Imperial College London

Digital Signal Processing

z-transform

Properties of the Region of Convergence (ROC)

DR TANIA STATHAKI
READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Imperial College London

Inverse z-transform

Welcome back to the DSP class!

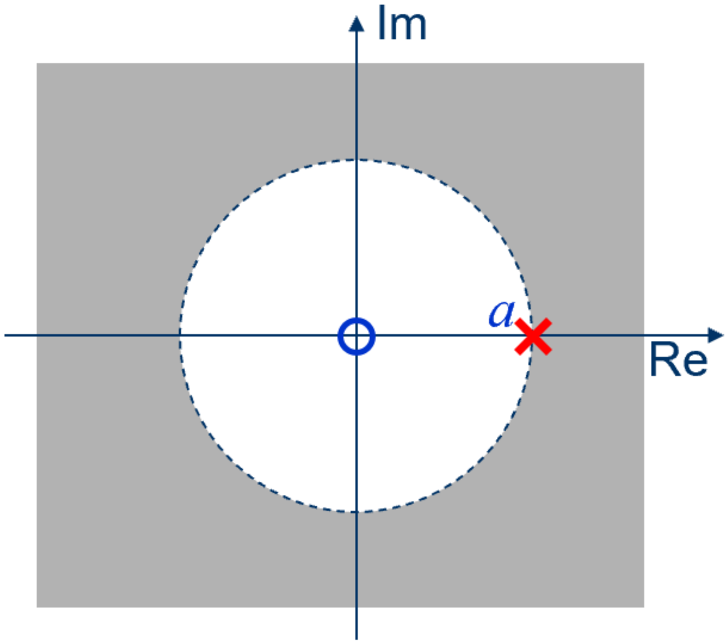
- In this class we will talk about the **properties of the Region of Convergence (ROC)** and the **location of the poles and zeros** of the z –transform.

- We will also solve some problems.

- Why are we interested?
  - Knowing the ROC of a transfer function of a discrete system enables us to **verify whether a system is stable or unstable.**
  - Knowing the ROC of a transfer function of a discrete system enables us to **verify whether a system is causal or non-causal.**
  - Knowing the location of poles and zeros helps us to extract information related to the **form of the frequency response**, as for example, the location of the peaks and the valleys.

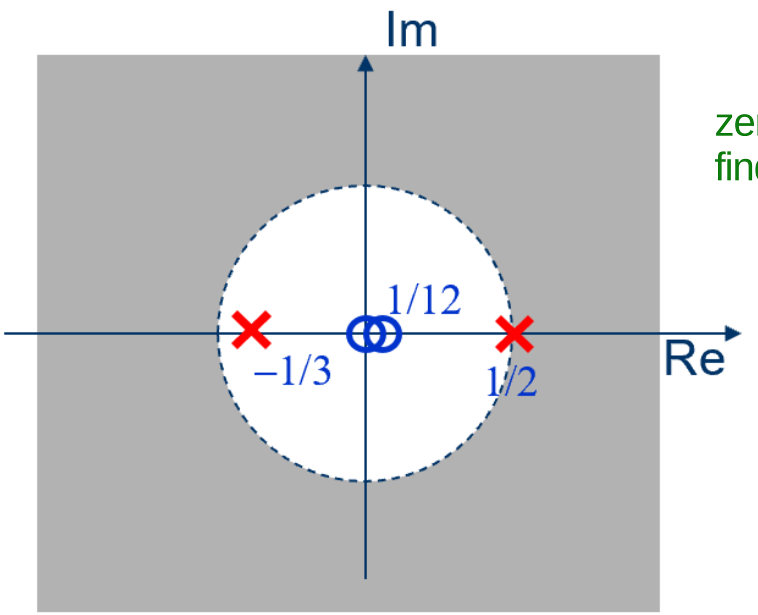
Right-sided sequences

- Consider the well known **causal** sequence $x(n) = a^n u(n)$ with z –transform $X(z) = \frac{z}{z-a}, |z| > |a|$.
- The above sequence is a so-called **right-sided sequence**.
- Observe that the ROC is bounded by the pole and is **the exterior of a circle** located on the z –plane.
- **The ROC does not include the pole.**



Example: Sum of two right sided sequences

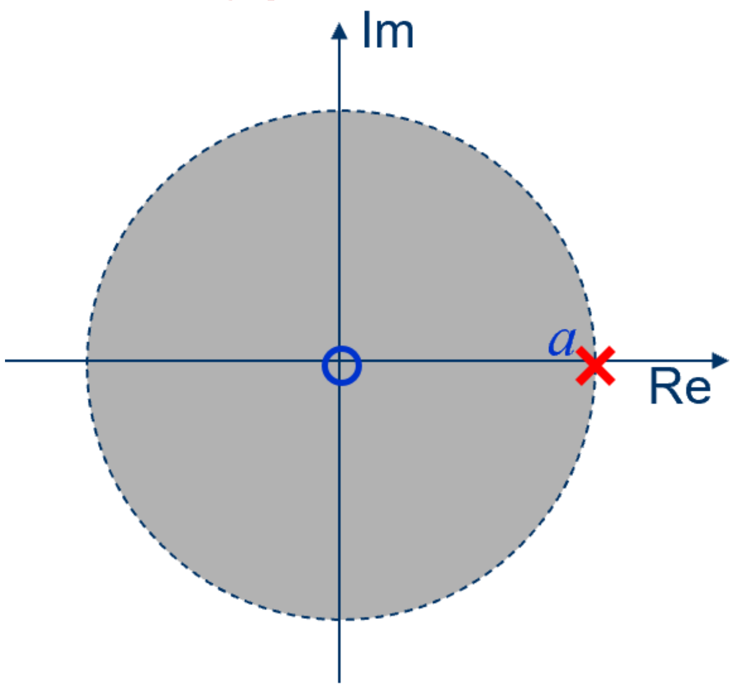
- Consider a sum of two causal sequences $x(n) = (1/2)^n u(n) + (-1/3)^n u(n)$ with z −transform $X(z) = \frac{z}{z-1/2} + \frac{z}{z+1/3}, |z| > |1/2| \cap |z| > |1/3| \Rightarrow |z| > |1/2|$.
- The above is also a **right-sided sequence**.
- Observe that the ROC is bounded by the maximum in magnitude pole and is again **the exterior of a circle** located on the z −plane.
- **The ROC does not include any poles.**



zeros are at 0 and 1/12; find why

Left-sided sequences

- Consider the well known **anti-causal sequence** $x(n) = -a^n u(-n-1)$ with z –transform $X(z) = \frac{z}{z-a}, |z| < |a|$.
- The above is called a **left-sided sequence**.
- Observe that the ROC is bounded by the pole and is **the interior of a circle** located on the z –plane.
- **The ROC does not include any poles.**



