1. (a) Consider the causal, real-coefficient transfer function

$$H_M(z) = \frac{\sum_{n=0}^{n=M} b[n]z^{-n}}{\sum_{n=0}^{n=M} a[n]z^{-n}}, M > 0$$

where $b[n] = a^*[M-n]$. The above transfer function is an allpass filter.

(i) Show that the phase of $H_M(e^{j\omega})$ is $\angle H_M(e^{j\omega}) = -\omega M - 2\angle A(e^{j\omega})$, where $\angle A(e^{j\omega})$ is the phase of the polynomial $A(e^{j\omega})$. **[3]**

Answer

We start with:

$$H_M(z) = \frac{\sum_{n=0}^{n=M} a^*[M-n]z^{-n}}{\sum_{n=0}^{n=M} a[n]z^{-n}}$$

If we denote the denominator polynomial of the allpass function as $A_M(z)$ with

$$H_M(z) = \frac{\sum_{r=0}^{r=M} a^*[r]z^{r-M}}{\sum_{n=0}^{n=M} a[n]z^{-n}}$$

$$H_M(z) = \frac{z^{-M}\sum_{r=0}^{r=M} a^*[r]z^{r}}{\sum_{n=0}^{n=M} a[n]z^{-n}} = \frac{z^{-M}\sum_{n=0}^{n=M} a^*[n]z^{n}}{\sum_{n=0}^{n=M} a[n]z^{-n}}$$

$$A_M(z) = \sum_{n=0}^{n=M} a[n]z^{-n}$$

$$A_M(e^{j\omega}) = \sum_{n=0}^{n=M} a[n]e^{-j\omega n}$$

$$H_M(e^{j\omega}) = \frac{e^{-j\omega M}A_M^*(e^{j\omega})}{A_M(e^{j\omega})}$$

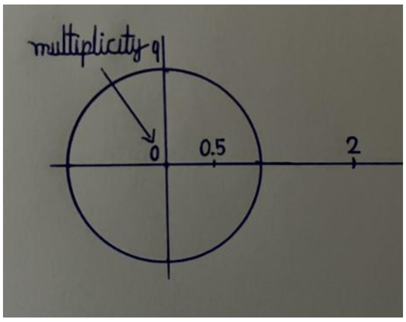
We see that $\angle H_M(e^{j\omega}) = -\omega M - 2\angle A(e^{j\omega})$.

(ii) Consider a causal, real, stable, allpass transfer function $H(z)$ given as:

$$H(z) = \frac{-pz^{-q} + z^{-1-q}}{1 - pz^{-1}}$$

Sketch the pole-zero plot of $H(z)$ if $p = 0.5$, where q is a positive integer. **[3]**

Answer



(iii) Determine the phase response of $H(z)$ as a function of the pole p and the parameter q using the result from part (a)(i) above. **[3]**

Answer

The phase response is $-\omega(q+1) - 2\tan^{-1}\left(\frac{p\sin(\omega)}{1-p\cos(\omega)}\right)$.

(b) Calculate the phase shift introduced by a second-order all-pass filter with a transfer function:

$$H(z) = \frac{a_2 + a_1 z^{-1} + a_0 z^{-2}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$$

Suppose that $a_0 = 1,\ a_1 = -3a/4$ and $a_2 = a^2/8$ where a is a real parameter.

(i) Determine the condition on a for the filter to be stable. **[2]**

Answer

$H(z) = \frac{(z^{-1}-\frac{a}{4})(z^{-1}-\frac{a}{2})}{(\frac{a}{4}z^{-1}-1)(\frac{a}{2}z^{-1}-1)}$

$\left|\frac{a}{4}\right| < 1 \Rightarrow |a| < 4$

$\left|\frac{a}{2}\right| < 1 \Rightarrow |a| < 2$

Combining the above we conclude to $|a| < 4$.

