AE 616/236: End-sem. 19/11/2024. Total=40. Duration=3 hrs.

Answers without detailed explanation will be heavily penalized.
Answer Q1, and any 6 of the remaining questions.

1. (a) (1 point) Discuss the concept and utility of the 'diabatic-sonic' state in Rayleigh flow.
   (b) (1 point) Sketch the $h - s$ diagram for a Fanno flow, and discuss its implication.
   (c) (1 point) Discuss the phenomenon of Mach reflection?
   (d) (1 point) What is a free boundary, and how and why does an oblique shock reflect from it?
   (e) (2 points) Briefly describe the operation of a pressure-driven shock tube, along with plots of the typical variation of pressure, temperature and velocity vs. position in the tube, as well as a relevant $x - t$ diagram.
   (f) (1 point) Consider a cone with half angle β. What is its drag coefficient, as predicted by classical Newtonian theory in the context of hypersonic aerodynamics?
   (g) (1 point) Briefly describe the modelling of vibrational mode of excitation in a diatomic gas, and what it implies for temperature changes across a normal shock at high speeds.
   (h) (2 points) Briefly describe the modelling procedure that accounts for equilibrium chemical reactions in case of flow through a normal shock.

Solution:

(a) Must mention that heat addition (not removal) must be in reversible manner. This excludes, in particular, friction at the wall as well as viscous effects in the core. Further, no area change is allowed.

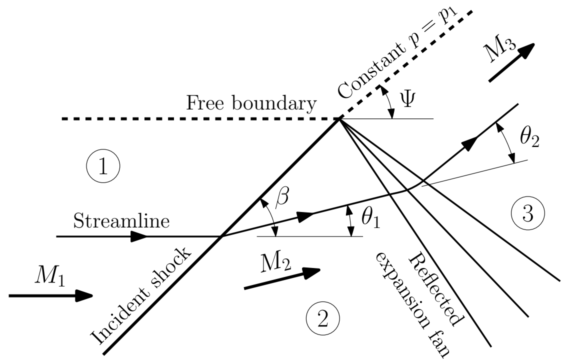
For utility, one should discuss how and why the diabatic-sonic state is consistent throughout the Rayleigh flow as long as there are no shocks (as in supersonic inlet) or readjustment of the inlet flow (as in subsonic inlet). Further, one should discuss how the heat addition required to achieve the sonic state is calculated for the inlet state, the actual heat addition (or removal) is subtracted (or added) therefrom, the consequent heat addition required to achieve the sonic state is calculated for the exit state, and the exit state is determined therefrom.

(b) Should show the proper shape of the diagram. Also, discuss how it implies that – for entropy to increase in the flow per 2nd law of thermodynamics – the flow is driven towards sonic state irrespective of whether the inlet is subsonic or supersonic. Should also discuss how it implies that enthalpy (and hence temperature) decreases (resp. increases) in subsonic (resp. supersonic) Fanno flow.

(c) Discussed in class.

(d) Free boundary is a material demarcation that forms automatically to equalize pressure but allow all other flow properties – velocity, temperature, entropy, etc. – to change across it. For example, this is seen at the exit of an imperfectly-expanded supersonic jet.

An oblique fan reflects as an expansion fan because the pressure is necessarily atmospheric before the oblique shock, higher after it, and hence must come back down to atmospheric through an expansion fan. Preferably draw a sketch to show this, as follows.



Free-boundary reflection of an oblique shock.

(e) A verbal description of the operation of the shock tube is mandatory. Must show proper plots as mentioned.

(f) $C_D = 2\sin^2\beta$.

(g) Must discuss modelling of case where vibrational mode is partially excited, as well as how it affects the equations for a normal shock. Also, must discuss how it reduces the temperature behind a normal shock w.r.t. the case where this mode is not excited.

(h) Need to talk about a specific chemical model – e.g. the 5-species model. Need to mention the relevant chemical reactions, how the composition of the gas is modelled using (a) Dalton's law of partial pressures, (b) principle of mass action to relate the ratio of partial pressures of the chemical species (raised to appropriate indices) to the temperature, and (c) conservation of number of nuclei of all element involved. Also, talk about how this enters into the governing equations for a normal shock, in terms of enthalpy of formation and sensible enthalpy changes, etc.

2. Air flows through a constant area frictionless duct with inlet temperature of 20°C.

(a) (1 point) If the inlet Mach number is 0.5, what is the maximum possible exit stagnation temperature?

(b) (4 points) It is desired to transfer heat such that the stagnation temperature at the duct's exit is 1180 K. For this condition what must be the limiting subsonic inlet Mach number?

Solution:

(a) From the isentropic flow table, the temperature ratio corresponding to $M_1 = 0.5$ is $T_{01}/T_1 = 1.050$. Since the inlet (static) temperature is given as $T_1 = 20°\text{C} = 293$ K, the inlet stagnation temperature is found as $T_{01} = 1.050T_1 = 1.050 \times 293 = 307.7$ K.

