LAB 2- EMPRICAL INTERPOLATION FOR NON-LINEAR PROBLEMS

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1 Introduction

The Non-linear problems have many significances in several engineering problems. Solving these non-linear terms still requires several operations that depend on the number of degrees of freedom of the original model. The Non-Linear terms are calculated using several techniques such as fixed-point iterations, etc. Due to this, the computational time taken by the system is more and the cost of computation also increases.

The model order reduction method, Proper Orthogonal Decomposition is performed on the solution and non-linear term to reduce the complexity of the problem. But the computation complexity of non-linear terms is the same as the original model. So, the Emprical Interpolation method is implemented. Based on the interpolation points obtained, the non-linear terms are evaluated. In this report, This method is being implemented on a non-linear 2D transient heat diffusion problem and the non-linear term is conductivity as it depends non-linearly on the temperature.

2 2D Full order FEM Solution

The problem we are studying is a non-linear 2D transient heat diffusion problem. The domain of the problem is shown in the figure below. The problem is non-linear because the thermal conductivity depends non-linearly on the temperature. There is a heat source in the domain. The boundary conditions are prescribed as follows: Homogeneous Dirichlet condition at both Up and Right boundaries, adiabatic at Left and Bottom boundaries, and there is a heat flux at the hole boundary.

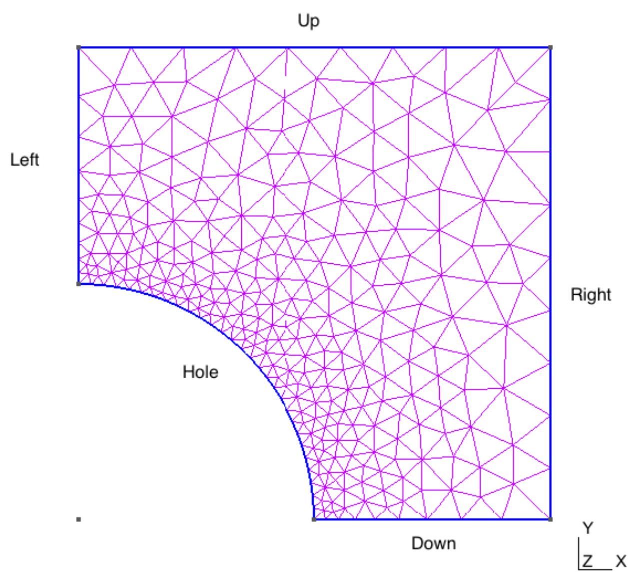

Figure 1: Problem Domain

The non-linear conductivity is defined as follows:

$$k(u) = k_o(1 + 0.5((u+1)^5 - 1))$$

The 2D full order FEM solution is simulated.The full solution is stored.It is used to calculate the reduced basis for the solution and the non-linear term.

The 2D FEM solution contour plot is obtained as shown in the figure.

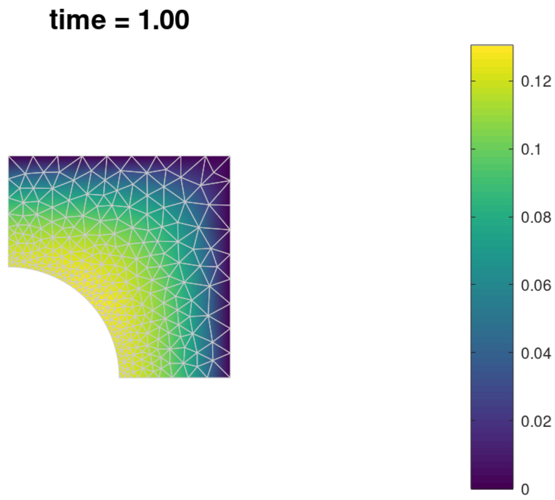

Figure 2: Contour plot for 2D FEM solution

3 Model Order Reduction

3.1 Reduced Basis for Solution and Non-Linear Term

The Reduced basis are extracted from the stored 2D FEM solution. Depending on the number of Snapshots, a matrix is obtained from the full solution. The single value decomposition (svd)is then performed on the obtained matrix. Based on the threshold set (in this case = 1.e-4), only certain singular values are considered. Corresponding eigenvectors are chosen to form the reduced basis. In the same way, the reduced basis is obtained from the non-linear term. Then the reduced mass matrix and diffusion operator matrix is assembled by Galerkin projection.It is done for the mass matrix (M) as follows:

