Ordinary Differential Equation of Higher Order

Unit: I

Subject Name
Mathematics-II

Course Details
B .Tech 2nd Sem

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Department
Mathematics

Content

- Introduction
- Linear differential equation of nth order with constant coefficients
- Simultaneous linear differential equations
- Second order linear differential equations with variable coefficients
- Solution by changing independent variable
- Reduction of order, Normal form
- Method of variation of parameters
- Cauchy-Euler equation, Series solutions (Frobenius Method)

Course Objective

- The objective of this course is to familiarize the engineering students with techniques of solving Ordinary differential equations, Fourier series expansion, Laplace transform and vector calculus and its application in real world. It aims to equip the students with adequate knowledge of mathematics that will enable them in formulating problems and solving problems analytically.

Course Outcome

Course Name: Mathematics-II	
CO1	**Apply the concept of differentiation to solve differential equations.**
CO2	Apply the concept of convergence of sequence and series to evaluate Fourier series
CO3	Apply the Laplace transform to solve ordinary differential equations
CO4	Apply the concept of vector calculus to evaluate line, surface and volume integrals.
CO5	Demonstrate the basic concept Proportion & Partnership, Problem of ages, Allegation & Mixture, Direction, Blood relation , Simple & Compound interest

Program Outcomes (POs)

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis:** Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

Program Outcomes (POs)

5. **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

Program Outcomes (POs)

9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication:** Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

CO-PO Mapping

Course Name: Mathematics-II

CO	PO-1	PO-2	PO-3	PO-4	PO-5	PO-6	PO-7	PO-8	PO-9	PO-10	PO-11	PO-12
CO1	**3**	**3**	**3**	**3**	**3**	**1**	**2**	-	-	**2**	**2**	**3**
CO2	3	3	3	2	2	-	-	-	-	2	2	1
CO3	3	2	3	2	3	1	-	-	-	2	2	2
CO4	3	2	3	2	3	1	1	-	-	2	-	3
CO5	1	1	1	1	1	-	-	-	-	2	-	2
Mean	**2.6**	**2.2**	**2.6**	**2**	**2.4**	**1.0**	**1.5**	-	-	**2**	**2**	**2.2**

Prerequisite and Recap

- Basic Knowledge of Mathematics-I (B.Tech)
- Idea about differential coefficients.
- Differential Equation of First Order First Degree.

Unit Objective (CO1)

- ➢ Understand the concept of differentiation and apply for solving differential equations.
- ➢ Differential equation use in solving problem of electrical circuit, mechanical vibrations and other field in engineering.
- ➢ Solution of differential equation in series form which produce special functions.

Differential Equation(CO1)

- **Definition**

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Differential Equation

Ordinary Differential Equation(ODE)

Partial Differential Equation(PDE)

Differential Equation(CO1)

Examples:

$$dy = (x + \sin x)\ dx \qquad --(1)$$

$$\frac{d^4 x}{dt^4} + \frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \qquad --(2)$$

$$y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{dy/dx} \qquad --(3)$$

Differential Equation(CO1)

$$k\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad --(4)$$

$$\frac{\partial^2 v}{\partial t^2}=k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \qquad --(5)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \qquad --(6)$$

Differential Equation(CO1)

- **Order:** The order of the highest order derivative involved in a differential equation is called the *order of the differential equation*.

- **Degree:** The *degree of a differential equation* is the degree of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fraction as far as the derivatives are concerned.

- **Linear and non-linear differential equation:**

A differential is called *linear* if

(i) Every dependent variable and every derivative involved occurs in the first degree only, and

(ii) No product of dependent variable and/or derivatives occur.

*** A differential equation which is not linear is called a ***non-linear differential equation.***

Differential Equation(CO1)

Differential Equation → Solution

General (or Complete) solution of an *n*th order D.E. will have *n* arbitrary constants.

Order(D.E) = Number of constants in solution

Formation of differential equations(CO1)

- Differential equations are formed by elimination of arbitrary constants.
- To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order.
- Elimination of n arbitrary constants lead us to nth order derivatives and hence a differential equation of the nth order.

Example

1. Eliminate the arbitrary constant a and b from the equation $y = e^x(a\cos x + b\sin x)$ and obtain the differential equation.

Sol. We have the relation $y = e^x(a\cos x + b\sin x)$ ..(1)

Differentiating equation (1) w.r.t. *x*, we have

$$y' = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x)$$

$$y' = y + e^x(-a\sin x + b\cos x) \qquad ..(2)$$

Differentiating equation (2) w.r.t. *x*, we have

Continued....

$$y'' = y' + e^{x}(-a\sin x + b\cos x) + e^{x}(-a\cos x - b\sin x)$$

$$y'' = y' + (y' - \mathrm{y}) - \mathrm{y}$$ Using (1) and (2)

or $$y'' - 2y' + 2\,\mathrm{y} = 0$$

Which is the required differential equation.

The Wronskian's (CO1)

Definition

The Wronskian of n functions $y_1(x), y_2(x), y_3(x)$ ………… $y_n(x)$

is denoted by W and is defined to be determinant

$$W(x) = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ . & . & . & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

The Wronskian's (CO1)

Linearly dependent and independent set of functions

Functions $y_1(x)$, $y_2(x)$, $y_3(x)$ ………… $y_n(x)$ are linearly dependent if

$$W(x) = 0$$

Functions $y_1(x)$, $y_2(x)$, $y_3(x)$ ………… $y_n(x)$ are linearly independent if

$$W(x) \neq 0$$

Example

1. Prove that the function 1, x, x^2 are linearly independent. Hence form the differential equation.

Sol. Let $y_1 = 1$, $y_2 = x$ and $y_3 = x^2$

The Wronskian is given by

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

Therefore y_1 , y_2 and y_3 are linearly independent.

Continued...

The general solution of required differentail equation written as

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 = c_1 + c_2 x + c_3 x^2 \qquad ..(1)$$

Differentiating (1), we get $\quad y' = c_2 + 2c_3 x \qquad ..(2)$

Differentiating (2), we get $\quad y'' = 2c_3 \qquad ..(3)$

Diff (3), we get $\quad y''' = 0 \quad or \quad \dfrac{d^3 y}{dx^3} = 0 \qquad ..(4)$

Equation (4) is the required solution.

Exercise

Q. Obtain the differential equation from each of the following functions by elimination of arbitrary constants:

$$(i) \quad y = c_1 e^{x} + c_2 e^{-2x}$$

$$(ii) \quad y = a\cos(\log x) + \mathrm{b}\sin(\log x)$$

$$(iii) \quad y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$(iv) \quad y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

FAQ

Q1. Show that the following pair of functions are linearly independent (with W≠0) and form the differential from:

$$(i)\quad \left\{e^{x}, xe^{x}\right\}$$

$$(ii)\quad \left\{e^{ax}\sin bx, e^{ax}\cos bx\right\}$$

Q2. Determine the differential equation whose set of independent solution is $\left\{e^{x}, xe^{x}, x^{2}e^{x}\right\}$

Daily Quiz

1. $y = cx - c^2$, is the general solution of the differential equation............

2. Order and degree of DE $\dfrac{d^2y}{dx^2} + \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = 0$ is..........

3. If $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$ then differential equation is............

Recap

- ✓ Introduction of Differential Equation.
- ✓ Order and Degree of Differential Equation.
- ✓ Formation of Differential Equation.

Objective of topic(CO1)

Linear differential equation with constant coefficients:

- Complete Solution of linear differential equation with constant coefficients.
- By Different method finding Particular Integral.
- General Method to finding solution.

Linear differential equation of n^{th} order with constant coefficients(CO1)

Linear differential equation with constant coefficients:

A differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots\ldots\ldots\ldots + a_n y = X \qquad (1)$$

Where X is a function of x only and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are constants.

$$or \qquad (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots a_n) y = X \qquad (2)$$

$$or \qquad f(D) y = X \qquad (3)$$

Linear differential equation with constant coefficients (CO1)

Thus, the general solution of (1) is y = C.F. + P.I.,

where C.F. involves n arbitrary constant and P.I. does not involve any arbitrary constant.

C.F. = Complementary function

P.I. = Particular Integral

Complementary function(CO1)

➢Working rule for finding complementary function

We first write the equation in symbolic form *f(D)y = X*

Write auxiliary equation, namely, *f(m) = 0* which will be an algebraic equation of nth degree.

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots\dots + a_n = 0$$

On solving the auxiliary equation we shall get *n* roots. Three cases arise, according as the roots of the auxiliary equations.

