Imperial College London

Digital Signal Processing

Topic 3

Introduction to Digital Filters Part 1

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Filters in digital signal processing

- In signal processing, a **filter** (the term **system** is also used) is a device or algorithm that removes some unwanted frequency components from a signal.
- In other words, a filter removes some frequencies or frequency bands.
- However, filters do not exclusively act in the frequency domain; we can operate on a signal in time domain in such a way that implies the suppression or removal of certain frequencies.
- Filters are widely used in:
  - Electronics and telecommunication
  - Radio and television
  - Audio recording
  - Radar
  - Music synthesis
  - Image processing and computer graphics

All filters we deal with in this course are Linear Time Invariant (LTI) systems.

Discrete LTI systems

- Most useful LTI systems can be described by a difference equation as follows:

$$y[n] = \sum_{r=0}^{M} b[r]x[n-r] - \sum_{r=1}^{N} a[r]y[n-r]$$

x[n]: input

y[n]: output

$$\Leftrightarrow \sum_{r=0}^{N} a[r]y[n-r] = \sum_{r=0}^{M} b[r]x[n-r],\ \boldsymbol{a[0]=1}$$

$$\Leftrightarrow a[n]*y[n] = b[n]*x[n] \Leftrightarrow A(z)Y(z) = B(z)X(z)$$

$$\Leftrightarrow Y(z) = \frac{B(z)}{A(z)}X(z) \Rightarrow Y\left(e^{j\omega}\right) = \frac{B\left(e^{j\omega}\right)}{A\left(e^{j\omega}\right)}X\left(e^{j\omega}\right)$$

  - The above system is **causal**.
  - The **order** of system is $\max(M,N)$, or $\max(r)$ with $a[r] \neq 0$ or $b[r] \neq 0$.
  - We assume that $\boldsymbol{a[0]=1}$; if not, we divide $A(z)$ and $B(z)$ by $a[0]$.
  - The above filter is **BIBO stable if and only if** the roots of $A(z)$ which are the **poles** of the transfer function $Y(z)$ lie within the unit circle. this is because the system is causal!
  - If a system is BIBO stable, then the output will be bounded for every input to the system that is bounded. (BIBO=Bounded Input Bounded Output)

FIR filters

- In case $A(z) = 1$ we have a so-called **Finite Impulse Response (FIR)** filter as follows:

$$y[n] = \sum_{r=0}^{M} b[r]x[n-r]$$
$$\Leftrightarrow y[n] = b[n] * x[n]$$
$$\Leftrightarrow Y(z) = B(z)X(z)$$
$$\Leftrightarrow Y\left(e^{j\omega}\right) = B\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

  - From the relationship $y[n] = \sum_{r=0}^{M} b[r]x[n-r] = b[n] * x[n]$, we see that the **impulse response** $h[n]$ is the sequence $b[n]$, i.e., $h[n] = b[n]$.
  - The **frequency response** of the above filter is $B\left(e^{j\omega}\right)$ and it is the **Discrete Time Fourier Transform (DTFT)** of the **impulse response** sequence $\{b[n]\}$.
  - $B\left(e^{j\omega}\right) = \sum_{r=0}^{M} b[r]e^{-jr\omega}$

FIR filters cont.

- The behaviour of the frequency response $B(e^{j\omega})$ is determined by the zeros of $B(z) = \sum_{r=0}^{M} b[r]\, z^{-r}$.
- The above is very often written as $z^M B(z) = \sum_{r=0}^{M} b[r]\, z^{M-r}$. This is done if we prefer to work with positive powers of z.
- The following **very** important properties hold:
  - For real coefficients $b[n]$, the zeros of $B(z)$ are real or occur in complex conjugate pairs, i.e., if z_0 is a zero of $B(z)$ then z_0^* is also a zero of $B(z)$.
  - For symmetric coefficients $b[n] = b[M-n]$, the zeros of $B(z)$ occur in reciprocal pairs, i.e., if z_0 is a zero of $B(z)$ then z_0^{-1} is also a zero of $B(z)$.
  - For real and symmetric coefficients $b[n]$ both of the above conditions must hold, i.e.:
    - there are groups of four conjugate and reciprocal zeros or
    - we have reciprocal pairs on the real axis or
    - we have complex conjugate pairs with magnitude equal to 1
  - Note that if $z_0 = re^{j\theta}$, then $z_0^* = re^{-j\theta}$ and $z_0^{-1} = \frac{1}{r} e^{-j\theta}$

FIR filter. Some proofs.