(ii) Find the phase response of $H(z)$ as a function of the poles of $H(z)$. **[3]**

Answer

$$\angle H(e^{j\omega}) = -2\omega - 2\tan^{-1}\left(\frac{\frac{a}{2}\sin(\omega)}{1-\frac{a}{2}\cos(\omega)}\right) - 2\tan^{-1}\left(\frac{\frac{a}{4}\sin(\omega)}{1-\frac{a}{4}\cos(\omega)}\right)$$

(iii) Calculate the phase shift introduced by this filter at $\omega = 0$ and $\omega = \pi$. **[2]**

Answer

$$\angle H(e^{j0}) = -2\tan^{-1}\left(\frac{\frac{a}{2}\sin(0)}{1-\frac{a}{2}\cos(0)}\right) - 2\tan^{-1}\left(\frac{\frac{a}{4}\sin(0)}{1-\frac{a}{4}\cos(0)}\right) = 0$$

$$\angle H(e^{j\pi}) = -2\pi - 2\tan^{-1}\left(\frac{\frac{a}{2}\sin(\pi)}{1-\frac{a}{2}\cos(\pi)}\right) - 2\tan^{-1}\left(\frac{\frac{a}{4}\sin(\pi)}{1-\frac{a}{4}\cos(\pi)}\right) = -2\pi$$

(c) The average group delay of an IIR stable system is $n - m$ where n is the number of poles and m is the number of zeros of the system located inside the unit circle.

(i) Demonstrate that the factors $(z - q)$ and $(qz - 1)$ have the same amplitude response. Use this result to show that inverting a zero originally located inside the unit circle to a position outside the unit circle increases the group delay by 1. **[3]**

Answer

Both terms have amplitude $1 + q^2 - 2q\cos(\omega)$.

(ii) Using the conclusion from part c(i) above, identify which filter, among all filters with the same amplitude response, achieves the minimum group delay. Such a filter is referred to as a minimum-phase filter. **[3]**

Answer

The term with the zero outside the unit circle has group delay one unit less.

(iii) Explain why linear-phase filters cannot be categorized as minimum-phase filters. **[3]**

Answer

Because linear phase filters have zeros outside the unit circle because of the (anti)symmetry property.

2. (a) (i) Demonstrate that antisymmetric linear phase Finite Impulse Response (FIR) filters of order N with a transfer function $H(z)$ (**Type 4**), satisfy the Antimirror Image Polynomial relationship (AIP) given below:

$$H(z) = -z^{-(N-1)}H(z^{-1})$$

[4]

Answer

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = -\sum_{n=0}^{N-1} h[N-1-n]z^{-n}$$
$$= -\sum_{r=0}^{N-1} h[r]z^{-(N-1-r)} = -z^{-(N-1)}H(z^{-1})$$

(ii) Investigate the types of filters (e.g., low-pass, high-pass) that can be implemented using **Type 4** linear-phase Finite Impulse Response (FIR) transfer functions. **[4]**

Answer

We know that N is even and therefore, $N-1$ is odd. Thus,
$H(1) = -H(1) \Rightarrow H(1) = 0$

(iii) Prove that a Type 4 linear-phase Finite Impulse Response (FIR) transfer function, $H(z)$, can be represented in the following form: **[4]**

$$H(z) = \left(\sum_{n=0}^{\frac{N}{2}-1} h[n](z^{-n} - z^{n-N+1})\right)$$

Answer

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n} - \sum_{n=n/2}^{N-1} h[N-1-n]z^{-n}$$
$$= \sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n} - \sum_{r=0}^{\frac{N}{2}-1} h[r]z^{r-(N-1)} = \sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n} - \sum_{n=0}^{\frac{N}{2}-1} h[n]z^{n-(N-1)}$$
$$H(z) = \left(\sum_{n=0}^{\frac{N}{2}-1} h[n](z^{-n} - z^{n-N+1})\right)$$

(b) Let $H(z)$ represent the transfer function of a real-coefficient, linear-phase FIR filter. The filter has zeros located at specific points in the z −plane, with a subset of these zeros given as:

$$\left\{1, 0.5e^{j\frac{\pi}{6}}, -\frac{1}{5}, j\right\}$$

(i) Since the zeros of a linear-phase FIR filter must exist in particular configurations, we deduce that the filter $H(z)$ must include additional zeros beyond those provided.

Determine the additional zeros of $H(z)$ for each possible type of linear-phase FIR filter with the minimum allowable length. **[6]**

Answer

For all types of linear phase FIR filters, the following observations are valid:
The zero $0.5e^{j\frac{\pi}{6}}$, is associate with 3 additional zeros, i.e.,
$0.5e^{j\frac{\pi}{6}}, 0.5e^{-j\frac{\pi}{6}}, 2e^{j\frac{\pi}{6}}, 2e^{-j\frac{\pi}{6}}$ (4 zeros).
The zero $\frac{1}{5}$ is associated with a zero at -5 (2 zeros).
The zero j is associated with a zero at $\frac{1}{j} = -j$ (2 zeros).
Therefore, all 4 types in that case have at least 9 zeros.