Complementary function(CO1)

Case 1: Real roots

(i) First suppose that the auxiliary equation has *n* distinct roots $m_1, m_2, \ldots\ldots., m_n$ then

C.F. = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots\ldots\ldots + c_n e^{m_n x}$

(ii) Suppose that the auxiliary equation has *n* equal roots $m_1 = m_2 = \ldots\ldots., = m_n$ then

C.F. = $(c_1 + c_2 x + c_3 x^2 + \ldots\ldots\ldots\ldots + c_n x^{n-1}) e^{m_1 x}$

For Example: Let us solve $\left(D^2 - 5D + 6\right) y = 0$

Here auxiliary equation is $m^2 - 5m + 6 = 0$

Which gives m = 2, 3 (distinct roots)

Complementary function(CO1)

Hence CF is $c_1e^{2x} + c_2e^{3x}$

Since here X= 0, so the general solution is

$$y = c_1e^{2x} + c_2e^{3x}$$

Example: Solve the following

$(i)\ \left(D^2 - 3D + 2\right)y = 0$

$(ii)\ \left(D^3 - 3D + 2\right)y = 0$

Complementary function(CO1)

Case 2: Complex roots

(i) Let be a $\alpha \pm i\beta$ complex roots. Then

C.F. = $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$$or \;\; c_1 e^{\alpha x} \cos(\beta x + c_2), \;\; or \;\; c_1 e^{\alpha x} \sin(\beta x + c_2)$$

(ii) Repeated roots. Let α±iβ , α±iβ be two equal complex roots. Then

C.F = $e^{\alpha x}\left\{(c_1 + xc_2)\cos \beta x + (c_3 + xc_4)\sin \beta x\right\}$,

Example: Solve $\left(D^2 - 2D + 5\right)y = 0$

Here auxiliary equation is $m^2 - 2m + 5 = 0$

Complementary function(CO1)

Which gives m = *-1± i* (complex roots)

CF is $e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$

Since here X = 0, so genereal solution is

$$y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

Example: Solve the following:

(i) $\left(D^2 - 2D + 3\right) y = 0$

(ii) $\left(D^3 - 2D^2 + 4D - 8\right) y = 0$

Complementary function(CO1)

Case 3: Surds roots

(i) Let $\alpha \pm \sqrt{\beta}$ be a surds roots. Then

$$\text{C.F.} = e^{\alpha x}[c_1 \cosh(\sqrt{\beta}\, x) + c_2 \sinh(\sqrt{\beta}\, x)]$$

(ii) Repeated roots. Let $\alpha \pm \sqrt{\beta}$ $\alpha \pm \sqrt{\beta}$ be two equal surds roots. Then

$$\text{C.F} = e^{\alpha x}\left\{(c_1 + xc_2)\cosh(\sqrt{\beta}\, x) + (c_3 + xc_4)\sinh(\sqrt{\beta}\, x)\right\},$$

Example: Solve $\left(D^2 - 4D + 1\right)y = 0$

Here auxiliary equation is $m^2 - 4m + 1 = 0$

Complementary function(CO1)

which gives $m = 2 \pm \sqrt{3}$ (surds)

Hence CF is $e^{2x}[c_1 \cosh(\sqrt{3}\,x) + c_2 \sinh(\sqrt{3}\,x)]$

Since X = 0, so the general solution is

$$y = e^{2x}[c_1 \cosh(\sqrt{3}\,x) + c_2 \sinh(\sqrt{3}\,x)]$$

Example: Solve the following:

$(i)\ \left(D^2 + 6D + 4\right)y = 0$

$(ii)\ \left(D^3 - 5D^2 + 5D - 1\right)y = 0$

Particular Integral Topic(CO1)

➢Determination of the Particular Integral (P.I.)

Let the given differential equation be

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ...a_n)y = X$$

$$or \qquad f(D)y = X$$

Then $\quad P.I. = \dfrac{1}{f(D)} X$

Working rule for P.I.

Formula I: When $X = e^{ax}$

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \ \ where \ \ f(a) \neq 0.$$

$$If \ \ f(a) = 0, then \ \ P.I. = x \frac{1}{f'(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}, \ \ where \ \ f'(a) \neq 0.$$

Example

1. Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$

Sol. The given equation in symbolic form is

$$\left(D^2 - 3D + 2\right)y = e^{3x}$$

Whose auxiliary equation is

$$\left(m^2 - 3m + 2\right) = 0 \qquad m = 1, 2$$

$$\therefore \text{ CF is } c_1 e^x + c_2 e^{2x}$$

Continued.....

$$P.I. = \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{3^2 - 3.3 + 2} e^{3x}$$

$$= \frac{1}{2} e^{3x}$$

$$\therefore \text{ Solution is } \quad y = \text{C.F.+P.I.} = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Example

2. Solve $(D^3-5D^2+7D-3)y=e^{2x}\cosh x.$

Sol. The auxiliary equation is

$$\left(m^3-5m^2+7m-3\right)=0$$

$$\therefore\ m=1,1,3$$

$$\therefore\ \text{CF is } (c_1+c_2x)e^x+c_3e^{3x}$$

$$PI=\frac{1}{(D^3-5D^2+7D-3)}e^{2x}\cosh x$$

$$=\frac{1}{(D^3-5D^2+7D-3)}e^{2x}\left(\frac{e^x+e^{-x}}{2}\right)$$

Continued.....

$$=\frac{1}{(D^3-5D^2+7D-3)}\left(\frac{e^{3x}}{2}+\frac{e^x}{2}\right)$$

$$=\frac{1}{(D^3-5D^2+7D-3)}\frac{e^{3x}}{2}+\frac{1}{(D^3-5D^2+7D-3)}\frac{e^x}{2}$$

$$=x\frac{1}{(3D^2-10D+7)}\frac{e^{3x}}{2}+x\frac{1}{(3D^2-10D+7)}\frac{e^x}{2}$$

$$=x\frac{1}{(3.3^2-10.3+7)}\frac{e^{3x}}{2}+x^2\frac{1}{(6D-10)}\frac{e^x}{2}$$

$$=x\frac{e^{3x}}{8}+x^2\frac{1}{(6.1-10)}\frac{e^x}{2}=x\frac{e^{3x}}{8}-x^2\frac{e^x}{8}$$

$\therefore$ Solution is $y=C.F.+P.I.=(c_1+c_2x)e^x+c_3e^{3x}+x\dfrac{e^{3x}}{8}-x^2\dfrac{e^x}{8}$

Exercise

Solve the following:

$$(i)\ (D^2 - 3D + 2)y = e^{5x}.$$

$$(ii)\ (D-2)^3 y = e^{2x}.$$

$$(iii)\ (D^2 - 1)y = \cosh x.$$

$$(iv)\ (D^3 + 1)y = (e^x + 1)^2.$$

Formula II: When $X = \sin(ax+b)\, or\ \cos(ax+b)$

$$P.I. = \frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \text{ provided } f(-a^2) \neq 0.$$

$$If\ \ f(-a^2) = 0, then\ \ P.I. = x\frac{1}{f'(D^2)}\sin(ax+b) = x\frac{1}{f'(-a^2)}e^{ax}, \text{ provided } f^{'}(-a^2) \neq 0.$$

Example

1. Solve $(D^2 - 2D + 1)y = \cos 3x$

The auxiliary equation is

$$\left(m^2 - 2m + 1\right) = 0, \quad m = 1,1$$

$$\therefore \text{ CF is } (c_1 + c_2 x)e^x$$

$$PI = \frac{1}{(\mathrm{D}^2 - 2\,\mathrm{D} + 1)} \cos 3x$$

$$= \frac{1}{(-3^2 - 2\,\mathrm{D} + 1)} \cos 3x \qquad \text{Replace } D^2 \; by - 3^2$$

Continued....

$$= -\frac{1}{2}\frac{1}{(\mathrm{D}+4)}\cos 3x$$

$$= -\frac{1}{2}\frac{1}{(\mathrm{D}+4)} \text{ x } \frac{(D-4)}{(\mathrm{D}-4)}\cos 3x$$

$$= -\frac{1}{2}\frac{(D-4)}{(\mathrm{D}^2-16)}\cos 3x$$

$$= -\frac{1}{2}\frac{(D-4)}{(-3^2-16)}\cos 3x$$

$$= \frac{1}{50}(\mathrm{D}-4)\cos 3x$$

Continued…

$$= \frac{1}{50}(D\cos 3x - 4\cos 3x)$$

$$= \frac{1}{50}(-3\sin 3x - 4\cos 3x)$$

$$\therefore \text{ Solution is } \quad y = (c_1 + c_2 x)e^x + \frac{1}{50}(-3\sin 3x - 4\cos 3x)$$

Example

2. Solve $(D^2+4)y=\sin^2 x$

The auxiliary equation is

$$\left(m^2+4\right)=0, \quad m=\pm 2i$$

$$\therefore \text{ CF is } (c_1\cos 2x+c_2\sin 2x)$$

$$PI=\frac{1}{(\mathrm{D}^2+4)}\sin^2 x$$

$$=\frac{1}{2}\left[\frac{1}{(\mathrm{D}^2+4)}(1-\cos 2x)\right]$$

Continued...

$$=\frac{1}{2}\left[\frac{1}{(\mathrm{D}^2+4)}e^{0x}-\frac{1}{(\mathrm{D}^2+4)}\cos 2x\right]$$

$$=\frac{1}{2}\left[\frac{1}{(0+4)}e^{0x}-x\left(\frac{1}{2D}\cos 2x\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{4}-x\left(\frac{1}{4}\sin 2x\right)\right] \quad =\frac{1}{8}[1-x\sin 2x]$$

Hence the complete solution is

$$y=C.F.+P.I.=c_1\cos 2x+c_2\sin 2x+\frac{1}{8}[1-x\sin 2x]$$

Exercise

Solve the following:

$(i)\ (D^2 - 3D + 2)y = \sin 3x.$

$(ii)\ (D^3 + D^2 - D - 1)y = \cos 2x.$

$(iii)\ (D^2 + 1)y = \sin x \sin 2x.$

$(iv)\ \dfrac{d^4 y}{dx^4} - m^4 y = \cos mx$

Particular Integral Topic(CO1)

Formula III: *When* $X = x^m$ or polynomial of degree m, m being +ve integer.

