LAB 1-THE PROPER ORTHOGONAL DECOMPOSITION

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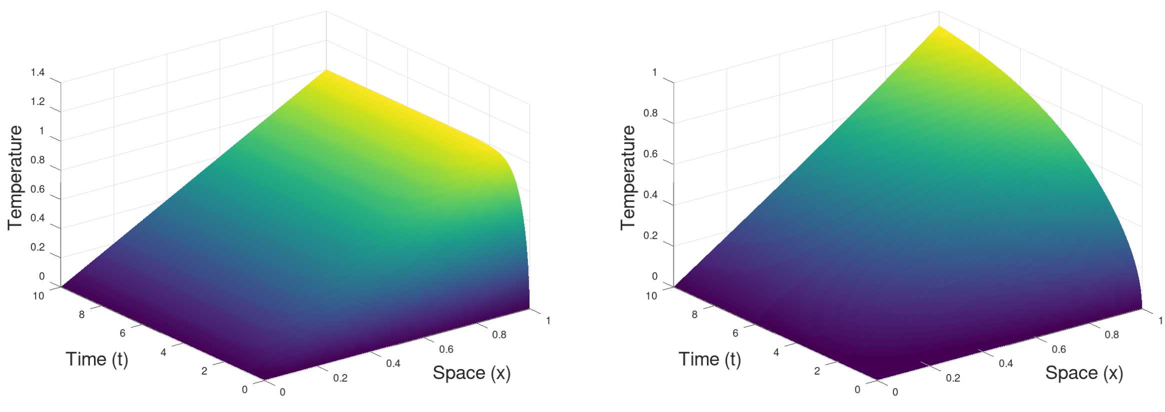
1 Introduction

Numerical Simulations in Engineering have ever-increasing importance in the field of aerospace and mechanical engineering to life sciences. Due to the progressive improvement in available computational power, nowadays its possible for numerical simulation of a complex problem with the help of Full-order approximation techniques such as Finite Element Method, Finite Difference Methods, and Finite Volume Method. To solve the problem of higher-order complexity, the computation time will be in several hours or days. Full order approximation techniques can become prohibitive when expected to quickly and efficiently process an iterative solution of PDE.So when the PDEs depend on the parameters,i.e, Parameterized PDEs. It is computationally expensive to evaluate such PDEs.So, Model order reduction can be used to replace the Full-order problem with a problem of lower complexity without simplifying the underlying physics.

Proper Orthogonal Decomposition (POD) is a model reduction method that is based on the posteriori approach. In this approach, the data is collected from the Full-order evaluation of the problem, and a reduced basis is obtained from it. The original variables are transformed into a new set of uncorrelated variables. Then, A lower-dimensional representation of the data is obtained by truncating the new basis to retain the more significant modes. In this report, the POD is being implemented on the transient heat equation.

2 Testing the FEM script

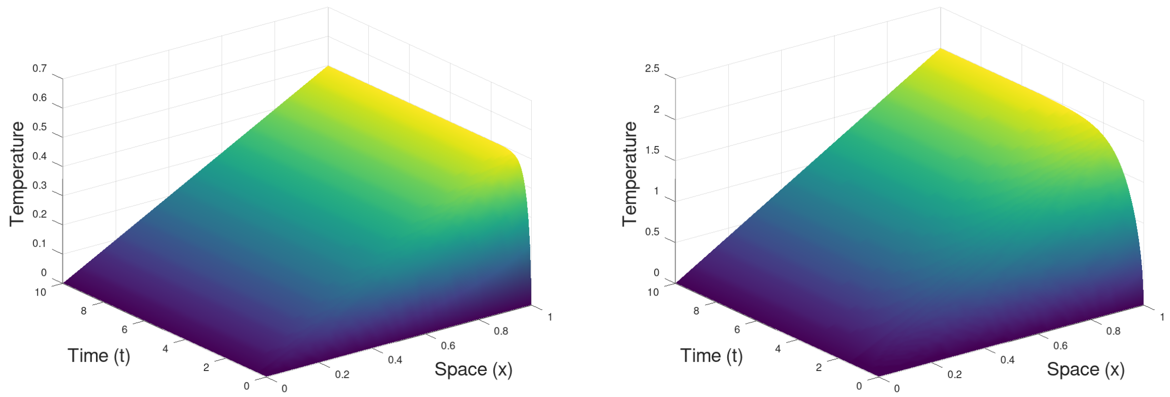
The FEM script was ran using different input arguments to check the correctness of the script.The problem is ran over a unit domain and over a period of 10 seconds.The boundary conditions are set as,Dirichlet condition is set to zero on the left end.Neumann condition is set to unit flux on the right end.The initial condition is set to zero uniform temperature.Theta(θ) is set to 1,that is implicit integration is used.Then,the values of k,ρ and cp are set as shown in the figure.Then the following surface plots are obtained.



