Imperial College London

Digital Signal Processing

Lecture

Design of IIR filters and Problems

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This lecture

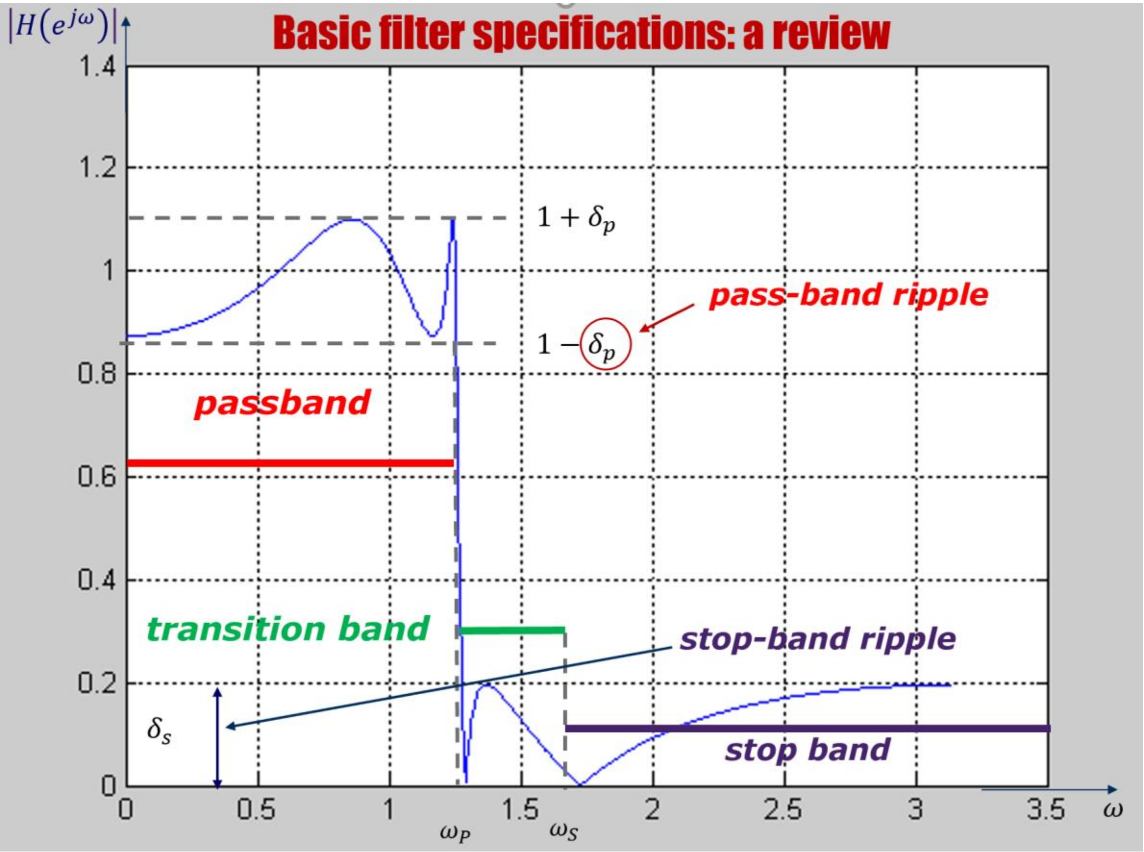
- In this lecture we will see how to transform a filter's transfer function to another filter's transfer function.
- We will see THREE types of transformations:
  - We can transform an **analogue filter to** a **digital filter** and vice versa. This will be done through the so-called **bilinear transformation**.
  - We can transform an **analogue** **lowpass** filter **to** an **analogue** filter **of** **any type**: highpass, bandpass or bandstop. This is an **analogue-to-analogue transformation**.
  - We can transform a **digital** **lowpass** filter to a **digital** filter **of** **any type**: highpass, bandpass or bandstop. This is a **digital-to-digital transformation**.

IIR filter design. Why using an IIR filter?

- **Advantages**
  - **Low implementation cost**: require less coefficients and memory than FIR filters in order to satisfy a similar set of specifications.
  - **analogue equivalent**: IIR digital filters can be designed using analogue filters by employing specific transformations from the $s-$plane to the $z-$plane and vice versa.
- **Disadvantages**
  - **Non-linear phase characteristics**: The phase characteristics of an IIR filter are generally nonlinear, especially near the cut-off frequencies. Allpass equalisation filters can be used in order to improve the passband phase characteristics.
  - **Stability**: With IIR filters we have to face the issue of stability.

Basic IIR filter specifications: a review

- In most practical applications, the problem of interest is the development of a realizable approximation to a given **magnitude only** response specification.
  - The phase can be corrected by cascading the system with an allpass filter.



IIR filter design

- As indicated in the previous slide's figure, in the passband, defined by $0 \leq \omega \leq \omega_p$, we require that $\left|G(e^{j\omega})\right| \cong 1$ with an error $\pm\delta_p$, i.e., $1 - \delta_p \leq \left|G(e^{j\omega})\right| \leq 1 + \delta_p$.
- In the stopband, defined by $\omega_s \leq \omega \leq \pi$, we require that $G(e^{j\omega}) \cong 0$ with an error δ_s, i.e., $G(e^{j\omega}) \leq \delta_s, \omega_s \leq \omega \leq \pi$.
- **In practical filter design problems, very often the passband edge frequency F_p and the stopband edge frequency F_p are specified in Hz.**
- The normalized band edge frequencies need to be computed from the filter specifications in Hz using the following relationships:

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

IIR filter design

- In IIR filter design, the most common practice is to convert an analogue **prototype** filter transfer function $H_a(s)$ to a **desired** digital filter transfer function $G(z)$.
  - Analogue filter approximation techniques are highly advanced.
  - They usually yield closed-form solutions.
  - Extensive tables are available for analogue filter design.
- A digital transfer function $G(z)$ is derived from the corresponding analogue $H_a(s)$.

$$H_a(s) = \frac{P_a(s)}{D_a(s)} \Rightarrow G(z) = \frac{P(z)}{D(z)}$$

- The basic idea behind the conversion of an analogue prototype transfer function $H_a(s)$ to a desired digital filter transfer function $G(z)$ is to apply a mapping from the s –domain to the z – domain, so that the essential properties of the analogue frequency response are preserved.
- Requirements for the mapping are:
  - The imaginary axis $j\Omega$ of the the s –plane is mapped onto the unit circle in the z –plane.
  - A stable $H_a(s)$ must be transformed into a stable $G(z)$.
  - Most widely used transformation: **Bilinear Transformation and its inverse.**

Inverse bilinear transformation

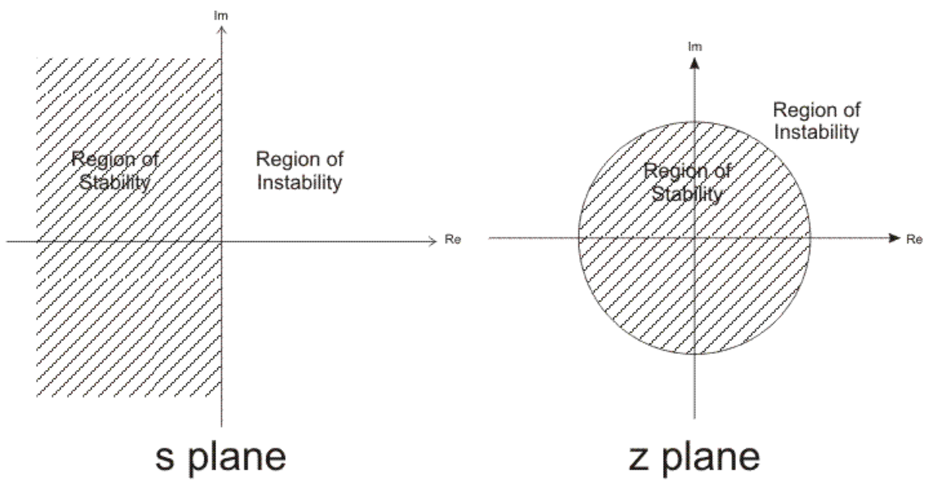
- **Inverse bilinear transformation $z \rightarrow s$**

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right),\ T \text{ is the sampling period}$$

- The above transformation maps a single point in the z –plane to a unique point in the s –plane and vice-versa.
- The relation between $G(z)$ and $H_a(s)$ is then given by

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

- There is also a mapping of the s –plane into the z –plane. The proof is given in the next slide.



causal systems are assumed in these figures

Bilinear transformation cont.

- **Bilinear transformation $s \rightarrow z$**

$$z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}$$

- For $s_0 = \sigma_0 + j\Omega_0$

$$z = \frac{1 + \frac{T}{2}(\sigma_0 + j\Omega_0)}{1 - \frac{T}{2}(\sigma_0 + j\Omega_0)}$$

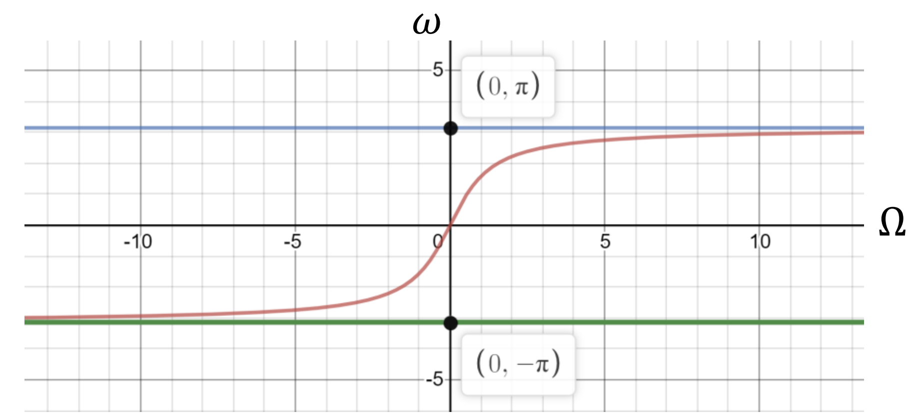
$$|z|^2 = \frac{(1 + \frac{T}{2}\sigma_0)^2 + (\frac{T}{2}\Omega_0)^2}{(1 - \frac{T}{2}\sigma_0)^2 + (\frac{T}{2}\Omega_0)^2}$$

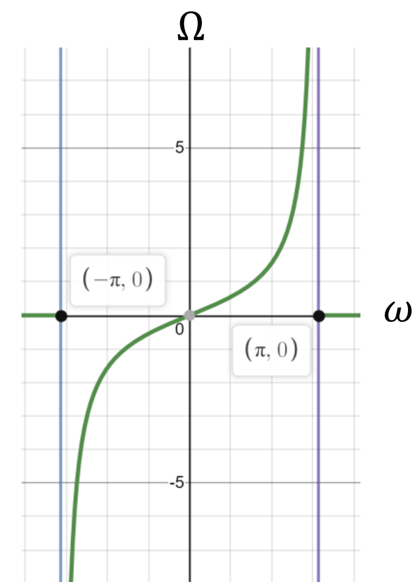
  - $\sigma_0 = 0 \Rightarrow |z|^2 = \frac{1+(\frac{T}{2}\Omega_0)^2}{1+(\frac{T}{2}\Omega_0)^2} = 1 \Rightarrow |z| = 1 \Rightarrow$ the imaginary axis is mapped on the unit circle.
  - $\sigma_0 < 0 \Rightarrow |z|^2 < 1 \Rightarrow |z| < 1$ the left-half plane is mapped inside the unit circle.
  - $\sigma_0 > 0 \Rightarrow |z|^2 > 1 \Rightarrow |z| > 1$ the right-half plane is mapped outside the unit circle.

