Differential Calculus-II

Unit: 2

Subject Name

Mathematical Foundation-I

Course Details

(B Tech Ist Sem.)

Faculty Name

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Department

Mathematics

Content

- Successive Differentiation(nth order derivatives)
- Leibnitz’s theorem and its application
- Asymptotes.
- Curve tracing: Cartesian and Polar Co-ordinates
- Partial derivatives
- Total derivatives
- Euler’s theorem for homogeneous functions

Successive Differentiation,(CO2..)

The process of finding the differential coefficient of a function again and again is called successive differentiation.

If $y = f(x)$ then, First differential coefficient is $\frac{dy}{dx}$

Second differential coefficient is $\frac{d^2y}{dx^2}$

Third differential coefficient is $\frac{d^3y}{dx^3}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

n^{th}differential coefficient of y $\frac{d^ny}{dx^n}$

Successive Differentiation,(CO2..)

Thus, if $y = f(x)$,the successive differential co-efficients of $f(x)$ are

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots \dots \quad \dots \frac{d^ny}{dx^n}$$

$$\boldsymbol{y_1, y_2, y_3, \dots \dots \dots \dots y_n}$$

$$\boldsymbol{y', y'', y''' \dots \dots \dots \dots y'^n}$$

$$\mathbf{Dy}, \boldsymbol{D^2y, D^3y \dots \dots \dots \dots D^ny}$$

$$\boldsymbol{f'(x), f''(x), f'''(x) \dots \dots \dots \dots f'^n(x)}$$

Successive Differentiation,(CO2..)

➢ **nth derivative of some elementary functions-**

(1) If $y = e^{ax}$ then $y_n = a^n . e^{ax}$

(2) If $y = a^x$ then $y_n = a^x (\log a)^n$

(3) If $y = \frac{1}{ax+b}$ then $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

(4) If $y = \log(ax+b)$ then $y_n = \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$

Successive Differentiation,(CO2..)

(5) If $y = \sin(ax+b)$ then $y_n = a^n \sin(ax+b+n\frac{\pi}{2})$

(6) If $y = \cos(ax+b)$ then $y_n = a^n \cos(ax+b+n\frac{\pi}{2})$

(7) If $y = e^{ax}\sin(\mathrm{b}\,x+c)$ then

$$y_n = (a^2+\mathrm{b}^2)^{n/2}\ \mathrm{e}^{ax}\sin(\mathrm{b}\,x+c+n\tan^{-1}\frac{b}{a})$$

(8) If $y = e^{ax}\cos(\mathrm{b}\,x+c)$ then

$$y_n = (a^2+\mathrm{b}^2)^{n/2}\ \mathrm{e}^{ax}\cos(\mathrm{b}\,x+c+n\tan^{-1}\frac{b}{a})$$

Successive Differentiation,(CO2..)

(9) if $y=(ax+b)^m$

then $y_n = m(m-1)(m-2)(m-3)\ldots\ldots(m-n+1)\,(\mathrm{ax+b})^{m-n}\,a^n$

Case (i) If m is a positive integer

$$y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

Case (ii) If m = n

then $y_n = n!\mathrm{a}^n$ = a constant

Case(iii) From case (ii)

$$y_{n+1} = 0,\, y_{n+2} = 0$$

$i.e\ y_n = 0$ when $n > m$

Successive Differentiation,(CO2..)

(10) If $y = (ax+b)^{-m}$ then

$$y_n = (-1)^n \frac{(m+n-1)!}{(m-1)!}(ax+b)^{-m-n}a^n$$

Successive Differentiation,(CO2..)

$$1.\ \text{If } y=\frac{1}{1-5x+6x^2}, \text{find } y_n \qquad (UPTU\ 2005)$$

$$Ans: (-1)^n n!\left[\frac{2^{n+1}}{(2x-1)^{n+1}}-\frac{3^{n+1}}{(3x-1)^{n+1}}\right]$$

$$2.\ \text{If } y=\frac{2x+1}{(2x-1)(2x+3)}, \text{find } y_n \qquad (UPTU\ 2015)$$

$$Ans: (-1)^n n!2^{n-1}\left[\frac{1}{(2x-1)^{n+1}}+\frac{1}{(2x+3)^{n+1}}\right]$$

$$3.\ \text{If } y=\frac{ax+b}{cx+d}, \text{find } y_n \qquad (MTU\ 2013)$$

$$Ans: \frac{bc-ad}{c}\left[\frac{(-1)^n n!c^n}{(cx+d)^{n+1}}\right]$$

Successive Differentiation,(CO2..)

$$4.\ If\ \ y = x\log(1+x),\ \text{Prove that } y_n = (-1)^{n-2}(n-2)!\left[\frac{(x+n)}{(x+1)^n}\right]$$

$$5.\ If\ \ y = \sin px + \cos px, \text{ Prove that } y_n = p^n[1+(-1)^n \sin 2px]^{1/2}$$

Hence show that $y_8(\pi) = (1/2)^{31/2}$

where $p = 1/4.$

$$\text{Hint: } y_n = p^n[\sin(px + \frac{n\pi}{2}) + \cos(px + \frac{n\pi}{2})]$$

$$= p^n[\{\sin(px + \frac{n\pi}{2}) + \cos(px + \frac{n\pi}{2})\}^2]^{1/2}$$

$$= p^n[1 + 2\sin(px + \frac{n\pi}{2})\cos(px + \frac{n\pi}{2})]^{1/2}$$

Successive Differentiation,(CO2..)

- 6.Find y_n if $y = \frac{x^n - 1}{x - 1}$. (MTU2012)
- 7.The (n-1)th derivative of x^n is…….. (MTU2012)
- 8.$y_n = \frac{d^n}{dx^n}(x^n logx)$ then prove that $y_n = ny_{n-1} + (n-1)$
  And Hence show that $y_n = n!\,(logx + 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n})$
- 9.$y = e^x sin^2 x$ then find y_n.
- 10.If $y = xlog\frac{x-1}{x+1}$,
  show that $y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$
  (UPTU2003)

Leibnitz theorem,(CO2..)

This helps us to find the n^{th} differential co-efficient of the product of two functions in terms of the successive derivatives of the functions.

Statement:

If u and v are two functions of x, having derivatives of the n^{th} order, then

$$\frac{d^n}{dx^n}(u\ v) = u_n\ v + {}^nc_1 u_{n-1}\ v_1 + {}^nc_2 u_{n-2}\ v_2 + {}^nc_3 u_{n-3}\ v_3 +$$

$$\ldots\ldots + {}^nc_r u_{n-r}\ v_r + \ldots\ldots + {}^nc_n u\ v_n\ ,$$

where suffixes of u and v denote the differentiation w.r.t. x.

Leibnitz theorem,(CO2..)

1. Find the n[th] derivative of $x^3 \cos x$.

$$y = x^3 \cos x$$

Differentiating n times by Leibnitz theorem

$$\mathrm{D}^n y = (D^n \cos x).x^3 + {}^nC_1(D^{n-1}\cos x).Dx^3 + {}^nC_2(D^{n-2}\cos x).D^2x^3$$

$$+ {}^nC_3(D^{n-3}\cos x).D^3x^3$$

$$= x^3\cos(x + n\pi/2) + 3nx^2\cos(x + (n-1)\pi/2) +$$

$$6x\frac{n(n-1)}{2!}\cos(x + (n-2)\pi/2) + 6\frac{n(n-1)(n-2)}{3!}\cos(x + (n-3)\pi/2)$$

$$= x^3\cos(x + n\pi/2) + 3nx^2\cos(x + (n-1)\pi/2) + 3xn(n-1)\cos(x + (n-2)\pi/2)$$

$$+ n(n-1)(n-2)\cos(x + (n-3)\pi/2).$$

Leibnitz theorem,(CO2..)

2. If $y = \sin(m\sin^{-1} x)$, *prove that*

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$

$$y_1 = \cos(m\sin^{-1} x)\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = m\cos(m\sin^{-1} x)$$

squaring both sides

$$(1-x^2)y_1^{\,2} = m^2\cos^2(m\sin^{-1} x) = m^2(1-\sin^2(m\sin^{-1} x))$$

$$(1-x^2)y_1^{\,2} = m^2(1-y^2)$$

again differentiating w.r.t. x

Leibnitz theorem,(CO2..)

