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Digital Signal Processing
Multirate Digital Signal Processing

Theory and Problems

DR TANIA STATHAKI

READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

Multirate Digital Signal Processing

- Until now we have come across solely with single-rate systems.
  - The sampling-rate at the input, output and all internal nodes are the same.
- There are many applications where a signal at a given sampling rate needs to be converted into another signal with a different sampling rate.
- Discrete-time systems with unequal sampling rates at various parts of the system are called **multirate systems**.

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Multirate Digital Signal Processing: Digital audio

- In digital audio three different sampling rates are presently employed.
  - $32kHz$ in broadcasting
  - $44.1kH$ in digital CD
  - $48kHz$ in DVD
- Conversion of sampling rates of audio signals among the above rates is necessary in many applications.

Basic sampling rate alteration devices

- **Up-sampler** – it is used to increase the sampling rate by an integer factor.
- **Down-sampler** – it is used to decrease the sampling rate by an integer factor.
- Conventional elements such as **adders**, **multiplies**, **delays** are employed.
- Sampling periods will not be shown in the block-diagram representations of the up-sampler and the down-sampler.
  - The mathematical theory of multirate systems can be understood without bringing the sampling period or the sampling frequency into the picture.

Up-sampler
Time-domain definition

- An up-sampler with an **up-sampling factor L**, where L is a positive integer, develops an output sequence $x_u[n]$ with a sampling rate that is L times larger than that of the input sequence $x[n]$.

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

- The up-sampling operation is performed by inserting equidistant zero-valued samples between two consecutive samples of $x[n]$:

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

Note that $x_u[0] = x\left[\frac{0}{L}\right] = x[0], x_u[1] = 0, \ldots, x_u[L] = x\left[\frac{L}{L}\right] = x[1], \ldots$

- If sampling rates were explicitly depicted, the block-diagram would look as follows:

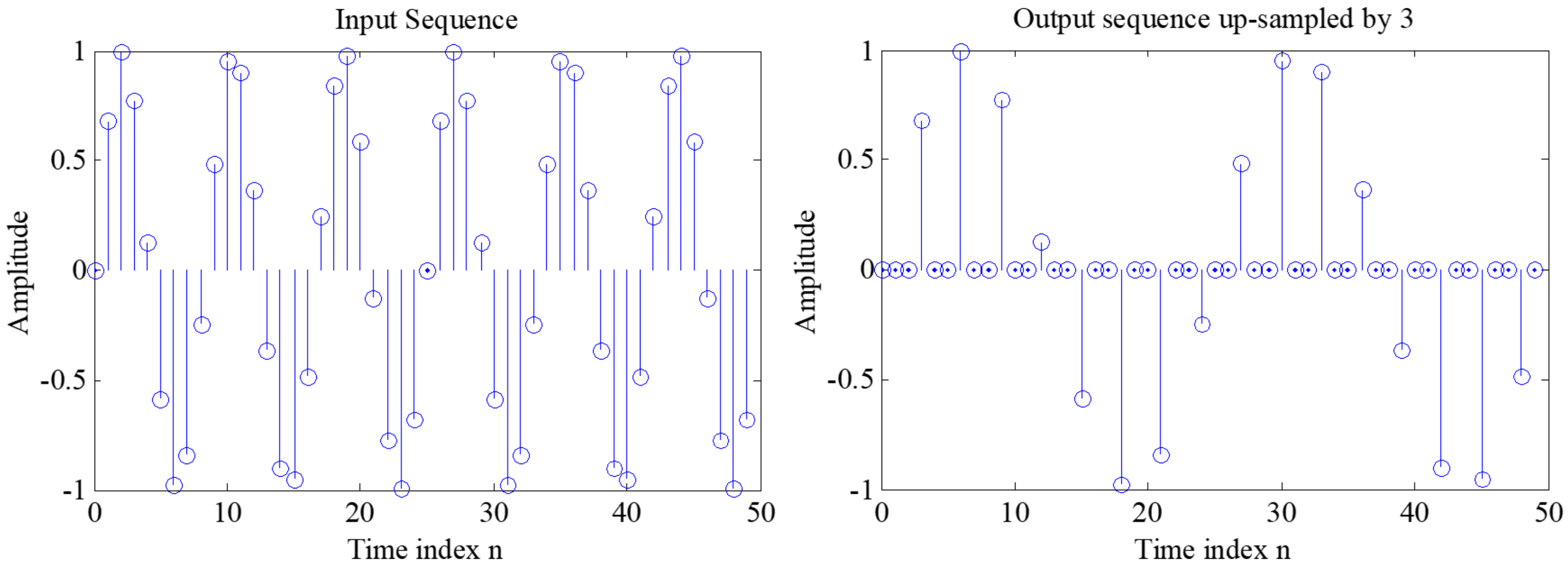
$$x[n] = x_a(nT) \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n] = \begin{cases} x_a(nT/L), & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Input sampling frequency $F_T = \frac{1}{T}$ *Output sampling frequency* $F_T' = LF_T = \frac{L}{T}$

Up-sampler Illustration

- Figure below shows the up-sampling by a factor of 3 of a sinusoidal sequence with a frequency of 0.12Hz (cycle duration 8.33 units of time).

6 cycles are shown)



- In practice, the zero-valued samples inserted by the up-sampler are replaced with appropriate nonzero values using some type of filtering process. This process is called **interpolation**.

Down-sampler
Time-domain definition

- A down-sampler with a **down-sampling factor M**, where M is a positive integer, develops an output sequence $x_d[n]$ with a sampling rate that is $\left(\frac{1}{M}\right)^{\text{th}}$ of that of the input sequence $x[n]$.

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

- The down-sampling operation is performed by keeping every M^{th} sample of the input $x[n]$ and ignoring the $M-1$ in-between samples:

$$x_d[n] = x[Mn]$$

Note that $x_d[0] = x[M \cdot 0] = x[0]$, $x_d[1] = x[M \cdot 1] = x[M]$, ...

- If sampling rates were explicitly depicted, the block-diagram would look as follows:

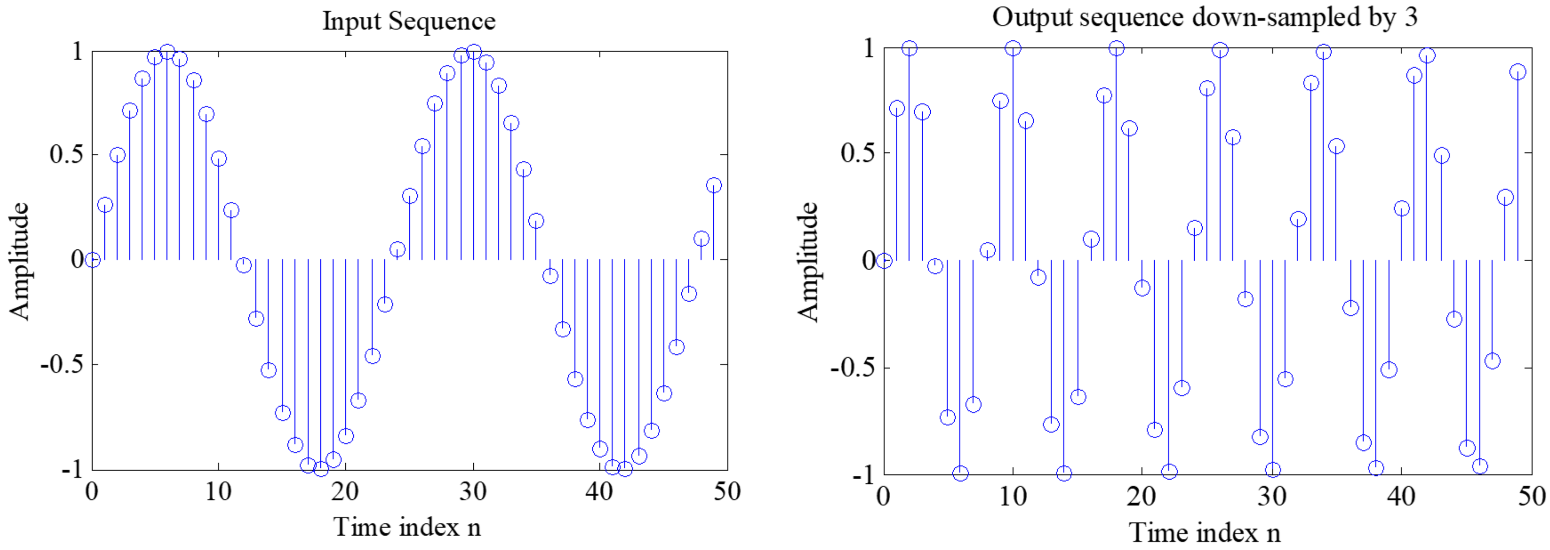
$$x[n] = x_a(nT) \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n] = x_a(nMT)$$

Input sampling frequency $F_T = \frac{1}{T}$ *Output sampling frequency* $F'_T = \frac{F_T}{M} = \frac{1}{MT}$

Down-sampler Illustration

- Figure below shows the down-sampling by a factor of 3 of a sinusoidal sequence with a frequency of $0.042Hz$.

T=23.8



Basic sampling rate alteration devices Time-variance property

- The up-sampler and the down-sampler are linear but **time-varying** discrete-time systems.
- We illustrate the time-varying property of a down-sampler.
- **The time-varying property of an up-sampler can be proven in a similar manner.**
- **From now on we will replace $x_d[n]$ with $y[n]$ to be consistent with the textbook.**
- Consider a factor-of-M down-sampler defined by $y[n] = x[Mn]$.
- Its output $y_1[n]$ for an input $x_1[n] = x[n - n_0]$ is then given by $y_1[n] = x_1[Mn] = x[Mn - n_0]$.
- From the input-output relation of the down-sampler we obtain $y[n - n_0] = x[M(n - n_0)] = x[Mn - Mn_0] \neq y_1[n] = x[Mn - n_0]$ **(time-variance proved)**.

Up-sampler
Frequency-domain characterisation

- Consider first a factor-of-2 up-sampler whose input-output relation in the time-domain is given by

$$x_u[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \dots \\ 0, & \text{otherwise} \end{cases}$$

- Its z –transform is:

$$X_u(z) = \sum_{n=-\infty}^{\infty} x_u[n]z^{-n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2]z^{-n} = \sum_{m=-\infty}^{\infty} x[m]z^{-2m} = X(z^2)$$

- Consider now a factor-of-L up-sampler. It can be shown immediately that:

$$X_u(z) = X(z^L)$$

- On the unit circle, i.e., for $z = e^{j\omega}$, the input-output relation is given by $X_u(e^{j\omega}) = X(e^{j\omega L})$.

