LAB 5 - SEPARATED REPRESENTATION OF VECTOR FIELD PROBLEMS

BY

SACHIN SRINIVASA SHETTY

Contents

List of Figures

1 Introduction

In this lab,we are implementing the Proper Generalized Decomposition(PGD) for vector field problems.In the previous labs,we studied how the PGD can be used to solve a scalar field problem.Compared to the scalar field problems,vector field problems doesn't introduce any conceptual difficulty.We can use the Separated representation to compute each one of the vector field components.

We are the studying the plane-stress elasticity problem.The displacement field are defined over the 2-D domain,the horizontal and vertical components of the diplacement field.

2 Separated Representations of Vector field problems

The weak form of the plane-stress elasticity problem in a body Ω is as follows,

$$\int_{\Omega} \varepsilon^{*t} D \varepsilon d\Omega = \int_{\Omega} u^{*t} b d\Omega$$

$$\varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$D = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ V & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}, b = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

The weak form of the left hand side in terms of the primary variables,u and v is as follows,

$$\int_{\Omega} \varepsilon^{*t} D \varepsilon d\Omega = \frac{E}{1-v^2} \int_{\Omega} [\varepsilon_x^* \varepsilon_x + v \varepsilon_x^* \varepsilon_y + v \varepsilon_y^* \varepsilon_x + \varepsilon_y^* \varepsilon_y + \frac{1-v}{2} \gamma_{xy}^* \gamma_{xy}] d\Omega$$

$$\int_{\Omega} \varepsilon^{*t} D \varepsilon d\Omega = \frac{E}{1-v^2} \int_{\Omega} [\frac{\partial u^*}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial u^*}{\partial x} \frac{\partial v}{\partial y} + v \frac{\partial v^*}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial v^*}{\partial y} \frac{\partial v}{\partial y} + \frac{1-v}{2} [(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x})(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})]]$$

$$\begin{aligned} \int_{\Omega} \varepsilon^{*t} D \varepsilon d\Omega = \frac{E}{1-v^2} \int_{\Omega} [\frac{\partial u^*}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial u^*}{\partial x} \frac{\partial v}{\partial y} + v \frac{\partial v^*}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial v^*}{\partial y} \frac{\partial v}{\partial y} \\ + \frac{1-v}{2} [\frac{\partial u^*}{\partial y} \frac{\partial u}{\partial y} + \frac{\partial u^*}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial v^*}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v^*}{\partial x} \frac{\partial v}{\partial x}]] \end{aligned}$$

The displacement field in separated representations is as follows,

$$u(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \sum_{i=1}^{M} \begin{bmatrix} X_u^i(x) Y_u^i(y) \\ X_v^i(x) Y_v^i(y) \end{bmatrix} = \sum_{i=1}^{M} \begin{bmatrix} X_u^i(x) \\ X_v^i(x) \end{bmatrix} \circ \begin{bmatrix} Y_u^i(y) \\ Y_v^i(y) \end{bmatrix} = \sum_{i=1}^{M} X^i(x) \circ Y^i(y)$$

where, $\circ$ stands for Hadamard (Component-wise) product The Right hand side terms are,

$$\int_{\Omega} u^{*t} b d\Omega = \int_{\Omega} u^* b_x + v^* b_y d\Omega$$

2.1 Space X Problem

The domain is defined as $\Omega = [0,L]x[O,H]$ in the Cartesian domain.The displacement are defined as $u = X_uY_u$ and $v = X_vY_v$.The test function are defined as $u^* = X_u^*Y_u + X_uY_u^*$ and $v^* = X_v^*Y_v + X_vY_v^*$.For the Space x problem,the Y_u and Y_v are known. The terms in Left hand side of the equation are computed as follows.

$$\int_\Omega \frac{\partial u^*}{\partial x}\frac{\partial u}{\partial x} = \int_\Omega \frac{\partial X_u^*}{\partial x}Y_u\frac{\partial X_u}{\partial x}Y_u = \int_0^L \frac{\partial X_u^*}{\partial x}dx\int_0^H Y_uY_udy = (X_u^{*t}K_xX_u)\otimes(Y_u^tM_yY_u)$$

where,

$$(X_u^{*t}K_xX_u)\otimes(Y_u^tM_yY_u) = ([X_u^{*t} \quad X_v^{*t}]\begin{bmatrix}K_x & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}X_u\\ X_v\end{bmatrix})\otimes([[Y_u^t \quad Y_v^t]\begin{bmatrix}M_y & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}Y_u\\ Y_v\end{bmatrix})$$