From the Rayleigh flow table, the stagnation temperature ratio corresponding to $M_1 = 0.5$ is $T_{01}/T_0^* = 0.6914$. Using the known stagnation temperature at the inlet, the diabatic sonic total temperature is found as $T_0^* = T_{01}/0.6914 = 307.7/0.6914 = 445.0$ K. The maximum value of this ratio is of course unity, corresponding to sonic condition. If heat is to be added to the flow reversibly, then T_0^* must be preserved throughout the flow, and prevail at the exit too. Thus, the maximum possible exit stagnation temperature can be determined as $T_{02,\max} = T_0^* = 445.0 \text{ K} = 172.0°\text{C}$.

(b) In the limiting case, the diabatic sonic total temperature must be $T_0^* = 1180$ K throughout the duct, including at the inlet. If the Mach number is higher than this limiting value, then the flow will become choked. Thus, the limiting subsonic Mach number is an upper limit. It is clear from the previous part that the subsonic inlet Mach number must be less than 0.5, since that Mach number resulted in an exit stagnation temperature that is well below the target.

Since the inlet Mach number is unknown in this case, we cannot directly find the inlet stagnation temperature from our knowledge of the inlet static temperature (and the available columns of the tables). Thus, we have to take an iterative approach. Our target is $T_1/T_0^* = 293/1180 = 0.2483$.

Let us start by guessing $M_1 = 0.4$. Then, from the isentropic flow table, the temperature ratio is $T_{01}/T_1 = 1.032$. Moreover, from the Rayleigh flow table, corresponding to $M_1 = 0.4$, we have $T_{01}/T_0^* = 0.5290$, so that $T_1/T_0^* = 0.5290/1.032 = 0.5126$. Clearly, this is still well above the target of 0.2483, suggesting that we have to lower our guess of M_1.

We tabulate the further steps below. The next step uses an arbitrary guess. However, further steps are determined using the secant approach, limited by the resolution of the table.

$M_{1,des}$	$M_{1,table}$	T_{01}/T_1	T_{01}/T_0^*	T_1/T_0^*	**Error: 0.2483** $-\ T_1/T_0^*$
0.40	0.40	1.032	0.5290	0.5126	−0.2643
0.20	0.20	1.008	0.1736	0.1722	0.1121
0.2447	0.24	1.012	0.2395	0.2367	0.0116
0.2447	0.26	1.014	0.2745	0.2707	−0.0224

To arrive at the final limiting value, we interpolate between the last two results that bracket our solution:

$$M_{1,\text{lim}} = \frac{0 - 0.0116}{-0.0224 - 0.0116}(0.26 - 0.24) + 0.24 = \boxed{0.2468}.$$

Aliter: If one happens to remember the functional form for T_0/T_0^*, then an analytical solution may also be obtained. Indeed, it may be recalled that

$$\frac{T_0}{T_0^*} = \frac{(\gamma+1)M^2\{2 + (\gamma-1)M^2\}}{(1+\gamma M^2)^2}.$$

Further, the isentropic flow relation for the temperature ratio may be readily recalled as $T_0/T = 1 + 0.5(\gamma-1)M^2$. But, the desired value for the ratio T_0^*/T is known, viz. $T_0^*/T = 1180/293 = 4.027$. Then, we have

$$\frac{T_0^*}{T} = \frac{T_0/T}{T_0/T_0^*} = \left(1 + \frac{\gamma-1}{2}M^2\right)\frac{(1+\gamma M^2)^2}{(\gamma+1)M^2\{2+(\gamma-1)M^2\}} = \frac{1 + 2\gamma M^2 + \gamma^2 M^4}{2(\gamma+1)M^2}$$
$$\implies M^4 - 2\underbrace{\left[\frac{\gamma+1}{\gamma^2}\frac{T_0^*}{T} - \frac{1}{\gamma}\right]}_{=:B} M^2 + \underbrace{\frac{1}{\gamma^2}}_{=:C} = 0.$$

This is a quadratic equation in M^2. It will have real solution(s) if the discriminant is non-negative. For this, we have the discriminant as

$$D := B^2 - C = \frac{(\gamma+1)^2}{\gamma^4}\left(\frac{T_0^*}{T}\right)^2 - 2\frac{\gamma+1}{\gamma^3}\frac{T_0^*}{T}.$$

For this to be non-negative, we must have

$$\frac{T_0^*}{T} \geq \frac{2\gamma}{\gamma+1} = 1.167.$$

This is indeed the case for us.

Further, the two possible solutions for M^2 are

$$M^2 = B \pm \sqrt{B^2 - C}.$$

Since $B > 0$ (as $T_0^*/T > \gamma/(\gamma+1)$) and $C = 1/\gamma^2 > 0$, both signs of the radical result in positive solutions for M^2 (and hence, real and positive solutions for M).

We have $B = (\gamma+1)/\gamma^2(T_0^*/T) = (1.4+1)/1.4^2 \times 4.027 = 4.217$. Thus, the two solutions for M^2 are $(8.373, 0.0609)$. Evidently, the first solution yields a supersonic inlet, which is not desirable. So, the desired limiting subsonic inlet Mach number is $M = \sqrt{0.0609} = \boxed{0.2468}$. (This is identical to the answer obtained by the iterative approach.)