Up-sampler
Frequency-domain characterization cont.

- Consider again a factor-of-L up-sampler. It was shown that:

$$X_u(z) = X(z^L)$$

- We see that up-sampling by a factor-of-L causes periodic repetition of the basic spectrum.

- We see easily that we now have L "spectral replications" within the original period of the spectrum which is 2π.

- Therefore, we must use a lowpass filter with cutoff at π/L called an **interpolation filter** to remove the $L-1$ unwanted images in the spectra of the up-sampled signal $x_u(n)$.

Up-sampler example
Frequency-domain characterization cont.

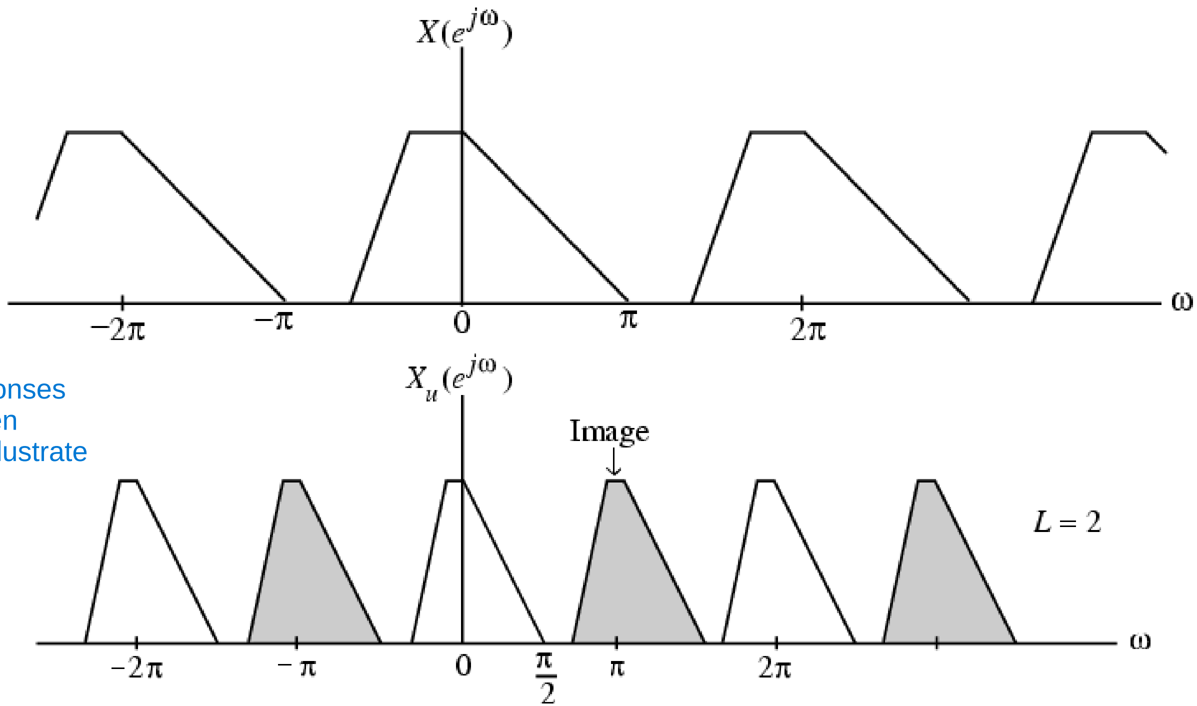
- Consider a factor-of-2 up-sampler. As it can be seen below, a factor-of-2 sampling rate expansion leads to a compression of $X(e^{j\omega})$ by a factor of 2 and a 2-fold repetition in the baseband $[0, 2\pi]$.
- Similarly in the case of a factor-of-L sampling rate expansion, there will be $L-1$ additional images of the input spectrum in the baseband.
- This process is called **imaging** as we get additional "**images**" of the input spectrum.

The DTFT shown is not an even function of the frequency.

This implies that the discrete function in time is a complex sequence.

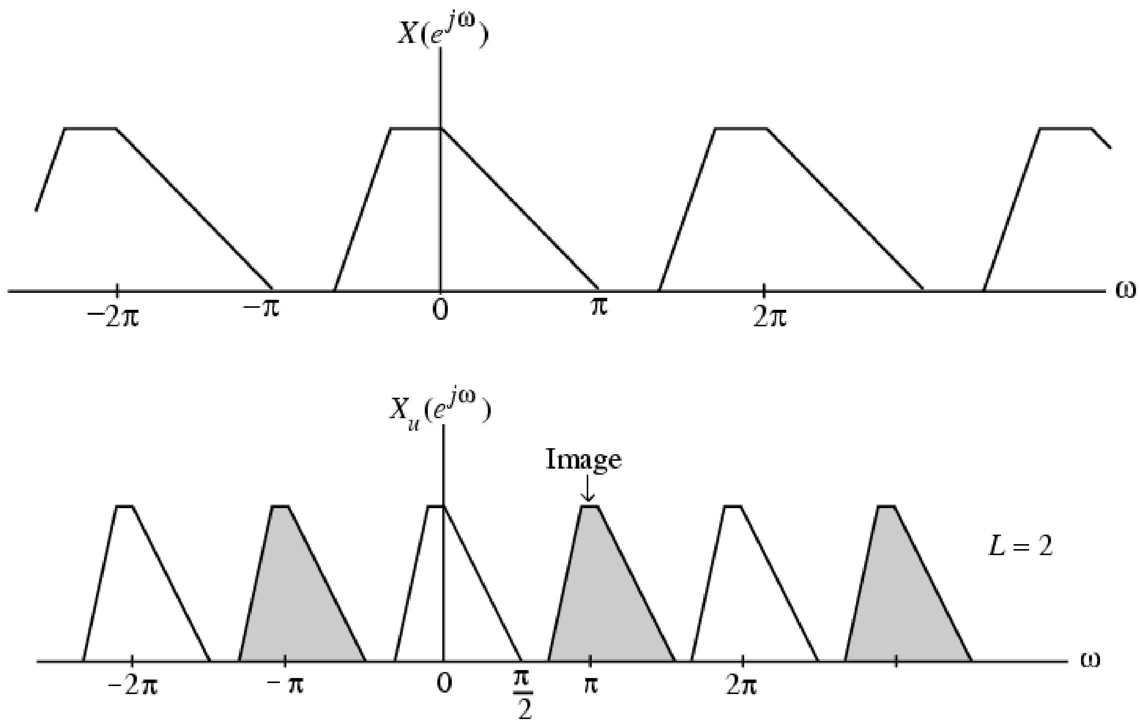
We do not often use complex sequences.

However, asymmetric responses have been purposely chosen throughout this lecture, to illustrate more clearly the effects of sampling rate alteration.



Up-sampler example for $L = 2$
Frequency-domain characterization cont.

- We have assumed that the frequency response (DTFT) is real and asymmetric to illustrate more clearly the effect of up-sampling.
- The fact that the DTFT $X(e^{j\omega})$ is **not even** implies that the corresponding discrete sequence in time, $x[n]$, is complex.



Down-sampler
Frequency-domain characterisation

- This is now less straightforward! From now on we will denote the down-sampled signal and its z –transform with $y[n]$ and $Y(z)$, respectively.
- The z –transform of the down-sampled signal is given by:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[Mn]z^{-n}$$

- Define:

$$x_{\text{int}}[n] = \begin{cases} x[n], & n = 0, \pm M, \pm 2M, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$c[n] = \begin{cases} 1, & n = 0, \pm M, \pm 2M, \dots \\ 0, & \text{otherwise} \end{cases}$$

  - We see that $x_{\text{int}}[n] = c[n]x[n]$
  - We can prove that $c[n] = \frac{1}{M}\sum_{k=0}^{M-1} W_M^{kn}$ with $W_M = e^{j2\pi/M}$ **(HOMEWORK)**

    Use $\frac{1}{M}\sum_{k=0}^{M-1} a^k = \begin{cases} \frac{1-a^M}{1-a} & a \neq 1 \\ 1 & a = 1 \end{cases}$

- Based on the above we obtain:

$$Y(z) = \sum_{n=-\infty}^{\infty} x\,[Mn]z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn]z^{-n} = \sum_{k=-\infty}^{\infty} x_{\text{int}}[k]z^{-k/M} = X_{\text{int}}(z^{\frac{1}{M}})$$

Down-sampler
Frequency-domain characterization cont.

- $Y(z) = \sum_{n=-\infty}^{\infty} x\,[Mn]z^{-n} = \sum_{n=-\infty}^{\infty} x_{\text{int}}[Mn]z^{-n} = \sum_{k=-\infty}^{\infty} x_{\text{int}}[k]z^{-k/M} = X_{\text{int}}(z^{\frac{1}{M}})$

- Based on the definitions of previous slide:

$$X_{\text{int}}(z) = \sum_{n=-\infty}^{\infty} x_{\text{int}}[n]z^{-n} = \sum_{n=-\infty}^{\infty} c[n]x[n]z^{-n} = \frac{1}{M}\sum_{n=-\infty}^{\infty} \Big(\sum_{k=0}^{M-1} W_M^{kn}\Big)\, x[n]z^{-n}$$

$$= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x[n]W_M^{kn}\, z^{-n} = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x[n](W_M^{-k}z)^{-n} = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^{-k}z)$$

This looks like a z transform

- Therefore,

$$Y(z) = X_{\text{int}}\left(z^{\frac{1}{M}}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^{-k}z^{\frac{1}{M}})$$

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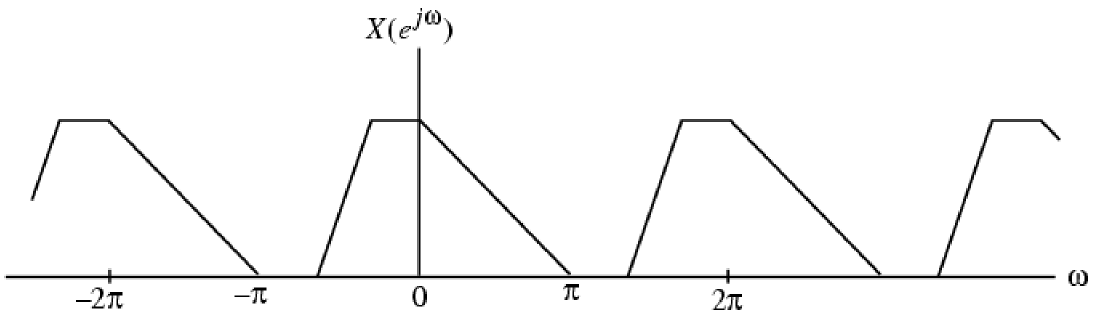
Down-sampler example for $M = 2$ Frequency-domain characterization cont.