Bilinear transformation cont.

- For $z = e^{j\omega}$ with $T = 2$ we have

$$j\Omega = \frac{2}{T}\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}\right)}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}} + e^{-\frac{j\omega}{2}}\right)} = \frac{2j\sin(\frac{\omega}{2})}{2\cos(\frac{\omega}{2})} = j\tan\frac{\omega}{2} \Rightarrow \Omega = \tan\frac{\omega}{2}$$





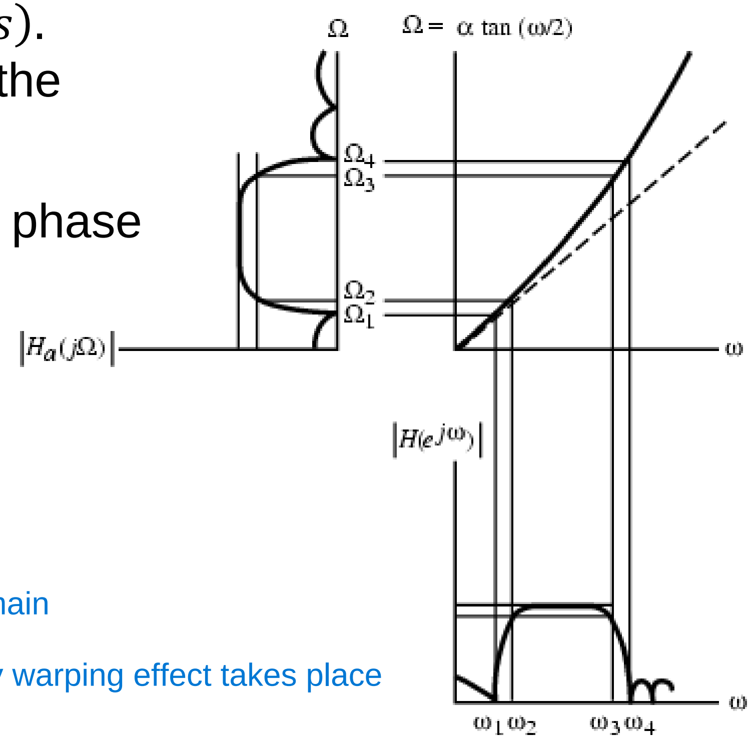
- The positive imaginary axis in the s −plane is mapped into the upper half of the unit circle in the z −plane or $\Omega \in [0, +\infty] \Rightarrow \omega \in [0, +\pi] \Rightarrow z \in [1, -1]$.
- The negative imaginary axis in the s −plane is mapped into the lower half of the unit circle in the z −plane or $\Omega \in [-\infty, 0] \Rightarrow \omega \in [-\pi, 0] \Rightarrow z \in [-1,1]$

IIR filter design steps

- IIR filter design consists of 3 steps:
  - Define the specifications of the **desired** digital filter $G(z)$.
  - Develop the specifications of the **prototype** analogue filter $H_a(s)$ by applying the bilinear transformation to specifications of $G(z)$.
  - Design $H_a(s)$.
  - Determine $G(z)$ by applying the inverse bilinear transformation to $H_a(s)$.
- As a result, the parameter T has no effect on $G(z)$ and $T = 2$ is chosen for convenience. In that case $\Omega = \tan\frac{\omega}{2}$.

Frequency warping

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**. Effect of warping are shown in the figure below.
- Steps in the design of an IIR digital filter with specified magnitude response:
  - Prewarp the critical bandedge frequencies (ω_p, ω_s) to find their analogue equivalents (Ω_p, Ω_s).
  - Design the analogue prototype filter $H_a(s)$.
  - Design the digital filter $G(z)$ by applying the inverse bilinear transformation to $H_a(s)$.
- Bilinear transformations do not preserve the phase response of the analogue filter.



Advantages of bi-linear transformation method:
- The mapping is one to one
- There is no aliasing effect
- Stable analog filter is transformed into the stable digital filter
- There is no restriction one type of filter that can be transformed
- There is one to one transformation from the s-domain to the Z- domain

Disadvantages of bi-linear transformation method:
- The mapping is non-linear in this method because of this frequency warping effect takes place

First order Butterworth lowpass analogue filter

- Consider the first order Butterworth lowpass analogue filter, with $3 - dB$ cutoff frequency at Ω_c:

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

- Applying inverse bilinear transformation to the above we get the transfer function of a first-order Butterworth lowpass digital filter $G_{LP}(z)$.
- The relation between $G_{LP}(z)$ and $H_a(s)$ is then given by

$$G_{LP}(z) = H_a(s)\Big|_{s=\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{\Omega_c(1+z^{-1})}{(1-z^{-1})+\Omega_c(1+z^{-1})}$$

- Rearranging terms we get:

$$G_{LP}(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1-az^{-1}}$$

where

you can prove these expressions easily

$$a = \frac{1-\Omega_c}{1+\Omega_c} = \frac{1-\tan\frac{\omega_c}{2}}{1+\tan\frac{\omega_c}{2}}$$

Design of IIR highpass, bandpass, and bandstop digital filters

- We now wish to design the other three types of filters, namely, highpass, bandpass and bandstop filters.
- There are so-called **frequency transformations** available, which convert an **analogue lowpass** filter **to** an **analogue** filter **of any other type**.

Two approaches can be followed:

Regarding the first point: sometimes analogue specifications are given in which case we can get digital specifications using the normalizations of Slide 5.

- **First Approach:**
  - Prewarp the digital frequency specifications of the desired digital filter $G_D(z)$ to arrive at frequency specifications of an analogue filter $H_D(s)$ of the same type.
  - Convert the frequency specifications of $H_D(s)$ into those of a prototype **analogue** lowpass filter $H_{LP}(s)$ using an appropriate frequency transformation.
    - **We will learn these transformations in the subsequent slides.**
  - Design the analogue lowpass filter $H_{LP}(s)$.
  - Convert $H_{LP}(s)$ into $H_D(s)$ using the inverse of the frequency transformation used above.
  - Obtain the filter $G_D(z)$ by applying bilinear transformation to $H_D(s)$.

$$G_D(z) \rightarrow H_D(s) \rightarrow \boldsymbol{H_{LP}(s)} \rightarrow H_D(s) \rightarrow G_D(z)$$

these filters are not yet realized

Design

Design of IIR highpass, bandpass, and bandstop digital filters

- **Second Approach:**
  - Prewarp the digital frequency specifications of the desired digital filter $G_D(z)$ to arrive at frequency specifications of an analogue filter $H_D(s)$ of the same type.
  - Convert the frequency specifications of $H_D(s)$ into those of a prototype analogue lowpass filter $H_{LP}(s)$ using a frequency transformation.
  - Design the analogue lowpass filter $H_{LP}(s)$.
  - Convert $H_{LP}(s)$ into an IIR digital transfer function $G_{LP}(z)$ using the bilinear transformation.
  - Transform $G_{LP}(z)$ into the desired digital transfer function $G_D(z)$ with an appropriate spectral transformation.
  - There are so-called **digital filter frequency transformations** available, which convert a **digital lowpass** filter **to** a **digital filter of any other type**.

$$G_D(z) \rightarrow H_D(s) \rightarrow \underset{\uparrow \atop \textbf{Design}}{\boldsymbol{H_{LP}(s)}} \rightarrow G_{LP}(z) \rightarrow G_D(z)$$

The two approaches in one slide

$$G_D(z) \rightarrow H_D(s) \rightarrow \boldsymbol{H_{LP}(s)} \rightarrow \boldsymbol{H_D(s)} \rightarrow G_D(z)$$

$$G_D(z) \rightarrow H_D(s) \rightarrow \boldsymbol{H_{LP}(s)} \rightarrow \boldsymbol{G_{LP}(z)} \rightarrow G_D(z)$$

Design

Design of analogue highpass, bandpass, and bandstop filters

Frequency transformations

- Recall the intermediate steps involved in the first design process:
  - Convert the frequency specifications of $H_D(s)$ into those of a prototype **analogue** lowpass filter $H_{LP}(s)$ using an appropriate **frequency transformation**.
  - Design the analogue lowpass filter $H_{LP}(s)$.
  - Convert $H_{LP}(s)$ into $H_D(s)$ using the inverse of the frequency transformation used above.
    - **Do NOT confuse frequency transformations to the bilinear transformation. They are different!**
- Let $\boldsymbol{s = \sigma + j\Omega}$ denote the Laplace transform variable of the prototype analogue lowpass filter $H_{LP}(s)$ and $\boldsymbol{\hat{s} = \hat{\sigma} + j\hat{\Omega}}$ denote the Laplace transform variable of the desired analog filter $H_D(\hat{s})$.
  - The mapping from s −domain to $\hat{s}$ −domain is given by the invertible transformation $s = F(\hat{s})$.
  - $H_D(\hat{s}) = H_{LP}(s)|_{s=F(\hat{s})}$
  - $H_{LP}(s) = H_D(\hat{s})|_{\hat{s}=F^{-1}(s)}$

$$G_D(z) \rightarrow \boldsymbol{H_D(s)} \rightarrow \boldsymbol{H_{LP}(s)} \rightarrow \boldsymbol{H_D(s)} \rightarrow G_D(z)$$

