AE 616/236: Mid-sem. 18/9/2024. Total=20. Duration=2 hrs.

Answers without detailed explanation will be heavily penalized.
Answer Q1 OR Q2, and all of Q3 through Q8.

1. (4 points) In Rayleigh flow, it is known that

$$\frac{T}{T^*} = \frac{(1+\gamma)^2 M^2}{(1+\gamma M^2)^2}, \quad \text{and} \quad \frac{T_0}{T_0^*} = \frac{(1+\gamma)M^2\{2+(\gamma-1)M^2\}}{(1+\gamma M^2)^2}.$$

Using only these and the energy equation, quantitatively show that there is a range of Mach numbers where cooling causes the static temperature to rise. Characterize the maximum achievable temperature in such a flow when air is the medium.

Solution:

In a Rayleigh flow, by definition the diabatic static and total temperatures (i.e., T^* and T_0^*, respectively) are constant. Thus, changes of static temperature in the flow are directly proportional to changes of T/T^*, and likewise for the total temperature. Moreover, since Mach number appears as square everywhere, and is never negative, it is much simpler to change variables to $m := M^2$. We find out the rate of change of static temperature with square of Mach number:

$$\begin{aligned}\frac{\mathrm{d}(T/T^*)}{\mathrm{d}m} &= (1+\gamma)^2 \frac{\mathrm{d}}{\mathrm{d}m}\left\{\frac{m}{(1+\gamma m)^2}\right\} = (1+\gamma)^2\left[(1+\gamma m)^{-2} - 2\gamma m(1+\gamma m)^{-3}\right] \\ &= \frac{(1+\gamma)^2}{(1+\gamma m)^3}[1+\gamma m - 2\gamma m] = \frac{(1+\gamma)^2(1-\gamma m)}{(1+\gamma m)^3}.\end{aligned}$$

Thus, T/T^* reaches an extremum at $m = 1/\gamma$ or $M = 1/\sqrt{\gamma} < 1$.

Heuristically, we know that this must be a maximum. However, we can show this more rigorously by evaluating the second derivative:

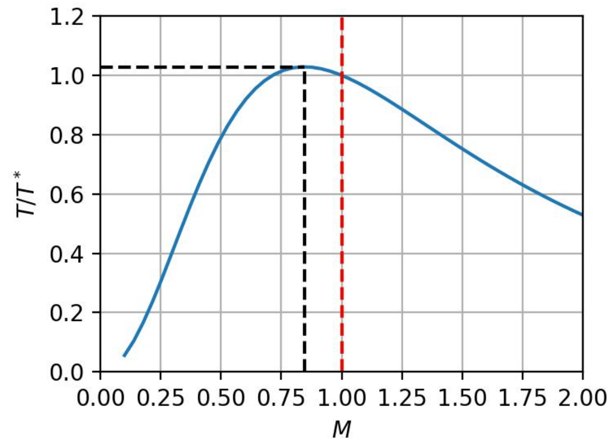
$$\begin{aligned}\frac{\mathrm{d}^2(T/T^*)}{\mathrm{d}m^2} &= \frac{\mathrm{d}}{\mathrm{d}m}\left[\frac{(1+\gamma)^2(1-\gamma m)}{(1+\gamma m)^3}\right] = (1+\gamma)^2\left[-\gamma(1+\gamma m)^{-3} - 3\gamma(1-\gamma m)(1+\gamma m)^{-4}\right] \\ &= \frac{(1+\gamma)^2}{(1+\gamma m)^4}[-\gamma(1+\gamma m) - 3\gamma(1-\gamma m)] = \frac{2(1+\gamma)^2}{(1+\gamma m)^4}(-4\gamma + 2\gamma^2 m).\end{aligned}$$

At the extremum, we have

$$\left.\frac{\mathrm{d}^2(T/T^*)}{\mathrm{d}m^2}\right|_{m=1/\gamma} = \frac{(1+\gamma)^2}{(1+1)^4}(-4\gamma+2\gamma) = -\frac{\gamma(1+\gamma)^2}{8} < 0$$

Thus, the extremum identified above is indeed a maximum.