- The spectrum of the input is as follows:



- Based on the definitions of previous slide:

$$Y(z) = \frac{1}{2}\sum_{k=0}^{1} X(W_2^{-k} z^{\frac{1}{2}}) = \frac{1}{2}\left(X\left(W_2^0 z^{\frac{1}{2}}\right) + X\left(W_2^{-1} z^{\frac{1}{2}}\right)\right) = \frac{1}{2}\left(X\left(z^{\frac{1}{2}}\right) + X\left(-z^{\frac{1}{2}}\right)\right)$$

$$W_2 = e^{j2\pi/2} = e^{j\pi} = -1,\ W_2^0 = 1,\ W_2^{-1} = -1$$

- Therefore,

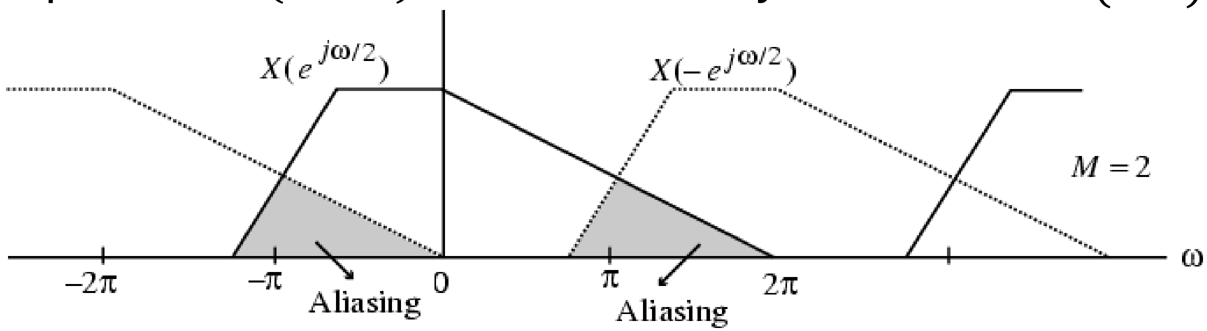
$$Y(e^{j\omega}) = \frac{1}{2}\left(X\left(e^{j\omega/2}\right) + X\left(-e^{j\omega/2}\right)\right) = \frac{1}{2}\left(X\left(e^{j\omega/2}\right) + X\left(e^{-j\pi}e^{j\omega/2}\right)\right)$$

or

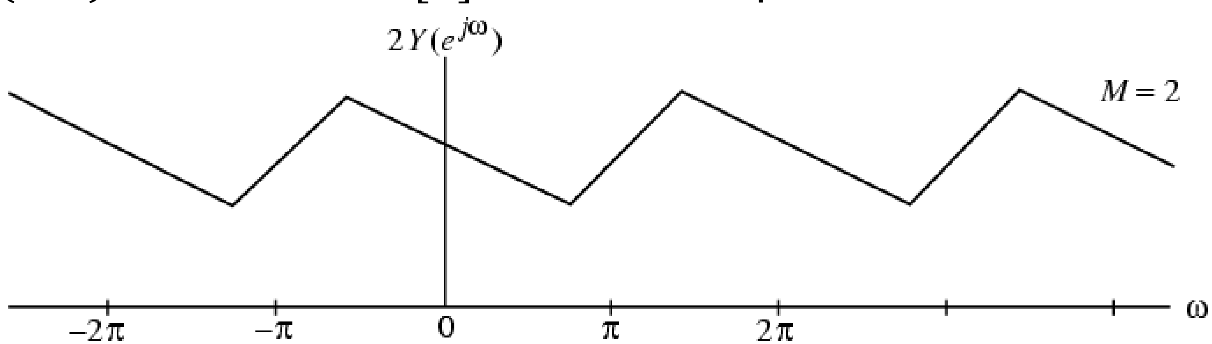
$$Y(e^{j\omega}) = \frac{1}{2}\left(X\left(e^{j\omega/2}\right) + X\left(e^{j(\omega-2\pi)/2}\right)\right)$$

Down-sampling and aliasing

- $X(-e^{j\omega/2}) = X(e^{j(\omega-2\pi)/2})$ implies that the second term $X(-e^{j\omega/2})$ in the previous equation is simply obtained by shifting the first term $X(e^{j\omega/2})$ to the right by an amount 2π as shown below.
- Note that the spectrum $X(e^{j\omega/2})$ is a stretched-by-2 version of $X\left(e^{j\omega}\right)$.



- The plots of the two terms have an overlap, and hence, in general, the original "shape" of $X(e^{j\omega})$ is lost when $x[n]$ is down-sampled as indicated below.



Down-sampler
Frequency-domain characterization cont.

- Suppose that $x[n]$ is a bandlimited signal, i.e., $X(e^{j\omega}) \neq 0,\ -\omega_c \leq \omega \leq \omega_c$
- When we decimate the signal by a factor of M we have

$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(W_M^{-k} z^{\frac{1}{M}}\right), W_M = e^{j2\pi/M}$

The first two consecutive replications of the spectrum are:

- $X\left(W_M^{-0} e^{j\omega/M}\right) = X\left(e^{j\omega/M}\right),\ W_M^{-0} = 1$

  $X\left(e^{j\omega/M}\right) \neq \mathbf{0}, -\omega_c \leq \omega/M \leq \omega_c \Rightarrow \boldsymbol{-M\omega_c \leq \omega \leq M\omega_c}$
- $X\left(W_M^{-1} e^{j\omega/M}\right) = X\left(e^{-j2\pi/M} e^{j\omega/M}\right) = X\left(e^{j(\omega-2\pi)/M}\right), W_M^{-1} = e^{-j2\pi/M}$

  $X\left(e^{j(\omega-2\pi)/M}\right) \neq \mathbf{0}, -\omega_c \leq (\omega - 2\pi)/M \leq \omega_c \Rightarrow -M\omega_c \leq \omega - 2\pi \leq M\omega_c$

  $\Rightarrow \boldsymbol{-M\omega_c + 2\pi \ \leq \omega \leq M\omega_c + 2\pi}$
- For the two replications above to not overlap, the upper bound of the first replication must be smaller or equal to the lower bound of the second replication:

  $M\omega_c \leq -M\omega_c + 2\pi \Rightarrow 2M\omega_c \leq 2\pi \Rightarrow M\omega_c \leq \pi \Rightarrow \boldsymbol{\omega_c \leq \pi/M}$
- The same result would hold if we took any two consecutive replications.

Down-sampling summary

- The overlapping of the stretched spectra causes the **aliasing** that takes place due to under-sampling.

- There is no overlap, i.e., no aliasing, only if $X(e^{j\omega}) = 0$ for $|\omega| \geq \pi/M$.

- The relation between the DTFTs of the output and the input of a factor-of-M down-sampler is given by

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

- Observe that $Y(e^{j\omega})$ is a sum of M uniformly shifted and stretched versions of $X(e^{j\omega})$ and scaled by a factor of $1/M$.

- Observe that $Y(e^{j\omega})$ is periodic with a period 2π, even though the stretched versions of $X(e^{j\omega})$ are periodic with a period $2M\pi$ **(HOMEWORK)**.

- Down-sampling is also called **decimation** and up-sampling is called **interpolation**.

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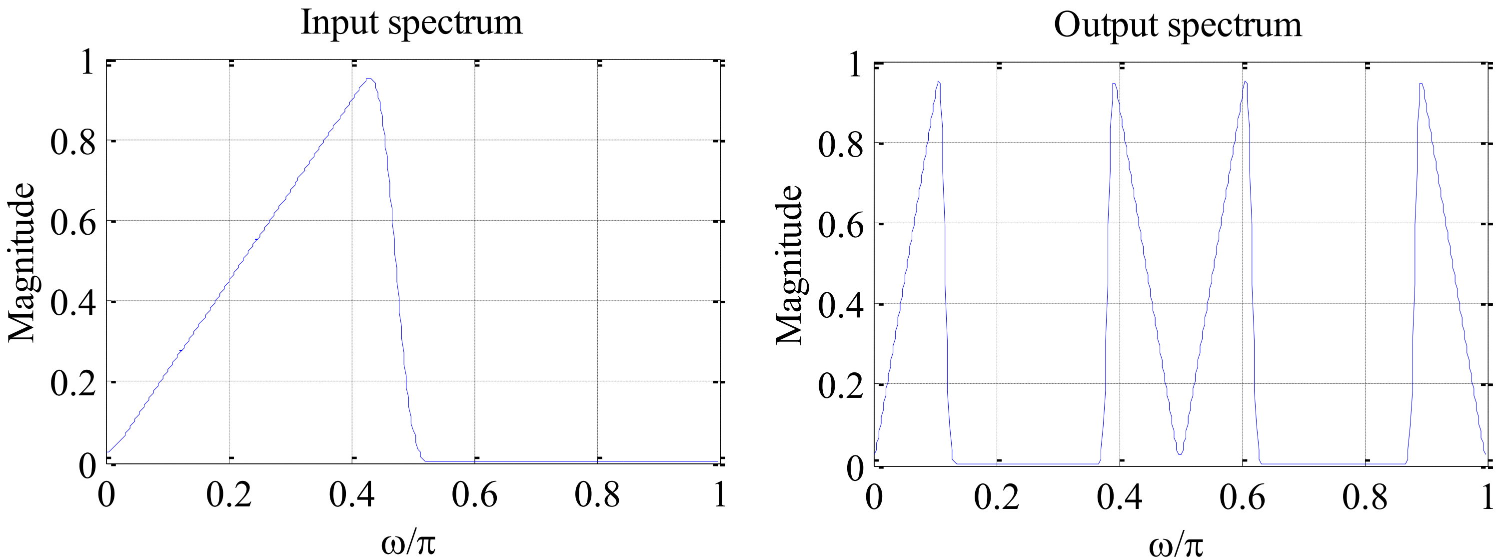
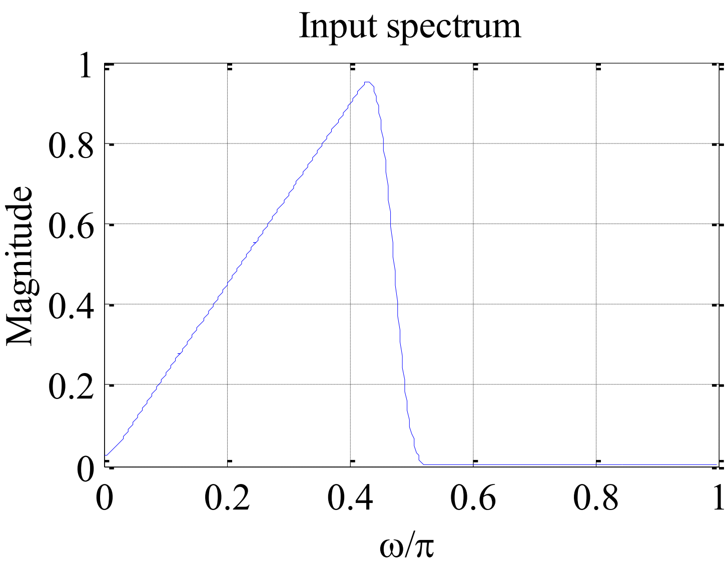
Up-Sampler: MATLAB implementation

