# Differential Calculus II

**Unit: 3**

**MATHEMATICS – 1 (KAS – 103)**

**B Tech 1st Sem**

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# Content

- Taylor’s and Maclaurin’s Theorems for a function of one variables
- Taylor’s and Maclaurin’s Theorems for a function of two variables
- Jacobian
- Approximation of errors
- Maxima and Minima of functions of several variables
- Lagrange's method of multipliers

# Course Objective

- The objective of this course is to familiarize the engineering students with techniques of solving matrices, differential calculus, multivariable calculus problems. It aims to equip the students with adequate knowledge of mathematics that will enable them in formulating problems and solving problems analytically.

# Unit Objective

- The use of differential calculus to solve physics, geometry, and optimization problems.
- Apply calculus concepts to solve real-world problems such as optimization and related rates problems. For example, students will be expected to construct related rate equations to compute the value of a variable and interpret the resulting value in the context of the given problem.
- The students will learn the essential concepts of series expansion as an Taylor’s & Maclaurin’s series, Jacobian, maxima-minima and approximation of errors in a comprehensive manner.

# Course Outcomes 2020-21 (B. Tech. – 1st Sem)

| Course Name: Mathematics-I (KAS 103) | |
|---|---|
| CO1 | Apply the concept of matrices to solve linear simultaneous equations |
| CO2 | Apply the concept of successive differentiation and partial differentiation to solve problems of Leibnitz theorems and total derivatives . |
| **CO3** | Apply partial differentiation for evaluating maxima, minima, Taylor's series and Jacobians. |
| CO4 | llustrate multiple integral to find area, volume, centre of mass and centre of gravity. |
| CO5 | Demonstrate the basic concept of Profit, Loss, Number & Series, Coding & decoding. |

# Program Outcomes (PO)

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis:** Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

# Program Outcomes (PO)

5. **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

# Program Outcomes (PO)

9. **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication:** Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance:** Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

# CO-PO Mapping 2020-21 (B. Tech. – 1st Sem)

| Course Name: Mathematics-I (KAS 103) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **CO** | **PO-1** | **PO-2** | **PO-3** | **PO-4** | **PO-5** | **PO-6** | **PO-7** | **PO-8** | **PO-9** | **PO-10** | **PO-11** | **PO-12** |
| CO1 | 3 | 2 | 1 | 1 | 3 | 2 | - | - | - | 2 | 2 | 3 |
| CO2 | 3 | 3 | 2 | 3 | 3 | - | - | - | - | 2 | 3 | 3 |
| **CO3** | **3** | **2** | **3** | **3** | **3** | **2** | **-** | **-** | **-** | **2** | **3** | **3** |
| CO4 | 3 | 2 | 3 | 3 | 2 | 2 | - | - | - | 2 | 2 | 3 |
| CO5 | 1 | 1 | 1 | 1 | 1 | - | - | - | - | 2 | - | 3 |
| **Mean** | **2.6** | **2** | **2** | **2.2** | **2.4** | **2.0** | **-** | **-** | **-** | **2** | **2.5** | **3** |

# Prerequisite and Recap

- Basic knowledge of function of one and two variables

- Basic knowledge of differentiation

# Topic Objective

## Taylor’s and Maclaurin’s Theorem (CO 3)

- Students will be able to understand the concepts of expansion of a function of several variables.

- Taylor’s Theorem is used in physics when it’s necessary to write the value of a function at one point in terms of the value of that function at a nearby point.

## Taylor's Theorem

If the *f(x + h)* be a function of h (x being independent of h) which can be expended in powers of h and the expansion be differentiable any number of times, then

$$f(\mathrm{x}+\mathrm{h}) = \mathrm{f(x)} + \mathrm{hf}^{'}(\mathrm{x}) + \frac{h^2}{2!} f^{''}(\mathrm{x}) + \dots\dots\dots + \frac{h^n}{n!} f^{n}(\mathrm{x}) + \dots.. \quad \text{-------(1)}$$

**Case 1:** When we put *x = a in* equation (1) then

$$f(\mathrm{a}+\mathrm{h}) = \mathrm{f(a)} + \mathrm{hf}^{'}(\mathrm{a}) + \frac{h^2}{2!} f^{''}(\mathrm{a}) + \dots\dots\dots + \frac{h^n}{n!} f^{n}(\mathrm{a}) + \dots.. \quad \text{--------(2)}$$

# Expansion of function of one variable (CO3)

**Case 2:** If we put $h = x - a$ in equation (2) then

$$f(x) = \mathrm{f(a)} + (x - \mathrm{a})\,\mathrm{f}^{'}(\mathrm{a}) + \frac{(x-a)^2}{2!} f^{''}(\mathrm{a}) + \ldots\ldots\ldots + \frac{(x-a)^n}{n!} f^{n}(\mathrm{a}) + \ldots\ldots$$

**Case 3:** If we Put $a = 0$ and $h = x$ in equation (2) then

$$f(x) = \mathrm{f(0)} + x\,\mathrm{f}^{'}(0) + \frac{x^2}{2!} f^{''}(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^{n}(0) + \ldots\ldots$$

$$= \sum_{n=0}^{n=\infty} \frac{x^n}{n!} f^{n}(0)$$

which is known as **Maclaurin's theorem.**

# Example 1

**Question:** Expand $\log(1+x)$ in powers of $x$ . Then find the series for $\log\left(\frac{1+x}{1-x}\right)$ and hence determine the value of $\log\left(\frac{11}{9}\right)$ upto five places of decimal. **(MTU 2012)**

**Solution:** Let $f(x) = log\ (1+x)$ $\quad f'(x) = \frac{1}{1+x}$ $\quad f''(x) = -\frac{1}{(1+x)^2}$

$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = -1$

We know by Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots + \frac{x^n}{n!} f^n(0) + \ldots\ldots$$

Then we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

# Example 1 Contd....

Replacing *x* by *-x* in it then

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots.$$

Then we can find

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right]$$

by putting *x = 1/10,* we get

$$\log\left(\frac{11}{9}\right) = 0.20067$$

# Example 2

**Question:** Use Taylors theorem to Express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x - 2)$.

**Solution:** Let $f(x) = 2x^3 + 7x^2 + x - 6$ $\qquad$ $f(2) = 40$

$f'(x) = 6x^2 + 14x + 1$ $\qquad$ $f'(2) = 53$

$f''(x) = 12x + 14$ $\qquad$ $f''(2) = 38$

$f'''(x) = 12$ $\qquad$ $f'''(2) = 12$

By Taylors theorem ,we know

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots\ldots + \frac{(x-a)^n}{n!}f^n(a) + \ldots..$$

$$= 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

# Example 3

**Question:** Expand $e^x$ in powers of x by Maclaurin's theorem

**Solution:**

Let $f(x) = e^x$ $\quad f'(x) = e^x$ $\quad f''(x) = e^x$ $\quad f^n(x) = e^x$

$f(0) = 1$ $\quad f'(0) = 1$ $\quad f''(0) = 1$ $\quad f^n(0) = 1$

$$f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

# FAQ

1. Expand $e^{2x}\sin x$ in ascending powers of $x$ up to $x^4$.

**Ans:** $x+2x^2+\frac{11}{3!}x^3+------+\frac{5^{n/2}}{n!}x^n\sin\left(n\tan^{-1}\frac{1}{2}\right)$ **(UPTU 2014)**

2. Show that $$\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\right)$$

**(UPTU2015)**

3. Calculate the approximate value of √10 to four decimal places by taking the first four terms of an appropriate Taylor's expansion.

