Digital Signal Processing

Topic 4

Introduction to Digital Filters Part 2

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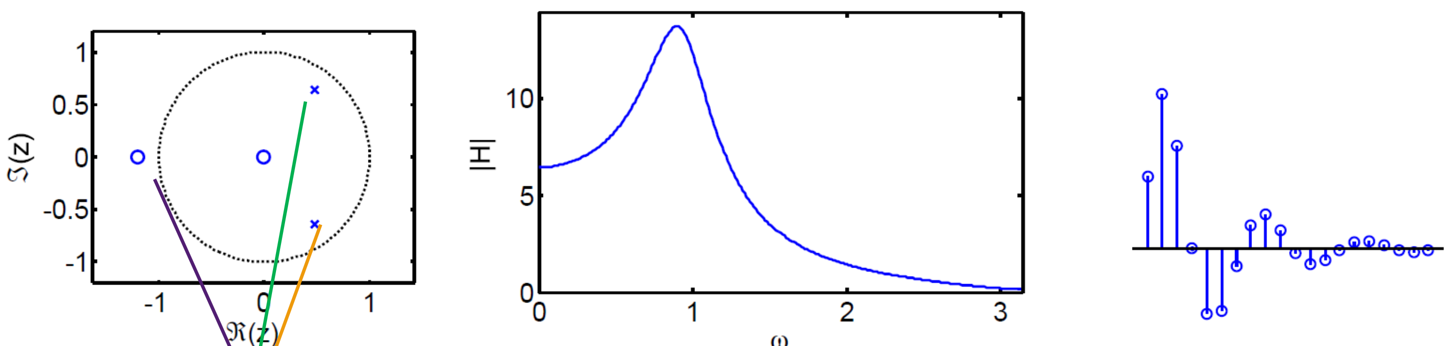
Transformations of z: Negating z

- Suppose that $H_{\text{Neg}}(z) = H(-z)$. Then $H_{\text{Neg}}(z)$ has the following two properties:
  - **Pole and zero locations are negated.**
    - If z_0 is a zero of $H(z)$, then $H_{\text{Neg}}(-z_0) = H(z_0) = 0$. Hence, $-z_0$ is a zero of $H_{\text{Neg}}(z)$.
    - We can write $-z_0 = e^{-j\pi}z_0$
    - Suppose that $z_0 = re^{j\theta} \Rightarrow -z_0 = e^{-j\pi}re^{j\theta} = re^{j(-\pi+\theta)}$
    - Therefore, each zero z_0 is transformed to a zero with the same magnitude and phase shifted by π.
    - The same statement applies to poles.
  - **The frequency response is flipped and conjugated.**
    - The frequency response is given by $H_{\text{Neg}}(e^{j\omega}) = H(-e^{j\omega}) = H(e^{-j\pi}e^{j\omega}) = H(e^{j(\omega-\pi)})$. This corresponds to shifting the frequency response by $\pi \text{rads/sample}$ or equivalently $-\pi \text{rads/sample}$.
    - If all the coefficients $a[n]$ and $b[n]$ are real-valued, then the frequency response has conjugate symmetry, i.e., $H(e^{-j\omega}) = H^*(e^{j\omega})$. In this case we can write $H_{\text{Neg}}(e^{j\omega}) = H(e^{j(\omega-\pi)}) = H^*(e^{j(\pi-\omega)})$. make sure you understand why

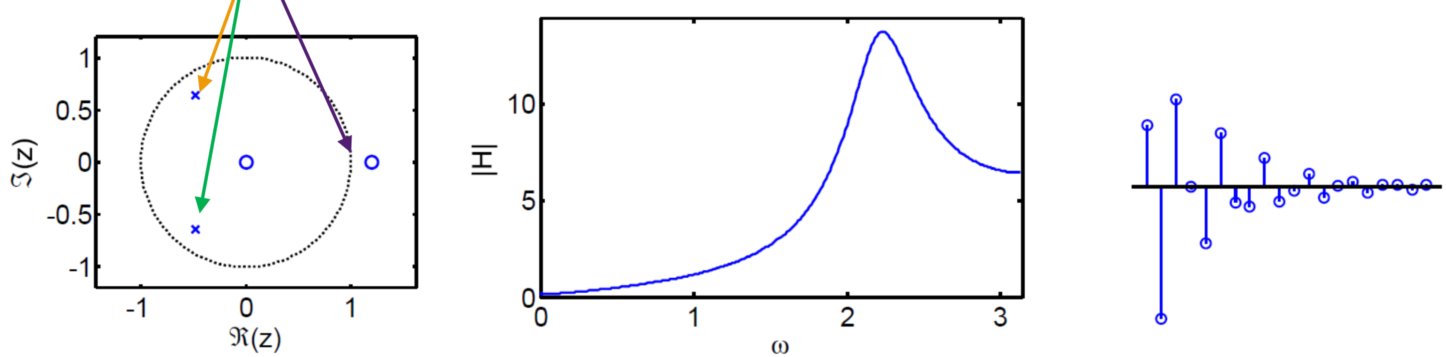
Example: Negating z

- Given the filter $H(z)$ below, we form $H_{\text{Neg}}(z) = H(-z)$. Observe that **we negate only the odd powers of z**, yielding in negating $a[n]$ and $b[n]$ of odd n.

Example: $H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}} = \frac{2z(z+1.2)}{(z-0.48-j0.64)(z-0.48+j0.64)}$



Negate z: $H_{\text{Neg}}(z) = \frac{2-2.4z^{-1}}{1+0.96z^{-1}+0.64z^{-2}}$ **Negate odd coefficients**



Problem: Negating z

Problem: Find the impulse response that corresponds to the transfer function

$$H(z) = \frac{z(2z + 2.4)}{z^2 - 0.96z + 0.64}$$

Use the property $\frac{z(Az+B)}{z^2+2az+|\gamma|^2} \Leftrightarrow r|\gamma|^n \cos(\beta n + \theta)\, u[n]$.