Differentiating n times by Leibnitz theorem

$$(1-x^2)y_{n+2}+{}^nC_1(-2x)y_{n+1}+{}^nC_2(-2)y_n-(xy_{n+1}+{}^nC_1y_n)+m^2y_n=0.$$

$$(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2!}(-2)y_n-(xy_{n+1}+ny_n)+m^2y_n=0.$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0.$$

Leibnitz theorem,(CO2..)

$$\text{Determine } y_n(0) \text{ where } y = e^{m\cos^{-1}x}. \qquad (1)$$

$$y_1 = \frac{-my}{\sqrt{1-x^2}} \qquad (2)$$

$$(1-x^2)y_2 - xy_1 = m^2 y. \qquad (3)$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0. \qquad (4)$$

On putting $x = 0$ in above equations

$$y(0) = e^{m\pi/2}; \qquad y_1(0) = -me^{m\pi/2}$$

$$y_2(0) = m^2 e^{m\pi/2};$$

$$y_{n+2}(0) = (n^2+m^2)y_n(0) \qquad (5)$$

Leibnitz theorem,(CO2..)

From equation (5), $for\ n = 1, 2, 3, 4$............

$$y_3(0) = (1^2 + m^2)y_1(0) = -m(1^2 + m^2)e^{\frac{m\pi}{2}}$$

$$y_4(0) = (2^2 + m^2)y_2(0) = m^2(2^2 + m^2)e^{\frac{m\pi}{2}}$$

$$y_5(0) = (1^2 + m^2)y_3(0) = -m(1^2 + m^2)(3^2 + m^2)e^{\frac{m\pi}{2}}$$

$$y_6(0) = (2^2 + m^2)y_2(0) = m^2(2^2 + m^2)(4^2 + m^2)e^{\frac{m\pi}{2}}$$

when n is even

$$y_n(0) = m^2(2^2 + m^2)(4^2 + m^2)\ldots\ldots\ldots\ldots\ldots\left((\mathrm{n}-2)^2 + m^2\right)e^{\frac{m\pi}{2}}$$

Leibnitz theorem,(CO2..)

when n is odd

$$y_n(0) = -m(1^2+m^2)(3^2+m^2)\dots\dots\dots\dots\dots\left((n-2)^2+m^2\right)e^{\frac{m\pi}{2}}$$

1. If $y = (\sin^{-1}x)^2$, find $y_n(0)$.

   (UPTU-2006, 09,GBTU-2011)

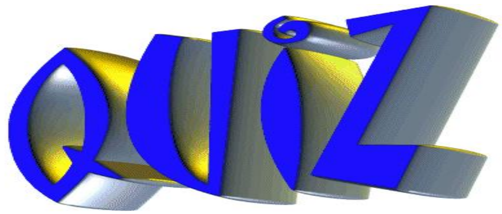
2. If $y = (x+\sqrt{1+x^2})^m$, find $y_n(0)$.

   $(GBTU-2013)$

3. If $y = \sin(a\sin^{-1}x)$, find $y_n(0)$.

   (UPTU-2007,09 GBTU-2011)

Leibnitz theorem,(CO2..)



Q.1. nthderivative of $y = \frac{1}{ax+b}$.

Q.2 nth derivative of $y = \log(ax + b)$

Q.3.4thderivative of $y = x^3$

Q.4.write nthterm of Leibnitz's theorem

Leibnitz theorem,(CO2..)

FAQ

Q.1.If $y = a.\cos(logx) + b.\sin(logx)$ then prove that

$x^2 y_{n+2} + (2n+1)xy_{n+1} + \left(n^2+1\right)y_n = 0.$

Q.2.If $(y)^{\frac{1}{m}} + (y)^{\frac{-1}{m}} = 2x$, prove that

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + \left(n^2-m^2\right)y_n = 0.$

Q.3.If $y = x^n logx$ prove that

(i) $y_{n+1} = \frac{n!}{x}$

(ii) $x^2 y_{p+2} + (2p-2n+1)xy_{p+1} + (p-n)^2 y_p = 0.$

Recap

In this chapter, we discussed the following points

1. **Successive Differentiation** i.e.

The process of finding the differential coefficient of a function again and again is called successive differentiation.

2. **Leibnitz theorem i.e.**

$$\frac{d^n}{dx^n}(u\ v) = u_n\ v + {}^n c_1 u_{n-1}\ v_1 + {}^n c_2 u_{n-2}\ v_2 + {}^n c_3 u_{n-3}\ v_3 + \ldots\ldots + {}^n c_r u_{n-r}\ v_r + \ldots\ldots + {}^n c_n u\ v_n\ ,$$

Objective of the topic,(CO2..)

Asymptotes and Curve Tracing

An **asymptote** of a **curve** is a line such that the distance between the **curve** ... information about the behavior of curves in the large, and determining the **asymptotes** of a function is an important step in **sketching** its graph.

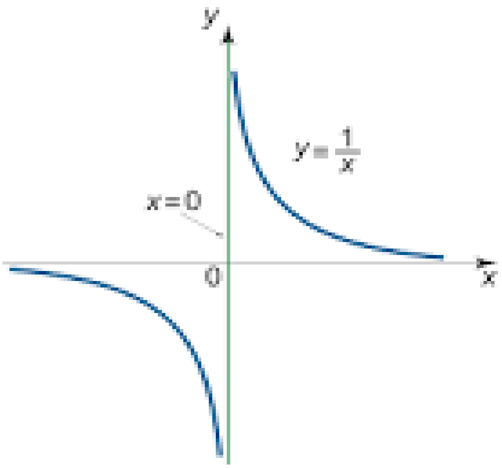
Asymptotes,(CO2..)

➢ Asymptotes-

An asymptotes of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the X or Y coordinates tends to infinity.

OR

Asymptotes is a straight line which touches the curve $y = f(x)$ at infinity.



Asymptotes,(CO2..)

Working Rule-

1. Let $f(x, y) = 0$, be the given curve of nth degree.

2. Find the polynomial $\emptyset_n(m), which\ can\ be\ obtained\ by\ putting\ x = 1, y = m$ in the highest degree terms of the given equation of the curve.

3. Put $\emptyset_n(m) = 0$ and solve for m,say $m = m_1$,m_2,…$m_n$are its roots.
4. Find the polynomial $\emptyset_{n-1}(m), which\ can\ be\ obtained\ by\ putting\ x = 1, y = m$ in the next highest degree terms of the given equation of the curve.

Asymptotes,(CO2..)

5.Find c by using the formula$c = -\frac{\emptyset_{n-1}(m)}{\emptyset'_n(m)}$, (provided $\emptyset'_n(m) \neq 0$).

6.Using the formula, we find c_1,c_2.....c_n.

7.Put the value of m_1, m_2,...m_n and c_1, c_2..... $c_n$in$y = mx + c$ then we get all asymptotes of the given curve$y = f(x)$.

Hence the asymptotes are

$$y = m_1x + c_1, y = m_2x + c_2, \ldots\ldots\ldots, y = m_nx + c_n$$

Asymptotes,(CO2..)

Q.1.Find the asymptotes of the curve

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$$

Solution- we get

$$\emptyset_3(m) = 1 + 2m - m^2 - 2m^3 = 0$$

$$\Rightarrow m = -\frac{1}{2}, 1, -1$$

And $$\emptyset_3'(m) = 2 - 2m - 6m^2$$

And $$\emptyset_2(m) = m - m^2$$

Therefore $$c = -\frac{\emptyset_{n-1}(m)}{\emptyset_n'(m)}$$

$$c = -\frac{m - m^2}{2 - 2m - 6m^2}$$

$c = \frac{m^2 - m}{2 - 2m - 6m^2}$ then

If $m_1 = -\frac{1}{2}$ and $c_1 = \frac{1}{2}$

If $m_2 = 1$ and $c_2 = 0$

If $m_3 = -1$ and $c_3 = -1$

Therefore, the asymptotes are

$$y = m_1 x + c_1 \Rightarrow y = -\frac{1}{2}x + \frac{1}{2}$$

$$y = m_2 x + c_2 \Rightarrow y = x$$

$$y = m_3 x + c_3 \Rightarrow y = -x - 1$$

Procedure for Tracing Curves having Cartesian Equation-

- **I. Symmetry-**

1.If the equation of the curve contains only even powers of y then curve is symmetrical about the X axis.