$$M_{reduction} = ReducedBasis^T * M * ReducedBasis$$

3.2 Basis Vectors for Solution and Non-Linear Term

The below figure represents the first four dominant vectors of the solution respectively.

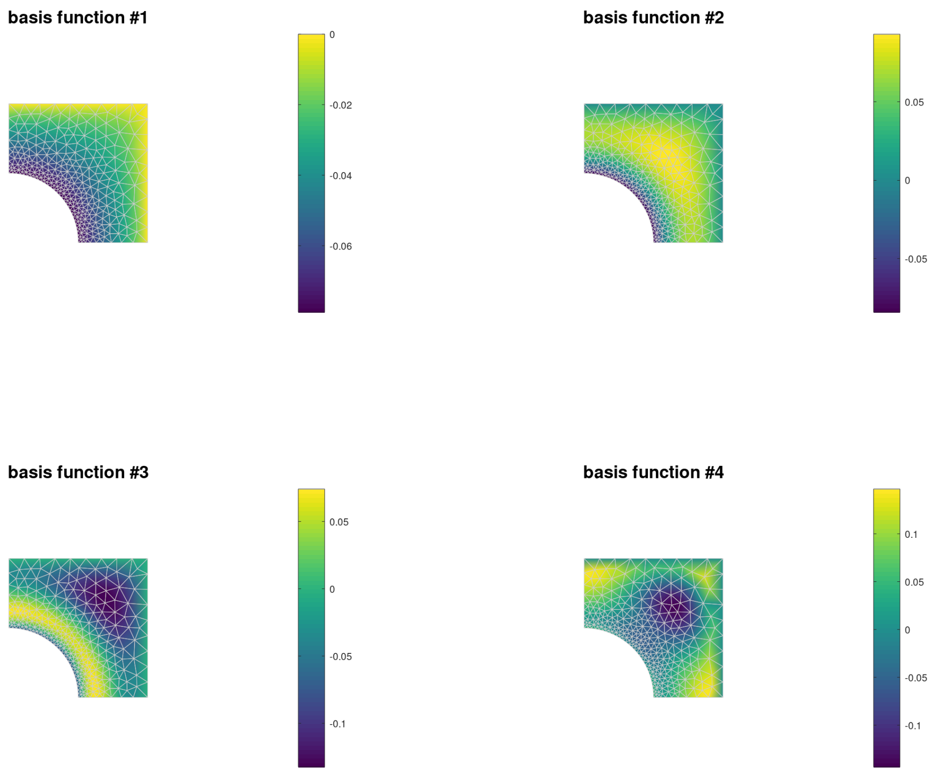

Figure 3: First Four Dominant vectors of the Solution

The below figure represents the first four dominant vectors of the non-linear term respectively.

3.3 Empirical Interpolation Method

The non-linear coefficients (β) is evaluated using the Emprical Interpolation method (EIM).The interpolation points (x_p) are obtained as follows:

