IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2022

# MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS

# SOLUTIONS

1. (a) i. The vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ are linearly independent. Moreover, since the solution allows for an arbitrary linear combination of them, this means that they are in the null space of $\mathbf{A} \in \mathbb{R}^{3\times 3}$. So rank of $\mathbf{A}$ is one.

ii. For $a = 1$ and $b = -2$ we have

$$\mathbf{A} \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 9 \\ -4.5 \\ -4.5 \end{bmatrix}$$

So the last column of $\mathbf{A}$ is $[1, -0.5, -0.5]^T$. For $a = 1$ and $b = 2.5$ we have

$$\mathbf{A} \begin{bmatrix} 4.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -4.5 \\ -4.5 \end{bmatrix}$$

so the first column of $\mathbf{A}$ is $[2, -1, -1]^T$. Finally the second column can be found by ensuring that $[1, 1, 0]^T$ is in the null space of $\mathbf{A}$. In conclusion we have:

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & -0.5 \\ -1 & 1 & -0.5 \end{bmatrix}$$

iii. One way to find the minimum norm solution is by finding the vector $\mathbf{x}$ that is orthogonal to both $\mathbf{n}_1 = [1, 1, 0]^T$ and $\mathbf{n}_2 = [1, 0, -2]^T$, and that solves $\mathbf{Ax} = \mathbf{b}$. So the two orthogonality constraints lead to $\mathbf{x} = [a, -a, a/2]$. Moreover, the condition $\mathbf{Ax} = \mathbf{b}$ yields $a = 2$ and so $\mathbf{x}_{MN} = [2, -2, 1]^T$.

(b) i. Assuming $p(t) = at^2 + bt + c$, we need to solve:

$$\begin{bmatrix} y_1 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} t_1^2 & t_1 & 1 \\ t_4^2 & t_4 & 1 \\ t_5^2 & t_5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The matrix

$$\mathbf{A} = \begin{bmatrix} t_1^2 & t_1 & 1 \\ t_4^2 & t_4 & 1 \\ t_5^2 & t_5 & 1 \end{bmatrix}$$

is a Vandermonde matrix therefore its determinant is $(t_1 - t_4)(t_1 - t_5)(t_4 - t_5)$ and is full rank when $t_i \neq t_j$. So the solution is unique.

ii. We need to find $a_1, b_1$ satisfying

$$\begin{bmatrix} y_1 \\ y_4 \end{bmatrix} = \begin{bmatrix} t_1 & 1 \\ t_4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

which yields $a = 0.75$ and $b = -1$.

iii. We are after coefficient $a_2, b_2$ such that $\sum_i |y_i - (a_2 t_i + b_2)|^2$ is minimized. We build the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 4 & 1 \\ 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The least-squares solution is given by $\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{y}$ where $\mathbf{c} = [a_2, b_2]^T$ and $\mathbf{y} = [1.5, 2, 0.5, 0.5, -1]^T$. We have that:

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 46 & 12 \\ 12 & 5 \end{bmatrix}.$$

and

$$(\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{\det(\mathbf{A}^T\mathbf{A})} \begin{bmatrix} 5 & -12 \\ -12 & 46 \end{bmatrix}.$$

Consequently

$$\mathbf{c} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}.$$

iv. The solution is unique since $\mathbf{A}$ is full column rank.

v. $e_2 = 1$ and $e_2 < \sum_{i=1}^{5} |y_i - a_1 t_i - b_1|^2 = 2.125$

2. (a) i. The matrix has rank two so $\sigma_3 = 0$. We then find

$$\sigma_2 = (trace(\mathbf{A^T A}) - \sigma_1^2)^{1/2} = 3.$$

ii. We have

$$\mathbf{u}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{v}_2 = [1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]^T.$$

Moreover, $\mathbf{u}_3$ must have norm one and must be orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. Consequently $\mathbf{u}_3 = [1/\sqrt{2}, 0, -1/\sqrt{2}]^T$. In this case we also have $\mathbf{v}_1 = \mathbf{u}_1$ and $\mathbf{v}_3 = \mathbf{u}_3$.

iii. The solution is $\mathbf{x} = \mathbf{A}^\dagger \mathbf{b}$ with $\mathbf{A}^\dagger = \mathbf{V}\mathbf{\Sigma}^\dagger \mathbf{U}^T$ and

$$\mathbf{\Sigma}^\dagger = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $\mathbf{b} = [-0.5, 2, -0.5]^T$.

(b) i. When $c = 0.5$, the matrix is not full row rank, so we cannot guarantee that a solution exists.

ii. We know that $\mathbf{x}_{MN} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{b}$

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

and

$$(\mathbf{A}\mathbf{A}^T)^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

Consequently, $\mathbf{x}_{MN} = [-3, 0, 5]^T$.

