IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2023

Mathematics for Signals and Systems

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates:

For questions involving matrix manipulations or calculations, please provide answers showing intermediate calculations steps.

Given a matrix $\mathbf{A}$ with full column rank, the orthogonal projection onto the space spanned by the columns of $\mathbf{A}$ is given by: $\mathbf{P} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$.

The inverse of an upper triangular matrix is also an upper triangular matrix.

The Questions

1. (a) Consider the system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{3\times 3}$ and $\mathbf{b} = [4, 8, 4]^T$. The complete solution to this system is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

for arbitrary a and b.

i. Find the rank of $\mathbf{A}$ [2]

ii. Find $\mathbf{A}$ [2]

iii. Find the minimum norm solution. [2]

(b) Assume that $\mathbf{A} \in \mathbb{R}^{n\times n}$. If $\mathbf{Ax} = \mathbf{0}$ has a non-zero solution, show that $\mathbf{A}^T\mathbf{y} = \mathbf{b}$ fails to be solvable for some choices of $\mathbf{b}$. Construct an example of $\mathbf{A}$ and $\mathbf{b}$. [2]

(c) Determine the four fundamental subspaces of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[4]

Question 1 continues on the next page

(d) Consider the system $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

and $\mathbf{b} = [1, 7]^T$. A basis for the null space of $\mathbf{A}$ is $\mathbf{n} = [0, -1, 2]^T$ and a possible solution to the system is $\mathbf{x}_1 = [1, 2, 3]^T$. Find an alternative solution $\mathbf{x}_2$ such that $\|\mathbf{x}_2\| < \|\mathbf{x}_1\|$. [2]

(e) The linear combination of $\mathbf{v} = [1, 1, 0]^T$ and $\mathbf{w} = [0, 1, 1]^T$ fills a plane in $\mathbb{R}^3$.

i. Find a vector $\mathbf{z}$ that is perpendicular to the plane generated by $\mathbf{v}$ and $\mathbf{w}$. [2]

ii. Find a vector $\mathbf{u}$ that is not on the plane and that is different from $\alpha\mathbf{z}$ for an arbitrary constant α. [2]

(f) Show that if a matrix $\mathbf{S}$ is symmetric and orthogonal then $\mathbf{S}^2 = \mathbf{I}$. [2]

2. (a) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 12 & 4 \end{bmatrix}.$$

i. Find the matrix $\mathbf{P}_C$ that models the orthogonal projection onto the space spanned by the columns of $\mathbf{A}$. [3]

ii. Now, find the projection matrix $\mathbf{P}_R$ onto the row space of the above matrix. [3]

iii. Assume $\mathbf{x} = [22, 33, 11]^T$. Is the vector $\hat{\mathbf{x}} = [12, 36, 12]^T$ the orthogonal projection of $\mathbf{x}$ onto the space spanned by the rows of $\mathbf{A}$? Briefly justify your answer. [3]

(b) Consider again the matrix $\mathbf{A}$ of part (a).

i. Can you guarantee that the system $\mathbf{A}\mathbf{x} = \mathbf{b}$, has always a solution? Briefly justify your answer. [2]

ii. Can you guarantee that the system $\mathbf{A}\mathbf{x} = \mathbf{P}_C\mathbf{b}$, has always a solution? Briefly justify your answer. [2]

iii. Assume $\mathbf{b} = [1, 0]^T$, find a solution to $\mathbf{A}\mathbf{x} = \mathbf{P}_C\mathbf{b}$. [2]

(c) i. If $\mathcal{P}$ is a projection matrix, prove so is $\mathbf{I} - \mathcal{P}$, where $\mathbf{I}$ is the identity matrix. [5]

3. (a) Let V be the vector space consisting of all functions of the form

$$f(x) = \alpha + \beta x + \gamma x^2,$$

for any $\alpha, \beta, \gamma \in \mathbb{R}$. Consider the linear transformation

$$L = \frac{df}{dx} - f(x).$$

i. Find the matrix representing the linear mapping L with respect to the basis $\{1, x, x^2\}$. [3]

ii. Use the answer from part i. to find one solution to the following differential equation:

$$\frac{df}{dx} - f(x) = 3 - x + 2x^2.$$

[3]

(b) Find a basis for the following subspaces of $\mathbb{R}^4$

i. The vectors for which $x_1 = x_3 + 2x_4$ [1]

ii. The vectors for which $x_1 + x_2 = 0$ [1]

Question 3 continues on the next page

(c) Consider the matrix

$$\mathbf{Q} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ a & 0.7 & 0.5 \\ 0 & b & 0.3 \end{bmatrix}.$$

i. Find the values of a and b so that $\mathbf{Q}$ models a Markov or "transition" matrix. Briefly justify your answer. [3]

ii. Assume $\mathbf{Q}$ models the fraction of population that moves every year across three different towns. Find the steady state of $\mathbf{Q}$. [3]

iii. Assume the total population at the beginning of your observation is $80,000$. What is the population in each town at the steady state? [3]

(d) Calculate the singular values of the following matrix

$$\mathbf{A} = \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right].$$

[3]