For Type 1 linear phase FIR filters there can be even number of zeros at $z = 1$ or at $z = -1$ or at both locations or no zeros at $z = 1$ and at $z = -1$.
Therefore, we need at least an extra zero at 1 (2 zeros at 1).
We do not need zeros at -1.
On total 10 zeros at least.

For Type 2 linear phase FIR filters there can be an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$.
Therefore, we need at least an extra zero at 1 (2 zeros at 1).
We need at least one zero at -1 (1 zero on total).
On total 11 zeros at least.

For Type 3 linear phase FIR filters there can be an odd number of zeros at $z = 1$ and an odd number of zeros at $z = -1$.
Therefore, we don't need extra zeros at 1 (1 zero at 1).
We need at least one zero at -1 (1 zero on total).
On total 10 zeros at least.

For Type 4 linear phase FIR filters there can be an odd number of zeros at $z = 1$ and an even number or no zeros at $z = -1$.
Therefore, we don't need extra zeros at 1 (1 zero at 1).
We don't need zeros at -1.
On total 9 zeros at least.

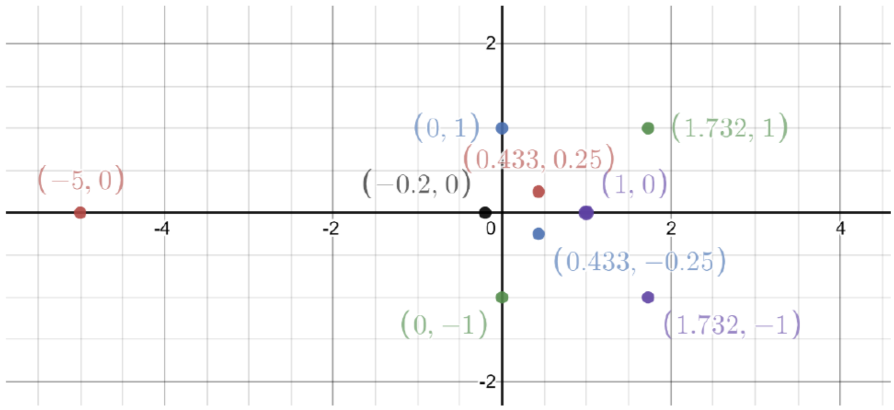
(ii) What is the filter type of the minimum length? **[3]**

Answer

This will be Type 4 with minimum 9 zeros.

(iii) Sketch the zero diagram of the filter identified in (ii) above on the z–plane. **[4]**
(iv)

Answer


(0, 1)
(1.732, 1)
(0.433, 0.25)
(−5, 0)
(−0.2, 0)
(1, 0)
(0.433, −0.25)
(0, −1)
(1.732, −1)
2
-2
-4
0
4

3. (a) The bilinear transformation from the s −plane to the z −plane is given by

$$z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

Here, T is the sampling period of the signal.

(i) Illustrate, using mathematical relationships, how the bilinear transformation and its inverse map each point $s = \sigma + j\Omega$ in the s −plane to a corresponding point $z = re^{j\omega}$ in the z −plane, and vice versa. **[3]**

Answer

$$z = \frac{1+s}{1-s}$$

For $s = j\Omega_0$ (a point on the imaginary axis), we have that $z = \frac{1+j\Omega_0}{1-j\Omega_0}$ which has a magnitude equal to 1. This implies that a point on the imaginary axis in the s −plane is mapped onto a point on the unit circle in the z −plane where $|z| = 1$. In the general case, for $s = \sigma_0 + j\Omega_0$

$$z = \frac{1 + \sigma_0 + j\Omega_0}{1 - \sigma_0 - j\Omega_0} \Rightarrow |z|^2 = \frac{(1+\sigma_0)^2 + {\Omega_0}^2}{(1-\sigma_0)^2 + {\Omega_0}^2}$$

A point in the left half s −plane with $\sigma_0 < 0$ is mapped onto a point inside the unit circle in the z −plane as $|z| < 1$. Likewise, a point in the right half s −plane with $\sigma_0 > 0$ is mapped onto a point outside the unit circle in the z −plane as $|z| > 1$.

(ii) Prove that the relationship between continuous-time angular frequency Ω and discrete-time angular frequency ω under the bilinear transformation is generally non-linear. Additionally, demonstrate that this mapping becomes approximately linear at low frequencies.
[Hint: Use the small-angle approximation for the tangent function, $\tan(x) \approx x$ when $x \to 0$]. **[3]**

Answer

From

$$z = \frac{1+s}{1-s}, z - zs = 1 + s \Rightarrow z - 1 = zs + s = s(z+1) \Rightarrow s = \frac{z-1}{z+1} = \frac{1-z^{-1}}{1+z^{-1}}$$

We see that $j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j\tan\left(\frac{\omega}{2}\right) \Rightarrow \Omega = \tan\left(\frac{\omega}{2}\right)$ which is a non-linear relationship.