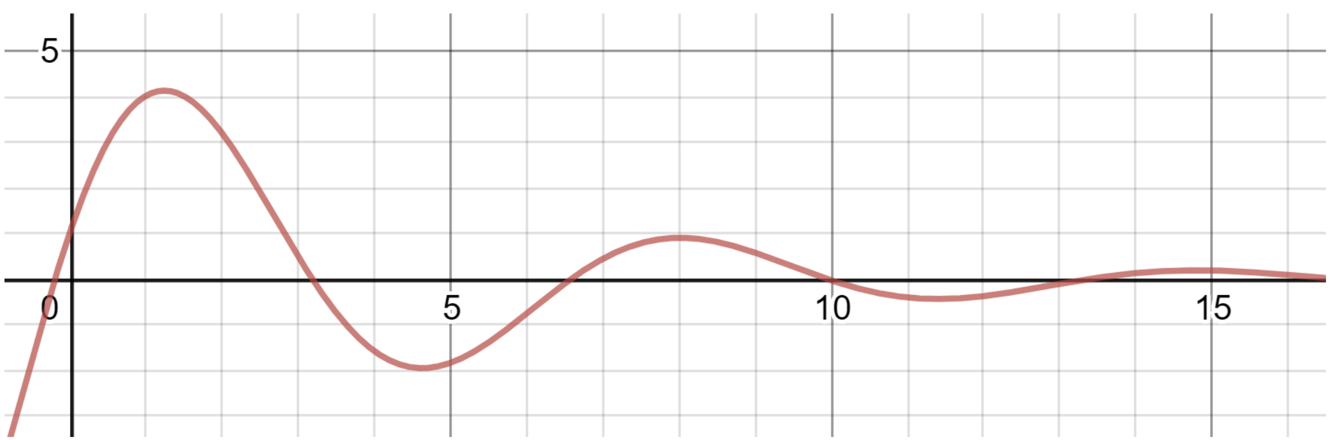
Then, negate z and repeat the task. This is shown in the next slide.

Solution:

$A = 2,\ B = 2.4,\ a = -0.48,\ |\gamma| = 0.8,\ \ r = \sqrt{\frac{A^2|\gamma|^2+B^2-2AaB}{|\gamma|^2-a^2}} = \sqrt{\frac{4\cdot 0.64+5.76-2\cdot 2\cdot(-0.48)\cdot 2.4}{0.64-0.2304}} = 5.618$

$\beta = \cos^{-1}\frac{-a}{|\gamma|} = 0.927\text{rad},\ \ \theta = \tan^{-1}\frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}} = -1.38\text{rad}.$

Therefore, $h[n] = 5.618 \cdot 0.8^n \cdot \cos(0.927n - 1.38)\, u[n]$. Observe the decaying cosine.



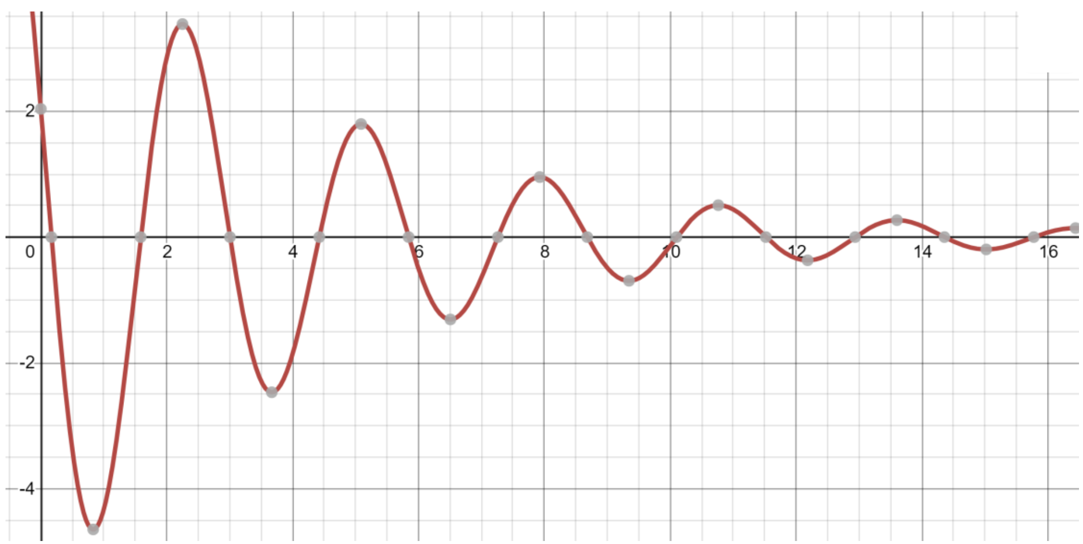
Problem: Negating z cont.