3. Air is flowing into an insulated duct with a velocity of 153 m/s. The temperature and pressure at the inlet are 280°C and 28 bar respectively. The exit pressure is 15.7 bar.
   (a) (3½ points) Find the temperature at the exit of the duct.
   (b) (1½ points) If the duct diameter is 15 cm and the Fanning friction factor is 0.005, find its length.

Solution:

(a) The speed of the sound at the inlet is $a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \times 287 \times (280+273)} = 477.3$ m/s. Thus, the inlet Mach number is $M_1 = V_1/a_1 = 153/477.3 = 0.320$.

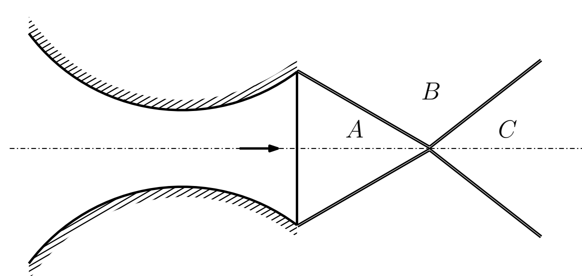
From the Fanno flow table, the relevant quantities at $M_1 = 0.32$ are $T_1/T^\# = 1.176$, $p_1/p^\# = 3.389$ and $4fL_1^\#/D_H = 4.447$. Thus, for this flow, $p^\# = p_1/3.389 = 28/3.389 = 8.262$ bar. But, it is given that the exit pressure is $p_2 = 15.7$ bar. Then, at the exit, the relevant pressure ratio is $p_2/p^\# = 15.7/8.262 = 1.900$. Since this is greater than one, we can be assured that the entire flow is on the same Fanno line.

In particular, given that the inlet is subsonic, we are assured that so is the exit. Thus, we have to look up the subsonic part of the Fanno table for the pressure ratio $p_2/p^\# = 1.900$. The nearest entry is 1.898, corresponding to $M_2 = 0.56$, $T_2/T^\# = 1.129$ and $4fL_2^\#/D_H = 0.6736$. Finally, the requisite exit temperature is

$$T_2 = \frac{T_2/T^\#}{T_1/T^\#} T_1 = \frac{1.129}{1.176}(280+273) = \boxed{528.1 \text{ K} = 255.1 \text{ K}}.$$

(b) The difference in the non-dimensional friction fractions is $4fL/D_H = 0.6736 = 4fL_1^\#/D_H - 4fL_2^\#/D_H = 4.447 - 0.6736 = 3.7734$. Thus, the length of the duct is $L = 3.7734/(4f/D_H) = 3.7734/(4 \times 0.005/0.15) = \boxed{28.3 \text{ m}}$.

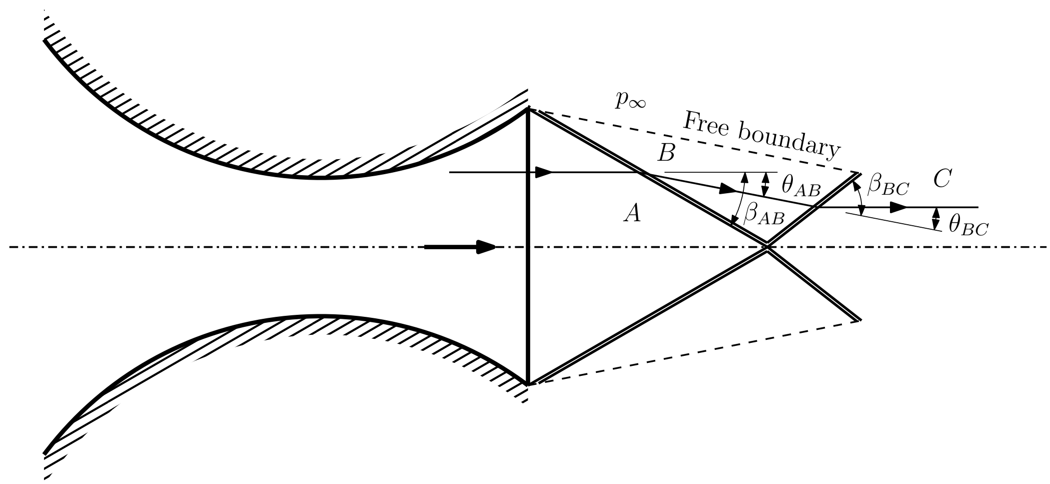
4. (5 points) A supersonic flow leaves a two-dimensional nozzle as a parallel, one-dimensional flow (region A) with a Mach number of 2.6 and static pressure (in region A) of 50 kPa. The pressure of the atmosphere into which the jet discharges is 100 kPa. Find the pressures in regions B and C of the figure. Also describe qualitatively, quantitatively and diagrammatically the actual wave system that separates region B from A and region C from B.



Solution:

Since the exit pressure $p_e = p_A = 50$ kPa is less than the back pressure $p_b = p_\infty = 100$ kPa, the C-D nozzle must be operating in the overexpanded state. Thus, an oblique shock must be formed at the exit – this is what separates region B from A. Further, the centreplane of the flow acts as a plane solid surface such that the primary oblique shock reflects off of it as another oblique shock, that now separates region C from B.

