Assignment 2
AIP

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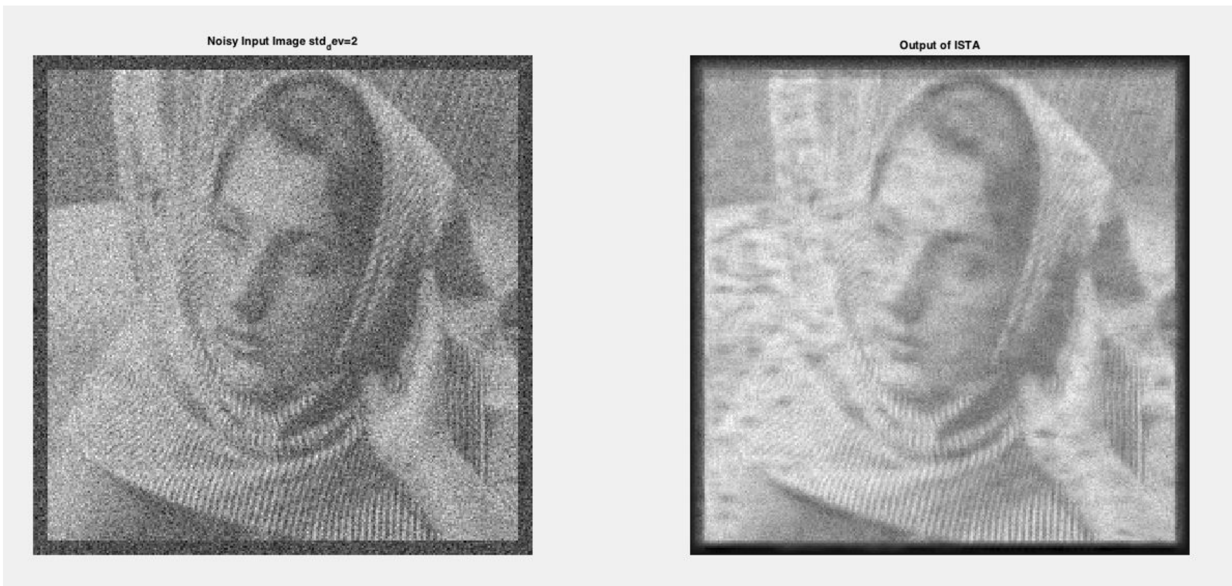
Question 1)

Part a)

We used identity matrix as the “phi”. “Psi” was made using the kron delta function.

Results of denoising ->



The RMSE error of the output image was 0.14

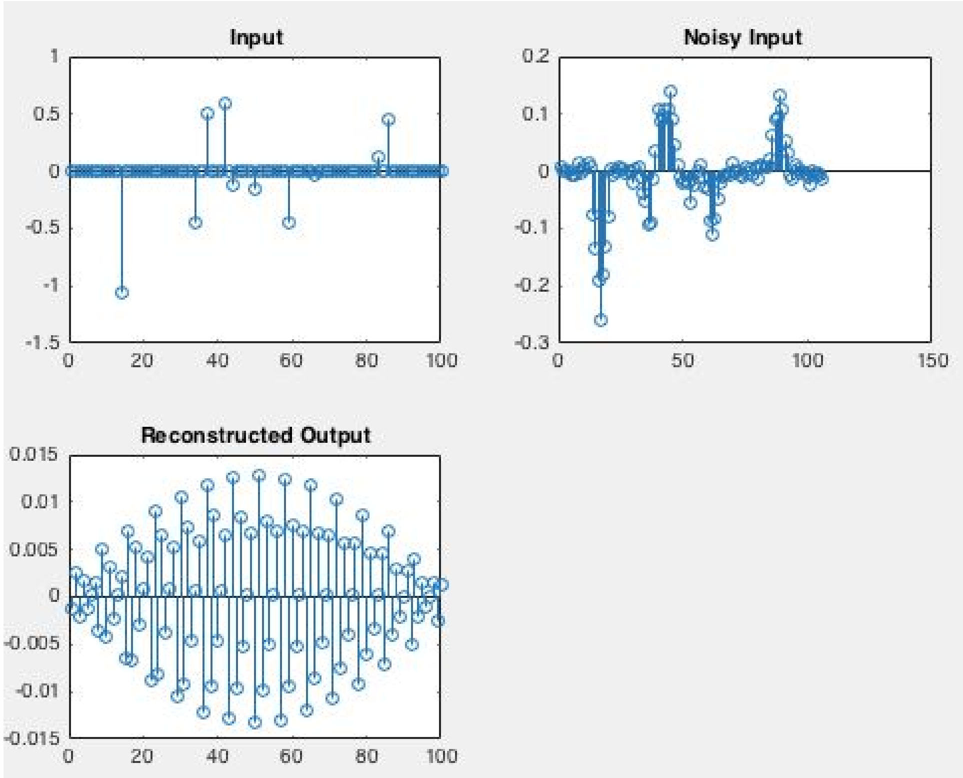
Part b) The phi matrix was generated randomly.