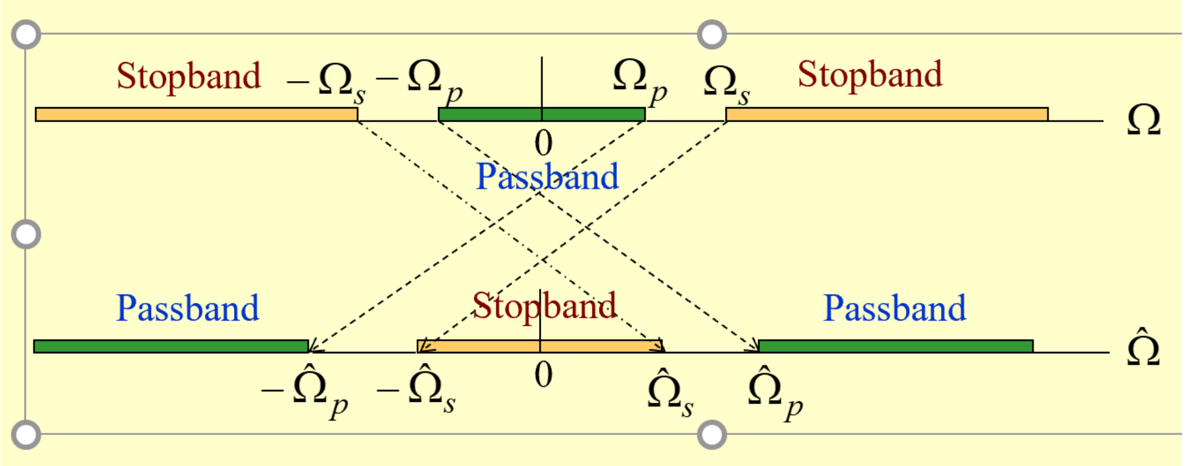
Analogue highpass filter design

- LP to HP spectral transformation

$$s = \frac{\Omega_p \widehat{\Omega}_p}{\hat{s}}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\widehat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$.

- On the imaginary axis the transformation is $j\Omega = \frac{\Omega_p \widehat{\Omega}_p}{j\widehat{\Omega}} \Rightarrow \Omega = -\frac{\Omega_p \widehat{\Omega}_p}{\widehat{\Omega}}$.
  - $0 \le \Omega \le \Omega_p \Rightarrow -\infty < \widehat{\Omega} \le -\widehat{\Omega}_p$
  - $-\Omega_p \le \Omega \le 0 \Rightarrow \widehat{\Omega}_p \le \widehat{\Omega} \le \infty$
  - $\Omega_s \le \Omega \le \infty \Rightarrow -\widehat{\Omega}_s \le \widehat{\Omega} \le 0$
  - $-\infty \le \Omega \le -\Omega_s \Rightarrow 0 \le \widehat{\Omega} \le \widehat{\Omega}_s$



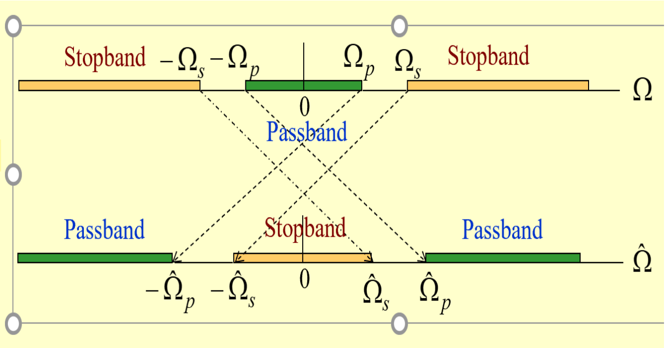
Problem: analogue Butterworth highpass filter design

- Design an analogue Butterworth highpass filter with the specifications: $\hat{F}_p = 4kHz, \hat{F}_s = 1kHz, a_p = 0.1dB,$ minimum stopband attenuation $a_s = 40dB$. For the prototype analogue lowpass filter choose $\Omega_p = 1$.

Solution

- $\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}} \Rightarrow \boldsymbol{\Omega_s} = -\frac{\Omega_p \hat{\Omega}_p}{-\hat{\Omega}_s} = \frac{1 \cdot 2\pi \hat{F}_p}{2\pi \hat{F}_s} = \frac{\hat{F}_p}{\hat{F}_s} = \frac{4000}{1000} = 4.$
- Analogue lowpass filter specifications: $\Omega_p = 1$, $\Omega_s = 4$, $a_p = 0.1dB$, $a_s = 40dB$
- We first use the function **buttord** to determine the order **N** and the $3 - dB$ cutoff frequency **Wn** of the lowpass filter $H_{LP}(s.)$
- Next, we use the function **butter** to determine the transfer function $H_{LP}(s)$ of the prototype lowpass filter.
- Finally, the lowpass filter is transformed into the desired highpass filter $H_{HP}(s)$ using the function **lp2hp**.
- The code fragments used are:

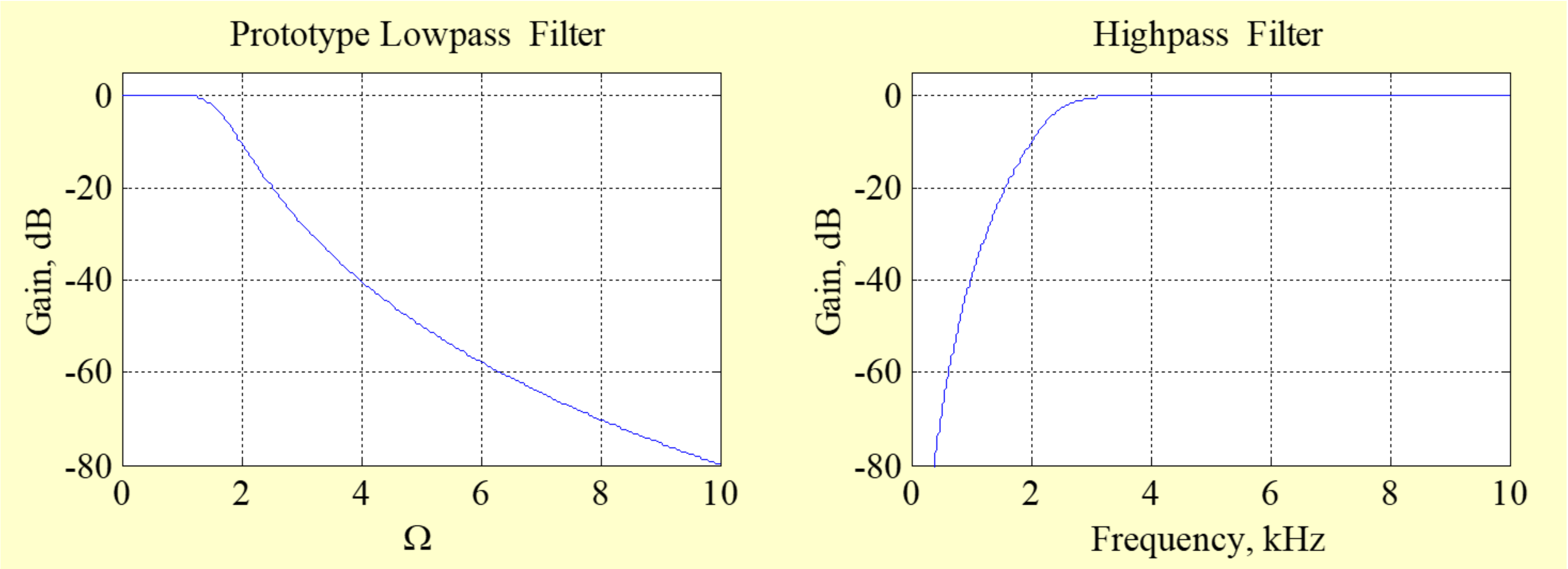
```
[N, Wn] = buttord(1, 4, 0.1, 40, 's');
[B, A] = butter(N, Wn, 's');
[num, den] = lp2hp(B, A, 4);
```



For explanation of the routines look at file: Topic 7 Filter CODES

Problem: analogue Butterworth highpass filter design cont.

- The gain responses of the two filters are shown below.



Problem: analogue Butterworth highpass filter design cont.

- The transfer function of the analogue lowpass filter $H_{LP}(s)$ can be obtained by displaying the numerator coefficient vector **B** and the denominator coefficient vector **A** and is given by

$$H_{LP}(s) = \frac{10.2405}{s^5 + 5.1533s^4 + 13.278s^3 + 21.1445s^2 + 20.8101s + 10.2405}$$

- The transfer function of the desired analogue highpass filter $H_{HP}(s)$ can be obtained by displaying the numerator coefficient vector **num** and the denominator coefficient vector **den**.
  - Note that the desired highpass filter can be designed directly using the code fragment below:

    ```
    [N, Wn] = buttord(4, 1, 0.1, 40, 's');
    [num, den] = butter(N, Wn, 'high', 's');
    ```
  - However, the LP to HP transformation is worth learning because it gives insight into the relationship between the filter coefficients and the frequency responses.

Analogue bandpass filter design

- LP to BP spectral transformation

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_0^2}{\hat{s}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$$

The transformation effectively maps the single passband of a lowpass filter to two symmetrical passbands in the bandpass filter.

where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\hat{\Omega}_{p_1}, \hat{\Omega}_{p_2}$ are the lower and upper passband edge frequencies of the desired bandpass filter $H_{BP}(\hat{s})$.