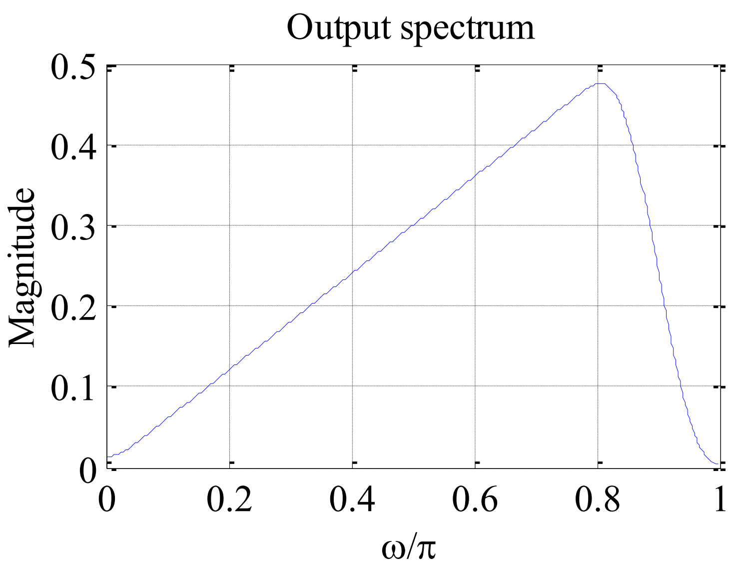
- MATLAB can be used to illustrate the frequency-domain properties of the up-sampler shown below for $L = 4$.



Down-Sampler: MATLAB implementation

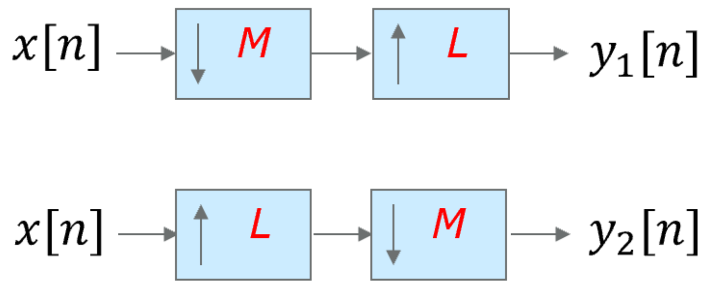
- MATLAB can be used to illustrate the frequency-domain properties of the down-sampler shown below for $M = 2$.





Cascade equivalences

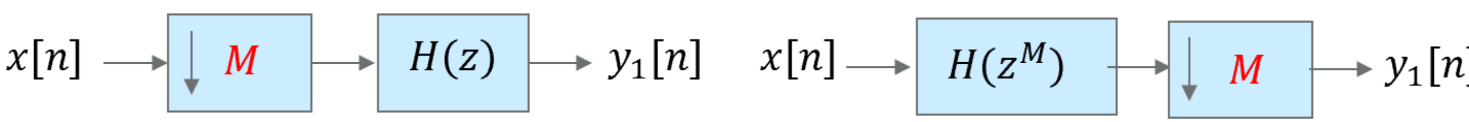
- A complex multirate system is formed by proper interconnections of up-samplers, down-samplers and LTI digital filters.
- In many applications these devices appear in a **cascade** form.
- To implement a fractional change in the sampling rate we need to employ a cascade of an up-sampler and a down-sampler.
- Consider the two cascade connections shown below:



- **It can be proven** that a cascade of a factor-of-M down-sampler and a factor-of-L up-sampler is **interchangeable**, i.e., $y_1[n] = y_2[n]$, **if and only if** M and L are **relatively prime** (M and L do not have any common factor that is an integer $k > 1$).

Noble identities

- Two other cascade equivalences are shown below. They are called **noble identities**.
- These types of rules are very useful in multirate signal processing.
- Note that $H(z)$ represents a transfer function of an LTI system.
- **Cascade equivalence #1**



- **Cascade equivalence #2**

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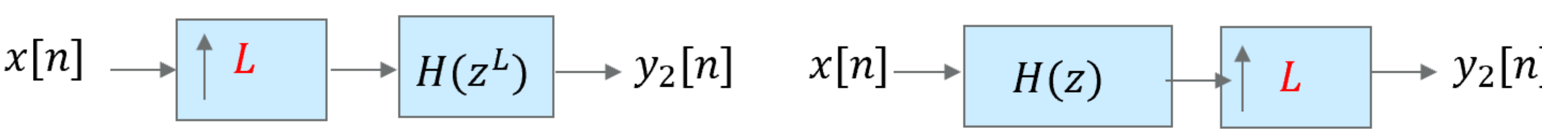

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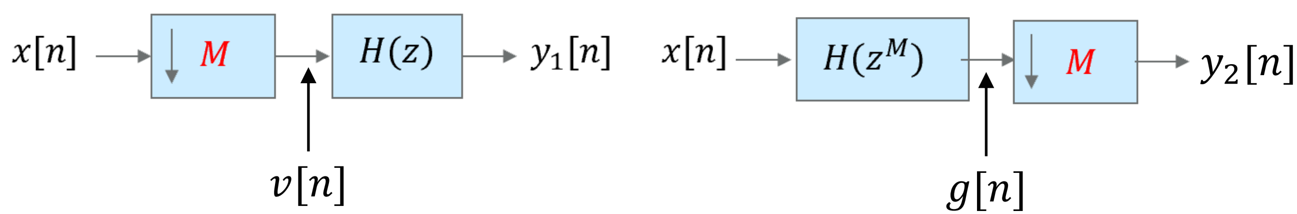
Problem 1

- Verify the cascade equivalence:

Cascade equivalence #1



Solution

$V(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^{-k} z^{\frac{1}{M}})$

$Y_1(z) = H(z)V(z) = \boldsymbol{\frac{1}{M} H(z) \sum_{k=0}^{M-1} X(W_M^{-k} z^{\frac{1}{M}})}$

$G(z) = H\left(z^M\right)X(z)$

$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} G(W_M^{-k} z^{\frac{1}{M}}) = \frac{1}{M}\sum_{k=0}^{M-1} H\left(W_M^{-kM} z\right) X\left(W_M^{-k} z^{\frac{1}{M}}\right)$

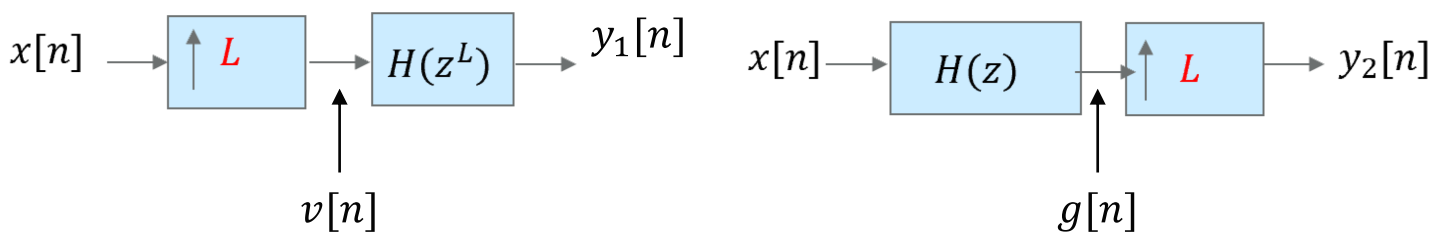
$= \frac{1}{M}\sum_{k=0}^{M-1} H(z)\, X\left(W_M^{-k} z^{\frac{1}{M}}\right)$

$= \boldsymbol{\frac{1}{M} H(z) \sum_{k=0}^{M-1} X\left(W_M^{-k} z^{\frac{1}{M}}\right)}$, $W_M^{-kM} = e^{-j2\pi kM/M} = e^{-j2\pi k} = 1$

Problem 2

- Verify the cascade equivalence:

Cascade equivalence #2



Solution

$V(z) = X(z^L)$

$Y_1(z) = H(z^L)V(z) \Rightarrow \boldsymbol{Y_1(z) = H(z^L)X(z^L)}$

$Y_2(z) = G(z^L)$

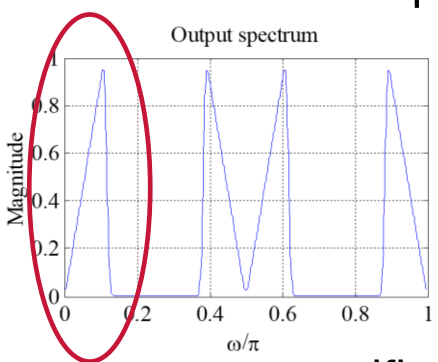
$G(z) = H(z)X(z)$

$\boldsymbol{Y_2(z) = H(z^L)X(z^L)}$

For simplicity, from this slide up to Slide 37 we denote the signal x(t) sampled at period T with x[n] and not with x[nT].