$$H_{\text{Neg}}(z) = \frac{2z(z-1.2)}{z^2 + 0.96z + 0.64}$$

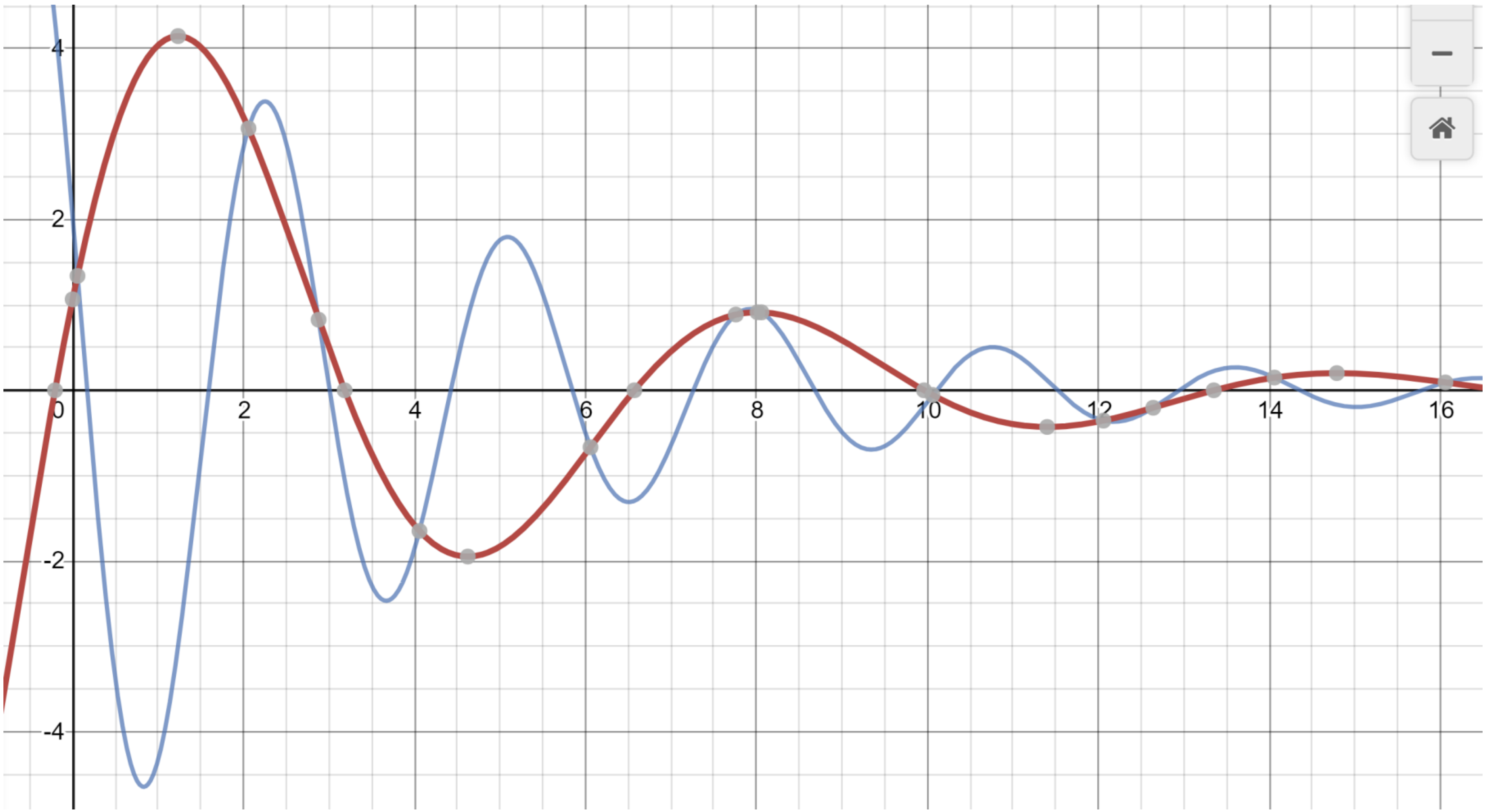
We use the same property again with $A = 2$, $B = -2.4$, $a = 0.48$, $|\gamma| = 0.8$.

$r = \sqrt{\frac{A^2|\gamma|^2+B^2-2AaB}{|\gamma|^2-a^2}} = \sqrt{\frac{4\cdot 0.64+5.76-2\cdot 2\cdot 0.48\cdot(-2.4)}{0.64-0.2304}} = 5.618$, $\beta = \cos^{-1}\frac{-a}{|\gamma|} = 2.214\text{rad}, \theta = \tan^{-1}\frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}} = -1.2\text{rad}.$

Therefore, $h_{\text{Neg}}(n) = 5.618\, 0.8^n \cos(2.214n + 1.2)\, u[n]$



Problem: Negating z cont. $h[n]$ (red) and $h_{\mathrm{Neg}}(n)$ (blue)



Transformations of z: Raising the power of z

Cubing z

- Suppose that $H_C(z) = H(z^3)$. This is equivalent to inserting two zeros between each $a[n]$ and $b[n]$. Then $H_C(z)$ has the following properties:
  - **Pole and zero** numbers are multiplied by 3

    If z_0 is a zero of $H(z)$, then $H_C(\sqrt[3]{z_0}) = H\left(\sqrt[3]{z_0}^3\right) = H(z_0) = 0$. Thus, the cube roots of z_0 are zeros of $H_C(z)$. The same statements applies to poles. Any z_0 has three cube roots in the complex plane whose magnitudes have the same value of $\sqrt[3]{|z_0|}$ and whose phases are $\frac{\angle z_0}{3} + \frac{2\pi k}{3}$, $k = 0,1,2$ or $\frac{\angle z_0}{3} + \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$.

  - **The frequency response is replicated three times**

    The frequency response is given by $H_C(e^{j\omega}) = H(e^{j3\omega})$. This corresponds to **shrinking** the response horizontally by a factor of 3.

    Also $H_C\left(e^{j\left(\omega \pm \frac{2\pi}{3}\right)}\right) = H\left(e^{j3\left(\omega \pm \frac{2\pi}{3}\right)}\right) = H\left(e^{j(3\omega \pm 2\pi)}\right) = H_C(e^{j\omega})$ meaning that there are three replications of the frequency response spaced $\frac{2\pi}{3}$ apart.

    [Consider $z_o = re^{j\theta}$ and thus, ${z_o}^{1/3} = r^{1/3}e^{j\theta/3}$. The cube of the points $r^{1/3}e^{j\left(\frac{\theta}{3}+\frac{2\pi}{3}\right)}$ and $r^{1/3}e^{j\left(\frac{\theta}{3}+\frac{4\pi}{3}\right)}$ is also equal to z_o.]