$$x_p = arg \max_{x \varepsilon \Omega} |V^p - \sum_{q=1}^{p-1} V^q c_q|$$

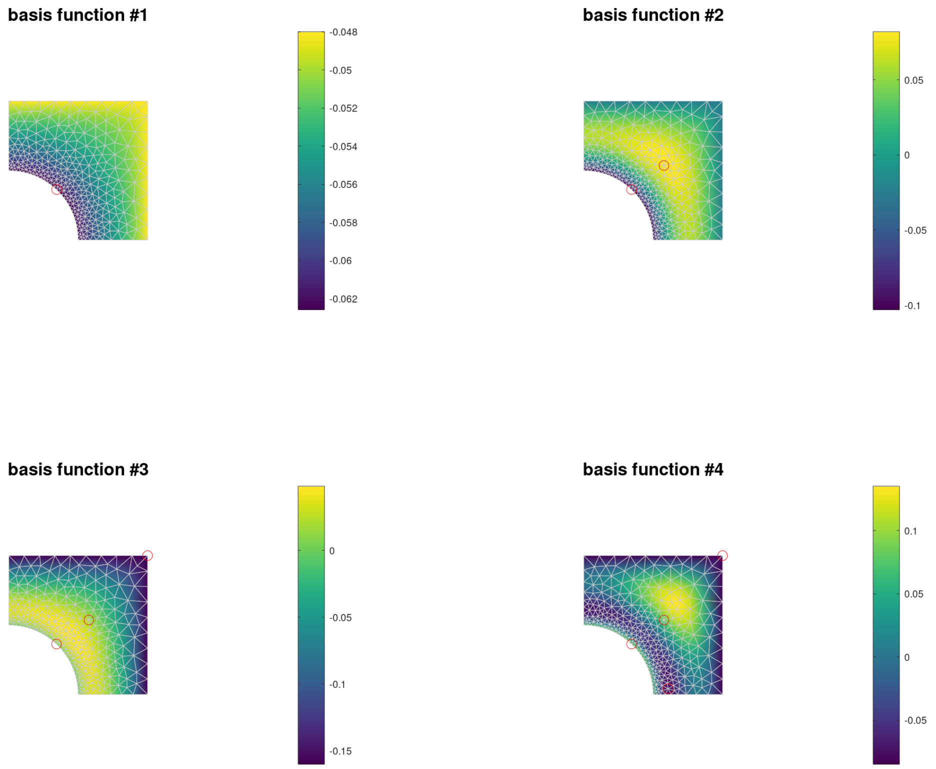

Figure 4: First Four Dominant vectors of the Non-Linear Term

3.4 Reduced Order Solution

From the interpolation points obtained,the non-linear co-efficients (β) are calculated as follows:

$$\alpha_{in}^{l-1} \rightarrow u_n^{l-1}(x_p) \approx \sum_{i=1}^{d} B^i(x_p)\alpha_{in}^{l-1}$$

$$k^{l-1}(x_p) = k(u_n^{l-1}(x_p))$$

$$k^{l-1}(x_p) = \sum_{l=1}^{d'} V^l(x_p)\beta_{ln}^{l-1}$$

$$\beta = V^{-1}(x_p)k(x_p)$$

After the values of the non-linear co-efficients(β) are obtained.The reduced terms are arranged in terms of affine decomposition.Then the solution for the reduced system is obtained.The contour plot is obtained as follows:

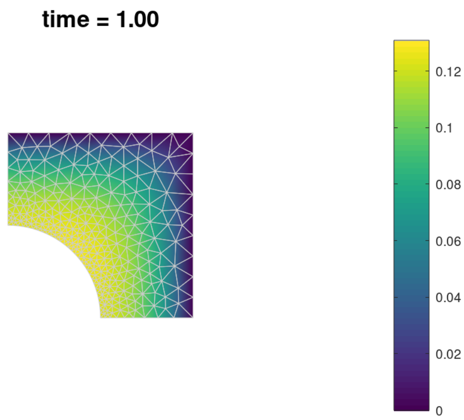

Figure 5: Contour Plot of the Reduced Order Solution

4 Spectral Properties

The spectral properties of original FEM and reduced conductivity matrices are studied in this section.The original FEM conductivity matrix is symmetric.All the eigen values of this matrix are positive .i.e $\lambda_i > 0$ $\forall i$.So,this matrix is a Symmetric Positive Definite matrix .i.e $x^T Kx > 0$ $\forall x \neq 0$.

