Maths for Signals and Systems

Problem Sheet 3

Singular Value Decomposition (SVD)

Consider a matrix A of dimension $m \times n$ with $m < n$ and rank r. Recall from the lectures that $r \leq m$. The matrix AA^T is square, symmetric and of dimension $m \times m$. The matrix $A^T A$ is square, symmetric and of dimension $n \times n$. The following properties hold:

- Both AA^T and $A^T A$ have rank r (the same rank as the original matrix A).
- The m eigenvalues of $A^T A$ are identical to the eigenvalues of AA^T and the rest $n-m$ eigenvalues are 0.
- The so called **singular values** of A are the square roots of the non-zero eigenvalues of AA^T (or $A^T A$).
- Matrix A has a so called **Singular Value Decomposition (SVD)** of the form $A = U\Sigma V^T$ where U is of dimension $m \times m$, Σ is of dimension $m \times n$ and V is of dimension $n \times n$. Furthermore, U contains the eigenvectors of AA^T in its columns, V contains the eigenvectors of $A^T A$ in its columns and $\Sigma_{ij} = \begin{cases} \sigma_i = \sqrt{\lambda_i} & i = j, i \leq r \\ 0 & \text{otherwise} \end{cases}$ with $\lambda_i, i = 1, \ldots, r$ the non-zero eigenvalues of AA^T (or $A^T A$).
- The above comments imply that $AA^T = U\Sigma^2 U^T$ and $A^T A = V\Sigma^2 V^T$.

The above analysis is straightforward in the case of $m > n$.
To understand better the structure of Σ, in the case of a 3×4 matrix of rank 2 we have $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, whereas in the case of a 4×3 matrix of rank 2 we have $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Problems

1. Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$.

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2. Find the singular values of the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

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3. Find the SVD of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

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4. Find the projection matrix onto the subspace of R^4:

$$S = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Use this to compute the projection of vector $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ onto W.

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5. Find the best fit line $y = cx + d$ through the points $(0,0), (1,1), (2,3)$.

6. Consider the following orthonormal vectors $u_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $u_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Let $x = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$. If S is the span of u_1 and u_2, then:
   a. Find the projection y of x onto S.
   b. Verify that $w = x - y$ is orthogonal to S.
   c. Show that $||x||^2 = ||y||^2 + ||w||^2$.
   d. Compute the distance from x to the subspace S
7. Consider the system of equations:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Find the minimum norm solution