Problem: Cubing z

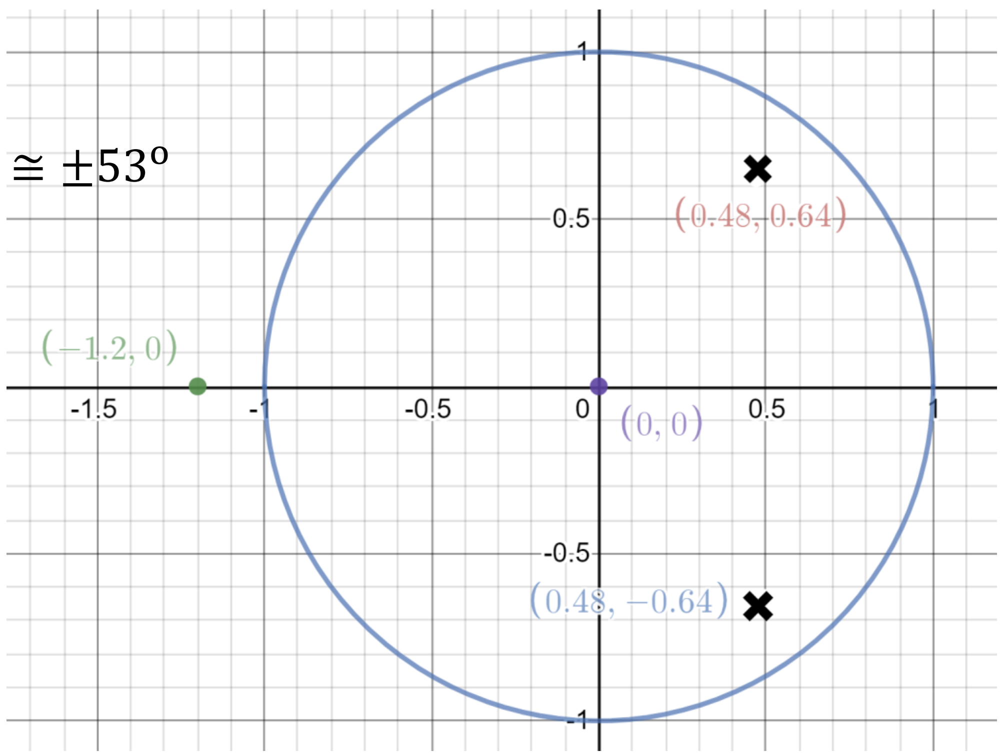
- Given the filter $H(z)$ of the previous example, we can form a new one $H_C(z) = H(z^3)$.
- This is equivalent to inserting two zeros between each $a[n]$ and $b[n]$.

Example: $\boldsymbol{H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}} = \frac{z^{-1}2(z+1.2)}{z^{-2}(z^2-0.96z+0.64)} = \frac{2z(z+1.2)}{z^2-0.96z+0.64}}$

Poles: $0.48 \pm j0.64$

- Magnitude: $\sqrt{0.48^2 + 0.64^2} = 0.8$
- Phase: $\pm\tan^{-1}\frac{0.64}{0.48} = \pm 0.9273\text{rads} \cong \pm 53^{o}$

Zeros : $0, -1.2$



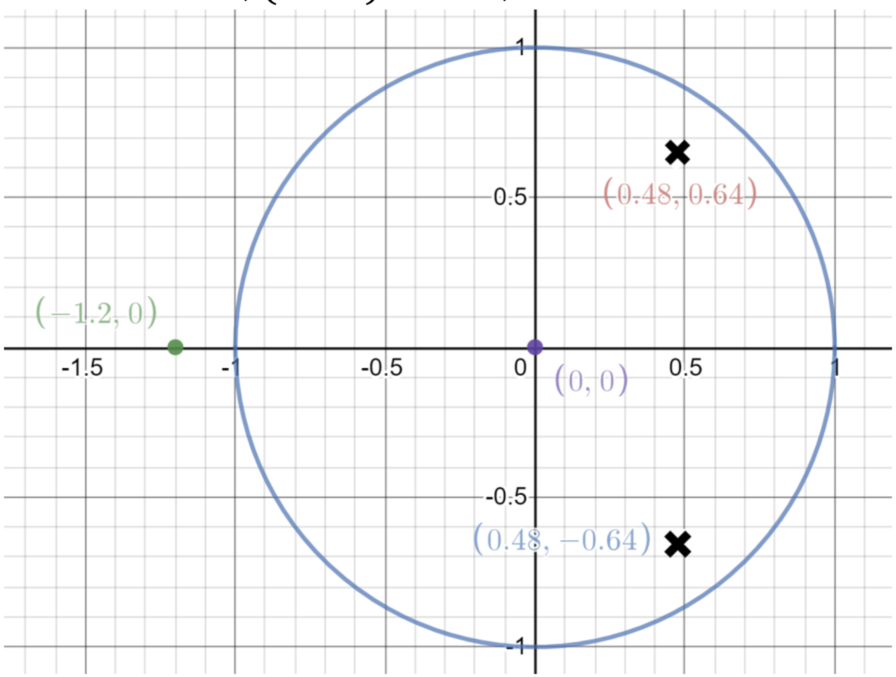
Problem: Cubing z

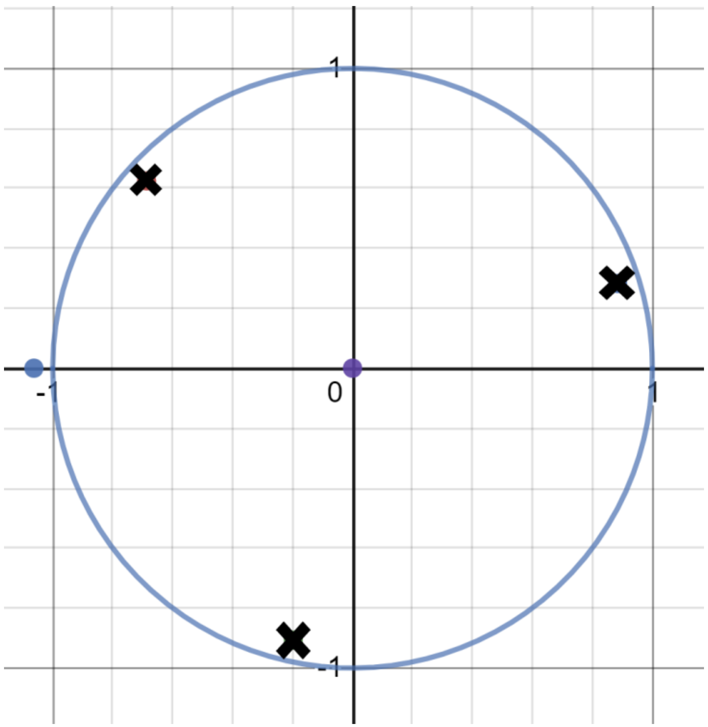
- Cube z: $H_C(z) = \boldsymbol{H}(\boldsymbol{z}^3) = \frac{2+2.4z^{-3}}{1-0.96z^{-3}+0.64z^{-6}}$

  Poles: $(0.48 \pm j0.64)^{1/3}$
  - Magnitude: $0.8^{1/3}$ = 0.9283
  - Phase: $\frac{\tan^{-1}\frac{0.64}{0.48}}{3} + \{0, 120^o, 240^o\} \cong \{18^o, 138^o, 258^o\}$

  Zeros : $0, (-1.2)^{1/3} = 0, -1.06$

Observe that the cubing moves the poles closer to the unit circle.





