SET, RELATION AND FUNCTIONS

Single Type

1. Let $f: N \to N$ be defined as $f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$. Then f is

(A) one-to-one (B) one-to-one and onto.

(C) onto (D) none of these

2. For the sets of real numbers given by

$R_1 = \{(x, y) : x \in R,\ y \in R \quad x^2 + y^2 \le 25\}$

$R_2 = \{(x, y) : x \in R,\ y \in R,\ 9y \ge 4x^2\}$,

$R_1 \cap R_2$ is

(A) one-to-one (B) onto

(C) one-to-one and onto (D) none of these

3. If f : [0, ∞) → [0, ∞) and f (x) = $\frac{x}{1+x}$, then f is

(A) one-one and onto (B) one-one but not onto

(C) onto but not one-one (D) neither one-one nor onto

4. Range of $\sin^{-1}\left(\frac{x^2+1}{x^2+2}\right)$ is:

(A) [0, π/2] (B) (0, π/6)

(C) [π/6, π/2) (D) none of these

5. Let f : [-10,10] $\to$ R, where f(x) = sin x + $[x^2/a]$ be an odd function. Then set of values of parameter 'a' is/are:

(A) (-10, 10) - {0} (B) (0, 10)

(C) [100,∞) (D) (100, ∞)

6. Let f : [-π/3, 2π/3] $\to$ [0,4] be a function defined as f(x) = $\sqrt{3}$ sin x- cosx+2. Then $f^{-1}(x)$ is given by

(A) $\sin^{-1}\left(\frac{x-2}{2}\right)-\frac{\pi}{6}$ (B) $\sin^{-1}\left(\frac{x-2}{2}\right)+\frac{\pi}{6}$

(C) $\frac{2\pi}{3}+\cos^{-1}\left(\frac{x-2}{2}\right)$ (D) none of these

7. If fog = |sin x| and gof = $\sin^2\sqrt{x}$ then f(x) and g(x) are:

(A) f(x) = $\sqrt{\sin x}$, g(x) = x^2 (B) f(x) = |x|, g(x) = sin x

(C) f(x) = $\sqrt{x}$, g(x) = $\sin^2 x$ (D) f(x) = sin $\sqrt{x}$, g(x) = x^2

8. For a real valued function f(x) satisfying $2f(xy) = (f(x))^y + (f(y))^x$ for all x, y $\in$R and f(1) = a, (a $\neq$ 1), the expression (a - 1).$\sum_{i=1}^{n} f(i)$ + a is equal to

(A) $2a^n$ (B) a^{n+1}

(C) $2a^{n+1}$ (D) none of these

9. The domain of f(x) = $\sqrt{\log_{1/4}\left(\frac{5x-x^2}{4}\right)}$ + ${}^{10}C_x$ is

(A) (0, 1]U [4, 5) (B) (0, 5)

(C) {1, 4} (D) none of these

10. If $f(x) + 2f(1- x) = x^2 + 2 \ \forall \ x \in R$, then f(x) is given as

(A) $\frac{(x-2)^2}{3}$ (B) $x^2 - 2$

(C) 1 (D) none of these

11. If $f(x) = \begin{cases} x & when \quad x \quad is \quad rational \\ 1-x & when \quad x \quad is \quad irrational \end{cases}$, then fof (x) is given as

(A) 1 (B) x

(C) 1 + x (D) none of these

12 For $|x - 1| + |x - 2| + |x - 3| \leq 6$, x belongs to

(A) $0 \leq x \leq 4$ (B) $x \leq 0$ or $x \geq 4$

(C) $x \leq -2$ or $x \geq 4$ (D) none of these

13. The range of the function $y = \sqrt{-2\cos^2 x + 3\cos x - 1}$ is

(A) (0, 2) (B) $\left[\frac{1}{2\sqrt{2}}, 1\right]$

(C) $\left[0, \frac{1}{2\sqrt{2}}\right]$ (D) none of these

14. Given $f(x) = \ln\frac{1+x}{1-x}$ and $g(x) = \frac{3x + x^3}{1 + 3x^2}$. Then f(g(x)) is equal to

(A) – f(x) (B) 3f(x)

(C) $[f(x)]^2$ (D) none of these

15. f (x) = $\frac{\cos x}{\left[\frac{2x}{\pi}\right]+\frac{1}{2}}$, where x is not an integral multiple of π and [.] denotes the greatest integer function is

