LAB - COUPLED 1D LINEARIZED THERMO-ELASTIC PROBLEM

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1 Introduction

In a coupled system, physically or computationally heterogeneous mechanical components interact dynamically. In the sense that the response must be obtained in more than one way. As the interaction is multiway, Heterogeneity is used in the sense of solving the coupled equations that model the system at the same time.

Coupling effects can be classified as mild or powerful depending on the influence of one system on the other. The coupling system is termed weak, if one subsystem influences the evolution of another, with no discernible reverse effect, but a strong coupling is such that the growth of subsystems is influenced by each other in both directions.

In extreme-scale computational environments, the promise of coupled multiphysics simulation will not be realized hence the computation is performed by divide-and-conquer operator splitting. This is the proposed approach (ArmeroSimon1992) associated with a two-phase operator split of the full non-linear system of thermoelasticity into an adiabatic elastodynamics phase, followed by a heat conduction phase at fixed configuration. Monolithic techniques, while appearing to be more realistic in terms of the physics accounted for, have shown to be an unreliable solution for large-scale challenges. They don't appear to be the advantage of the many time scales available in the sub-problems, squandering computational resources in the process. Staggered techniques based on an operator split are investigated as viable substitutes for monolithic approaches.

We are going to study the 1-D linearized thermoelastic model, as it is a strongly coupled problem, we are going to investigate the monolithic, Isothermal, and Adiabatic Staggered techniques.

1.1 Problem Statement

The length of the bar is considered to be L, with a cross-sectional area S. The bar is made up of material obeying the linear thermo-elastic behavior with Young's Modulus (E), density (ρ), specific heat capacity (C), thermal expansion coefficient (α) and thermal conductivity (K). The bar is experiencing Dirichlet boundary conditions on both ends. The behavior of the bar is described by the axial displacement field u(x,t) and temperature variations θ(x,t).

We Solve this system using a variety of techniques namely monolithic, staggered isothermal and staggered adiabatic.

2 Boundary Value Problem

This section highlights the thermoelasticity governing equations and covers a number of major results on the linearized problem's mathematical structure. The numerical analysis relies heavily on these findings. In fact, the algorithms' structure is influenced by the theoretical results presented below.

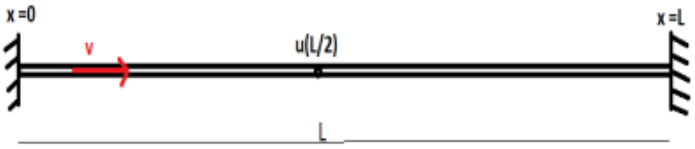


Figure 1: 1D bar fixed at both ends

2.1 1D Linearized thermo-elastic case:

2.1.1 Constitutive equation:

The equation governing Linear thermo-elastic materials can be described through the following free energy density:

$$W(\varepsilon,\theta)=\frac{1}{2}(\varepsilon-\alpha\theta):H:(\varepsilon-\alpha\theta)-\frac{C_\sigma\theta^2}{2T_0}$$

where,
C_σ is constant-stress heat capacity
T_0 reference temperature
θ is the relative temperature
α thermo-elastic dilatancy tensor

The stress tensor and the entropy density are then given by:

$$\sigma=\frac{\partial W}{\partial\varepsilon}=H:(\varepsilon-\alpha\theta)\eta=-\frac{\partial W}{\partial\theta}=C_\varepsilon\frac{\theta}{T_0}+\alpha:H:\varepsilon$$

where C_ε is the constant strain heat capacity.
The thermal conduction given by Fourier's law is,

$$q=-k.\nabla\theta$$

where k is the thermal conductivity tensor.