**Ans:** 3.16227

# Expansion of function of several variable (CO3)

If $f(x+h, y+k)$ be a given function which can be expanded into a series of positive ascending powers of $h$ and $k$ then

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \cdots$$

------------(1)

This is known as Taylor's series expansion of $f(x+h, y+k)$ in powers of $h$ and $k$.

# Expansion of function of several variable (CO3)

**Case 1:** Putting h = x–a ,x = a, k = y – b, y = b in equation (1) then

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

**Case 2:** Putting $h = x, x = 0, k = y, y = 0$ in equation (1) then

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

Which is known as Maclaurin's series.

# Example 1

**Question:** Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and$(y+2)$ up to second degree terms.

**Solution:**We have $f(x,y) = x^2y + 3y - 2$, where $a = 1, b = -2$

| | | x = 1, y = -2 |
|---|---|---|
| $f(x,y)$ | $x^2y + 3y - 2$ | -10 |
| $f_x(x,y)$ | 2xy | -4 |
| $f_y(x,y)$ | $x^2 + 3$ | 4 |
| $f_{xx}(x,y)$ | 2y | -4 |
| $f_{xy}(x,y)$ | 2x | 2 |
| $f_{yy}(x,y)$ | 0 | 0 |
| $f_{xxx}(x,y)$ | 0 | 0 |
| $f_{xxy}(x,y)$ | 2 | 2 |
| $f_{xyy}(x,y)$ | 0 | 0 |
| $f_{yyy}(x,y)$ | 0 | 0 |

# Example 1 Contd...

By Taylor's expansion

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

$$f(x,y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 \quad +2(x-1)(y+2)+..$$

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# Example 2

**Question:** Find theTaylor's series expansion of $f(x, y) = x^3 + xy^2$ about the point (2,1)

**Solution:** we have

| | | x = 2, y = 1 |
|---|---|---|
| $f(x, y)$ | $x^3 + xy^2$ | 10 |
| $f_x(x,y)$ | $3x^2+y^2$ | 13 |
| $f_y(x,y)$ | $2xy$ | 4 |
| $f_{xx}(x,y)$ | $6x$ | 12 |
| $f_{xy}(x,y)$ | $2y$ | 2 |
| $f_{yy}(x,y)$ | $2x$ | 4 |
| $f_{xxx}(x,y)$ | 6 | 6 |
| $f_{xxy}(x,y)$ | 0 | 0 |
| $f_{xyy}(x,y)$ | 2 | 2 |
| $f_{yyy}(x,y)$ | 0 | 0 |

# Example 2 Contd...

By Taylor theorem for a function of two variables,

$$f(x,y) = f(2,1) + (x-2)\, f_x(2,1) + (\, y-1\,)\, f_y(2,1)$$
$$+\frac{1}{2!}\left\{(x-2)^2 f_{xx}(2,1) + \ 2(x-2)(y-1) f_{xy}(2,1) + \ (y-1)^2 f_{yy}(2,1)\right\} \ldots$$

$$f(x,y)$$
$$= 10 + (x-2)13 + (y-1)4$$
$$+ \ \frac{1}{2}\{(x-2^2)12 + 2(x-2)(y-1)2 + \ (y-1)^2 4\} + \ \ldots$$

# Example 3

**Question:** Find theTaylor's series expansion of $f(x,y) = e^x \sin y$ in powers of x and y as far as terms of third degree.

**Solution:** we have

| | | x = 0, y = 0 |
|---|---|---|
| $f(x,y)$ | $e^x \sin y$ | 0 |
| $f_x(x,y)$ | $e^x \sin y$ | 1 |
| $f_y(x,y)$ | $e^x \cos y$ | 0 |
| $f_{xx}(x,y)$ | $e^x \sin y$ | 0 |
| $f_{xy}(x,y)$ | $e^x \cos y$ | 1 |
| $f_{yy}(x,y)$ | $- e^x \sin y$ | 0 |
| $f_{xxx}(x,y)$ | $e^x \sin y$ | 0 |
| $f_{xxy}(x,y)$ | $e^x \cos y$ | 1 |
| $f_{xyy}(x,y)$ | $-e^x \sin y$ | 0 |
| $f_{yyy}(x,y)$ | $-e^x \cos y$ | -1 |

# Example 3 Contd…

By Taylor theorem for a function of two variables,

$$\mathrm{f(x,y)} = \mathrm{f(0,0)} + (\mathrm{x}-0)\,\mathrm{f_x(2,1)} + (\,\mathrm{y}-0\,)\,\mathrm{f_y(0,0)}$$

$$+\frac{1}{2!}\left\{(x-0)^2\mathrm{f_{xx}}(0,0) + 2(x-0)(y-0)\mathrm{f_{xy}}(0,0) + (y-0)^2\mathrm{f_{yy}}(0,0)\right\}\ldots$$

$$\mathrm{f(x,y)} = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \ldots\ldots\ldots\ldots$$

# FAQ

**1.** Expand $e^x \log(1+y)$ in Taylor's series about the origin up to the terms of degree three. **[UPTU 2014]**

**Ans:** $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \cdots \dots \dots \dots$

**2.** Expand $e^x \sin y$ in the powers of x and y in the neighborhood of $(0,\frac{\pi}{4})$ up to the third degree terms.**[U.P.T.U.2008]**

**Ans**:

$$\frac{1}{\sqrt{2}}\left[\begin{array}{c}1 + x + \left(y - \frac{\pi}{4}\right) + \frac{x^2}{2!} + x\left(y - \frac{\pi}{4}\right) - \frac{1}{2!}\left(y - \frac{\pi}{4}\right)^2 + \frac{x^3}{3!} - \frac{1}{3!}\left(y - \frac{\pi}{4}\right)^3 \\ +3\frac{x^2}{3!}\left(y - \frac{\pi}{4}\right) - \frac{3x}{3!}\left(y - \frac{\pi}{4}\right)^2\end{array}\right]$$

# Quiz

- Expansion $of\ e^{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

- Expansion of $f(x, y) = f(a, b) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