Ex. $y^2 = 4ax$

2.If the equation of the curve contains only even powers of x then curve is symmetrical about the Y axis.

Ex. $x^2 = 4ay$

3. If the equation of the curve contains even powers of both x and y then curve is symmetrical about the both axis.

Ex. $x^2 + y^2 = a^2$

Curve Tracing,(CO2..)

4. If the equation of the curve remains unchanged when x is changed to y and y is changed to x then curve is symmetrical about the line $y = x$

Ex. $x^3 + y^3 = 3axy$

5. If the equation of the curve remains unchanged when x is changed to -y and y is changed to -x then curve is symmetrical about the line $y = -x$

Ex. $x^4 + y^4 = 3a^2xy$

6. If the equation of the curve remains unchanged when x is changed to -x and y is changed to -y then curve is symmetrical about in opposite quadrants.

Ex. $x^5 + y^5 = 5ax^2y$

Curve Tracing,(CO2..)

- **II.Origin-**

If the constant term is missing from the equation of the curve then it passes through the origin.

- **IF CURVE PASSES THROUGH THE ORIGIN THEN**

Find the tangents at origin for this we equate to zero the lowest degree terms appear in equation.

1. If the two tangents are real and distinct then origin is NODE.
2. If the two tangents are real and coincide then origin is CUSP.
3. If the two tangents are conjugate (or isolated) then origin are IMAGINARY.

Curve Tracing,(CO2..)

NOTE-

1. A cusp is called a single cusp or a double cusp according as the two branches of the curve lie entirely on one side or both sides of the common normal.

2. A cusp single or double is said to be of first kind or second kind according as the two branches of the curve, lie on opposite or same side of the common tangent.

- **III. Asymptotes-**

Find all the asymptotes of the curve. The curve will not go beyond its asymptotes.

Curve Tracing,(CO2..)

1.Asymptotes parallel to x-axis-

We equate to zero the coefficients of the highest power of x . If the coefficients of the highest power of x is constant then there is no asymptotes parallel to x-axis.

2.Asymptotes parallel to y-axis-

We equate to zero the coefficients of the highest power of y. If the coefficients of the highest power of y is constant then there is no asymptotes parallel to y-axis.

3.Oblique asymptotes- write the equation of the form

$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \cdots$, then $y = mx + c$ is the asymptote to the curve.

- **IV. Points of intersections-** Find intersections of the curve

(i)With the x-axis (put y = 0)

(ii)With the y-axis (put x = 0)

(iii)With the line y = x particularly if the curve is symmetrical aboutit.

(iv) With the line y = - x particularly if the curve is symmetrical about it.

(v) With the asymptotes (if necessary)

Curve Tracing,(CO2..)

- Note-

If curve cuts the axis at the point (a,0) then find the equation of the tangent at the point (a,0).

For this we put x = X + a and y = Y + 0 in given equation of curve and equate to zero the lowest degree terms.

- V. Region-Find regions in the four quadrants to which the curve is limited.
- VI. **Result-** Considering all these points ,found the rough shape of curve .

Curve Tracing,(CO2..)

Q.1.Trace the curve $y^2(2a-x)=x^3$.

Solution- Given equation of curve$y^2 2a - y^2 x = x^3$

(i)Symmetry-Clearly the given curve is only even power of y. Therefore curve is symmetrical about x-axis.

(ii)Position of origin-Clearly the given curve passes through origin (0,0) then equation of tangent at origin (0,0),for this we equate to zero the lowest degree term i.e.

$$2ay^2 = 0 \text{ i.e.} y = 0,0$$

$\Longrightarrow$ there are two real tangent but identical so origin is a cusp.

(iii) Asymptotes- Asymptotes parallel to y-axis i.e.

$2a - x = 0 \implies x = 2a$,thus$x = 2a$ is only real Asymptotes of the curve.

(iv) Intersection with the axes- Clearly curve does not intersect axes any where except at origin.

(v)Region- We have $y = \sqrt{\frac{x^3}{(2a-x)}}$

(a) When $x < 0, y$ is imaginary therefore no portion of the curve lies to the left of the line $x = 0$ i.e. y axis.

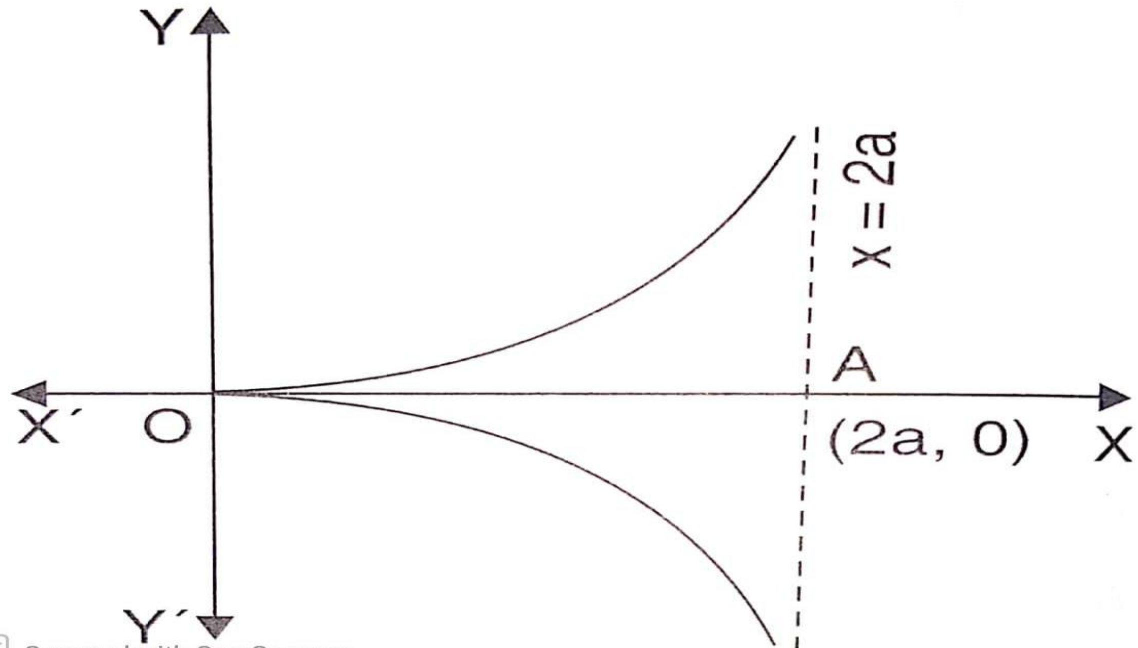
Curve Tracing,(CO2..)

(b) When $0 < x < 2a$,y is real.

(c) When $x > 2a$,y is imaginary therefore no portion of the curve lies to the right of the line $x = 2a$.