(A) an odd function (B) even function
(C) neither odd nor even (D) none of these

16. Domain of the function f (x) = $\frac{1}{\sqrt{{}^{10}C_{x-1} - 3.{}^{10}C_x}}$ contains the points

(A) 9, 10, 11 (B) 9, 10, 12
(C) all natural numbers (D) none of these

17. If a function satisfies the condition $f\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, $x \neq 0$, then domain of f (x) is

(A) [–2, 2] (B) $(-\infty, -2] \cup [2, \infty)$
(C) $(0, \infty)$ (D) none of these

18. Domain of the function f (x) = $\frac{x}{\sqrt{\sin(\ln x) - \cos(\ln x)}}$ is

(A) $(e^{2n\pi}, e^{(3n+1/2)\pi})$ (B) $(e^{(2n+1/4)\pi}, e^{(2n+5/4)\pi})$
(C) $(e^{(2n+1/4)\pi}, e^{(3n - 3/4)\pi})$ (D) none of these

19. If the functions f(x) and g(x) are defined on $R \rightarrow R$ such that

$f(x) = \begin{cases} 0, & x \in rational \\ x, & x \in irrational \end{cases}$, $g(x) = \begin{cases} 0, & x \in irrational \\ x, & x \in rational \end{cases}$, then $(f - g)(x)$ is

(A) one-one and onto (B) neither one-one nor onto

(C) one-one but not onto (D) onto but not one-one

20. Let $f : R \rightarrow (0,1]$ be defined by $f(x) = \frac{1}{x^2 + 1}$, then f is

(A) many one (B) into

(C) many one and into (D) one-one and onto

21. The number of functions *f* from the set A={0,1,2} in to the set B={0,1,2,3,4,5,6,7} such that $f(i) \le f(j)$ for i<j and $i, j \in A$ is

(A) ${}^{8}C_{3}$ (B) ${}^{8}C_{3} + 2({}^{8}C_{2})$

(C) ${}^{10}C_{3}$ (D) ${}^{10}C_{4}$

22. Let $f : \left[0, \sqrt{3}\right] \rightarrow \left[0, \frac{\pi}{3} + \log_e 2\right]$ defined by $f(x) = \log_e \sqrt{x^2 + 1} + \tan^{-1} x$ then $f(x)$ is

(A) one – one and onto (B) one – one but not onto

(C) onto but not one – one (D) neither one – one nor onto

Integer Type

23. If f is a function such that f(0) = 2, f(1) = 3 and f(x+2) = 2f(x) – f(x+1) for every real x then f(5) is

24. If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ and given that F(5) = 5, then F(10) is equal to

25. If f (x + y) = f(x) + f(y) – xy – 1 for all x, y, and f(1) = 1 then the number of Solutions of f(n) = n, $n \in N$, is

26. The period of the function $f(x) = 3\sin\frac{\pi x}{3} + 4\cos\frac{\pi x}{4}$ is

27. If the period of the function $f(x) = 3\sin\frac{\pi x}{3} + 4\cos\frac{\pi x}{4}$ is 3k then k, is

28. If the function f (x) = $x^3 + e^{\frac{x}{2}}$ *and* $g(x) = f^{-1}(x),$ then the value of $g'(1)$ *is*

29. The inverse of the function $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is $\frac{1}{m}\log_{10}\left(\frac{1+x}{1-x}\right)$, $m \neq 0$, then the value of m is

30. If $\log_2 x + \log_x 2 = \frac{5}{2} = \log_2 y + \log_y 2$ and $x \neq y$, then the value of $x + y - \sqrt{2}$ is

SOLUTIONS

Single Type

1. (C)

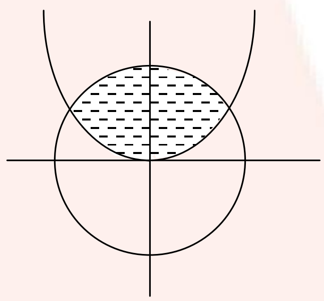
Here $f(3) = 2, f(4) = 2$. Hence f is not one-to-one. Also

$f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 2, f(5) = 3, f(6) = 3, ...$

$f(2n - 1) = n$ and f(2n) = n $\Rightarrow$ Range of f = N.

Hence (C) is correct answer.