- Maclaurian's expansion of $\sin x = \ldots\ldots\ldots$

- **Taylor's theorem**

- NPTEL-Lecture 17: Taylor's Theorem
https://www.youtube.com/watch?v=jiEaKYI0ATY

- NPTEL-Lecture 16: Taylor's Theorem for function of two variable
https://www.youtube.com/watch?v=r6lDwJZmfGA

- **Maclaurin's Theorem**

- Lecture 27- Taylor's Theorem and Maclaurin's Series
https://www.youtube.com/watch?v=Jk9xMY4mPH8

# Weekly Assignment (CO3)

## Taylor and Maclaurin Series

1. Expand $e^{2x} sinx$ in ascending power of $x$ up to $x^5$. **Ans:** $x + 2x^2 + \frac{11}{6}x^3 + x^4 + \frac{41}{120}x^5$
2. Expand $\log(1 + x)$ in powers of $x$. Then find series for $log_e\left[\frac{1+x}{1-x}\right]$ and hence determine the value of $log_e\left(\frac{11}{9}\right)$ upto five places of decimal. **Ans:** $\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$, $log_e\left[\frac{1+x}{1-x}\right] = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right]$, 0.20067
3. Expand $x^y$ in the powers of $(x - 1)$ and $(y - 1)$up to the third degree terms. **Ans:** $1 + (x - 1) + (x - 1)(y - 1) + \frac{1}{2}(x - 1)^2(y - 1) + \cdots$
4. Expand $e^x \sin y$ in the powers of x and y in the neighborhood of $\left(0, \frac{\pi}{4}\right)$ up to the third degree terms. **Ans:** $\frac{1}{\sqrt{2}}\left[1 + x + \left(y - \frac{\pi}{4}\right) + \frac{x^2}{2!} + x\left(y - \frac{\pi}{4}\right) - \frac{1}{2!}\left(y - \frac{\pi}{4}\right)^2 + \frac{x^3}{3!} - \frac{1}{3!}\left(y - \frac{\pi}{4}\right)^3 + 3\frac{x^2}{3!}\left(y - \frac{\pi}{4}\right) - \frac{3x}{3!}\left(y - \frac{\pi}{4}\right)^2\right] + \cdots$
5. Expand $e^x \log(1 + y)$ in Taylor's series about the origin up to the terms of degree three.

**Ans:** $y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 + \cdots \ldots \ldots \ldots$

6. Expand $tan^{-1}\frac{y}{x}$ in the neighbourhood of (1, 1) upto and inclusive of second degree terms. Hence compute f(1.1, 0.9) approximately. **Ans:** $\frac{\pi}{4} - \frac{1}{2}(x - 1) + \frac{1}{2}(y - 1) + \frac{1}{4}(x - 1)^2 - \frac{1}{4}(y - 1)^2$; 0.6857 approx.

# Recap

- ✓ Taylor's & Maclaurin's Theorems for one variable.
- ✓ Expansion of function of several variables
- ✓ Discussion on assignments.

# Topic Objective

## Jacobians (CO3)

- Students will be able to evaluate of Jacobians of two and more variables.

- In our physical world many things can be represented as a vector field etc.

- The Jacobian matrix of such a field is just how the field changes with the input (position, time, etc.), to the first order.

# Jacobians (CO3)

If u and v continuous and differentiable function of two variables x and y, i.e.,

$u = f_1(x, y)$ $and$ $v = f_2(x, y)$ then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the jacobian of u and v with respect to x,y and is denoted as

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

# Properties of Jacobians (CO3)

## Property-1

If u and v are functions of x and y then $J.J^* = 1$ where
$J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J^* = \frac{\partial(x,y)}{\partial(u,v)}$.

Similarlyif u , v and w are functions of x, y and z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

# Properties of Jacobians (CO3)

## Property-2

If u , v are functions of r, s and r, s are function of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}\ .$$

Similarly, If u , v, w are functions of r , s , t and r, s , t are functions of *x* , y and z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$$

# Properties of Jacobians (CO3)

## Property-3

If functions u, v of two independent variables x, y are dependent,

then $\frac{\partial(u,v)}{\partial(x,y)} = 0.$

# Jacobian of Implicit Functions (CO3)

If u, v are implicit functions of x, y connected by $f_1, f_2$ such that $f_1(u, v, x, y) = 0$, and $f_2\,(u, v, x, y) = 0$

then $$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

If u, v and w are implicit functions of x, y and z connected by $f_1, f_2$ and $f_3$ such that
$f_1(u, v, w, x, y, z) = 0$, $f_2\,(u, v, w, x, y, z) = 0$ and $f_3\,(u, v, w, x, y, z) = 0$

then $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

# Example 1

**Question**: If $x = u(1+v)$ $and$ $y = v(1+u), find\,\frac{\partial(x,y)}{\partial(u,v)}$.

**Solution:** we have $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + u + v$$

# Example 2

**Question:** If $u = xyz$ , $v = x^2 + y^2 + z^2$ $and$ $w = x + y + z$ then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

**Solution:**

We have $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2(x-y)(y-z)(z-x).$$

Also, we know that $\frac{\partial(u,v,w)}{\partial(x,y,z)} \times \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \frac{-1}{2(x-y)(y-z)(z-x)}.$$

# Example 3

**Question:** If $x = \sqrt{vw},\ y = \sqrt{uw}, z = \sqrt{uv}\ and\ u = rsin\theta\ cos\varphi,$
$v = rsin\theta sin\varphi,\ w = rcos\vartheta$. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$.**(UPTU2008)**

**Solution:**

We have $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix} = \frac{1}{4}$

and $\frac{\partial(u,v,w)}{\partial(r,\theta,\varphi)} = r^2 sin\theta$

Thus $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \frac{1}{4} r^2 sin\theta$

# Example 4

**Question:** If $u, v, w$ are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ , find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

**Solution:** We have $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

$$\lambda^3 - 3(x+y+z)\lambda^2 + 3(x^2+y^2+z^2)\lambda - (x^3+y^3+z^3)\lambda = 0$$

If *u* ,*v*, *w* are the roots of the equation

$$u + v + w = x + y + z$$

$$uv + vw + wu = x^2 + y^2 + z^2$$

$$uvw = \frac{1}{3}(x^3 + y^3 + z^3)$$

# Example 4 Contd…

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

After solving, we get $\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = -2(x-y)(y-z)(z-x)$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

# Example 4 Contd…

We get

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = (w-u)(w-v)(u-v)$$

then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

Thus

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -\frac{-2(x-y)(y-z)(z-x)}{(w-u)(w-v)(u-v)}$$

# Example 5

**Question:** Determine whether the following functions are functionally dependent or not. If functionally dependent, find the relation between them.

$$u = sin^{-1}x + sin^{-1}y \; and \; v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

**Solution:**

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & -\frac{xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix} = 0$$

i.e., u and v are functionally dependent.