(a) k=1,ρ=1,cp=1

(b) k=1,ρ=1,cp=10

Figure 1: FEM Solutions for varying specific heat (cp)



(a) k=2,ρ=1,cp=1 (b) k=0.5,ρ=1,cp=1

Figure 2: FEM Solutions for varying conductivity (k)

3 Solving the Full Order Model

The transient heat equation problem is solved using FEM. The solution of the Full order solution will be used as a reference and to compare the error with the POD solution. It is simulated for t=40 secs and the domain of unitary length is discretized using 51 nodes. The time step is 0.08 sec and the space step is 0.02. The material values are prescribed as k=1,ρ=1, and cp=50. The initial condition is uniform temperature everywhere. The boundary condition is prescribed as follows, heat flux is adiabatic on the left end and as shown in the figure below on the right end. The simulation is then ran and is compared with the reference plot to ensure the resemblance.

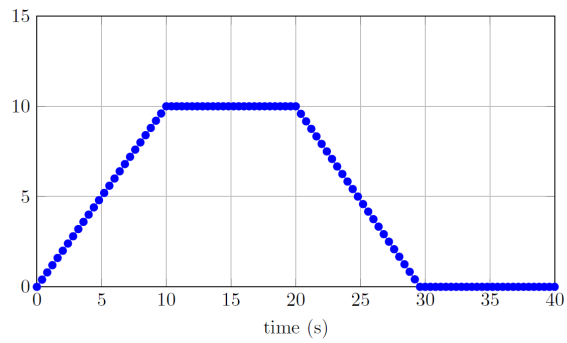


Figure 3: Heat flux on the right end of the bar

The Full-order solution surface plot obtained was:

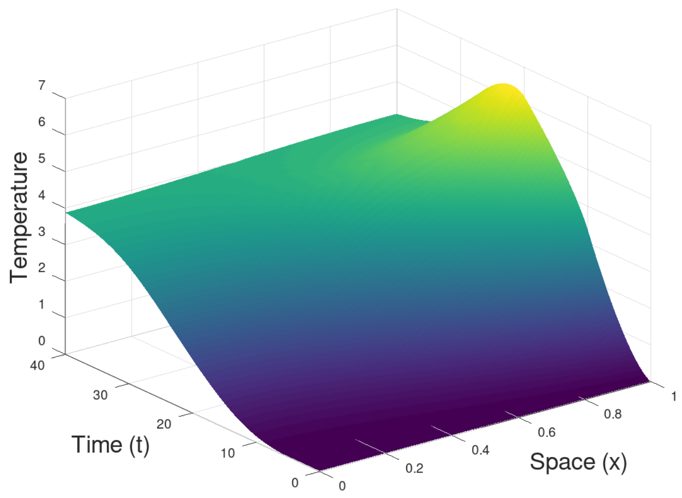


Figure 4: Full Order - FEM Solution

4 POD Model Order Reduction

4.1 Initial Sample

The full order model is then solved for 5 seconds using FEM.This solution will be the Initial sample.

4.2 Reduced Basis

The Reduced basis are extracted using the solution of the initial sample. Depending on the number of Snapshots(in this case = 10), a matrix is obtained from the Initial sample. The single value decomposition (svd)is then performed on the obtained matrix. Based on the threshold set (in this case = 1.e-6), only certain singular values are considered. Corresponding eigen vectors are chosen to form the reduced basis.

$$\mathsf{u} = \sum_{i=1}^{d} \alpha_i B^i \qquad \equiv \qquad \mathsf{u} = B\alpha$$

4.3 Building Reduced System

With the determined reduced basis, the reduced system is built with Galerkin projection. The concept of the Galerkin projection is shown below. The solution for t=40seconds is obtained from the reduced basis. This solution is the POD solution.

$$A(p)\mathrm{u} = b \quad \rightarrow \quad B^{\mathsf{T}}A(p)B\alpha \quad \rightarrow \quad \tilde{A}(p)\alpha = \tilde{b}$$