The RMSE error of the output image was 0.14

Part c) This part took a bit of effort. dwt2(X,'db1') function was used to model haar wavelets for psi. I created a separate function get_psi which basically returns the haar coefficients of the input. The dwt2 function returns 4 matrices of half the size of input
[cA, cH, cV, cD] .
Let us say input is of size (64,1), the output will be 4 vectors of (32,1). I take the average of cH, cV, cD and append them to cA to get an output vector of (64,1). This is then multiplied by the psi matrix.
Alpha value was chosen to be same as the part b

Unfortunately, my code was not converging and hence I failed to get a desired output.

Part d) This part involved a lot of parameter tuning. The A matrix was chosen as a circulant matrix. But a lot of parameter tuning was required.
Outputs ->



Q2 We start with proowing Theorem ~~1,2~~ 3.

Question in theorem 3 is →

$$\min_{\tilde{x} \in \mathbb{R}^n} \|\tilde{x}\|_{\ell_1} \text{ subject to } \|y - \phi\tilde{x}\|_{\ell_2} < \varepsilon$$

ε is noisy contrib

→ Now let x^* be the optimal actual solution

$$\|\phi(x^* - x)\|_{\ell_2} = \|\phi x^* - y + y - \phi x\|_{\ell_2}$$

Applying triangle inequality $[\|v+w\|_2 < \|v\|_2 + \|w\|_2]$

$$\|\phi(x^* - x)\|_2 \le \|\phi x^* - y\|_2 + \|y - \phi x\|_2 \le 2\varepsilon$$ (1)

The 2nd inequality follows from the problem posed in theorem 3, i.e. $\|y - \phi x\|_2 < \varepsilon$ & x^* & x are solutions to this.

$y = \phi x + \varepsilon$ & x^* is the optimal soln.

* $x^* = x + h$, decompose h to h_{T_0} h_{T_1} . . .

$$\|h_{T_j}\|_2 = \sqrt{h_{T_{j1}}^2 + h_{T_{j2}}^2 + \dots} \le \sqrt{\Delta \times h_{T_j \max}^2}$$

$$= \Delta^{1/2} h_{T_j \max}$$ (2)

$$= \Delta^{1/2} \|h_{T_j}\|_\infty$$

($h_{T_{ji}}$ = i^{th} element

$h_{T_j \max}$ = max element)

* All elements of h_{T_j} are smaller then $h_{T_{j-1}}$

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$$\Rightarrow h_{T_j\,max} \le \frac{h_{T_{j-1}1} + h_{T_{j-1}2} + \dots}{s}$$

(sum of all elements of $h_{T_{j-1}}$ divided by number of non zero elements "s").

$$\Rightarrow \|h_{T_j}\|_\infty \le \frac{\|h_{T_{j-1}}\|_{\ell_1}}{s} \qquad (2)$$

$$\Rightarrow \|h_{T_j}\|_2 \le s^{1/2}\|h_{T_j}\|_\infty \le s^{-1/2}\|h_{T_{j-1}}\|_1$$

summing both sides $j \ge 2$

$$\sum_{j=2} \|h_{T_j}\|_2 \le s^{-1/2}\|h_{T_1}\|_1 + s^{-1/2}\|h_{T_2}\|_1 + \dots \cancel{s^{-1/2}}$$

$$= s^{-1/2}\Big[\underbrace{\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \|h_{T_3}\|_1 + \dots}_{\text{sum of all element not in } h_{T_0}}\Big] \qquad (3)$$

$$= s^{-1/2}\|h_{T_0^c}\|_1$$

4th question follows simply from repeated application of triangle ineq.