Similarly,for other terms,

$$\int_\Omega \frac{\partial u^*}{\partial x}\frac{\partial v}{\partial y} = \int_\Omega \frac{\partial X_u^*}{\partial x}Y_uX_v\frac{\partial Y_v}{\partial y} = \int_0^L \frac{\partial X_u^*}{\partial x}X_vdx\int_0^H Y_u\frac{\partial Y_v}{\partial y}dy = (X_u^{*t}C_{x_1}X_v)\otimes(Y_u^tC_{y_1}Y_v)$$

$$\int_\Omega \frac{\partial v^*}{\partial y}\frac{\partial u}{\partial x} = \int_\Omega X_v^*\frac{\partial Y_v}{\partial y}\frac{\partial X_u}{\partial y}Y_u = \int_0^L X_v^*\frac{\partial X_u}{\partial x}dx\int_0^H \frac{\partial Y_v}{\partial y}Y_udy = (X_v^{*t}C_{x_1}X_u)\otimes(Y_v^tC_{y_1}Y_u)$$

$$\int_\Omega \frac{\partial v^*}{\partial y}\frac{\partial v}{\partial y} = \int_\Omega X_v^*\frac{\partial Y_v}{\partial y}X_v\frac{\partial Y_v}{\partial y} = \int_0^L X_v^*X_vdx\int_0^H \frac{\partial Y_v}{\partial y}\frac{\partial Y_v}{\partial y}dy = (X_v^{*t}M_xX_v)\otimes(Y_v^tK_yY_v)$$

$$\int_\Omega \frac{\partial u^*}{\partial y}\frac{\partial u}{\partial y} = \int_\Omega X_u^*\frac{\partial Y_u}{\partial y}X_u\frac{\partial Y_u}{\partial y} = \int_0^L X_u^*X_udx\int_0^H \frac{\partial Y_u}{\partial y}\frac{\partial Y_u}{\partial y}dy = (X_u^{*t}M_xX_u)\otimes(Y_u^tK_yY_u)$$

$$\int_\Omega \frac{\partial u^*}{\partial y}\frac{\partial v}{\partial x} = \int_\Omega X_u^*\frac{\partial Y_u}{\partial y}\frac{\partial X_v}{\partial x}Y_v = \int_0^L X_u^*\frac{\partial X_v}{\partial x}dx\int_0^H \frac{\partial Y_u}{\partial y}Y_vdy = (X_u^{*t}C_{x_2}X_v)\otimes(Y_u^tC_{y_2}Y_v)$$

$$\int_\Omega \frac{\partial v^*}{\partial x}\frac{\partial u}{\partial y} = \int_\Omega \frac{\partial X_v^*}{\partial x}Y_vX_u\frac{\partial Y_u}{\partial y} = \int_0^L \frac{\partial X_v^*}{\partial x}X_udx\int_0^H Y_v\frac{\partial Y_u}{\partial y}dy = (X_v^{*t}C_{x_1}X_u)\otimes(Y_v^tC_{y_1}Y_u)$$

$$\int_\Omega \frac{\partial v^*}{\partial x}\frac{\partial v}{\partial x} = \int_\Omega \frac{\partial X_v^*}{\partial x}Y_v\frac{\partial X_v}{\partial x}Y_v = \int_0^L \frac{\partial X_v}{\partial x}\frac{\partial X_v}{\partial x}dx\int_0^H Y_vY_vdy = (X_v^{*t}K_xX_v)\otimes(Y_v^tM_yY_v)$$

The terms in Right hand side of the equation are computed as follows.

$$\int_\Omega u^*b_xd\Omega = \int_\Omega X_u^*Y_ub_{xx}b_{xy}d\Omega = \int_0^L X_u^*b_{xx}dx\int_0^H Y_ub_{xy}dy = (X_u^{*t}M_xb_{xx})\otimes(Y_u^tM_yb_{xy})$$

$$\int_\Omega v^*b_yd\Omega = \int_\Omega X_v^*Y_vb_{yx}b_{yy}d\Omega = \int_0^L X_v^*b_{xy}dx\int_0^H Y_vb_{yy}dy = (X_v^{*t}M_xb_{yx})\otimes(Y_v^tM_yb_{yy})$$

2.2 Space Y Problem

For the Space y problem,the X_u and X_v are known. The terms in the equation are computed as follows.

