IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2019

MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS

SOLUTIONS

1. (a) Here, we consider complex vectors to provide the more general solution. However, full mark will be provided also to the students who worked out the correct derivation assuming real vectors.
We have:

$$0 \leq \|\mathbf{x} - \alpha\mathbf{y}\|^2 = \langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y}\rangle = \|\mathbf{x}\|^2 - \langle \mathbf{x}, \alpha\mathbf{y}\rangle - \langle \alpha\mathbf{y}, \mathbf{x}\rangle + |\alpha|^2\|\mathbf{y}\|^2.$$

Now assuming $\alpha = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{y}\|^2}$ and using the fact that if $c = \langle \mathbf{x}, \mathbf{y}\rangle$ then $\langle \mathbf{y}, \mathbf{x}\rangle = c^*$, we obtain

$$0 \leq \|\mathbf{x} - \alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \alpha^*\langle \mathbf{x}, \mathbf{y}\rangle - \alpha\langle \mathbf{y}, \mathbf{x}\rangle + \frac{|\langle \mathbf{x}, \mathbf{y}\rangle|^2}{\|\mathbf{y}\|^2} = \|\mathbf{x}\|^2 - 2\frac{|\langle \mathbf{x}, \mathbf{y}\rangle|^2}{\|\mathbf{y}\|^2} + \frac{|\langle \mathbf{x}, \mathbf{y}\rangle|^2}{\|\mathbf{y}\|^2}.$$

Consequently

$$\frac{|\langle \mathbf{x}, \mathbf{y}\rangle|^2}{\|\mathbf{y}\|^2} \leq \|\mathbf{x}\|^2,$$

which then yields the Cauchy-Schwarz inequality: $|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\|\mathbf{y}\|$.

(b) i. We apply L to $1, x$ and e^x.

$$L(1) = -1$$
$$L(x) = 2 - x$$
$$L(e^x) = e^x$$

The first corresponds to the vector $[-1, 0, 0]^T$ with respect to the given basis, the second corresponds to the vector $[2, -1, 0]^T$ and the third to $[0, 0, 1]^T$ Thus the matrix representing L with respect to the given basis is:

$$\mathbf{L} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ii. To find a solution to the differential equation we need to find a $\mathbf{f}$ such that

$$\mathbf{Lf} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$$

The inverse of $\mathbf{L}$ is given by:

$$\mathbf{L^{-1}} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So $\mathbf{f} = [-5, -1, -2]^T$ and the solution is $f(x) = -5 - x - 2e^x$.

(c) i. The vectors span a four dimensional space and a possible selection of basis vectors is $[1, 0, 1, 0, 0]^T$,$[2, 0, 0, 0, 1]^T$,$[0, 1, 0, 0, 0]^T$, $[0, 0, 0, 1, 0]^T$

ii. The vectors cover a four dimensional space and the basis vectors are $[1, 0, 0, -1, 0]^T$,$[0, 1, 0, 0, 0]^T$, $[0, 0, 1, 0, 0]^T$ $[0, 0, 0, 0, 1]^T$.

iii. The space has dimension two and a possible basis is: $[1, 1, 1, 1, 1]^T$, $[2, 4, -6, 0, 6]^T$

(d) We put the 4 vectors along the rows of the matrix $\mathbf{A}$ and row reduce:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 8 & 16 \\ 3 & 5 & 9 & 17 \\ 2 & 3 & 5 & 9 \\ 3 & 4 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & -1 & -3 & -7 \\ 2 & 3 & 5 & 9 \\ 3 & 4 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & -1 & -3 & -7 \\ 0 & -1 & -3 & -7 \\ 3 & 4 & 6 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 8 & 16 \\ 0 & -1 & -3 & -7 \\ 0 & -1 & -3 & -7 \\ 0 & -2 & -6 & -14 \end{bmatrix}.$$

Therefore, without further reduction, we see that S has dimension 2 and a possible basis is: $[2,4,8,16]^T, [0,-1,-3,-7]^T$.