A free boundary will be formed between the jet plume and the ambient, as shown in the sketch below. This will require that the pressure in region B be equal to the ambient pressure (across the free boundary). Thus, $\boxed{p_B = p_\infty = 100 \text{ kPa}}$.



The pressure ratio across the first oblique shock (the one between regions A and B) is $p_2/p_1 = p_B/p_A = 100/50 = 2.0$. From the normal shock tables, the nearest entry for static pressure ratio is 1.991 corresponding to $M_{n,A\to B} = 1.36$ and $M_{n,B\to A} = 0.7572$. Thus, the shock angle $\beta_{AB} = \sin^{-1}(M_{n,A\to B}/M_A) = \sin^{-1}(1.36/2.6) = 31.5°$. From the $\beta - \theta - M$ chart, the corresponding flow turning angle is $\theta_{AB} = 10.8°$. Then, the Mach number in region B is $M_B = M_{n,B\to A}/\sin(\beta_{AB} - \theta_{AB}) = 0.7572/\sin(31.5° - 10.8°) = 2.14$.

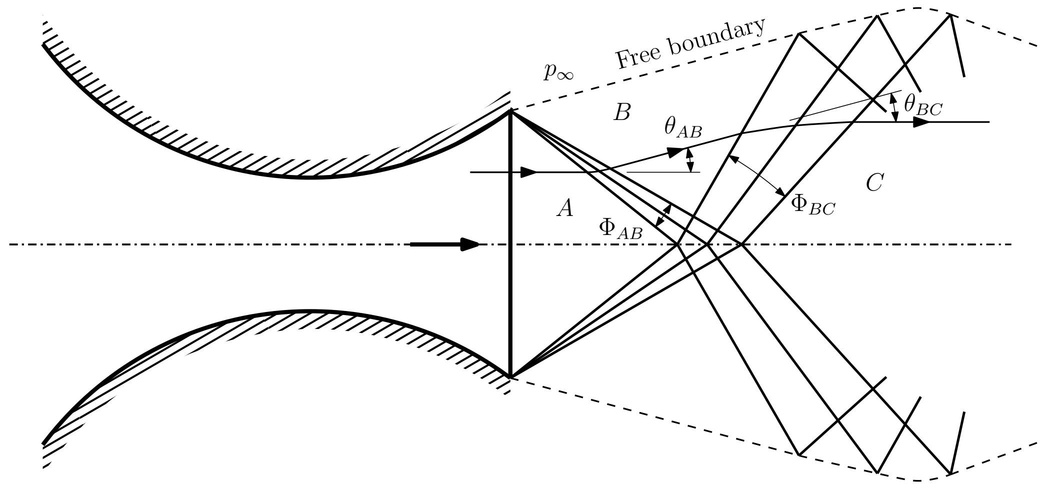
Next we come to the reflected (or refracted) shock. Due to the symmetry of the shock cell structure, the flow in region C must be parallel to the centerline (and the flow direction in region A). That is, $\theta_{BC} = \theta_{AB} = 10.8°$. Then, from the $\beta - \theta - M$, corresponding to $M = 2.14$ and $\theta = 10.8°$ we have $\beta_{BC} = 37.5°$. Thus, the pre-shock shock-normal Mach number is $M_{n,B\to C} = M_B \sin\beta_{BC} = 2.14 \sin 37.5° = 1.30$. From the normal shock table, the corresponding static pressure ratio is $p_C/p_B = 1.805$. Then, we have the requisite pressure in region C as $p_C = 1.805 p_B = 1.805 \times 100 = \boxed{180.5 \text{ bar}}$.

5. (5 points) A reservoir containing air at 2 MPa is connected to ambient air at 100 kPa through a two-dimensional converging-diverging nozzle designed to produce one-dimensional flow at Mach 2.0 (see figure above; same as that for previous question). Find the pressures in regions A, B and C of the figure. Also describe qualitatively, quantitatively and diagrammatically the actual wave system that separates region B from A and region C from B.

Solution:

The pressure at the exit can be found from isentropic flow relations. In particular, from the isentropic flow table, corresponding to $M = 2.0$ we have $p_0/p = 7.824$. Then, with the reservoir pressure given to be $p_0 = 2$ MPa $= 2000$ kPa, the exit pressure is $p_e = p_0/7.824 = 2000/7.824 = 255.6$ kPa. This is the pressure in region A, so that $\boxed{p_A = 255.6 \text{ kPa}}$.

Since the exit pressure $p_e = p_A = 255.6$ kPa is greater than the back pressure $p_b = p_\infty = 100$ kPa, the C-D nozzle must be operating in the underexpanded state. Thus, an expansion fan must be formed at the exit – this is what separates region B from A. Further, the centreplane of the flow acts as a plane solid surface such that the primary expansion fan reflects off of it as another expansion fan, that now separates region C from B. This flow is sketched in the figure below.