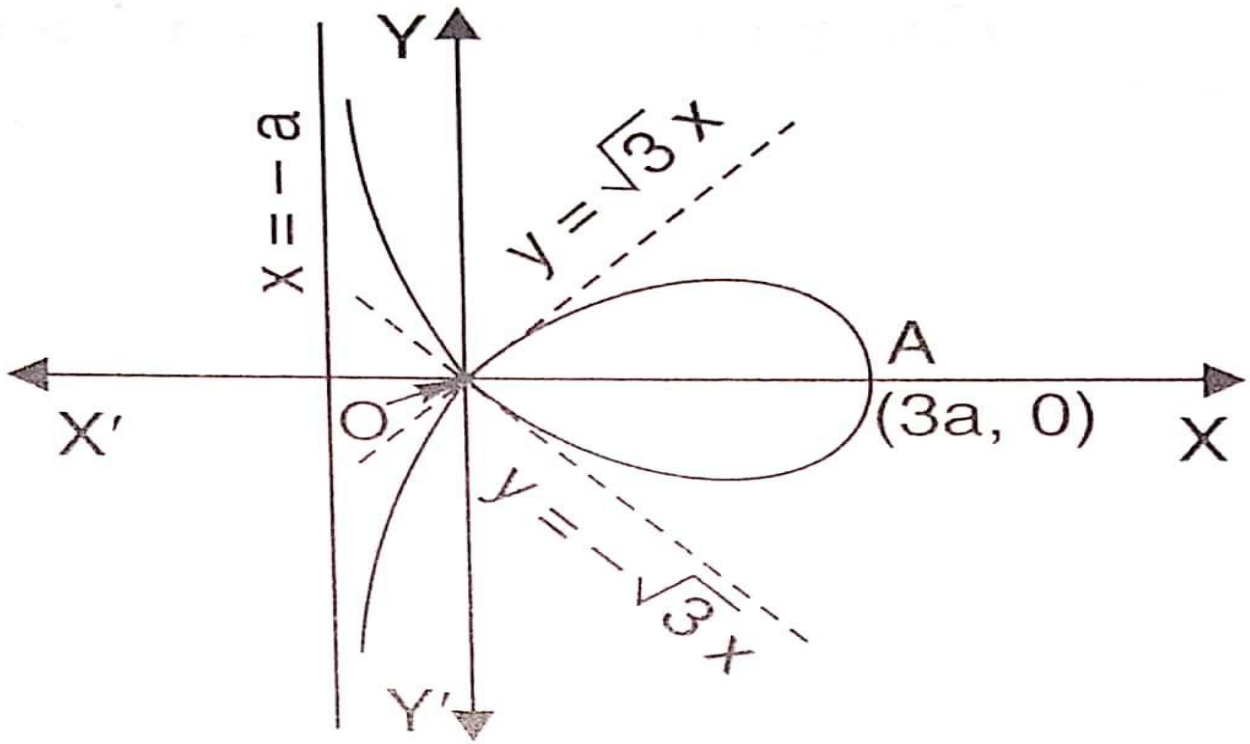
(vi) Result- Considering all these points ,found the rough shape of curve is as shown in figure.

Curve Tracing,(CO2..)



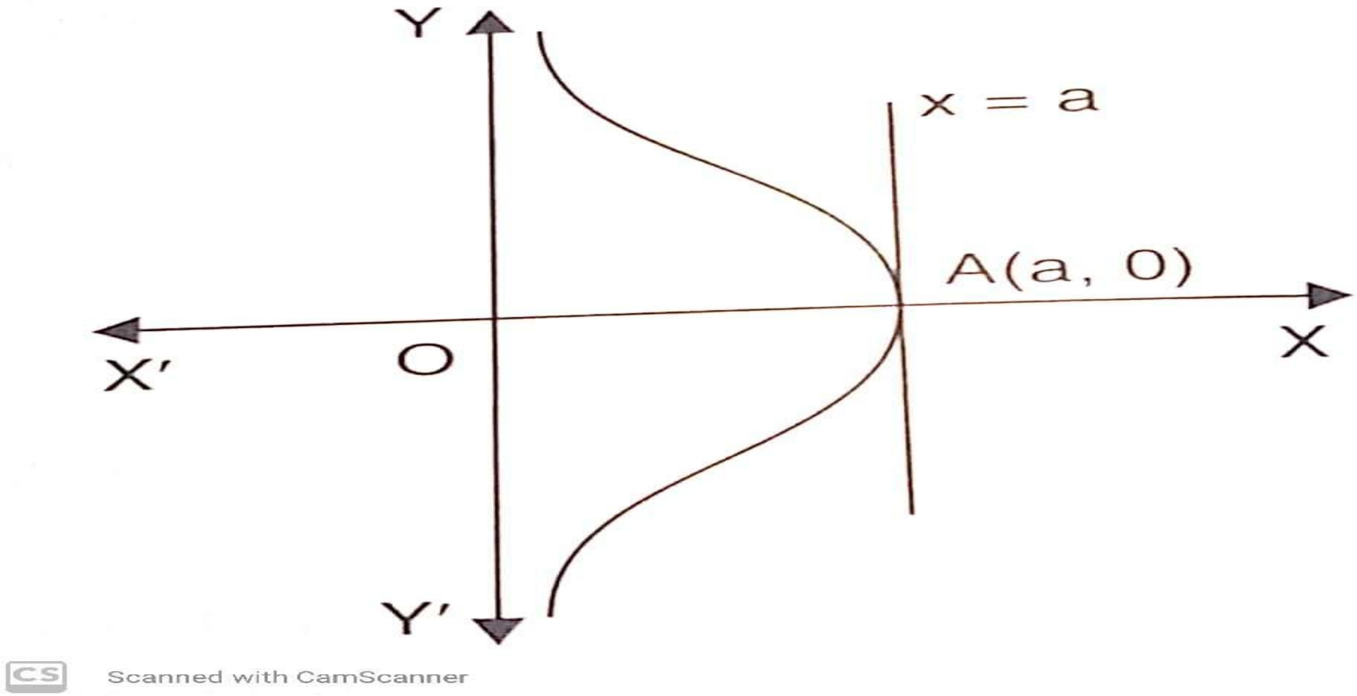
Curve Tracing,(CO2..)

Q.2. Trace the curve $y^2(a+x) = x^2(3a-x)$.



Curve Tracing,(CO2..)

Q.3. Trace the curve $xy^2 = a^2(a - x)$.



- **Procedure for Tracing Curves having Cartesian Equation-**

1.Symmetry-

(i) The equation of the curve does not change by changing the sign of θ,curve is symmetrical about the initial line i.e. x-axis.

(ii) The equation of the curve does not change by putting$r = -r$, curve is symmetrical about the pole.

(iii) The equation of the curve does not change by putting

$\theta = \pi - \theta$ curve is symmetrical about the line$\theta = \frac{\pi}{2}$ i.e. y-axis.

(iv) The equation of the curve does not change by putting

$\theta = \frac{\pi}{2} - \theta$ curve is symmetrical about the line$\theta = \frac{\pi}{4}$ i.e. y = x line.

Curve Tracing(CO2..)

2.Pole or Origin- Find whether the curve passes through the pole or not.

For this we put r = 0 then we get some real value of θ,then curve passes through the pole.

- (i) Find the tangents at pole-

For this we put r = 0, the real value of θ gives the tangents at the pole.

- (ii)Find the points ,where the curve cuts the initial line and the line$\theta = \frac{\pi}{2}$.

3.Asymptotes-If r$\rightarrow \infty$ as$\theta \rightarrow \theta$, then there is an asymptote . We find all the asymptote of the curve.

Curve Tracing(CO2..)

4.Points of intersection-

Find some points on the curve for convenient values ofθ.

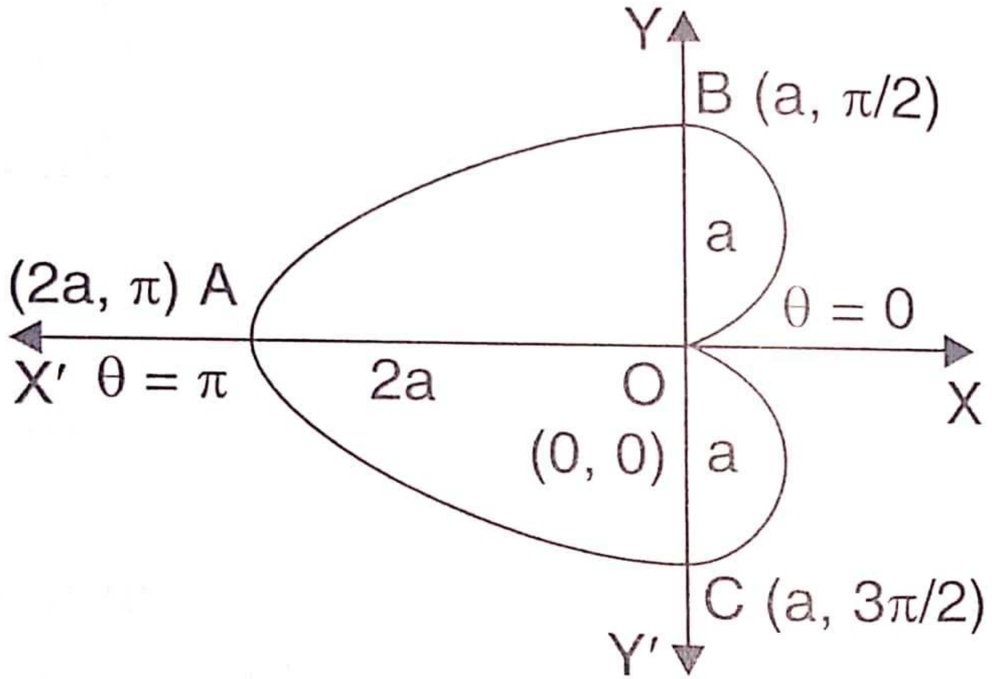
5.Region-

Solve the given equation for r and θ(if possible).Find the region in which the curve does not lie.