In all of the proofs below, we assume that z_0 is a root (zero) of $B(z)$ so that $B(z_0) = \sum_{r=0}^{M} b[r]\, z_0{}^{-r} = 0$ and then we prove that some other specific values related to z_0 also satisfy $B(z) = 0$.

- **Real $\boldsymbol{b[r]}$**

$$B\left(z_0^*\right) = \sum_{r=0}^{M} b[r]\left(z_0^*\right)^{-r} = \sum_{r=0}^{M} b^*[r]\left(z_0^*\right)^{-r} = \left(\sum_{r=0}^{M} b[r]\, z_0^{-r}\right)^* = (0)^* = 0$$

- **Symmetric $\boldsymbol{b[n] = b[M-n]}$**

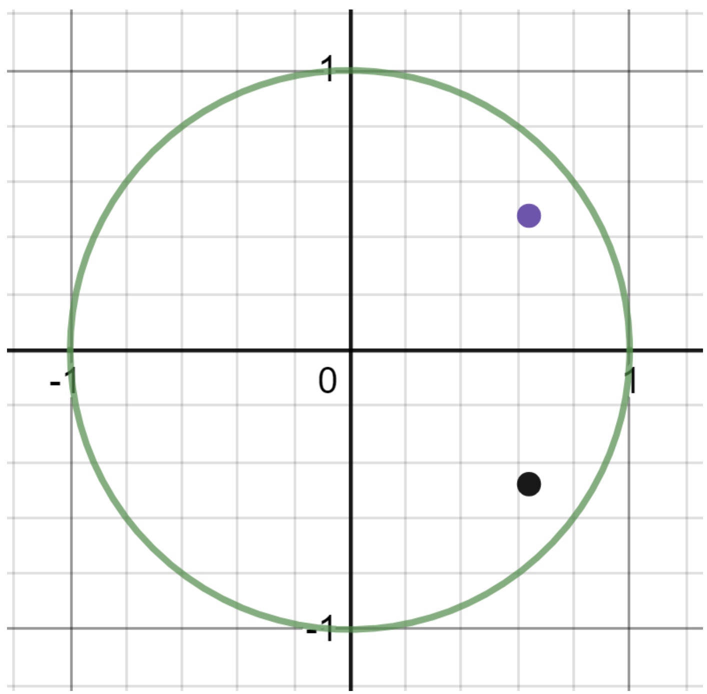
$$B\left(z_0^{-1}\right) = \sum_{r=0}^{M} b[r]\left(z_0^{-1}\right)^{-r} = \sum_{r=0}^{M} b[r]\, z_0^{r} = \sum_{n=0}^{M} b[M-n]\, z_0^{M-n}$$

$$= z_0^M \sum_{n=0}^{M} b[M-n]\, z_0^{-n} = z_0{}^M \sum_{n=0}^{M} b[n]\, z_0^{-n} = z_0^M \cdot 0 = 0$$

$\boldsymbol{[r = M - n]}$

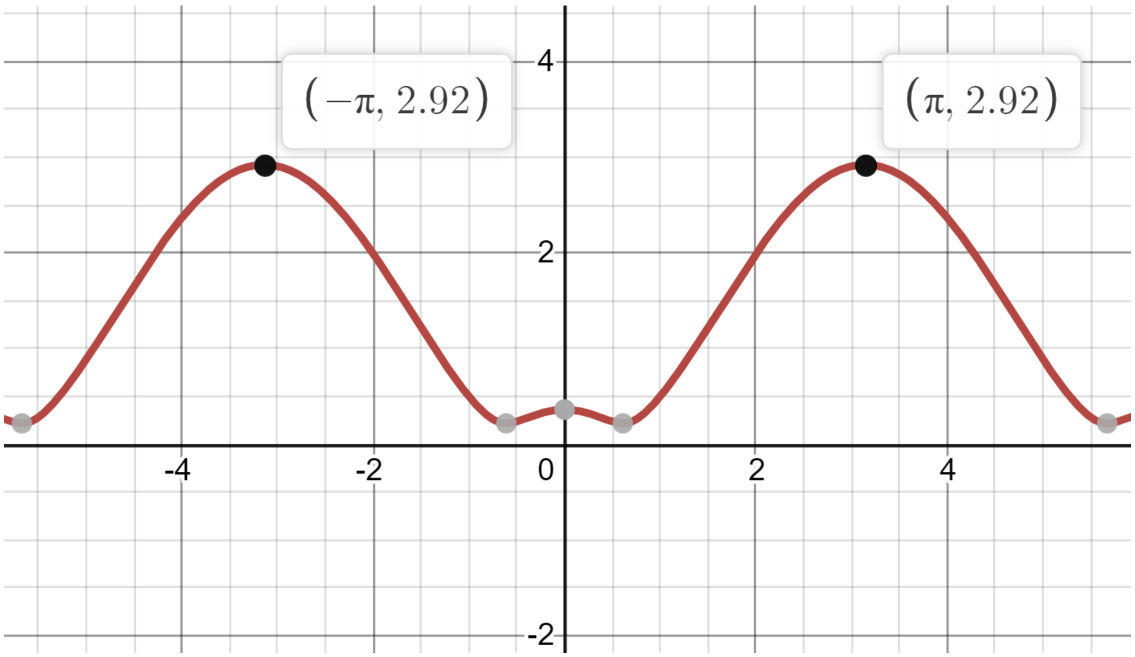
FIR filters with real coefficients: Example 1