# Example 5 Contd...

Relation between $u$ and $v$

Let $sin^{-1}x = \alpha$ $and$ $sin^{-1}y = \beta$

then $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$= sin\alpha\ cos\beta + sin\beta\ cos\alpha$$

$$= sinu.$$

# Example 6

**Question:** If $u^3+v^3=x+y,\ \ u^2+v^2=x^3+y^3$ find $\frac{\partial(u,v)}{\partial(x,y)}$.

**Solution:**

$$Let \qquad f_1 = u^3+v^3-x-y=0 \qquad\qquad f_2=u^2+v^2-x^3-y^3=0.$$

$$\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\frac{\partial(f_1,f_2)}{\partial(x,y)}\div\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_1}{\partial y}\\ \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_2}{\partial y}\end{vmatrix}\div\begin{vmatrix}\frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_1}{\partial u}\\ \frac{\partial\phi_2}{\partial u} & \frac{\partial\phi_2}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}-1 & -1\\ -3x^2 & -3y^2\end{vmatrix}\div\begin{vmatrix}3u^2 & 3v^2\\ 2u & 2v\end{vmatrix}$$

$$=\frac{3\left(y^2-x^2\right)}{6uv(u-v)} \quad =\frac{1}{2}\frac{y^2-x^2}{uv(u-v)}$$

# FAQ

**1.** If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}$ $and$ $y_3 = \frac{x_1 x_2}{x_3}$ then show that $\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = 4.$

**2.** If $x + y + z = u,\ y + z = uv\ \ and\ \ z = uvw$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.**(GBTU2013)**

**3.** If $x = rsin\theta cos\varphi, y = rsin\theta sin\varphi\ \ and\ z = rcos\theta,$

show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2 sin\theta$ . and find $\frac{\partial(r,\theta,\varphi)}{\partial(x,y,z)}$. **(GBTU2010)**

**4.** If $u_1 = x_{1+} x_{2+} x_{3+} x_4, u_1 u_2 = x_{2+} x_{3+} x_4,\ u_1 u_2 u_3 = x_{3+} x_4$ $u_1 u_2 u_3 u_4 = x_4$ then show that $\frac{\partial(x_1 x_2 x_3 x_4)}{\partial(u_1 u_2 u_3 u_4)} = u_1^3 u_2^2 u_{3.}$ **(MTU2013)**

**5.** Verify the chain rule for the Jacobians if $x = u, y\ \ =\ \ u \tan v,, z = w.$ **(UPTU 2009)**

# FAQ

**6.**If $u = x + 2y + z, v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between $u, v$ and$w$. **(UPTU2007, GBTU2011)**

**7**. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy$, prove that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{x^2 - y^2}{u^2 + v^2}$

**8.** If $u^3 + v^3 = x + y,\ u^2 + v^2 = x^3 + y^3$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{y^2 - x^2}{2uv(u-v)}$.**(UPTU2007)**

**9.** If $u = x(1 - r^2)^{-\frac{1}{2}}, v = y(1 - r^2)^{-\frac{1}{2}}, w = z(1 - r^2)^{-\frac{1}{2}}$ where $r^2 = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1 - r^2)^{-\frac{5}{2}}$. **(UPTU2009)**

# Quiz

- Calculate $\dfrac{\partial(u,v)}{\partial(x,y)}$ for $x = e^u \cos v$ and $y = e^u \sin v.$

- What is the condition for two functions u(x, y) and v(x, y) to be functinally dependent?

- The Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$ for the function

  $u = e^x \sin y, v = x + \log \sin y$ is.........

- If $x = u(1+v),\ y = v(1+u),$ the Jaconian $\dfrac{\partial(u,v)}{\partial(x,y)} = \dots\dots\dots\dots$

# Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details

- **Jacobian**

- Khan Acad- The Jacobian determinant
- https://www.youtube.com/watch?v=p46QWyHQE6M

- NPTEL-Lecture
- https://www.youtube.com/watch?v=fqq_UR4zhfI

# Recap

- ✓ Jacobian and their properties.
- ✓ Jacobian of Implicit Function.
- ✓ Discussion on assignments

# Topic Objective

## Errors and Approximations (CO3)

- Student will be able to understand the concepts of error and evaluate the approximations.

# Errors and Approximations (CO3)

- An approximation error can occur because:
- The measurement of the data is not precise due to the instruments.
- Approximations are used instead of the real data.
- The relative error is often used to compare approximations of numbers of widely differing size.

# Errors and Approximations (CO3)

Let $z = f(x, y)$. If $\delta x, \delta y$ are small increments in $x, y$ and $\delta z$, the corresponding increment in $z$ then

$$z + \delta z = f(x + \delta x, y + \delta y)$$

$$\delta z = f(x + \delta x, y + \delta y) - f(x, y)$$

$$\delta z = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$$

If $\delta x$ and $\delta y$ are small changes (or errors) in x and y respectively, then an approximate change (or error) in $z$ is $\delta z$. Replacing $\delta x, \delta y, \delta z$ by $dx, dy, dz$ respectively, we have

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

# Errors and Approximations (CO3)

If $\delta x$ is the error in $x$ then

- $\delta x$ is known as absolute error in $x$

- $\frac{\delta x}{x}$ is known as relative error in $x$.

- $\frac{\delta x}{x} \times 100$ is known as percentage error in $x$.

# Example 1

**Question:** Find the percentage error in the area of an ellipse when errors of 2% and 3% are made in measuring its major and minor axes respectively. **(GBTU2011)**

**Solution:** We have, Area of an ellipse, $A = \pi ab$

$$logA = log\pi + loga + logb$$

$$\frac{1}{A}\delta A = 0 + \frac{1}{a}\delta a + \frac{1}{b}\delta b$$

$$\frac{1}{A}\delta A \times 100 = 2\% + 3\% = 5\%$$

# Example 2

## Question:

Find approximate value of $[(0.98)^2+(2.01)^2+(1.94)^2]^{1/2}$ **(MTU2012)**

**Solution:** We have $u(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$

by taking $x = 1.0, y = 2, and\ z = 2$

Therefore we get , $x + \delta x = 0.98$

$\delta x = -0.02$

Similarly $\delta y = 0.01\ and\ \delta z = -0.06$

When $x = 1.0, y = 2, z = 2$ then $u = 3.$

Corresponding error $\delta u\ in\ u$ is given by

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z = -0.04$$

Required approximate value$= u + \delta u = 2.96$

# FAQ

**1.** Find the possible percentage error in computing the parallel resistance $r$ of two resistances $r_1$ and $r_2$ from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ where $r_1, r$ are each in error by 2%. **(UKTU2011)**

**2.** In estimating the number of bricks in a pile which is measured to be $(5m \times 10m \times 5m)$, the count of bricks is taken as 100 bricks per $m^3$.Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of the bricks is Rs.2000 per thousand bricks.**(UKTU2010)**

**3.** A balloon is in the form of a right cylinder of radius 1.5m and length 4.0m surmounted by hemispherical ends. If the radius is increased by 0.01m and length by 0.05m ,find the percentage . **(UPTU2006)**

**4.** Evaluate$(1.99)^2(3.01)^3(0.98)^{1/10}$ using approximation.**(UPTU2008)**

# Quiz

- If an error of 1% is made in measuring its length and breadth. what is the percentage error in the area of a rectanle?

