1. (a) Consider the causal transfer function:

$$H_M(z) = \frac{B(z)}{A(z)} = \frac{\sum_{n=0}^{n=M} b[n]z^{-n}}{\sum_{n=0}^{n=M} a[n]z^{-n}}, M > 0$$

where $b[n] = a^*[M-n]$. The above transfer function is an allpass filter.

(i) Show that the phase of $H_M(e^{j\omega})$ is $\angle H_M(e^{j\omega}) = -\omega M - 2\angle A(e^{j\omega})$, where $\angle A(e^{j\omega})$ is the phase of the polynomial $A(e^{j\omega})$. **[3]**

(ii) Consider a causal, real, stable, allpass transfer function $H(z)$ given as:

$$H(z) = \frac{-pz^{-q} + z^{-1-q}}{1 - pz^{-1}}$$

Sketch the pole-zero plot of $H(z)$ if $p = 0.5$, where q is a positive integer. **[3]**

(iii) Determine the phase response of $H(z)$ as a function of the pole p and the parameter q using the result from part (a)(i) above. **[3]**

(b) Consider a second-order all-pass filter with a transfer function:

$$H(z) = \frac{a_2 + a_1 z^{-1} + a_0 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

Suppose that $a_0 = 1,\ a_1 = -3a/4$ and $a_2 = a^2/8$ where a is a real parameter.

(i) Determine the condition on a for the filter to be stable. **[2]**

(ii) Find the phase response of $H(z)$ as a function of the poles of $H(z)$. **[3]**

(iii) Calculate the phase introduced by this filter at $\omega = 0$ and $\omega = \pi$. **[2]**

(c) The average group delay of an IIR stable system is $n - m$ where n is the number of poles and m is the number of zeros of the system located inside the unit circle.

(i) Demonstrate that the factors $(z - q)$ and $(q^* z - 1)$ have the same amplitude response. Use this result to show that inverting a zero originally located inside the unit circle to a position outside the unit circle increases the group delay by 1. Here, q is a complex number and q^* represents its complex conjugate. **[3]**

(ii) Using the conclusion from part c(i) above, identify which filter, among all filters with the same amplitude response, achieves the minimum group delay. Such a filter is referred to as a minimum-phase filter. **[3]**

(iii) Explain why linear-phase filters cannot be categorized as minimum-phase filters. **[3]**

2. (a) (i) Demonstrate that antisymmetric linear-phase Finite Impulse Response (FIR) filters of order N with a transfer function $H(z)$ (**Type 4**), satisfy the Antimirror Image Polynomial relationship (AIP) given below:

$$H(z) = -z^{-(N-1)}H(z^{-1})$$

[4]

(ii) Investigate the types of filters (e.g., low-pass, high-pass) that can be implemented using **Type 4** linear-phase Finite Impulse Response (FIR) transfer functions. **[4]**

(iii) Prove that a Type 4 linear-phase Finite Impulse Response (FIR) transfer function, $H(z)$, can be represented in the following form:

$$H(z) = \left(\sum_{n=0}^{\frac{N}{2}-1} h[n](z^{-n} - z^{n-N+1}) \right)$$

[4]

(b) Let $H(z)$ represent the transfer function of a real-coefficient, linear-phase FIR filter. The filter has zeros located at specific points in the z −plane, with a subset of these zeros given as:

$$\left\{1, 0.5e^{j\frac{\pi}{6}}, -\frac{1}{5}, j\right\}$$

(i) Since the zeros of a linear-phase FIR filter must exist in particular configurations, we deduce that the filter $H(z)$ must include additional zeros beyond those provided. Determine the additional zeros of $H(z)$ for each possible type of linear-phase FIR filter with the minimum allowable length. **[6]**

(ii) What is the filter type of the minimum length? **[3]**

(iii) Sketch the zero diagram of the filter identified in (ii) above on the z−plane. **[4]**

3. (a) The bilinear transformation from the s −plane to the z −plane is given by

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

Here, T is the sampling period of the signal.