2.1.2 Balance Equations:

In the interest of problem we consider linear momentum and conservation of energy subjected to no body forces or heat sources ,

$$\rho\ddot{u}=\frac{\partial}{\partial x}(E\frac{\partial u}{\partial x})-\alpha E\theta C\dot{\theta}=\frac{\partial}{\partial x}(k\frac{\partial\theta}{\partial x})-\alpha E\theta_{\text{ref}}\frac{\partial v}{\partial x}$$

Let,

$$\dot{u}=v,\quad \tilde{C}=\frac{C_\sigma}{\theta^2}\tilde{k}=\frac{k}{\theta_{\text{ref}}}\text{and}\quad M=H:\sigma$$

The equations above can be written in matrix form by using the operator split,

$$\begin{Bmatrix}\dot{u}\\ \dot{v}\\ \dot{\theta}\end{Bmatrix}=\begin{Bmatrix}v\\ \frac{1}{\rho}(\frac{\partial}{\partial x}(E\frac{\partial u}{\partial x})-M\theta)\\ \frac{1}{\tilde{C}}\frac{\partial}{\partial x}(k\frac{\partial\theta}{\partial x})-\alpha E\theta_{\text{ref}}\frac{\partial v}{\partial x}\end{Bmatrix}$$

2.1.3 Boundary Equations:

Initial Bounday conditions (Dirichlet):

$$u(0,t) = u(L,t) = 0$$
$$v(0,t) = u(L,t) = 0$$
$$\theta(0,t) = \theta(L,t) = 0$$

The velocity is assumed to be the first eigen mode corresponding to the free vibration of the mechanical system.

$$u(x,0) = sin(\frac{\pi}{L})$$

$$v(x,0) = 0$$
$$\theta(x,0) = 0$$

3 Monolithic approach:

3.1 Weak formulation and discretization:

We discretize the above matrix as ,

$$u(x,t) = \sum_{a}^{Nodes} N_a(x)u_a(t) \quad \partial u(x,t) = \sum_{a}^{Nodes} N_a(x)\partial u_a(t);$$

$$v(x,t) = \sum_{a}^{Nodes} N_a(x)v_a(t) \quad \partial v(x,t) = \sum_{a}^{Nodes} N_a(x)\partial v_a(t)$$

$$\theta(x,t) = \sum_{a}^{Nodes} N_a(x)u_a(t) \quad \partial \theta(x,t) = \sum_{a}^{Nodes} N_a(x)\partial \theta(t)$$

After the **Spatial** discretization, we derive the following discrete system by substituting, using the divergence theorem when necessary, and minimizing the energy potential:

$$M\dot{u} = Mv$$

$$M\dot{v} + Eu = B\theta$$
$$C\dot{\theta} + K\theta = -B_T v$$

3.2 Time discretization

We employ the approximate solution of the generalized mid-point scheme, which is given as follows, to perform numerical time integration of evolution differential equations:

$$Mu^{n+1} - h\beta Mv^{n+1} = Mv^n + h(1-\beta)Mv^n h\beta u^{n+1} E + Mv^{n+1} - h\beta B\theta^{n+1}$$

$$= -h(1-\beta)Eu^n + Mv^n + h(1-\beta)B\theta^n h\beta B^T v^{n+1} + (C + h\beta K)\theta^{n+1}$$

$$= -h(1-\beta)B^T v^n + Mv^n + (C - h(1-\beta)K)\theta^n$$

In a monolithic approach, the whole system of equations is solved at once (fully coupled approach). Using the generalized mid-point scheme,we obtain:

$$\begin{bmatrix} \mathbf{M} & -h\beta\mathbf{M} & \mathbf{0} \\ h\beta\mathbf{E} & \mathbf{M} & -h\beta\mathbf{B} \\ \mathbf{0} & h\beta\mathbf{B}^T & \mathbf{C}+h\beta\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \\ \theta^{n+1} \end{Bmatrix} = \begin{bmatrix} \mathbf{M} & h(1-\beta)\mathbf{M} & \mathbf{0} \\ -h(1-\beta)\mathbf{E} & \mathbf{M} & h(1-\beta)\mathbf{B} \\ \mathbf{0} & -h(1-\beta)\mathbf{B}^T & \mathbf{C}-h(1-\beta)\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{n} \\ \mathbf{v}^{n} \\ \theta^{n} \end{Bmatrix}$$

The choice of $\beta = 0.5$ to Crank-Nicolson method for both the stages, providing second order convergence.