6.**Result-**

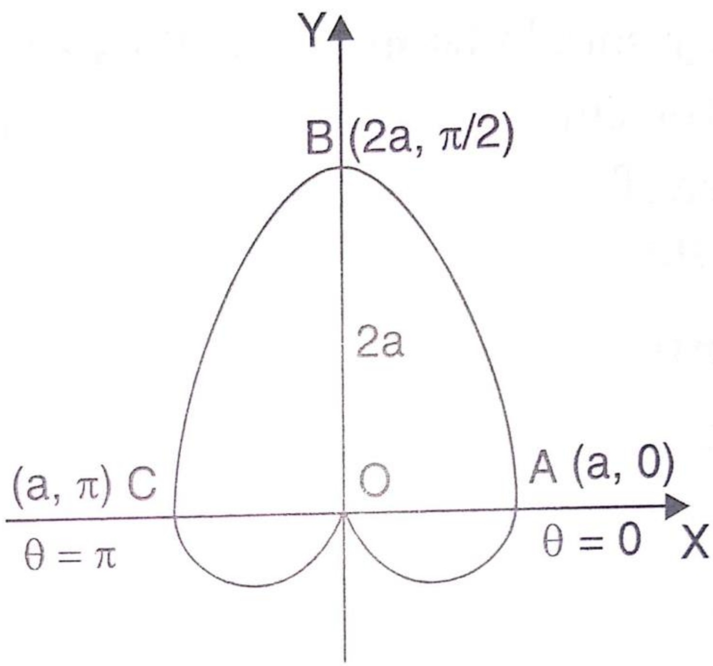
Considering all these points ,found the rough shape of curve .

Q.1. Trace the curve $r = a(1 - cos\theta)$.(Cardioid)

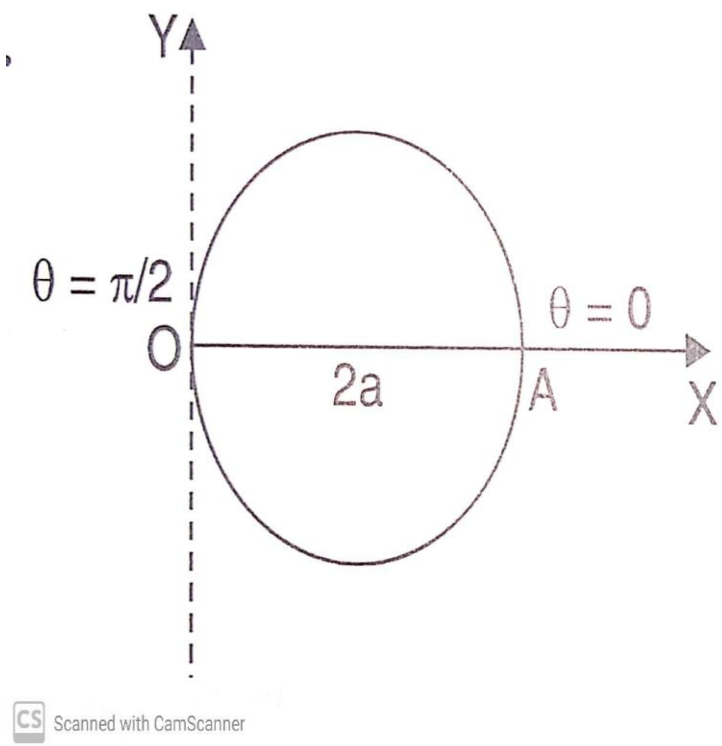


Curve Tracing(CO2..)

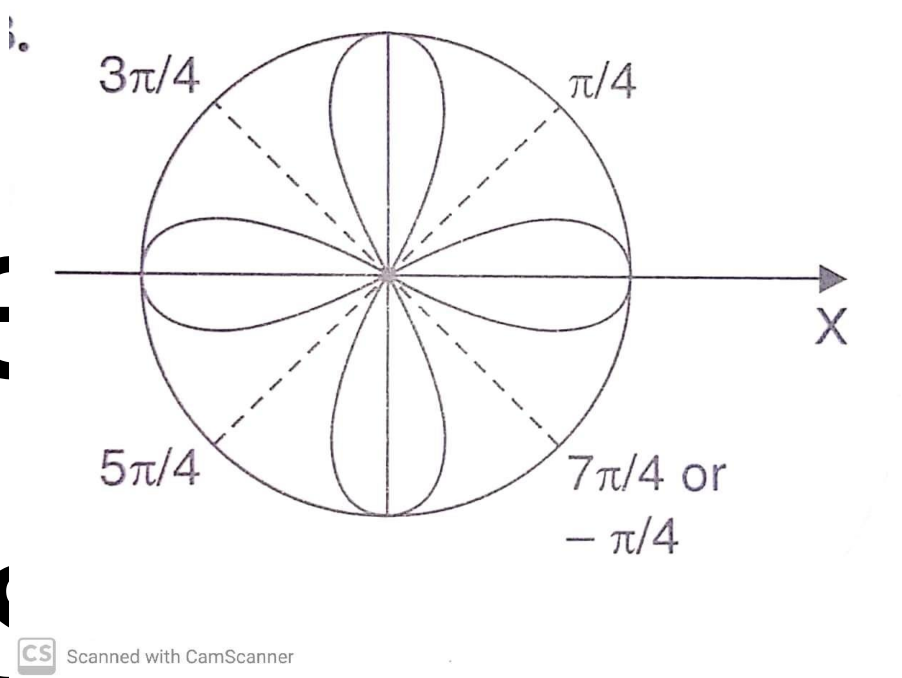
Q.2. Trace the curve $r = a(1 + cos\theta)$.(Cardioid)



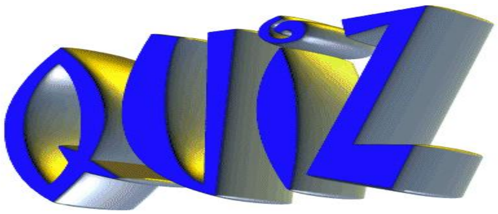
Q.3. Trace the curve $r = 2a\ cos\theta$.(Circle)



Q.4. Trace the curve $a \cos$



Curve Tracing(CO2..)



Q.1. If even power of y then curve is symmetrical…..
Q.2. If even power of x then curve is symmetrical…..
Q.3. If highest power of x are constant then the Asymptote….

FAQ

Q.1. Find the asymptotes of the curve
$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0.$
Ans- $2xy + y = 0, x - y + 1 = 0$ and$x + y + 1 = 0$

Q.2. Trace the curve $y^2(a - x) = x^2(a + x)$.

Recap

In this chapter, we discussed the following points

1. Asymptotes i.e. Asymptotes is a straight line which touches the curve $y = f(x)$ at infinity.

2. Curve Tracing **i.e.**

(i) Tracing Curves in Cartesian co-ordinates

(ii) Tracing Curves in Polar co-ordinates

Topic Objective

Partial derivatives (CO 2)

- Student will be able to understand partial derivatives and solve problems related to partial derivatives.
- Partial derivatives are used in solving the problem related to vector calculus and differential geometry.

Partial derivatives (CO2)

Function of two variables:

- If three variables *x, y, z* are so related that value of *z* depends upon the values of *x* and *y,* then z is called function of *x* and *y* and is denoted as

$$z = f(x, y)$$

- The set of points (*x, y*) in the *x*-*y* plane for which *f(x, y)* is defined is called the domain of *f(x, y)* and denoted as D.