Steps:

1. Take out lowest degree term from *f(D)* to make the first term unity.

   The remaining factor will be of the form *1+F(D) or 1-F(D).*

2. Take this factor in the numerator.

$$P.I. = \frac{1}{[1 \pm F(D)]} x^m = [1 \pm F(D)]^{-1} x^m$$

Particular Integral Topic(CO1)

3. Expand it in ascending power of D as far as the term containing D^m .

4. Operate on x^m term by term.

Note: Expansion with the help of binomial theorem.

$$(1+x)^{-1} = (1 - x + x^2 - x^3 + - - - - - - - -)$$

$$(1-x)^{-1} = (1 + x + x^2 + x^3 + - - - - - - - -)$$

Example

1. $\textit{Find the P.I. of } \left(D^2+5D+4\right)y = x^2+7x+9.$

Sol. $$P.I. = \frac{1}{D^2+5D+4}(x^2+7x+9)$$

$$= \frac{1}{4\left\{1+\frac{1}{4}\left(D^2+5D\right)\right\}}(x^2+7x+9)$$

$$= \frac{1}{4}\left\{1+\frac{1}{4}\left(D^2+5D\right)\right\}^{-1}(x^2+7x+9)$$

$$= \frac{1}{4}\left\{1-\frac{1}{4}\left(D^2+5D\right)+\frac{1}{16}\left(D^2+5D\right)^2-\ldots\ldots\ldots\ldots\right\}(x^2+7x+9)$$

Continued....

$$=\tfrac{1}{4}\left\{1-\frac{1}{4}\left(D^2+5D\right)+\frac{1}{16}\left(D^4+25D^2+10D^3\right)-\dots\dots\dots\dots\right\}(x^2+7x+9)$$

$$=\frac{1}{4}\left\{1(x^2+7x+9)-\frac{1}{4}D^2(x^2+7x+9)-\frac{5}{4}D(x^2+7x+9)+\frac{1}{16}25D^2(x^2+7x+9)\right\}$$

$$=\frac{1}{4}\left\{(x^2+7x+9)-\frac{1}{4}.2-\frac{5}{4}(2x+7)+\frac{1}{16}25.2\right\}$$

$$=\frac{1}{4}\left\{x^2+\frac{9}{2}x+\frac{23}{8}\right\}$$

Example

2. $Solve: \left(D^3 - D^2 - 6D\right)y = x^2 + 1.$

Sol. The auxiliary equation is $\left(m^3 - m^2 - 6m\right) = 0$

$$m = 0, 3, -2$$

and $\quad C.F. = c_1 + c_2 e^{3x} + c_3 e^{-2x}$

$$P.I. = \frac{1}{\left(D^3 - D^2 - 6D\right)}(x^2 + 1) = \frac{1}{-6D\left\{1 + \frac{1}{6}\left(D - D^2\right)\right\}}(x^2 + 1)$$

Continued...

$$=\frac{1}{-6D}\left\{1+\frac{1}{6}\left(D-D^2\right)\right\}^{-1}(x^2+1)$$

$$=\frac{1}{-6D}\left\{1-\frac{1}{6}\left(D-D^2\right)+\frac{1}{36}\left(D-D^2\right)^2-.......\right\}(x^2+1)$$

$$=\frac{1}{-6D}\left\{1-\frac{1}{6}\left(D-D^2\right)+\frac{1}{36}\left(D^2+D^4-2D^3\right)-.......\right\}(x^2+1)$$

$$=\frac{1}{-6D}\left\{(x^2+1)-\frac{1}{6}\left(D(x^2+1)-D^2(x^2+1)\right)+\frac{1}{36}\left(D^2(x^2+1)\right)\right\}$$

Continued....

$$= -\frac{1}{6D}\left\{x^2 - \frac{1}{3}x + \frac{25}{18}\right\}$$

$$= -\frac{1}{6}\left\{\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right\}$$

Hence the required solution is

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left\{x^3 - \frac{x^2}{2} + \frac{25}{6}x\right\}$$

Exercise

Solve the following:

$(i)\ (D^2 - 4)y = x^2.$

$(ii)\ (D^3 + 3D^2 + 2D)y = x^2 + 1.$

$(iii)\ (D^3 + 8)y = x^4 + 2x + 1.$

Particular Integral Topic(CO1)

Formula IV: (Exponential Shift) When X = $e^{ax}.V$

where V is function of x.

$$P.I. = \frac{1}{f(D)} e^{ax}.V = e^{ax}\left[\frac{1}{f(D+a)}V\right], \qquad \text{Replace } D \text{ by } D+a$$

$\frac{1}{f(D+a)}V$ is calucaled with the help of previous methods.

Example

1. $Solve \quad (D^2+3D+2)y = e^{2x}\sin x.$

Sol.

The auxiliary equation is

$$\left(m^2+3m+2\right)=0, \quad m=-2,-1$$

$$\text{So, we have} \quad C.F. = c_1e^{-2x}+c_2e^{-x}$$

$$PI = \frac{1}{(D^2+3D+2)}e^{2x}\sin x$$

$$= e^{2x}\frac{1}{(D+2)^2+3(D+2)+2}\sin x$$

Replace *D* by *D+2*

Continued...

$$= e^{2x} \frac{1}{D^2+7D+12} \sin x \quad = e^{2x} \frac{1}{-1^2+7D+12} \sin x$$

$$= e^{2x} \frac{1}{7D+11} \sin x = e^{2x} \frac{7D-11}{49D^2-11^2} \sin x$$

$$= e^{2x} \frac{7D-11}{-49-11^2} \sin x = e^{2x} \frac{1}{-170} 7D(\sin x) - 11(\sin x)$$

$$= \frac{e^{2x}}{170}(11\sin x - 7\cos x)$$

Hence the required solution $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^{2x}}{170}(11\sin x - 7\cos x)$

Example

2. $$Solve\ \frac{d^2y}{dx^2}-4\frac{dy}{dx}-5y=xe^{-x},\ given\ that\ y=0\ and$$

$$\frac{dy}{dx}=0\quad when\quad x=0.$$

Sol. Rewriting the given equation in symbolic form, we get

$$(D^2-4D-5)y=xe^{-x} \qquad ..(1)$$

Also given that : when *x = 0, y =0* *..(2)*

and when *x = 0, dy/dx = 0.* ..(3)

The auxiliary equation is $m^2-4m-5=0, \quad m=5,-1$

So, $C.F.=c_1e^{5x}+c_2e^{-x}$

Continued....

$$PI = \frac{1}{(D^2-4D-5)}xe^{-x} = e^{-x}\frac{1}{(D-1)^2-4(D-1)-5)}x$$

$$= e^{-x}\frac{1}{D^2-6D}x = e^{-x}\frac{1}{-6D\left(1-\frac{D}{6}\right)}x$$

$$= e^{-x}\frac{1}{-6D}\left(1-\frac{D}{6}\right)^{-1}x = e^{-x}\frac{1}{-6D}\left(1+\frac{D}{6}+\frac{D^2}{36}\right)x$$

$$= e^{-x}\frac{1}{-6D}\left(x+\frac{1}{6}\right) = e^{-x}\frac{1}{-6}\left(\frac{x^2}{2}+\frac{x}{6}\right)$$

$$\therefore\ y = c_1e^{5x}+c_2e^{-x}-\frac{e^{-x}}{12}\left(x^2+\frac{x}{3}\right) \qquad ..(4)$$

Continued....

Putting x = 0, y = 0 (due to (2)), in (4), we get

$c_1 + c_2 = 0$ ..(5)

$$From\,(4)\quad \frac{dy}{dx} = 5c_1e^{5x} - c_2e^{-x} + \frac{e^{-x}}{12}\left(x^2 + \frac{x}{3}\right) - \frac{e^{-x}}{12}\left(2x + \frac{1}{3}\right) \qquad ..(6)$$

Putting x = 0, dy/dx = 0 (due to (3)), in (6), we get

$5c_1 - c_2 - 1/36 = 0$ ..(7)

Solving (5) and (7), $c_1 = 1/216$ and $c_2 = -1/216$.