4 Staggered approach:

4.1 Isothermal Approach

In the isothermal staggered approach, the resolution of each sub-system done with some prediction of other field(s). The Prediction is updated at each time increment. Here the mechanical problem is first solved at given temperature (for a given time step), and the thermal problem is then solved, accounting for the computed displacements and velocities over the time step. Such an approach brakes the contractive nature of the problem thereby loosing the unconditional numerical stability.

In the mechanical stage, the temperature is fixed and the subproblem is resolved using a temperature predictor with source term. The linear system for the mechanical stage is,

$$\begin{bmatrix} \mathbf{M} & -h\beta_m\mathbf{M} \\ \beta_m h\mathbf{E} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & h(1-\beta_m)\mathbf{M} \\ -h(1-\beta_m)\mathbf{E} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n} \\ \mathbf{v}^{n} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ h\mathbf{B}\theta^n \end{bmatrix}$$

In thermal stage, the thermal problem is resolved for a fixed mechanical configuration with a source term accounting the displacements and velocities computed in the mechanical stage. The linear system for the thermal stage is,

$$(\mathbf{C} + h\beta_t\mathbf{K})\{\theta_{n+1}\} = (\mathbf{C} - (1-\beta_t)h\mathbf{K})\{\theta_n\} - h\mathbf{B}^T\{v_{n+1}\}$$

The global system of the staggered Isothermal approach is shown as below,

$$\begin{bmatrix} \mathbf{M} & -h\beta_m\mathbf{M} & \mathbf{0} \\ \beta_m\mathbf{E} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & h\beta_t\mathbf{B}^T & \mathbf{C}+h\beta_t\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \\ \theta^{n+1} \end{Bmatrix} = \begin{bmatrix} \mathbf{M} & h(1-\beta_m)\mathbf{M} & \mathbf{0} \\ -h(1-\beta_m)\mathbf{E} & \mathbf{M} & h\mathbf{B} \\ \mathbf{0} & -h(1-\beta_t)\mathbf{B}^T & \mathbf{C}-h(1-\beta_t)\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{n} \\ \mathbf{v}^{n} \\ \theta^{n} \end{Bmatrix}$$

4.2 Adiabatic Approach

In the adiabatic staggered scheme, the mechanical problem is first solved (over a time step) under adiabatic conditions (no conduction). The thermal problem is then solved, starting from the state resulting from the first stage. The choice of $\beta_m = 0.5$ corresponds to the Crank-Nicolson method for the adiabatic mechanical stage and $\beta_t = 1.0$ corresponds to the Euler scheme for the thermal stage are made.

The mechanical stage(Isothermal):

$$\begin{bmatrix} \mathbf{M} & -h\beta_m\mathbf{M} \\ \beta_m h\mathbf{E}_{ad} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & h(1-\beta_m)\mathbf{M} \\ -h\mathbf{E} + \beta_m h\mathbf{E}_{ad} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n} \\ \mathbf{v}^{n} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ h\mathbf{B}\theta^n \end{bmatrix}$$

where,$\mathbf{E}_{ad} = \mathbf{E} + \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ and θ^n is the temperature which is predicted at the end of this stage.

Then for the thermal stage,

$$\begin{aligned} \mathbf{C}(\{\theta_{n+1}\} - \{\theta_n\}) + h\mathbf{B}^T(\{u_{n+1}\} - \{u_n\}) = -h\mathbf{K}(\beta_t\{\theta_{n+1}\} + (1-\beta_t)\{\theta_n\}) \\ + (1-\beta_t)h\mathbf{K}\mathbf{C}^{-1}\mathbf{B}^T(\{u_{n+1}\} - \{u_n\}) \end{aligned}$$