- Consider the FIR filter with transfer function
$$B(z) = 1 - 1.28z^{-1} + 0.64z^{-2}$$
$$b[0] = 1, b[1] = -1.28, b[2] = 0.64$$
- The roots of the above polynomial are the complex conjugate pair $0.64 \pm j0.48$.
- Recall: For real coefficients $b[n]$, the zeros of $B(z)$ are real or occur in complex conjugate pairs, i.e., if z_0 is a zero of $B(z)$ then z_0^* is also a zero of $B(z)$.



FIR filters with real coefficients: Example 1 cont.

- $B(z) = 1 - 1.28z^{-1} + 0.64z^{-2}$, $b[0] = 1, b[1] = -1.28, b[2] = 0.64$
- The amplitude response of the above transfer function is shown below.
- Observe that the above transfer function behaves like a **high-pass filter** since the low frequencies are suppressed. (Useful frequencies are in the range $[0, \pi]$).
- By looking at the output of the filter $y[n] = x[n] - 1.28x[n-1] + 0.64x[n-2]$, can you think why is this filter a high-pass?

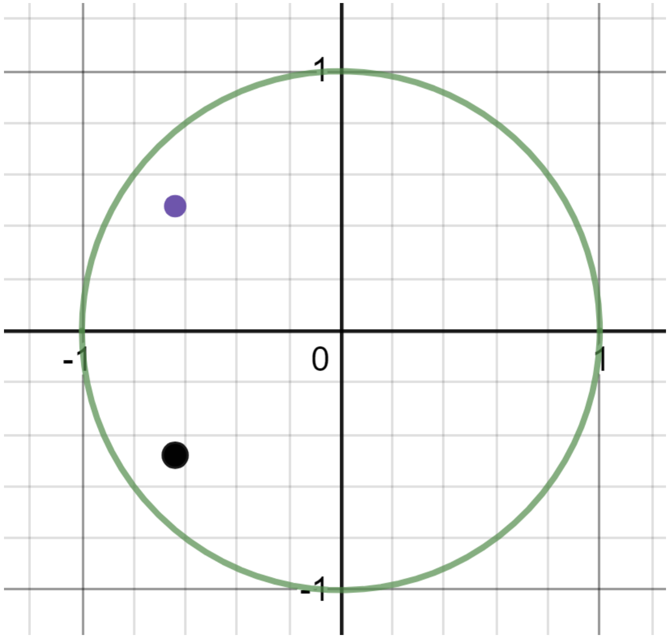


FIR filters with real coefficients: Example 2

- Consider the FIR filter with transfer function

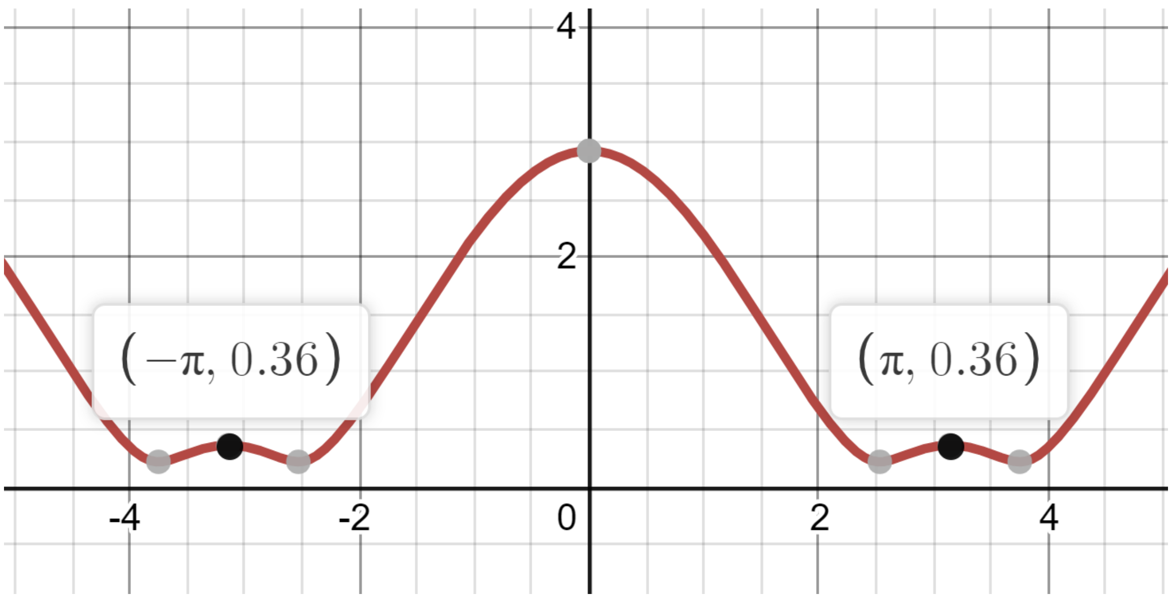
$$B(z) = 1 + \mathbf{1.28}z^{-1} + 0.64z^{-2}$$
$$b[0] = 1, b[1] = 1.28, b[2] = 0.64$$

- The roots of the above polynomial are the complex conjugate pair $-0.64 \pm j0.48$. They have the same imaginary parts as in the previous example, and real part with reversed sign.