With these values, (4) reduces to

$$y = (1/216)(e^{5x} - c_2e^{-x}) - \frac{e^{-x}}{12}\left(x^2 + \frac{x}{3}\right)$$

Exercise

Solve the following:

$$(i)\ (D^2+3D+2)y=e^{-2x}\cos x.$$

$$(ii)\ (D^2-5D+6)y=e^{4x}(x^2+9).$$

$$(iii)\ (D^3-3D-2)y=540\,x^3e^{-x}.$$

Particular Integral Topic(CO1)

Formula V : When X is $x^m V$, *where V* is some function of *x* and *m* is positive integer. In general *V* is *sinax* or *cosax*.

$$\frac{1}{f(D)}(x.V) = x\left\{\frac{1}{f(D)}V\right\} - \frac{f'(D)}{\{f(D)\}^2}V.$$

Example

1. $Solve: \ (D^2 - 2D + 1)y = x\,sinx.$

Sol. $A.E. \ is \ (m^2 - 2m + 1) = 0, \ so \ m = 1,1$

So, $C.F. = (c_1 + c_2 x)e^x$

$$PI = \frac{1}{(\mathrm{D}^2 - 2\,\mathrm{D}+1)} x \sin x$$

$$PI = x \frac{1}{(\mathrm{D}^2 - 2\,\mathrm{D}+1)} \sin x - \frac{2D-2}{(\mathrm{D}^2 - 2\,\mathrm{D}+1)^2} \sin x$$

$$= x \frac{1}{(-1 - 2\,\mathrm{D}+1)} \sin x - \frac{2D-2}{(-1 - 2\,\mathrm{D}+1)^2} \sin x$$

Continued....

$$= -\frac{x}{2}\left(\frac{1}{\mathrm{D}}\sin x\right) - \frac{1}{2}\frac{D-1}{D^2}\sin x$$

$$= -\frac{x}{2}\left(\int \sin x\, dx\right) - \frac{1}{2}\frac{D-1}{-1^2}\sin x$$

$$= \frac{x}{2}\cos x - \frac{1}{2}(-\sin x - \cos x)$$

$$\therefore \quad y = (c_1 + c_2 x)e^x + \frac{x}{2}\cos x + \frac{1}{2}(\sin x + \cos x)$$

Exercise

Solve the following:

$(i)\ (D^2 + D)y = x\cos x.$

$(ii)\ (D^2 - 1)y = x^2 \cos x.$

$(iii)\ (D^2 - 2D + 1)y = xe^x \sin x.$

Particular Integral Topic(CO1)

General Method for P.I.:

$$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.$$

$$\frac{1}{D+a}X = e^{-ax}\int Xe^{ax}dx.$$

The above general method, however, must be used for problems in which X is of the forms *secax, cosecax, cotax* or any other form not covered by previous method.

Example

1. $\quad Solve \quad (D^2 + a^2)y = \sec ax$

Sol. Here $\quad C.F. = c_1 \cos ax + c_2 \sin ax$

$$P.I. = \frac{1}{(D^2 + a^2)} \sec ax = \frac{1}{2ai}\left[\frac{1}{(D - ia)} - \frac{1}{(D + ia)}\right] \sec ax \qquad ..(1)$$

$$Now, \qquad \frac{1}{(D - ia)} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$$

$$= e^{iax} \int (\cos ax - i \sin ax) \frac{1}{\cos ax} dx$$

Continued….

$$= e^{iax}\int (1 - i\tan ax)\,dx$$

$$= e^{iax}\int (1 - i\tan ax)\,dx$$

$$= e^{iax}(x + \frac{i}{a}\log\cos ax) \qquad ..(2)$$

Replacing *i* by –*i* in (2), we get

$$\frac{1}{(D + ia)}\sec ax = e^{-iax}(x - \frac{i}{a}\log\cos ax) \qquad ..(3)$$

Continued....

From (1), (2) and (3), we get

$$P.I. = \frac{1}{2ai}\left[e^{iax}(x + \frac{i}{a}\log\cos ax) - e^{-iax}(x - \frac{i}{a}\log\cos ax) \right]$$

$$= \frac{1}{2ai}\left[x(e^{iax} - e^{-iax}) + \frac{i}{a}\log\cos ax(e^{iax} + e^{-iax}) \right]$$

$$= (x/a)\sin ax + (1/a^2)\cos ax \log\cos ax$$

Solution is

$$y = c_1 \cos ax + c_2 \sin ax + (x/a)\sin ax + (1/a^2)\cos ax \log\cos ax$$

Exercise

Solve the following:

$$(i)\ (D^2 + a^2)y = \tan ax$$

$$(ii)\ (D^2 + 3D + 2)y = e^{e^x}$$

$$(iii)\ (D^2 + 2D + 2)y = e^{-x}\sec^3 x$$

FAQ

$$Q1.\ \ Solve: (2D-1)^3 y = 0$$

$$Q2.\ Solve\ (D^2-1)y = xe^x + \cos^2 x + x^2$$

$$Q3.\ (D^2-4D+4)y = 8x^2e^{2x}\sin 2x.$$

$$Q4.\ \ (D^2-2D+2)y = e^x \tan x$$

Daily Quiz

$Q1.$ Find the solution of the equation $(D^2 - 1)y = 1,$ which vanishes when $x = 0$ and tends to a finite limit as $x \to -\infty$ and D stands for d/dx.

$Q2.$ Find the solution of $(d^2 i/dt^2) + (R/L)(di/dt) + (i/LC) = 0$

$where\, R^2 C = 4L$ and $R, \mathrm{C}, \mathrm{L}$ are constant.

Recap

- ✓ Auxiliary equation of Differential equation.
- ✓ Method to find C.F. according nature of roots.
- ✓ Different method to find P.I.
- ✓ General method to finding P.I.

Objective of topic(CO1)

- Solution when differential equation have variable coefficients.
- Solution of homogeneous linear equation.
- Differential equations have two dependent variables.

Cauchy-Euler Equations(CO1)

A linear differential equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots\dots + a_n y = X \qquad (1)$$

$$i.e. \qquad (a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \dots\dots\dots a_n) y = X \qquad (2)$$

Where *X* is a function of *x* or constant and and

$a_0, a_1, a_2, \dots\dots\dots, a_n$ are constants.

Cauchy-Euler Equations(CO1)

Working rule:

1. Let $x = e^z$ so that $\log x = z$

2. $xD = D_1,\ x^2D^2 = D_1(D_1 - 1),\ x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$

where $D = \dfrac{d}{dx},\ D_1 = \dfrac{d}{dz}$

Example

1. $Solve\ \ (x^2D^2 - xD + 2)y = x\log x$

Sol. $Let\ \ x = e^z\ \ so\,that\ \ z = \log x\ \ and\,let\ \ D_1 = d/dz$

Then the given equation becomes

$$\left[D_1(D_1 - 1) - D_1 + 2\right]y = ze^z$$

or $$\left[D_1^{\,2} - 2D_1 + 2\right]y = ze^z$$

which is linear differential equation with constant coefficients.

Continued....

Its auxiliary equation is $m^2 - 2m + 2 = 0 \quad so \quad m = 1 \pm i$

$$\therefore \quad C.F. = e^z (c_1 \cos z + c_2 \sin z) = x(c_1 \cos(\log x) + c_2 \sin(\log x))$$

$$P.I. = \frac{1}{D_1^2 - 2D_1 + 2} ze^z = e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z$$

$$= e^z \frac{1}{D_1^2 + 1} z = e^z \left(D_1^2 + 1\right)^{-1} z = e^z \left(1 - D^2 + ...\right) z = ze^z = x \log x$$

Required solution is $y = x(c_1 \cos(\log x) + c_2 \sin(\log x)) + x(\log x)$

Example

2. *Solve* $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Sol. Multiplying by x^2 , the given equation gives

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$$

$$or \quad \left(x^2D^2 + xD\right)y = 12\log x$$

$$Let \;\; x = e^z \;\; so\,that \;\; z = \log x \;\; and\; let \;\; D_1 = d/dz$$

Then the given equation becomes

$$\left[D_1(D_1 - 1) + D_1\right]y = 12z \quad or \quad D_1^{\,2}y = 12z$$

Continued...

Its auxiliary equation is $m^2 = 0 \quad \therefore \quad m = 0, 0$

$$\therefore \quad C.F. = (c_1 + c_2 z) = c_1 + c_2 \log x$$

and $$P.I. = \frac{1}{D_1^2} 12z = 12 \frac{1}{D_1} \frac{z^2}{2} = 12.\frac{z^3}{6} = 2(\log x)^3$$

$$\therefore \quad y = c_1 + c_2 \log x + 2(\log x)^3$$

Exercise

Solve the following:

$$(i)\quad x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\sin(\log x^2)$$

$$(ii)\quad x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log x$$

Legendre's Homogeneous Differential Equation(CO1)

➢Reducible to Homogeneous

$$a_0(ax+b)^n\frac{d^ny}{dx^n}+a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots\ldots+a_ny=X$$

Steps to solve the problems –

1. *Choose* $ax+b=e^z$

2. *put* $(ax+b)D=aD_1$

$$(ax+b)^2D^2=a^2D_1(D_1-1)$$

$$(ax+b)^3D^3=a^3D_1(D_1-1)(D_1-2)$$

Exercise

Solve the following:

1. $[(1+2x)^2 D^2 - 6(1+2x)D + 16]y = 8(1+2x)^2$
2. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$

Simultaneous Linear Differential Equations(CO1)

In this topic we shall consider ordinary differential equations involving two or more dependent variables. For a complete determination of the dependent variables, there must be as many equations as there are dependent variables. ***Such equations are called ordinary simultaneous differential equations.***

Let the given simultaneous equations be

$$f_1(D)x + f_2(D)y = f(t)$$

$$g_1(D)x + g_2(D)y = g(t), \quad \text{where } D = \frac{d}{dt}$$

in which x and y are function of t.

