LAB 3- Space-time separated representations with Proper Generalized Decomposition

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1 Introduction

The Proper Generalized Decomposition (PGD) is a priori approach to reduced-order modeling. In Proper Orthogonal Decomposition (POD) which is a posteriori approach, the full order solution was computed to extract the reduced basis. But in cases where the problem is of higher complexity, the computational cost will be much more to compute the full order solution. So, in cases such as this PGD method can be implemented where there is no requirement of solving the full order problem.

The PGD is an iterative numerical method, where the solution is approximated by successive enrichment. A new term is added to the approximation in each iteration. The efficiency of the PGD solution can be increased by increasing the number of enrichment terms. Scalar product separability property is used to separate the space-time problem into two sub-problems. The alternating directions algorithm is implemented to find the non-linear unknowns.The finite difference method is used to approximate the modes. The output of a PGD problem is a Parametric solution. In this lab, the 1D transient heat equation is studied. The PGD method is implemented on this problem and is compared with the exact solution.

2 Proper Generalized Decomposition

The 1D transient heat equation is considered and the PGD formulation is performed for this problem.The rank-d space time separated is represented as,

$$u^{(d)}(x,t) = \sum_{i=1}^{d} \alpha_i B_0^i(x) B_1^i(t)$$

The space-time weighted residual of the 1D transient heat equation,

$$\rho C_p \nabla u(t) - k\Delta u(x) - f(x,t) = 0 \tag{1}$$

Computing the new term by introducing

$$u(x,t) = u^{(d)}(x,t) + \tilde{u}(x,t)$$

The equation (1) becomes,

$$\rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \nabla B_1^i + \rho C_p B_0 \nabla B_1 - k \sum_{i=1}^{d} \alpha_i B_1^i \Delta B_0^i - k B_1 \Delta B_0 - f(x,t) = 0$$

The test function in terms of $B_0(x), B_1(t)$ and the respective variations,$B_0^*(x), B_1^*(t)$ is

$$v = B_0^*(x) B_1(t) + B_0(x) B_1^*(t)$$

$$v = B_0^* B_1 + B_0 B_1^*$$

Introducing the test function into the above equation and integrating over the space-time domain,

$$(\rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \nabla B_1^i, B_0^* B_1 + B_0 B_1^*) + (\rho C_p B_0 \nabla B_1, B_0^* B_1 + B_0 B_1^*) - (k \sum_{i=1}^{d} \alpha_i B_1^i \Delta B_0^i, B_0^* B_1 + B_0 B_1^*)$$
$$- (k B_1 \Delta B_0, B_0^* B_1 + B_0 B_1^*) - (f(x,t), B_0^* B_1 + B_0 B_1^*) = 0 \tag{2}$$

2.1 Space Problem

The Alternating Directions algorithm is applied to the above obtained equation, B_1 is considered known and the test function reduces to

$$v = B_0^* B_1$$

So the equation (2) reduces to

$$(\rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \nabla B_1^i, B_0^* B_1) + (\rho C_p B_0 \nabla B_1, B_0^* B_1) - (k \sum_{i=1}^{d} \alpha_i B_1^i \Delta B_0^i, B_0^* B_1) \\ - (k B_1 \Delta B_0, B_0^* B_1) - (f(x,t), B_0^* B_1) = 0 \quad (3)$$

The above equation in integral form is

$$\int \rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \nabla B_1^i B_0^* B_1 d\Omega dt + \int \rho C_p B_0 \nabla B_1 B_0^* B_1 d\Omega dt - \int k \sum_{i=1}^{d} \alpha_i B_1^i \Delta B_0^i B_0^* B_1 d\Omega dt \\ - \int k B_1 \Delta B_0 B_0^* B_1 d\Omega dt - \int f(x,t) B_0^* B_1 d\Omega dt = 0$$