Generalization: Raising the power of z

- The previous manipulation is extended to raising z to any positive integer power.
- The number of replications is equal to the power concerned.

Transformations of z: Scaling z

- Suppose that $H_S(z) = H(z/\alpha)$ where α is a non-zero real number. Then $H_S(z)$ has the following properties:

  Values of poles and zeros **are multiplied by α**

  If z_0 is a zero of $H(z)$, then $H_S(\alpha z_0) = H(z_0) = 0$ so αz_0 is a zero of $H_S(z)$. The phase of the zero is unchanged since $\angle \alpha z_0 = \angle z_0$. The magnitude of the zero is multipled by α. The same statement applies to poles. If $\alpha > 1$ then the pole positions will move closer to the unit circle. If α is large enough to make any pole cross the unit circle then the filter $H_S(z)$ will be unstable.

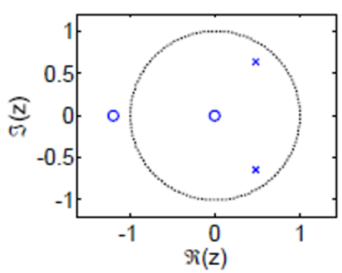
  The bandwidth of peaks in the response changes

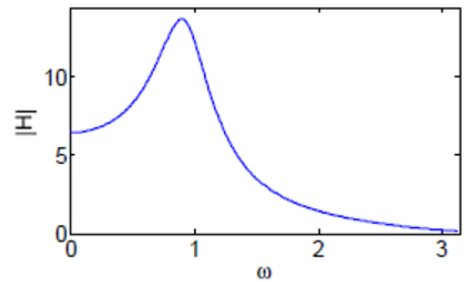
  It is proven that if $|\alpha| > 1$ the peak in $H_S(z)$ will have a higher amplitude and a smaller bandwidth. On the contrary, if $\alpha < 1$ then the peak will have a lower amplitude and a larger bandwidth.

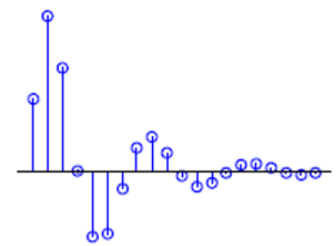
Example: Scaling z

- Given a filter $H(z)$ we can form a new one $H_S(z) = H(z/\alpha)$.
- This is equivalent to multiplying $a[n]$ and $b[n]$ by α^n.

Example: $H(z) = \frac{2+2.4z^{-1}}{1-0.96z^{-1}+0.64z^{-2}}$

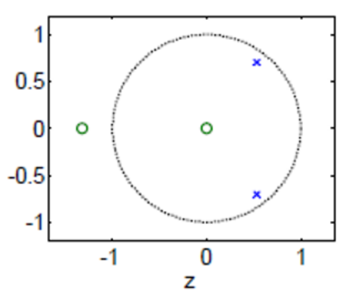


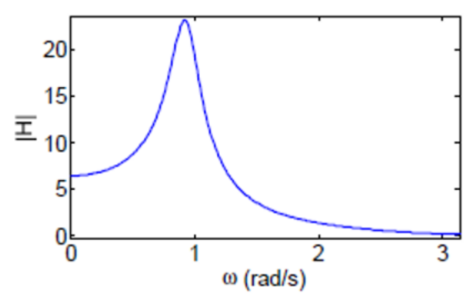


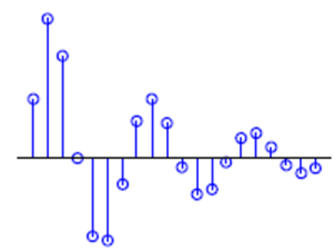


$$H_S(z) = H\left(\frac{z}{1.1}\right) = \frac{2+2.64z^{-1}}{1-1.056z^{-1}+0.7744z^{-2}}$$

The transformation variable z is divided by 1.1







- Poles and zeros are multiplied by α. For $|\alpha| > 1$ the peaks are sharpened.

Allpass filters

- If $H(z)=\frac{B(z)}{A(z)}=\frac{\sum_{n=0}^{M} b[n]z^{-n}}{\sum_{n=0}^{M} a[n]z^{-n}}$, with $b[n]=a^*[M-n]$ then we have a so-called **allpass filter**.
- $H(z)=\frac{B(z)}{A(z)}=\frac{\sum_{n=0}^{M} a^*[M-n]z^{-n}}{\sum_{n=0}^{M} a[n]z^{-n}}=\frac{\sum_{r=0}^{M} a^*[r]z^{r-M}}{\sum_{n=0}^{M} a[n]z^{-n}}=\frac{z^{-M}\sum_{r=0}^{M} a^*[r]z^{r}}{\sum_{n=0}^{M} a[n]z^{-n}}=\frac{z^{-M}\bar{A}(z^{-1})}{A(z)}$, $r=M-n$
  - The coefficients of $\bar{A}(z)$ are the conjugates of the coefficients of $A(z)$.
- $H(e^{j\omega})=\frac{B(e^{j\omega})}{A(e^{j\omega})}=\frac{e^{-j\omega M}\sum_{r=0}^{M} a^*[r]e^{j\omega r}}{\sum_{n=0}^{M} a[n]e^{-j\omega n}}=\frac{e^{-j\omega M}A^*(e^{j\omega})}{A(e^{j\omega})}$