Imperial College London

Two-sided (non-causal) exponential sequence example

- Consider the **two-sided sequence** $x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$.
  - $\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{1}{1-\left(-\frac{1}{3}z^{-1}\right)} = \frac{1}{1+\frac{1}{3}z^{-1}}$

    this is the well-known geometric series formula

    it is valid only if the argument inside the sum has magnitude lower than 1

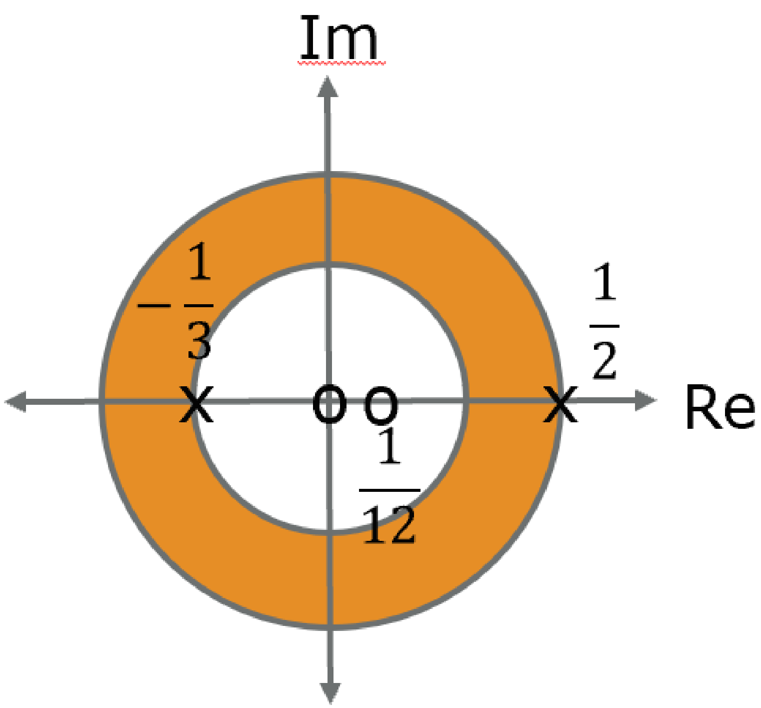
    $$\text{ROC: } \left|-\frac{1}{3}z^{-1}\right| < 1 \Rightarrow \left|\frac{1}{3z}\right| < 1 \Rightarrow \frac{1}{3} < |z|$$
  - $\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}$

    try to prove this; use the original geometric series formula

    $$\text{ROC: } \left|\frac{1}{2}z^{-1}\right| > 1 \Rightarrow \frac{1}{2} > |z|$$
  - $X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$
  - Final ROC: $\frac{1}{3} < |z| < \frac{1}{2}$
- The above transfer function is non-causal and unstable (**why?**)

Two-sided exponential sequence example cont.

- Observe that the ROC is bounded by the poles and it is a **ring**.



Finite length sequences

- Consider the **finite length** sequence:

  $$x[n] = \begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$$

  We assume that a is real and $|a| < 1$.

- The z –transform is

  $$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}.$$

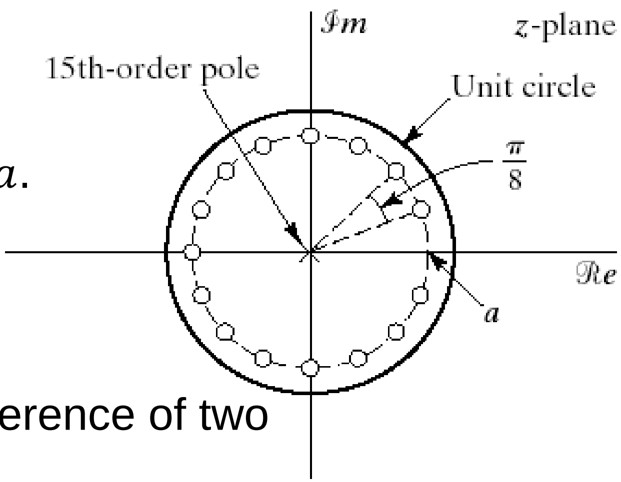
  - Suppose that $N = 16$.
  - We have **15 poles** at 0.
  - To find the zeros we set

    $$z^N - a^N = 0 \Rightarrow z = ae^{j\frac{2\pi}{N}k}, k = 0,1,\ldots,N-1.$$

  - For $k = 0$, the zero at a cancels with the pole at a.
    Therefore, we have $N - 1$ zeros.
    In the case $N = 16$ we have 15 zeros at

    $$ae^{j\frac{2\pi}{16}k} = ae^{j\frac{\pi}{8}k}, k = 1, \ldots, N-1.$$

  - The zeros have equal magnitude. The phase difference of two consecutive zeros is $\frac{2\pi}{N}$. For $N = 16$, $\frac{2\pi}{N} = \frac{\pi}{8}$.

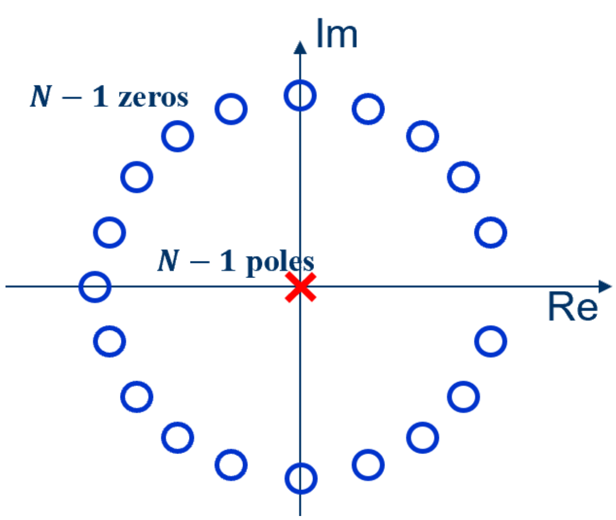


Finite length sequences cont.