Example

1. Solve the simultaneous differential equations

$$\frac{dx}{dt} = 3x + 2y, \quad \frac{dy}{dt} = 5x + 3y$$

Sol. Let *d/dt = D* then the given system of equations become

$$(D-3)x - 2y = 0 \qquad ..(1)$$

$$-5x + (D-3)y = 0 \qquad ..(2)$$

Operating eqt. (1) by (D-3) and multiplying eqt. (2) by 2 then adding, we get

$$[(D-3)^2 - 10]x = 0$$

$$(D^2 - 6D - 1)x = 0$$

Continued....

Auxiliary equation is $m^2 - 6m - 1 = 0 \Rightarrow m = 3 \pm \sqrt{10}$

$$C.F. = e^{3t}(c_1 \cosh\sqrt{10}\,t + c_2 \sinh\sqrt{10}\,t), \quad \text{P.I.} = 0$$

$$\therefore \qquad x = e^{3t}(c_1 \cosh\sqrt{10}\,t + c_2 \sinh\sqrt{10}\,t) \qquad ..(3)$$

From (1), $$2y = \frac{dx}{dt} - 3x$$

Continued....

$$= e^{3t}\sqrt{10}(c_1 \sinh\sqrt{10}\,t + c_2 \cosh\sqrt{10}\,t) + 3e^{3t}(c_1 \cosh\sqrt{10}\,t + c_2 \sinh\sqrt{10}\,t)$$
$$-3e^{3t}(c_1 \cosh\sqrt{10}\,t + c_2 \sinh\sqrt{10}\,t)$$

$$y = \frac{\sqrt{10}}{2}e^{3t}(c_1 \sinh\sqrt{10}\,t + c_2 \cosh\sqrt{10}\,t) \qquad ..(4)$$

The required solution is given by (3) and (4)

Example

2. Solve the simultaneous differential equations

$$\frac{dy}{dx} + y = z + e^x, \quad \frac{dz}{dx} + z = y + e^x$$

Sol. Writing *D = d/dt* , the given equations become

$$(D+1)\,y - z = e^x \qquad ..(1)$$

$$-y + (D+1)\,z = e^x \qquad ..(2)$$

Operating eqt. (1) by (D+1) , we get

$$(D+1)^2\,y - (D+1)z = (D+1)e^x = 2e^x \qquad ..(3)$$

Continued...

Adding (2) and (3), we get

$$[(D+1)^2-1]y=e^x+e^{3x} \quad \Rightarrow D(D+2)y=e^{3x}$$

$$m(m+2)=0 \quad gives\ m=0,-2$$

$$\therefore\ C.F.=c_1+c_2e^{-2x} \qquad Now\, P.I.=3\frac{1}{D(D+2)}e^x=e^x$$

$$\therefore\ y=c_1+c_2e^{-2x}+e^x \qquad ..(4)$$

Continued…

From (1) $z=\frac{dy}{dx}+y-e^{x}=-2c_2e^{-2x}+e^{x}+c_1+c_2e^{-2x}+e^{x}-e^{x}$

$$z=c_1-c_2e^{-2x}+e^{x} \qquad ..(5)$$

The required solution is given by (4) and (5)

Exercise

1. Solve $\frac{dx}{dt}+x-2y=0,\quad \frac{dy}{dt}+x+4y=0$; such that $x(0)=y(0)=1$.

2. Solve $\frac{dx}{dt}+2x-3y=t,\quad \frac{dy}{dt}-3x+2y=e^{2t}$.

3. Solve $\frac{dx}{dt}+\frac{dy}{dt}-2y=2\cos t-7\sin t,\quad \frac{dx}{dt}-\frac{dy}{dt}+2x=4\cos t-3\sin t$.

FAQ

Solve the following:

$$Q1. \quad x\frac{d^2y}{dx^2}-3\frac{dy}{dx}+\frac{y}{x}=\frac{\log x\sin(\log x)}{x}$$

$$Q2. \quad (3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1$$

$$Q3. \quad Solve\ \ x^2y''+xy'-y=x^3e^x$$

Daily Quiz

$$Q1.\ \ Solve\ \left(\frac{d}{dx}+\frac{1}{x}\right)^2 y = x^{-4}$$

$$Q3.\ \ Solve\ \ \frac{dx}{dt}-y=t,\ \ \frac{dy}{dt}+x=1$$

Recap

- ✓ Homogeneous Linear Differential Equation.
- ✓ Legendre’s Linear Differential Equation
- ✓ Simultaneous Linear Differential Equations.

Objective of topic(CO1)

- Second order differential equation with variable coefficients.
- Reduction method when one solution known
- By Changing independent variables.

Second Order Linear Differential Equations with Variable Coefficients (CO1)

The general(standard) form of the linear equations of second order:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Rule 1: *One solution known (or Reduction of Order):*

Working Rule:

STEP I. Put the given equation in standard form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ..(1)$$

in which the coefficients of $\frac{d^2y}{dx^2}$ is unity.

Reduction of Order (CO1)

STEP II. Find an integral *u* of C.F. By using the following table.

Condition satisfied	**An Integral of C.F.**
$1+P+Q=0$	$u=e^{x}$
$1-P+Q=0$	$u=e^{-x}$
$P+Qx=0$	$u=x$

Reduction of Order (CO1)

STEP III: Assume complete solution of given equation is *y = uv*, where *u* has been obtained in **STEP II**.

where *v* is obtained from equation

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=\frac{R}{u} \qquad ..(2)$$

take $\dfrac{dv}{dx}=q \;\; and \;\; \dfrac{d^2v}{dx^2}=\dfrac{dq}{dx}$

put in equation (2) and solve using First order linear equation method.

Example

1. $Solve \;\; xy''-(2x-1)y'+(x-1)y=0.$

Sol. Putting the given equation in standard form, we get

$$\frac{d^2y}{dx^2}-\left(2-\frac{1}{x}\right)\frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=0 \qquad ..(1)$$

Comparing (1) with $y''+Py'+Qy=R, \quad we\, have$

$$P=-\left(2-\frac{1}{x}\right), \quad Q=\left(1-\frac{1}{x}\right), \quad R=0 \qquad ..(2)$$

$$\text{Here}, \qquad 1+P+Q=0,$$

Continued...

showing that $\qquad y = u = e^x \qquad ..(3)$

is a part of C.F. of the equation of (1).

Let the complete solution of (1) be $\qquad y = uv \qquad ..(4)$

Then v is given $\qquad \dfrac{d^2v}{dx^2} + \left[P + \dfrac{2}{u}\dfrac{du}{dx}\right]\dfrac{dv}{dx} = \dfrac{R}{u}$

$$\frac{d^2v}{dx^2} + \left[-2 + \frac{1}{x} + \frac{2}{e^x}\frac{de^x}{dx}\right]\frac{dv}{dx} = 0, \quad \text{using (2) and (3)}$$

Continued...

$$\frac{d^2v}{dx^2}+\left[-2+\frac{1}{x}+2\right]\frac{dv}{dx}=0 \qquad or \qquad \frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0 \qquad ..(5)$$

$$Let \quad dv/dx=q \quad so\,that \quad d^2v/dx^2=dq/dx \qquad ..(6)$$

Then (5) becomes $\frac{dq}{dx}+\frac{q}{x}=0 \qquad or \qquad \frac{dq}{q}=-\frac{dx}{x}$

Integrating, $\log q=\log c_1-\log x \qquad or \qquad q=c_1/x$

$$or \quad \frac{dv}{dx}=\frac{c_1}{x} \qquad or \qquad dv=\frac{c_1dx}{x} \qquad \left[\because q=\frac{dv}{dx}\right]$$

Continued...

Integrating, $v = c_1 \log x + c_2$, c_1, c_2 being arbitrary constants ..(7)

From (3), (4) and (7), the required solution is

$$y = uv \qquad or \qquad y = e^{x}(c_1 + c_2 \log x).$$

Exercise

Solve the following differential equations:

$$(i) \quad \frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)y = e^x \sin x$$

$$(ii) \quad x^2 y_2 - 2x(1+x)y_1 + 2(1+x)y = x^3$$

Reduction to Normal Form (CO1)

Rule 2:

Working Rule:

STEP I. Put the given equation in standard form

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \qquad ..(1)$$

in which the coefficients of $\dfrac{d^2y}{dx^2}$ is unity.