Since the expansion fan is an isentropic wave, the stagnation pressure in region B must be same as in region A, which in turn is identical to the reservoir pressure of $p_0 = 2000$ kPa. But the static pressure in region B must equalize with the ambient pressure (or back-pressure) $p_B = p_\infty = \boxed{100 \text{ kPa}}$. Thus, in region B, the pressure ratio is $p_{0B}/p_B = p_0/p_\infty = 2000/100 = 20.0$. From the isentropic flow table, the nearest entry for the pressure ratio is $p_0/p = 19.95$ corresponding to a Mach number of $M_B = 2.60$.

From the P-M table, the P-M function values corresponding to $M_A = 2.0$ and $M_B = 2.60$ are $\nu_A = 26.38°$ and $\nu_B = 41.41°$. Thus, the flow turning angle is $\theta_{AB} = \nu_B - \nu_A = 41.41° - 26.38° = 15.03°$. Also, the corresponding Mach angles are $\mu_A = 30°$ and $\mu_B = 22.62°$. Thus, the fan angle is $\Phi_{AB} = \mu_A - \mu_B = 30° - 22.62° = 7.38°$.

Since the centreplane behaves like a plane solid surface in reflecting the incident expansion fan, the flow from region B is turned back by $\theta_{BC} = \theta_{AB} = 15.03°$. Then, the P-M function value for region C is $\nu_C = \nu_B + \theta_{BC} = 41.41° + 15.03° = 56.44°$. From the P-M table, the nearest entries for P-M function values are $56.07°$ and $56.91°$ corresponding to $M = 3.35$ and 3.40, respectively. We perform a linear interpolation to arrive at $M_C = (56.44° - 56.07°)/(56.91° - 56.07°) \times (3.40 - 3.35) + 3.35 = 3.372$. Further, the Mach angle corresponding to this is obtained by linear interpolation as $\mu_C = (56.44° - 56.07°)/(56.91° - 56.07°) \times (17.10° - 17.37°) + 17.37° = 17.25°$. So, the fan angle for the reflected expansion fan is $\Phi_{BC} = \mu_B - \mu_C = 22.62° - 17.25° = 5.37°$.

The stagnation pressure in region C continues to be $p_{0C} = p_0 = 2000$ kPa. Further, from the isentropic flow table, the pressure ratio corresponding to $M_C = 3.372$ is obtained by linear interpolation as $p_{0C}/p_C = (3.372 - 3.35)/(3.40 - 3.35) \times (66.12 - 61.52) + 61.52 = 63.54$. Then, the pressure in region C is found as $p_C = p_{0C}/63.54 = 2000/63.54 = \boxed{31.48 \text{ kPa}}$.

6. Air maintained at 150 kPa and 40°C in a large reservoir flows through a converging-only nozzle having 40 mm diameter at exit.

   (a) (½ point) What is the back pressure that just chokes the nozzle?

   (b) (2½ points) What is the maximum possible mass flow rate?

   (c) (2 points) Find the mass flow rate when the back-pressure is 94.5 kPa.

Do not use any formula for mass flow rate, other than its basic definition.

Solution:

The stagnation pressure and temperature are $p_0 = 150$ kPa and $T_0 = 40°\text{C} = 313$ K, respectively.

(a) When the nozzle is choked, sonic condition exists at the exit. From the isentropic flow table, the pressure ratio corresponding to sonic flow in air is $p_0/p^* = 1.893$. So, the requisite back pressure for just choking the nozzle is $p^* = p_0/1.893 = 150/1.893 = 79.24$ kPa.

(b) The maximum possible mass flow rate occurs when the nozzle is choked, as well as for all lower back pressures (keeping the reservoir conditions fixed).

Using the ideal gas law, we find the stagnation (reservoir) density as $\rho_0 = p_0/(RT_0) = 150 \times 10^3/(287 \times 313) = 1.670$ kg/m^3. At choked condition, the isentropic flow table gives the density ratio as $\rho_0/\rho^* = 1.577$. Thus, the exit density is $\rho_e = \rho^* = \rho_0/1.577 = 1.670/1.577 = 1.059$ kg/m^3.

Further, the exit speed equals the exit sound speed, so that $V_e = a_e = \sqrt{\gamma p_e/\rho_e} = \sqrt{1.4 \times 79.24 \times 10^3/1.0}$ 323.7 m/s.

Finally, the exit area is $A_e = \pi D_e^2/4 = \pi \times 0.04^2/4 = 1.257 \times 10^{-3}$ m^2.

Thus, the maximum possible mass flow rate through the nozzle is $\dot{m}_{\max} = \rho_e A_e V_e = 1.059 \times 1.257 \times 10^{-3} \times 323.7 = \boxed{0.431 \text{ kg/s}}$.

(c) When the back-pressure is 94.5 kPa (which is greater than the critical value of 79.24 kPa), the nozzle is not choked. Thus, the exit Mach number can be found from the isentropic flow table by looking up the prevalent pressure ratio of $p_0/p_e = p_0/p = 150/94.5 = 1.587$, which is $M_e = 0.84$. The corresponding density ratio is $\rho_0/\rho_e = 1.391$, so that the exit density is $\rho_e = 1.670/1.391 = 1.201$ kg/m^3.

