IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2019

Mathematics for Signals and Systems

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates: none

The Questions

1. (a) Prove Cauchy-Schwarz inequality, i.e., show that $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|$, where the norm is defined as $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$. Hint, use the inequality $\|\mathbf{x} - \alpha \mathbf{y}\|^2 \geq 0$ and assume

$$\alpha = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2}.$$

[4]

(b) Let V be the vector space consisting of all functions of the form

$$f(x) = \alpha + \beta x + \gamma e^x,$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$. Consider the linear transformation

$$L = 2\frac{df}{dx} - f(x).$$

i. Find the matrix representing the linear mapping L with respect to the basis $\{1, x, e^x\}$. [2]

ii. Use the answer from part i. to find one solution to the following differential equation:

$$2\frac{df}{dx} - f(x) = 3 + x - 2e^x.$$

[2]

(c) Find a basis for the following subspaces of $\mathbb{R}^5$

i. The vectors for which $x_1 = x_3 + 2x_5$ [1]

ii. The vectors for which $x_1 + x_4 = 0$ [1]

iii. $V = \text{span}\{[1, 1, 1, 1, 1]^T, [2, 4, -6, 0, 6]^T, [0, -1, 4, 1, -2]^T\}$ [1]

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(d) Consider the following subspace of $\mathbb{R}^4$:

$$S = \text{ span} \left\{ \begin{bmatrix} 2 \\ 4 \\ 8 \\ 16 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 9 \\ 17 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 6 \\ 10 \end{bmatrix} \right\}.$$

Find a basis for S. [3]

(e) Consider the linear mapping $\mathbf{A}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$\mathbf{y}_1 = \mathbf{A}\mathbf{x_1} = [2, 3, 5]^T,$$

$$\mathbf{y}_2 = \mathbf{A}\mathbf{x_2} = [1, 0, 0]^T,$$

and

$$\mathbf{y}_3 = \mathbf{A}\mathbf{x_3} = [0, 1, -1]^T,$$

where $\mathbf{x}_1 = [0, 0, 1]^T$, $\mathbf{x}_2 = [0, 1, 1]^T$ and $\mathbf{x}_3 = [1, 1, 1]^T$.

i. Determine the matrix $\mathbf{A}$ that describes this linear mapping [3]

ii. Determine the dimension and a basis for the range space and the null space of $\mathbf{A}$. [3]

2. (a) Find the dimension and construct a basis for the four subspaces associated with
$$\mathbf{A} = \begin{bmatrix} 6 & 5 & 6 & 3 \\ 2 & 3 & 0 & 1 \\ 4 & 4 & 3 & 2 \end{bmatrix}.$$
[5]

(b) Consider the system of linear equations $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ given by:
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

i. Can you claim that if a solution exist, it is unique? Briefly justify your answer. [2]

ii. Assume that $\mathbf{y} = [3, 6, -3]^T$, find the least-square solution. That is, find the $\mathbf{x}$ that minimizes $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2$. [3]

iii. You now want to find the least square solution to the problem above but you also want the norm of $\mathbf{x}$ to stay as small as possible. So find the $\mathbf{x}$ that minimizes $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \|\mathbf{x}\|^2$. [4]

(c) Consider the following matrix:
$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}.$$

Find the rank one matrix $\mathbf{B}$ that minimizes $\|\mathbf{A} - \mathbf{B}\|^2$ where $\|\cdot\|$ denotes the Frobenius norm of a matrix. [6]

3. (a) Let

$$\mathbf{p}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

i. Determine the vector $\hat{\mathbf{x}} \in \text{span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ that minimes $\|\mathbf{x} - \hat{\mathbf{x}}\|$. [4]

ii. Verify that the error vector $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ is orthogonal to $S = \text{span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$. [2]

(b) Consider the vectors $\mathbf{x} = [2, 1, 4]^T$ and $\hat{\mathbf{x}} = [3, 1, 3]^T$ with $\hat{\mathbf{x}} \in S$ and $S = \text{span}\{[1, 1, 1]^T, [1, 0, 1]^T\}$. Can you claim that $\hat{\mathbf{x}}$ is the unique vector in S that minimizes $\|\mathbf{x} - \hat{\mathbf{x}}\|$? Justify your answer. [3]

(c) Consider the matrix:

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Is $\mathbf{P}$ an orthogonal projection matrix? Justify your answer. [3]

(d) Assume $\mathbf{P}_1$ and $\mathbf{P}_2$ are two matrices representing orthogonal projections. Under which condition you can claim that $\mathbf{P}_1 + \mathbf{P}_2$ is an orthogonal projection? [3]

Question 3 continues on the next page

(e) Given the matrix

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 4 \\ 1 & -1 & -2 \end{bmatrix},$$

state whether $\mathbf{B}$ is the pseudo-inverse of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Justify your answer. [5]