STEP II. To remove the first derivative, we choose

$$u=e^{-\frac{1}{2}\int Pdx}$$

Reduction to Normal Form (CO1)

STEP II. We now assume that the complete solution of the given equation is $y = uv$. Then the given equation reduces to normal form

$$\frac{d^2v}{dx^2} + Iv = S, \quad where \;\; I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} \;\; and \;\; S = \frac{R}{u}$$

Note: This method is applicable when I = Constant or (Constant)/ x^2

Example

1. $Solve\ \ y''-2\tan x\, y'+5y=\sec x\, e^{x}$

Sol. Given $y''-2\tan x\, y'+5y=\sec x\, e^{x}$ ..(1)

Comparing with $y'' + Py' + Qy = R$, $P = -2\tan x$, $Q = 5$, $R = \sec x\, e^{x}$

To remove the first derivative from (1), we choose

$$u=e^{-\frac{1}{2}\int Pdx}=e^{-\frac{1}{2}\int(-2\tan x)dx}=e^{\log\sec x}=\sec x \qquad ..(2)$$

Let the required general solution be $y = uv$ ..(3)

Continued....

Then v is given by normal form $\dfrac{d^2v}{dx^2} + Iv = S$ ..(4)

Where, $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$= 5 - \frac{1}{4}4\tan^2 x - \frac{1}{2}(-2\sec^2 x)$

$= 5 - \tan^2 x + \sec^2 x = 6$

and $S = R/u = (\sec x\, e^x)/\sec x = e^x$

Continued….

Then (4) becomes $\frac{d^2v}{dx^2} + 6v = e^x$ or $(D^2+6)v = e^x$ ..(5)

Its auxiliary equation is $m^2 + 6 = 0$ so that $m = \pm i\sqrt{6}$

$\therefore$ $C.F. = c_1 \cos(\sqrt{6}\,x) + c_2 \sin(\sqrt{6}\,x)$

and $\text{P.I.} = \frac{1}{D^2+6} e^x = \frac{1}{1^2+6} e^x = \frac{1}{7} e^x$

Hence the solution of (5) is $v = c_1 \cos(\sqrt{6}\,x) + c_2 \sin(\sqrt{6}\,x) + \frac{e^x}{7}$ ..(6)

From (2), (3) and (6) the required solution is

$$y = \sec x[c_1 \cos(\sqrt{6}\,x) + c_2 \sin(\sqrt{6}\,x) + \frac{e^x}{7}].$$

Exercise

Solve the following differential equations:

$$(i)\quad \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$$

$$(ii)\quad \frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + (1 + \frac{2}{x^2})y = xe^x$$

Solution by Changing Independent Variable(CO1)

Rule 3:

Working Rule:

STEP I: Make coefficients of $\frac{d^2y}{dx^2}$ unity, *i.e.*

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

STEP II: We assume a relation between the new independent variable *z* and the old independent variable *x* given by

$$\left(\frac{dz}{dx}\right)^2 = Q \quad \Rightarrow \left(\frac{dz}{dx}\right) = +\sqrt{Q} \quad \text{(Rejecting -ve sign)}$$

$$dz = +\sqrt{Q}\,dx \quad \text{so that} \quad z = \int \sqrt{Q}\,dx$$

Solution by Changing Independent Variable(CO1)

STEP III: With relation between *z* and *x*, we transform (1) to get an equation of the form

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad (2)$$

$$\text{where } P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2},\ Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2},\ R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

Note: P_1 *and* Q_1 will be constant and R_1 in terms of z.

Example

1. By changing independent variable, solve

$$\cos x\, y'' + \sin x\, y' - 2y\cos^3 x = 2\cos^5 x$$

Sol. Dividing by *cosx*, given equation in standard form is

$$y'' + \tan x\, y' - 2y\cos^2 x = 2\cos^4 x \qquad ..(1)$$

Comparing (1) with y'' + Py' + Qy = R, we have

$$P = \tan x,\ \ Q = -2\cos^2 x,\ \ R = 2\cos^4 x$$

Continued…

$$\text{Choose z such } \left(\frac{dz}{dx}\right)^2 = \cos^2 x \quad or \quad \frac{dz}{dx} = \cos x$$

$$dz = \cos x\, dx \qquad so\, that \quad z = \sin x$$

With this z, (1) transform to

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \qquad ..(2)$$

Continued...

$$\text{where } P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{-\sin x + \tan x \cos x}{\cos^2 x} = 0$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = \frac{-2\cos^2 x}{\cos^2 x} = -2$$

$$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{2\cos^4 x}{\cos^2 x} = 2\cos^2 x = 2(1 - z^2)$$

Putting in (2), we get

Continued...

$$\frac{d^2y}{dz^2} - 2y = 2(1-z^2) \quad \text{or} \quad (D_1^2 - 2)y = 2(1-z^2)$$

Auxiliary equation is

$$m^2 - 2 = 0 \;\Rightarrow\; m = \pm\sqrt{2} \quad C.F. = c_1 \cosh\sqrt{2}\,z + c_2 \sinh\sqrt{2}\,z$$

$$\text{P.I.} = \frac{1}{D_1^2 - 2} 2(1-z^2) = -\left(1 - \frac{D_1^2}{2}\right)^{-1}(1-z^2) = -\left(1 + \frac{D_1^2}{2}\right)(1-z^2) = z^2$$

$$\therefore \qquad y = c_1 \cosh\sqrt{2}\,z + c_2 \sinh\sqrt{2}\,z + z^2$$

Complete solution is

$$y = c_1 \cosh\sqrt{2}\,(\sin x) + c_2 \sinh\sqrt{2}\,(\sin x) + \sin^2 x$$

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Example

2. By changing independent variable, solve

$$x^6 y'' + 3x^5 y' + a^2 y = \frac{1}{x^2}$$

Sol. Dividing by x^6, given equation in standard form is

$$y'' + \frac{3}{x} y' + \frac{a^2}{x^6} y = \frac{1}{x^8} \qquad ..(1)$$

Comparing (1) with y'' + Py' + Qy = R, we have

$$P = 3/x,\ Q = a^2/x^6,\ R = 1/x^8$$

Continued...

$$\text{Choose z such } \left(\frac{dz}{dx}\right)^2 = \frac{a^2}{x^6} \quad or \quad \frac{dz}{dx} = \frac{a}{x^3} \qquad \text{so that } z = -\frac{a}{2x^2}$$

With this z, (1) transform to $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad ..(2)$$

$$\text{where } P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-\frac{3a}{x^4} + \frac{3}{x}.\frac{a}{x^3}}{\left(\frac{dz}{dx}\right)^2} = 0$$

Continued...

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1$$

$$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{1/x^8}{a^2/x^6} = \frac{1}{a^2x^2} = -\frac{2z}{a^3}$$

Putting in (2), we get $$\frac{d^2y}{dz^2} + y = -2z/a^3$$

$$(D_1^2 + 1)y = -2z/a^3$$

Continued...

Auxiliary equation is

$$m^2+1=0 \;\Rightarrow m=\pm i \quad C.F.=c_1\cos z+c_2\sin z$$

$$\text{P.I.}=\frac{1}{D^2+1}\left(-\frac{2z}{a^3}\right)=-\frac{2}{a^3}\left(1+D_1{}^2\right)^{-1}z=-\frac{2}{a^3}\left(1-D_1{}^2....\right)z=-\frac{2z}{a^3}$$

$$\therefore \qquad y=c_1\cos z+c_2\sin z-\frac{2z}{a^3}$$

Complete solution is

$$y=c_1\cos z+c_2\sin z-\frac{2z}{a^3}$$

Continued...

$$or \qquad y = c_1 \cos\left(-\frac{a}{2x^2}\right) + c_2 \sin\left(-\frac{a}{2x^2}\right) + \left(\frac{1}{a^2x^2}\right)$$

$$or \qquad y = c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right) + \left(\frac{1}{a^2x^2}\right)$$

Exercise

1. By changing independent variable, solve
$$x\,y''+(2x^2-1)\,y'-24x^2y=8x^3\sin x^2.$$

2. By changing independent variable, solve
$$y''-\cot x\,y'-y\sin^2 x=\cos x-\cos^3 x$$

Method of Variation of Parameters(CO1)

Rule 4:

Working Rule: $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

STEP I: Find C.F. of equation (1). Suppose $C.F. = au + bv$

STEP II: Let $y = Au + Bv$ be complete solution of (1).

$$where\ A = -\int \frac{Rv}{W}dx + c_1 \qquad B = \int \frac{Ru}{W}dx + c_2$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

Note: Apply when asked.