Further, the exit sound speed is $a_e = \sqrt{\gamma p_e/\rho_e} = \sqrt{1.4 \times 94.5 \times 10^3/1.201} = 331.9$ m/s. So, the exit speed is $V_e = M_e a_e = 0.84 \times 331.9 = 278.8$ m/s.

Thus, the corresponding mass flow rate is $\dot{m} = \rho_e A_e V_e = 1.201 \times 1.257 \times 10^{-3} \times 278.8 = \boxed{0.421 \text{ kg/s}}$.

7. Air flows through a converging-diverging nozzle. At some section in the nozzle, the pressure, temperature, velocity and cross-sectional area are 2 bar, 200°C, 174.4 m/s and 1000 mm^2, respectively. Assuming isentropic flow conditions, determine:

(a) (1 point) sonic velocity and Mach number at this section,

(b) (1 point) stagnation temperature and pressure,

(c) (1½ points) Mach number, velocity, and flow area at the exit section where the pressure is 1.1 bar, and

(d) (1½ points) pressure, temperature, velocity and flow area at the throat section.

Solution:

We will denote the given section with the subscript '1', and the exit by the subscript 'e'.

The temperature at section '1' is $T_1 = 200°\text{C} = 473$ K.

(a) At section '1', the sonic velocity can be found from the static temperature as $a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \times 287 \times 473} = \boxed{435.9 \text{ m/s}}$. Thus, the Mach number at this section is $M_1 = V_1/a_1 = 174.4/435.9 = \boxed{0.400}$.

(b) From the isentropic flow table, corresponding to $M_1 = 0.4$, we have $T_{01}/T_1 = 1.032$ and $p_{01}/p_1 = 1.117$. Since the entire flow is isentropic (and thus definitely adiabatic), the *constant* stagnation temperature and pressure are $T_0 = T_{01} = 1.032T_1 = 1.032 \times 473 = \boxed{488.1\text{ K}}$ and $p_0 = p_{01} = 1.117p_1 = 1.117 \times 2 = \boxed{2.234\text{ bar}}$.

(c) With isentropic flow throughout, the pressure ratio at the exit is $p_0/p_e = 2.234/1.1 = 2.031$. From the isentropic flow table, the nearest entry for the pressure ratio is 2.033, corresponding to $\boxed{M_e = 1.06}$. The temperature ratio at this Mach number is $T_0/T_e = 1.225$, so that the exit temperature is $T_e = T_0/1.225 = 488.1/1.225 = 398.4$ K. Then the speed of sound at the exit is $a_e = \sqrt{\gamma R T_e} = \sqrt{1.4 \times 287 \times 398.4} = 400.1$ m/s, so that the exit velocity is $V_e = M_e a_e = 1.06 \times 400.1 = \boxed{424.1\text{ m/s}}$.

The Mach number is known at section ‘1’ and at the exit. From the isentropic flow table, the corresponding area ratios are $A_1/A^* = 1.590$ and $A_e/A^* = 1.003$. Note that we have used the same sonic area notation in both cases, because it is indeed consistent throughout the isentropic flow. Then, with the knowledge of the area at station ‘1’ (viz. $A_1 = 1000\text{ mm}^2$), we find the exit area as

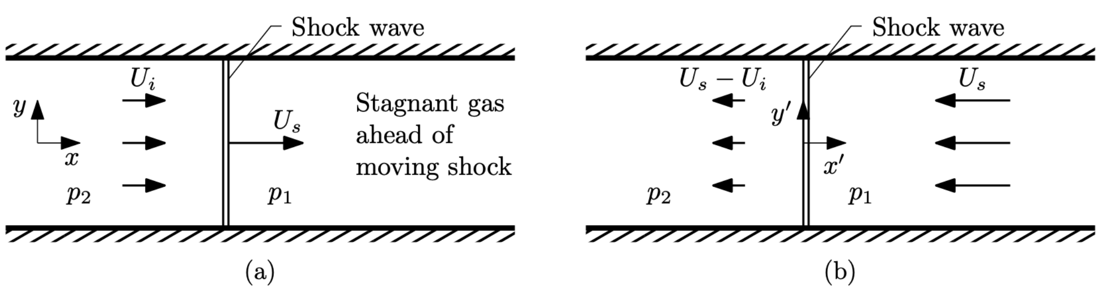
$$A_e = \frac{A_e/A^*}{A_1/A^*}A_1 = \frac{1.003}{1.590} \times 1000 = \boxed{630.8\text{ mm}^2}.$$

(d) At the throat, sonic condition is prevailing. Thus, from the isentropic flow table, we find that $p_0/p_t = 1.893$, $T_0/T_t = 1.200$, and $A_t/A^* = 1.000$ (of course!). With our knowledge of the (constant) total pressure and temperature, we can find the static pressure and temperature prevailing at the throat as $p_t = p_0/1.893 = \boxed{1.180\text{ bar}}$, and $T_t = T_0/1.200 = 488.1/1.200 = 406.7$ K. The flow being sonic, the throat velocity equals the sound speed thereat, so that $V_t = a_t = \sqrt{\gamma R T_t} = \sqrt{1.4 \times 287 \times 406.7} = \boxed{404.2\text{ m/s}}$. Finally, the throat area equals the (constant) isentropic sonic area, and hence can be found from knowledge of the parameters at station ‘1’ (or the exit) as

$$A_t = A^* = \frac{A_1}{A_1/A^*} = \frac{1000}{1.590} = \boxed{628.9\text{ mm}^2}.$$

