IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2017

Mathematics for Signals and Systems

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates: none

The Questions

1. (a) Find a basis for the subspace $V = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 0\}$. [2]

(b) Consider the following subspace of $\mathbb{R}^4$:

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 3 \\ -4 \end{bmatrix} \right\}.$$

Find a basis for S. [3]

(c) Find the dimension and construct a basis for the four subspaces associated with

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 0 & 1 \\ 2 & 6 & -1 & 1 \end{bmatrix}.$$

[5]

(d) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 2 & 0 & 2 & 4 \end{bmatrix}.$$

Find a basis for its null space. [5]

(e) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix},$$

where $t_1, t_2, t_3 \in \mathbb{R}$.

    i. Find the determinant of $\mathbf{A}$ [3]

    ii. For what values of t_1, t_2 and t_3 is $\mathbf{A}$ invertible? [2]

2. (a) Compute the trace of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

[3]

(b) Consider the set of data: $x = \{0, 1, 2, 4, 5\}$ and $y = \{-1.5, 0, 0, 1.5, 1\}$

   i. Make a plot of the data

[2]

   ii. Determine the best least-squares line that fits this data and plot the line

[3]

   iii. Compute the error $e_1 = \sum_{i=1}^{5} |y_i - a_1 x_i - b_1|^2$, where a_1, b_1 are the coefficients computed in part ii.

[2]

   iv. Approximate the data with the line $a_2 x + b_2$ that goes through the points $(x, y) = (0, -1.5)$ and $(x, y) = (5, 1)$. Compare the error $e_2 = \sum_{i=1}^{5} |y_i - a_2 x_i - b_2|^2$ with e_1.

[3]

   v. You have now been told that the points $(x, y) = (0, -1.5)$ and $(x, y) = (1, 0)$ are more reliable than the other three points, for this reason you want to find a_3, b_3 that minimize

$$e_3 = \sum_{i=1}^{5} |w_i(y_i - a_3 x_i - b_3)|^2, \tag{1}$$

where w_i, $i = 1, .., 5$ are properly chosen weights. Find the line $a_3 x + b_3$ that minimizes e_3 for weights $w_1 = w_2 = 2$ and $w_3 = w_4 = w_5 = 1$.

[3]

(c) Given a full column rank matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $m \geq n$, prove that $\mathbf{A}^H \mathbf{A}$ is square and non-singular.

[4]

3. (a) Consider the matrix:
$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

Is $\mathbf{P}$ an orthogonal projection matrix? Justify your answer.

[2]

(b) Suppose that $\tilde{\mathbf{P}}$ is an orthogonal projection matrix,

i. Can you claim that $(\mathbf{I} - \tilde{\mathbf{P}})$ is also an orthogonal projection? Justify your answer. Here $\mathbf{I}$ is the identity matrix.

[3]

ii. Determine the null space of $(\mathbf{I} - \tilde{\mathbf{P}})$.

[3]

(c) Consider the systems of linear equations $\mathbf{y} = \mathbf{Ax}$ with $\mathbf{A}$ given by:

$$\mathbf{A} = \begin{bmatrix} 5\sqrt{2} & 0 & 0 \\ -5\sqrt{2} & 0 & 0 \end{bmatrix}.$$

Given this $\mathbf{A}$, if a solution exists it is not unique.

i. Can you claim that the system has always at least one solution? Justify your answer.

[2]

ii. Assume that $\mathbf{y} = [2, 1]^T$, find an (exact or approximate) solution to $\mathbf{y} = \mathbf{Ax}$ and justify your answer.

[3]

Question 3 continues on the next page

(d) Find the singular values of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

[5]

(e) You want to find a rank-one matrix that best approximates the matrix

$$\mathbf{E} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

where the approximation error is measured using the Frobenius norm. Which of the following matrices represents the best rank-one approximation of $\mathbf{E}$? Briefly justify your answer.

i.
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

ii.
$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

iii.
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

[2]