CS663
Assignment 4 - Q6

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Q6 a) We have that A is a (m,n) matrix. $P = A^T A$ is a (m,m) matrix. Hence y is a $(m,1)$ vector. We have,

$$\begin{aligned} y^t P y &= y^t A^T A y \\ &= (Ay)^T (Ay) \\ &= ||Ay||_2^2 \\ &\geq 0 \end{aligned}$$

Similarly, for $Q = AA^T$,

$$\begin{aligned} z^t Q z &= z^t A A^T z \\ &= (A^T z)^T (A^T z) \\ &= ||A^T z||_2^2 \\ &\geq 0 \end{aligned}$$

Hence both P and Q are positive semi-definite. For an eigenvalue λ and eigenvector v (assume $||v||_2 = 1$),

$$\begin{aligned} \lambda v &= P v \\ \lambda v^T v &= v^T P v \\ \lambda &= v^T P v \geq 0 \end{aligned}$$

Hence all eigenvalues are non-negative.

Q6 b) We have,

$$\begin{aligned} \lambda u &= P u \\ \lambda A u &= A P u \\ \lambda (Au) &= A A^T (Au) \\ \lambda (Au) &= Q (Au) \end{aligned}$$

Hence Au is an eigenvector for Q, with the same eigenvalue. Similarly,

$$\begin{aligned} \mu v &= Q v \\ \mu A^T v &= A^T Q v \\ \mu (A^T v) &= A^T A (A^T v) \\ \mu (A^T v) &= P (A^T v) \end{aligned}$$

Hence $A^T v$ is an eigenvector for P with same eigenvalue. u is a $(n, 1)$ sized vector and v is a $(m, 1)$ sized vector.

Q6 c) We have,

$$\begin{aligned}
& \mu v_i = Qv_i \text{ (definition of eigenvector)} \\
\Rightarrow \quad & \mu v_i = AA^T v_i \\
& Au_i = \frac{1}{||A^T v_i||} A(A^T v_i) \\
\Rightarrow \quad & \quad = \frac{\mu}{||A^T v_i||} v_i = \gamma_i v_i
\end{aligned}$$

Clearly, $\gamma_i \geq 0$ since $\mu \geq 0$ and $||A^T v_i|| \geq 0$.

Q6 d) From the previous analysis we have,

$$\begin{aligned}
& Au_i = \gamma_i v_i \\
\Rightarrow \quad & A[u_1 u_2 ... u_m] = [\gamma_1 v_1 \gamma_2 v_2 ... \gamma_m v_m] \\
\Rightarrow \quad & AV = U\Gamma \\
\Rightarrow \quad & A = U\Gamma V^{-1} \\
\Rightarrow \quad & A = U\Gamma V^T \text{ since } V \text{ is orthonormal}
\end{aligned}$$

Hence we show the existence of SVD. In the case $m > n$, the last few γ_i are zero. In the case $n > m$, the remaining u_i can be formed using Gram Schmidt process.