(e) One way to determine $\mathbf{A}$ is to work out the linear mapping of the elements of the canonical basis $\mathbf{e}_i = [0,0,...,1,...,0]^T$ where the one is at location i and in this case i=1,2,3. We now observe that

$$\mathbf{Ae_3} = [2,3,5]^T,$$

since $\mathbf{x}_1 = \mathbf{e}_3$. Moreover,

$$\mathbf{Ae_2} = \mathbf{Ax_2} - \mathbf{Ae_3} = [1,0,0]^T - [2,3,5]^T = [-1,-3,-5]^T$$

and finally

$$\mathbf{Ae_1} = \mathbf{Ax_3} - \mathbf{Ax_2} = [0,1,-1]^T - [1,0,0]^T = [-1,1,-1]^T.$$

Consequently,

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & 2 \\ 1 & -3 & 3 \\ -1 & -5 & 5 \end{bmatrix}.$$

This matrix is full rank, so the null space is trivial and the range space has dimension 3 with a possible basis being the three columns of $\mathbf{A}$.

2. (a) We first row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 5 & 6 & 3 \\ 2 & 3 & 0 & 1 \\ 4 & 4 & 3 & 2 \end{bmatrix}$$

to find the dimension and the basis of its null space and range space. We have

$$\begin{bmatrix} 6 & 5 & 6 & 3 \\ 2 & 3 & 0 & 1 \\ 4 & 4 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 6 & 3 & 6 & 5 \\ 0 & 4/3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Column one and three are linearly independent while the other two columns can be obtained from one and three so range space has dimension 2 and the null space has dimension $n - rank(A) = 4 - 2 = 2$. A basis for the range space is given by the first and third column of $\mathbf{A}$, that is, $\mathbf{u}_1 = [6, 2, 4]^T, \mathbf{u}_2 = [6, 0, 3]^T$. We then find a basis for the null space by solving $\mathbf{Ax} = \mathbf{0}$ with $\mathbf{A}$ in echelon form. We have

$$\begin{bmatrix} -\frac{x_4}{2} - \frac{3}{2}x_2 \\ x_2 \\ \frac{2}{3}x_2 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 2/3 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for the null space is $[-3/2, 1, 3/2, 0]^T, [-1/2, 0, 0, 1]^T$. The range space of $\mathbf{A^T}$ has dimension two since the null space had dimension two and a basis is, e.g., the first two rows of $\mathbf{A}$ which are linearly independent, i.e., $[6, 5, 6, 3]^T$, $[2, 3, 0, 1]^T$. Finally the dimension of the null space of $\mathbf{A^T}$ is $3 - rank(A) = 1$. We find its single basis element $\mathbf{n_1}$ by imposing $\langle u_1, n_1\rangle = 0$ and $\langle u_2, n_1\rangle = 0$ which yields a basis $n_1 = [1, 1, -2]^T$.

(b) i. Yes, because the system is full column rank and so the null space is trivial.

ii. We find the least-square solution by computing $\hat{\mathbf{x}} = (\mathbf{A^TA})^{-1}\mathbf{A^Ty}$ which in this case yields:

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

iii. The best way to proceed is by computing the derivative over the two entries x_1 and x_2 of $\mathbf{x}$ and then we equate the derivatives to zero. This leads to the following system of equations:

$$-\mathbf{A^T}(\mathbf{y} - \mathbf{Ax}) + \mathbf{x} = 0$$

which implies:

$$\hat{\mathbf{x}}_M = (\mathbf{A^TA} + \mathbf{I})^{-1}\mathbf{A^Ty}$$

which yields $\hat{\mathbf{x}}_M = [1.8, 0.6]^T$.

It is of interest to note that the equation that leads to the solution is the same as for the least-square case but for the presence of the identity matrix in the inverse which is due to the regularization term.