Filters in sampling rate alteration systems

- From the sampling theorem it is known that the sampling rate of a **critically sampled** discrete-time signal with a spectrum occupying the full Nyquist range cannot be reduced any further since such a reduction will introduce **aliasing**.
- Hence, the bandwidth of a critically sampled signal must be reduced by lowpass filtering before its sampling rate is reduced by a down-sampler. The filter which attempts this process is called a **decimator**.
- Since up-sampling causes periodic repetition of the basic spectrum, the unwanted images in the spectrum of the up-sampled signal must be removed by using a lowpass filter. **This is equivalent to the zero-valued samples introduced by an up-sampler being interpolated to more meaningful values for an effective sampling rate increase.** The filter which attempts this process is called an **interpolator**.



- We will develop the frequency response specifications of the above lowpass filters.

Input-output of an interpolator

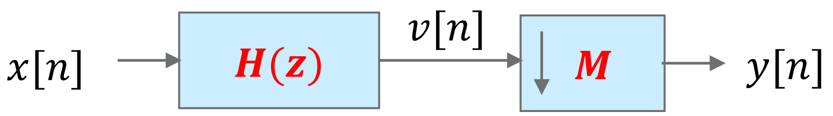
- We now consider the development of the input-output relation of the interpolation structure shown below:

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{x_u[n]} \boxed{H(z)} \longrightarrow y[n]$$

- In the time-domain the input-output relation of the factor-of-L up-sampler is given by:
  $x_u[Lm] = x[m], m = 0, \pm 1, \pm 2, \dots$
  We previously wrote
  $x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases}$
- We assume that $H(z)$ is an LTI system, in which case we have:
  $$y[n] = \sum_{l=-\infty}^{\infty} h[n-l]\, x_u[l] = \sum_{m=-\infty}^{\infty} h[n-Lm]\, x_u[Lm] = \sum_{m=-\infty}^{\infty} h[n-Lm]\, x[m]$$
  $[l = Lm]$
- The z –domain representation of the above system is:
  $$Y(z) = H(z)X(z^L)$$

Input-output of a decimator

- We now consider the development of the input-output relation of the decimation structure shown below:



- In the time-domain the input-output relation of the factor-of-M down-sampler is:

$$y[n] = v[Mn]$$

- $H(z)$ is an LTI system and therefore, we have:

$$v[n] = \sum_{l=-\infty}^{\infty} h[n-l]x[l],\ V(z) = H(z)X(z)$$

- Furthermore,

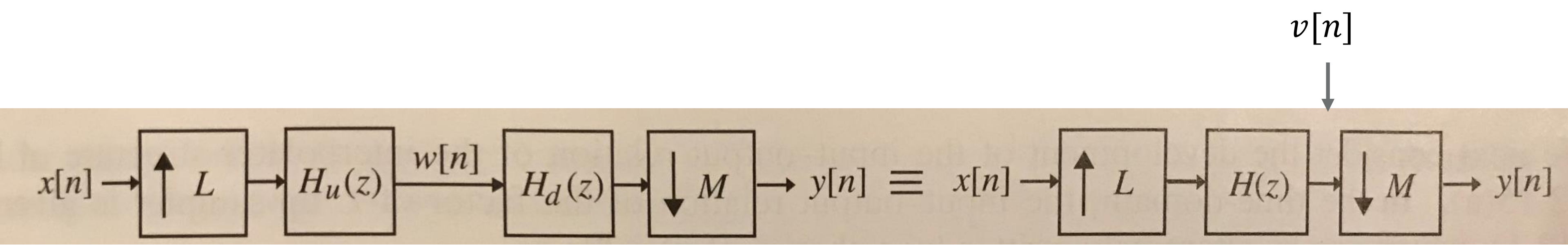
$$y[n] = v[Mn] = \sum_{l=-\infty}^{\infty} h[Mn-l]\,x[l],\ Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(W_M^{-k} z^{\frac{1}{M}})$$

- Combining the above:

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} H(W_M^{-k} z^{\frac{1}{M}})\,X(W_M^{-k} z^{\frac{1}{M}})$$

Interpolation filter specifications

- A fractional change in the sampling rate (change by a rational factor) can be achieved by cascading a factor-of-L interpolator with a factor-of-M decimator.
- **It is proven that the interpolator must precede the decimator.**
- The interpolation filter $H_u(z)$ and the decimation filter $H_d(z)$ can be replaced with **a single filter $H(z)$ designed to jointly avoid the aliasing that is caused by down-sampling and eliminate the images resulting from up-sampling.**

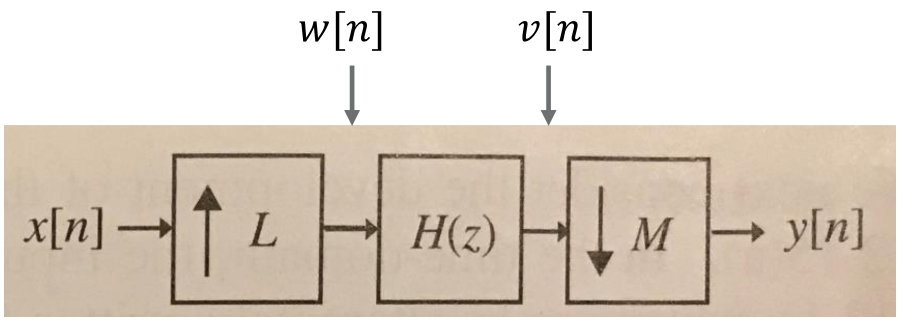


- For the fractional rate converter of figure above on the right, the input-output relationship can be derived in both time and z –domain as follows (next 2 slides):

$$y[n] = \sum_{m=-\infty}^{\infty} h[Mn - Lm]\, x[m],$$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} H\left(W_M^{-k} z^{\frac{1}{M}}\right) X\left(W_M^{-kL} z^{\frac{L}{M}}\right)$$

Proof of previous relationships



$y[n] = v[Mn]$

$$v[n] = \sum_{k=-\infty}^{\infty} h[n-k]\,w[k] \Rightarrow y[n] = v[Mn] = \sum_{k=-\infty}^{\infty} h[Mn-k]\,w[k]$$

$$w[k] = x[k/L] \Rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[Mn-k]\,x[k/L]$$

Define $\frac{k}{L} = m \Rightarrow k = Lm \Rightarrow y[n] = \sum_{m=-\infty}^{\infty} h[Mn - Lm]\,x[m]$

$W(z) = X(z^L), V(z) = H(z)W(z)$

$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(W_M^{-k} z^{\frac{1}{M}}) = \frac{1}{M}\sum_{k=0}^{M-1} H(W_M^{-k} z^{\frac{1}{M}})\, W(W_M^{-k} z^{\frac{1}{M}}) \Rightarrow$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} H(W_M^{-k} z^{\frac{1}{M}})\, X(W_M^{-kL} z^{\frac{L}{M}})$$

Interpolation filter specifications

- Assume that $x[n]$ has been obtained by sampling a band-limited continuous-time signal $x_a(t)$ at the Nyquist rate.
- If $X_a(j\Omega)$ and $X(e^{j\omega})$ denote the Fourier transforms of $x_a(t)$ and $x[n]$, respectively, we already known that

$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi n}{T_s}\right) \quad (1)$$

where T_s is the sampling period.
- Since the sampling is being performed at the Nyquist rate, there is no overlap between the shifted spectra of $X_a(j\omega/T_s)$.
- If we instead sample $x_a(t)$ at a much higher rate $f = LT_s \Rightarrow T = \frac{T_s}{L}$ yielding $\boldsymbol{y[n]}$, its Fourier transform $Y(e^{j\omega})$ is related to $X_a(j\Omega)$ as shown below:

$$Y(e^{j\omega}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi n}{T}\right) = \frac{L}{T_s}\sum_{n=-\infty}^{\infty} X_a\left(\frac{j\omega - j2\pi n}{T_s/L}\right) \quad (2)$$

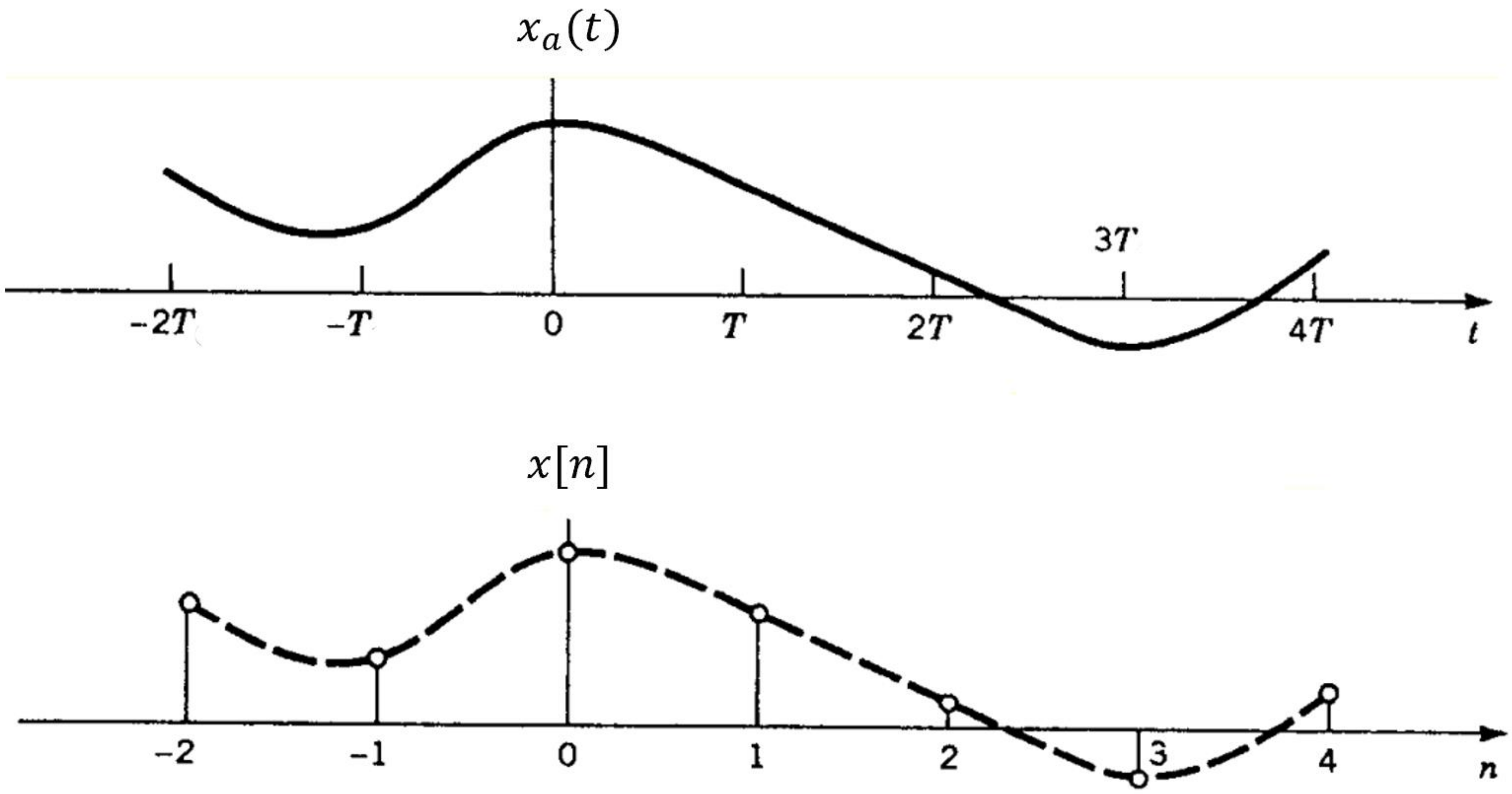
$$= L\frac{1}{T_s}\sum_{n=-\infty}^{\infty} X_a\left(\frac{j\boldsymbol{\omega L} - j2\pi nL}{T_s}\right)$$