2. (D)



Here $x^2 + y^2 \le 25$ are the elements of R_1 lying with in and on the circle $x^2 + y^2 = 25$, where as $4x^2 \le 9y$ are the elements of R_2 lying with in and on the parabola $4x^2 = 9y$.

Hence relation $R_1 \cap R_2$ is not a function.

Hence (D) is the correct answer.

3. (B)

The given function is f (x) = $\frac{x}{1+x} = 1 - \frac{1}{1+x}$.

Hence its range is [0, 1) which is a subset of [0, ∞).

Also the function is one-one.

Hence (B) is the correct answer.

4. (C)

Here, $\frac{x^2+1}{x^2+2} = 1-\frac{1}{x^2+2}$

Now, $2 \le x^2 + 2 < \infty$ for all $x \in R$

$\Rightarrow \frac{1}{2} \ge \frac{1}{x^2+2} > 0$

$\Rightarrow -\frac{1}{2} \le \frac{-1}{x^2+2} < 0$

$\Rightarrow \frac{1}{2} \le 1-\frac{1}{x^2+2} < 1$

$\Rightarrow \frac{\pi}{6} \le \sin^{-1}\left(1-\frac{1}{x^2+2}\right) < \frac{\pi}{2}$.

Hence (C) is the correct answer.

5. (D)

Since f(x) is an odd function,

$\left[\frac{x^2}{a}\right] = 0$ for all $x \in [-10,10]$

$\Rightarrow 0 \le \frac{x^2}{a} < 1$ for all $x \in [-10,10] \Rightarrow a > 100$.

Hence (D) is the correct answer.

6. (B)

$f(x) = \sqrt{3}\sin x - \cos x + 2 = 2\sin\left(x-\frac{\pi}{6}\right) + 2$.

Since f(x) is one-one and onto, f is invertible.

Now $fof^{-1}(x) = x \Rightarrow 2\sin\left(f^{-1}(x)-\frac{\pi}{6}\right) + 2 = x$

$\Rightarrow \sin\left(f^{-1}(x)-\frac{\pi}{6}\right) = \frac{x}{2} - 1 \Rightarrow f^{-1}(x) = \sin^{-1}\left(\frac{x}{2}-1\right)+\frac{\pi}{6}$, because $\left|\frac{x}{2}-1\right| \le 1$ for all $x \in [0, 4]$.

Hence (B) is the correct answer.

7. (C)

fog = $f(g(x)) = |\sin x| = \sqrt{\sin^2 x}$.

Also gof = $g(f(x)) = \sin^2\sqrt{x}$.

Obviously, $\sqrt{\sin^2 x} = \sqrt{g(x)}$

and $\sin^2\sqrt{x} = \sin^2(f(x))$

i.e. $g(x) = \sin^2 x$ and $f(x) = \sqrt{x}$.

Hence (C) is the correct answer.

8. (B)

We have $2f(xy) = (f(x))^y + (f(y))^x$.

Replacing y by 1, we get

$2f(x) = f(x) + (f(1))^x \Rightarrow f(x) = (f(1))^x \Rightarrow \sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} (f(1))^i = \sum_{i=1}^{n} a^i$

$= a + a^2 + a^3 + \dots + a^n = \frac{a-a^{n+1}}{1-a} \Rightarrow (a-1)\sum_{i=1}^{n} f(i) = a^{n+1} - a$

$\Rightarrow (a-1)\sum_{i=1}^{n} f(i) + a = a^{n+1}$.

Hence (B) is the correct answer.

9. (C)

Let $f_1 = \sqrt{\log_{1/4}\left(\frac{5x - x^2}{4}\right)}$ and $f_2 = {}^{10}C_x$.

Clearly f_1 is defined for $\log_{1/4}\left(\frac{5x - x^2}{4}\right) \geq 0$

$\Rightarrow 0 < \frac{5x - x^2}{4} \leq 1 \qquad \Rightarrow \frac{5x - x^2}{4} > 0$ and $\frac{5x - x^2}{4} \leq 1$

$\Rightarrow x\,(x - 5) < 0$ and $x^2 - 5x + 4 \geq 0$

$\Rightarrow x \in (0, 5)$ and $x \in (-\infty, 1] \cup [4, \infty)$

$\Rightarrow f_1$ is defined for $x \in (0, 1] \cup [4, 5)$ and

f_2 is defined for $x \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$\Rightarrow f(x)$ is defined for $x \in D_{f1} \cap D_{f2} = \{1, 4\}$.