- **The two transfer functions differ in the sign of the middle coefficient.**



FIR filters with real coefficients: Example 2 cont.

- $B(z) = B(z) = 1 + \mathbf{1.28}z^{-1} + 0.64z^{-2}$, $b[0] = 1, b[1] = 1.28, b[2] = 0.64$
- The amplitude response of the above transfer function is shown below.
- Observe that the above transfer function behaves like a **low-pass filter** since the high frequencies are suppressed.
- By looking at the output of the filter $y[n] = x[n] + 1.28x[n-1] + 0.64x[n-2]$, can you think why is this filter a low-pass?



Comparison of filters of Example 1 and Example 2

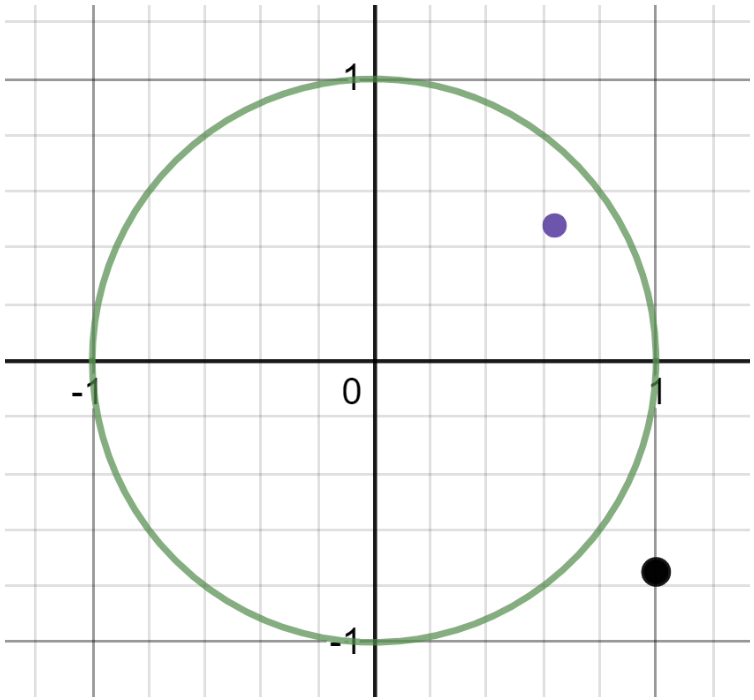
- Example 1: $B(z) = 1 - \mathbf{1.28}z^{-1} + 0.64z^{-2}$, $b[0] = 1, b[1] = \mathbf{-1.28}, b[2] = 0.64$
- Example 2: $B(z) = 1 + \mathbf{1.28}z^{-1} + 0.64z^{-2}$, $b[0] = 1, b[1] = \mathbf{1.28}, b[2] = 0.64$
- Observe that by changing the sign of one of the coefficients in a high-pass filter we transformed it into a low-pass filter!
- In principle:
  - Replacing a signal sample with a lineal combination of local neighbours with positive weights “blurs” the signal.
  - Replacing a signal sample **including some** differences of local neighbours can “sharpen” the signal. A high-pass FIR filter would require always some negative coefficients.

