1) Markov's inequality :

$$E(X) = \int_0^\infty x f_X(x)dx \geq \int_a^\infty x f_X(x)dx \geq \int_a a f_X(x)\,dx$$

$$E(X) \geq a \int_a^\infty f_X(x)dx = a\,P(X \geq a)$$

2) Chebyshev's inequality :

$$P(|X-E(X)| \geq \epsilon) \leq \frac{var(X)}{\epsilon^2}$$

non-trivial if $\epsilon^2 > var(X)$

Proof : $P(|X-E(X)| \geq \epsilon) = P(|X-E(X)|^2 \geq \epsilon^2)$

$= P(\underbrace{(X-E(X))^2}_{} \geq \epsilon^2)$ → $var(x) = Z$

Now apply Markov's inequality

$$P(Z \geq \epsilon^2) \leq \frac{E(Z)}{\epsilon^2}$$

$$P((X-E(X))^2 \geq \epsilon^2) \leq \frac{var(X)}{\epsilon^2}$$

3) Tighter Upper Bounds :

$$P(X \geq a) \leq \frac{E(f(x) \geq f(a))}{f(a)}$$ $f \to$ monotonically increasing, one-one

$f(x) = e^{tx}$ ← chernoff Bound.

$$P(X \geq a) = \frac{E(e^{tX} \geq e^{ta})}{e^{ta}}$$

for tightest bound ⇒ optimize wrt t.

eg. :-

X : exp λ ⇒ $E(e^{tX}) = \lambda/_{\lambda-t}$ if $t<\lambda$

RHS of chernoff bound $= \dfrac{\lambda}{e^{at}(\lambda-t)}$ ← minimize for $t = (\lambda - 1/a)$

Alternate proof of Markov's inequality:

if $X \geq 0$ & $a > 0$ then $P(X \geq a) \leq \frac{E(X)}{a}$

Proof: $1_{\{X \geq a\}} \leq \frac{X}{a}$

$E(1_{\{X \geq a\}}) \leq \frac{E(X)}{a} \Rightarrow P(X \geq a) \leq \frac{E(X)}{a}$

① Imp result: $E(X^2) = 0 \Rightarrow X = 0$ w.p. 1

② Imp result $P(|X - E(X)| \geq \epsilon) \leq \frac{var(X)}{\epsilon^2}$ $\quad$ $E(X^2) - [E(X)]^2 = 0$

$E(X^2) = [E(X)]^2$

$\Rightarrow X = E(X)$ w.p. 1

Characteristic fns. (helps in calc. moments)

$\Phi_X(\omega) = E(e^{j\omega X})$ $\qquad E(X+jY) = E(X) + jE(Y)$

$E(e^{j\omega X}) = E(\cos \omega X) + j\,E(\sin \omega X)$

→ For a complex fn. $g(X)$, $E(g(X)) = ?$

$$E(e^{j\omega x}) = \int_{-\infty}^{\infty} \cos(\omega x)\, f_X(x)\,dx + j\int_{-\infty}^{\infty} \sin(\omega x)\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} [\cos(\omega x) + j\sin(\omega x)]\, f_X(x)\,dx = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

⇒ this hold for complex $g(X)$.

F = fourier transform of $f_X(x)$ then $\Phi_X(\omega) = F(-\omega)$

Examples : (1) X: Binomial (n, p)

$$\Phi_X(\omega) = E(e^{j\omega X}) = \sum_{k=0}^{n} e^{j\omega k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} (e^{j\omega} p)^k \binom{n}{k} (1-p)^{n-k} = [pe^{j\omega} + 1 - p]$$

(2) $Y: N(0,1)$

$$\Phi_Y(\omega) = \int_{-\infty}^{\infty} e^{j\omega y} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega y - y^2/2}\,dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2 - 2j\omega y)}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y - j\omega)^2 - \frac{1}{2}\omega^2}\,dy$$

$$= \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}}\; e^{-\omega^2/2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-j\omega)^2}{2}}\,dy \right]$$

this turns out to be 1 even if the mean is replaced by a complex quantity.