Considering,the $\beta_t = 1$,we get the global system as,

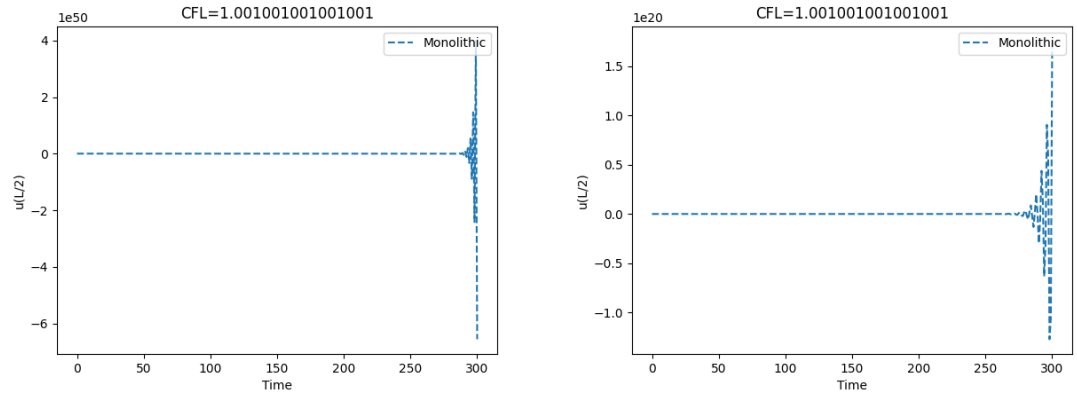
$$\begin{bmatrix} \mathbf{M} & -h\beta_m\mathbf{M} & \mathbf{0} \\ \beta_m h\mathbf{E}_{ad} & \mathbf{M} & \mathbf{0} \\ h\mathbf{B}^T & \mathbf{0} & \mathbf{C} + h\mathbf{K} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \\ \theta^{n+1} \end{Bmatrix} = \begin{bmatrix} \mathbf{M} & h(1-\beta_m)\mathbf{M} & \mathbf{0} \\ -h\mathbf{E} + h(\beta_m)\mathbf{E}_{ad} & \mathbf{M} & h\mathbf{B} \\ h\mathbf{B}^T & \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{u}^{n} \\ \mathbf{v}^{n} \\ \theta^{n} \end{Bmatrix}$$

5 Results and Discussions:

At CFL = 1, all the coupled numerical approaches are in good agreement with each other giving a similar solution field for both evolution of displacement and temperature. While for CFL = 3 it's seen that only the monolithic approach is observed to be stable.

5.1 Monolithic

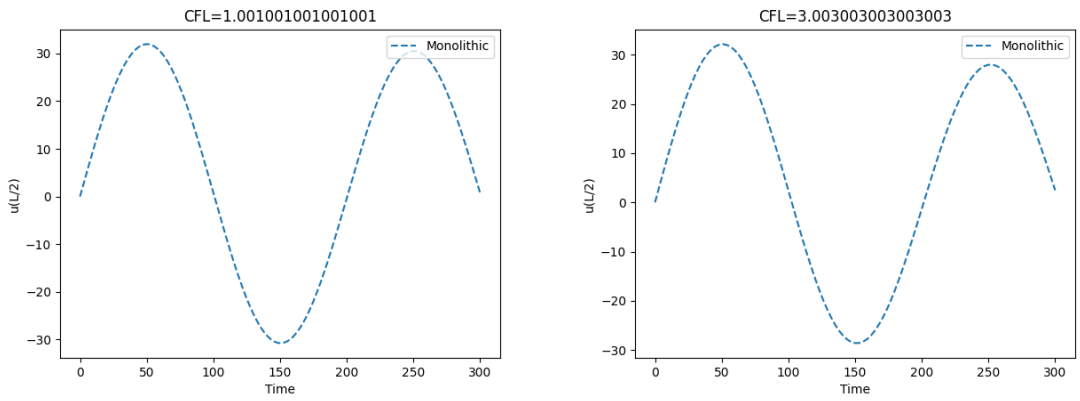
The computations were performed for CFL = 1 and 3.To check the numerical stability of the system,the values of β were set to 0.3 and 0.4 for CFL=1. The obtained results of evolution of displacement with time is shown in the figure 1. We can observe that the system exhibits very stable behaviour until T= 250(approximately) and it gradually grows yet very fast development of fluctuations. But for the $\beta = 0.5$,the numerical stability can be observed.



