Department of Mechanical Engineering, IIT Bombay

ME 782 Design Optimization Quiz 4 November 6, 2024 Maximum marks: **12**

Part 2 (Paper and pen, Open notes exam) 7:30 pm to 8:25 pm

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1. Consider the following function:

$f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1$ subject to $x_1 + x_2 = 4$

$$f(x_1, x_2) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 \tag{1}$$

Using an appropriate penalty function, analytically determine the values of variables corresponding to a minimum bound by the constraint. **[4 marks]**

The penalty function for the above problem can be written as

$$\phi(x_1, x_2, R) = 4x_1^2 + 3x_2^2 - 5x_1x_2 - 8x_1 + R\left(x_1 + x_2 - 4\right)^2 \tag{2}$$

Equating the gradient of equation (2) to zero gives the stationary value of equation (1) and is given by

$$\begin{aligned} 8x_1 - 5x_2 - 8 + R\left(2x_1 + 2x_2 - 8\right) &= 0 \\ 6x_2 - 5x_1 + R\left(2x_1 + 2x_2 - 8\right) &= 0 \end{aligned} \tag{3}$$

Simplifying equation (3)

$$\begin{bmatrix} 8+2R & -5+2R \\ -5+2R & 6+2R \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 8R+8 \\ 8R \end{Bmatrix} \tag{4}$$

Solving the linear non-homogenous equations of (4)

$$\begin{aligned} x_1 &= \frac{8\left(13R+6\right)}{48R+23} \\ x_2 &= \frac{8\left(11R+5\right)}{48R+23} \end{aligned} \tag{5}$$

The limiting values of equation (5) are obtained as the controlling parameter R tends to infinity. Since R is present in both the numerator and denominator of equations in (5), applying L'Hospital rule

$$x_1 = \frac{13 \times 8}{48} = 2.17$$ (6)

$$x_2 = \frac{8 \times 11}{48} = 1.83$$

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2. Consider the following function:

$f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6$ subject to $\begin{array}{l} 4 - x_1 - x_2 = 0 \\ x_1 + 2x_2 \le 10 \end{array}$

Using appropriate penalty functions, analytically determine the values of variables corresponding to a minimum bound by the constraints. **[8 marks]**

$$f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6 \tag{1}$$

The penalty function for the above problem can be written as

$$\phi(x_1, x_2, R) = f(x_1, x_2) = x_1^2 + x_2^2 - 4x_1 - 2x_2 + 6 + \frac{1}{R}\left(x_1 + x_2 - 4\right)^2 - R\left(\frac{1}{x_1 + 2x_2 - 10}\right) \tag{2}$$

Equating the gradient of equation (2) to zero gives the stationary value of equation (1) and are given by

$$2x_1 - 4 + \frac{2x_1 + 2x_2 - 8}{R} + \frac{R}{\left(x_1 + 2x_2 - 10\right)^2} = 0 \tag{3a}$$

$$2x_2 - 2 + \frac{2x_1 + 2x_2 - 8}{R} + \frac{2R}{\left(x_1 + 2x_2 - 10\right)^2} = 0 \tag{3b}$$

From equation (3) it is clear that the equation cannot be solved through hand calculations. Since the penalty function, with a common controlling parameter for both constraints, is written such that the limiting value of R tends to zero in the final solutions. Hence, the fourth term of equations (3a) and (3b) can assumed to be much smaller than the second terms. Therefore, the approximation of equations (3a) and (3b) becomes

$$2x_1 - 4 + \frac{2x_1 + 2x_2 - 8}{R} = 0 \tag{4}$$

$$2x_2 - 2 + \frac{2x_1 + 2x_2 - 8}{R} = 0$$

Equation (4) can be simplified as

$$\begin{bmatrix} 2R+2 & 2 \\ 2 & 2R+2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 4R+8 \\ 2R+8 \end{Bmatrix} \tag{5}$$

Solving the linear non-homogenous equations of (5)

$$\begin{aligned} x_1 &= \frac{2R+5}{R+2} \\ x_2 &= \frac{R+3}{R+2} \end{aligned} \tag{6}$$

The limiting values of equation (6) are obtained as the controlling parameter R tends to zero. Thus, the stationary values can be obtained as

$$\begin{aligned} x_1 &= \frac{5}{2} = 2.5 \\ x_2 &= \frac{3}{2} = 1.5 \end{aligned} \tag{7}$$

Since we had neglected the inequality constraint while determining the stationary values, the above stationary values can be checked for the inequality constraint: $x_1 + 2x_2 = 2.5 + 2 \times 1.5 = 5.5 \le 10$. In addition, the minimum occurs away from the boundary of the inequality.