- On the imaginary axis the transformation reduces to (prove this, it's really easy)

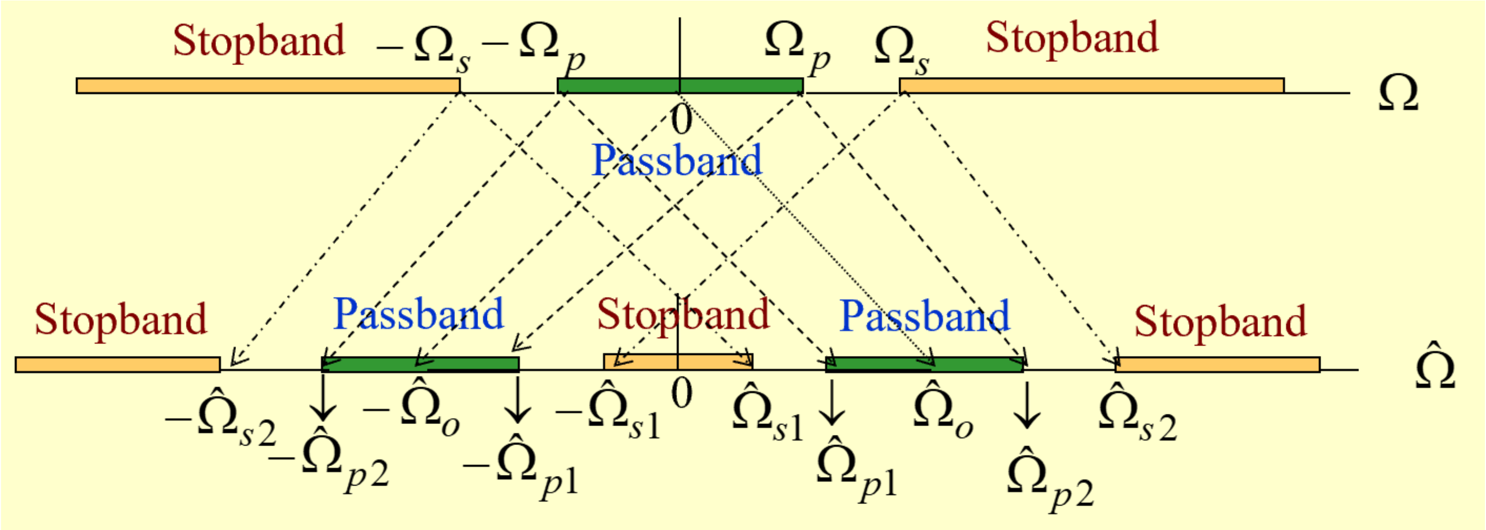
$\Omega = -\ \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$, where $\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1}$ is the **width of the passband**.

- $\Omega = 0 \Rightarrow \hat{\Omega} = \hat{\Omega}_0$ which is called the **passband centre frequency** of the BP filter.
- The passband edge frequency $\pm\Omega_p$ is mapped into $\mp\hat{\Omega}_{p_1}$ and $\pm\hat{\Omega}_{p_2}$, lower and upper passband edge frequencies.
- The stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s_1}$ and $\pm\hat{\Omega}_{s_2}$, lower and upper stopband edge frequencies.
- $-\Omega_p \le \Omega \le \Omega_p$ is mapped to $-\hat{\Omega}_{p_2} \le \Omega \le -\hat{\Omega}_{p_1}$ and $\hat{\Omega}_{p_1} \le \Omega \le \hat{\Omega}_{p_2}$.
- The following holds: $\hat{\Omega}_0^2 = \hat{\Omega}_{p_1}\hat{\Omega}_{p_2} = \hat{\Omega}_{s_1}\hat{\Omega}_{s_2} \Rightarrow \log(\hat{\Omega}_0) = \frac{\log(\hat{\Omega}_{p_1}) + \log(\hat{\Omega}_{p_2})}{2}$
  - If band edge frequencies do not satisfy the above condition, one of the frequencies must be changed to a new value so that the condition is satisfied.

For the proofs of this slide you can look at the file: Topic 7 OPTIONAL LH2BP notes

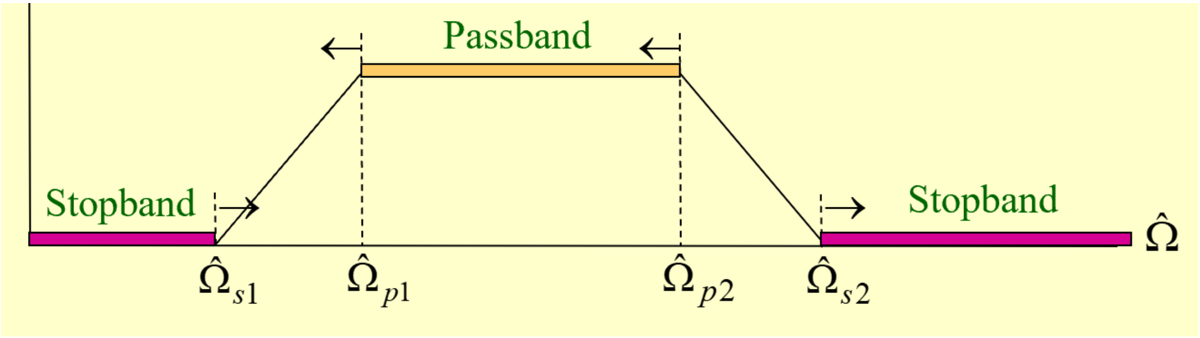
Analogue bandpass filter design cont.

$$\Omega = -\ \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$$



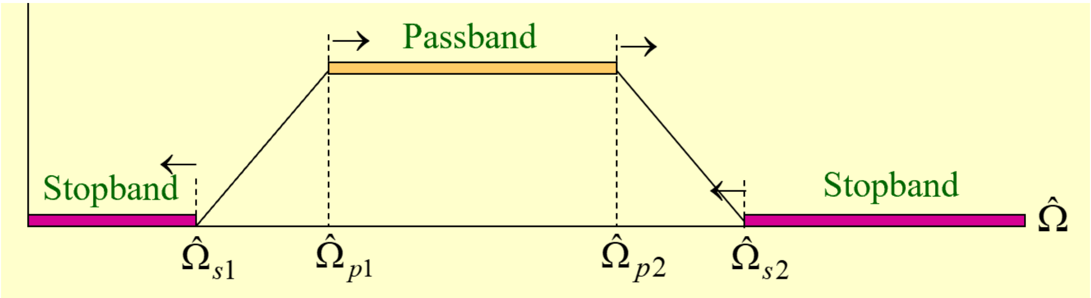
Analogue bandpass filter design cont.

- **Case 1:** $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} > \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$
  To make $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ we can either increase any of the stopband edges or decrease any of the passband edges, for example:
  - Decrease $\widehat{\Omega}_{p_1}$ to $\widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}/\widehat{\Omega}_{p_2}$ which makes a larger passband and shorter leftmost transition band.
  - Increase $\widehat{\Omega}_{s_1}$ to $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2}/\widehat{\Omega}_{s_2}$ which makes no change in passband and shorter leftmost transition band.
    - **Note that the condition $\widehat{\Omega}_0^2 = \widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ can also be satisfied by decreasing $\widehat{\Omega}_{p_2}$ which is not acceptable as the passband is reduced from the desired value.**
    - **Alternatively, the condition can be satisfied by increasing $\widehat{\Omega}_{s_2}$ which is not acceptable as the rightmost transition band is increased.**



Analogue bandpass filter design cont.

- **Case 2:** $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} < \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$
  To make $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ we can either decrease any of the stopband edges or increase any of the passband edges as shown below:



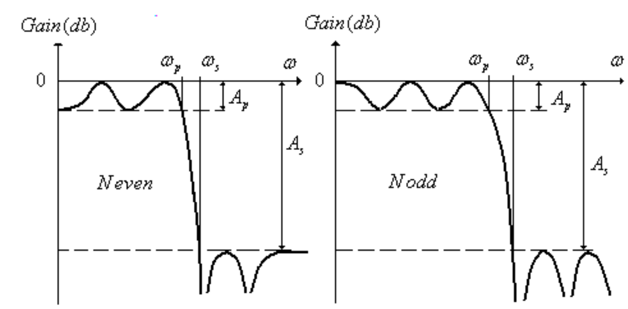
  - Increase $\widehat{\Omega}_{p_2}$ to $\widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}/\widehat{\Omega}_{p_1}$ which makes a larger passband and shorter rightmost transition band.
  - Decrease $\widehat{\Omega}_{s_2}$ to $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2}/\widehat{\Omega}_{s_1}$ which makes no change in passband and shorter rightmost transition band.
    - **Note that the condition $\widehat{\Omega}_0^2 = \widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ can also be satisfied by increasing $\widehat{\Omega}_{p_1}$ which is not acceptable as the passband is reduced from the desired value.**
    - **Alternatively, the condition can be satisfied by decreasing $\widehat{\Omega}_{s_1}$ which is not acceptable as the leftmost transition band is increased.**

Problem: analogue elliptic bandpass filter design

- Design an analogue elliptic bandpass filter with the specifications: $\hat{F}_{p_1} = 4kHz, \hat{F}_{p_2} = 7kHz, \hat{F}_{s_1} = 3kHz, \hat{F}_{s_2} = 8kHz, a_p = 1dB, a_s = 22dB$. For the prototype analogue lowpass filter choose $\Omega_p = 1$.