$$\|h_{T_0 \cup T_1^c}\|_2 = \|\sum_{j\geq 2} h_{T_j}\|_2 \leq \sum_{j\geq 2} \|h_{T_j}\|_2 \underbrace{\leq}_{\text{from } 1^{st} \,\&\, 3^{rd} \text{ Term of previous inequality}} \Delta^{-1/2} \|h_{T_0^c}\|_1$$

(5) $$\|x+h\|_1 = \underbrace{\sum_{i\in T_0} |x_i + h_i|}_{\text{from reverse triangle inequality}} + \sum_{i\in T_0^c} |x_i + h_i|$$

$$\|v-w\|_2 \geq \|v\|_2 - \|w\|$$

$$\cancel{\|x+h\| \geq \|x_{T_0}\|_1 - \|}$$

$$\|x+h\|_1 = \|x_{T_0} - (-h_{T_0})\|_1 + \|h_{T_0^c} - (-x_{T_0^c})\|_1$$

$$\|x+h_i\|_1 \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|$$

(6) From previous inequality, rearranging 1st & last part.

$$\|x\|_1 \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1$$

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + \|x_{T_0^c}\|_1 + \underbrace{\|x\|_1 - \|x_{T_0}\|_1}_{\text{apply triangular ineq.}}$$

$$\|h_{T_0^c}\| \leq \|h_{T_0}\|_1 + \|x_{T_0^c}\|_1 + \|x - x_{T_0}\|_1$$

$$\|h_{T_0^c}\| \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$

(7) Using eq^n 12 in eq^n 11.

$$\|h_{(T_0 \cup T_1)^c}\| \leq \Delta^{-1/2} \|h_{T_0^c}\|_1$$

$$= \Delta^{-1/2} \|h_{T_0}\|_1 + 2\Delta^{-1/2} \|x_{T_0^c}\|_1$$

(Cauchy Schwarz, for any vector v, $\|v\|_1 < \sqrt{n}\, \|v_2\|$

$\Rightarrow \|h_{T_0}\|_1 < \sqrt{\Delta}\, \|h_{T_0}\|_2$

$$\Rightarrow \|h_{(T_0 \cup T_1)^c}\| \leq \|h_{T_0}\|_2 + 2\varepsilon_0$$

where $\varepsilon_0 = \Delta^{-1/2} \|x - x_\Delta\|_1$

$\|_2$

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(8) Proceeding to 2^{nd} part of proof

$\|\phi h_{T_0 \cup T_1}\|_2$, from RIP we have

$$\|\phi x\|_2 \leq \sqrt{1+\delta_s}\,\|x\|_2$$

where "s" is the ~~huge~~ sparsity of x.

$h_{T_0 \cup T_1}$ has a sparsity of $2s$, since T_0 are the largest coefficient of h & T_1 are the next s largest.

$\therefore$ ~~$\|\phi$~~ $\|\phi h_{T_0 \cup T_1}\|_2 \leq \sqrt{1+\delta_{2s}}\,\|h_{T_0 \cup T_1}\|_2$

also $\|\phi h\|_2 = \|\phi(x^* - x)\| \leq 2\varepsilon$ as proved in Qn 1.

$\therefore \|\phi h_{T_0 \cup T_1}\|_2 \|\phi h\|_2 \leq 2\varepsilon\sqrt{1+\delta_{2s}}\,\|h_{T_0 \cup T_1}\|$

(9) Lemma 2.1

$$|\langle \phi x, \phi x' \rangle| \leq \delta_{s+s'}\,\|x\|_2\,\|x'\|_2$$

where s & s' are sparsities.

$\therefore |\langle \phi h_{T_0}, \phi h_{T_j} \rangle| \leq \delta_{s+s}\,\|h_{T_0}\|_2\,\|h_{T_j}\|_2$

~~where~~ because both h_{T_0} & h_{T_j} are s sparse from definition.