FIR filters with symmetric coefficients: Example 3

- Consider the FIR filter with transfer function

$$B(z) = 1 + (-1.64 + j0.27)z^{-1} + z^{-2}$$
$$b[0] = 1, b[1] = -1.64 + j0.27, b[2] = 1$$

- The roots of the above polynomial are $\mathbf{0.64 + j0.48}$ and $\frac{1}{\mathbf{0.64+j0.48}} = \mathbf{1 - j0.75}$.
- Recall: For symmetric coefficients $b[n] = b[M-n]$, the zeros of $B(z)$ occur in reciprocal pairs, i.e., if z_0 is a zero of $B(z)$ then z_0^{-1} is also a zero of $B(z)$.



FIR filters with symmetric coefficients: Example cont.

- $B(z) = 1 + (-\mathbf{1.64} + j0.27)z^{-1} + z^{-2}, b[0] = 1, b[1] = -1.64 + j0.27, b[2] = 1$
- The amplitude response of the above transfer function is shown below.
- **Exercise:** Verify that the above transfer function behaves like a **high-pass filter** by calculating and plotting the amplitude response.

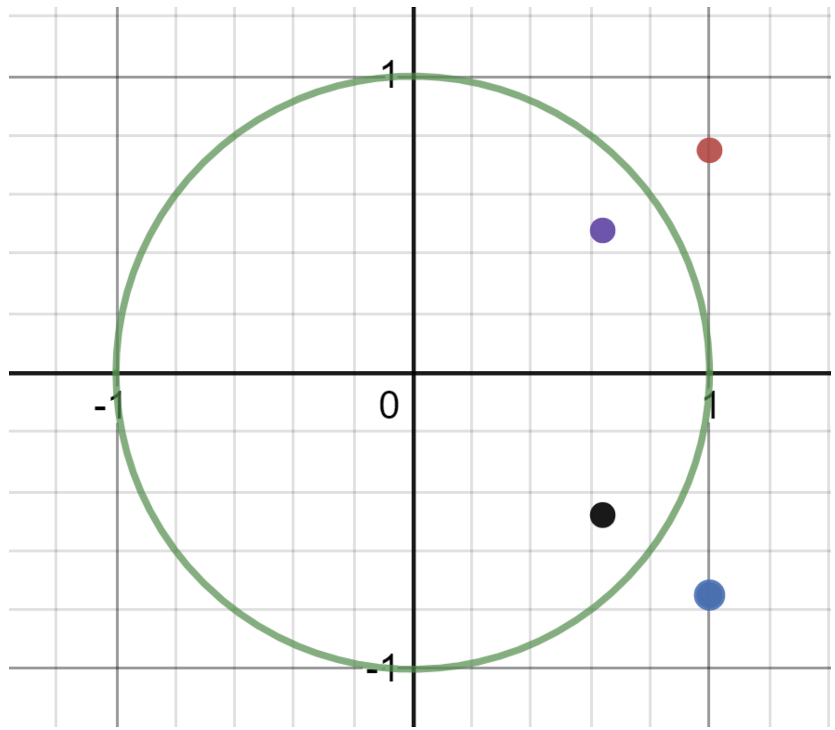
FIR filters with real and symmetric coefficients: Example 4