  Solution

  - $\hat{F}_{p_1}\hat{F}_{p_2} = 28 \times 10^6 \ \ Hz, \hat{F}_{s_1}\hat{F}_{s_2} = 24 \times 10^6 \ \ Hz$
  - Since $\hat{F}_{p_1}\hat{F}_{p_2} > \hat{F}_{s_1}\hat{F}_{s_2}$ we choose $\hat{F}'_{p_1} = \frac{\hat{F}_{s_1}\hat{F}_{s_2}}{\hat{F}_{p_2}} = \frac{24}{7} = 3.42857kHz.$
  - The width of the passband increases from $7kHz - 4kHz = \mathbf{3kHz}$ to $7kHz - \frac{24}{7}kHz = \mathbf{\frac{25}{7} = 3.571428kHz}$
  - $\hat{\Omega}_0^2 = \hat{\Omega}_{s_1}\hat{\Omega}_{s_2} = 4\pi^2 24 \Rightarrow \hat{F}_0 = \frac{\hat{\Omega}_0}{2\pi} = \sqrt{24} = 4.8989795kHz.$
  - $\Omega_s = - \ \Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}_{s_1}^2}{-\hat{\Omega}_{s_1}(\hat{\Omega}_{p_2} - \hat{\Omega}'_{p_1})} = \frac{\hat{F}_0^2 - \hat{F}_{s_1}^2}{\hat{F}_{s_1}(\hat{F}_{p_2} - \hat{F}'_{p_1})} = \frac{24-9}{3\times(7-\frac{24}{7})} = 1.4$
  - The analogue lowpass filter specifications are: $\Omega_p = 1, \Omega_s = 1.4, a_p = 1dB, a_s = 22dB$



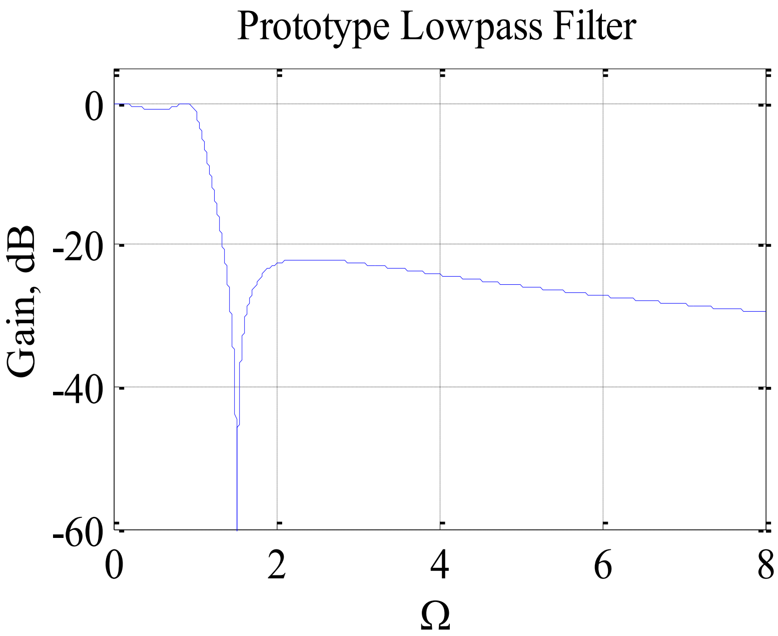
Problem: analogue bandpass filter design

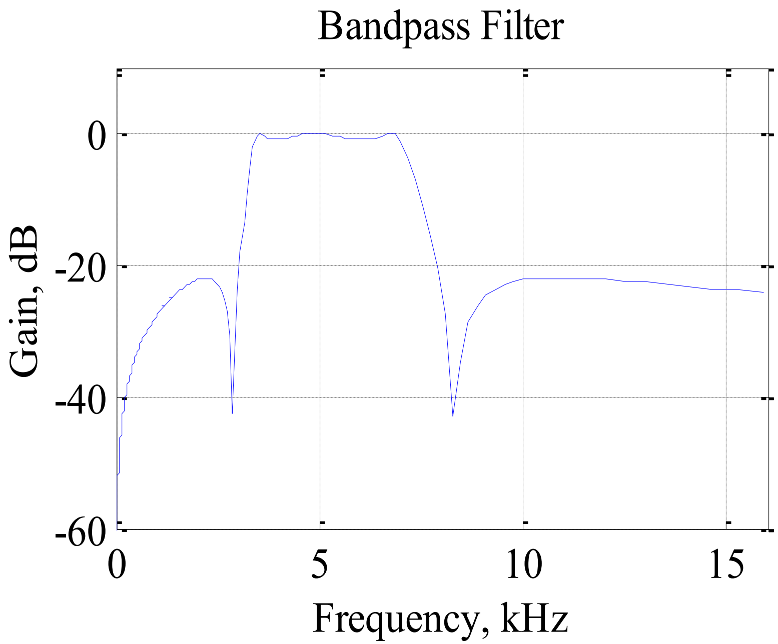
- The analogue lowpass filter specifications are:
  $\Omega_p = 1, \Omega_s = 1.4, a_p = 1dB, a_s = 22dB$
  - We first use the function **ellipord** to determine the order **N** and the passband edge angular frequency **Wn** of the prototype analogue lowpass filter $H_{LP}(s)$
  - Next, we use the function **ellip** to determine the transfer function $H_{LP}(s)$ of the prototype lowpass filter.
  - Finally, the lowpass filter is transformed into the desired bandpass filter $H_{BP}(s)$ using the function **lp2bp**.
  - Code fragments used

```
[N, Wn] = ellipord(1, 1.4, 1, 22, 's');
[B, A] = ellip(N, 1, 22, Wn, 's');
[num, den] = lp2bp(B, A, 4.8989795, 25/7);
```

Problem: analogue bandpass filter design cont.

- The gain responses of the two filters are shown below.





Problem: analogue bandpass filter design cont.

- The transfer function of the analogue lowpass filter $H_{LP}(s)$ can be obtained by displaying the numerator coefficient vector **B** and the denominator coefficient vector **A** and is given by

$$H_{LP}(s) = \frac{0.275s^2 + 0.63845}{s^3 + 0.965577s^2 + 1.243426s + 0.63844976}$$

- The transfer function of the desired analogue bandpass filter $H_{BP}(s)$ can be obtained by displaying the numerator coefficient vector **num** and the denominator coefficient vector **den**.
  - Note that the desired bandpass filter can be designed directly using the code fragment below:

```
Wp=[3.42857 7]*2*pi; Ws=[3 8 ]*2*pi
[N, Wn] = ellipord(Wp, Ws, 1, 22, ‘s’);
[num, den] = ellip(N, 1, 22, Wn, ‘s’);
```

Analogue bandstop filter design

- LP to BS Spectral Transformation

$$s = \Omega_s \frac{\hat{s}(\widehat{\Omega}_{s_2} - \widehat{\Omega}_{s_1})}{\hat{s}^2 + \widehat{\Omega}_0^2}$$

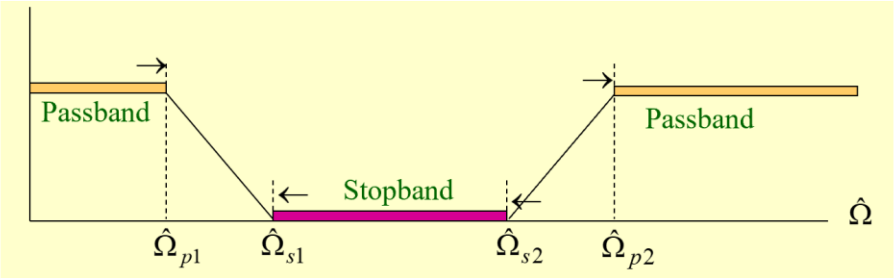
  where Ω_s is the stopband edge frequency of $H_{LP}(s)$ and $\widehat{\Omega}_{s_1}, \widehat{\Omega}_{s_2}$ are the lower and upper stopband edge frequencies of the desired bandstop filter $H_{BS}(\hat{s})$.
- On the imaginary axis the transformation is

$$\Omega = \Omega_s \frac{\widehat{\Omega}(\widehat{\Omega}_{s_2} - \widehat{\Omega}_{s_1})}{\widehat{\Omega}_0^2 - \widehat{\Omega}^2}$$

  where $\widehat{\Omega}_{s_2} - \widehat{\Omega}_{s_1}$ is the width of the stopband of the bandstop filter and $\widehat{\Omega}_0$ is the stopband centre frequency of the bandstop filter.
- The passband edge frequency $\pm\Omega_p$ is mapped into $\mp\widehat{\Omega}_{p_2}$ and $\pm\widehat{\Omega}_{p_1}$, lower and upper passband edge frequencies.
- The stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\widehat{\Omega}_{s_2}$ and $\pm\widehat{\Omega}_{s_1}$, lower and upper stopband edge frequencies.
- $\widehat{\Omega}_0^2 = \widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$
- If bandedge frequencies do not satisfy the above condition, then one of the frequencies needs to be changed to a new value so that the condition is satisfied.

Analogue bandstop filter design cont.

- **Case 1:** $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} > \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$
  To make $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ we can either increase any of the stopband edge or decrease any of the passband edge frequencies as shown below:

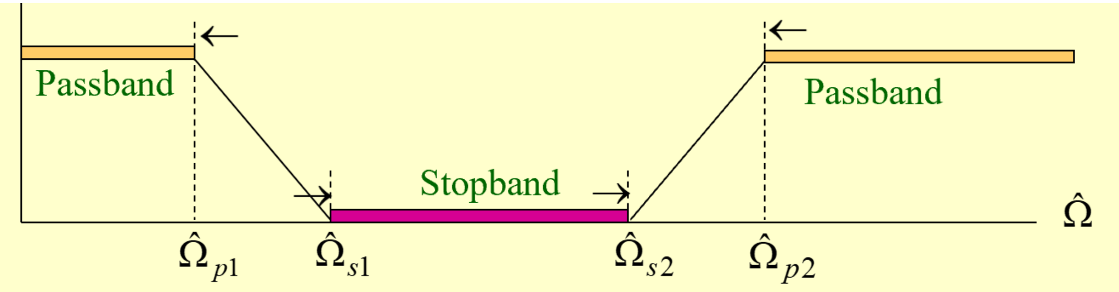


- Decrease $\widehat{\Omega}_{p_2}$ to $\widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}/\widehat{\Omega}_{p_1}$ which makes a larger high frequency passband and shorter rightmost transition band.
- Increase $\widehat{\Omega}_{s_2}$ to $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2}/\widehat{\Omega}_{s_1}$ which makes no change in passband and shorter rightmost transition band.
  - ✓ Note that the condition $\widehat{\Omega}_0^2 = \widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ can also be satisfied by decreasing $\widehat{\Omega}_{p_1}$ which is not acceptable as the low frequency passband is reduced from the desired value.
  - ➢ Alternatively, the condition can be satisfied by increasing $\widehat{\Omega}_{s_1}$ which is not acceptable as the stopband is decreased.

Analogue bandstop filter design cont.

- **Case 2:** $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} < \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$

  To make $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ we can either decrease any of the stopband edges or increase any of the passband edges as shown below:



  - Increase $\widehat{\Omega}_{p_1}$ to $\widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}/\widehat{\Omega}_{p_1}$ which makes a larger passband and shorter leftmost transition band.
  - Decrease $\widehat{\Omega}_{s_1}$ to $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2}/\widehat{\Omega}_{s_1}$ which makes no change in passband and shorter leftmost transition band.
    - Note that the condition $\widehat{\Omega}_0^2 = \widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2}$ can also be satisfied by increasing $\widehat{\Omega}_{p_2}$ which is not acceptable as the passband is reduced from the desired value.
    - Alternatively, the condition can be satisfied by decreasing $\widehat{\Omega}_{s_2}$ which is not acceptable as the stopband is decreased.