Interpolation filter specifications

- On the other hand, if we pass $x[n]$ through a factor-of- L up-sampler generating $x_u[n]$, the relation between the Fourier transforms $X_u(e^{j\omega})$ and $X(e^{j\omega})$ is:
$$X_u(e^{j\omega}) = X(e^{j\omega L})$$
- Therefore, it follows that if $x_u[n]$ is passed through an ideal lowpass filter with a cutoff at π/L and a gain of L, the output of the filter will be precisely $y[n]$.
- In practice, a transition band is provided to ensure the realizability of the lowpass interpolation filter $H(z)$.
- Hence, the desired lowpass filter should have a stopband edge at $\omega_s = \pi/L$ and a passband edge ω_p close to ω_s to reduce the distortion of the spectrum of $x[n]$.
- If ω_c is the highest frequency that needs to be preserved in $x[n]$, then $\omega_p = \omega_c/L$.

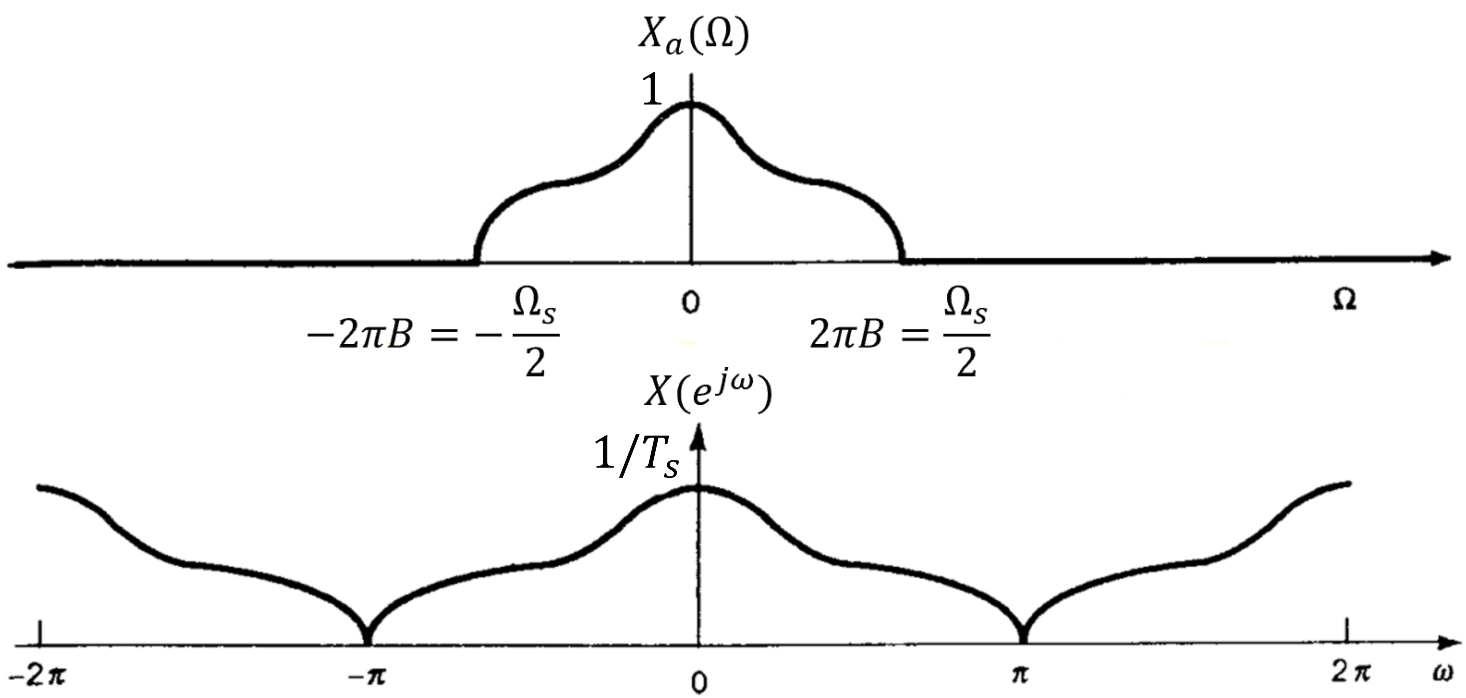
  Summarizing, the specifications of the lowpass interpolation filter are given by
$$|H(e^{j\omega})| = \begin{cases} L, & |\omega| \le \omega_c/L \\ 0, & \pi/L \le |\omega| \le \pi \end{cases}$$
- In a similar manner, we can develop the specifications for the lowpass decimation filter that are given by
$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_c/M \\ 0, & \pi/M \le |\omega| \le \pi \end{cases}$$

Continuous and sampled (discrete) signal

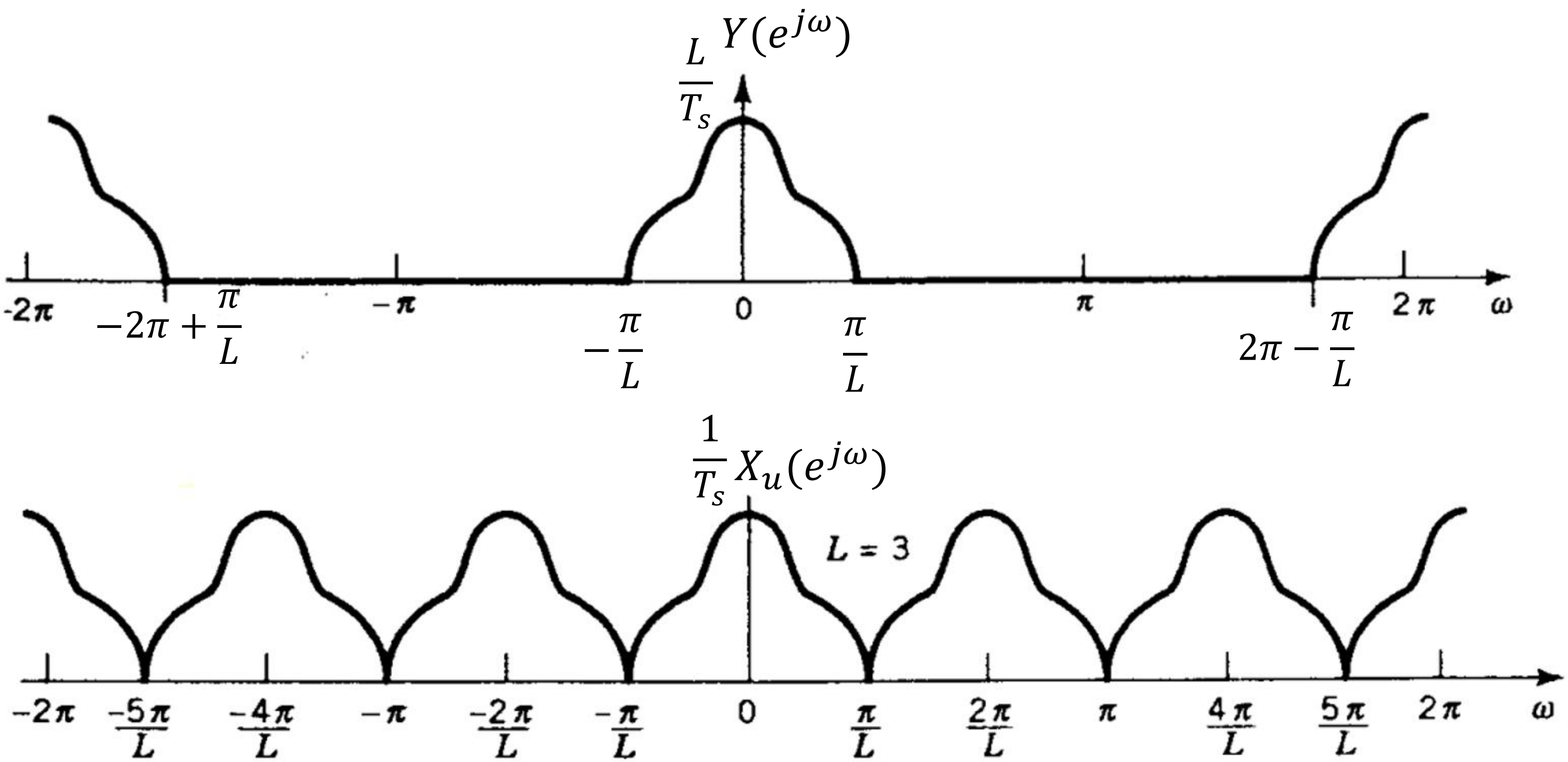


Spectrum of continuous-time signal
Spectrum of sampled signal wrt normalized angular frequency