(i) Illustrate, using mathematical relationships, how the bilinear transformation and its inverse map each point $s = \sigma + j\Omega$ in the s −plane to a corresponding point $z = re^{j\omega}$ in the z −plane, and vice versa. **[3]**

(ii) Prove that the relationship between continuous-time angular frequency Ω and discrete-time angular frequency ω under the bilinear transformation is generally non-linear. Additionally, demonstrate that this mapping becomes approximately linear at low frequencies.
[**Hint:** Use the small-angle approximation for the tangent function, $\tan(x) \approx x$ when $x \to 0$]. **[3]**

(b) The transfer function of an ideal analogue integrator is given by:

$$H_a(s) = \frac{1}{s}$$

A digital integrator can be derived from an analogue integrator by applying the bilinear transformation.

(i) Determine the amplitude and phase response of the analogue integrator and provide a rough sketch to illustrate the results. **[3]**

(ii) Derive the input-output relationship for the digital integrator in the time domain. For simplicity use $T = 2$ for the sampling period. **[3]**

(iii) Estimate the frequency response of the digital integrator. **[3]**

(iv) Create a rough sketch of the amplitude and phase responses of the digital integrator. **[3]**

(v) Compare the amplitude and phase responses of the analogue and digital integrators, with a focus on demonstrating that the amplitude responses are more closely aligned at low frequencies compared to high frequencies. **[3]**

(vi) Demonstrate that the pole of the digital integrator, located on the unit circle, causes its response to become unbounded for inputs with a DC component. Use the input signal $x[n] = u[n]$, where $u[n]$ is the unit step function, to illustrate this behaviour. **[4]**

4. (a) Examine the multirate configuration illustrated in **Figure 4.1** below. In this setup, $H_0(z), H_1(z)$, $H_2(z)$ and $H_3(z)$ are filters, each possessing ideal, zero-phase characteristics and real coefficients. Their corresponding frequency responses are outlined as follows:

$$H_0(e^{j\omega}) = u(\omega) - u(\omega - \frac{\pi}{6})$$

$$H_1(e^{j\omega}) = u(\omega - \frac{\pi}{6}) - u(\omega - \frac{\pi}{2})$$

$$H_2(e^{j\omega}) = u\left(\omega - \frac{\pi}{2}\right) - u\left(\omega - \frac{5\pi}{6}\right)$$

$$H_3(e^{j\omega}) = u\left(\omega - \frac{5\pi}{6}\right) - u(\omega - \pi)$$

The function $u(\omega)$ is the unit step function, defined as

$$u(\omega) = \begin{cases} 1 & \omega \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

If the input is a real sequence with a Discrete Time Fourier Transform

$$X(e^{j\omega}) = (\frac{\pi}{6} - \frac{6\omega^2}{\pi})\left(u(\omega) - u\left(\omega - \frac{\pi}{6}\right)\right)$$

sketch the Discrete Time Fourier Transform of the outputs $y_0[n], y_1[n]$, $y_2[n]$ and $y_3[n]$. It's important to note that all frequency domain representations mentioned are limited to the range $[0, \pi]$. **[15]**

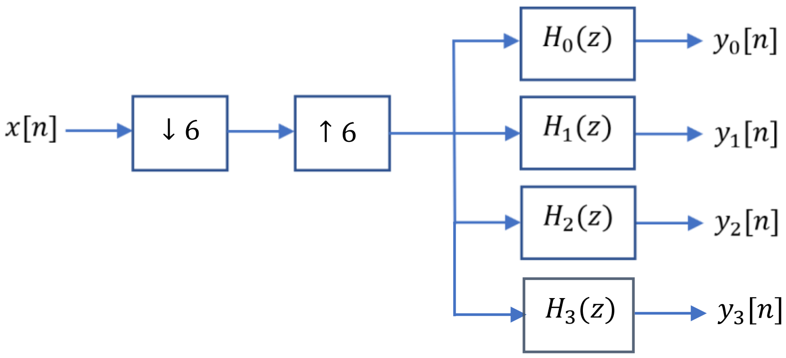

Figure 4.1

(b) Analyse the structure of **Figure 4.2** below and determine its input-output relations. **[10]**

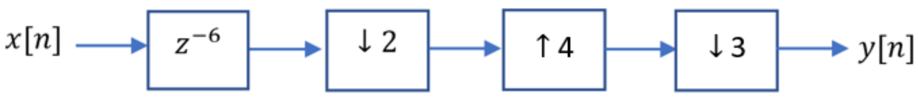

Figure 4.2