- For the z –transform

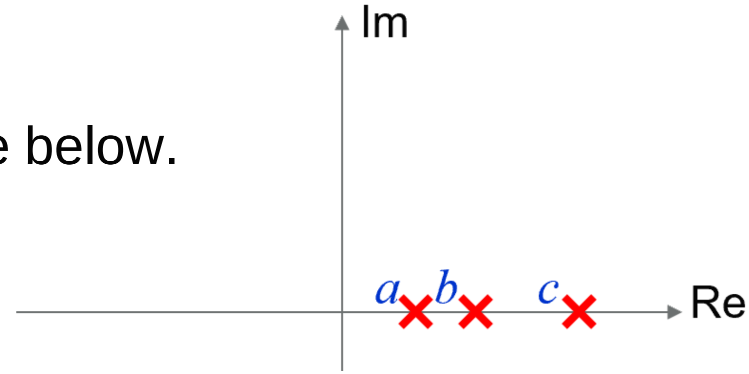
$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

  - $X(0) = \infty$. The 0 is a multiple pole.
  - All other values of z apart from 0 are permitted.
  - Therefore, the ROC is: $0 < |z|$
  - Observe again:the ROC does not include any pole.

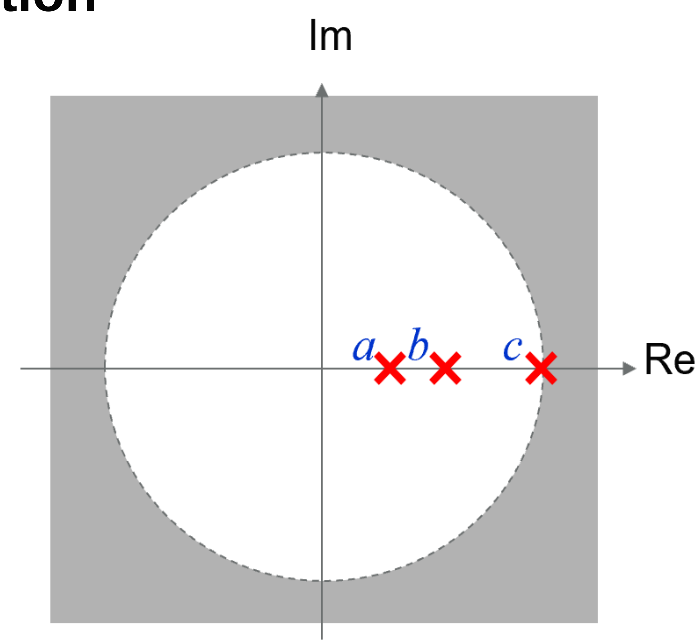


Problem

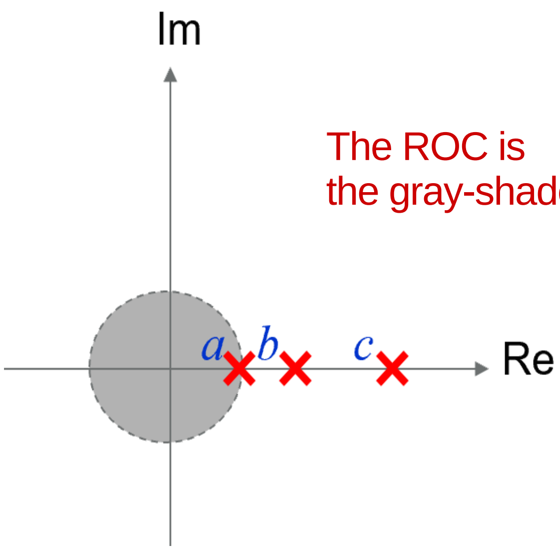
- Consider a rational z –transform with the pole-zero pattern shown in the figure below. Find the possible ROCs.