Example

1. Solve by method of variation of parameter:

$$\frac{d^2y}{dx^2}+y=\frac{2}{1+e^x}$$

Sol. Here $u = e^x$, $v = e^{-x}$ which are part of C.F. Also, $R=\frac{2}{1+e^x}$

Let $y = Ae^x + Be^{-x}$ be the complete solution of given equation, where A and B are determined as:

$$A=-\int\frac{Rv}{W}dx+c_1=-\int\frac{2e^{-x}/(1+e^x)}{-2}dx+c_1$$

Continued...

$$= \int \frac{e^{-x}}{1+e^{x}}dx + c_1 = \int \frac{e^{-2x}}{e^{-x}+1}dx + c_1 = \log\left(\frac{1+e^{x}}{e^{x}}\right) - e^{-x} + c_1$$

$$B = \int \frac{Ru}{W}dx + c_2 = \int \frac{2e^{x}/(1+e^{x})}{-2}dx + c_2$$

$$= -\int \frac{e^{x}}{1+e^{x}}dx + c_2 = -\log(1+e^{x}) + c_2$$

Hence the complete solution is

$$y = [\log\left(\frac{1+e^{x}}{e^{x}}\right) - e^{-x} + c_1]e^{x} + [-\log(1+e^{x}) + c_2]e^{-x}$$

Exercise

Solve following differential equation by by method of variation of parameter:

1. $y'' + y = \tan x$

2. $x^2 y'' + x y' - y = x^2 e^x$

FAQ

$Q1.$ Solve by changing the independent variable

$$xy''+(4x^2-1)y'+4x^3y=2x^3$$

$$Q2.\quad Solve\ y''-6y'+9y=\frac{e^{3x}}{x^2}$$

Daily Quiz

1. Part of the solution of $xy'' - y' + (1-x)y = x^2 e^{-x}$

   by the method of reduction of order is..........

Recap

- ✓ Second Order Linear Differential Equations with Variable Coefficients.
- ✓ Reduction of order method.
- ✓ Normal Form Method.
- ✓ Solution by changing independent variable.
- ✓ Method of Variation of Parameters.

Objective of topic(CO1)

- ➢Solution of differential equation in series form.

- ➢ Special function can be obtained as Bessel and Legendre polynomial.

Series solution about regular singular point *x =0*: [Frobenius Method] (CO1)

If *x* = 0 is regular point, we shall use the Frobenius method for finding series solution about x = 0.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \qquad (1)$$

Steps for solutions:

1. Assume $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + ... = \sum_{m=0}^{\infty} a_m x^{m+k}, \quad a_0 \neq 0 \qquad ..(2)$

2. Differentiating (2) and obtain y’ and y’’. Putting *y*, *y’* and *y’’* in (1).

Series solution about regular singular point $x = 0$: [Frobenius Method] (CO1)

3. Step2. Equating in zero the coefficient of the smallest power of *x* in the identity obtained step 2 above, we obtain a quadratic equation in *k*. The quadratic equation so obtained is called the **indicial equation.**

Caes1. when Roots are distinct and do not differ by an integer

Let m_1 and m_2 be the roots then complete solution is $y = c_1(y)_{\mathrm{m}_1} + c_2(y)_{\mathrm{m}_2}$

Caes2. when Roots are equal e.g., $\mathrm{m}_1 = \mathrm{m}_2$

Complete solution is $y = c_1(y)_{\mathrm{m}_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{\mathrm{m}_2}$

Series solution about regular singular point *x =0*: [Frobenius Method] (CO1)

Caes3. when Roots are distinct, differ by integer and makinga coefficient of y infinite.

In this case, if some of coefficient of y becomes infinite when m = m_2, we modify the form of y by replacing a_0 by $b_0(m-m_2)$.

$$\text{Complete solution is } y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$$

Case 4. When roots are distinct, differ by integer and making one coefficient indeterminate.

Let the roots me m_1 and m_2. If one of the coefficients (suppose a_1) becomes indeterminate when $m = m_2$, the complete solution is given by putting $m = m_2$ in y which then contains two arbitrary constants.

Exercise

Solve in series the differential equation:

1. $9x(1-x)y''-12y'+4y=0$

2. $xy''+y'+x^2y=0$

3. $x^2y''+xy'+(x^2-1)y=0$

4. $xy''+2y'+xy=0$

Daily Quiz

1. What will be roots of the indicial for the power series solution of the differential equation $2x^2y'' + (2x^2 - x)y' + y = 0$

Recap

- ✓ Series solution about regular singular point .
- ✓ Method to find indicial roots.

Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details

- **Self Made Video Link:**

  Simultaneous Linear Differential Equations:
- https://www.youtube.com/watch?v=Ql42qcOLKfo&t=7s

  Method of variation of parameters:
- https://www.youtube.com/watch?v=0CbAY6OaDWI&t=13s

- **Youtube/other Video Links**

  Oeder and Degree of Differential Equation:
- https://www.youtube.com/watch?v=qIyx1kFTqT8

  Simultaneous Linear equation:
- https://www.youtube.com/watch?v=n_3ZmnVnrc4

  Series solution by Frobenius Method:
- https://www.youtube.com/watch?v=19Vt7ds8Lvw

Weekly Assignment(CO1)

Assignment1.1

Q. Find the solution of following ordinary differential equations:

(a) $(D^3 - 5D^2 + 7D - 3)y = e^{2x} coshx.$ $Ans: y = (c_1 + c_2 x)e^x + c_3 e^{2x} - \frac{1}{8}x^2 e^x + \frac{1}{8}xe^{3x}.$

(b) $(D^2 + 5D + 6)y = 3e^{2x} + 6e^{-3x} - 3log4 + 2^x.$ $PI = \frac{3}{20}e^{2x} + 6xe^{-3x} - 3\frac{log4}{6} + \frac{2^x}{(log2)^2 + 5log2 + 6}.$

(c) $(D^2 - 2D + 3)y = cosx + x^2.$ $PI = \frac{1}{4}(cosx - sinx) + (9x^2 + 12x + 2)/27.$

(d) $(D^2 - 1)y = xe^x + cos^2 x.$ $PI = \frac{1}{4}e^x(x^2 - x) - \frac{1}{2} - \frac{1}{10}cos2x.$

(e) $(D^2 - 1)y = xsinx + e^x(1 + x^2).$ $PI = \frac{1}{12}e^x(9x - 3x^2 + 2x^3) - \frac{1}{2}(xsinx + cosx).$

(f) $(D^2 - 2D + 1)y = xe^x sinx.$ $PI = -e^x(xsinx + 2cosx).$

(g) $(x^2 D^2 - 3xD + 5)y = x^2 sinlogx.$ $Ans: y = x^2(c_1 coslogx + c_2 sinlogx) - \frac{1}{2}x^2 logxcoslogx.$

(h) $[(1 + 2x)^2 D^2 - 6(1 + 2x)D + 16]y = 8(1 + 2x)^2.$ $PI = (1 + 2x)^2\{\log(1 + 2x)\}^2.$

Weekly Assignment(CO1)

Assignment1.2

Q. Solve the following simultaneous differential equations:

(a) $\frac{dx}{dt} = 3x + 8y,\ \frac{dy}{dt} = -x - 3y,\ x(0) = 6, y(0) = -2,\ Ans: x = 4e^{t} + 6e^{-t}, y = e^{t} - 3e^{-t}.$

(b) $\frac{dx}{dt} + x - 2y = 0,\ \frac{dy}{dt} = x + 4y = 0,\ x(0) = y(0) = 1,\ Ans: x = 4e^{-2t} - 3e^{-3t}, y = 2e^{-2t} + 3e^{-3t}.$

(c) $\frac{dx}{dt} - y = e^{t}, \frac{dy}{dt} + x = sint, x(0) = 1, y(0) = 0,\ Ans: x = 2sint + \frac{3}{2}cost + \frac{t}{2}cost - \frac{1}{2}e^{t}, y = \frac{1}{2}cost - \frac{3}{2}sint + \frac{t}{2}sint - \frac{1}{2}e^{t}.$

(d) $\frac{dx}{dt} + 4x + 3y = t,\ \frac{dy}{dt} + 2x + 5y = e^{t},\ Ans: x = c_1e^{-2t} + c_2e^{-7t} + \frac{5}{14}t - \frac{1}{8}e^{t} - \frac{31}{196}, y = -\frac{2}{3}c_1e^{-2t} - c_2e^{-7t} - \frac{1}{7}t + \frac{5}{24}e^{t} + \frac{9}{98}.$

Assignment1.3

1. Using change of independent variable solve the following ODE:

(a) $x^6\frac{d^2y}{dx^2}+3x^5\frac{dy}{dx}+a^2y=\frac{1}{x^2}$, $Ans: y=c_1cos\left(\frac{a}{2x^2}\right)+c_2sin\left(\frac{a}{2x^2}\right)+\frac{1}{a^2x^2}$.

(b) $cosxy''+sinxy'-2cos^3x\, y=2cos^5x$, $Ans: c_1e^{\sqrt{2}x}+c_2e^{-\sqrt{2}x}+z^2,\ z=sinx$.

(c) $xy''+(4x^2-1)y'+4x^3y=2x^3$, $Ans: (c_1+zc_2)e^{-z}+\frac{1}{2},\ z=x^2$.

(d) $y''-cotx\, y'-sin^2x\, y=cosx-cos^3x$, $Ans: c_1e^{-z}+c_2e^{z}+z,\ z=-cosx$.

2. Using variation of parameter solve following ODE:

(a) $y''-3y'+2y=\frac{e^x}{1+e^x}$, $Ans: y=[\log(e^{-x}+1)+c_1]e^x+[\log(e^{-x}+1)-(e^{-x}+1)+c_2]e^{2x}$.