Problem: IIR highpass digital filter design

- Design a Type 1 Chebychev IIR digital highpass filter $F_p = 700Hz, F_s = 500Hz, a_p = 1dB, a_s = 32dB, F_T = 2kHz$. Choose $\Omega_p = 1$ for the prototype analogue lowpass filter.

Solution

- $\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = \frac{2\pi\times 700}{2000} = 0.7\pi$
- $\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = \frac{2\pi\times 500}{2000} = 0.5\pi$
- Prewarping the above frequencies we get
  - $\hat{\Omega}_p = \tan\frac{\omega_p}{2} = \tan\frac{0.7\pi}{2} = 1.9626105$
  - $\hat{\Omega}_s = \tan\frac{\omega_s}{2} = \tan\frac{0.5\pi}{2} = 1.0$

angular edge frequencies of the analog highpass filter

- Using $\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}} = -\frac{\hat{\Omega}_p}{\hat{\Omega}}$ we get $\Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = 1.9626105$
- Analogue lowpass filter specifications:

$$\Omega_p = 1, \Omega_s = 1.9626105, a_p = 1dB, a_s = 32dB$$

Return to Slide 13 to see a verbal description of the sequence of steps

Problem: IIR highpass digital filter design cont.

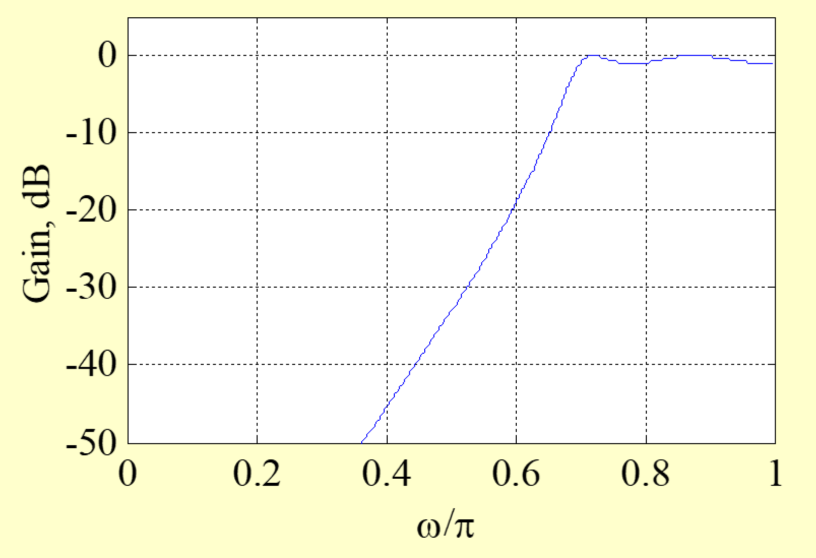
- The analogue lowpass filter specifications are: $\Omega_p = 1, \Omega_s = 1.9626105, a_p = 1dB, a_s = 32dB$
  - We first use the function **cheb1ord** to determine the order **N** and the $3-dB$ cutoff frequency **Wn** of the lowpass filter $H_{LP}(s)$.
  - Next, we use the function **cheby1** to determine the transfer function $H_{LP}(s)$ of the prototype lowpass filter.
  - After that, the lowpass filter is transformed into the desired highpass filter $H_{HP}(s)$ using the function **lp2hp**.
  - Finally, by using the routine **bilinear**, the desired digital IIR highpass filter $G_{HP}(z)$ is designed by applying the bilinear transformation to $H_{HP}(s)$.
  - Code fragments used

```
[N, Wn] = cheb1ord(1, 1.9626105, 1, 32, 's')
[B, A] = cheby1(N, 1, Wn, 's');
[BT, AT] = lp2hp(B, A, 1.9626105);
[num, den] = bilinear(BT, AT, 0.5);
```

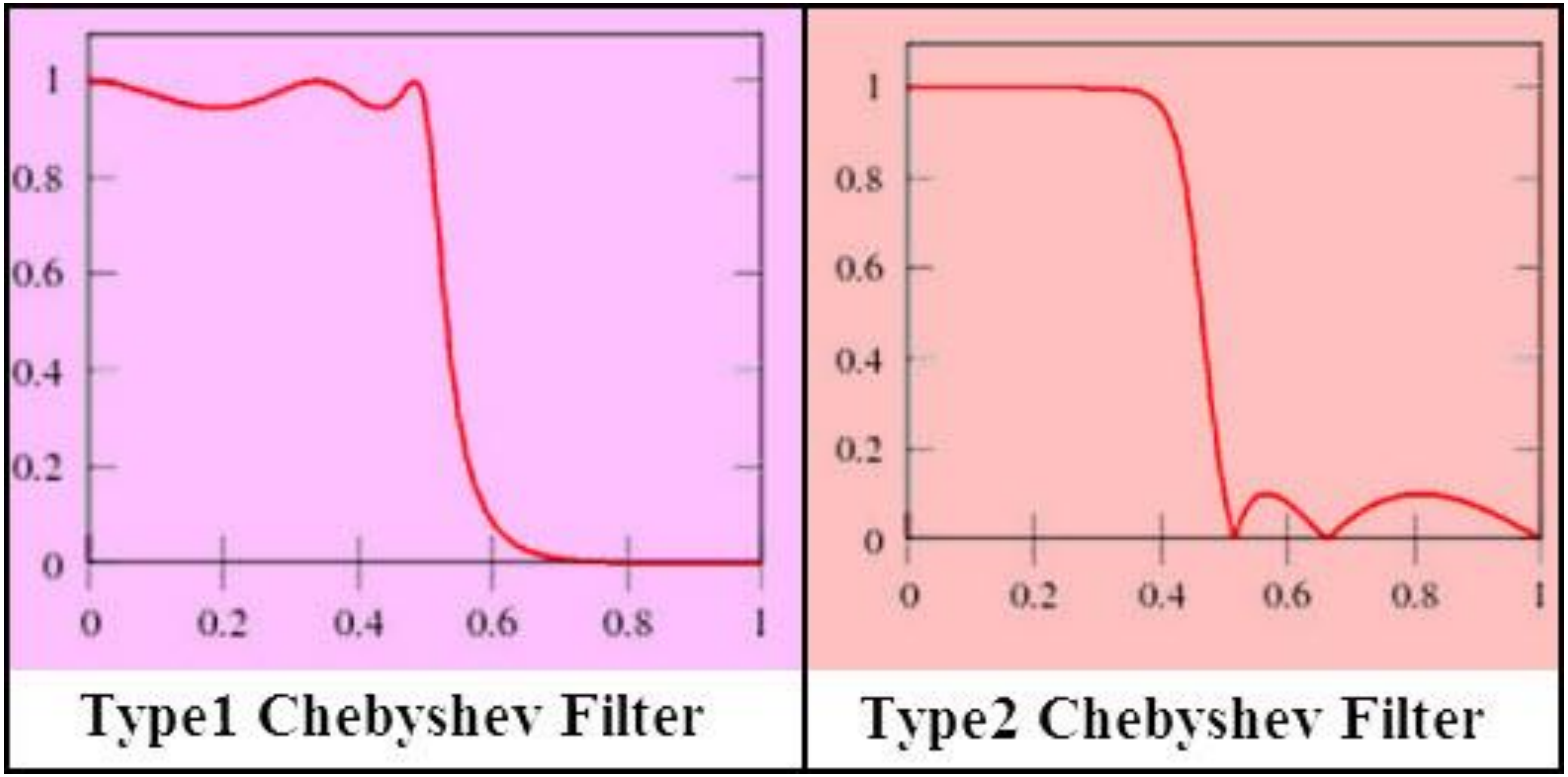
Return to Slide 13 to see a verbal description of the sequence of steps

Problem: IIR highpass digital filter design cont.

- The transfer function of the analogue lowpass filter $H_{LP}(s)$ can be obtained by displaying the numerator coefficient vector **B** and the denominator coefficient vector **A**.
- The transfer function of the desired analogue highpass filter $H_{HP}(s)$ can be obtained by displaying the numerator coefficient vector **BT** and the denominator coefficient vector **AT**.
- Finally, the transfer function of the desired digital highpass filter $G_{HP}(z)$ can be obtained by displaying the numerator coefficient vector **num** and the denominator coefficient vector **den**.
  - Note that the desired digital highpass filter can be designed directly using the routines **cheb1ord** and **cheby1**.
- The gain response of the filter is shown in the figure on the right.



Chebyshev Type I and Type II filters



Problem: IIR bandpass digital filter design

- Design of a Butterworth IIR digital bandpass filter with the specifications: $\omega_{p_1} = 0.45\pi, \omega_{p_2} = 0.65\pi, \omega_{s_1} = 0.3\pi, \omega_{s_2} = 0.75\pi,\ a_p = 1dB, a_s = 40dB$. For the prototype analogue lowpass filter we choose $\Omega_p = 1$.

  Solution

  - Using prewarping we get:
    - $\hat{\Omega}_{p_1} = \tan\frac{\omega_{p_1}}{2} = \tan\frac{0.45\pi}{2} = 0.8540807$
    - $\hat{\Omega}_{p_2} = \tan\frac{\omega_{p_2}}{2} = \tan\frac{0.65\pi}{2} = 1.6318517$
    - $\hat{\Omega}_{s_1} = \tan\frac{\omega_{s_1}}{2} = \tan\frac{0.3\pi}{2} = 0.5095254$
    - $\hat{\Omega}_{s_2} = \tan\frac{\omega_{s_2}}{2} = \tan\frac{0.75\pi}{2} = 2.41421356$
  - Width of passband: $\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1} = 0.777771$
  - $\hat{\Omega}_0^2 = \hat{\Omega}_{p_1}\hat{\Omega}_{p_2} = 1.393733$ and $\hat{\Omega}_{s_1}\hat{\Omega}_{s_2} = 1.23010325 \neq \hat{\Omega}_0^2$.
  - We set $\hat{\Omega}'_{s_1} = \hat{\Omega}_{p_1}\hat{\Omega}_{p_2} / \hat{\Omega}_{s_2} = 0.5773031$.
  - Using $\Omega = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}^2}{\hat{\Omega}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})}$ we get:

$$\Omega_s = -\Omega_p \frac{\hat{\Omega}_0^2 - \hat{\Omega}'^{2}_{s_1}}{-\hat{\Omega}'_{s_1}(\hat{\Omega}_{p_2} - \hat{\Omega}_{p_1})} = -\frac{1.393733 - 0.3332788}{-0.5773031 \times 0.777771} = 2.3617627$$

Problem: IIR bandpass digital filter design cont.