8. (5 points) Approximate knocking in an auto engine as the occurrence of a normal shock wave traveling at 1004 m/s downward, into the unburned mixture at 700 kPa and 500 K. Compared to this high speed of the shock, the piston’s speed (say around 10 m/s upward) can be safely neglected. Determine the pressure acting on the piston face after the shock reflects from it. Assume the gas has the properties of air ($R = 287$ J/kg · K), and acts as a perfect gas with $\gamma = 1.4$. Use appropriate sketches to explain your approach.

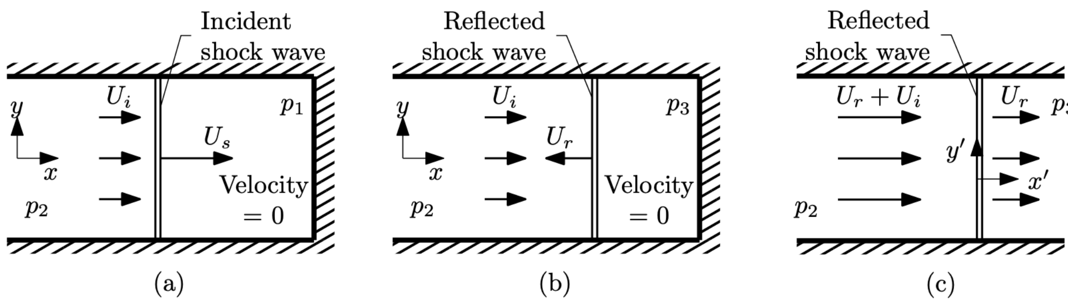
Solution:



Relation between flow (a) relative to lab, and (b) relative to shock, in a case of a normal shock wave moving into a stationary medium.

We first address the incident shock. Relative to the moving shock, the speed of the pre-shock flow is $u_1 = U_s = 1004$ m/s. Also, the speed of sound in this region can be obtained from the given static temperature thereat – viz. $T_1 = 500$ K – as $a_1 = \sqrt{\gamma R T_1} = \sqrt{1.4 \times 287 \times 500} = 448.2$ m/s. Thus, the pre-shock Mach number (relative to the shock) is $M_1 = u_1/a_1 = 1004/448.2 = 2.24$. From the normal shock table, the corresponding pressure ratio is $p_2/p_1 = 5.6872$, temperature ratio is $T_2/T_1 = 1.8924$ and relative speed change is $\Delta V/a_1 = 1.495$.

Since the gas ahead of the incident shock is at rest, we can directly find the speed of the flow induced behind the shock (in the engine frame) as $U_i = \Delta V = (\Delta V/a_1)a_1 = 1.495 \times 448.2 = 670.06$ m/s. Further, from the knowledge of the pre-shock static pressure and temperature, we can now find their post-shock counterparts as $p_2 = 5.6872 p_1 = 5.6872 \times 700 = 3981.0$ kPa $= 3.9810$ MPa, and $T_2 = 1.8924 T_1 = 1.8924 \times 500 = 946.2$ K. Then, the speed of sound in the gas behind the shock is $a_2 = \sqrt{\gamma R T_2} = \sqrt{1.4 \times 287 \times 946.2} = 616.59$ m/s.



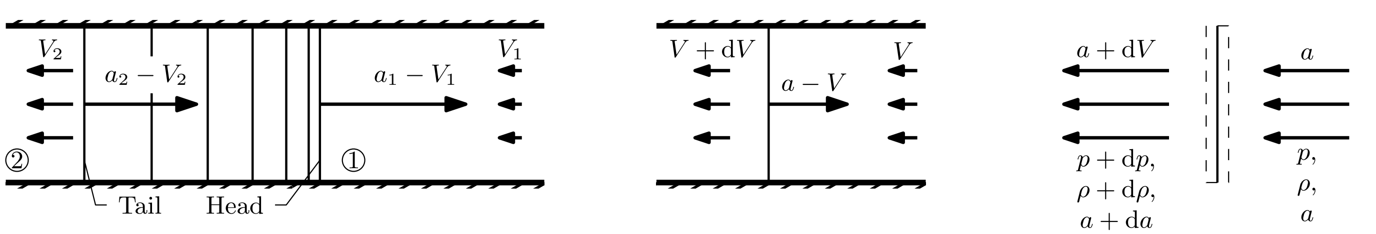
Reflection of a moving shock from an end wall. Flow relative to wall in case of (a) incident shock, and (b) reflected shock. (c) Flow relative to shock in latter case.