(c) We solve this problem from first principles. First of all we can write the constraint as follows $\mathbf{C}\mathbf{x} = 1$ with $\mathbf{C} = [1/3\ 1/3\ 1/3]$. The matrix $\mathbf{C}$ has null space given by span($\mathbf{n}_1 = [1, -1, 0]^T, \mathbf{n}_2 = [1, 0, -1^T]$). We then find all the possible solutions to this linear system which are given by $[1, 1, 1]^T + a\mathbf{n}_1 + b\mathbf{n}_2$. We finally find the constants $a, b$ that minimise $\|\mathbf{y} - \mathbf{x}\|^2$. So we need to find $a, b$ that minimise $(y_1 - x_1 - a - b)^2 + (y_2 - x_2 + a)^2 + (y_3 - x_3 + b) = (-a - b)^2 + a^2 + (b - 2)^2$. We take the derivative with respect to $a$ and $b$ and equal to zero which yields $a = 2/3$ and $b = 4/3$. Finally $\mathbf{x} = [5/3, 5/3, -1/3]^T$.

3. (a) The matrix $\mathbf{Q}$ is symmetric and after row reduction we have:

$$\mathbf{Q} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1.5 & -1 \\ 0 & 0 & 4/3 \end{bmatrix}$$

All pivots are positive therefore $\mathbf{Q}$ is positive definite.

(b) Assume the vectors are linearly dependent, that means that we have found a set of non-zero coefficients $\alpha_0, \alpha_1, ..., \alpha_k$ such that

$$\alpha_0 \mathbf{d}_0 + \alpha_1 \mathbf{d_1} + ... + \alpha_k \mathbf{d}_k = 0. \tag{1}$$

We now multiply (1) with $\mathbf{d}_i^T \mathbf{Q}$, leading to

$$\mathbf{d}_i^T \mathbf{Q}(\alpha_0 \mathbf{d}_0 + \alpha_1 \mathbf{d_1} + ... + \alpha_k \mathbf{d}_k) = \alpha_i \mathbf{d}_i^T \mathbf{Q} \mathbf{d}_i,$$

where we have used the fact that for all $i \neq j$, $\boldsymbol{d}_i^T \boldsymbol{Q} \boldsymbol{d}_j = 0$. However, we also know that $\boldsymbol{d}_i^T \boldsymbol{Q} \boldsymbol{d}_i \neq 0$. Therefore (1) cannot be satisfied and so the vectors are linearly independent.

(c) Given $\mathbf{d}_0 = [1, 0, 0]$, we find $\mathbf{d}_1 = [d_{1,1}, d_{1,2}, d_{1,3}]^T$ by imposing $\mathbf{d}_1^T \mathbf{Q} \mathbf{d}_0 = 0$. This yields the constraint $3d_{1,1} + d_{1,3} = 0$, so we pick $\mathbf{d}_1 = [1, 0, -3]^T$. Finally $\mathbf{d}_2$ must satisfy $\mathbf{d}_2^T \mathbf{Q} \mathbf{d}_0 = 0$ and $\mathbf{d}_2^T \mathbf{Q} \mathbf{d}_1 = 0$ leading to $\mathbf{d}_2 = [1, 4, -3]^T$.

(d) i. we have $\mathbf{Qx} = \sum_{i=0}^{n-1} \beta_i \mathbf{Q} \mathbf{d}_i$, and so $\sum_{i=0}^{n-1} \beta_i \mathbf{Q} \mathbf{d}_i = \mathbf{b}$. We now multiply both side of this last equation with $\mathbf{d}_j^T$ and use $\boldsymbol{d}_i^T \boldsymbol{Q} \boldsymbol{d}_j = 0$ for $i \neq j$ to obtain $\beta_j \mathbf{d}_j^T \mathbf{Q} \mathbf{d}_j = \mathbf{d}_j^T \mathbf{b}$ yielding

$$\beta_j = (\mathbf{d}_j^T \mathbf{Q} \mathbf{d}_j)^{-1} \mathbf{d}_j^T \mathbf{b}$$

ii. We note that $\mathbf{d}_1^T \mathbf{b} = \mathbf{d}_2^T \mathbf{b} = 0$, therefore the solution is simply given by $\mathbf{x} = \beta_0 \mathbf{d}_0 = [2, 0, 0]^T$.

(e) i. The adjacency matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

ii. We use the first iteration to compute the degree of the nodes. Therefore $\mathbf{u}_0 = [1, 1, 1, 1, 1, 1]^T$ leading to $\mathbf{u}_1 = \mathbf{A}\mathbf{u}_0 = [1, 3, 2, 3, 2, 1]^T$. The second iteration then gives us $\mathbf{u}_2 = \mathbf{A}\mathbf{u}_1 = [3, 6, 6, 7, 4, 2]^T$. Therefore the node 4 is the central node in the graph. We note that in theory we should normalise the vectors at each iteration for stability reasons. However, given the small number of iterations this is not necessary. Students get full mark if they compute another iteration and if they claim that node 2 and node 4 have the same centrality.