(b) The transfer function of an ideal analogue integrator is given below:

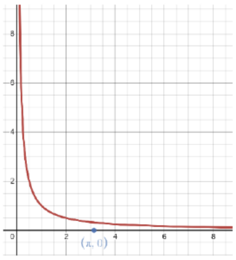
$$H_a(s) = \frac{1}{s}$$

A digital integrator can be obtained from an analogue integrator using the bilinear transformation.

(i) Find the amplitude and phase response of the analogue integrator and provide a rough sketch. **[3]**

Answer

$$H_a(j\Omega) = \frac{1}{j\Omega} = -\frac{j}{\Omega} = \frac{1}{\Omega}(-j) = \frac{1}{\Omega}e^{-\frac{j\pi}{2}},\ |H_a(j\Omega)| = \frac{1}{\Omega},\ \angle H_a\left(e^{j\Omega}\right) = -\frac{\pi}{2}$$



The sketch for the phase response is straightforward.

(ii) Derive the input-output relationship for the digital integrator. For simplicity use $T = 2$ for the sampling period. **[3]**

Answer

From $H_a(s) = \frac{1}{s}$ using the inverse bilinear transformation we get

$$H(z) = \frac{1+z^{-1}}{1-z^{-1}} \Rightarrow \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-z^{-1}} \Rightarrow Y(z) - z^{-1}Y(z) = X(z) + z^{-1}X(z)$$

$$\Rightarrow y[n] - y[n-1] = x[n] + x[n-1] \Rightarrow y[n] = y[n-1] + x[n] + x[n-1]$$

(iii) Estimate the frequency response of the digital integrator. **[3]**

Answer

$$H\left(e^{j\omega}\right) = \frac{1+e^{-j\omega}}{1-e^{-j\omega}} = \frac{e^{-j\omega/2}}{e^{-j\omega/2}}\frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}} = \frac{2\cos(\omega/2)}{2j\sin(\omega/2)} = j\cot(\omega/2)$$

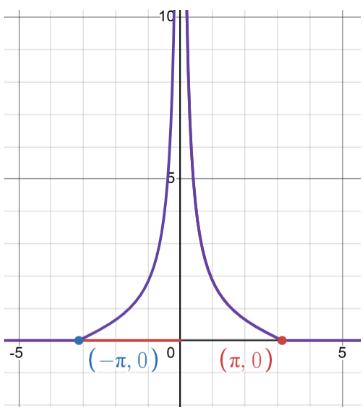
(iv) Create a rough sketch of the amplitude and phase responses of the digital integrator. **[3]**

Answer

We write the frequency response as:

$$H\left(e^{j\omega}\right) = \cot(\omega/2)e^{-j\pi/2}$$

The plot for the magnitude response is given below for the range $(-\pi, \pi)$.



(v) Compare the amplitude and phase responses of the analogue and digital integrators, with a focus on demonstrating that the amplitude responses are more closely aligned at low frequencies compared to high frequencies. **[3]**

Answer

For low frequencies the magnitude response of the digital integrator is $\cot(\omega/2) \approx 2/\omega$. We know that $\Omega = \omega/2$, therefore $\frac{2}{\omega} = 1/\Omega$ which is equal to the magnitude response of the analogue integrator.
The phase responses are equal.

(vi) Demonstrate that the pole of the digital integrator, located on the unit circle, causes its response to become unbounded for inputs with a DC component. Use the input signal $x[n] = u[n]$, where $u[n]$ is the unit step function, to illustrate this behaviour. **[4]**

$$y[n] = y[n-1] + x[n] + x[n-1]$$
$$y[0] = y[-1] + x[0] + x[-1] = x[0] = 1$$
$$y[1] = y[0] + x[1] + x[0] = x[0] = 3$$
$$y[2] = y[1] + x[2] + x[1] = x[0] = 5$$

We see that $y[n] = 2n + 1$ and therefore, $y[n]$ increases without limit as n increases.