If we consider each of the reduced-order conductivity matrices, the matrices obtained are not positive definite. The matrices have at least one negative eigenvalues. But if we consider the final reduced-order conductivity matrices, the matrix is symmetric. The eigenvalues are positive.So,this matrix is a Symmetric Positive Definite matrix .i.e $x^T Kx > 0$ $\forall x \neq 0$.

5 Error Analysis

The error is calculated between the 2D FEM solution and Reduced Order solution using the L2 norm.The threshold considered is 1.e-4.The error is calculated at each time step.First,the absolute error is taken at each time step as follows:

$$variable = abs(fem.solution - rom.solution)$$

Then the L2-norm is calculated at each time step as follows:

$$error_i = \sqrt{\sum |variable|_i^2}$$

where,i represents the column of each time step.

The below figure represents the plot of Error vs Time. As seen from the graph, it can be seen that the error increases to a certain maximum value and decreases as the time step increases. This behavior of the L2 norm may be due to the chosen threshold or also due to the interpolation points considered to calculate the conductivity.

The log graph of the relative error vs time is shown below. As seen from the graph of relative error,the order of error is decreased as the time is increased.The relative error was in the order of 10^{-3}.

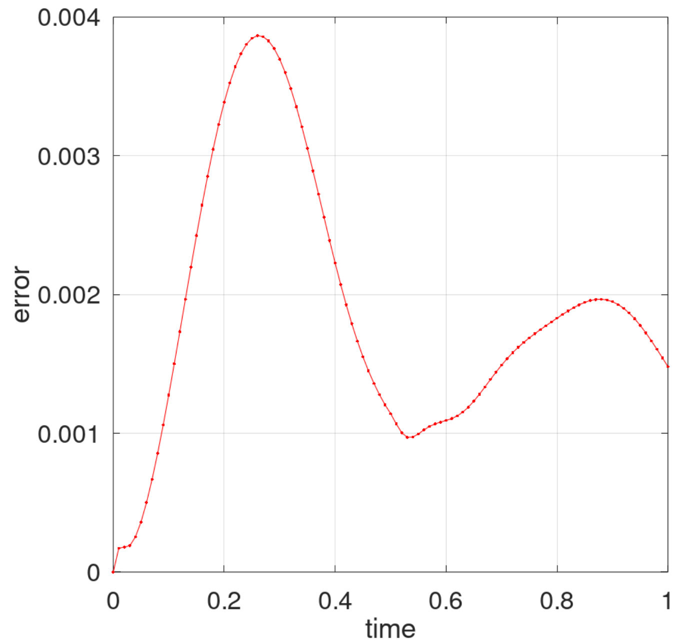

Figure 6: Evolution of L2 norm error for threshold=1.e-4

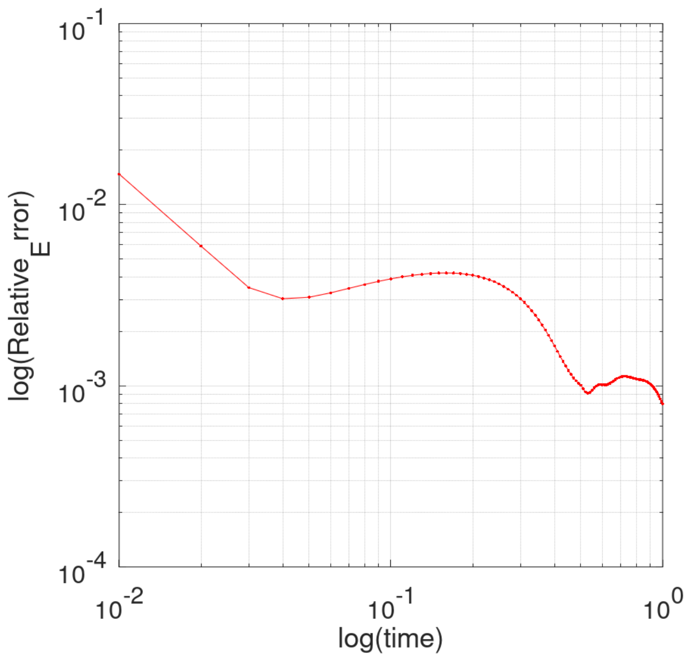

Figure 7: Evolution of Relative Error for threshold=1.e-4

6 Computational Time

The computational time of the both FEM solver and reduced order solver for various time span is shown in the table and plot below