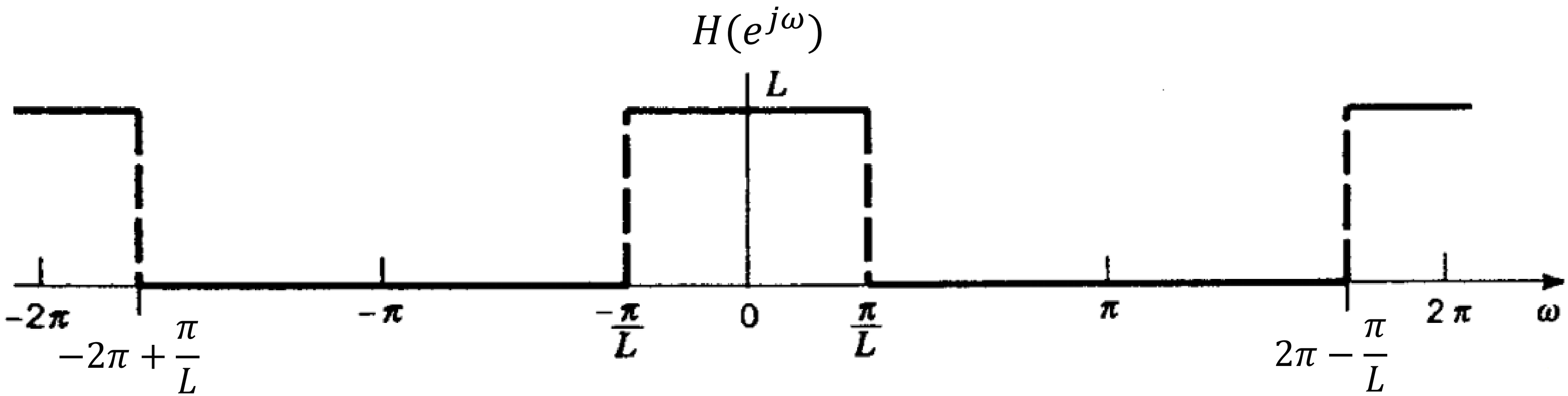
- We denote the normalized angular frequency as $\omega = \frac{\Omega}{f_s} \Leftrightarrow \omega = \Omega T_s \Leftrightarrow \boldsymbol{\Omega = \frac{\omega}{T_s}}$
- $\Omega = \frac{\Omega_s}{2} \Rightarrow \omega = \frac{\Omega_s}{2f_s} = \frac{2\pi f_s}{2f_s} = \pi$



Sampling with 3 times higher sampling rate
Spectrum of interpolated-by-3 signal

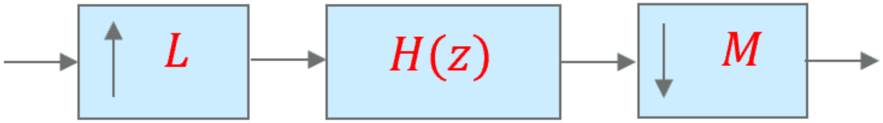


Interpolator filter



Filters for fractional sampling rate alteration

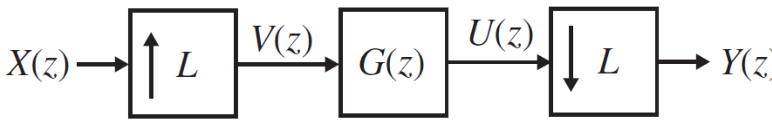
- As mentioned, a fractional change in the sampling rate can be achieved by cascading a factor-of-M decimator with a factor-of-L interpolator, where M and L are positive integers, using the following configuration.



- It can be proven that such a cascade is equivalent to a **decimator with a decimation factor of $\boldsymbol{M/L}$** or **an interpolator with an interpolation factor of $\boldsymbol{L/M}$**.
- Hence, in the above configuration for the fractional sampling rate alteration, the lowpass filter $H(z)$ has a stopband edge frequency given by $\omega_s = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right)$.
- The lowpass decimation or interpolation filter can be designed either as an FIR or an IIR digital filter.
- IIR digital filters are, in general, computationally more efficient than equivalent FIR digital filters, and are therefore preferred where computational cost needs to be minimized.

Problem 3

- Show that the structure below is an LTI system and find its transfer function.



Solution

$V(z) = X(z^L)$

$U(z) = G(z)V(z)$

$Y(z) = \frac{1}{L}\sum_{k=0}^{L-1} U(W_L^{-k} z^{\frac{1}{L}}) \ = \frac{1}{L}\sum_{k=0}^{L-1} G(W_L^{-k} z^{\frac{1}{L}})\, V(W_L^{-k} z^{\frac{1}{L}})$

$$= \frac{1}{L}\sum_{k=0}^{L-1} G(W_L^{-k} z^{\frac{1}{L}})\, X(W_L^{-kL} z)$$

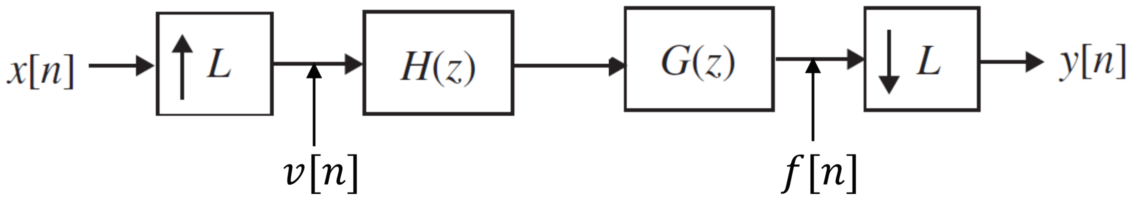
$W_L = e^{j2\pi/L} \Rightarrow W_L^{-kL} = e^{-j2\pi kL/L} = e^{-j2\pi k} = 1$

$Y(z) = \frac{1}{L}\sum_{k=0}^{L-1} G(W_L^{-k} z^{\frac{1}{L}})\, X(z) \Rightarrow \frac{Y(z)}{X(z)} = \frac{1}{L}\sum_{k=0}^{L-1} G(W_L^{-k} z^{\frac{1}{L}})$

The system's transfer function is the ratio between output and input z –transforms and therefore, the overall system is an LTI system.

Problem 4

- Find a condition which would make the system below an identity system.



Solution

$V(z) = X(z^L)$

$F(z) = G(z)H(z)V(z)$

$$Y(z) = \frac{1}{L}\sum_{k=0}^{L-1} F(W_L^{-k} z^{\frac{1}{L}}) = \frac{1}{L}\sum_{k=0}^{L-1} G\left(W_L^{-k} z^{\frac{1}{L}}\right) H(W_L^{-k} z^{\frac{1}{L}}) V(W_L^{-k} z^{\frac{1}{L}})$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} G\left(W_L^{-k} z^{\frac{1}{L}}\right) H(W_L^{-k} z^{\frac{1}{L}}) X(W_L^{-kL} z) = \frac{1}{L}\sum_{k=0}^{L-1} G\left(W_L^{-k} z^{\frac{1}{L}}\right) H\left(W_L^{-k} z^{\frac{1}{L}}\right) X(z)$$

$Y(z) = X(z)$ if $\frac{1}{L}\sum_{k=0}^{L-1} G\left(W_L^{-k} z^{\frac{1}{L}}\right) H\left(W_L^{-k} z^{\frac{1}{L}}\right) = 1.$

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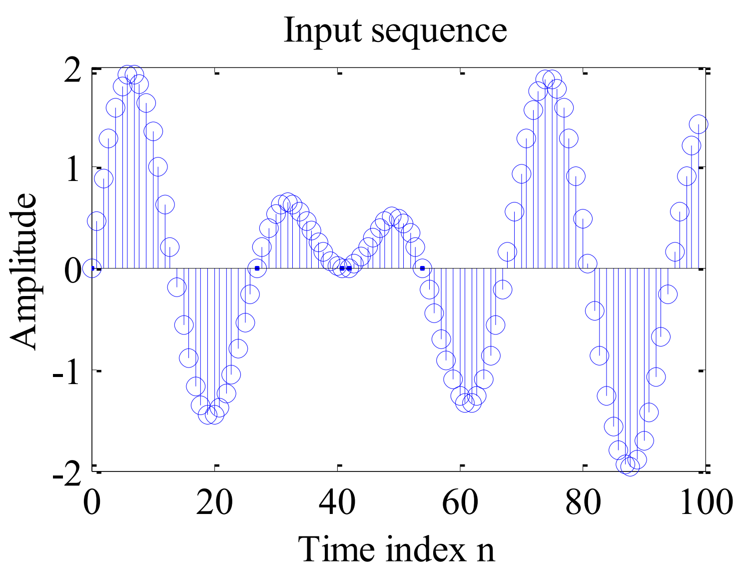
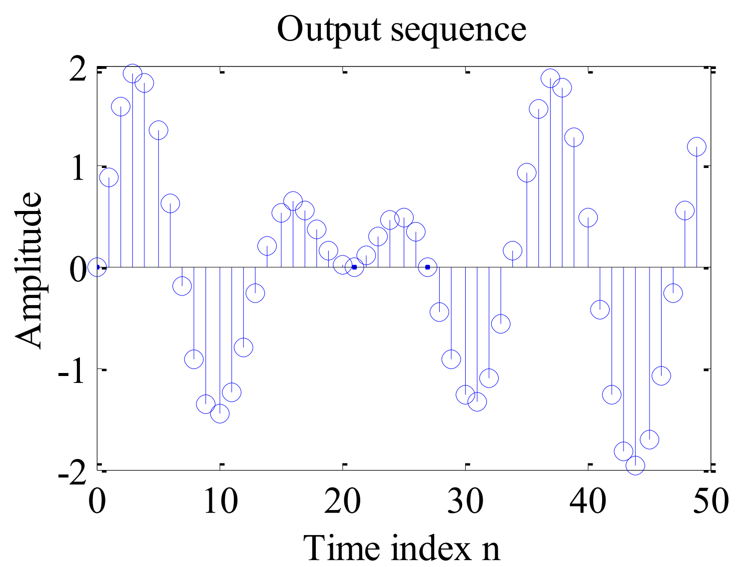
Sampling Rate Alteration Using MATLAB

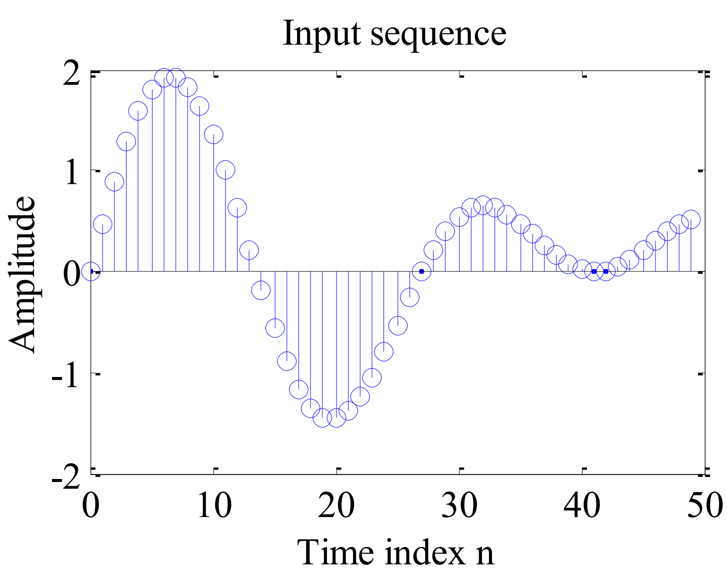
- The function `decimate` can be employed to reduce the sampling rate of an input signal vector `x` by an integer factor `M` to generate the output signal vector `y`.
- The input and output plots of a factor-of-2 decimator designed using MATLAB.