Formal proof : split $e^{j\omega x}$ into $\cos \omega x + j\sin \omega x$

(3) $Y = aX + b$ $\qquad \Phi_Y(\omega) = ?$ $\qquad \Phi_Y(\omega) = E(e^{j\omega(aX+b)}) = e^{j\omega b} \cdot \Phi_X(a\omega)$

(4) $Y: N(\mu, \sigma^2)$ we can make it standard normal by the transformation:

$X = \dfrac{Y - \mu}{\sigma}$ $\qquad X: N(0,1)$ $\qquad \Phi_Y(\omega) = e^{j\omega\mu}\, e^{-\frac{\sigma^2}{2}\omega^2}$

Does $\phi_X(\omega)$ always exists?

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

similar to fourier transform

$\therefore$ existance is guaranteed if ~~the~~ absolute integrability.

and $\int_{-\infty}^{\infty} |e^{j\omega x}| f_X(x) dx < \infty$

$\Rightarrow$ $\phi_X(\omega)$ exists

$\phi_Z(\omega) = ?$ if $Z = X+Y$ X, Y are independant

$$\phi_Z(\omega) = E\left(e^{j\omega(X+Y)}\right) = E\left(e^{j\omega X} \cdot e^{j\omega Y}\right)$$

$$= E(e^{j\omega X})\, E(e^{j\omega Y}) = \phi_X(\omega) \cdot \phi_Y(\omega)$$

$E(g(X)\, h(Y)) = E(g(X))\, E(h(Y))$ if X, Y are independant

$f_Z(z) = f_X(x) * f_Y(y) \leftarrow$ similar to the F.T. property.

Ex. 1 X: Binomial (n, p) Y: Binomial (m, p)

By calc. $\phi_Z(\omega) = (pe^{j\omega} + 1 - p)^{n+m}$

$\Rightarrow$ Z : Binomial $(m+n, p)$

This follows from the property of characteristic functions to be unique.

2 $Z = aX + bY + c$ X, Y are ind. Gaussian.

$$\phi_Z(\omega) = e^{j\omega c}\, \phi_X(a\omega)\, \phi_Y(bY)$$

$X : N(\mu_1, \sigma_1^2)$ $Y : N(\mu_2, \sigma_2^2)$

$\phi_X(\omega) = \exp\left(j\omega\mu_1 - \frac{1}{2}\sigma_1^2\omega^2\right)$ ||ly $\phi_Y(\omega) = \exp\left(j\omega\mu_2 - \frac{1}{2}\sigma_2^2\omega^2\right)$

$$\phi_Z(\omega) = e^{j\omega c} \exp\left(ja\omega\mu_1 - \frac{1}{2}\sigma_1^2 a^2 \omega^2\right) \cdot \exp\left(jb\omega\mu_2 - \frac{1}{2}\sigma_2^2 b^2\omega^2\right)$$

$$= \exp\left[j\omega(a\mu_1 + b\mu_2 + c) - \frac{(a^2\sigma_1^2 + b^2\sigma_2^2)\,\omega^2}{2}\right]$$

$\Rightarrow$ Z is Gaussian with parameters.

Moments : $E(X^n) \leftarrow$ nth

$E(X) \leftarrow$ mean

Skewness = $E((X$

$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x$

Ex. 1 $Y : N(\mu, \sigma^2)$

tedious to fi

characteristic f

Moment generating f

$L(s) = \int_{-\infty}^{\infty}$

$\Lambda_X(s) = E($

$\Lambda_X(s) = \int_{-\infty}$

$\Lambda_X'(s) =$

we see that

ex. 1 $Y : N$

first fi

$\phi_X($

||ly

$E(X^4) = 3$

$E(Y^3) = ?$

... exists

... $f_X(x)\,dx$

... Gaussian

... exist if the absolute integrability ...
... $f_X(x)\,dx < \infty$

X, Y are independent

$E(e^{j\omega X} \cdot e^{j\omega Y})$
$= \Phi_X(\omega) \cdot \Phi_Y(\omega)$
... if X, Y are independent
... ← similar to the F.T. property