- The percentage error in computing the area of an ellipse when an error of 1% is made in measuring the major and minor axes is..........

# Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details

- **Approximation of Error**

- NPTEL-Lecture 18: Error Approximation
- https://www.youtube.com/watch?v=G0V_yp0jz5c

# Recap

- ✓ Errors
- ✓ Approximation

# Topic Objective

**<u>Maxima and minima of Functions of Several Variables (CO 3)</u>**

- Students will be able to evaluate maxima and minima of a function of two and several variables.

# Extreme of Functions of Several Variables (CO 3)

## Introduction

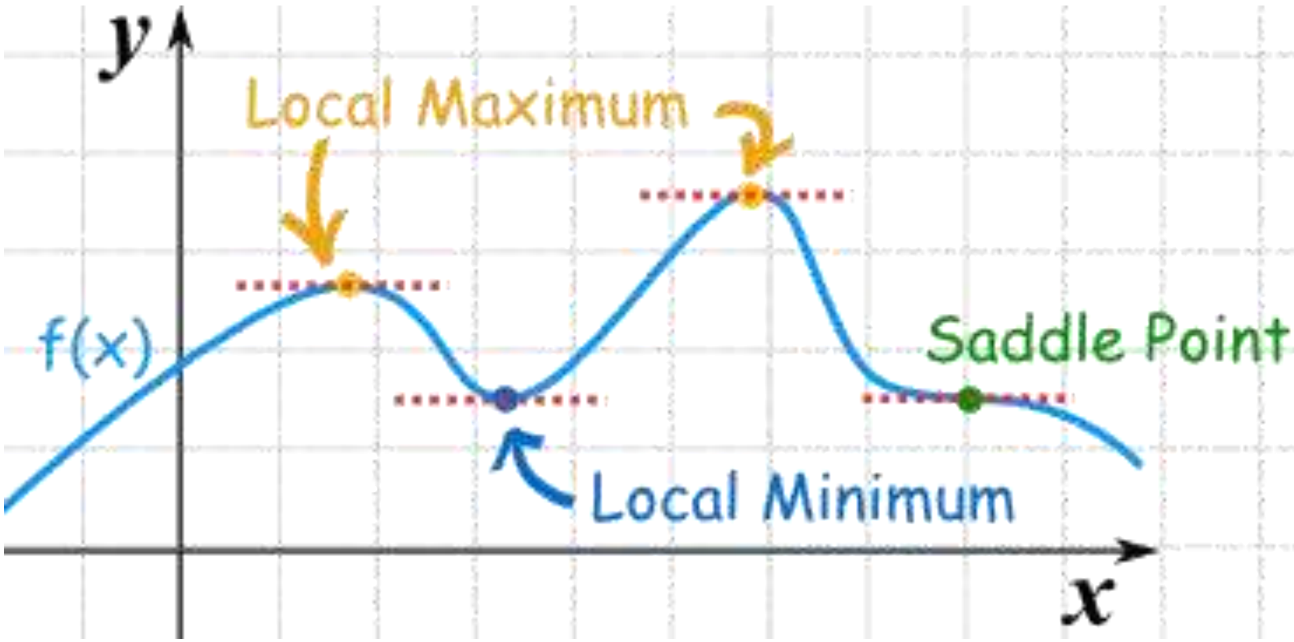
- A maximum is a high point and a minimum is a low point.

- The terms maxima and minima refer to extreme values of a function, that is, the maximum and minimum values that the function attains. Maximum means upper bound or largest possible quantity. The absolute maximum of a function is the largest number contained in the range of the function.

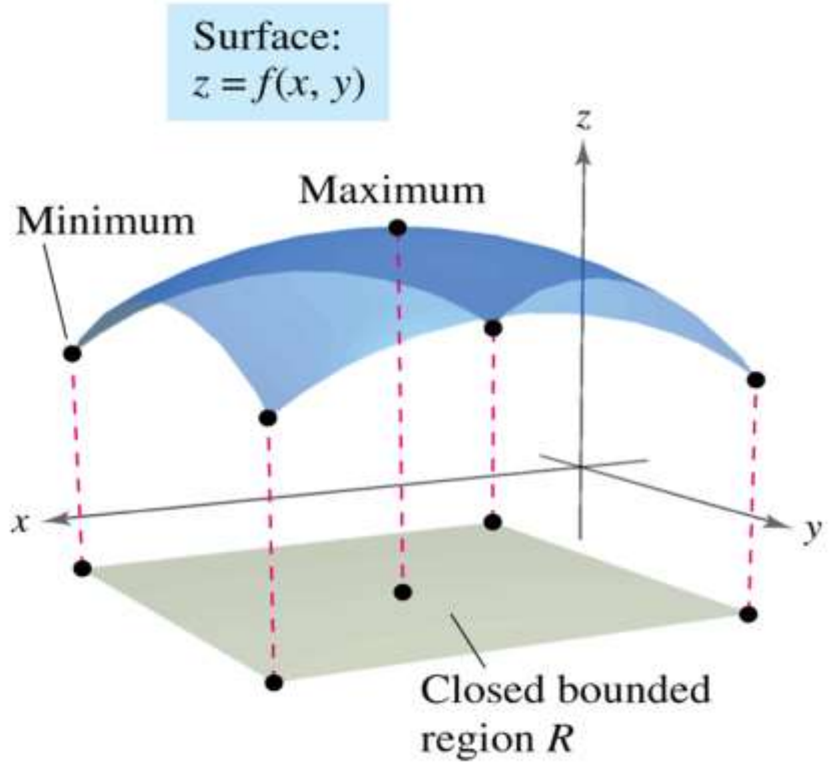
- Example: A ball is thrown in the air. Its height at any time t is given by, $h = 3 + 14t - 5t^2$ . What is its maximum height?

# Extreme of Functions of Several Variables (CO 3)

## For single variable



## For several variable variable



## Working rule to find extreme value

**Step 1:** Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y},$ and $\frac{\partial^2 f}{\partial y^2}$

**Step 2:**

Find stationary point by solving $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously.

Let (a,b), (c,d) .......are the solutions.