$$\int_{\Omega}\frac{\partial u^*}{\partial x}\frac{\partial u}{\partial x}=\int_{\Omega}\frac{\partial X_u}{\partial x}Y_u^*\frac{\partial X_u}{\partial x}Y_u=\int_0^L\frac{\partial X_u}{\partial x}dx\int_0^H Y_u^*Y_u dy=(X_u^tK_xX_u)\otimes(Y_u^{*t}M_yY_u)$$

$$\int_{\Omega}\frac{\partial u^*}{\partial x}\frac{\partial v}{\partial y}=\int_{\Omega}\frac{\partial X_u}{\partial x}Y_u^*X_v\frac{\partial Y_v}{\partial y}=\int_0^L\frac{\partial X_u}{\partial x}X_v dx\int_0^H Y_u^*\frac{\partial Y_v}{\partial y}dy=(X_u^tC_{x_1}X_v)\otimes(Y_u^{*t}C_{y_1}Y_v)$$

$$\int_{\Omega}\frac{\partial v^*}{\partial y}\frac{\partial u}{\partial x}=\int_{\Omega}X_v\frac{\partial Y_v^*}{\partial y}\frac{\partial X_u}{\partial y}Y_u=\int_0^L X_v\frac{\partial X_u}{\partial x}dx\int_0^H\frac{\partial Y_v^*}{\partial y}Y_u dy=(X_v^tC_{x_2}X_u)\otimes(Y_v^{*t}C_{y_2}Y_u)$$

$$\int_{\Omega}\frac{\partial v^*}{\partial y}\frac{\partial v}{\partial y}=\int_{\Omega}X_v\frac{\partial Y_v^*}{\partial y}X_v\frac{\partial Y_v}{\partial y}=\int_0^L X_vX_v dx\int_0^H\frac{\partial Y_v^*}{\partial y}\frac{\partial Y_v}{\partial y}dy=(X_v^tM_xX_v)\otimes(Y_v^{*t}K_yY_v)$$

$$\int_{\Omega}\frac{\partial u^*}{\partial y}\frac{\partial u}{\partial y}=\int_{\Omega}X_u\frac{\partial Y_u^*}{\partial y}X_u\frac{\partial Y_u}{\partial y}=\int_0^L X_uX_u dx\int_0^H\frac{\partial Y_u^*}{\partial y}\frac{\partial Y_u}{\partial y}dy=(X_u^tM_xX_u)\otimes(Y_u^{*t}K_yY_u)$$

$$\int_{\Omega}\frac{\partial u^*}{\partial y}\frac{\partial v}{\partial x}=\int_{\Omega}X_u\frac{\partial Y_u^*}{\partial y}\frac{\partial X_v}{\partial x}Y_v=\int_0^L X_u\frac{\partial X_v}{\partial x}dx\int_0^H\frac{\partial Y_u^*}{\partial y}Y_v dy=(X_u^tC_{x_2}X_v)\otimes(Y_u^{*t}C_{y_2}Y_v)$$

$$\int_{\Omega}\frac{\partial v^*}{\partial x}\frac{\partial u}{\partial y}=\int_{\Omega}\frac{\partial X_v}{\partial x}Y_v^*X_u\frac{\partial Y_u}{\partial y}=\int_0^L\frac{\partial X_v}{\partial x}X_u dx\int_0^H Y_v^*\frac{\partial Y_u}{\partial y}dy=(X_v^tC_{x_1}X_u)\otimes(Y_v^{*t}C_{y_1}Y_u)$$

$$\int_{\Omega}\frac{\partial v^*}{\partial x}\frac{\partial v}{\partial x}=\int_{\Omega}\frac{\partial X_v}{\partial x}Y_v^*\frac{\partial X_v}{\partial x}Y_v=\int_0^L\frac{\partial X_v}{\partial x}\frac{\partial X_v}{\partial x}dx\int_0^H Y_v^*Y_v dy=(X_v^tK_xX_v)\otimes(Y_v^{*t}M_yY_v)$$

Similarly,The terms in Right hand side of the equation are computed as follows.

$$\int_{\Omega}u^*b_x d\Omega=\int_{\Omega}X_uY_u^*b_{xx}b_{xy}d\Omega=\int_0^L X_ub_{xx}dx\int_0^H Y_u^*b_{xy}dy=(X_u^tM_xb_{xx})\otimes(Y_u^{*t}M_yb_{xy})$$

$$\int_{\Omega}v^*b_y d\Omega=\int_{\Omega}X_vY_v^*b_{yx}b_{yy}d\Omega=\int_0^L X_vb_{xy}dx\int_0^H Y_v^*b_{yy}dy=(X_v^tM_xb_{yx})\otimes(Y_v^{*t}M_yb_{yy})$$