(a) $\beta = 0.3$ (b) $\beta = 0.4$

Figure 2: Evolution of Displacement

For value of β is set 0.5 for L/2, figure 2 shows the displacement evolution at the middle of the bar for CFL = 1 and CFL = 3 using the two ways to better showcase the accuracy of a monolithic approach.



(a) CFL=1 (b) CFL=3

Figure 3: Evolution of Displacement

From the figure-4 representing in the monolithic approach, for the parameter $\beta = 0.5$ and CFL=1 there is no temperature difference throughout the time span.

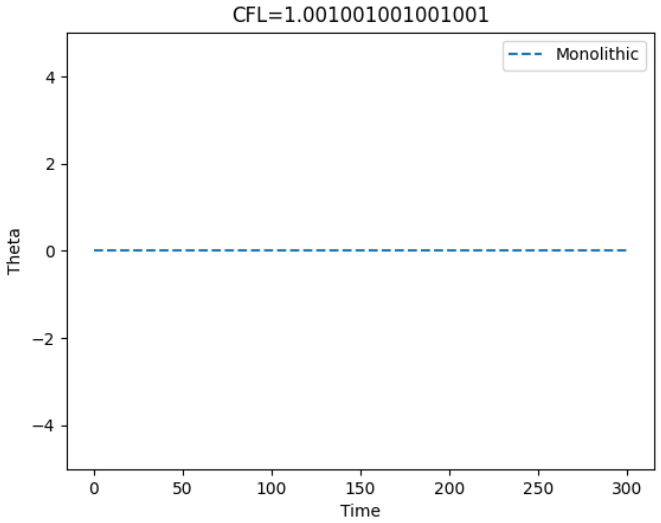
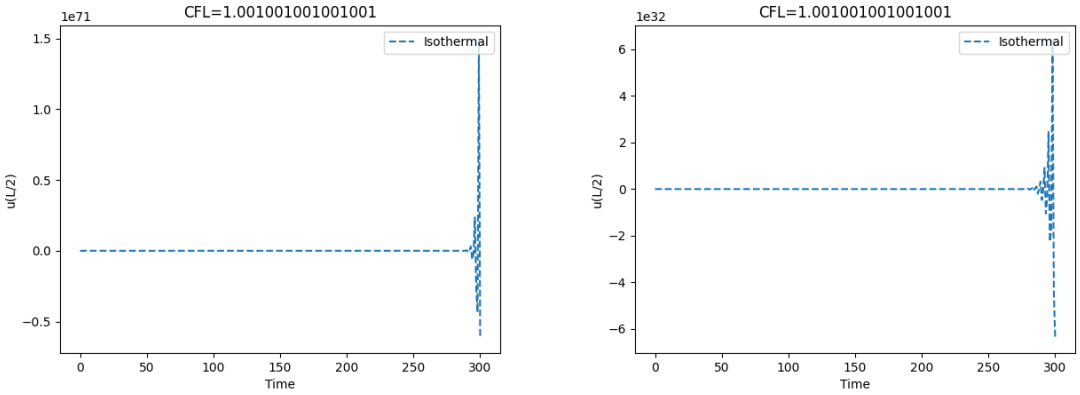


Figure 4: Evolution of Temperature for $\beta = 0.5$

5.2 Staggered-Isothermal

From the figure 5, depicting the evolution of displacement for the parameter CFL=1.0, $\beta = 0.3$ and $\beta = 0.4$, it can be infered that the displacement at L/2 appears to be oscillatory, While from the figure 6, the stability can be observed for $\beta = 0.5$ which in turn confirms that staggered isothermal is conditionally stable.But for CFL = 3, the solution was not attained for the considered von Neumann stability condition.