... (n, p)
... $e^{j\omega} + (1-p))^{n+m}$

... of characteristic

ind. Gaussian
$\Phi_Y(bY)$

... $\Phi_Y(\omega) = \exp\left(j\omega\mu_Y - \frac{1}{2}\sigma_Y^2\omega^2\right)$
$\exp\left(j b\omega\mu_2 - \frac{1}{2}\sigma_2^2 b^2\omega^2\right)$

$\frac{(a^2\sigma_1^2 + b^2\sigma_2^2)\ldots}{2}$

Moments: $E(X^n)$ = n^{th} moment of X. measure of asymmetry of f about m

$E(X)$ = mean $E(X^2)$ = spread $E(X^3)$ = skewness

$$\text{Skewness} = \frac{E((X-\eta_X)^3)}{\sigma^3}$$

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\,dx \quad \text{or} \quad \sum_i x_i^n p_i$$

Ex. (1) $Y: N(\mu, \sigma^2)$ $E(Y) = \mu$ $E(Y^2) = \mu^2 + \sigma^2$

tedious to find $E(Y^n)$ for $n \geq 3$ we can use characteristic functions.

Moment generating functions are similar to Laplace transform. generalisation to characteristic functions.

$$L(s) = \int_{-\infty}^{\infty} f(t) e^{-st}\,dt$$

$$\Lambda_X(s) = E(e^{sX}) \Rightarrow \Lambda_X(s) = L(-s)$$

$$\Lambda_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$$

$$\Lambda_X'(s) = \int_{-\infty}^{\infty} e^{sx} x f_X(x)\,dx$$ we see $\Lambda_X'(0) = E(X)$

we see that $\left(\frac{d}{ds}\right)^n \Lambda_X(0) = E(X^n)$

ex. (1) $Y: N(\mu, \sigma^2)$ $E(Y^3) = ?$ $E(Y^4) = ?$

first find $E(X^3)$ & $E(X^4)$ where $X = \frac{Y-\mu}{\sigma}$

$\Phi_X(\omega) = e^{-\omega^2/2}$

lly $\Lambda_X(s) = e^{s^2/2}$ $\Rightarrow E(X^3) = \left(\frac{d}{ds}\right)^3 \Lambda_X(s)\Big|_{s=0}$

$= \left(\frac{d}{ds}\right)^2 e^{s^2/2} \cdot s\Big|_{s=0}$ $= e^{s^2/2} + s^2 e^{s^2/2}$
$= (s^2+1) e^{s^2/2}$

$E(X^4) = 3$ lly

$= \frac{d}{ds}\left[(s^2+1)e^{s^2/2}\right]\Big|_{s=0} = 0$

$E(Y^3) = ?$ $Y^3 = (\sigma X + \mu)^3$

$Y^3 = \sigma^3 X^3 + \mu^3 + 3\sigma^2 X^2 \mu + 3\sigma X \mu^2$

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Order Statistics

min $(X_1, \dots, X_n)$; max$(X_1, \dots, X_n)$

Seller needs to select price P_n. Doesn't know others' prices.

Expected profit if $P = x$

$$\text{profit} = \begin{cases} x-c & \text{if good sold} \\ 0 & \text{if not sold} \end{cases}$$

Expected profit $= (x-c)\, \underbrace{P(\text{good is sold})}$

$$\underline{\underline{P(P_{n} > x)}} = 1 - F_{P_{n}}(x)$$

Let $X_1, \dots, X_n$ be iid and CDF being $F(\cdot)$

$X_{(k)}$: k^{th} smallest of $X_1, \dots, X_n$

$X_{(1)}, \dots, X_{(n)}$ are called order statistics of $X_1, \dots, X_n$

$P(X_{(k)} \le x) = ?$ $= P(k \text{ or more of } X_1, \dots, X_n \text{ are } \le x)$

$$= \sum_{i=k}^{n} P(\text{exactly } i \text{ out of } X_1, \dots, X_n \text{ are } \le x)$$