Solution



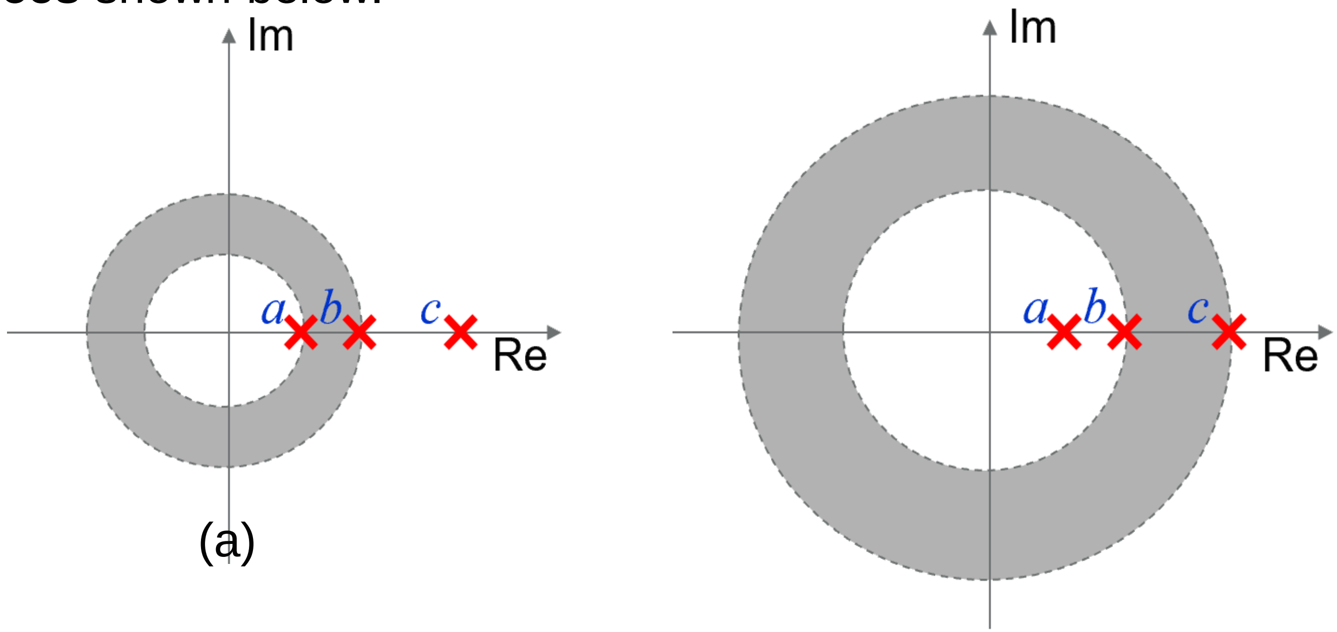
Right-sided sequence



Left-sided sequence

Solution cont.

- For the previous pole-zero plot we also have two possibilities of **two-sided sequences** shown below.



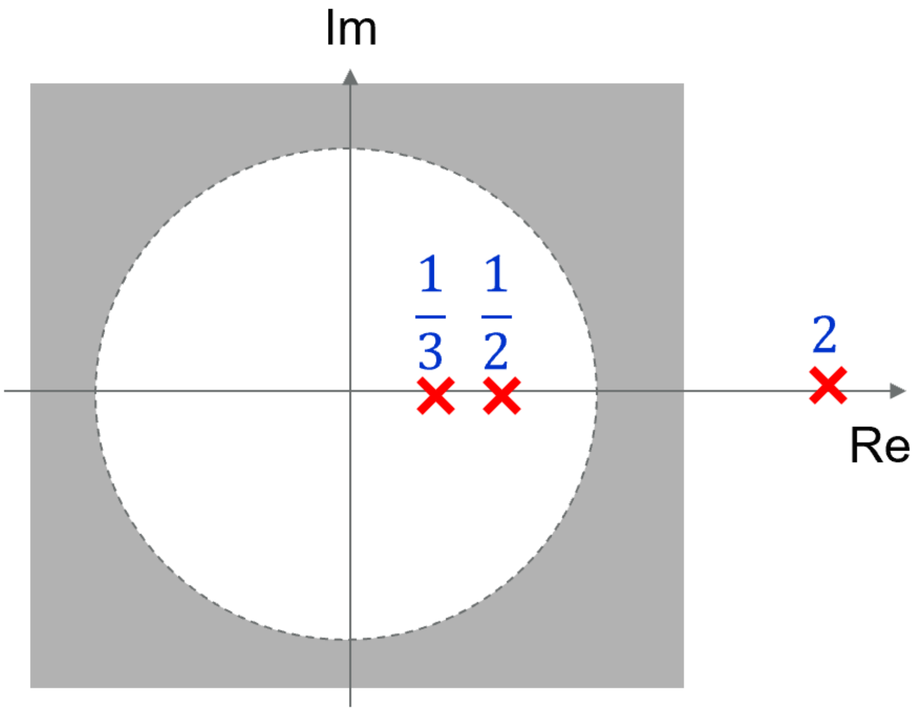
- Right-sided $x[n] = (a^n + b^n + c^n)u[n]$
- Left−sided $x[n] = -(a^n + b^n + c^n)u[-n-1]$
- Two-sided $x[n] = a^n u[n] - (b^n + c^n)u[-n-1]$
- Two-sided $x[n] = (a^n + b^n)u[n] - c^n u[-n-1]$

Properties of the ROC of z-transform

- The ROC is a **ring** or **disk** in the $z-$plane centered at the origin.
- DTFT exists if and only if the ROC includes the unit circle.
- The ROC cannot contain any poles.
- The ROC for finite-length sequences is the entire $z-$plane except possibly $z=0$ and $z=\infty$. (Non-causal systems might have poles at +oo.)
- The ROC for a right-sided sequence extends outwards from the outermost pole possibly including $z=\infty$. please make sure you understand why
- The ROC for a left-sided sequence extends inwards from the innermost pole possibly including $z=0$.
- The ROC of a two-sided sequence is a ring bounded by poles or it doesn't exist.
- The ROC must be a connected region.
- A $z-$transform does not uniquely determine a sequence without specifying the ROC.

Problem

- Shown in figure below is the pole-zero plot for the z –transform $X(z)$ of a sequence. (There are no zeros shown).
- Determine what can be inferred about the associated ROC from each of the following statements.
  - $x[n]$ is right-sided.
  - $x[n]$ is left-sided.
  - The DTFT of $x[n]$ converges.
  - The DTFT of $x[n]$ does not converge.



causal: right-sided
anti-causal: left-sided
non-causal: two-sided

Solution

- $x[n]$ is right-sided. In that case, the ROC extends outwards from the outermost pole possibly including $z = \infty$. Therefore, ROC: $|z| > 2$.

  The sequence is $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]$. causal and unstable

- $x[n]$ is left-sided. In that case, the ROC extends inwards from the innermost pole possibly including $z = 0$. Therefore, ROC: $|z| < 1/3$.

  The sequence is $x[n] = -\left(\frac{1}{3}\right)^n u[-n-1] - \left(\frac{1}{2}\right)^n u[-n-1] - 2^n u[-n-1]$. anti-causal and unstable

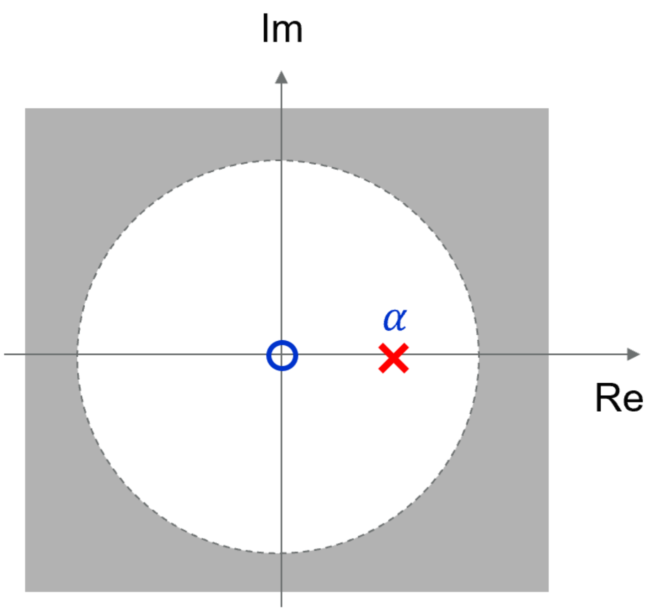
- The DTFT of $x[n]$ converges. In that case the ROC must include the unit circle. Therefore, ROC: $\frac{1}{2} < |z| < 2$.