- Consider the FIR filter with transfer function

$$B(z) = 1 - 3.28z^{-1} + 4.7625z^{-2} - 3.28z^{-3} + z^{-4}$$

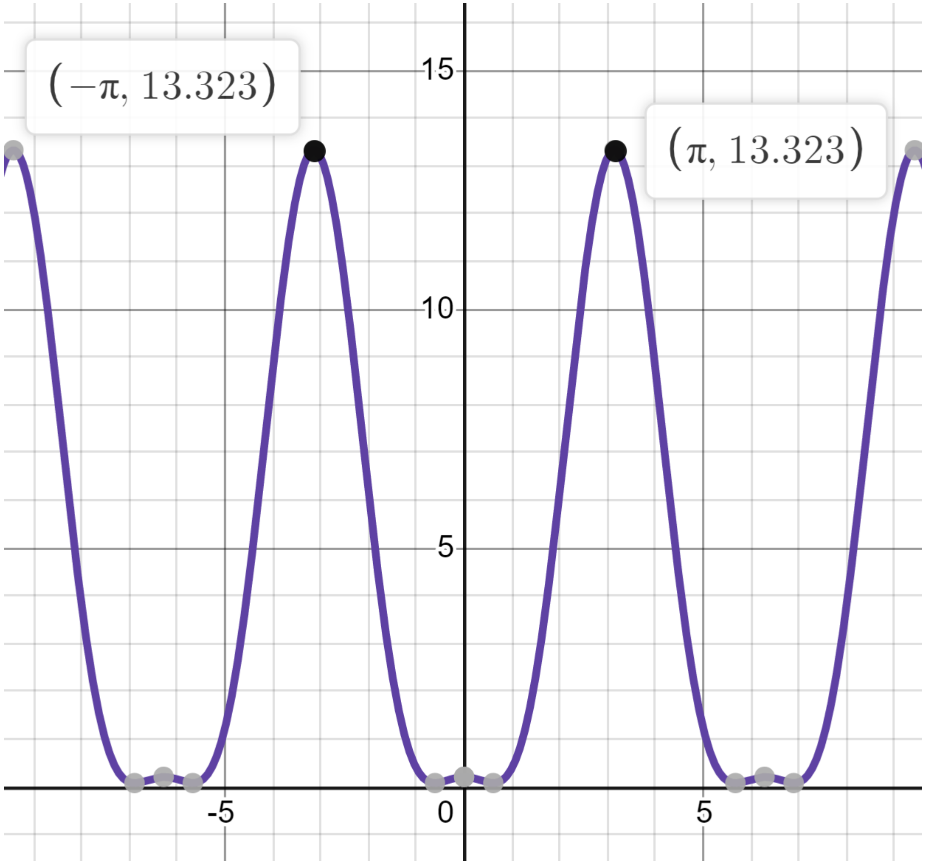
$$b[0] = b[4] = 1, b[1] = b[3] = -3.28, b[2] = 4.7625$$

- The roots of the above polynomial are $0.64 + j0.48$ and $\frac{1}{0.64+j0.48} = 1 - j0.75$ and their conjugates $0.64 - j0.48$ and $1 + j0.75$.



FIR filters with real and symmetric coefficients: Example 4

- $B(z) = 1 - 3.28z^{-1} + 4.7625z^{-2} - 3.28z^{-3} + z^{-4}$
  $b[0] = b[4] = 1, b[1] = b[3] = -3.28, b[2] = 4.7625$
- The amplitude response of the above transfer function is shown below.
- It is a high pass filter.



IIR filters

- Let us recall the original general form:

$$y[n] = \sum_{r=0}^{M} b[r]x[n-r] - \sum_{r=1}^{N} a[r]y[n-r]$$

- From the above relationship we have:

$$Y(z) = \sum_{r=0}^{M} b[r]z^{-r}X(z) - \sum_{r=1}^{N} a[r]z^{-r}Y(z) \Rightarrow$$

$$Y(z) = X(z)\sum_{r=0}^{M} b[r]z^{-r} - Y(z)\sum_{r=1}^{N} a[r]z^{-r} \Rightarrow$$

$$Y(z) = X(z)\boldsymbol{b[0]}\sum_{r=0}^{M} \frac{b[r]}{b[0]}z^{-r} - Y(z)\sum_{r=1}^{N} a[r]z^{-r} \Rightarrow$$

$$Y(z)\left(1 + \sum_{r=1}^{N} a[r]z^{-r}\right) = X(z)\boldsymbol{b[0]}\left(1 + \sum_{r=1}^{M} \frac{b[r]}{b[0]}z^{-r}\right)$$

IIR filters cont.