2.3 Definition of Cell Arrays for *easyPGD*

2.3.1 Definition of AA

For this problem,AA is defined by 2 rows and 8 columns.The 8 columns corresponds to the 8 terms in the LHS of the equations.The AA cell array is defined as follows,

$$AA\{1,1\} = (\frac{E}{1-v^2})\begin{bmatrix} K_x & 0 \\ 0 & 0 \end{bmatrix} \quad AA\{1,2\} = (\frac{Ev}{1-v^2})\begin{bmatrix} 0 & C_{x_1} \\ 0 & 0 \end{bmatrix}$$

$$AA\{1,3\} = (\frac{Ev}{1-v^2})\begin{bmatrix} 0 & 0 \\ C_{x_2} & 0 \end{bmatrix} \quad AA\{1,4\} = (\frac{E}{1-v^2})\begin{bmatrix} 0 & 0 \\ 0 & M_x \end{bmatrix}$$

$$AA\{1,5\} = (\frac{E}{2(1+v)})\begin{bmatrix} M_x & 0 \\ 0 & 0 \end{bmatrix} \quad AA\{1,6\} = (\frac{E}{2(1+v)})\begin{bmatrix} 0 & C_{x_2} \\ 0 & 0 \end{bmatrix}$$

$$AA\{1,7\} = (\frac{E}{2(1+v)})\begin{bmatrix} 0 & 0 \\ C_{x_1} & 0 \end{bmatrix} \quad AA\{1,8\} = (\frac{E}{2(1+v)})\begin{bmatrix} 0 & 0 \\ 0 & K_x \end{bmatrix}$$

$$AA\{2,1\} = \begin{bmatrix} M_y & 0 \\ 0 & 0 \end{bmatrix} \quad AA\{2,2\} = \begin{bmatrix} 0 & C_{y_1} \\ 0 & 0 \end{bmatrix}$$

$$AA\{2,3\} = \begin{bmatrix} 0 & 0 \\ C_{y_2} & 0 \end{bmatrix} \quad AA\{2,4\} = \begin{bmatrix} 0 & 0 \\ 0 & K_y \end{bmatrix}$$

$$AA\{2,5\} = \begin{bmatrix} K_y & 0 \\ 0 & 0 \end{bmatrix} \quad AA\{2,6\} = \begin{bmatrix} 0 & C_{y_2} \\ 0 & 0 \end{bmatrix}$$

$$AA\{2,7\} = \begin{bmatrix} 0 & 0 \\ C_{y_1} & 0 \end{bmatrix} \quad AA\{2,8\} = \begin{bmatrix} 0 & 0 \\ 0 & M_y \end{bmatrix}$$

2.3.2 Definition of BB

For this problem,BB is defined by 2 rows and 1 column.The BB cell array is defined as follows.

$$BB\{1,1\} = \begin{bmatrix} M_x b_{xx} & 0 \\ 0 & M_x b_{yx} \end{bmatrix}$$

$$BB\{2,1\} = \begin{bmatrix} M_y b_{xy} & 0 \\ 0 & M_y b_{yy} \end{bmatrix}$$

2.3.3 Definition of Bord

For this problem,Bord is defined by 2 rows and 1 column.Symmetry boundary conditions on both bottom and left edges,i.e, in the bottom edge the vertical component of the displacement is zero,while in the left edge the horizontal component of the displacement is zero.The Bord cell array is defined as follows.

$$Bord\{1,1\} = [1]$$

$$Bord\{2,1\} = [N_y + 1]$$

3 PGD solution for the body force $b_x = b_y = 1$

In this section,the PGD solution for the body force $b_x = b_y = 1$ is discussed.The Horizontal and Vertical components of the displacement are shown below.