  The sequence is $x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[n] - 2^n u[-n-1]$. non-causal and stable

- The DTFT of sequence $x[n]$ does not converge. In that case the ROC does not include the unit circle. We have 3 scenarios.
  - $|z| < \frac{1}{3}$ (sequence given above)
  - $|z| > 2$ (sequence given above)
  - $1/3 < |z| < 1/2$ and $x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1] - 2^n u[-n-1]$. non-causal and unstable

Problem

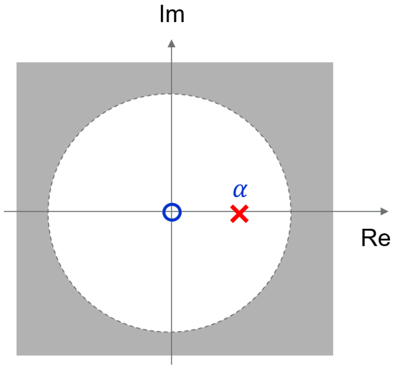
- The transfer function $H(z)$ of a system has a zero-pole configuration shown below. In this plot one zero and one pole are shown.
  1. Sketch $\left|H(e^{j\omega})\right|$ as the number of zeros at $z=0$ increases from 1 to 5.
  2. How does the number of zeros affect the phase of $H(e^{j\omega})$? Justify your answer.
  3. Find the region on the z –plane for which $|H(z)|=1$.



Solution

1. From the zero-pole location we immediately formulate the z –transform as:

$$H(z) = \frac{z}{z-a}$$



and the frequency response as:

$$H(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega}-a} = \frac{e^{j\omega}}{\cos(\omega)+j\sin(\omega)-a}$$

Therefore, the amplitude response is:

$$|H(e^{j\omega})| = \left|\frac{e^{j\omega}}{e^{j\omega}-a}\right| = \frac{|e^{j\omega}|}{|e^{j\omega}-\alpha|} = \frac{1}{|\cos(\omega)+j\sin(\omega)-\alpha|} = \frac{1}{\sqrt{(\cos(\omega)-\alpha)^2+(\sin(\omega))^2}}$$

$$\frac{1}{\sqrt{1+\alpha^2-2\cos(\omega)\cdot\alpha}}$$

In case of $n > 1$ zeros $H(z) = \frac{z^n}{z-a}$

$$|H(e^{j\omega})| = \left|\frac{e^{jn\omega}}{e^{j\omega}-a}\right| = \frac{|e^{jn\omega}|}{|e^{j\omega}-\alpha|} = \frac{1}{|\cos(\omega)+j\sin(\omega)-\alpha|} = \frac{1}{\sqrt{1+\alpha^2-2\cos(\omega)\cdot\alpha}}$$

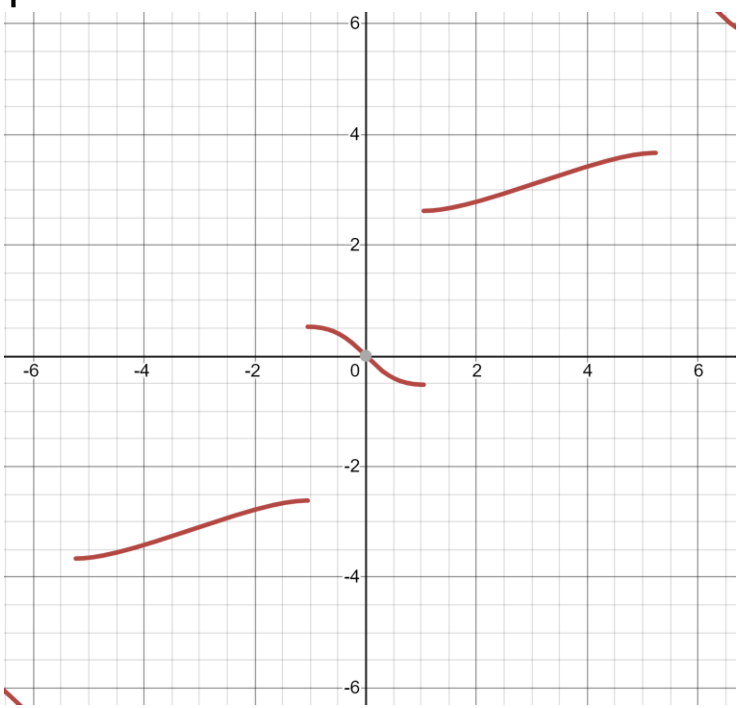
We observe that incorporating extra zeros at $z = 0$ into the transfer function does not affect the amplitude response.

Solution cont.