The POD solution plot obtained was

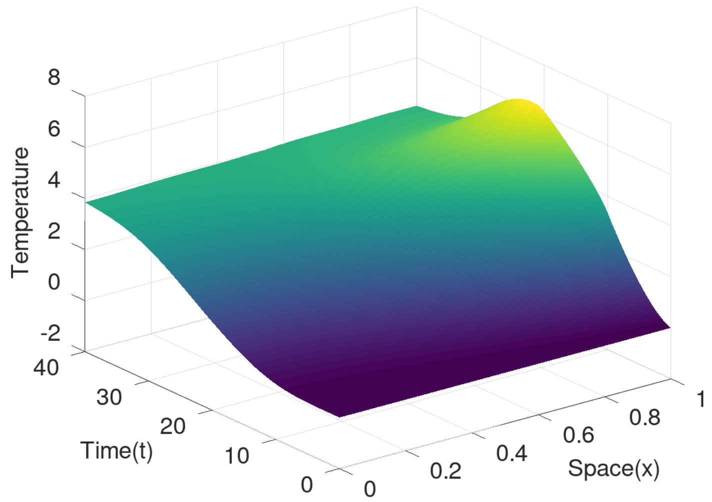


Figure 5: POD Solution

The evolution of temperature with time along the length of the bar obtained from FEM (Full order), and POD (Reduced order) solvers can be viewed in comparison with each other as shown in the figure.It can be observed that both Full-order and POD solution are similar at all the time steps.

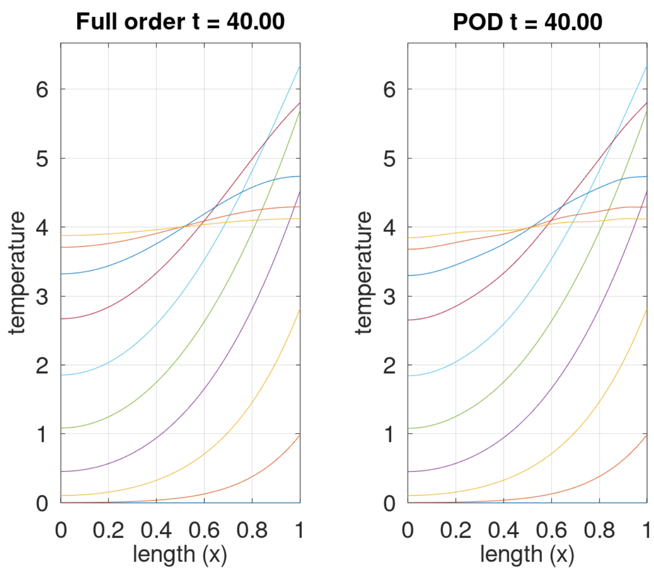
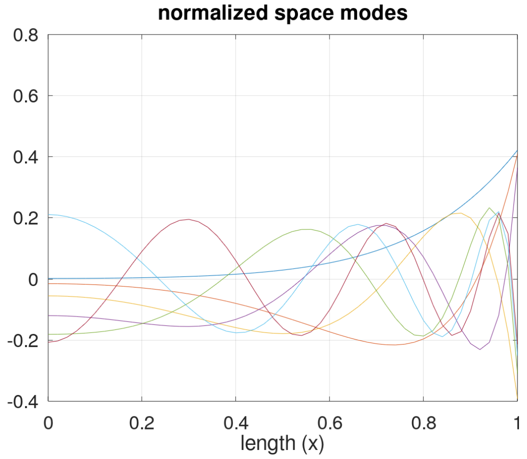
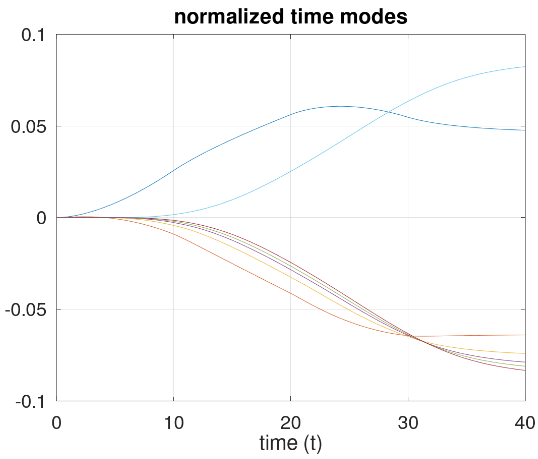


Figure 6: Evolution of Temperature in respect to time along the length of the bar

The normalised space and time modes for the POD solution obtained as shown in the figure.It can be observed that there are 7 space and time modes for the number of snapshots =10 and threshold values =1.e-6.The space-time modes can be used to construct POD model.The higher number of space-time modes used,higher the accuracy of the POD model.