- Analogue Butterworth lowpass filter specifications:

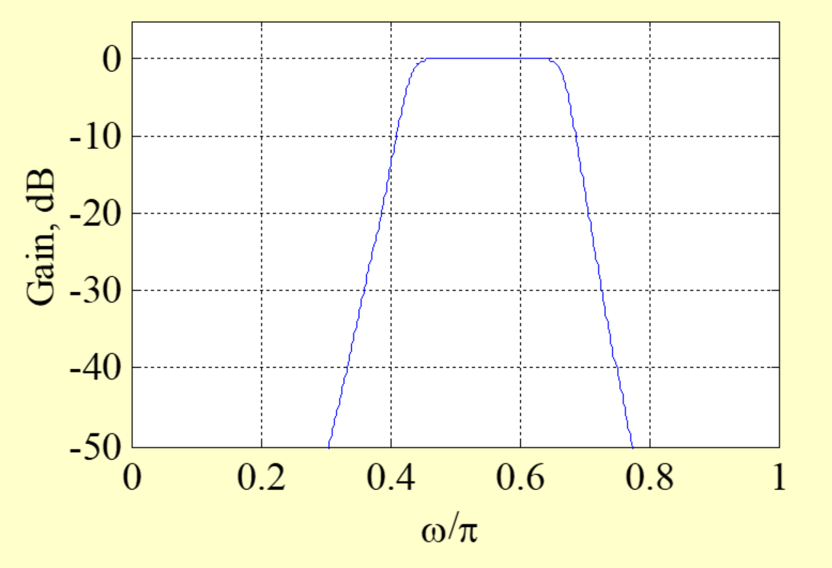
$$\Omega_p = 1, \Omega_s = 2.3617627, a_p = 1dB, a_s = 40dB$$

  - We first use the function **buttord** to determine the order **N** and the passband edge angular frequency **Wn** of the prototype analogue lowpass filter $H_{LP}(s)$.
  - Next, we use the function **butter** to determine the transfer function $H_{LP}(s)$ of the prototype lowpass filter.
  - Then, the lowpass filter is transformed into the desired bandpass filter $H_{BP}(s)$ using the function **lp2bp**.
  - Finally, by using the routine **bilinear**, the desired digital IIR bandpass filter $G_{BP}(z)$ is designed by applying the bilinear transformation to $H_{BP}(s)$.
- MATLAB code fragments used for the design

```
[N, Wn] = buttord(1, 2.3617627, 1, 40, 's')
[B, A] = butter(N, Wn, 's');
[BT, AT] = lp2bp(B, A, 1.1805647, 0.777771);
[num, den] = bilinear(BT, AT, 0.5);
```

Problem: IIR bandpass digital filter design cont.

- The transfer function of the analogue lowpass filter $H_{LP}(s)$ can be obtained by displaying the numerator coefficient vector **B** and the denominator coefficient vector **A**.
- The transfer function of the desired analogue bandpass filter $H_{BP}(s)$ can be obtained by displaying the numerator coefficient vector **BT** and the denominator coefficient vector **AT**.
- Finally, the transfer function of the desired digital bandpass filter $G_{BP}(z)$ can be obtained by displaying the numerator coefficient vector **num** and the denominator coefficient vector **den**.
  - Note that the desired digital bandpass filter can be designed directly using the routines **buttord** and **butter**.
- The gain response of the filter is shown.



Problem: IIR bandstop digital filter design

- Design of an elliptic IIR digital bandstop filter with the specifications: $\omega_{p_1} = 0.3\pi, \omega_{p_2} = 0.75\pi, \omega_{s_1} = 0.45\pi, \omega_{s_2} = 0.65\pi,\ a_p = 1dB, a_s = 40dB$. For the prototype analogue lowpass filter we choose $\Omega_s = 1$.

Solution

- Using prewarping we get:
  - $\widehat{\Omega}_{p_1} = \tan\frac{\omega_{p_1}}{2} = \tan\frac{0.3\pi}{2} = 0.5095254$
  - $\widehat{\Omega}_{p_2} = \tan\frac{\omega_{p_2}}{2} = \tan\frac{0.75\pi}{2} = 2.4142136$
  - $\widehat{\Omega}_{s_1} = \tan\frac{\omega_{s_1}}{2} = \tan\frac{0.45\pi}{2} = 0.8540806$
  - $\widehat{\Omega}_{s_2} = \tan\frac{\omega_{s_2}}{2} = \tan\frac{0.65\pi}{2} = 1.6318517$
- Width of stopband: $\widehat{\Omega}_{s_2} - \widehat{\Omega}_{s_1} = 0.777771$
- $\widehat{\Omega}_0^2 = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2} = 1.393733$ and $\widehat{\Omega}_{p_1}\widehat{\Omega}_{p_2} = 1.23010325 \neq \widehat{\Omega}_0^2$
- We therefore modify $\widehat{\Omega}_{p_1}$ so that $\widehat{\Omega}'_{p_1}$ and $\widehat{\Omega}_{p_2}$ exhibit geometric symmetry with respect to $\widehat{\Omega}_0^2$.
- We set $\widehat{\Omega}'_{p_1} = \widehat{\Omega}_{s_1}\widehat{\Omega}_{s_2} / \widehat{\Omega}_{p_2} = 0.5773031$.

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Problem: IIR bandstop digital filter design cont.

- Using $\Omega = \Omega_s \frac{\hat{\Omega}(\hat{\Omega}_{s_2} - \hat{\Omega}_{s_1})}{\hat{\Omega}_0^2 - \hat{\Omega}^2}$ we get

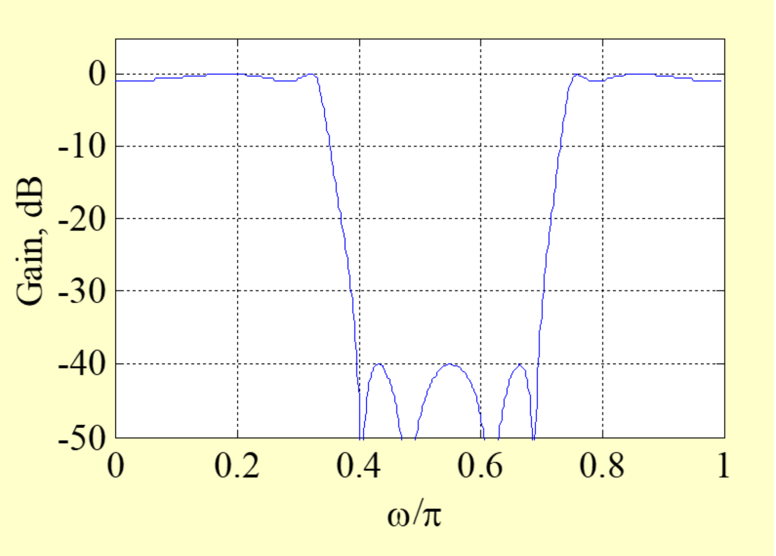
$$\Omega_p = \Omega_s \frac{\hat{\Omega}'_{p_1}\ (\hat{\Omega}_{s_2} - \hat{\Omega}_{s_1})}{\hat{\Omega}_0^2 - {\hat{\Omega}'_{p_1}}^2} = 1 \cdot \frac{0.5773031 \times 0.777771}{1.393733 - 0.3332787} = 0.4234126$$

- Analogue elliptic lowpass filter specifications:

$$\Omega_p = 0.4234126, \Omega_s = 1, a_p = 1dB, a_s = 40dB$$

- MATLAB code fragments used for the design

```
[N, Wn] = ellipord(0.4234126, 1, 1, 40, 's');
[B, A] = ellip(N, 1, 40, Wn, 's');
[BT, AT] = lp2bs(B, A, 1.1805647, 0.777771);
[num, den] = bilinear(BT, AT, 0.5);
```



Spectral transformations of IIR digital filters: Objective

- The objective of this section is to be able to modify the characteristics of a digital IIR filter to meet some new specifications.
  - For example, after a lowpass filter with a passband edge at $2kHz$ has been designed, it may be required to move the passband edge to $2 \pm f_0\ kHz$.
  - It is possible to transform a given lowpass digital IIR transfer function $G_L(z)$ into another digital transfer function $G_D(\hat{z})$ that could be another lowpass, highpass, bandpass or bandstop filter.
  - We describe here the spectral transformations that can be used to implement the above type of transformations.

Spectral transformations of IIR digital filters

- z^{-1} is used to denote the unit delay in the prototype lowpass filter $G_L(z)$ and $\hat{z}^{-1}$ will be used to denote the unit delay in the transformed filter $G_D(\hat{z})$ to avoid confusion.
- The unit circles in the $z-$ and $\hat{z}-$ planes are defined by $z = e^{j\omega}$, $\hat{z} = e^{j\hat{\omega}}$.
- The transformation from the z –domain to the $\hat{z}-$ domain is given by

$$z = F(\hat{z})$$

- The transformation from $G_L(z)$ to $G_D(\hat{z})$ is given by

$$G_D(\hat{z}) = G_L\big(F(\hat{z})\big)$$

- Properties of the function $F(\hat{z})$:
  - $F(\hat{z})$ must be a rational function of $\hat{z}$.
  - The inside of the $z-$ plane must be mapped to the inside of the $\hat{z}-$ plane in order to ensure stability of the new filter.
  - The unit circle of the $z-$ plane must be mapped to the unit circle of the $\hat{z}-$ plane.