- The domain may be entire *x*-*y* plane or a part of the *x*-*y* plane

- The collection of corresponding values of z is called the range of *f(x, y)*

First Order Partial Derivatives (CO2)

- Let z = f(x, y) be a function of two independent variables x and y.

- The partial derivative of z with respect to x treating y as constant is denoted as

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } z_x \text{ or } f_x$$

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

First Order Partial Derivatives (CO2)

- Let z = f(x, y) be a function of two independent variables x and y.

- The partial derivative of z with respect to y treating x as constant is denoted as

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } z_y \text{ or } f_y$$

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Higher Derivatives (CO2)

- If $z = f(x, y)$, then for second order partial derivatives we use the following notation:

$$f_{xx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Mixed Partial Derivatives (CO2)

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

- Partial derivatives of order 3 or higher can also be defined.
- For instance,and using Clairaut's Theorem we can show that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$

Example 1

- $z=\sqrt{1-x^2-y^2}$; z is real.
- The domain is $(1-x^2-y^2)\geq 0 \Rightarrow x^2+y^2\leq 1$

  $i.e.$ all x, y such that $x^2+y^2\leq 1$ is domain.

- The range is set of all real positive numbers.

Example 2

Question: If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution: Holding *y* constant and differentiating with respect to *x*, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

$$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding *x* constant and differentiating with respect to *y*, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example 3

Question: If $f(x,y)=\sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Solution: Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x}=\cos\left(\frac{x}{1+y}\right).\frac{1}{1+y}$$

$$\frac{\partial f}{\partial y}=\cos\left(\frac{x}{1+y}\right).\left[-\frac{x}{(1+y)^2}\right]$$

Example 4

Question: Calculate f_{xxyz} $\quad if \quad f(x, y, z) = sin(3x + yz).$

Solution:

$$f_x = 3\cos(3x + yz)$$

$$f_{xx} = -9\sin(3x + yz)$$

$$f_{xxy} = -9z\cos(3x + yz)$$

$$\boxed{f_{xxyz} = -9\left[\cos(3x + yz) - yz\sin(3x + yz)\right]}$$

Example 5

Question: Find the second partial derivatives of $f(x, y) = x^3 + x^2 y^2 - 2y^2$

Solution:

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2 y^2 - 4y$$

Therefore

$$f_{xx}(x, y) = 6x + 2y^3$$

$$f_{yy}(x, y) = 6x^2 y - 4$$

$$f_{xy}(x, y) = 6xy^2$$

$$f_{yx}(x, y) = 6xy^2$$

Example 6

Question: if $u=\left(x^2+y^2+z^2\right)^{-1/2}$, Then evaluate $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$

Solution: $\dfrac{\partial u}{\partial x}=-\dfrac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}.2x$

$$\Rightarrow \frac{\partial u}{\partial x}=-x\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2}=(-1)\left(x^2+y^2+z^2\right)^{-3/2}+(-x)\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2}.2x$$

$$=-\left(x^2+y^2+z^2\right)^{-3/2}+3x^2\left(x^2+y^2+z^2\right)^{-5/2}$$

Example 6 Contd...

$$\frac{\partial^2 u}{\partial y^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$\frac{\partial^2 u}{\partial z^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3z^2\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -3\left(x^2 + y^2 + z^2\right)^{-3/2} + 3(x^2 + y^2 + z^2)\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -3\left(x^2 + y^2 + z^2\right)^{-3/2} + 3\left(x^2 + y^2 + z^2\right)^{-3/2} \qquad = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

FAQ

1. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right)^2 u = \frac{-9}{(x+y+z)^2}$

2. If $v = \left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}$, show that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

3. If $z(x+y) = \left(x^2 + y^2\right)$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

4. If $x^x y^y z^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

5. Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, where $u = \log\left(\frac{x^2 + y^2}{xy}\right)$

Quiz

- If $u = e^{x^2+y^2+z^2}$, then $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$.........
- $\dfrac{\partial^2 u}{\partial x \partial y} = \ldots\ldots\ldots,$ if $u = x^2 + y^2.$
- If $w = \dfrac{y}{z} + \dfrac{z}{x}$, then $xw_x + yw_y + zw_z = \ldots\ldots\ldots\ldots$

Composite Function (CO2)

If $u = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then,

- u is said to be a composite function of t.

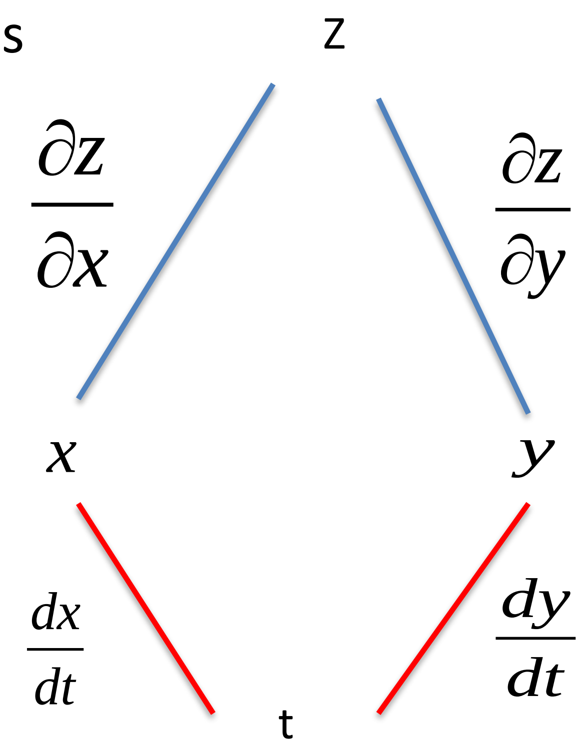
If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(u,v)$ and $y = h(u,v)$ are both differentiable functions of u,v . Then,

- z is said to be a composite function of u,v.

Differentiation of Composite Function (CO2)

Chain Rule - Case 1

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then, z is a differentiable function of t and



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- $\frac{dz}{dt}$ is called the total differential coefficient of z with regard to t.

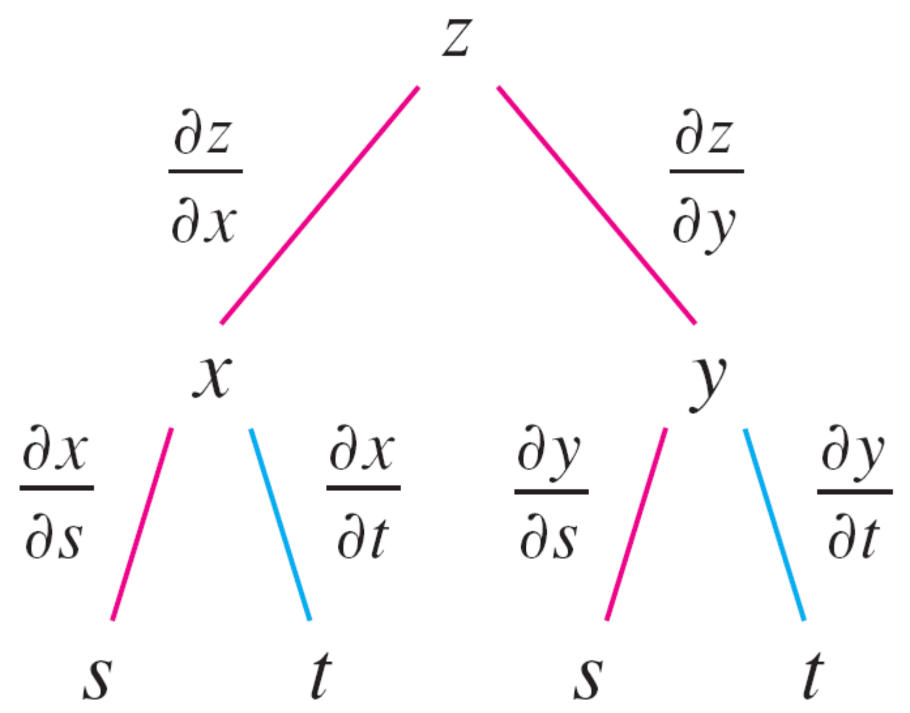
Differentiation of Composite Function (CO2)

Chain Rule - Case 2

- Suppose $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Deduction (CO2)

- If both *x* and *y* are functions of *x then*,

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

- We obtain:

$$\frac{dy}{dx}=-\left(\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y}\right)=-\frac{F_x}{F_y}$$

Example 1

Question: If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt .

Solution: The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

Example 2

Question: If $u = x^4 y + y^2 z^3$, where $x = rse^t, y = rs^2 e^{-t}, z = r^2 s \sin t$ find the value of $\partial u / \partial s$ when $r = 2, s = 1, t = 0$

Solution:

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3 y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t)$$

When r = 2, s = 1, and t = 0, we have: x = 2, y = 2, z = 0

Thus,

$$\frac{\partial u}{\partial s} = (64)(2) + (16)(4) + (0)(0) = 192$$

Example 3

Question: Find dy/dx, if $x^3 + y^3 = 6xy$.

Solution: Let $F(x, y) = x^3 + y^3 - 6xy = 0$

By deduction, $$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial x} / \frac{\partial F}{\partial y}\right) = -\frac{F_x}{F_y}$$

So, $$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Example 4

Question: If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Solution: Let $e^{y-z} = r, \quad e^{z-x} = s, \quad e^{x-y} = t$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} \qquad = \frac{\partial u}{\partial r}\cdot 0 + \frac{\partial u}{\partial s} e^{z-x}(-1) + \frac{\partial u}{\partial t} e^{x-y}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} e^{z-x} + \frac{\partial u}{\partial t} e^{x-y} \text{..........}(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} \qquad = \frac{\partial u}{\partial r}\cdot e^{y-z} + \frac{\partial u}{\partial s}\cdot 0 + \frac{\partial u}{\partial t} e^{x-y}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot e^{y-z} - \frac{\partial u}{\partial t} e^{x-y} \text{..........}(2)$$

Example 4 Contd...

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z} \quad =\frac{\partial u}{\partial r}\cdot e^{y-z}(-1)+\frac{\partial u}{\partial s}\cdot e^{z-x}+\frac{\partial u}{\partial t}\cdot 0$$

$$\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial r}\cdot e^{y-z}+\frac{\partial u}{\partial s}\cdot e^{z-x}\ldots\ldots\ldots..(3)$$

On adding eq. (1) , (2) and (3)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

FAQ

1. If $u = f(r)$, where $r^2 = x^2 + y^2$,

prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

(UPTU-2006, 2007, GBTU-2010)

2. If $z = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$, show that

$$e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$ (GBTU-2013)

3. If $u = f(y-z, z-x, x-y)$ prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

(GBTU-2010)

Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details

- **Total derivatives**
- Lecture 09 – Chain rule 1

https://www.youtube.com/watch?v=McT-UsFx1Es

- Lecture 10 – Chain rule 2

https://www.youtube.com/watch?v=_1TNtFqiFQo

- **Change of variables**

- Lecture 10 – Chain rule 2

https://www.youtube.com/watch?v=X6kp2o3mGtA

Recap

- ✓ Composite function
- ✓ Total Derivatives
- ✓ Chain rule case 1
- ✓ Chain rule case 2

Topic Objective

Euler's Theorem for Homogeneous Fn. (CO 2)

- Student will be able to understand the concepts homogeneous function and apply the Euler's theorem for homogeneous function.
- It is useful to study the mathematical theorem that establishes a relationship between a homogeneous function and its partial derivatives.

Homogeneous Function (CO2)

- Consider the function

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots\dots\dots + a_n y^n$$

The degree of each term in x and y is n.

- A function $f(x, y)$ of two independent variables x and y is said to be homogenious of degree n if $f(x, y)$ can be written in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$

- Examples

(1) $F(x, y) = x^n \sin(\frac{y}{x})$ (2) $F(x, y) = x^3 - 3xy^2 + y^3$

(3) $F(x, y) = \frac{\left(\sqrt{y} - \sqrt{x}\right)}{y - x}$

Euler's Theorem on Homogeneous Function (CO2)

- If $z = F(x, y)$ be a homogeneous function of x, y

of degree n in x and y then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$.

- Corollary

If z = $f(x, y)$ is a homogeneous function of x and y of degree n,

then $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Deductions From Euler's Theorem (CO2)

If $f(u) = V(x, y)$ be a homogeneous function of x, y of degree n then

- $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}.$$

- $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \varphi(u)[\varphi'(u) - 1]$$

  where $\varphi(u) = n\dfrac{f(u)}{f'(u)}$

Example 1

Question: If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{x}{y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Solution: Let $v = \sin^{-1}\dfrac{x}{y}$ and $w = \tan^{-1}\dfrac{x}{y}$

$$\therefore \qquad u = v + w$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(x\frac{\partial v}{\partial x} + x\frac{\partial w}{\partial x}\right) + \left(y\frac{\partial v}{\partial y} + y\frac{\partial w}{\partial y}\right)$$

$$= \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + \left(x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right)$$

= 0 + 0 (since are homogenous functions of degree zero)

Example 2

Question: If $u=\frac{x^2y^2}{x+y}$, Then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

Solution:

$$u(tx,ty)=\frac{(tx)^2(ty)^2}{tx+ty}=t^3\frac{x^2y^2}{x+y}$$

Here u is a homogenous functions of degree 3

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u=3\frac{x^2y^2}{x+y}$$

Example 3

Question:

If u= $\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, $x \neq y$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\left(1-4\sin^2 u\right)\sin 2u$$

Solution: We have $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$

$$Let\ z = \frac{x^3+y^3}{x-y} \quad \text{then} \quad \tan u = z$$

$$\text{where z} = \frac{x^3+y^3}{x-y} = x^2\,\frac{1+\dfrac{y^3}{x^3}}{1-\dfrac{y}{x}}$$

is a homogeneous function of degree two.

Example 3 Contd...

By Eular's theorem ,we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \text{n z} = 2\text{ z}$

but $\frac{\partial z}{\partial x}=\sec^2 u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$

Also $\frac{\partial^2 z}{\partial x^2}=\sec^2 u\frac{\partial^2 u}{\partial x^2}+2\ \sec^2 u\ \tan \text{u}\left(\frac{\partial u}{\partial x}\right)^2$

$$\frac{\partial^2 z}{\partial y^2}=\sec^2 u\frac{\partial^2 u}{\partial y^2}+2\ \sec^2 u\ \tan \text{u}\left(\frac{\partial u}{\partial y}\right)^2$$

$$\frac{\partial^2 z}{\partial x\partial y}=\sec^2 u\frac{\partial^2 u}{\partial x\partial y}+2\ \sec^2 u\ \tan \text{u}\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

Also by corollary of Eular's theorem,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=2(2-1)z$$

Example 3 Contd…

$$\Rightarrow \sec^2 u\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)$$

$$+2\sec^2 u\ \tan u\left(x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2\right)=2\tan u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2\ \tan u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)^2=2\sin u\ \cos u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u-2\ \tan u\sin^2 2u$$

$$=\sin 2u\left(1-2\ \tan u\sin 2u\right)$$

$$=\sin 2u\left(1-4\sin^2 u\right)$$

FAQ

1. Verify Euler's theorem if $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$.

(MTU -2011)

2. If u= $\sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

(UPTU2007)

3. If $z = \log\dfrac{x^2 + y^2}{x + y}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 1$

(UPTU-2009)

Quiz

- *If* $u(x,y)=\left(\sqrt{x}+\sqrt{y}\right)^5$, then value of $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$ is ....

- If $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=$ ....

- If $u=\log\left(\dfrac{x^2}{y}\right)$, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=$ ....

Faculty Video Links, Youtube & NPTEL Video Links and Online Courses Details

- **Euler's theorem**

- Lecture 12: Euler's theorem for homogeneous function
  https://www.youtube.com/watch?v=btLWNJdHzSQ

Weekly Assignment,co2

Assignment-2.1

Q.1. Find y_n, where $y = \frac{ax+b}{cx+d}$

Q.2. Find nth derivative of $y \ = \ x^2e^x$at x = 0.