(a) $\beta = 0.3$ (b) $\beta = 0.4$

Figure 5: Evolution of Displacement

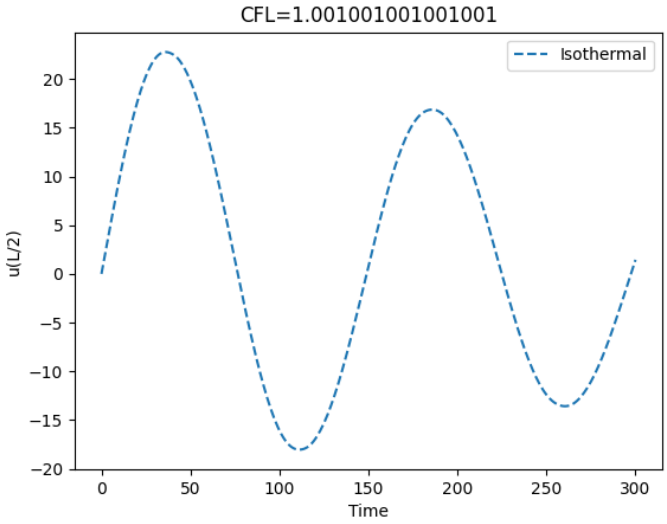


Figure 6: Evolution of Displacement for $\beta = 0.5$

The obtained results of evolution of temperature with time is shown in the figure 7. We can observe that unlike monolithic and adiabatic, here the system exhibits very noisy behaviour but also maintaining some close level of oscillations with the neighbouring time sets.

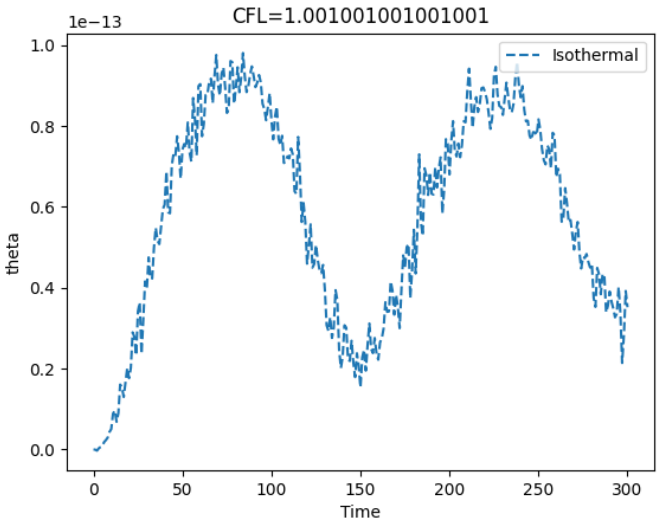


Figure 7: Evolution of Temperature for $\beta = 0.5$

5.3 Staggered-Adiabatic

It can be seen that the adiabatic solution figure 8 agrees fairly well with the monolithic solution. This level of accuracy is acceptable when considering the computational savings of the adiabatic technique.

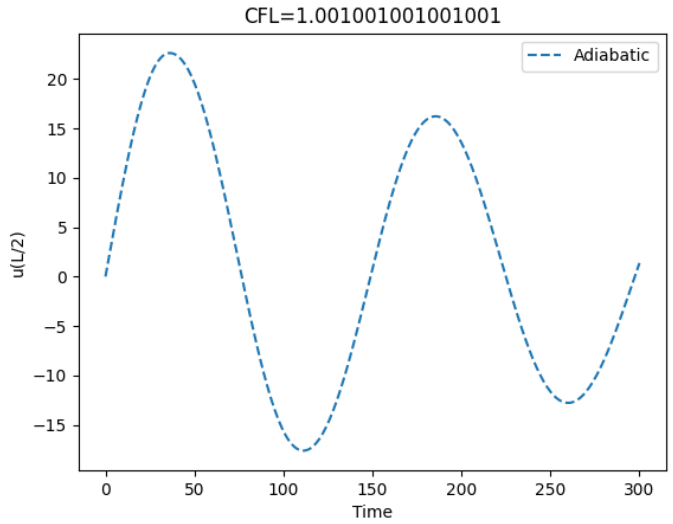


Figure 8: Evolution of Displacement for $\beta = 0.5$

For temperature at L/2, the Monolithic solution is stable while the Adiabatic and Isothermal solutions are both oscillatory (figure 9).