Alternative method for proving maxima: We have identified that there is only one extremum of temperature. Thus, to show that it is indeed the maximum, we only need to show that the temperature at this extremum is higher than that at $M \to 0$ and $M \to \infty$. At $M \to 0$, we have $T/T^* \to 0$. At $M \to \infty$, we have $T/T^* \to 0$. Further, at $m = 1/\gamma$, we have $T/T^* = 0.25(1+\gamma)^2/\gamma > 0$. Thus, the value of T/T^* at the identified *sole* extremum is indeed greater than its value at the extremities of M, which proves that the identified extremum is a maximum.



With the above guidelines, we can sketch the static temperature vs. Mach number graph for Rayleigh flow, as shown above, for $\gamma = 1.4$.

To find out the effect of heating/cooling, we have to start from the energy equation. We have

$$T_{02} = T_{01} + \frac{q}{C_p}.$$

Or, in differential form,

$$\mathrm{d}T_0 = \frac{\mathrm{d}q}{C_p}.$$

Thus, heating (resp. cooling) increases (resp. decreases) the total temperature. Thus, to relate the heating/cooling to the static temperature, we have to find out the changes in Mach number wrought by heating/cooling from the relation of total temperature to Mach number, and then use the previously derived relation between changes of static temperature and Mach number.

We find out the rate of change of total temperature (normalized by the diabatic sonic total temperature) with square of Mach number:

$$\begin{aligned}
\frac{\mathrm{d}(T_0/T_0^*)}{\mathrm{d}m} &= \frac{\mathrm{d}}{\mathrm{d}m}\left\{\frac{(1+\gamma)m\{2+(\gamma-1)m\}}{(1+\gamma m)^2}\right\} \\
&= (1+\gamma)\left[\{2+2(\gamma-1)m\}(1+\gamma m)^{-2} - 2\gamma\{2m+(\gamma-1)m^2\}(1+\gamma m)^{-3}\right] \\
&= \frac{2(1+\gamma)}{(1+\gamma m)^3}\left[1+\gamma m+(\gamma-1)m+\gamma(\gamma-1)m^2-2\gamma m-\gamma(\gamma-1)m^2\right] = \frac{2(1+\gamma)(1-m)}{(1+\gamma m)^3}.
\end{aligned}$$

This result, along with the differential energy equation, leads to the conclusion that:

- For subsonic flow, heat addition (resp. removal) causes increase (resp. decrease) of total temperature, which in turn causes increase (resp. decrease) of Mach number.
- For supersonic flow, heat addition (resp. removal) causes increase (resp. decrease) of total temperature, which in turn causes decrease (resp. increase) of Mach number.

Focusing on the first conclusion, we can say that cooling of a subsonic flow leads to decrease of its Mach number. But, we are noticing above that, for the range of Mach numbers $M \in [1/\sqrt{\gamma}, 1]$, the static temperature is increasing with decrease in Mach number (i.e., cooling), which was to be proved.

The maximum achievable temperature, starting with a certain diabatic sonic temperature T^*, can be found by substituting the extremal Mach number in the given expression. We have

$$\frac{T_{\max}}{T^*} = \frac{(1/\gamma)(1+\gamma)^2}{\{1+\gamma(1/\gamma)\}^2} = \frac{(1+\gamma)^2}{4\gamma}$$

For air, we have $\gamma = 1.4$, whence the above expression evaluates to $\boxed{T_{\max} = 1.029T^*}$.

OR

2. (4 points) At the beginning of a duct carrying *nitrogen*, the pressure is 1.5 bar, the temperature is 248.2 K, and the Mach number is 0.80. After some heat transfer, the pressure is 2.5 bar. Determine the direction and amount of heat transfer, and the downstream temperature and Mach number.

Solution:

Since nitrogen is a diatomic gas, its specific heat ratio is $\gamma = 1.4$. So, we can re-use the gas tables for air.

From the isentropic flow tables, for Mach 0.8 we have $T_0/T = 1.128$. Thus, the total temperature at the beginning of the duct is $T_{01} = 1.128T_1 = 1.128 \times 248.2$ K $= 280$ K.