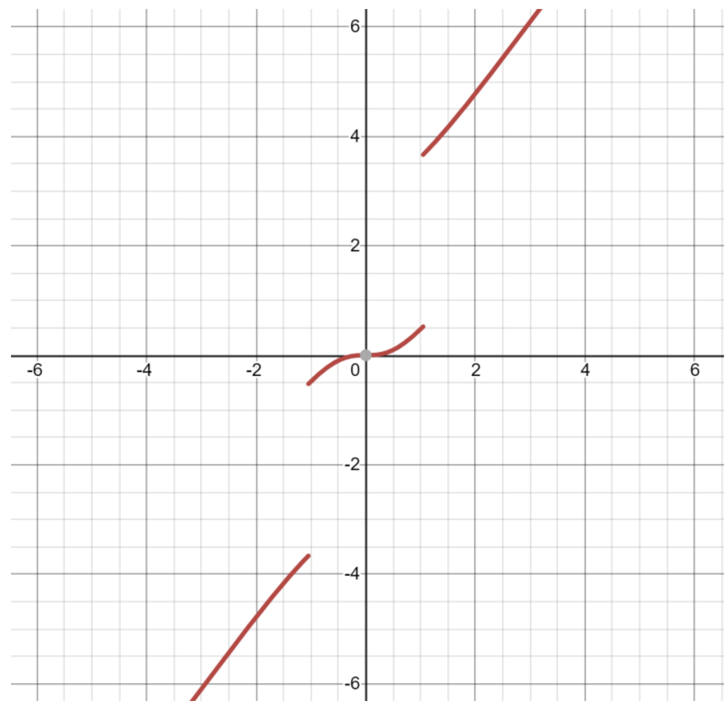
2. The phase response for $n > 1$ zeros is:

$$\measuredangle H\left(e^{j\omega}\right) = n\omega - \arctan\left(\frac{\sin(\omega)}{\cos(\omega) - \alpha}\right)$$

We observe that, by introducing k additional zeros, we add the linear term $k\omega$ to the phase.

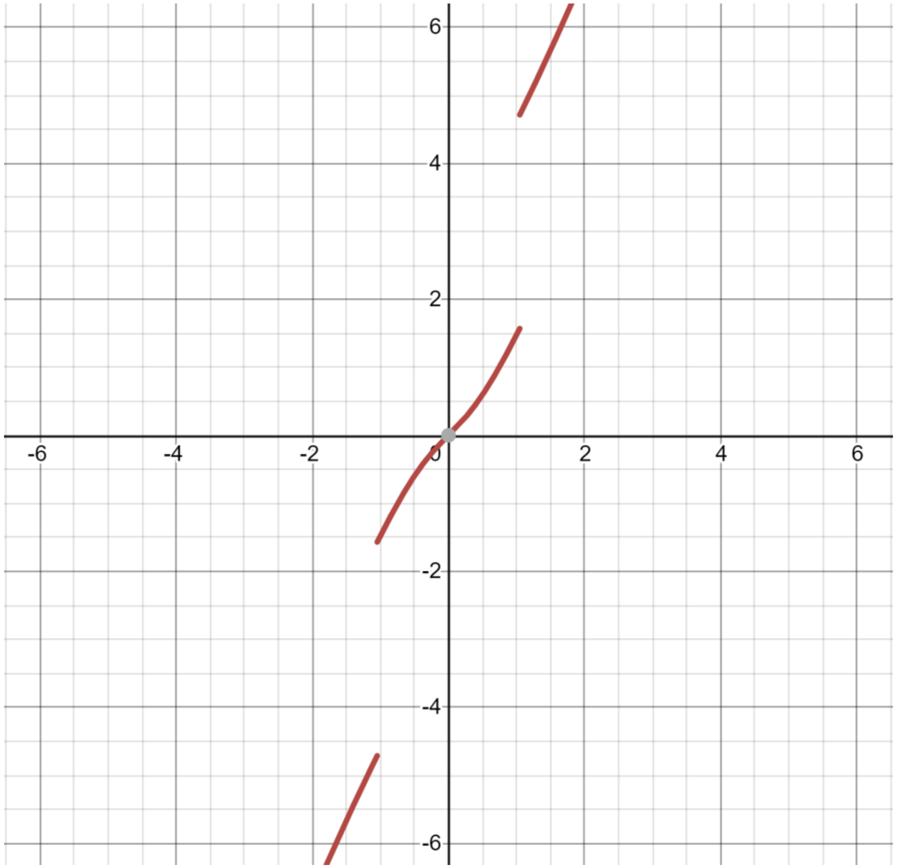


$n = 1$

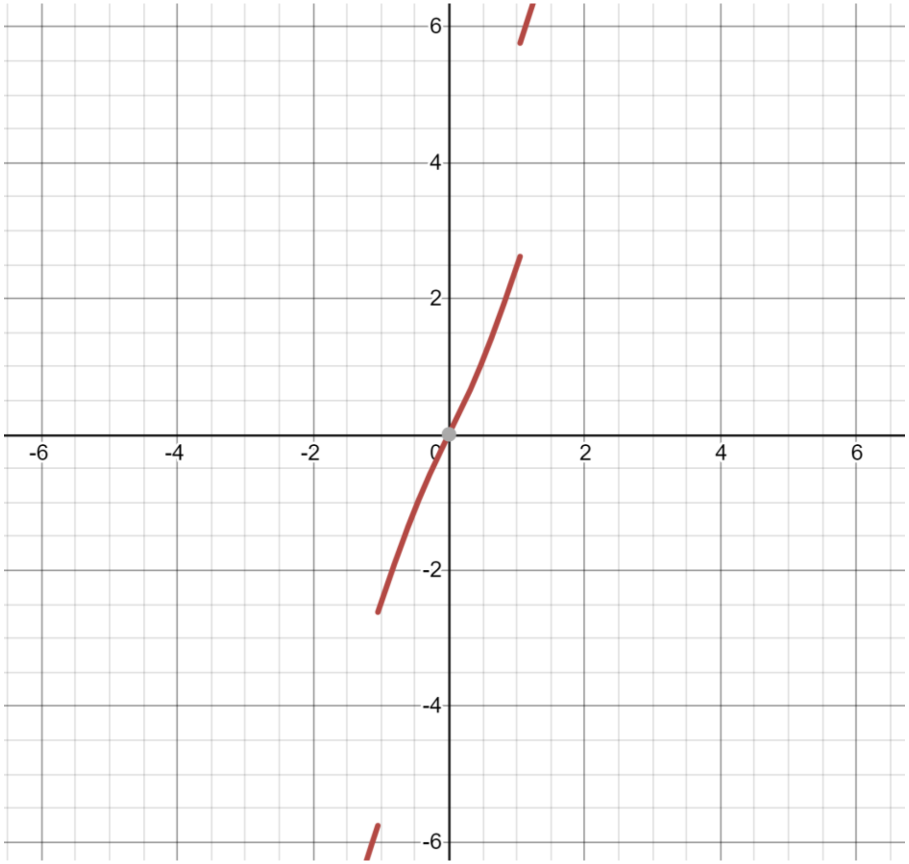


$n = 2$

Solution cont.