Using Separation properties of the Integrals,we get

$$\int_\Omega B_0^* B_0 d\Omega \int_t \rho C_p \nabla B_1 B_1 dt - \int_\Omega \Delta B_0 B_0^* d\Omega \int_t k B_1 B_1 dt = \int_\Omega f(x) B_0^* d\Omega \int_t f(t) B_1 dt \\ - \int_\Omega \sum_{i=1}^{d} B_0^i B_0^* d\Omega \int_t \rho C_p \alpha_i \nabla B_1^i B_1 dt + \int_\Omega \sum_{i=1}^{d} \Delta B_0^i B_0^* d\Omega \int_t k \alpha_i B_1^i B_1 dt$$

The strong form of the above equation is then obtained.This is because the discretization used in this lab is Finite Differences.

$$\rho C_p B_0 \int_t \nabla B_1 B_1 dt - k \Delta B_0 \int_t B_1 B_1 dt = f(x) \int_t f(t) B_1 dt \\ - \rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \int_t \nabla B_1^i B_1 dt + k \sum_{i=1}^{d} \Delta B_0^i \int_t k \alpha_i B_1^i B_1 dt \quad (4)$$

2.2 Time Problem

In the time problem ,B_0 is considered known and the test function reduces to

$$v = B_0 B_1^*$$

So the equation (2) reduces to

$$(\rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \nabla B_1^i, B_0 B_1^*) + (\rho C_p B_0 \nabla B_1, B_0 B_1^*) - (k \sum_{i=1}^{d} \alpha_i B_1^i \Delta B_0^i, B_0 B_1^*) \\ - (k B_1 \Delta B_0, B_0 B_1^*) - (f(x,t), B_0 B_1^*) = 0 \quad (5)$$

The above equation in integral form is

$$\int \rho C_p \sum_{i=1}^{d} \alpha_i B_0^i \nabla B_1^i B_0 B_1^* d\Omega dt + \int \rho C_p B_0 \nabla B_1 B_0 B_1^* d\Omega dt - \int k \sum_{i=1}^{d} \alpha_i B_1^i \Delta B_0^i B_0 B_1^* d\Omega dt \\ - \int k B_1 \Delta B_0, B_0 B_1^* d\Omega dt - \int f(x,t) B_0 B_1^* d\Omega dt = 0$$

Using Separation properties of the Integrals,we get

$$\int_t \rho C_p \nabla B_1 B_1^* dt \int_\Omega B_0 B_0 d\Omega - \int_t k B_1 B_1^* dt \int_\Omega \Delta B_0 B_0 d\Omega = \int_t f(t) B_1^* dt \int_\Omega f(x) B_0 d\Omega$$
$$- \int_t \rho C_p \sum_{i=1}^{d} \alpha_i \nabla B_1^i B_1^* dt \int_\Omega B_0^i B_0 d\Omega + \int_t k \sum_{i=1}^{d} \alpha_i B_1^i B_1^* dt \int_\Omega \Delta B_0^i B_0 d\Omega$$

The strong form of the above equation is then obtained.

$$\rho C_p \nabla B_1 \int_\Omega B_0 B_0 d\Omega - k B_1 \int_\Omega \Delta B_0 B_0 d\Omega = f(t) \int_\Omega f(x) B_0 d\Omega$$
$$- \rho C_p \sum_{i=1}^{d} \alpha_i \nabla B_1^i dt \int_\Omega B_0^i B_0 d\Omega + k \sum_{i=1}^{d} \alpha_i B_1^i \int_\Omega \Delta B_0^i B_0 d\Omega \quad (6)$$

2.3 Finite Difference Method

The Finite Difference is used to discretize.The first order time derivatives are approximated using the backward scheme.
For a certain time-step j,

$$(\frac{\partial B_1}{\partial t})_j \approx \frac{(B_1)_j - (B_1)_{j-1}}{\Delta t}$$

for the whole time domain t,

$$(\frac{\partial B_1}{\partial t}) = G_1 B_1$$

and Similarly for the time derivative $(\frac{\partial B_1^i}{\partial t})$,it is approximated at a certain time-step j as,

$$(\frac{\partial B_1^i}{\partial t})_j \approx \frac{(B_1^i)_j - (B_1^i)_{j-1}}{\Delta t}$$

for the whole time domain t,

$$(\frac{\partial B_1^i}{\partial t}) = G_1 B_1^i$$

The second order space derivatives are approximated using the centre scheme.
For a certain time-step j,