  The two sums are complex conjugates and therefore, they have the same magnitude.
- Hence, $|H(e^{j\omega})|=1,\ \forall\,\omega$. A filter with the above property is called an allpass filter.
- The phase is NOT constant: $\angle H(e^{j\omega})=-\omega M+\angle A^*(e^{j\omega})-\angle A(e^{j\omega})=-\omega M-2\angle A(e^{j\omega})$

Example

- A 1st order allpass filter with real coefficients can be written as:

  $$H(z)=\frac{B(z)}{A(z)}=\frac{-p+z^{-1}}{1-pz^{-1}}=-p\,\frac{1-p^{-1}z^{-1}}{1-pz^{-1}}=\frac{1-zp}{z-p}=-p\,\frac{z-1/p}{z-p}$$

  - There is a pole at p and a zero at p^{-1} $\Rightarrow$ in an allpass filter the zeros and the poles are reciprocal.
  - $|e^{j\omega}-p|=|p|\,|e^{j\omega}-1/p|\Rightarrow\frac{|e^{j\omega}-p|}{|e^{j\omega}-1/p|}=|p|,\ \forall\,\omega$. The ratio of the distances of a pole and its corresponding zero from any point on the unit circle is constant.

Allpass filters properties

- If the coefficients $a[n]$ are all real, then the conjugation has no effect and the numerator coefficients are identical to the denominator but in reverse order.
- The z –transform of an allpass filter can also be written as:

$$H(z) = \prod_{i=1}^{M} \frac{(1-p_i^* z)}{(z-p_i)}$$

- $|H(z)|^2 = H(z)H^*(z) = \prod_{i=1}^{M} \frac{(1-p_i^* z)(1-p_i z^*)}{(z-p_i)(z^*-p_i^*)} = \prod_{i=1}^{M} \frac{1-p_i z^*-p_i^* z+p_i p_i^* z z^*}{z z^*-p_i z^*-p_i^* z+p_i p_i^*}$

$$= \prod_{i=1}^{M} \left(1 + \frac{1+p_i p_i^* z z^*-z z^*-p_i p_i^*}{z z^*-p_i z^*-p_i^* z+p_i p_i^*}\right) = \prod_{i=1}^{M} \left(1 + \frac{1+|p_i|^2|z|^2-|z|^2-|p_i|^2}{z z^*-p_i z^*-p_i^* z+p_i p_i^*}\right) = \prod_{i=1}^{M} \left(1 + \frac{(1-|z|^2)(1-|p_i|^2)}{|z-p_i|^2}\right)$$

- If all the $|p_i| < 1$, then each term in the product is $\overset{>}{\underset{<}{=}} 1$ according to whether $|z| \overset{<}{\underset{>}{=}} 1$.

Group delay from the frequency response

- $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega}$. The group delay assesses the linearity of the phase.
- $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})}$
- We may extract the phase using $\ln H(e^{j\omega}) = \ln|H(e^{j\omega})| + j\angle H(e^{j\omega})$. **This is popular in DSP applications.**
- $\tau_H(e^{j\omega}) = -\frac{d\left(\mathbf{Im}\left(\ln H(e^{j\omega})\right)\right)}{d\omega} \Rightarrow \boldsymbol{\tau_H(e^{j\omega}) = \mathbf{Im}\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right)}$
- We can also show that $\boldsymbol{\tau_H(e^{j\omega}) = \mathbf{Re}\left(\frac{j}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right)}$

Proof

- Consider a complex number $a + jb$.
  $\mathbf{Re}\{j(a+jb)\} = \mathbf{Re}\{-b+ja\} = -b = \mathbf{Im}\{-(a+jb)\}$
- $\mathbf{Re}\left(\frac{j}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right) = \mathbf{Im}\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right) = \boldsymbol{\tau_H(e^{j\omega})}$
- Furthermore, $\mathbf{Re}\left(\frac{j}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right) = \mathbf{Re}\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}}$
  $\left(\frac{-z}{H(z)}\frac{dH(z)}{dz}\right)\Big|_{z=e^{j\omega}} = \left(\frac{-z}{H(z)}\frac{dH(z)}{je^{j\omega}d\omega}\right)\Big|_{z=e^{j\omega}} = \left(\frac{-z}{H(z)}\frac{dH(z)}{jzd\omega}\right)\Big|_{z=e^{j\omega}} = \left(\frac{j}{H(z)}\frac{dH(z)}{d\omega}\right)\Big|_{z=e^{j\omega}} =$
  $\frac{j}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}$

Group delay from the impulse response

- The frequency response is related to the impulse response throught the relationship:

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} h[n]e^{-jn\omega} = \mathcal{F}(h[n])$$

$\mathcal{F}(\cdot)$ denotes the Discrete Time Fourier Transform (DTFT)

- $\frac{dH(e^{j\omega})}{d\omega} = -j\sum_{n=0}^{\infty} nh[n]e^{-jn\omega} = -j\mathcal{F}(nh[n])$

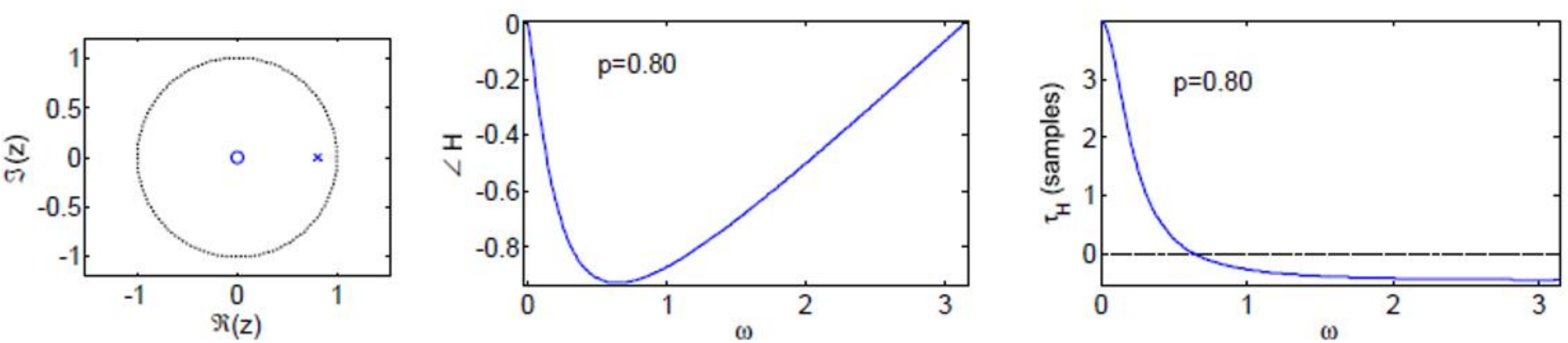
- In that case, $\tau_H(e^{j\omega}) = \mathbf{Im}\left(\frac{-1}{H(e^{j\omega})}\frac{dH(e^{j\omega})}{d\omega}\right) = \mathbf{Im}\left(\frac{j\mathcal{F}(nh[n])}{\mathcal{F}(h[n])}\right) = \mathbf{Re}\left(\frac{\mathcal{F}(nh[n])}{\mathcal{F}(h[n])}\right)$