(c) We know that the rank-one matrix $\mathbf{B}$ that minimizes $\|\mathbf{A} - \mathbf{B}\|^2$ is $\mathbf{B} = \sigma_1\mathbf{u_1}\mathbf{v_1}^H$, where u_1, v_1 are the singular vectors of $\mathbf{A}$ related to the largest singular value.

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We therefore first find σ_1 by finding the eigenvalues of:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \sqrt{2}/2 \\ 1 & 0 & -\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$$

We can then find the two eigenvalues by using the trace and determinant formulas which tell us that $\lambda_1 + \lambda_2 = 3$ and $\lambda_1 * \lambda_2 = 2$. Therefore $\lambda_1 = 2$ and $\lambda_2 = 1$ and so $\sigma_1 = \sqrt{2}$.

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The eigenvector of $\mathbf{T}$ related to λ_1 gives us one singular vector: $\mathbf{v}_1 = [\sqrt{2}/2, \sqrt{2}/2]^T$. The second relevant singular vector is

$$\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{v}_1 = [1, 0, 0]^T.$$

Therefore

$$\mathbf{B} = \sigma_1 \mathbf{u_1} \mathbf{v_1}^H = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. (a) i. The projection operator is:

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

where $\mathbf{A}$ is the matrix whose columns are $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. However, before computing $\mathbf{P}$, we realize that S has dimension two and so we need only two among the three vectors under consideration. We also note that $\mathbf{p}_1$ and $\mathbf{p}_2$ are orthonormal and so we pick these two. Because of orthogonality $(\mathbf{A}^T\mathbf{A})^{-1} = \mathbf{I}$ and $\mathbf{P}$ reduces to

$$\mathbf{P} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

and

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix}.$$

ii. The error vector is $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} = [-1, -1, 1, 1]^T$ and clearly $\mathbf{e} \perp S$ since $\langle \mathbf{e}, \mathbf{p}_1 \rangle = 0$ and $\langle \mathbf{e}, \mathbf{p}_2 \rangle = 0$

(b) The projection theorem states that the vector in S that minimizes the least-square error is unique and satisfies the orthogonality condition. So we just need to verify that $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ satisfies $\mathbf{e} \perp S$. In this case $\mathbf{e} = [-1, 0, 1]^T$ and clearly $\langle \mathbf{e}, [1, 1, 1]^T \rangle = 0$ and $\langle \mathbf{e}, [1, 0, 1]^T \rangle = 0$.

(c) The answer is yes because $\mathbf{P}$ satisfies $\mathbf{P} = \mathbf{P}^H$ and $\mathbf{P}^2 = \mathbf{P}$.

(d) We denote $\mathbf{P}_3 = \mathbf{P}_1 + \mathbf{P}_2$. Clearly $\mathbf{P}_3^H = \mathbf{P}_3$ since $\mathbf{P}_3^H = \mathbf{P}_1^H + \mathbf{P}_2^H = \mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3$ where we used the fact that $\mathbf{P}_1$ and $\mathbf{P}_2$ are orthogonal projections. However, normally, $\mathbf{P}_3^2 \neq \mathbf{P}_3$ since $\mathbf{P}_3^2 = \mathbf{P}_1^2 + \mathbf{P}_2^2 + \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_1 = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_1\mathbf{P}_2 + \mathbf{P}_2\mathbf{P}_1 \neq \mathbf{P}_3$. We have equality only when the cross products are equal to zero which happens when the range spaces of $\mathbf{P}_1$ and $\mathbf{P}_2$ are orthogonal.

(e) The answer is yes. To show this we can compute the pseudo-inverse of $\mathbf{A}$ and show that it is equal $\mathbf{B}$. Alternatively, we can simply verify that

i. $\mathbf{ABA} = \mathbf{A}$

ii. $\mathbf{BAB} = \mathbf{B}$

iii. $\mathbf{AB} = (\mathbf{AB})^T$

iv. $\mathbf{BA} = (\mathbf{BA})^T$