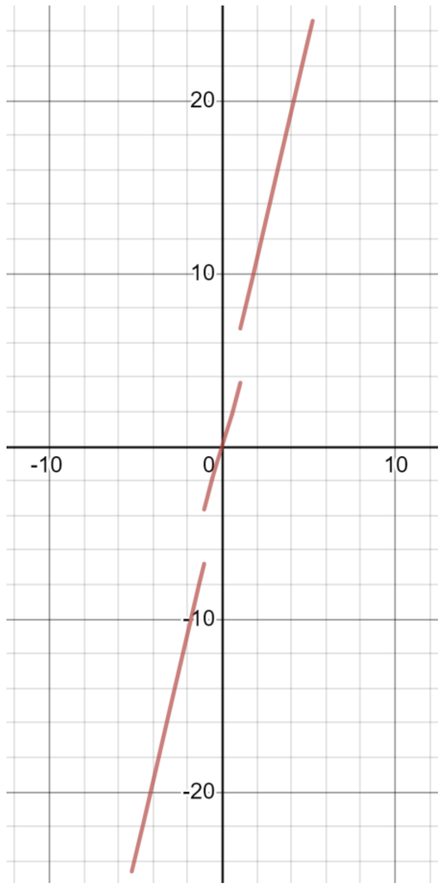
$n = 3$

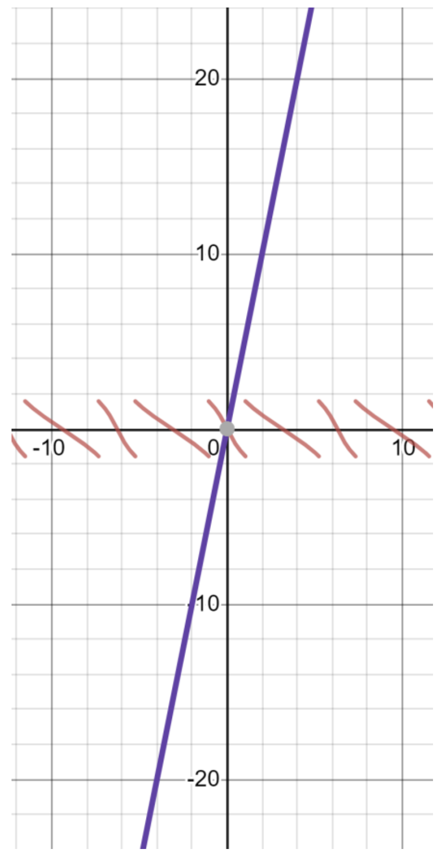


$n = 4$

Solution cont.

We observe that for $n = 5$ the phase is almost linear (left plot). On the right plot we depict separately the functions $n\omega$ **(purple)** and $-\arctan(\frac{\sin(\omega)}{\cos(\omega)-\alpha})$ **(red)**.





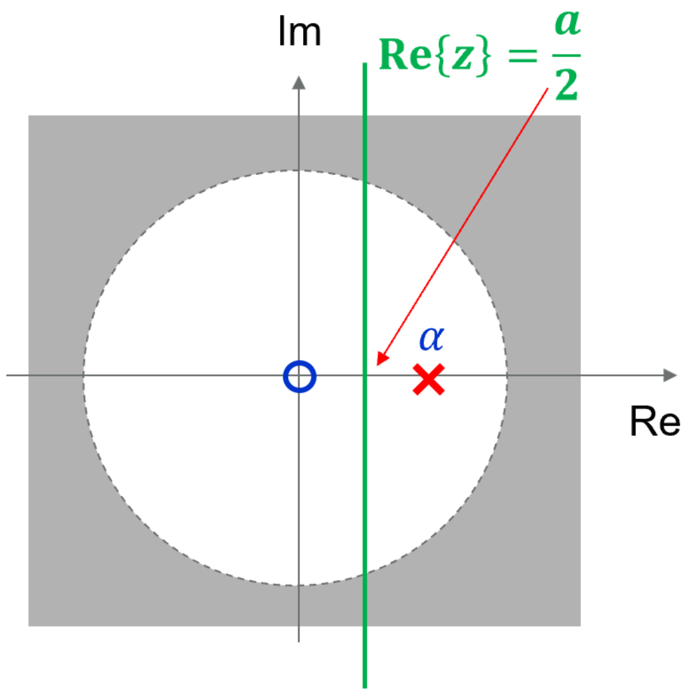
$n = 5$

Solution cont.

3. $H(z) = \frac{z}{z-a}$

$|H(z)| = 1 \Rightarrow |H(z)|^2 = H(z)H^*(z) = 1 \Rightarrow \frac{z}{z-a}\frac{z^*}{z^*-a} = \frac{|z|^2}{|z|^2+\alpha^2-az^*-az} =$
$\frac{|z|^2}{|z|^2+\alpha^2-2a\text{Re}\{z\}} = 1 \Rightarrow \alpha^2 - 2a\text{Re}\{z\} = 0 \Rightarrow \alpha^2 = 2a\text{Re}\{z\} \Rightarrow a = 2\text{Re}\{z\}$