**Step 3:** For each solution in step 2, find r, s and t as follows-

$$r = \frac{\partial^2 f}{\partial x^2}, \ s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}$$

## Working rule to find extreme value

**Step 4:** At the point (a,b)

(i) If $rt - s^2 > 0$ and $r < 0$ at $(a, b)$ Then f(x,y) is maximum at (a,b)

(ii) If $rt - s^2 > 0$ and $r > 0$ at $(a, b)$ Then f(x,y) is minimum at (a,b)

(iii) If $rt - s^2 < 0$ and $r > 0$ at $(a, b)$

Then f(x,y) is neither maximum nor minimum at (a, b). Such points is known as saddle point.

(iv) If $rt - s^2 = 0$ at $(a, b)$

Then no conclusion can be made about extreme values of f(x,y) And further investigation is required.

# Example 1

## Question:

Find the stationary value of $x^3 + y^3 - 3axy, \mathrm{a} > 0$ (UPTU2009,GBTU2012)

## Solution:

Step (i):- We have, stationary points are $(0,0)$ and $(a,a)$

Step (ii):- $r = 6x, s = -3a$ and $t = 6y$

Step(iii):- At (0,0), r = 0, s = -3a, t = 0, then $rt - s^2 = -9a^2 < 0$.

Hence f(x ,y) is niether maximum nor minimum at (0,0)

At (a , a), r = 6a, s = -3a, t = 6a

then $rt - s^2 = 27a^2 > 0$ also $r = 6a > 0$

Hence f(x,y) is minimum at(a,a)

Therefore $f_{\min} = -a^3$

# Example 2

**Question:** Find the extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.

**Solution:** Calculate $\frac{\partial \mathrm{f}}{\partial \mathrm{x}}, \frac{\partial \mathrm{f}}{\partial \mathrm{y}}$ and equating it with zero

$$f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0 => x = y = -2$$

Therefore, the point $(-2, -2)$ is the only point where $f$ may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1$$

$\mathrm{rt} - \mathrm{s}^2 = (-2)(-2) - (1)^2$ = 4 – 1= 3.

$r < 0$ and $rt - \mathrm{s}^2 > 0$ tells us that $f$ has a maximum at $(-2, -2)$ .

The value of $f$ at this point is $f(-2, -2) = 8$.

# Lagrange's Method of Undetermined Multipliers (CO3)

This method is useful to find the extreme values (i.e., maximum and minimum) for the given function, whenever the variables are three or more. To find the Maxima and Minima for the given function using Lagrange's Method, the following procedure must be followed:

**Step 1:** Let us consider given function to be $f(x, y, z)$ subject to the condition $\Phi(x, y, z) = 0$.

**Step 2:** Let us define a Lagrange's function $F = f + \lambda\Phi$ , where $\lambda$ is called the Lagrange multiplier.

# Lagrange's Method of Undetermined Multipliers (CO3)

**Step 3:** Find first order partial derivatives and equate to zero

*i.e.* $\frac{\partial F}{\partial x} = 0 \Longrightarrow \frac{\partial f}{\partial x} + \lambda \frac{\partial \Phi}{\partial x} = 0$.....(1)

$\frac{\partial F}{\partial y} = 0 \Longrightarrow \frac{\partial f}{\partial y} + \lambda \frac{\partial \Phi}{\partial y} = 0$.....(2)

$\frac{\partial F}{\partial z} = 0 \Longrightarrow \frac{\partial f}{\partial z} + \lambda \frac{\partial \Phi}{\partial z} = 0$.....(3)

Let the given condition be $\Phi(x, y, z) = 0$. ....(4)

**Step 4**: Solve (1), (2), (3) & (4) , eliminate $\lambda$ to get the values of *x, y, z*

# Lagrange's Method of Undetermined Multipliers (CO3)

**Step 5:** The values so obtained will give the stationary point of $f(x, y, z)$

**Step 6:** The minimum/maximum value will be obtained by substituting the values of $x, y, z$ in the given function.

**Note:** Lagrange's method does not enable us to find whether there is a maximum or minimum.

# Example 3

## Question:

Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$

**Solution:** Let us consider given function to be $f = x^2 + y^2 + z^2$ and $\Phi = xyz - a^3$ . Let us define Lagrange's function $F = f + \lambda\Phi$ , where $\lambda$ is called the Lagrange multiplier.

$$\Longrightarrow F = (x^2 + y^2 + z^2) + \lambda(xyz - a^3)$$

Now, $\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + \lambda yz = 0 \Longrightarrow \frac{\lambda}{2} = -\frac{x}{yz}$...... (1)

$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + \lambda xz = 0 \Longrightarrow \frac{\lambda}{2} = -\frac{y}{xz}$...... (2)

$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + \lambda xy = 0 \Longrightarrow \frac{\lambda}{2} = -\frac{z}{xy}$...... (3)

Solving (1), (2) & (3) $\Longrightarrow \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}$

# Example 3 Contd…

Now, consider $\frac{x}{yz} = \frac{y}{xz} \Longrightarrow x^2 = y^2$…… (4)

Again, consider $\frac{y}{xz} = \frac{z}{xy} \Longrightarrow z^2 = y^2$…… (5)

Again solving (4) & (5) $\Longrightarrow x^2 = y^2 = z^2 \quad \Longrightarrow x = y = z$

Given $g = xyz - a^3 = 0$ At $x = y = z \Longrightarrow x^3 = a^3 \quad \Longrightarrow x = a$

Similarly, we get $y = a,\ z = a$

Hence, the minimum value of the function is given by

$$(f)_{(a,a,a)} = a^2 + a^2 + a^2 = 3a^2$$

# Example 4

**Question:** Find the extreme value of $x^2 + y^2 + z^2$ ,subject to the condition $xy + yz + zx = p$ **(UPTU2008)**

**Solution:** Lagrange's equation ,

$$2x + \lambda(y+z) = 0$$

$$2y + \lambda(x+z) = 0$$

$$2z + \lambda(x+y) = 0$$

Solving these equation, we get $x = y = z$

$$x^2 = \frac{p}{3}, y^2 = \frac{p}{3}, z^2 = \frac{p}{3}$$

Extreme value of $f(x,y,z) = \frac{p}{3} + \frac{p}{3} + \frac{p}{3} = p$

# FAQ

**1.** Find the maximum value of the function $f(xyz) = (z - 2x^2 -$

**2.** A rectangular box open at the top is of given volume, what must be the dimensions so that the surface is minimum.

**[UPTU 2012, GBTU 2010]** **Ans:** $x = y = (2V)^{\frac{1}{3}}$

**3**. Find the maximum and minimum distances from the origin to the curve $x^2 + 4xy + 6y^2 = 140.$ **[MTU 2012]**
**Ans:** 21.6589(max), 4.5706(min)

# Quiz

- What is maximum value of function $f(x, y) = 1 - x^2 - y^2$?