(10) $\|h_{T_0}\|_2 + \|h_{T_1}\|_2$

Now, $\|h_{T_0 \cup T_1}\|_2 = \|h_{T_0} + h_{T_1}\|$ (since T_0 & T_1 are disjoint)

$$\|h_{T_0} + h_{T_1}\|_2 = \sqrt{h_{T_0}^2 + h_{T_1}^2 + \cancel{2h_{T_0}h_{T_1}}}$$

$$= \sqrt{h_{T_0}^2 + h_{T_1}^2}$$

Now since root mean square > arithmatic mean

$$\sqrt{\frac{h_{T_0}^2 + h_{T_1}^2}{2}} \geq \frac{\|h_{T_0}\|_2 + \|h_{T_1}\|_2}{2}$$

$$\Rightarrow \|h_{T_0}\|_2 + \|h_{T_1}\| \leq \sqrt{2}\sqrt{h_{T_0}^2 + h_{T_1}^2}$$

$$= \sqrt{2}\,\|h_{T_0} + h_{T_1}\|_2$$

$$= \sqrt{2}\,\|h_{T_0 \cup T_1}\|_2$$

(11) The ~~LHS~~ left inequality follows from restricted isometry property.

$h_{T_0 \cup T_1}$ is a $2s$ sparse vector.

$$\therefore \|\phi h_{T_0 \cup T_1}\|_2^2 \geq (1 - \delta_{2s})\|h_{T_0 \cup T_1}\|_2^2$$

→ For right, we have

$$\|\phi h_{T_0 \cup T_1}\|^2 = \underbrace{\langle \phi h_{T_0\cup T_1}, \phi h \rangle}_{\le 2\varepsilon\sqrt{1+\delta_{2\Delta}}\,\|h_{T_0\cup T_1}\|_2} - \langle \phi h_{T_0\cup T_1}, \sum_{j\ge 2} \phi h_{T_j} \rangle$$

$\le 2\varepsilon\sqrt{1+\delta_{2\Delta}}\,\|h_{T_0\cup T_1}\|_2$
(Qn 8) — (a)

Now,

$$\langle \phi h_{T_0\cup T_1}, \sum_{j\ge 2} \phi h_{T_j} \rangle = \langle \phi h_{T_0}, \sum_{j\ge 2} \phi h_{T_j} \rangle + \langle \phi h_{T_1}, \sum \phi h_{T_j} \rangle$$

$$\langle \phi h_{T_0}, \sum \phi h_{T_j} \rangle \le \delta_{2\Delta} \|h_{T_0}\|_2 \sum_{j\ge 2} \|h_{T_j}\|_2 \quad \text{— (b)}$$

from a & b

$$\Rightarrow \|\phi h_{T_0\cup T_1}\|_2^2 \le 2\varepsilon\sqrt{1+\delta_{2\Delta}}\,\|h_{T_0\cup T_1}\| + \delta_{2\Delta} \sum_{j\ge 2} \|h_{T_j}\|_2 \left(\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \right)$$

$$< 2\varepsilon\sqrt{1+\delta_{2\Delta}}\,\|h_{T_0\cup T_1}\| + \sqrt{2}\,\delta_{2\Delta} \sum_{j\ge 2} \|h_{T_j}\|_2 \, \|h_{T_0\cup T_1}\|$$

$$\Rightarrow \|\phi h_{T_0\cup T_1}\|_2^2 \le \|h_{T_0\cup T_1}\|_2 \left(2\varepsilon\sqrt{1+\delta_{2\Delta}} + \sqrt{2}\,\delta_{2\Delta} \sum_{j\ge 2} \|h_{T_j}$$

(12) From Qn 11 & Qn 4

comparing 1^{st} & last term

$$(1-\delta_{2\Delta})\|h_{T_0 \cup T_1}\|_2 \leq 2\varepsilon\sqrt{1+\delta_{2\Delta}} + \sqrt{2}\,\delta_{2\Delta}\underbrace{\sum_{j\geq 2}\|h_{T_j}\|_2}_{Qn\,4}$$

l_2

$$\|h_{T_0 \cup T_1}\|_2 \leq \frac{2\varepsilon\sqrt{1+\delta_{2\Delta}}}{1-\delta_{2\Delta}} + \frac{\sqrt{2}\,\delta_{2\Delta}}{1-\delta_{2\Delta}}\,\Delta^{-1/2}\|h_{T_0^c}\|_1$$

Hence Prooved.