(a) Normalised Space Modes



(b) Normalised Time Modes

Figure 7: Normalised Space and Time for the POD solution

5 Error Analysis

5.1 Relative Error

The Relative error is calculated between the full-order solution and POD solution.It is calculated as

$$RelativeError = \frac{||u_{FEM} - u_{POD}||_2}{||u_{FEM}||_2}$$

The Relative Error obtained was in the order of 10^{-3} magnitude i.e, 0.0022453.It can be seen that the magnitude of Relative Error is of lower order that is 0.22%,so the POD solution obtained is in agreement with the solution obtained from Full-order FEM solution.

5.2 Error Analysis based on Number of Snapshots and Threshold

The POD solution was obtained for different numbers of snapshots to study the variation of relative error. The below log-log plot was obtained from the observation made. It can be seen that as the number of snapshots increases the relative error decreases. This is because as the number of snapshots increases, more columns of the Full-order FEM Solution are considered for the construction of reduced basis, due to this the relative error of the POD solution concerning Full order solution decreases. But there are some fluctuations of relative error at a higher number of snapshots but the fluctuations stabilize to a certain point.

The POD solution was obtained for different orders of threshold to study the variation of relative error. The below log-log plot was obtained from the observation made.It can be seen that as the order of threshold is of lower order the relative error decreases for the same number of snapshots =10.So depending upon the accuracy required in the POD solution, the number of snapshots and the threshold can be fixed.

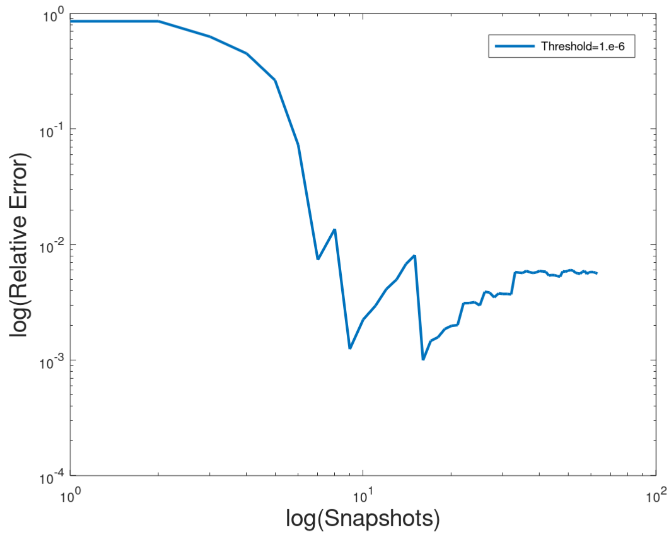


Figure 8: Evolution of Relative Error with respect to Snapshots

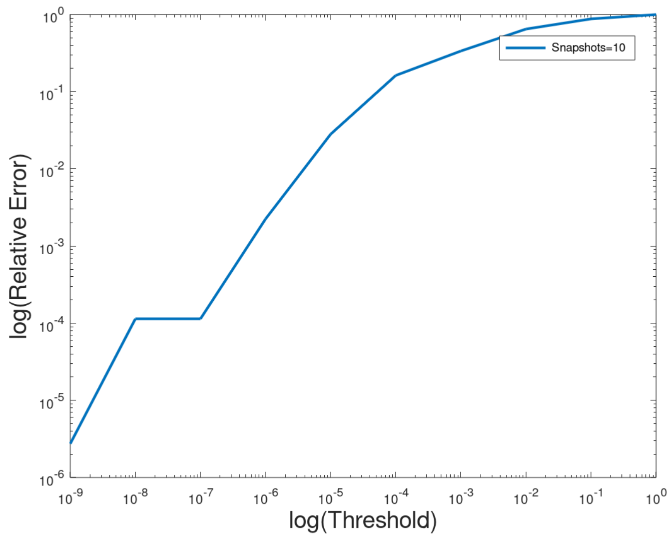


Figure 9: Evolution of Relative Error with respect to Threshold

6 Conclusion

In conclusion, The POD method is a good choice when simulating Parameterized problems. As it is not required to run the Full order simulation for various parameters changes. Due to this, the computational time can be reduced by a significant amount for complex problems. The computational time is reduced due to the decrease in degrees of freedom in the POD method. However, the full order FEM solution is required to construct the reduced model as the POD method is a model reduction based on the posteriori approach. So, while solving higher complexity problems, the POD method may not be reasonable.In such cases,a priori approach can be implemented.