From the Rayleigh flow tables, for $M_1 = 0.8$ we have $p_1/p^* = 1.266$, $T_1/T^* = 1.025$, and $T_{01}/T_0^* = 0.9639$. Since the inlet pressure, temperature and total temperature are 1.5 bar, 248.2 K and 315.84 K, respectively, we have $p^* = p_1/1.266 = 1.5/1.266 = 1.185$ bar, $T^* = T_1/1.025 = 248.2/1.025 = 242.1$ K, and $T_0^* = T_{01}/0.9639 = 280/0.9639 = 290.5$ K.

It is given that the downstream pressure is $p_2 = 2.5$ bar. Thus, for the downstream location, we have $p_2/p^* = 2.5/1.185 = 2.110$. The nearest entry in the Rayleigh flow table is $p/p^* = 2.099$ corresponding to $\boxed{M_2 = 0.32}$, $T_2/T^* = 0.4512$, and $T_{02}/T_0^* = 0.3837$. Thus, the downstream temperature is $T_2 = 0.4512T^* = 0.4512 \times 242.1 = \boxed{109.2 \text{ K}}$, and the downstream total temperature is $T_{02} = 0.3837T_0^* = 0.3837 \times 290.5$ K $= 111.5$ K.

The molecular weight of nitrogen is 28, so that its gas constant is $R = 8314/28 = 296.9$ J/kg $\cdot$ K. Given that the specific heat ratio of nitrogen is $\gamma = 1.4$, its specific heat at constant pressure is

$$C_p = \frac{\gamma}{\gamma - 1}R = \frac{1.4}{1.4 - 1} \times 296.9 = 1039.15 \text{ J/kg} \cdot \text{K}.$$

Since the total temperature is falling, heat is being removed. The amount of heat *added* per unit mass of the flow is found from the energy relation as

$$q = C_p(T_{02} - T_{01}) = 1039.15 \times (111.5 - 280) = -1.75 \times 10^5 \text{ J/kg}.$$

That is, $\boxed{1.75 \times 10^5 \text{ J/kg}}$ of heat is being removed per unit mass of the flow.

3. (2 points) The difference between the total and static pressure before a normal shock is 5 bar. What is the maximum static pressure that can exist at this point ahead of the shock? The gas is *oxygen*.

Solution:

Let us denote the total-to-static pressure difference as $\Delta p := p_0 - p$. We have from isentropic flow relations

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} \quad \Longrightarrow \quad \frac{\Delta p}{p} = \frac{p_0 - p}{p} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\gamma/(\gamma-1)} - 1.$$

Clearly, for a given Δp (i.e., numerator of LHS fraction), the static pressure (i.e., denominator of LHS fraction) will be maximum when the RHS is minimum, which in turn will happen when the Mach number is minimum.

The second law of thermodynamics dictates that the minimum Mach number upstream of a normal shock is 1.0, which sets the limiting condition.

For oxygen, the specific heat ratio is $\gamma = 1.4$, so that we can use the standard gas dynamics tables. From the isentropic flow tables (or the normal shock tables), for $M = 1.0$, we have $p_0/p = 1.893$, so that $\Delta p/p_{\max} = 0.893$. Then, the maximum static pressure is

$$p_{\max} = \frac{\Delta p}{0.893} = \frac{5 \text{ bar}}{0.893} = \boxed{5.6 \text{ bar}}.$$

4. (3 points) Consider the flow of a perfect gas (of gas constant R and specific heat ratio γ) through a duct of area A. Show that, if the mass flow rate through the duct is $\dot{m}$ and the total temperature of the gas is T_0, then the pressure at the Fanno-sonic state is given by the relation

$$p^{\#} = \frac{\dot{m}}{A}\left[\frac{2RT_0}{\gamma(\gamma+1)}\right]^{1/2}.$$

Solution:

From mass conservation, the mass flow through the duct may be related to the condition at the Fanno sonic state as

$$\dot{m} = A\rho^{\#}V^{\#} = A\rho^{\#}a^{\#} = A\rho^{\#}\sqrt{\gamma p^{\#}/\rho^{\#}} = A\sqrt{\gamma\rho^{\#}p^{\#}}.$$