(13) Substituting Qn 6 in previous result

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha\varepsilon + \rho\Delta^{-1/2}\|h_{T_0}\|_{l_1} + \rho\Delta^{-1/2} \times 2\underbrace{\|x_{T_0^c}\|_1}_{\|x-x_\Delta\|_1}$$

$$= \alpha\varepsilon + 2\rho e_0 + \rho\Delta^{-1/2}\underbrace{\|h_{T_0}\|_1}_{} \quad \text{— (a)}$$

k

root mean sq > arithmatic mean

$$\sqrt{\frac{a^2+b^2+c^2 \ldots}{\Delta}} > \frac{a+b+c \ldots}{\Delta}$$

$$\sqrt{a^2+b^2+c^2 \ldots} > \sqrt{\Delta}\,(a+b+c \ldots)$$

$\|h_{T_0 \cup T_1}\|_2 > \|h_{T_0}\|_2 > \sqrt{n}\,\|h_{T_0}\|_1$ — (b)

join (a) & (b)

$\|h_{T_0 \cup T_1}\|_2 < \alpha\varepsilon + 2\rho e_0 + \rho\|h_{T_0 \cup T_1}\|_2$

(14) First ineq., is simply applying Triangle ineq. on h (15)

$$h = h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}$$

$$\|h\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \underbrace{\|h_{(T_0 \cup T_1)^c}\|_2}_{\text{using result from Q7}}$$

$$\|h\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + 2e_0 + \|h_{T_0}\|_2$$

$$\leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0 \qquad (\|h_{T_0}\|_2 < \|h_{T_0 \cup T_1}\|_2)$$

Now applying result from previous qn.

$$\|h\|_2 \leq 2(1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0) + 2e_0$$

$$= 2(1-\rho)^{-1}\alpha\varepsilon + \frac{4\rho e_0}{1-\rho} + 2e_0$$

$$= 2(1-\rho)^{-1}\alpha\varepsilon + \frac{2\rho e_0 + 2e_0}{1-\rho}$$

$$= 2(1-\rho)^{-1}\left[\alpha\varepsilon + (1+\rho)e_0\right]$$

neq. on h

$\|_2 < \|h_{T_0 \cup T_1}\|_2)$

(15) From eq^n 15

$$\|h_{T_0}\|_1 \leq \rho \|h_{T_0^c}\|_1$$

also $$\|h_{T_0^c}\|_1 \leq 2(1-\rho)^{-1} \|x_{T_0^c}\|_1$$

$$\Rightarrow \|h\|_1 = \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1$$

$$\leq (1+\rho) \|h_{T_0^c}\|_1$$

$$\leq 2(1+\rho)(1-\rho)^{-1} \|x_{T_0^c}\|_1$$

Hence Prooved

Q.3] a) x is a purely sparse vector.

$y = \phi x + \eta$ s.t. $\|\eta\|_2 \leq \varepsilon$

Let oracular solution be represented by $\tilde{x}$.

we know the indices (set S) of non-zero elements of signal x.

$\therefore$ Let ϕ_s represent matrix where columns not belonging to set S are zero. (1)

$\therefore$ $y = \phi_s \tilde{x}$

multiplying on both sides by ϕ_s^T

$\therefore$ $\phi_s^T y = \phi_s^T \phi_s \tilde{x}$

$\therefore$

$\tilde{x} = (\phi_s^T \phi_s)^{-1} \phi_s^T y$ (given that $\phi_s^T \phi_s$'s inverse exists)

$\therefore$ $\boxed{\tilde{x} = \phi_s^{+} y}$ (2)

b) $\|\tilde{x} - x\|_2 = \|\phi_s^{+} y - x\|_2$ (from 2)

$= \|\phi_s^{+}(\phi x + \eta) - x\|_2$

$= \|(\phi_s^T \phi_s)^{-1} \phi_s^T \phi x + \phi_s^{+} \eta - x\|_2$ (3)

We know that x is purely sparse and indices (set S) where it is non zero.