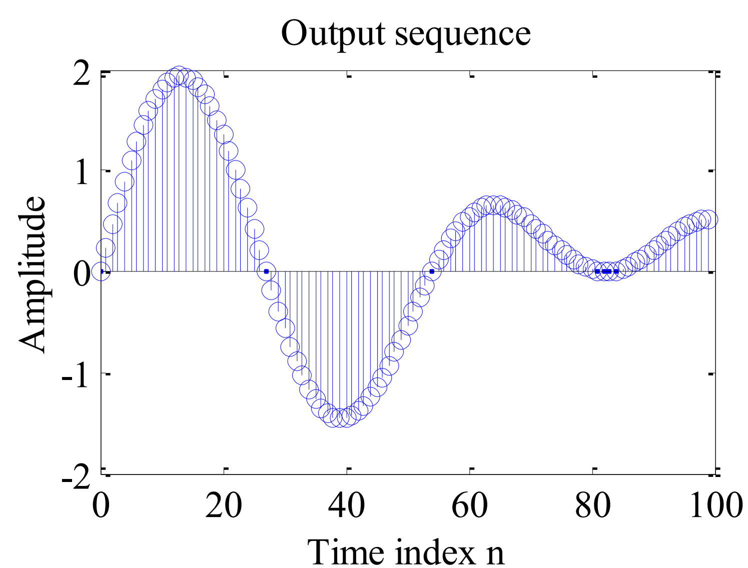




Sampling Rate Alteration Using MATLAB

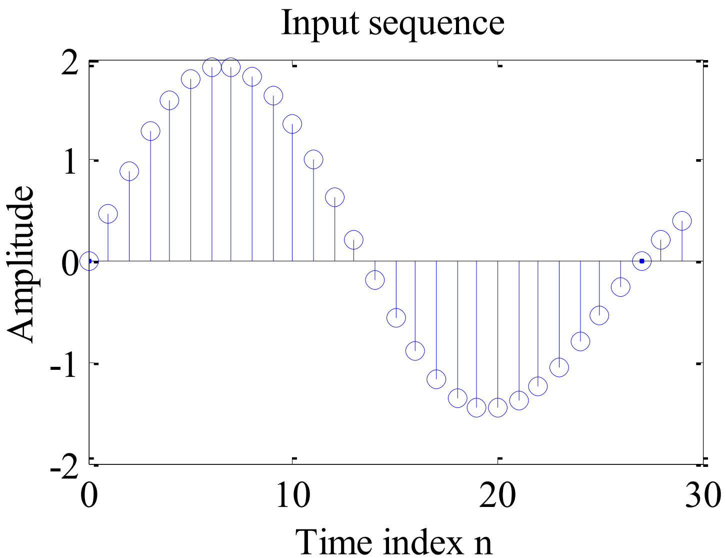
- The function `interp` can be employed to increase the sampling rate of an input signal `x` by an integer factor `L` generating the output vector y.
- The lowpass filter employed here is a symmetric FIR filter.
- The filter allows the original input samples to appear as is in the output and finds the missing samples by minimizing the mean-square errors between these samples and their ideal values.
- The input and output plots of a factor-of-2 interpolator designed using MATLAB are shown below.

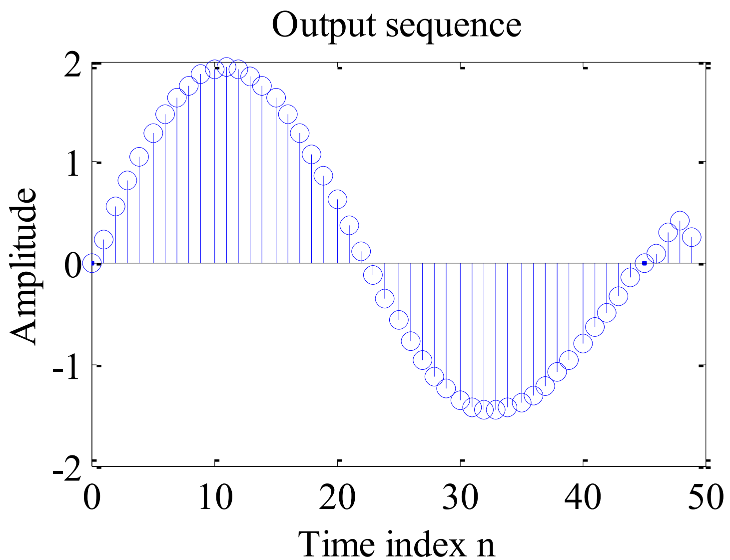




Sampling Rate Alteration Using MATLAB

- The function `resample` can be employed to increase the sampling rate of an input vector `x` by a ratio of two positive integers, `L/M`, generating an output vector `y`.
- The lowpass FIR filter is designed using `fir1` with a Kaiser window.
- The fractional interpolation of a sequence can be obtained using MATLAB which employs the function `resample`.
- The input and output plots of a factor-of-`5/3` interpolator designed using MATLAB are given below.





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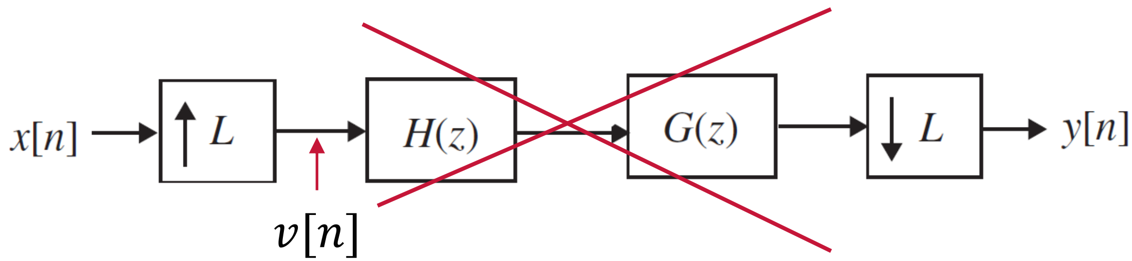
Problem 5

- Show that a system which consists of an interpolator followed by a decimator, both with the same factor L, acts as an identity system, $y[n] = x[n]$.

  This is equivalent to the problem given previously in Slide 39, illustrated in the figure below, with $H(z) = G(z) = 1$.



Solution

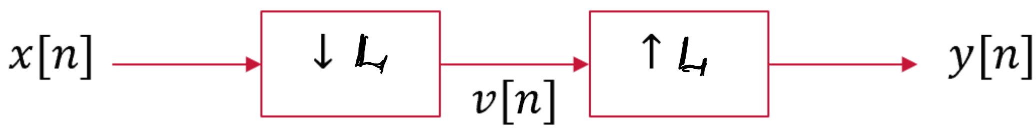
Output of the interpolator:

$V(z) = X(z^L)$

$$Y(z) = \frac{1}{L}\sum_{k=0}^{L-1} V(W_L^{-k} z^{\frac{1}{L}}) = \frac{1}{L}\sum_{k=0}^{L-1} X(W_L^{-kL} z^{\frac{L}{L}}) = \frac{1}{L}\sum_{k=0}^{L-1} X(z) = \frac{1}{L} L X(z) = X(z)$$

Problem 6

- Show that a system which consists of a decimator followed by an interpolator, both with the same factor L, **does not act as an identity system**.



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Solution

Output of the decimator:

$V(z) = \frac{1}{L}\sum_{k=0}^{L-1} X(W_L^{-k} z^{\frac{1}{L}})$

$$Y(z) = V\left(z^L\right) = \frac{1}{L}\sum_{k=0}^{L-1} X(W_L^{-k} z^{\frac{L}{L}}) = \frac{1}{L}\sum_{k=0}^{L-1} X(W_L^{-k} z) \neq X(z)$$

Problem 7

- Show that a system which consists of two decimators connected in series, with factors L and M, is equivalent to a decimator with factor $L \cdot M$.

Solution

Output of the first decimator:

$$V(z) = \frac{1}{L}\sum_{l=0}^{L-1} X(W_L^{-l} z^{\frac{1}{L}})$$

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} V(W_M^{-k} z^{\frac{1}{M}}) = \frac{1}{M}\sum_{k=0}^{M-1}\frac{1}{L}\sum_{l=0}^{L-1} X(W_L^{-l} W_M^{-k/L} z^{\frac{1}{ML}})$$

$$= \frac{1}{ML}\sum_{k=0}^{M-1}\sum_{l=0}^{L-1} X(W_{ML}^{-lM} W_{ML}^{-k} z^{\frac{1}{ML}}) = \frac{1}{ML}\sum_{k=0}^{M-1}\sum_{l=0}^{L-1} X(W_{ML}^{-(lM+k)} z^{\frac{1}{ML}})$$

$$= \frac{1}{ML}\sum_{n=0}^{ML-1} X(W_{ML}^{-n} z^{\frac{1}{ML}})$$

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Problem 8

- Show that a system which consists of two interpolators connected in series, with factors L and M, is equivalent to an interpolator with factor $L \cdot M$.

Solution

Output of the first decimator:

$V(z) = X(z^L)$

$Y(z) = V\left(z^M\right) = X(z^{LM})$