Q.3. If $y = x^n logx$, prove that $y_{n+1} = \frac{n!}{x}$

Q.4. Show that the function f defined by $f(x) = |x|^3 , x \in R$ is differentiable at $x = 0$ and $f'(0) = 0$.

Assignment-2.1

Q.5. If $y = sin(asin^{-1}x)$ show that $\left(1 - x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 - a^2\right)y_n = 0$.

Q.6. If $y = \ (sin^{-1}x)^2$then findy_n (0).

Q.7. If $y = e^{mcos^{-1}x}$Show that

$\left(1 - x^2\right)y_{n+2} - (2n+1)xy_{n+1} - \left(n^2 + m^2\right)y_n = 0\ and\ calculate\ y_n(0)$.

Q.8. If $y = \left[x + \sqrt{1+x^2}\right]^m\ find\ y_n(0)$.

Q.9. If $x = sin\sqrt{y}\ find\ y_n(0)$.

Weekly Assignment,co2

Assignment-2.2

Q.1.Find the asymptotes of the curve , $x^3 + y^3 = 3axy$

Ans- $x + y + a = 0$ is a real asymptote.

Q.2Trace the curve $r = a(1 + sin\theta)$.(Cardioid)

Q.3.Trace the curve $y^2(x - a) = x^2(x + a)$

Q.4.Trace the curve : $y^2(a + x) = x^2(3a - x)$.

Weekly Assignment (CO2)

Partial derivatives

1. If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, prove that
$\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2$

2. If $z = \log(e^x + e^y)$ show that $rt - s^2 = 0$ where $r = \dfrac{\delta^2 z}{\delta x^2}$, $t = \dfrac{\delta^2 z}{\delta y^2}$ and $s = \dfrac{\delta^2 z}{\delta x \delta y}$.

3. If $z = f(x + ct) + \emptyset(x - ct)$ show that $\dfrac{\partial^2 z}{\partial t^2} = c^2 \dfrac{\partial^2 z}{\partial x^2}$

4. If $u = e^{xyz}$ then prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)u$

5. If $e^{\frac{-z}{x^2-y^2}} = x - y$ then show that $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 - y^2$

Weekly Assignment (CO2)

Euler's theorem based question

1. Verify Euler's theorem for the functions :(i) $z = \frac{x^{1/3}+y^{1/3}}{\sqrt{x}+\sqrt{y}}$ (ii) $u = \log \frac{x^2+y^2}{xy}$.

2. If $u = x\sin^{-1}\left(\frac{x}{y}\right) + y\sin^{-1}\left(\frac{y}{x}\right)$, find the value of $x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2}$. Ans. 0.

3. If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\tan u$

4. If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that $xu_x + yu_y = \sin 2u$

Total derivatives and change of variable

1. If $v = f(2x-3y, 3y-4z, 4z-2x)$ prove that $6v_x + 4v_y + 3v_z = 0$.
2. $If\ u = f(e^{y-z}, e^{z-x}, e^{x-y})$ Then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$
3. $If\ u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

MCQ s

Q.1.Curve is symmetrical about x-axis then

(i)Even power of x (ii) Even power of y(iii) Even power of both

(iv) Odd power of x.

Q.2. Curve is symmetrical about y-axis then

(i)Even power of x (ii) Even power of y(iii) Even power of both

(iv) Odd power of x.

Q.3. Curve is symmetrical about both axes then

(i)Even power of x (ii) Even power of y(iii) Even power of both

(iv) Odd power of x.

MCQ s

Q.4.If $y = (ax + b)^m$ then y_n at$n = \mathrm{m}$

(i) $n!\,a^n$(ii) $n!\,n^n$(iii) $na!\,a^n$(iv) 0

Q.5.Ify $= sin3x.sin2x$ then y_n

(i) $\frac{1}{2}\left[cos\left(x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$

(ii) $\frac{1}{2}\left[cos\left(x + \frac{n\pi}{2}\right) + 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$

(iii)$\frac{1}{2}\left[cos\left(x - \frac{n\pi}{2}\right) - 5^n cos\left(5x - \frac{n\pi}{2}\right)\right]$

(iv) $\frac{1}{2}\left[cos\left(5x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$

Old Question Papers

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B. TECH.
(SEM I) THEORY EXAMINATION 2019-20
MATHEMATICS-I

Time: 3 Hours ***Total Marks: 100***

Note: 1. Attempt all Sections. If require any missing data; then choose suitably.

SECTION A

1. Attempt *all* questions.

Q. No.	Question	Marks	CO
a.	Show that vectors (1, 6, 4), (0, 2, 3) and (0, 1, 2) are linearly independent.	2	1
b.	Define Lagrange's mean value theorem.	2	2
c.	If $u = x(1-y), v = xy$, find $\frac{\partial(u,v)}{\partial(x,y)}$.	2	3
d.	Show that vector $\vec{V} = (x+3y)\hat{\imath} + (y-3z)\hat{\jmath} + (x-2z)\hat{K}$ is solenoidal.	2	5
e.	Find the value of 'b' so that rank of $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{bmatrix}$ is 2.	2	1
f.	Evaluate $\int_0^2 \int_0^1 (x^2 + 3y^2)\,dy\,dx$.	2	4
g.	Find grad $\emptyset$ at the point (2, 1, 3) where $\emptyset = x^2 + yz$	2	5
h.	If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.	2	3
i.	Find $\frac{du}{dt}$ if $u = x^3 + y^3, x = a\cos t, y = b\sin t$.	2	3
j.	Find the area lying between the parabola $y = 4x - x^2$ and above the line $y = x$.	2	4

Old Question Papers

SECTION B

2. Attempt any *three* of the following:

Q. No.	Question	Marks	CO
a.	Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and hence find A^{-1}.	10	1
b.	If $y = e^{m\cos^{-1}x}$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$. Hence find y_n when $x = 0$.	10	2
c.	If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ and $u + v + w = x^2 + y^2 + z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$.	10	3
d.	Evaluate the integral by changing the order of integration: $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$.	10	4
e.	Verify Stoke's theorem for the vector field $\vec{F} = (x^2 - y^2)\hat{\imath} + 2xy\hat{\jmath}$ integrated round the rectangle in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a, y = b$.	10	5

3. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	For what values of λ and μ the system of linear equations: $x+y+z=6$, $x+2y+5z=10$, $2x+3y+\lambda z=\mu$ has (i) a unique solution (ii) no solution (iii) infinite solution. Also find the solution for $\lambda=2$ and $\mu=8$.	10	1
b.	Find the rank of the matrix $A=\begin{bmatrix}1 & 3 & 4 & 2\\ 2 & -1 & 3 & 2\\ 3 & -5 & 2 & 2\\ 6 & -3 & 8 & 6\end{bmatrix}$ by reducing it to normal form.	10	1

4. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	Verify the Cauchy's mean value theorem for the function e^x and e^{-x} in the interval $[a,b]$. Also show that 'c' is the arithmetic mean between a and b.	10	2
b.	Trace the curve $r^2=a^2\cos 2\theta$.	10	2

5. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	If $u=f(2x-3y, 3y-4z, 4z-2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}=0$.	10	3
b.	Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.	10	3

5. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.	10	3
b.	Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.	10	3

6. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	Evaluate $\iint (x+y)^2 dx\, dy$, where R is the parallelogram in the xy-plane with vertices $(1,0), (3,1), (2,2), (0,1)$ using the transformation $u = x + y,\ v = x - 2y$.	10	4
b.	Find the volume of the region bounded by the surface $y = x^2,\ x = y^2$ and the planes $z = 0,\ z = 3$.	10	4

7. Attempt any *one* part of the following:

Q. No.	Question	Marks	CO
a.	Verify the divergence theorem for $\vec{F} = 4xz\hat{\imath} - y^2\hat{\jmath} + yz\hat{K}$ taken over the rectangular parallelepiped $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$.	10	5
b.	Find the directional derivative of $\emptyset(x, y, z) = x^2yz + 4xz^2$ at (1, -2, 1) in the direction of $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$. Find also the greatest rate of increase of $\emptyset$.	10	5

References

Thank You