$$(\frac{\partial^2 B_0}{\partial x^2})_j \approx \frac{1}{\Delta x^2}[(B_0)_{j+1} - 2(B_0)_j - (B_0)_{j-1}]$$

for the whole space domain Ω,

$$(\frac{\partial^2 B_0}{\partial x^2}) = K_0 B_0$$

and similarly for the space domain derivative $(\frac{\partial^2 B_0^i}{\partial t^2})$,it is approximated at a certain time-step j as,

$$(\frac{\partial^2 B_0^i}{\partial x^2})_j \approx \frac{1}{\Delta x^2}[(B_0^i)_{j+1} - 2(B_0^i)_j - (B_0^i)_{j-1}]$$

for the whole space domain Ω,

$$(\frac{\partial^2 B_0^i}{\partial x^2}) = K_0 B_0^i$$

The G_1 and K_0 are the finite difference matrices.The I_0 and I_1 are the identity matrix for space and time.

2.4 PGD Formulation

Using the Finite Difference in the equations (4) and (6),we get the PGD formulation of the problem.
For the Space Problem:

$$[\alpha_0 I_0 - \beta_0 K_0]B_0 = \gamma_0 f_0 - \sum_{i=1}^{d} \alpha_i [\alpha_0^i I_0 - \beta_0^i K_0] B_0^i$$

where,

- $\alpha_0 = \int_t \frac{\partial B_1}{\partial t} B_1 dt = \int_t G_1 B_1 B_1 dt$
- $\beta_0 = \int_t B_1 B_1 dt$
- $\gamma_0 = \int_t f(t) B_1 dt$
- $\alpha_0^i = \int_t \frac{\partial B_1^i}{\partial t} B_1 dt = \int_t G_1 B_1^i B_1 dt$
- $\beta_0^i = \int_t B_1^i B_1 dt$

For the Time problem:

$$[\alpha_1 G_1 - \beta_1 I_1]B_1 = \gamma_1 f_1 - \sum_{i=1}^{d} \alpha_i [\alpha_1^i G_1 - \beta_1^i I_1] B_1^i$$

where,

- $\alpha_1 = \int_\Omega B_0 B_0 d\Omega$
- $\beta_1 = \int_\Omega \frac{\partial^2 B_0}{\partial x^2} B_0 d\Omega = \int_\Omega K_0 B_0 B_0 d\Omega$
- $\gamma_1 = \int_t f(x) B_0 d\Omega$
- $\alpha_1^i = \int_\Omega B_0^i B_0 d\Omega$
- $\beta_0^i = \int_\Omega \frac{\partial^2 B_0^i}{\partial x^2} B_0 d\Omega = \int_\Omega K_0 B_0^i B_0 d\Omega$

3 Exact Solution

The exact solution is computed for the problem,the values are set to $k = 1, C_p = 1$ and $\rho = 1$.Then the problem becomes,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1 \quad in \quad (x,t) \in]0,1[x]0,0.1[$$

$$u(0,t) = 0, \quad u(1,t) = 0$$

$$u(x,0) = 0$$

This problem is simulated and the exact solution is obtained.

4 Comparison between PGD and Exact Solution

The PGD solution and the exact solution were simulated with space domain discretized as $\Delta x = 0.05$ and time domain discretized as $\Delta t = 0.05$.The PGD solution was computed using the first 5 terms.The fixed-point tolerance was set to $\varepsilon = 10^{-7}$ and the tolerance for the convergence was set to $\tilde{\varepsilon} = 10^{-7}$.
The below figure shows the Surface plot of the exact solution: The below figure shows the