Hence (C) is the correct answer.

10. (A)

By replacing x with (1 – x) in the given expression, we get

$f(1 - x) + 2\,f(1 - 1 + x) = (1 - x)^2 + 2$

$\Rightarrow f(1 - x) + 2\,f(x) = (1 - x)^2 + 2$

Now $f(x) + 2\,f(1 - x) - 2(f(1-x) + 2f(x))$

$= x^2 + 2 - 2((1 - x)^2 + 2) \Rightarrow -3\,f(x) = x^2 + 2 - 2(3 - 2x + x^2)$

$\Rightarrow 3\,f(x) = x^2 - 4x + 4 \Rightarrow f(x) = \frac{(x - 2)^2}{3}$.

Hence (A) is the correct answer.

11. (B)

$$\text{fof}(x) = \begin{cases} f(x) & \text{when } f(x) \text{ is rational} \\ 1-f(x) & \text{when } f(x) \text{ is irrational} \end{cases}$$

$$= \begin{cases} x & \text{when } x \text{ is rational} \\ 1-(1-x) & \text{when } x \text{ is irrational} \end{cases} = x.$$

Hence (B) is the correct answer.

12. (A)

We have $|x - 1+x - 2 + x - 3| \le |x - 1| + |x - 2| + |x - 3| \le 6$

$\Rightarrow |3x - 6| \le 6 \Rightarrow -6 \le 3x - 6 \le 6 \Rightarrow 0 \le x \le 4.$

Hence (A) is the correct answer.

13. (C)

Put $\cos x = t, -1 \le t \le 1$

$f(t) = y = \sqrt{-2t^2 + 3t - 1}$, $-2t^2 + 3t - 1 \ge 0 \Rightarrow \frac{1}{2} \le t \le 1$

$$\frac{dy}{dt} = \frac{-4t+3}{2\sqrt{-2t^2+3t-1}} = 0 \Rightarrow t = \frac{3}{4}$$

$$f\left(\frac{3}{4}\right) = \frac{1}{2\sqrt{2}} \text{ and } f\left(\frac{1}{2}\right) = f(1) = 0$$

Hence (C) is the correct answer.

14. (B)

$$f(g(x)) = \ln\frac{1+g(x)}{1-g(x)} = \ln\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}} = \ln\frac{(1+x)^3}{(1-x)^3} = 3\ln\frac{1+x}{1-x} = 3\, f(x).$$

Hence (B) is the correct answer.

15. (A)

Clearly $\left[-\frac{2x}{\pi}\right]+\frac{1}{2}=-\left(\left[\frac{2x}{\pi}\right]+\frac{1}{2}\right)$

$\Rightarrow$ f(x) is an odd function.

Hence (A) is the correct answer.

16. (D)

Given function is defined if $^{10}C_{x-1} > 3\ ^{10}C_x$

$\Rightarrow \frac{1}{11-x} > \frac{3}{x} \Rightarrow 4x > 33$

$\Rightarrow x \geq 9$ but $x \leq 10 \Rightarrow x = 9, 10.$

Hence (D) is the correct answer.

17. (B)

$f\left(x+\frac{1}{x}\right) = \left(x+\frac{1}{x}\right)^2 - 2 \Rightarrow f(y) = y^2 - 2$, where $y = x + \frac{1}{x}$

for $x > 0$, $y = x + \frac{1}{x} \geq 2$ and for $x < 0$, $y = x + \frac{1}{x} \leq -2$

Hence (B) is the correct answer.

18. (B)

For domain sin (ln x) > cos (ln x) and x > 0

$2n\pi + \frac{\pi}{4} < \ln x < 2n\pi + \frac{5\pi}{4}.\quad n \in I$

Hence (B) is the correct answer.

19. (A)

Let h(x) = f(x) – g(x) = $\begin{cases} x; & x \in \text{irrational} \\ -x; & x \in \text{rational} \end{cases}$

$\Rightarrow$ the function h(x) is one-one and onto.

Hence (A) is the correct answer.