Now, we apply the ideal gas law to the Fanno sonic state to have $\rho^{\#} = p^{\#}/(RT^{\#})$. With this, we have

$$\dot{m} = A\sqrt{\gamma(RT^{\#})}p^{\#} \qquad \Longrightarrow p^{\#} = \frac{\dot{m}}{A}\left[\frac{RT^{\#}}{\gamma}\right]^{1/2}.$$

Finally, we relate the Fanno-sonic static temperature to the (constant) total temperature of the (adiabatic) flow using the definition of the latter, viz. $T_0/T = 1 + 0.5(\gamma-1)M^2$. But, at the Fanno sonic state, the Mach number is unity. So, we find that

$$T^{\#} = \frac{T_0}{1 + 0.5(\gamma-1)} = \frac{2T_0}{\gamma+1}.$$

Substituting this in the previous expression for $p^{\#}$, we arrive at the desired result.

5. At section 'A' in a constant-area duct carrying air, the stagnation pressure is 147 kPa and the Mach number is 0.80. At section 'B' along the same 'Fanno line', the pressure is 131.93 kPa and the temperature is 50°C. The diameter and length of the duct are 20 cm and 4 m, respectively.
   (a) (1 point) Compute the Mach number at section 'B'.
   (b) (½ point) Compute the temperature at section 'A'.
   (c) (½ point) Which way is the air flowing?
   (d) (1 point) What is the Fanning friction factor of the duct?

Solution:

(a) From the isentropic flow table, for $M_A = 0.8$, we have $p_{0A}/p_A = 1.524$. Thus, the static pressure at section 'A' is $p_A = p_{0A}/1.524 = 147/1.524 = 96.46$ kPa. From the Fanno flow table, for $M_A = 0.8$, we have $p_A/p^{\#} = 1.289$. Therefore, the Fanno sonic (static) pressure for this flow is $p^{\#} = p_A/1.289 = 96.46/1.289 = 74.83$ kPa.

Then, for section 'B', we have $p_B/p^{\#} = 131.93/74.83 = 1.763$. The nearest entry for $p/p^{\#}$ in the Fanno flow table is $\boxed{M_B = 0.6}$, for which $p_B/p^{\#} = 1.763$.

(b) From the Fanno flow table, for $M = 0.8$ and 0.6, we have $T/T^{\#} = 1.064$ and 1.119, respectively. Then, the temperature at section 'A' can be found as

$$T_A = \frac{T_A/T^{\#}}{T_B/T^{\#}} T_B = \frac{1.064}{1.119}(50 + 273)\text{ K} = \boxed{307.12\text{ K} = 34.12°\text{C}}.$$

(c) There is subsonic flow in the duct, and the Mach number is increasing from section 'B' to section 'A'. Thus, <u>flow is from section 'B' to section 'A'</u>.

(d) From the Fanno flow table, for $M = 0.8$ and 0.6, we have $4fL^{\#}/D_H = 0.07229$ and 0.4908, respectively. Then, we have that $4fL/D_H = 0.4908 - 0.07229 = 0.4185$. Then, the Fanning friction factor for the duct is

$$f = 0.4185\frac{D_H}{4L} = 0.4185\frac{0.2}{4 \times 4} = \boxed{0.0052}.$$

6. A simple wedge with a total included angle of 34° is used to measure the Mach number of supersonic flows. When inserted into a wind tunnel and aligned with the flow, oblique shocks are observed at 38° angles to the free stream.

   (a) (1 point) What is the Mach number in the wind tunnel?

   (b) (1 point) Through what range of Mach numbers could this wedge be useful?

Solution:

(a) On either side of the wedge, the flow turning angle is $\theta = 17°$ and the oblique shock angle is $\beta = 38°$. On the $\beta - \theta - M_1$ chart, we see that this coordinate corresponds to $\boxed{M_1 = 2.6}$.

(b) If a detached shock is formed in front of the wedge, then it will not be able to indicate the flow Mach number. Thus, the *minimum* Mach number that the wedge can measure is the one for which $\theta_{\max} = 17°$. This happens to be $M_1 = 1.7$. Thus, the wedge is only useful for measuring the Mach number in case of $M_1 > 1.7$.