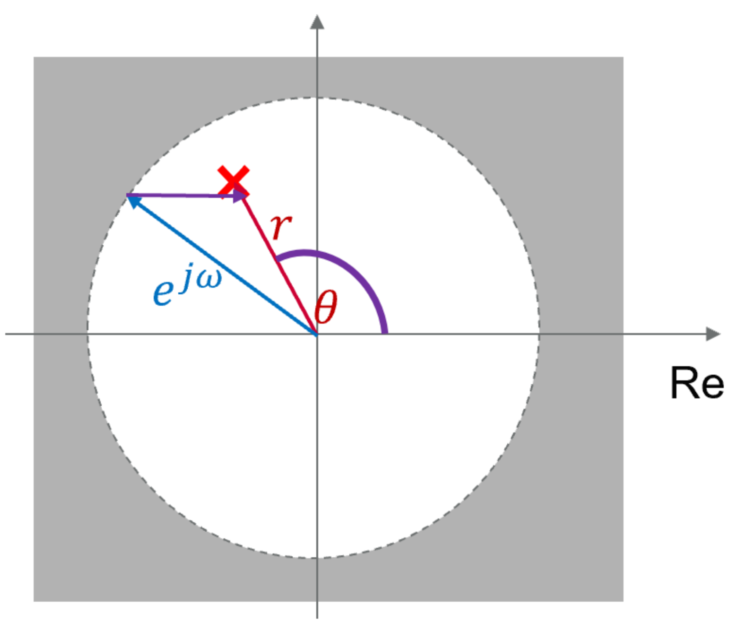
$\Rightarrow \text{Re}\{z\} = \frac{a}{2}$



Effect of a pole on the magnitude of the frequency response

- Consider a transfer function with a single complex pole at $\boldsymbol{a = re^{j\theta}}$. The transfer function is $H(z) = \frac{1}{z-a} = \frac{1}{z-re^{j\theta}}$. In that case the frequency response is $H(e^{j\omega}) = \frac{1}{e^{j\omega}-re^{j\theta}}$.
  The vector $\boldsymbol{e^{j\omega}} - \boldsymbol{re^{j\theta}}$ is shown with **purple** in the figure below.
  - We see that the magnitude of the function $H(e^{j\omega})$ is maximum when the magnitude of the vector $e^{j\omega} - re^{j\theta}$ is minimum. This happens when the vector $e^{j\omega}$ is on top on the vector $re^{j\theta}$ (i.e., they are parallel), or in other words, $\omega = \theta$.
  - Furthermore, for $\omega = \theta$, $\boldsymbol{e^{j\omega} - re^{j\theta}}$ **decreases** (and consequently the magnitude of $H(e^{j\omega})$ increases) **as r approaches the unit circle**.
  - Therefore, if we want a peak in the amplitude response at a certain frequency we must place a pole with a phase equal to that frequency and magnitude as close to 1 as possible.

There is analogy with continuous systems.
We will see this later.



Effect of a pole on the magnitude of the frequency response. Example

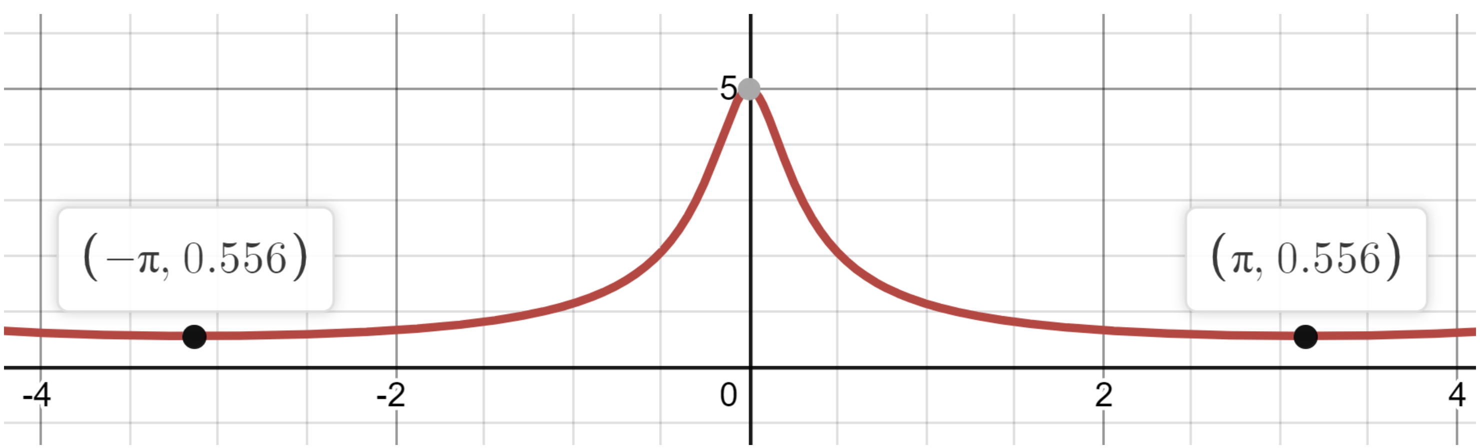
- Consider the transfer function of the last example $H(z) = \frac{z}{z-a}$. In that case the pole a is real and positive. The amplitude response is:

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1 + \alpha^2 - 2\cos(\omega) \cdot \alpha}}$$

- Based on the previous analysis, since the pole is real and positive, it can be written as $a = ae^{j0}$.
- We should expect that the maximum of the amplitude response occurs at $\omega = 0$ and is

$$\left|H(e^{j0})\right| = \frac{1}{1-a}$$

- The amplitude response is shown below for $a = 0.8$.



Effect of a zero on the magnitude of the frequency response

- Based on the comments of the previous slide, if we want a local minimum in the amplitude response at a certain frequency, we must place a zero with a phase equal to that frequency and magnitude as close to 1 as possible.
  - There is analogy with continuous systems.
  - Observe that zeros can be located outside the unit circle; **stability of a linear system is solely determined by the position of the poles.**

- **Example:** A system has transfer function:

$$H(z) = (1-2z^{-1})(1-3z^{-1}) = 1-5z^{-1}+6z^{-2}$$

By taking the inverse z –transform we obtain:

$$h[n] = \delta[n] - 5\delta[n-1] + 6\delta[n-2]$$

Problem:

- Create the pole-zero plot
- Find the ROC

Solution

- **Example:** A system has transfer function:

$$H(z) = \left(1 - 2z^{-1}\right)\left(1 - 3z^{-1}\right) = 1 - 5z^{-1} + 6z^{-2}$$

By taking the inverse z –transform we obtain:

$$h[n] = \delta[n] - 5\delta[n-1] + 6\delta[n-2]$$

Solution:

$$H(z) = \frac{(z-2)(z-3)}{z^2}$$

Poles: a double pole at 0 with multiplicity 2.

Zeros: at 2 and 3.

ROC: $|z| > 0$