Spectral transformations of IIR digital filters

- From $z = F(\hat{z})$, it is $|z| = |F(\hat{z})|$, where

$$|F(\hat{z})| \begin{cases} > 1 & \text{if} \quad |z| > 1 \\ = 1 & \text{if} \quad |z| = 1 \\ < 1 & \text{if} \quad |z| < 1 \end{cases}$$

- Recall that a stable allpass function $A(z)$ satisfies the condition.

$$|A(z)| \begin{cases} < 1 & \text{if} \quad |z| > 1 \\ = 1 & \text{if} \quad |z| = 1 \\ > 1 & \text{if} \quad |z| < 1 \end{cases}$$

- Therefore, $\frac{1}{F(\hat{z})}$ must be a stable allpass function whose general form is

$$\frac{1}{F(\hat{z})} = \pm \prod_{l=1}^{L} \left(\frac{1 - a_l^* \hat{z}}{\hat{z} - a_l} \right), \ |a_l| < 1$$

$$F(\hat{z}) = = \pm \prod_{l=1}^{L} \left(\frac{\hat{z} - a_l}{1 - a_l^* \hat{z}} \right)$$

Lowpass to lowpass spectral transformation

- To transform a lowpass filter $G_L(z)$ with a cutoff frequency ω_c to another lowpass filter $G_D(\hat{z})$ with a cutoff frequency $\widehat{\omega}_c$, the transformation is

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}} = \frac{1 - \alpha\hat{z}}{\hat{z} - \alpha}$$

  where α is a function of the two specified cutoff frequencies.

- On the unit circle we have

$$e^{-j\omega} = \frac{e^{-j\widehat{\omega}} - \alpha}{1 - \alpha e^{-j\widehat{\omega}}}$$

- From the above we get

$$e^{-j\omega} \mp 1 = \frac{e^{-j\widehat{\omega}} - \alpha}{1 - \alpha e^{-j\widehat{\omega}}} \mp 1 = (1 \pm \alpha)\,\frac{e^{-j\widehat{\omega}} \mp 1}{1 - \alpha e^{-j\widehat{\omega}}}$$

- Taking the ratio of the above two expressions

$$\frac{e^{-j\omega} - 1}{e^{-j\omega} + 1} = \frac{1 + \alpha}{1 - \alpha}\,\frac{e^{-j\widehat{\omega}} - 1}{e^{-j\widehat{\omega}} + 1} \Rightarrow \tan\left(\frac{\omega}{2}\right) = \frac{1 + \alpha}{1 - \alpha}\tan\left(\frac{\widehat{\omega}}{2}\right)$$

- Solving we get

$$\alpha = \frac{\sin\left(\frac{\omega_c - \widehat{\omega}_c}{2}\right)}{\sin\left(\frac{\omega_c + \widehat{\omega}_c}{2}\right)}$$

Example: Lowpass to lowpass spectral transformation

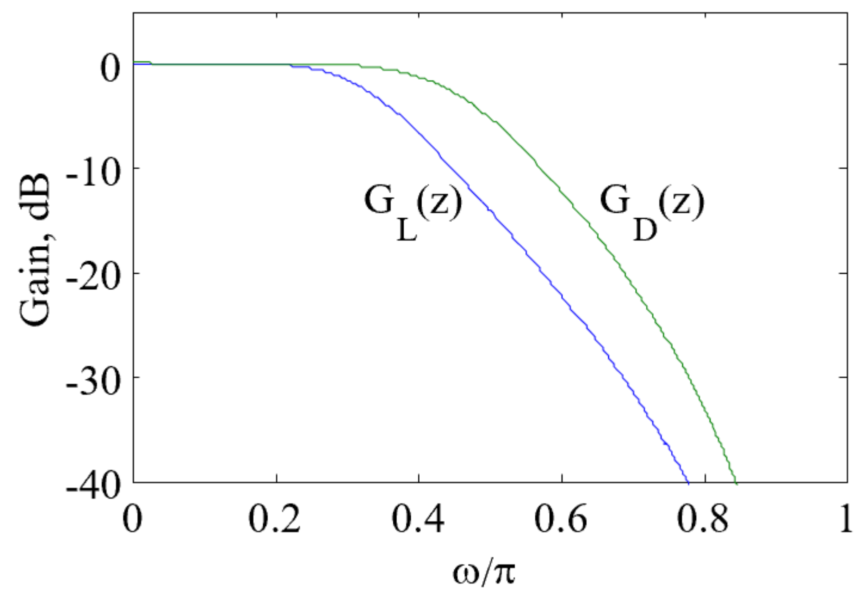
- Consider the lowpass digital filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$$

with a passband from DC to 0.25π, with a $0.5dB$ ripple. Redesign the above filter to move the passband edge to 0.35π.

Solution

- $\alpha = \frac{\sin(\frac{\omega_c-\widehat{\omega}_c}{2})}{\sin(\frac{\omega_c+\widehat{\omega}_c}{2})} = \frac{-\sin(\frac{0.1\pi}{2})}{\sin(\frac{0.6\pi}{2})} = -0.1934$
- $G_D(\hat{z}) = G_L(z)\Big|_{z^{-1}=\frac{\hat{z}^{-1}-a}{1-a\hat{z}^{-1}}=\frac{\hat{z}^{-1}+0.1934}{1+0.1934\hat{z}^{-1}}}$

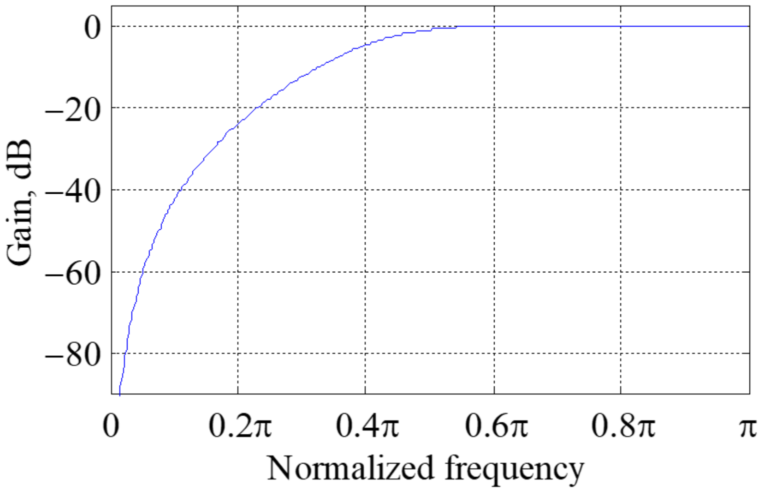


Lowpass to highpass spectral transformation

- Desired transformation $z^{-1} = -\frac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}}$.
- The transformation parameter α is given by $\alpha = -\frac{\cos(\frac{\omega_c+\widehat{\omega}_c}{2})}{\cos(\frac{\omega_c-\widehat{\omega}_c}{2})}$.
- ω_c is the cutoff frequency of the lowpass filter and $\widehat{\omega}_c$ is the cutoff frequency of the desired highpass filter.
- **Example:** Transform the lowpass filter $G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593z^{-1})(1-0.6763z^{-1}+0.3917z^{-2})}$ with a passband edge of 0.25π to a highpass filter with a passband edge at 0.55π.

Solution

- $\alpha = -\frac{\cos\left(\frac{0.8\pi}{2}\right)}{\cos\left(\frac{0.3\pi}{2}\right)} = -0.3468$
- $z^{-1} = -\frac{\hat{z}^{-1}-0.3468}{1-0.3468\hat{z}^{-1}}$
- $G_D(\hat{z}) = G(z)\Big|_{z^{-1}=-\frac{\hat{z}^{-1}-0.3468}{1-0.3468\hat{z}^{-1}}}$



Lowpass to bandpass spectral transformation

- Desired transformation $z^{-1} = -\frac{\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + \frac{\beta-1}{\beta+1}}{\frac{\beta-1}{\beta+1}\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + 1}$.
- The parameters α and β are given by
  - $\alpha = \frac{\cos((\widehat{\omega}_{c2}+\widehat{\omega}_{c1})/2)}{\cos((\widehat{\omega}_{c2}-\widehat{\omega}_{c1})/2)}$
  - $\beta = \cot((\widehat{\omega}_{c2} - \widehat{\omega}_{c1})/2)\tan(\omega_c/2)$
- ω_c is the cutoff frequency of the lowpass filter and $\widehat{\omega}_{c1}, \widehat{\omega}_{c2}$ are the desired lower and upper cutoff frequencies of the bandpass filter.
- **Special Case** - The transformation can be simplified if $\omega_c = \widehat{\omega}_{c2} - \widehat{\omega}_{c1}$.
  - Then the transformation reduces to:

$$z^{-1} = -\hat{z}^{-1}\frac{\hat{z}^{-1}-\alpha}{1-\alpha\hat{z}^{-1}}$$

with $\alpha = \widehat{\omega}_0$, with $\widehat{\omega}_0$ denoting the desired center frequency of the bandpass filter.

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Lowpass to bandstop spectral transformation

- Desired transformation $z^{-1} = -\frac{\hat{z}^{-2}-\frac{2\alpha}{\beta+1}\hat{z}^{-1}+\frac{\beta-1}{\beta+1}}{\frac{\beta-1}{\beta+1}\hat{z}^{-2}-\frac{2\alpha}{\beta+1}\hat{z}^{-1}+1}.$
- The parameters α and β are given by
  - $\alpha = \frac{\cos((\widehat{\omega}_{c2}+\widehat{\omega}_{c1})/2)}{\cos((\widehat{\omega}_{c2}-\widehat{\omega}_{c1})/2)}$
  - $\beta = \cot((\widehat{\omega}_{c2} - \widehat{\omega}_{c1})/2)\tan(\omega_c/2)$
- ω_c is the cutoff frequency of the lowpass filter and $\widehat{\omega}_{c1}, \widehat{\omega}_{c2}$ are the desired lower and upper cutoff frequencies of the bandpass filter.

Digital spectral transformations: general comments

- When you come across with the frequency ω_c, note that it is not uniquely defined. It is a frequency that marks the end of the passband. In some cases, it can be considered to be ω_p, in some cases ω_s, in other cases the $3dB$ cut-off frequency.
- It should be noted that digital spectral transformations can be used only to map one frequency point ω_c in the magnitude response of the lowpass prototype filter into a new position $\widehat{\omega}_c$ with the same magnitude response value for the transformed lowpass and highpass filters; or into two new positions, $\widehat{\omega}_{c_1}$ and $\widehat{\omega}_{c_2}$, with the same magnitude response values for the transformed bandpass and bandstop filters. **Hence, it is possible only to map either the passband edge or the stopband edge of the lowpass prototype filter onto the desired position(s) but not both.**
- Working out spectral transformations by hand can be extremely tedious and prone to errors.
- The Symbolic Toolbox of Matlab greatly simplifies this job.