20. (A)

f is many one as f is even. Further f is onto as range of f is (0, 1]

21. (C)

Hint: $0 < 1 < 2$

$\Rightarrow f(0) \le f(1) \le f(2)$

f(0)<f(1)<f(2) $\Rightarrow {}^8C_3$

f(0)<f(1)=f(2) $\Rightarrow {}^8C_2$

f(0)=f(1)<f(2) $\Rightarrow {}^8C_2$

f(0)=f(1)=f(2) $= {}^8C_1$

22. (A)

$f'(x) = \dfrac{x+1}{x^2+1} \Rightarrow f(x)$ *is increasing in* $\left[0, \sqrt{3}\right]$

Integer Type

23. (13)

For $x = 0$, $f(2) = 2f(0) – f(1) = 2 \times 2 – 3 = 1$,

for $x = 1$, $f(3) = 2f(1) – f(2) = 6 – 1 = 5$,

for $x = 2$, $f(4) = 2f(2) – f(3) = 2 \times 1 – 5 = -3$,

for $x = 3$, $f(5) = 2f(3) – f(4) = 2\times5 – (-3)= 13$.

24. (5)

$f''(x) = –f(x)$ and $f'(x) = g(x)$

$\Rightarrow \quad f''(x) \,.\, f'(x) + f(x) \,.\, f'(x) = 0$

$\Rightarrow f(x)^2 + (f'(x))^2 = c \;\Rightarrow\; (f(x)^2 + (g(x))^2 = c$

$\Rightarrow F(x) = c \;\Rightarrow\; F(10) = 5$.

25. (1)

Putting $y = 1$, $f(x + 1) = f(x) + f(1) – x – 1 = f(x) – x$.

$\therefore f(n + 1) = f(n) – n < f(n)$.

So, $f(n) < f(n – 1) < f(n – 2) < \ldots < f(1) = 1$

$\therefore f(n) = n$ holds for $n = 1$ only.

26. (24)

The period of $\sin\frac{\pi x}{3}$ is $\frac{2\pi}{\pi/3}$, i.e. 6. The period of $\cos\frac{\pi x}{4}$ is $\frac{2\pi}{\pi/4}$, i.e., 8.

LCM of 6 and 8 is 24. So, the period of $f(x) = 24$.

27. (8)

$3\sin\frac{\pi x}{4}$ has period 6

$4\cos \pi x/4$ has period 8

Net period 24

$3k = 24 \Rightarrow k = 8.$

28. (2)

Let $y = g(x) = f^{-1}(x)$. Then $x = 1 \Rightarrow y = 0\left(\sin ce\ f(0) = 1\right)$

$$g'(x) = \frac{1}{f'(y)} = \frac{1}{3y^2 + \frac{1}{2}e^{y/2}}$$

$\therefore g'(1) = 2$

29. (2)

$y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}} = \frac{10^{2x} - 1}{10^{2x} + 1}$ ... (i)

The function is one-one for if $y(x_1) = y(x_2)$

then $\frac{10^{2x_1} - 1}{10^{2x_1} + 1} = \frac{10^{2x_2} - 1}{10^{2x_2} + 1} \Rightarrow x_1 = x_2$

We have $\frac{1}{y} = \frac{10^{2x} + 1}{10^{2x} - 1}$

$\frac{1+y}{1-y} = \frac{10^{2x} + 1 + 10^{2x} - 1}{10^{2x} + 1 - 10^{2x} + 1} = 10^{2x}$

$\Rightarrow 2x = \log\left(\frac{1+y}{1-y}\right) \Rightarrow f^{-1}(y) = \frac{1}{2}\log_{10}\left(\frac{1+y}{1-y}\right)$

$\therefore m = 2$.

30. (4)

$\log_2 x + \log_x 2 = 5/2 = \log_2 y + \log_y 2$

$\Rightarrow t + \frac{1}{t} = \frac{5}{2}, s + \frac{1}{s} = \frac{5}{2}$ where $t = \log_2 x$,, $s = \log_2 y$

$\Rightarrow t = 2, 1/2$ & $s = 2, 1/2$ (as $t \neq s$, $\because x \neq y$)

$\therefore t = 2$ and $s = 1/2$ or $t = 1/2$ and $s = 3$

$\therefore \log_2 x = 2$ and $\log_2 y = 1/2$

$\therefore x = 4$ and $y = \sqrt{2}$

$\therefore x + y - \sqrt{2} = 4 + \sqrt{2} - \sqrt{2} = 4.$