Definition of ϕ_s is given at (1).

$\therefore$ $\phi_s x = \phi x$ (4)

$\therefore$ from (3) and (4), $\|\tilde{x} - x\|_2 = \|(\phi_s^T \phi_s)^{-1} \phi_s^T \phi_s x + \phi_s^{+} \eta - x\|_2$

$$\therefore \|\tilde{x}-x\|_2 = \|x - \phi^+\eta - x\|_2$$

$$= \|\phi^+\eta\|_2$$

$$\leq \|\phi^+_s\|_2 \|\eta\|_2$$ (by Cauchy-Schwarz inequality)

(c) for a matrix M, if $M = USV^T$

then, $M^{-1} = VS^{-1}U^T$

therefore if λ is minimum singular value of M,

$\lambda' = \frac{1}{\lambda}$ is maximum singular value of M^{-1}; (5)

Also, from (4), we have $\phi_s x = \phi x$ (6)

similarly, $\tilde{\phi}_s \tilde{x} = \phi \tilde{x}$ (7)

because $\tilde{x}$ is zero at indices in set S.

from RIP property,

$$(1-\delta_{2k})\|\tilde{x}-x\|_2^2 \leq \|\phi(\tilde{x}-x)\|_2^2 \leq (1+\delta_{2k})\|\tilde{x}-x\|_2^2$$

$$\therefore 1-\delta_{2k} \leq \left(\frac{\|\phi(\tilde{x}-x)\|_2}{\|\tilde{x}-x\|_2}\right)^2 \leq 1+\delta_{2k}$$

Taking minimum over all $\|\tilde{x}-x\|_2^2$,

$$1-\delta_{2k} \leq \lambda^2 \leq 1+\delta_{2k}$$

$$\therefore 1-\delta_{2k} \leq \frac{1}{\lambda'^2} \leq 1+\delta_{2k} \quad \text{(from 5)}$$

$$\therefore \boxed{\frac{1}{\sqrt{1+\delta_{2k}}} \leq \lambda' \leq \frac{1}{\sqrt{1-\delta_{2k}}}}$$

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(d) $\|x - \tilde{x}\|_2 = \|\phi^{+}\eta\|_2$

$= \dfrac{\|\phi^{+}\eta\|_2}{\|\eta\|_2} \|\eta_2\|$

taking max over $\|\eta\|_2$

$\max_{\|\eta\|_2} \|x - \tilde{x}\|_2 = \lambda' \varepsilon$

but

$\dfrac{1}{\sqrt{1+\delta_{2k}}} \le \lambda' \le \dfrac{1}{\sqrt{1-\delta_{2k}}}$

$\therefore \dfrac{1}{\sqrt{1+\delta_{2k}}} \le \lambda\varepsilon \le \dfrac{\varepsilon}{\sqrt{1-\delta_{2k}}}$

$\therefore \boxed{\dfrac{\varepsilon}{\sqrt{1+\delta_{2k}}} \le \|x - \tilde{x}\|_2 \le \dfrac{\varepsilon}{\sqrt{1-\delta_{2k}}}}$

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Qn.4] Quantifying the performance of compressive sensing on scalp EEG signals.

This paper compares implementation of various compressive sensing techniques on EEG signals.

EEG signals are electric signals generated by brain. They are recorded by placing electrodes on scalp. To make low battery, portable devices with less memory, it is quite useful to employ some compressing technique.
This paper explores various basis/dictionaries to see which one provides a sparser representation of EEG signals. They are as follows:
(1) Gabor dictionary (gaussian envelope sinusoidal pulse)
(2) Mexican hat
(3) Linear and Cubic Spline
(4) Linear and Cubic B-Spline

Optimization function is $\min_{s \in \mathbb{R}^N} \|s\|_{\ell_1}$ such that $y_i = \langle \phi_i, \Psi s \rangle$
(Same as basis pursuit problem ~~show~~ taught in class)
various techniques are compared. They are: basis pursuit (linear programming), matching pursuit and orthogonal matching pursuit.

It can be seen from results that BP algo performs better but takes more time as compared to MP and OMP.
Cubic B-spline and gabor dictionaries provide the the sparsest representation of EEG signals.