4. (a) Examine the multirate configuration illustrated in **Figure 4.1** below. In this setup, $H_0(z), H_1(z)$, $H_2(z)$ and $H_3(z)$ are filters, each possessing ideal, zero-phase characteristics and real coefficients. Their corresponding frequency responses are outlined as follows:

$$H_0(e^{j\omega}) = u(\omega) - u(\omega - \frac{\pi}{6})$$
$$H_1(e^{j\omega}) = u(\omega - \frac{\pi}{6}) - u(\omega - \frac{\pi}{2})$$
$$H_2(e^{j\omega}) = u\left(\omega - \frac{\pi}{2}\right) - u\left(\omega - \frac{5\pi}{6}\right)$$
$$H_3(e^{j\omega}) = u\left(\omega - \frac{5\pi}{6}\right) - u(\omega - \pi)$$

The function $u(\omega)$ is the unit step function, defined as

$$u(\omega) = \begin{cases} 1 & \omega \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

If the input is a real sequence with a Discrete Time Fourier Transform

$$X(e^{j\omega}) = (\frac{\pi}{6} - \frac{6\omega^2}{\pi})\left(u(\omega) - u\left(\omega - \frac{\pi}{6}\right)\right)$$

sketch the Discrete Time Fourier Transform of the outputs $y_0[n], y_1[n]$, $y_2[n]$ and $y_3[n]$. It's important to note that all frequency domain representations mentioned are limited to the range $[0, \pi]$. **[15]**

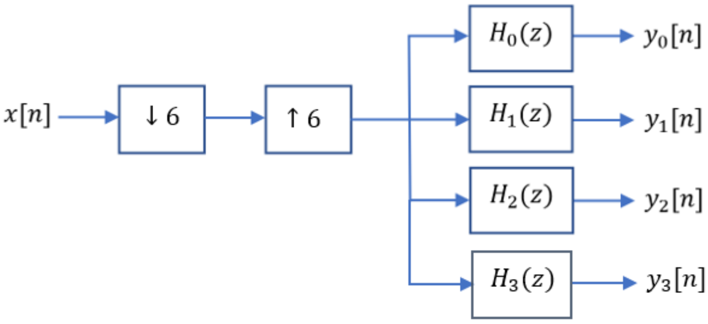
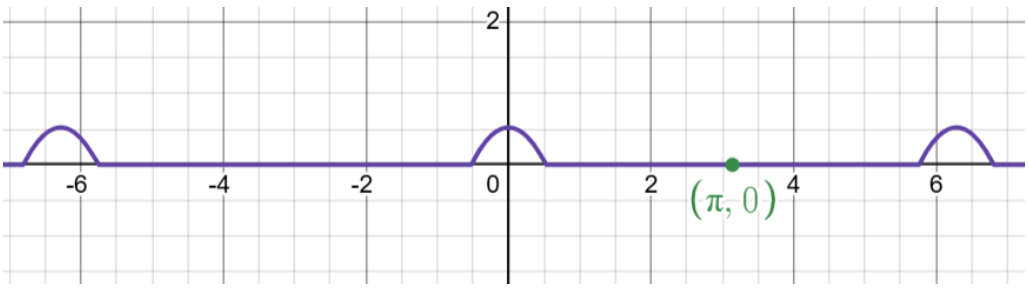


Figure 4.1

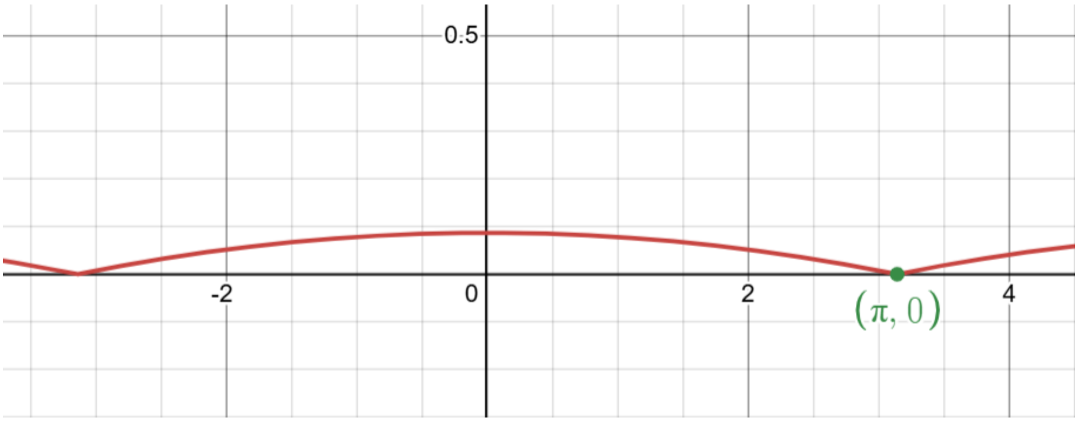
Answer

The input $x[n]$ is depicted below.