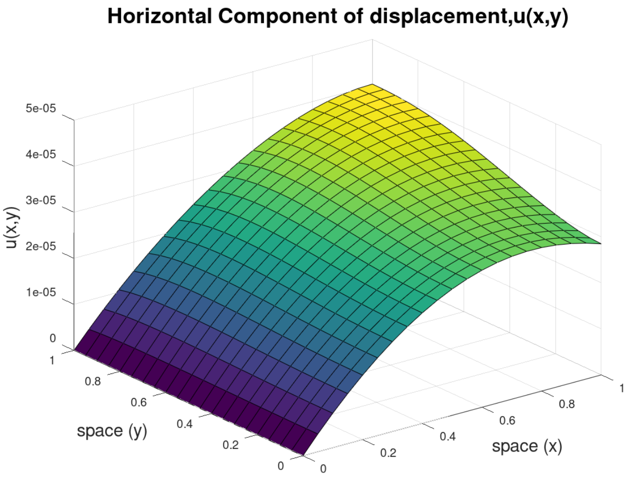

Figure 1: Horizontal Component of displacement

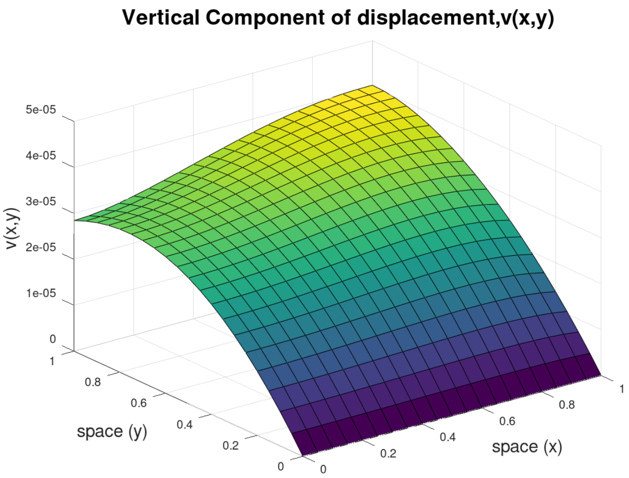

Figure 2: Vertical Component of displacement

The magnitude of the displacement is shown below,It is maximum at (1,1).

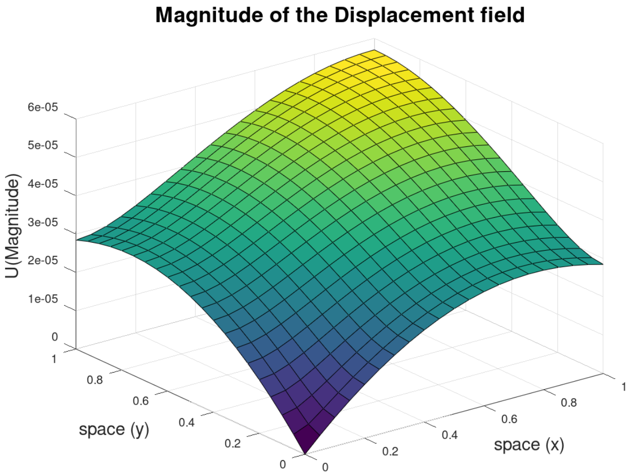

Figure 3: Magnitude of the displacement

The magnitude of the displacement field on deformed shape of the domain,using a scale factor of 5.10^3 is shown below.

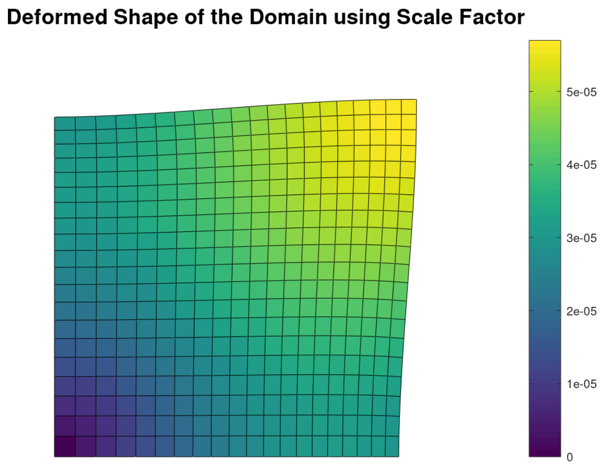

Figure 4: Deformed Shape of the domain

4 PGD solution for the body force $b_x = x^2y, b_y = (y-1)^2$

In this section,the PGD solution for the body force $b_x = x^2y, b_y = (y-1)^2$ is discussed.The Horizontal and Vertical components of the displacement are shown below.