Time(secs)	FEM(secs)	Reduced Order(secs)
0.5	6.2792	0.0326
1	12.9201	0.0683
1.5	18.7737	0.0951
2	24.6256	0.1297
5	60.3543	0.3176
8	89.8181	0.5326

Table 1: Computational time of FEM and Reduced order solvers for different time span

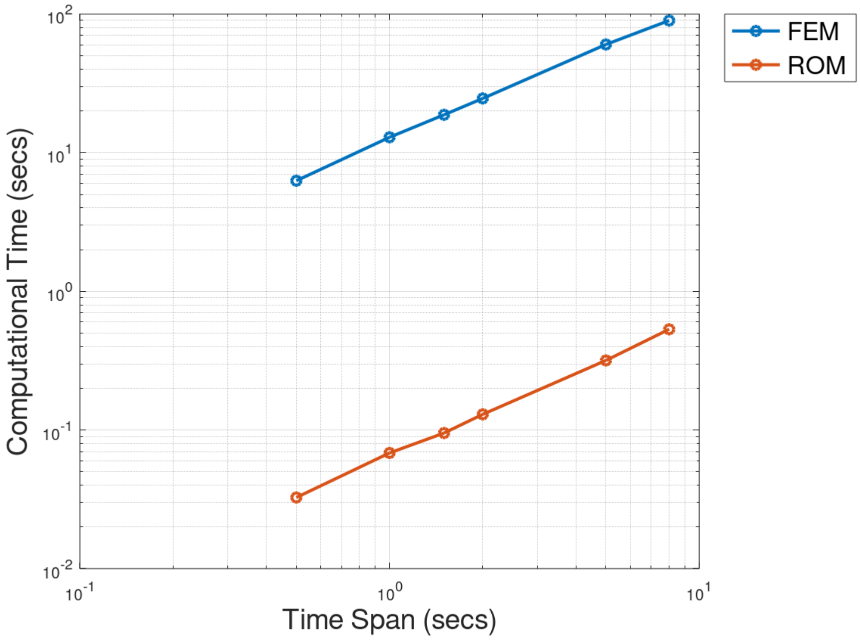

Figure 8: Plot for Computational time of FEM and ROM for different time span

As it can be observed from the table, the reduced-order solver is much more efficient than the Full order 2D FEM solver. But, to work in Reduced-order, the time taken to get the reduced basis must also be considered. Even with the inclusion of this time, the reduced-order solver is much efficient than the Full order FEM solver.

7 Conclusion

In conclusion, The full order FEM model was run. The non-linear term was evaluated using fixed-point iterations. The reduced basis was computed using SVD on the solution and non-linear terms. Then, the reduced vectors were used to compute the reduced-order system. Then using the Emprical interpolation method, the suitable interpolation points were obtained. Using the interpolation points, the co-efficient of thermal conductivity was computed. The reduced-order model system was constructed. Then the system was simulated.

It was observed that the computational time taken by the reduced-order was much less compared to the full-order FEM solution. It was more effective when working on a higher complexity system. The relative error was in the order of 10^{-3}. So, the Proper orthogonal Decomposition with Emprical Interpolation Method is an efficient model order reduced method with certain error bounds when compared to the Full order FEM.