(b) $y''+3y'+2y=e^{e^x}$, $Ans: y=c_1e^{-x}+c_2e^{-2x}+e^{-2x}e^{e^x}$.

(c) $y_2+4y=4tan2x$, $Ans: y=c_1cos2x+c_2sin2x-cos2x\log(sec2x+tan2x)$.

(d) $y''-y=e^{-x}sine^{-x}+cose^{-x}$, $Ans: y=c_1e^{-x}+c_2e^{x}-e^{x}sine^{-x}$.

(e) $x^2y''+xy'-y=x^3e^x$, $Ans: y=c_1x+\frac{c_2}{x}+(x-3+\frac{3}{x})e^x$.

Weekly Assignment(CO1)

Assignment1.4

Q. Find the Frobenius series solution:

(a) $2x^2y'' + xy' - (x+1)y = 0$ $Ans: y = Ax\left(1 + \frac{1}{5}x + \frac{1}{70}x + ..\right) + Bx^{-1/2}\left(1 - x - \frac{1}{2}x^2 +\right)$.

(b) $x^2y'' + x(x-1)y' - (1-x)y = 0$ $Ans: y = Ax + B\left(xlogx - x + \frac{1}{4}x^2 - \cdots\right)$.

(c) $x^2y'' + xy' + (x^2-1)y = 0$ $Ans: y = Ax\left(1 - \frac{x^2}{2.4} + \frac{x^4}{2.4^2.6} - ..\right) + Bx^{-1}logx\left(-\frac{1}{2}x^2 + \frac{x^4}{2.4^2} - \cdots\right) + Bx^{-1}\left(1 + \frac{x^2}{2^2} -\right)$.

(d) $(1-x^2)y'' - xy' + 4y = 0$ $Ans: y = a_0(1-2x^2) + a_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots\right)$.

(e) $(x-x^2)y'' + (1-x)y' - y = 0$ $Ans: y = A(1 + x + \frac{2}{4}x^2 + \frac{2.5}{4.9}x^3 + \cdots) + B\left(y_1 logx + a_0(-2x - x^2 - \frac{14}{27}x^3 - \cdots)\right)$.

MCQ s

1. Degree and order of the diff. Equation $x\frac{d^2y}{dx^2}+y\frac{dy}{dx}+4y^2=1$

(i) Ord = 2, Deg = 2

(ii) Ord = 1, Deg = 2

(iii) Ord = 2, Deg = 0

(iv) Ord = 2, Deg = 1

2. The solution of the diff Equation $\frac{dy}{dx}+\frac{3y}{x}=\frac{1}{x^4}$

(i) $\frac{x}{6}e^{2x}$ (ii) $\frac{x}{6}e^{-2x}$ (iii) x^2e^{2x} (iv) $\frac{x^2}{6}e^{2x}$

MCQ s

3. A differential equation from the relation $y = c_1e^{-x}$ is

(i) $y' - y = 0$

(ii) $y' - 2y = 0$

(iii) $y' + y = 0$

(iv) $y' + 2y = 0$

MCQ s

4. Solution of the diff. equation $(D^4 + D^2)\, y = 0$ is

(i) $y = (c_{1+}c_2 x)e^x + c_3 sinx + c_4 cosx$

(ii) $y = (c_{1+}c_2 x) + c_3 sinx + c_4 cosx$

(iii) $y = (c_{1+}c_2 x)sinx + (c_3 + c_4 x)cosx$

(iv) *None of them*

MCQ s

5. The P.I. of the diff. equation $(D^2 + 4)y = \text{Cos } 2x$ is

(i) $\frac{x}{4}\cos 2x$

(ii) $\frac{x}{4}\sin 2x$

(iii) $x\cos 2x$

(iv) $\frac{1}{4}\sin 2x$

MCQ

6. The Particular Integral of $\frac{1}{D-a} \; Sin2x$

(i) $e^{2x} \int e^{-2x} \sin 2x \, dx$

(ii) $e^{-2x} \int e^{2x} \sin 2x \, dx$

(iii) $e^{2x} \int e^{2x} \sin 2x \, dx$

(iv) None of these

MCQ

7. The roots of the indicial for the power series solution of the differential equation $y'' + 5x\,y' + x^2 y = 0$

(i) $0, 4$

(ii) $0, -4$

(iii) $1, 4$

(iv) $1, -4$

Old Question Papers

KAS203

Printed Pages: 02
Paper Id: 199243

Sub Code: KAS203
Roll No.

B. TECH.
(SEM II) THEORY EXAMINATION 2018-19
MATHEMATICS-II

Time: 3 Hours ***Total Marks: 100***

Note: Attempt all Sections. If require any missing data; then choose suitably.

SECTION A

1. Attempt *all* questions in brief. **2 x 10 = 20**

QNo.	Question	Marks	CO
a.	Find the P.I of $\frac{d^2y}{dx^2} + 4y = sin2x$	2	1
b.	Solve simultaneous equations $\frac{dx}{dt} = 3y$, $\frac{dy}{dt} = 3x$	2	1
c.	Find the volume of solid generated by revolving the circle $x^2 + y^2 = 25$ about y-axis.	2	2
d.	Evaluate $\Gamma\left(-\frac{5}{2}\right)$. where Γ is gamma function	2	2
e.	Find the Fourier constant a_1 of $f(x) = x^2$, $-\pi \leq x \leq \pi$	2	3
f.	Discuss the convergence of sequence $a_n = \frac{2n}{n^2+1}$.	2	3
g.	Show that complex function $f(z) = z^3$ is analytic.	2	4
h.	Define Conformal mapping.	2	4
i.	Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path $y = x$.	2	5
j.	Find residue of $f(z) = \frac{cosz}{z(z+5)}$ at $z = 0$	2	5

Old Question Papers

SECTION B

2. Attempt any *three* of the following:

QNo.	Question	Marks	CO
a.	Use Frobenius method to solve $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$	10	1
b.	Apply Dirichlet integral to find the volume of an octant of the sphere $x^2 + y^2 + z^2 = 25$.	10	2
c.	Find half range sine series of $f(x) = \begin{cases} x & 0 < x < 2 \\ 4 - x & 2 < x < 4 \end{cases}$	10	3
d.	Show that $u = x^4 - 6x^2y^2 + y^4$ is harmonic function. Find complex function $f(z)$ whose u is a real part.	10	4
e.	Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in regions (i) $1 < \lvert z \rvert < 2$ (ii) $2 < \lvert z \rvert$	10	5

SECTION C

3. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Solve $\frac{d^2y}{dx^2} + y = tanx$ by method of variation of parameter.	10	1

Page **1** of **2**

Old Question Papers

KAS203

b.	Solve $x^2\frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$ by Normal Form.	10	1

4. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Prove that $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$ where Γ is gamma function	10	2
b.	Use Beta and Gamma function to solve $\int_0^\infty \frac{1}{1+x^4}dx \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$	10	2

5. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Find the Fourier series of $f(x) = xsinx$, $-\pi \le x \le \pi$	10	3
b.	State D' Alembert's test. Test the series $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} \ldots\ldots + \frac{x^n}{n^2+1} + \ldots\ldots$	10	3

Old Question Papers

6. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Let $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$ $when\ z \neq 0,$ $f(z) = 0\ when\ z = 0.$ Prove that Cauchy Riemann satisfies at $z = 0$ but function is not differentiable at $z = 0$.	10	4
b.	Find Mobius transformation that maps points $z = 0, -i, 2i$ into the points $w = 5i, \infty, -\frac{i}{3}$ respectively.	10	4

7. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Using Cauchy Integral formula evaluate $\int_c \frac{\sin z}{(z^2+25)^2}dz$ where c is circle $\|z\| = 8$	10	5
b.	Apply residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$	10	5

Expected Questions for University Exam

Q1. $Solve\ (D^2-1)y = xe^x + \cos^2 x + x^2$

Q2. Solve $(D^3+3D^2+2D)y = x^2+1.$

Q3. $Solve\ (D^2-2D+1)y = xe^x \sin x.$

Q4. $Solve\ (D^2+3D+2)y = e^{e^x}$

Q5. $Solve\quad x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=\sin(\log x^2)$

Expected Questions for University Exam

Q6. Solve the simltaneous linear equation

$$\frac{dx}{dt}+2x-3y=t, \quad \frac{dy}{dt}-3x+2y=e^{2t}.$$

$Q7.$ Solve $\dfrac{d^2y}{dx^2}+y=\dfrac{2}{1+e^x}$ by method of variaton of parameter.

Q8. By changing independent variable, solve

$$y''-\cot x\, y'-y\sin^2 x=\cos x-\cos^3 x$$

Q9. *Solve in series* $9x(1-x)y''-12y'+4y=0$

Summary

- Course Objective
- COs and POs of subject
- Mapping of COs and Pos
- Prerequisite and Recap
- Linear differential equation of nth order with constant coefficients and Cauchy-Euler equation.
- Simultaneous linear differential equations
- Second order linear differential equations with variable coefficients.
- Series solutions (Frobenius Method)

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Thank You