Now, in case of the reflected shock, it is propagating into a flow that has speed $U_i = 670.06$ m/s and it is leaving behind a flow that is at rest. Thus, the change in speed across the reflected shock is $\Delta V_r = 670.06$ m/s. Further, the pre-shock gas has speed of sound $a_{r,1} = a_2 = 616.59$ m/s. Thus, $(\Delta V/a_1)_r = 670.06/616.59 = 1.0867$. From the normal shock table, the nearest entry for this quantity is 1.080 corresponding to $M_{r,1} = 1.84$ and $(p_2/p_1)_r = 3.7832$. Thus, the requisite pressure on the piston face behind the reflected shock is $p_{r,2} = 3.7832 p_{r,1} = 3.7832 p_2 = 3.7832 \times 3.9810 = \boxed{15.061 \text{ MPa}}$.

9. (5 points) Starting from first principles, derive the relation between the static pressure ratio across a moving expansion wave in a one-dimensional duct, and the flow velocities (in the laboratory frame) ahead of and behind it (considering that both velocities may be non-zero). Use appropriate sketches.

Solution:

The overall setup of the problem is shown in the left figure below. The speed of the gas ahead of the expansion wave (region '1') is V_1, whereas that behind the wave (region '2') is V_2. The head (resp. tail) of the wave is moving with speed equalling a_1 (resp. a_2) relative to medium '1' (resp. medium '2').



Overall setup — Single wavelet, lab frame — Moving CV w/ single wavelet

Let us consider a single wavelet of the moving expansion fan, as shown in the middle figure. Let the local flow velocity be V and the local sound speed be a. That means that, relative to the flow, the wavelet is moving with speed a. This also means that, relative to the lab frame, the wavelet is moving with speed $a - V$. Let the speed *increase* by $\mathrm{d}V$ across the wavelet.

It is evident from the preceding description that each wavelet moves with a constant speed in the lab frame. Thus, the governing equations can be (and will be) formulated in a frame moving with the wavelet (with speed $a - V$). This is shown in the right figure. In this frame, the speed of the pre-wave flow is $V + (a - V) = a$ and that of the post-wave flow is $(V + \mathrm{d}V) + (a - V) = a + \mathrm{d}V$. Let the pressure, density and sound speed change from p, ρ and a to $p + \mathrm{d}p$, $\rho + \mathrm{d}\rho$ and $a + \mathrm{d}a$, respectively, across the wavelet, as shown.

Applying mass and momentum balance, with the prevailing assumptions of inviscid one-dimensional flow with no body force, we have

$$\begin{aligned} &\text{Mass:} && (\rho + \mathrm{d}\rho)(a + \mathrm{d}V) - \rho a = 0, \\ &\text{Momentum:} && (\rho + \mathrm{d}\rho)(a + \mathrm{d}V)^2 - \rho a^2 + (p + \mathrm{d}p) - p = 0. \end{aligned}$$

Using the mass equation in the momentum equation, we have

$$\mathrm{d}p = \rho a^2 - \rho a(a + \mathrm{d}V) = -\rho a \mathrm{d}V.$$

(We are already finding that pressure reduces as the flow speed increases across the wavelet, the hallmark of an expansion wave.)

From the definition of the speed of sound, we have $\partial p/\partial\rho|_s = a^2$. However, since our flow is isentropic, we have $\mathrm{d}p = a^2 \mathrm{d}\rho$. Thus, the preceding momentum equation (in non-conservative form) can be rewritten as

$$\mathrm{d}V = -\frac{a}{\rho}\mathrm{d}\rho.$$

In lieu of the energy equation, we will invoke the isentropy assumption. In particular, we have

$$a \propto T^{0.5} \propto \left(\rho^{\gamma-1}\right)^{0.5} = \rho^{(\gamma-1)/2} \qquad \Longrightarrow \ln a = \frac{\gamma-1}{2}\ln\rho + \text{constant} \qquad \Longrightarrow \frac{\mathrm{d}a}{a} = \frac{\gamma-1}{2}\frac{\mathrm{d}\rho}{\rho}.$$

Using this in the preceding result, we end up with the very simple relation between changes in flow speed and changes in sound speed:

$$\mathrm{d}V = -\frac{2}{\gamma-1}\mathrm{d}a.$$

Integrating between the end points of the wave, we have the result

$$V_2 - V_1 = \frac{2}{\gamma-1}(a_1 - a_2) \qquad \Longrightarrow \frac{a_2}{a_1} = 1 - \frac{\gamma-1}{2}\frac{V_2 - V_1}{a_1}.$$

Thus, the sound speed indeed decreases across the expansion wave.

However, we wish to relate the change in velocity to the pressure ratio across the moving expansion wave. For this, we again invoke the isentropic flow relation

$$p \propto T^{\gamma/(\gamma-1)} \propto \left(a^2\right)^{\gamma/(\gamma-1)} \qquad \Longrightarrow \frac{p_2}{p_1} = \left(\frac{a_2}{a_1}\right)^{\frac{2\gamma}{\gamma-1}}.$$

Thus,

$$\boxed{\frac{p_2}{p_1} = \left(1 - \frac{\gamma-1}{2}\frac{V_2 - V_1}{a_1}\right)^{\frac{2\gamma}{\gamma-1}}.}$$