- We factorize both the numerator and denominator of the transfer function

$$H(z)=\frac{Y(z)}{X(z)}=\frac{B(z)}{A(z)}=\frac{b[0]\left(1+\sum_{r=1}^{M}\frac{b[r]}{b[0]}z^{-r}\right)}{\left(1+\sum_{r=1}^{N}a[r]z^{-r}\right)}=\frac{b[0]\prod_{i=1}^{M}(\mathbf{1-q_i z^{-1}})}{\prod_{i=1}^{N}(\mathbf{1-p_i z^{-1}})}$$

- This is also written as:

$$H(z)=\frac{B(z)}{A(z)}=\frac{b[0]z^{-M}\prod_{i=1}^{M}(z-q_i)}{z^{-N}\prod_{i=1}^{N}(z-p_i)}=\frac{b[0]z^{N}\prod_{i=1}^{M}(z-q_i)}{z^{M}\prod_{i=1}^{N}(z-p_i)}$$

- The roots of $A(z)$ and $B(z)$ are the **poles** $\{\boldsymbol{p_i}\}$ and **zeros** $\{\boldsymbol{q_i}\}$ of $H(z)$, respectively.
- We have an additional $N-M$ zeros at $z=0$ if $N>M$; these zeros are represented with the term z^{N-M} which affects only the phase of the transfer function.
- We have an additional $M-N$ poles at $z=0$ if $M>N$.
- Note that, unless $A(z)$ is a factor of $B(z)$, the division $H(z)=\frac{B(z)}{A(z)}$ has infinite terms, i.e., $H(z)=\frac{B(z)}{A(z)}=\sum_{n=0}^{\infty}h[n]z^{-n}$. By definition the sequence $\{h[n]\}$ is the impulse response of the filter. Since, the impulse response has infinite number of terms we call a filter with a rational transfer function an Infinite Impulse Response (IIR) filter.

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IIR filter frequency response: Example

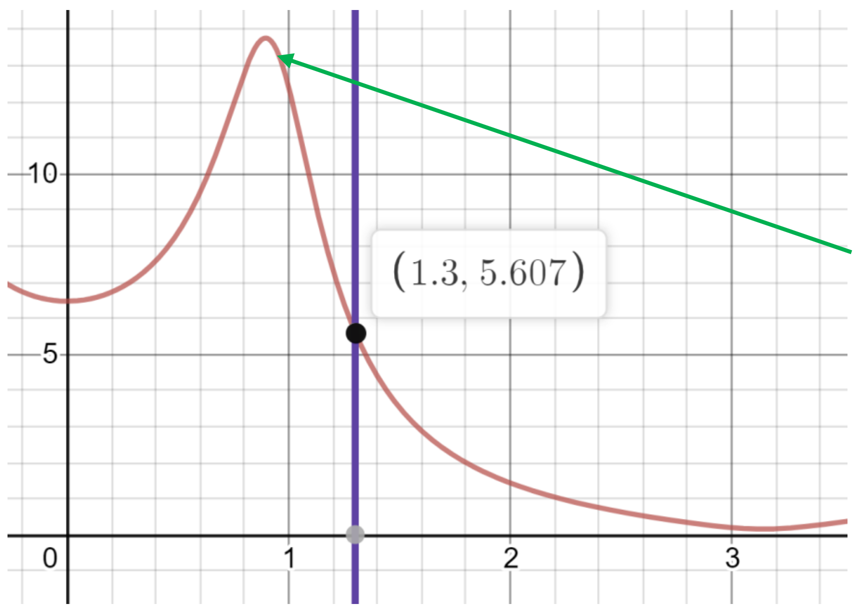
- Consider:

$$H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}} = \frac{2(1+1.2z^{-1})}{(1-(0.48-0.64j)z^{-1})(1-(0.48+0.64j)z^{-1})}$$

- At $\omega = 1.3\text{rads}\ (74.485\ \text{degrees})$

$$\left|H(e^{j\omega})\right|_{\omega=1.3} = \frac{2\sqrt{(1+1.2\cos(1.3))^2+(1.2\sin(1.3))^2}}{\sqrt{(1-0.96\cos(1.3)+0.64\cos(2\cdot1.3))^2+(0.96\sin(1.3)-0.64\sin(2\cdot1.3))^2}} = 5.607$$