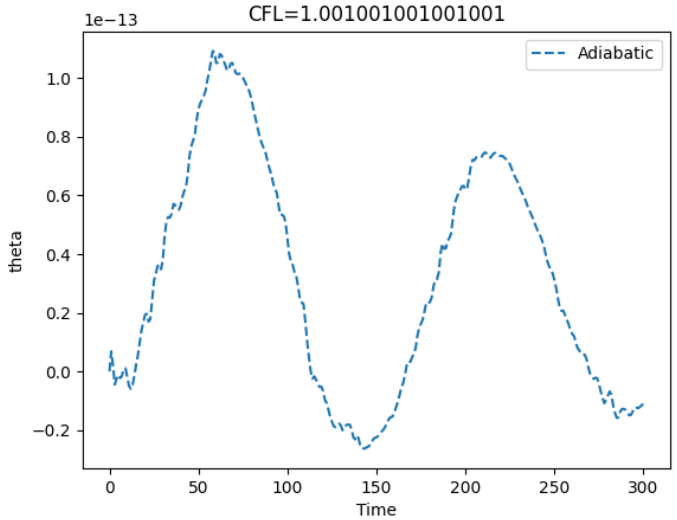


Figure 9: Evolution of Temperature for $\beta = 0.5$

5.4 Comparison

From the figure 10, depicting the comparison of the evolution of displacement for different approaches, for parameters CFL=1.0 and $\beta = 0.5$, it can be infered that the isothermal and adiabatic approaches followed the same trend but with a small change in the amplitude after T=100 which prevailed till T=300. Also there is difference in the amplitude peak between Monolithic and the other two approach. From the figure 11, depicting the comparison of the

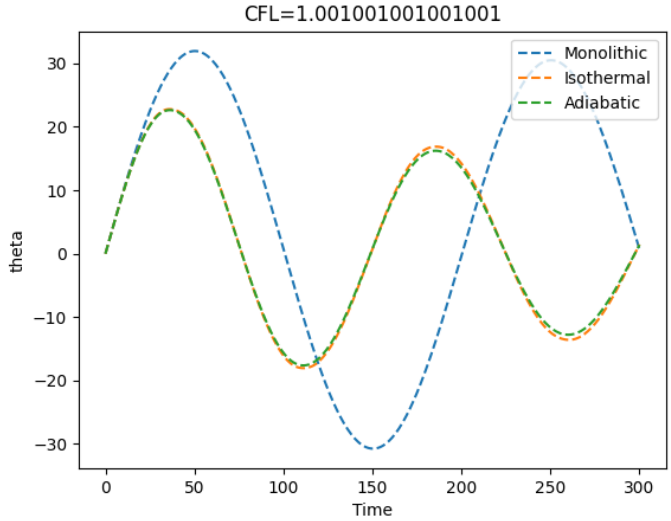


Figure 10: Comparison of the displacement evolution for different approaches

evolution of temperature for different approaches at L/2, for parameters CFL=1.0 and $\beta = 0.5$, it can be infered that there is a delay between isothermal and adiabatic approaches.

It can be observed that the Monolithic, Isothermal and Adiabatic solutions are all stable for CFL = 1. Whereas for CFL = 3, the Monolithic and Adiabatic solutions are stable, while the Isothermal split shows disastrous oscillations, confirming the split's conditional stability. The oscillations in the displacements are caused by this unsteady computation. Moreover we see that oscillations begins first in temparature followed by oscillations in displacement.