- Determine the point(s) where the function $u = x^2 + y^2 + 6x + 12$ has a maximum or minimum.

- Find the statonary points of $f(x, y) = 5x^2 + 10y^2 + 12xy - 4x - 6y + 1$.

# Extreme of functions of several variables

- NPTEL-Lecture 14: Extreme values- I
- https://www.youtube.com/watch?v=9-tir2V3vYY

- NPTEL-Lecture 17: Maxima and minima of functions of two variables
- https://www.youtube.com/watch?v=jGwA4hknYp4

# Weekly Assignment (CO3)

## Jacobian

1. If $u_1 = \frac{x_2 x_3}{x_1}$, $u_2 = \frac{x_3 x_1}{x_2}$ $and$ $u_3 = \frac{x_1 x_2}{x_3}$ , find the value of $\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}$. **Ans. 4**
2. If $x = \sqrt{vw}, y = \sqrt{uw}$ , $z = \sqrt{uv}$ and $u = r sin\ \theta cos\ \emptyset$ $and$ $v = r sin\ \theta \sin \emptyset$, $w = r \cos \theta$ find the value of $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)}$. **Ans.** $\frac{1}{4} r^2 \sin\theta$
3. If $u = \frac{x+y}{z}$, $v = \frac{z+y}{x}$, $w = \frac{y(x+y+z)}{xz}$, then show that u, v and w are not independent and find relation between them. **Ans.** Relation is $uv - w = 1$
4. .If $x + y + z = u, y + z = uv, z = uvw$. Then show $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^2 v$.
5. Prove that the functions $u = 3x + 2y - z, v = x - 2y + z, w = x^2 - 2xy - xz$ are not independent, and find the relation between them
6. If u, v and w are the roots of the equation $(x-a)^3 + (x-b)^3 + (x-c)^3 = 0$, then find $\frac{\partial(u,v,w)}{\partial(a,b,c)}$.

   **Ans.** $\frac{-2(a-b)(b-c)(c-a)}{(u-v)(v-w)(w-u)}$

## Maxima and Minima

1. A rectangular box open at the top is to have a volume of 32 cubic feet. Determine the dimensions of the box requiring least material for its construction. **Ans.** $l = 4, b = 4, h = 2$
2. Find the dimension of the rectangular box of maximum capacity whose surface area is given. When (i) box is open at the top (ii) box is closed. **Ans.** $(i) l = b = 2h \quad (ii) l = b = h$
3. Divide 24 into three parts such that the continued product of the first ,square of the second and the cube of the third may be maximum. **Ans. 4,8,12.**
4. Find the shortest and maximum distance from the point (1, 2, -1) to the sphere $x^2 + y^2 + z^2 = 24$. **Ans.** $\sqrt{6}, \sqrt{54}$
5. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

# Recap

- ✓ Extrema of functions of several variables
- ✓ Stationary points
- ✓ Rule to find maxima and minima of function of two variable
- ✓ Lagrange’s method of multipliers
- ✓ Discussion on assignment

# MCQ s

**1.** If $x = r\cos\theta$ , $y = r\sin\theta$ then the value of $\frac{\partial(x,y)}{\partial(r,\theta)}$ is

**(i)** $r$ **(ii)** 1/$r$**(iii)** 1 **(iv)** 2**Ans. (i)**

**2.** If $x = u, y = u\,tanv$ , $z = w$ then value of $\frac{\partial(x,y,z)}{\partial(u,v,w)}$is

**(i)** $\frac{x}{x^2+y^2}$**(ii)** $u\,sec^2v$ **(iii)** 1 **(iv)** 0 **Ans. (ii)**

**3.** Two functions are functionally dependent. If their jacobian with respect to independent variable is

**(i)** 1 **(ii)** 0 **(iii)** ½ **(iv)** None of these

**Ans. (ii)**

**4.** If $rt - s^2 > 0$ $and$ $r < 0$ at the point (a,b) then the value of function has

**( i )** Minimum **(ii)** Neither maximum nor minimum

**(iii)** Zero **(iv)** Maximum **Ans. (iv)**

# MCQ s

**5**. If $f(x,y) = 1 - x^2y^2$ then stationary point is

**(i)** (0,0) **(ii)** (1,1) **(iii)** None of these **(iv)** 1

**Ans.(iii)**

**6.** The minimum value of $f(x,y) = x^2 + y^2$ is

(i) 1**(ii)** 2 **(iii)** 5 **(iv)** 0 **Ans. (iv)**

**7.** Expand $e^{x+y}$ in power of $(x-1)$ $and$ $(y+1)$ up to first-degree is

**(i)** $f(x) = 1 + (x-1) + (y+1)$**(ii)** $f(x) = 1 + (x+1) + (y+1)$

**(iii)** $f(x) = 1 + (x-1) + (y-1)$ **(iv)** $f(x) = 1 + xy$ **Ans.(i)**

**8.** Maclaurin'sExpansion of $sinx$is

**(i)** $x - \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}} - \cdots$**(ii)** $x - \frac{x^3}{\sqrt{3}} - \frac{x^5}{\sqrt{5}} - \cdots$**(iii)** $x + \frac{x^3}{\sqrt{3}} + \frac{x^5}{\sqrt{5}}$ ...

**(iv)** $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ .... . **Ans.(iv)**

# MCQ s

**9.** In calculating the volume of a right circular cylinder, errors of 2% and 1% are found in measuring height and base radius respectively. Then the percentage error in volume of the cylinder is

**(i)** 2% **(ii)** 1% **(iii)** 4% **(iv)** 10% **Ans.(iii)**

**10.** If $\delta x$ is the error in $x$ then $\frac{\delta x}{x}$ x100 is called

**(i)** Percentage error **(ii)** relative error **(iii)** absolute error **(iv)** none of these **Ans(i)**

**11.**The errors in measuring the radius of the base of a cone and its volume are found to be 1% and 4%, then the error in the height of the cone is

**(i)** 1% **(ii)** 2% **(ii)** 3% **(iv)** 4% **Ans.(ii)**

**12.** Approximate value of $(1.04)^{3.01}$ is

**(i)** 1.00 **(ii)** 1.11 **(iii)** 1.12 **(iv)** 1.13 **Ans.(iii)**

# Old Year Paper 2019 (KAS-103)

## Differential Calculus 2 (CO3)

### Section A

h. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. 2

i. Find $\frac{du}{dt}$ if $u = x^3 + y^3, x = a\cos t, y = b\sin t$. 2

### Section B

If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ and $u + v + w = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$. 10

### Section C

a. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$. 10

b. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. 10

# Old Year Paper 2019 (KAS-103)

Printed Page 1 of 2 | Sub Code:KAS103

Paper Id: 199103 | Roll No:

**B. TECH.**
**(SEM I) THEORY EXAMINATION 2019-20**
**MATHEMATICS-I**

*Time: 3 Hours* | *Total Marks: 100*

**Note: 1.** Attempt all Sections. If require any missing data; then choose suitably.

## SECTION A

**1. Attempt *all* questions.**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | Show that vectors (1, 6, 4), (0, 2, 3) and (0, 1, 2) are linearly independent. | 2 | 1 |
| b. | Define Lagrange's mean value theorem. | 2 | 2 |
| c. | If $u = x(1-y), v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$. | 2 | 3 |
| d. | Show that vector $\vec{V} = (x+3y)\hat{\imath} + (y-3z)\hat{\jmath} + (x-2z)\hat{K}$ is solenoidal. | 2 | 5 |
| e. | Find the value of 'b' so that rank of $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{bmatrix}$ is 2. | 2 | 1 |
| f. | Evaluate $\int_0^2 \int_0^1 (x^2 + 3y^2)dydx$. | 2 | 4 |
| g. | Find grad ∅ at the point (2, 1, 3) where $\emptyset = x^2 + yz$ | 2 | 5 |
| h. | If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. | 2 | 3 |
| i. | Find $\frac{du}{dt}$ if $u = x^3 + y^3, x = a\cos t, y = b\sin t$. | 2 | 3 |
| j. | Find the area lying between the parabola $y = 4x - x^2$ and above the line $y = x$. | 2 | 4 |

# Old Year Paper 2019 (KAS-103)

## SECTION B

**2. Attempt any *three* of the following:**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and hence find $A^{-1}$. | 10 | 1 |
| b. | If $y = e^{m\cos^{-1}x}$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. Hence find $y_n$ when $x = 0$. | 10 | 2 |
| c. | If $u^3+v^3+w^3 = x+y+z$, $u^2+v^2+w^2 = x^3+y^3+z^3$ and $u+v+w = x^2+y^2+z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$. | 10 | 3 |
| d. | Evaluate the integral by changing the order of integration: $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$. | 10 | 4 |
| e. | Verify Stoke's theorem for the vector field $\vec{F} = (x^2-y^2)\hat{\imath} + 2xy\hat{\jmath}$ integrated round the rectangle in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a, y = b$. | 10 | 5 |

# Old Year Paper 2019 (KAS-103)

Printed Page 2 of 2 Sub Code:KAS103

Paper Id: 199103 Roll No:

**3. Attempt any *one* part of the following:**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | For what values of $\lambda$ and $\mu$ the system of linear equations: $x+y+z=6$, $x+2y+5z=10$, $2x+3y+\lambda z=\mu$ has (i) a unique solution (ii) no solution (iii) infinite solution. Also find the solution for $\lambda=2$ and $\mu=8$. | 10 | 1 |
| b. | Find the rank of the matrix $A=\begin{bmatrix}1 & 3 & 4 & 2\\ 2 & -1 & 3 & 2\\ 3 & -5 & 2 & 2\\ 6 & -3 & 8 & 6\end{bmatrix}$ by reducing it to normal form. | 10 | 1 |

**4. Attempt any *one* part of the following:**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | Verify the Cauchy's mean value theorem for the function $e^x$ and $e^{-x}$ in the interval$[a,b]$. Also show that 'c' is the arithmetic mean between a and b. | 10 | 2 |
| b. | Trace the curve $r^2=a^2\cos 2\theta$. | 10 | 2 |

**5. Attempt any *one* part of the following:**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | If $u=f(2x-3y,3y-4z,\ 4z-2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0$. | 10 | 3 |
| b. | Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. | 10 | 3 |

# Old Year Paper 2019 (KAS-103)

**6. Attempt any *one* part of the following:**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | Evaluate $\iint (x+y)^2 dx\, dy$, where R is the parallelogram in the $xy$-plane with vertices $(1,0),(3,1),(2,2),(0,1)$ using the transformation $u = x + y,\ v = x - 2y$. | 10 | 4 |
| b. | Find the volume of the region bounded by the surface $y = x^2,\ x = y^2$ and the planes $z = 0,\ z = 3$. | 10 | 4 |

**7. Attempt any *one* part of the following:**

| Q. No. | Question | Marks | CO |
|---|---|---|---|
| a. | Verify the divergence theorem for $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{K}$ taken over the rectangular parallelepiped $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$. | 10 | 5 |
| b. | Find the directional derivative of $\emptyset(x,y,z) = x^2yz + 4xz^2$ at (1, -2, 1) in the direction of $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$. Find also the greatest rate of increase of $\emptyset$. | 10 | 5 |

# Old Year Paper

## Links for old year papers

### Mathematics I, 2019-20 (KAS-103)

http://www.aktuonline.com/papers/btech-1-sem-mathematics-1-kas103-2020.html

### Mathematics I, 2018-19 (NAS-103)

http://www.aktuonline.com/papers/btech-1-sem-engineering-mathematics-1-nas-103-2018-19.html

### Mathematics I, 2017-18 (RAS-103)

http://www.aktuonline.com/papers/btech-1-sem-engineering-mathematics-ras-103-2017-18.html

# Expected Questions for University Exam

- If $u = e^{xyz}$ then prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$
- If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that $xu_x + yu_y = \sin 2u$
- If $v = f(2x - 3y, 3y - 4z, 4z - 2x)$ prove that $6v_x + 4v_y + 3v_z = 0$.
- Expand $e^x \log(1 + y)$ in Taylor's series about the origin up to the terms of degree three.
- If u, v and w are the roots of the equation $(\lambda - x)^3$+$(\lambda - y)^3 + (\lambda - z)^3 = 0$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.
- A rectangular box open at the top is to have a volume of 32 cubic feet. Determine the dimensions of the box requiring least material for its construction.
- Find the dimension of the rectangular box of maximum capacity whose surface area is given. When (i) box is open at the top (ii) box is closed.

# Summary

- **Taylor's and Maclaurin's Theorems**

  Expansion of a function of one variables

  Expansion of a function of two variables

- **Jacobian**

  Jacobian and their properties

  Jacobian of implicit function

- **Approximation of errors**

  Errors

  Approximations

- **Maxima and Minima of functions of several variables**

  Stationary points

  Rules to find maxima and minima for function of two variable

  Lagrange's method of multiplier

# References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- D. Poole, Linear Algebra : A Modern Introduction, 2nd Edition, Brooks/Cole, 2005.
- Veerarajan T., Engineering Mathematics for first year, Tata McGraw-Hill, New Delhi, 2008.
- Ray Wylie C and Louis C Barret, Advanced Engineering Mathematics, Tata Mc-Graw-Hill; Sixth Edition.
- P. Sivaramakrishna Das and C. Vijayakumari, Engineering Mathematics, 1st Edition, Pearson India Education Services Pvt. Ltd.

# Thank You