$$= \sum_{i=k}^{n} \binom{n}{i} [F(x)]^{i} [1 - F(x)]^{n-i}$$

independent

independent

$Y_1 = X_1$	$x_1 = y_1$	$f_{X_1, \dots, X_n}(y_1, \dots, y_n)$
$Y_2 = X_1 + X_2$	$x_2 = y_2 - y_1$	
$Y_3 = X_1 + X_2 + X_3$	$x_3 = y_3 - y_2$	$J(x_1, \dots, x_n) = \begin{vmatrix} 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 0 \cdots \\ \vdots \\ 1 \end{vmatrix}$
$\vdots$	$\vdots$	
$Y_n = X_1 + X_2 + \dots + X_n$	$x_n = y_n - y_{n-1}$	$= 1$

Mean & Variance of Binomial R.V.

Coupon collector's problem

$N \rightarrow$ types of coupons

equally likely

X: No. of coupons he needs to collect until a complete set of at least one of each type has been obtained.

$N = 10$ $\quad X = X_1 + X_2 + X_3 + \cdots + X_N$

$$\underbrace{5}_{X_1 = 1},\ \underbrace{5,\ 7}_{X_2 = 2},\ \underbrace{7,\ 5,\ 5,\ 7,\ 1}_{X_3 = 5},\ \underbrace{5,\ 7,\ 1,\ 3}_{X_4 = 4},\ \underbrace{8}_{X_5 = 1},\ \ldots$$

$$P(X_i = 1) = \frac{N-(i-1)}{N} \qquad P(X_i = 2) = \frac{(i-1)}{N} \cdot \frac{N-(i-1)}{N}$$

$\Rightarrow X_i$ is geometric r.v.

we know $E(X)$ for geometric r.v. $= 1/p$

so here $E(X) = N \underbrace{\left(\sum_{j=1}^{N} 1/j\right)}$

$\simeq \underbrace{N \log N}_{\text{penality due to randomness}}$ for large N

Variance of sum of r.v.s

$X_1, X_2, \ldots, X_n \rightarrow$ independant r.v.s

$\mu_1, \mu_2, \ldots, \mu_n \rightarrow$ means

$$var(\underbrace{X_1 + X_2 + \cdots + X_n}_{X}) = ?$$

$$E(X) = \mu_1 + \mu_2 + \cdots + \mu_n$$

$$X - E(X) = (X_1 - \mu_1) + (X_2 - \mu_2) + \cdots + (X_n - \mu_n)$$

$$var(X) = E\left[\left((X_1-\mu_1) + (X_2-\mu_2) + \cdots + (X_n - \mu_n)\right)^2\right]$$

$$= E\left[\sum_{i=1}^{n} (X_i - \mu_i)^2 + \sum_{1 \le i < j \le n} (X_i - \mu_i)(X_j - \mu_j)\right] \quad 0$$

covar of $X_i, X_j = 0$

$\therefore \Rightarrow var(X) = var(X_1) + var(X_2) + \cdots + var(X_n)$

a complete
een obtained.

$\frac{N-(i-1)}{N}$

rge N
e its randomnes

$var(cZ) = c^2 var(Z)$

Estimation

$\bar{x} = \frac{x_1 + \dots + x_n}{n}$

$x_1, \dots, x_n$ are i.i.d. $\rightarrow$ unknown distribution.

want to ~~find~~ ~~mean~~ estimate for mean & variance.

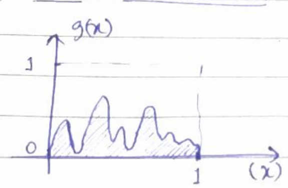
estimator for $\mu \rightarrow \bar{x} = \frac{x_1 + \dots + x_n}{n}$

$E(\bar{x}) = \mu$

$var(\bar{x}) = \frac{\sigma^2}{n}$ as $n \rightarrow \infty$, estimation becomes accurate.

estimator for $var(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Monte Carlo Simulation:



$x_i = \begin{cases} 1 & \text{if } i^{th} \text{ pt. is in shaded rgn.} \\ 0 & \text{o.w.} \end{cases}$

$E(x_i) = \int_0^1 g(x)\, dx$

use estimator and draw large no. of random values.