Group delay: example

- $H(z) = \frac{1}{1-pz^{-1}}$, p **is real**
- Note that if $H(z) = \frac{B(z)}{A(z)}$, then $\tau_H(e^{j\omega}) = \tau_B(e^{j\omega}) - \tau_A(e^{j\omega})$.
  In this case $\tau_B(e^{j\omega}) = 0$ so that $\tau_H(e^{j\omega}) = -\tau_A(e^{j\omega})$
- $\tau_H(e^{j\omega}) = -\mathbf{Re}\left(\frac{-z}{A(z)}\frac{dA(z)}{dz}\right)\Big|_{z=e^{j\omega}} = -\mathbf{Re}\left(\frac{-zpz^{-2}}{A(z)}\right)\Big|_{z=e^{j\omega}} = \mathbf{Re}\left(\frac{pz^{-1}}{A(z)}\right)\Big|_{z=e^{j\omega}}$

$$= \mathbf{Re}\left(\frac{pe^{-j\omega}}{1-pe^{-j\omega}}\right) = \mathbf{Re}\left(\frac{p}{e^{j\omega}-p}\right) = \frac{\mathbf{Re}\left(p(e^{-j\omega}-p)\right)}{(e^{j\omega}-p)(e^{-j\omega}-p)} = \frac{p\cos\omega - p^2}{1-2p\cos\omega + p^2}.$$

- **Average group delay over ω = #poles − #zeros within the unit circle.**
- **Note that zeros on the unit circle count $-\frac{1}{2}$!**
- In that case the average value of the group delay is #poles − #zeros $= 1 - 1 = 0$.



Group delay example without going to the frequency domain

- Consider the term $\frac{B(z)}{A(z)} = \frac{1}{1-pz^{-1}} = \frac{z}{z-p}$. (Pole is assumed real for simplicity).
- The phase of the above term is now $\omega - \tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega)-p}\right)$.
- The negative of the derivative of the phase, i.e., the group delay, is, therefore,

$$-\frac{d}{d\omega}\left[\omega - \tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega)-p}\right)\right] = \tau_B - \tau_A = -1 - \frac{p\cos(\omega) - 1}{p(2\cos(\omega) - p) - 1} = \frac{p\cos(\omega) - p^2}{1 - 2p\cos(\omega) + p^2}$$

Group delay properties

- A single pole contributes to the transfer function with the term $\frac{1}{z-p}$. (The pole is assumed real for simplicity).
- The phase of the above term is $-\tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega)-p}\right)$.
- The negative of the derivative of the phase, i.e., the group delay, is, therefore,

$$-\frac{d}{d\omega}\left[-\tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega)-p}\right)\right]=\frac{p\cos(\omega)-1}{p(2\cos(\omega)-p)-1}$$

  https://www.derivative-calculator.net
- The average group delay due to a single pole is given by the integral:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{p\cos(\omega)-1}{p(2\cos(\omega)-p)-1}\,d\omega=\begin{cases}1 & |p|<1\\ 0 & |p|>1\end{cases}$$

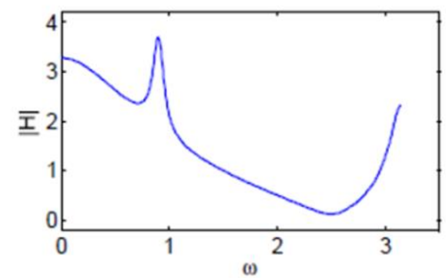
  Furthermore, $\int_{-\pi}^{\pi}\frac{1}{2\pi}\frac{p\cos(\omega)-1}{p(2\cos(\omega)-p)-1}\,d\omega=\frac{1}{2}$ if $|p|=1$