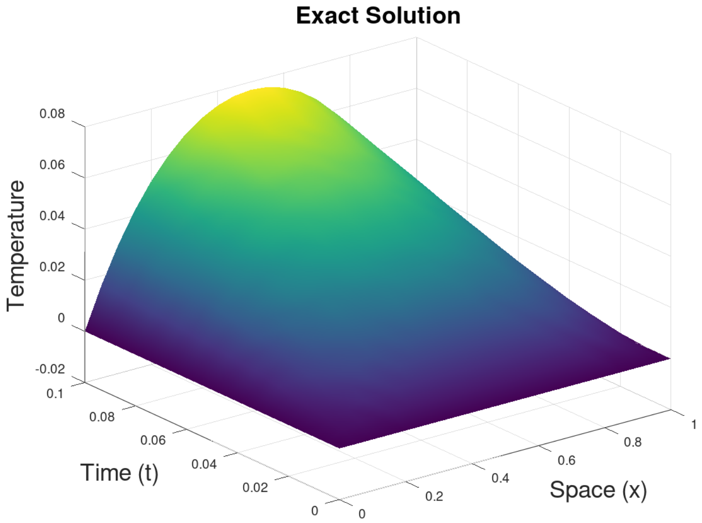

Figure 1: Surface Plot of the Exact Solution

Surface plot of the PGD solution:

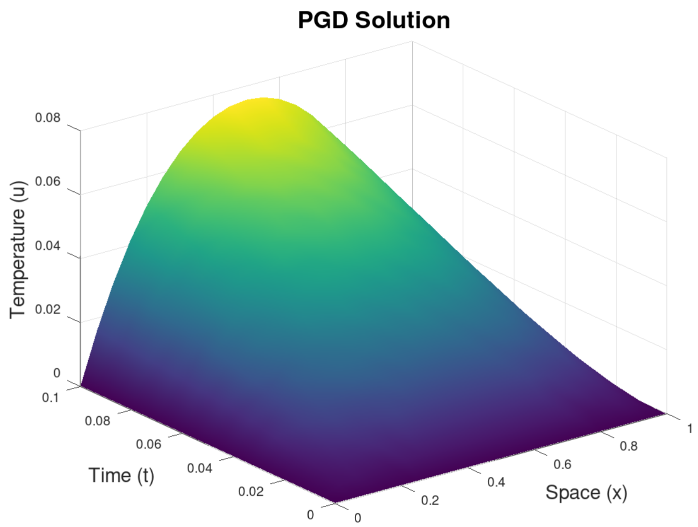

Figure 2: Surface Plot of the PGD Solution

5 Error Analysis

The Relative error was calculated for an increasing number of Enrichment terms. The following plot was obtained. It can be seen that as the number of enrichment terms is increased the relative

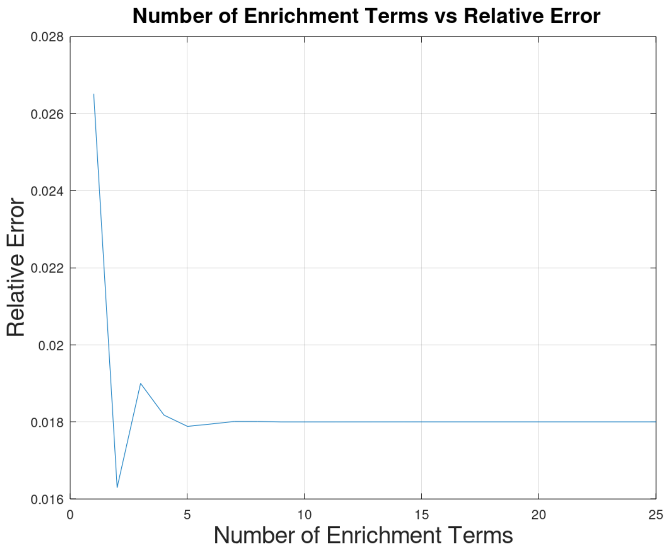

Figure 3: Number of Enrichment Terms vs Relative Error

error decreases rapidly. At a certain stage, the relative error stabilizes to a value irrespective of the number of enrichment terms.

6 Conclusion

In conclusion, The PGD method was implemented to the 1D transient problem and the solution obtained was studied with the exact solution. The PGD method provides a good approximation by using a fewer number of enrichment terms. The approximation obtained is in agreement with the exact solution. The complexity of the problem was reduced by using PGD. It can also be observed that after a certain number of enrichment the relative error does not change, so it doesn't affect the obtained solution. Hence, we can get a good approximation for less number of enrichment terms.