partial(a'z)}{\partial z} = \begin{bmatrix} \frac{\partial a'z}{\partial z_1} \\ \frac{\partial a'z}{\partial z_2} \\ \vdots \\ \frac{\partial a'z}{\partial z_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

3. Let there be a matrix $z'Az$ such that:

$$z'Az = [z_1 z_2 \dots z_n] \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

   where A is a symmetric matrix (that is, $a_{ij} = a_{ji}$). Then,

$$\frac{\partial(z'Az)}{\partial z} = 2Az.$$

Proof:

$$
\begin{aligned}
z'Az &= \begin{bmatrix} z_1 & z_2 & \dots & z_n \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} \sum_i^n z_i a_{i1} & \sum_i^n z_i a_{i2} & \dots & \sum_i^n z_i a_{in} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \sum_j^n \sum_i^n z_i a_{ij} z_j
\end{aligned}
$$

Thus, since $a_{ij} = a_{ji}$, for example $\frac{\partial(z'Az)}{\partial z_1} = 2a_{ii}z_1 + z_2 a_{21} + ... + z_2 a_{12} + ... = 2\sum_i^n a_{1i} z_i$. Therefore,

$$
\frac{\partial(z'Az)}{\partial z} = \begin{bmatrix} \frac{\partial(z'Az)}{\partial z_1} \\ \frac{\partial(z'Az)}{\partial z_2} \\ \vdots \\ \frac{\partial(z'Az)}{\partial z_n} \end{bmatrix} = \begin{bmatrix} 2\sum_i^n a_{1i} z_i \\ 2\sum_i^n a_{2i} z_i \\ \vdots \\ 2\sum_i^n a_{ni} z_i \end{bmatrix} = 2Az
$$