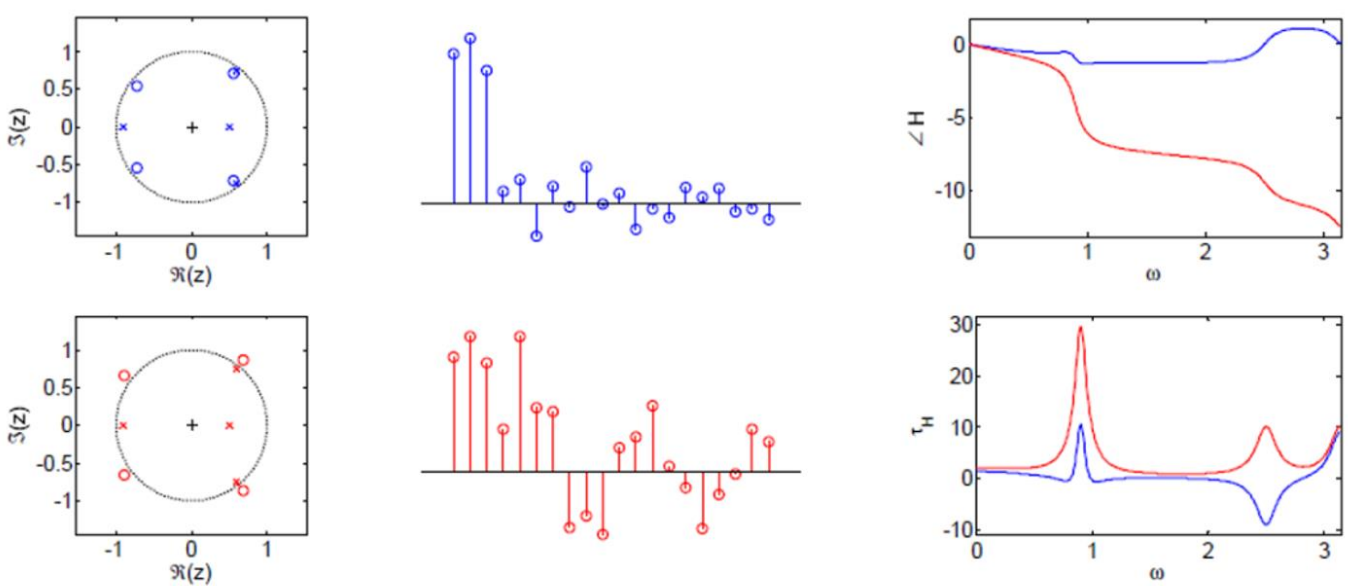
  https://www.integral-calculator.com
- Similarly, the phase due to a single zero term is $\tan^{-1}\left(\frac{\sin(\omega)}{\cos(\omega)-d}\right)$ and following the above analysis we conclude that the average group delay due to a single zero is -1, 0 or -1/2.
- Therefore, based on the above we can state:

  Average group delay over $\omega=\#\text{poles}-\#\text{zeros}$ within the unit circle. Zeros on the unit circle count $-\frac{1}{2}$.

Minimum phase filters

- ❑ Inverting an interior zero to the exterior multiplies $\left|H(e^{j\omega})\right|$ by a constant but increases average group delay be 1 sample.
- ❑ **A filter with all zeros inside the unit circle is a minimum phase filter.**
  - They have the lowest possible group delay for a given magnitude response.
  - Energy in $h[n]$ is concentrated towards $n = 0$.





Linear phase filters

- A FIR filter is linear-phase if and only if its coefficients are symmetrical or anti-symmetrical around the center coefficient, that is, the first coefficient is the same as the last; the second is the same as the next-to-last, etc. (A linear-phase FIR filter having an odd number of coefficients will have a single coefficient in the center which has no mate.)

- As seen **linear phase** refers to the condition where the phase response of the filter is a linear (straight-line) function of frequency (excluding phase wraps at $\pm\pi$). This results in the group delay through the filter being the same at all frequencies. Therefore, the filter does not cause "phase distortion" or "delay distortion". This can be a critical advantage of FIR filters over IIR filters in certain systems, for example, in digital data modems.

- Actually, the most popular alternative to linear phase is minimum phase. Minimum-phase filters (which might better be called minimum delay filters) have less delay than linear-phase filters with the same amplitude response, at the cost of a non-linear phase characteristic, a.k.a. phase distortion.

Linear phase filters: symmetric

- The phase of a linear phase filter is a linear function: $\angle H(e^{j\omega}) = \theta_0 - \alpha\omega$
- In that case the group delay is constant: $\tau_H(e^{j\omega}) = -\frac{d\angle H(e^{j\omega})}{d\omega} = \alpha$
- A filter has linear phase if and only if is **symmetric** or **antisymmetric**:
  $h[n] = h[M-n]\ \forall n$ or else $h[n] = -h[M-n]\ \forall n$
  If M is even there is a midpoint, whereas if M is odd there isn't a midpoint
- Suppose that M is even:

$$2H(e^{j\omega}) = \sum_0^M h[n]e^{-j\omega n} + \sum_0^M h[M-n]e^{-j\omega(M-n)}$$

$$= e^{-j\omega\frac{M}{2}}\left(\sum_0^M h[n]e^{-j\omega(n-\frac{M}{2})} + \sum_0^M h[M-n]e^{j\omega(n-\frac{M}{2})}\right)$$

$h[n]$ symmetric:

$$2H(e^{j\omega}) = e^{-j\omega\frac{M}{2}}\left(\sum_0^M h[n]e^{-j\omega(n-\frac{M}{2})} + \sum_0^M h[n]e^{j\omega(n-\frac{M}{2})}\right)$$

$$H(e^{j\omega}) = e^{-j\omega\frac{M}{2}}\sum_0^M h[n]\cos\left((n-\frac{M}{2})\omega\right)$$

Linear phase impulse responses always come up in a form like this. We will elaborate in detail in Part 6

Linear phase filters: antisymmetric

- Suppose that M is even:

$$2H(e^{j\omega}) = \sum_0^M h[n]e^{-j\omega n} + \sum_0^M h[M-n]e^{-j\omega(M-n)}$$

$$= e^{-j\omega\frac{M}{2}}\left(\sum_0^M h[n]e^{-j\omega(n-\frac{M}{2})} + \sum_0^M h[M-n]e^{j\omega(n-\frac{M}{2})}\right)$$

$h[n]$ antisymmetric

$$2H(e^{j\omega}) = e^{-j\omega\frac{M}{2}}\left(\sum_0^M h[n]e^{-j\omega(n-\frac{M}{2})} - \sum_0^M h[n]e^{j\omega(n-\frac{M}{2})}\right)$$

$$2H(e^{j\omega}) = -2je^{-j\omega\frac{M}{2}}\sum_0^M h[n]\sin\left(\left(n-\frac{M}{2}\right)\omega\right)$$

$$H(e^{j\omega}) = e^{-j\pi}e^{j\frac{\pi}{2}}e^{-j\omega\frac{M}{2}}\sum_0^M h[n]\sin\left(\left(n-\frac{M}{2}\right)\omega\right)$$

$$H(e^{j\omega}) = e^{-j(\frac{\pi}{2}+\omega\frac{M}{2})}\sum_0^M h[n]\sin\left(\left(n-\frac{M}{2}\right)\omega\right)$$

In a subsequent lecture we will get into more detail into linear systems.