$\left|H(e^{j\omega})\right|$



$\arctan\left(\frac{0.64}{0.48}\right) \cong 0.93\text{rads}$

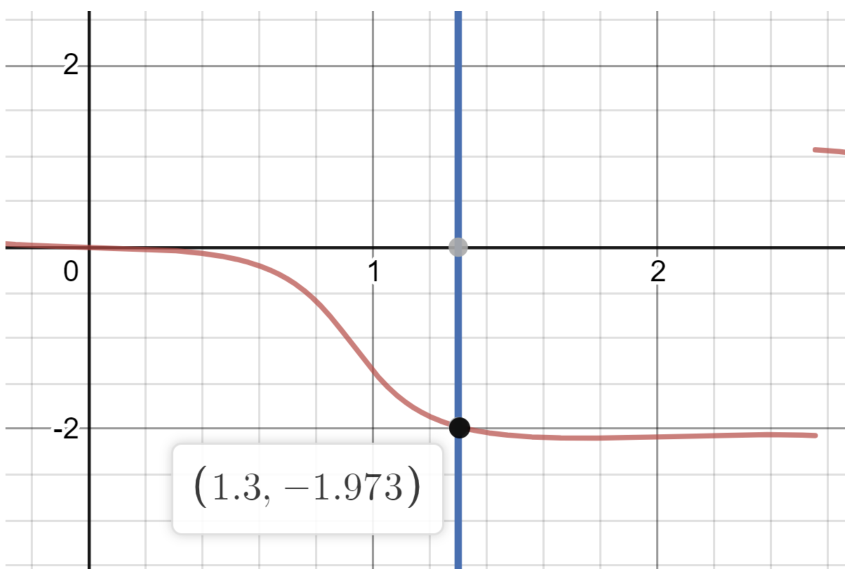
The peak of the amplitude response occurs around that frequency.

IIR filter frequency response: Example cont.

- At $\omega = 1.3$:

$$\angle H(e^{j\omega}) = \arctan\left(\frac{-2.4\sin(\omega)}{2+2.4\cos(\omega)}\right) - \arctan\left(\frac{0.96\sin(\omega)-0.64\sin(2\omega)}{1-0.96\cos(\omega)+0.64\cos(2\omega)}\right)$$

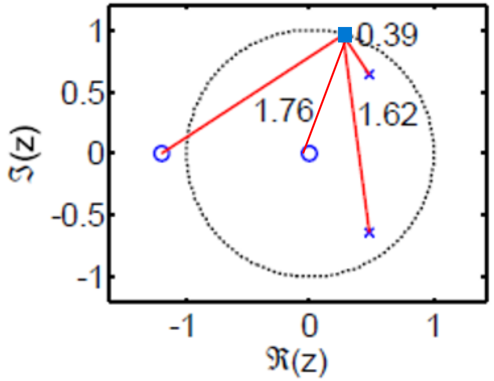
$$\angle H(e^{j\omega})\Big|_{\omega=1.3} = -1.973$$

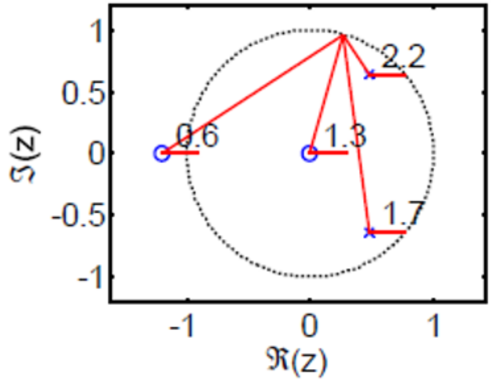


IIR filter frequency response: Example cont.

- The figure on the left displays the distances $\left|e^{j\omega} - p_i\right|\Big|_{\omega=1.3}$, $\left|e^{j\omega} - q\right|\Big|_{\omega=1.3}$ of poles and zeros respectively, from the point $\omega = 1.3\text{rads}\ (74.5\text{degrees})$.
- The figure on the right displays the phases $\angle(e^{j\omega} - q_i)\big|_{\omega=1.3}$, $\angle(e^{j\omega} - p_i)\big|_{\omega=1.3}$ of the same vectors from the point $\omega = 1.3$.

The tiny blue square
shows the point 1.3 rads.





$N - M = 2 - 1 = 1$
(one zero at the origin)