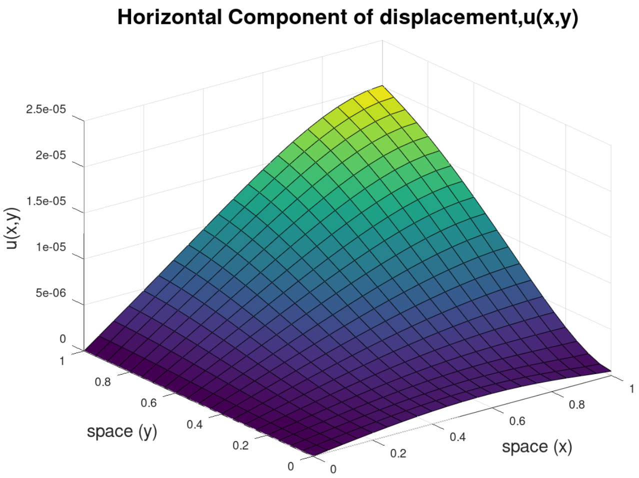

Figure 5: Horizontal Component of displacement

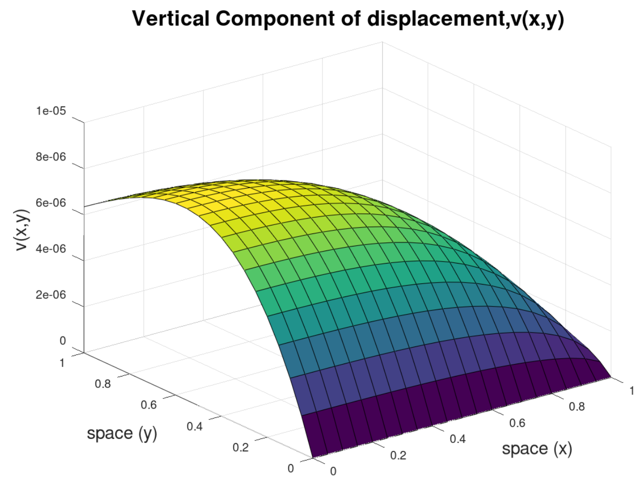

Figure 6: Vertical Component of displacement

The magnitude of the displacement is shown below,It is maximum at (1,1). The magnitude

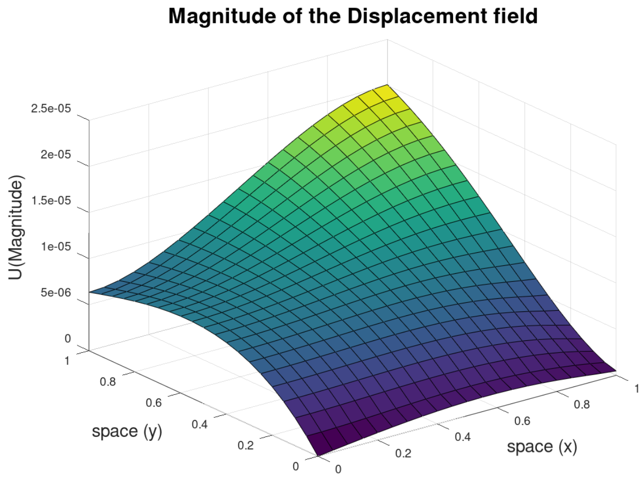

Figure 7: Magnitude of the displacement

of the displacement field on deformed shape of the domain,using a scale factor of 5.10^3 is shown below.

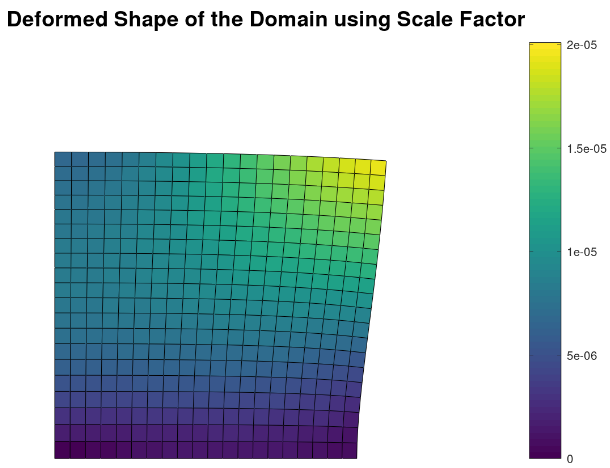

Figure 8: Deformed Shape of the domain

5 Conclusion

In conclusion,the Proper Generalised Decomposition (PGD) was implemented on the plane-stress elasticity problem with body force ($b_x = b_y = 1$) and body force ($b_x = x^2y$ and $b_y = (y-1)^2$).The PGD to the vector field problem doesn't introduce any conceptual difficulty.The generalized PGD code *easyPGD* which is a generalised PGD code was used to obtain the solution.