7. In the flow past a compression comer, the upstream Mach number and pressure are 3.5 and 1 atm., respectively. Downstream of the corner, the pressure is 5.48 atm..

   (a) (2 points) Calculate the deflection angle of the corner.

   (b) (1 point) What is the post-shock Mach number?

Solution:

(a) The static pressure ratio across the shock is $p_2/p_1 = 5.48$. Looking up the normal shock table, we see the nearest (actually exact) entry for this pressure ratio has $M_{n,1} = 2.2$ and $M_{n,2} = 0.5471$.

Then, with the upstream Mach number given as $M_1 = 3.5$, we have the oblique shock wave angle as

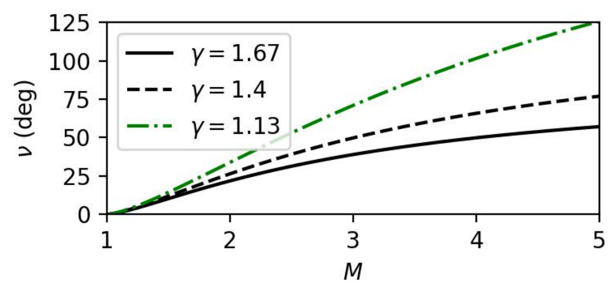
$$\beta = \sin^{-1}\frac{M_{n,1}}{M_1} = \sin^{-1}\frac{2.2}{3.5} = 38.9°.$$

Looking up $\beta - \theta - M_1$ chart, we have the turning angle as $\boxed{\theta = 23.6°}$.

(b) The downstream Mach number is straightforwardly found as

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.5471}{\sin(38.9° - 23.6°)} = \boxed{2.073}.$$

8. (3 points) The variation of the Prandtl-Meyer function $\nu(M)$ with Mach number M is shown in the graph below for three different specific heat ratios. Consider a supersonic *air* flow at a particular Mach number M_1 encountering a convex corner of a particular angle θ. Then comment on the changes to the downstream Mach number and the expansion fan angle, if the air stream is replaced by *helium* (keeping M_1 and θ fixed).



Solution:

Air has specific heat ratio 1.4, whereas helium, being monatomic, has specific heat ratio 1.67. It is observed from the graph above that (a) $\nu_{air}(M) > \nu_{He}(M)$ for the same Mach number M, and (b) $(\mathrm{d}\nu/\mathrm{d}M)_{air} > (\mathrm{d}\nu/\mathrm{d}M)_{He}$ for the same Mach number. That is, the gap between the two curves increases with Mach number.

Thus, for a particular upstream Mach number M_1, we will have $\nu_{1,air} > \nu_{1,He} = \nu_{1,air} - \Delta\nu_1$, say. Then, for the same turning angle θ, we will have $\nu_{2,air} = \nu_{1,air} + \theta$ and $\nu_{2,He} = \nu_{1,He} + \theta = \nu_{1,air} + \theta - \Delta\nu_1 = \nu_{2,air} - \Delta\nu_1$. That is, $\nu_{2,air} - \nu_{2,He} = \Delta\nu_1$.

Due to the increased separation of the two curves at the higher downstream Mach number (say, $M = M_{2,air}$), the downstream Mach number for helium has to be greater than that for air to obtain the same difference in P-M function values, viz. $\Delta\nu_1$. So, we conclude that, $\boxed{M_{2,He} > M_{2,air}}$.

The fan angle depends on the Mach angles, which are themselves independent of specific heat ratio. Since the upstream Mach numbers are the same, we have $\mu_{1,He} = \mu_{1,air}$. Moreover, since Mach angle is inversely related to the Mach number, and the downstream Mach number for helium is higher, we have $\mu_{2,He} < \mu_{2,air}$. But, the fan angle is $\Phi = \mu_1 - \mu_2 + \theta$. Thus, with the first and third terms being same, the lower downstream Mach angle for helium results in a higher fan angle. That is, $\boxed{\Phi_{He} > \Phi_{air}}$.