CS663
Assignment 4 - Q5

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Our minimizing objective function is J such that (ignoring the terms not having $\boldsymbol{e}$),

$$J(\boldsymbol{e}) = -\boldsymbol{e}^T S \boldsymbol{e}$$

We have two constraints,

$$\boldsymbol{e}^T \boldsymbol{e} = 1$$
$$\boldsymbol{e}^T \boldsymbol{e_1} = 0$$

Writing a Lagrange multiplier equation (which we try to maximize),

$$J(\boldsymbol{e}) = \boldsymbol{e}^T S \boldsymbol{e} - \lambda(\boldsymbol{e}^T \boldsymbol{e} - 1) - \mu(\boldsymbol{e}^T \boldsymbol{e_1})$$

Taking derivative with respect to $\boldsymbol{e}$ and equating to zero,

$$\begin{aligned}
0 &= 2S\boldsymbol{e} - 2\lambda\boldsymbol{e} - \mu\boldsymbol{e_1} \\
0 &= 2\boldsymbol{e_1}^T S\boldsymbol{e} - 2\lambda\boldsymbol{e_1}^T\boldsymbol{e} - \mu \text{ (Multiplying } \boldsymbol{e_1}^T \text{ on both sides)} \\
0 &= 2\boldsymbol{e_1}^T S^T\boldsymbol{e} - 0 - \mu \text{ (Since S is symmetric)} \\
0 &= 2\lambda_1\boldsymbol{e_1}^T\boldsymbol{e} - 0 - \mu \\
\therefore \mu &= 0 \\
\therefore \lambda &= \boldsymbol{e}^T S\boldsymbol{e}
\end{aligned}$$

Hence, once again we need λ to be an eigenvalue and $\boldsymbol{e}$ to be an eigenvector. Since we cannot choose the largest eigenvalue (due to $\boldsymbol{e_1}^T\boldsymbol{e} = 0$), we use the second largest eigenvalue. $\boldsymbol{e}$ is the unit eigenvector for the second largest eigenvalue.