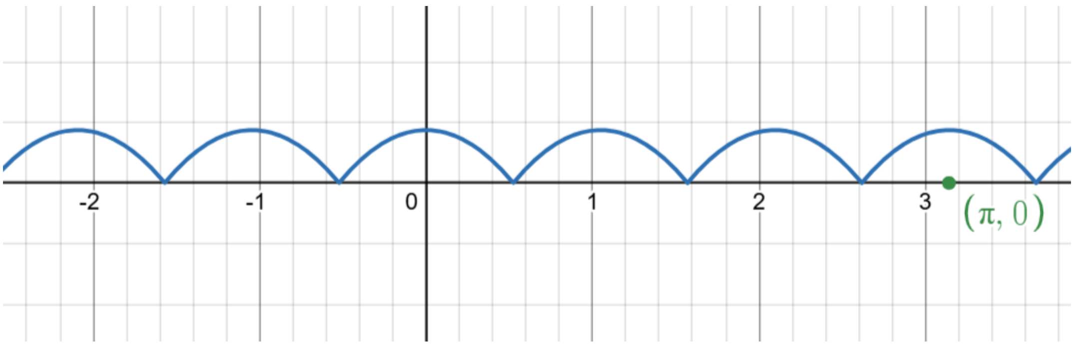
After decimation by 6, we obtain the following function in frequency domain, which is illustrated in the following figure:

$$Y\left(e^{j\omega}\right) = \frac{1}{6}\sum_{k=0}^{5} X\left(e^{-j\frac{2\pi}{6}} e^{j\frac{\omega}{6}}\right)$$

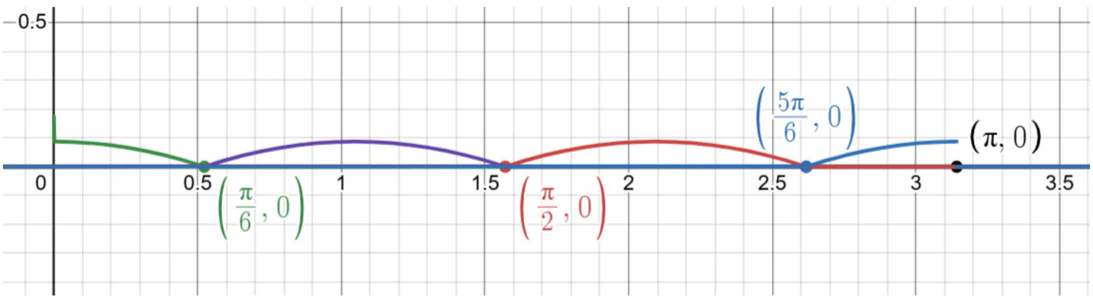


After interpolation by 6, we obtain the following function in frequency domain, which is illustrated in the following figure:

$G\left(e^{j\omega}\right) = Y\left(e^{j6\omega}\right)$



The outputs of the 3 filters are obvious (shown in the figure below with green, purple, red and blue).



(b) Develop an expression for the output $y[n]$ as a function of the input $x[n]$ for the multirate structure of **Figure 2** below.

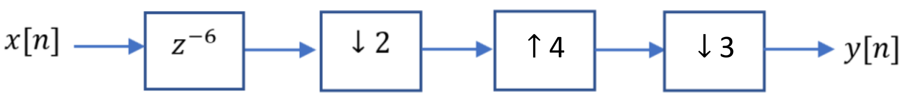


Figure 2

[10]

Answer

The output of the first component of the system which is z^{-6} is $x_1[n] = x[n-6]$.

The output of the second box is a **decimation-by-2** and therefore, it is

$$x_2[n] = x_1[2n] = x[2n-6]$$

The output of the second box is an **interpolation-by-4** and therefore, it is

$$x_3[n] = \begin{cases} x_2[n/4] = x_1[n/2] = x[n/2-6] & n \text{ is a multiple of } 4 \\ 0 & \text{otherwise} \end{cases}$$

The output of the third box is a **decimation-by-3** and therefore, it is

$$x_4[n] = \begin{cases} x_3[3n] = x_2[3n/4] = x_1[3n/2] = x[3n/2-6] & n \text{ is a multiple of } 4 \\ 0 & \text{otherwise} \end{cases}$$