1. (a) Consider the causal, complex coefficient transfer function shown below:

$$H_M(z) = \frac{\sum_{n=0}^{n=M-1} d^*[M-n]z^{-n} + z^{-M}}{1 + \sum_{n=1}^{n=M} d[n]z^{-n}}, M > 1 \quad (\mathbf{1})$$

(i) Show that the transfer function in **(1)** above is an allpass filter. **[5]**

Answer

The condition for an allpass transfer function is that $|H_M(z)| = 1$, for $|z| = 1$, which implies that $|H_M(e^{j\omega})| = 1$ for all ω. The given function is of the form:

$$H_M(z) = \frac{\sum_{n=0}^{n=M-1} d^*[M-n]z^{-n} + z^{-M}}{1 + \sum_{n=1}^{n=M} d[n]z^{-n}} = \frac{\sum_{n=0}^{n=M} d^*[M-n]z^{-n}}{\sum_{n=0}^{n=M} d[n]z^{-n}}, d^*[0] = 1$$
$$= \frac{z^{-M}\sum_{n=0}^{n=M} d^*[M-n]z^{M-n}}{\sum_{n=0}^{n=M} d[n]z^{-n}} = \frac{z^{-M}\sum_{r=0}^{r=M} d^*[r]z^{r}}{\sum_{n=0}^{n=M} d[n]z^{-n}},$$
$r = M - n, d^*[0] = 1$

If we denote the denominator polynomial of the allpass function as $A_M(z)$ with

$$A_M(z) = \sum_{n=0}^{n=M} d[n]z^{-n}$$

then it follows that the given $H_M(z)$ can be written as:

$$H_M(z) = \frac{z^{-M}\bar{A}_M(z^{-1})}{A_M(z)}$$

$\bar{A}_M(z)$ is a polynomial that is obtained from the polynomial $A_M(z)$ if we complex conjugate the coefficients of $A_M(z)$ but we make no changes to the variable z.
For $z = e^{j\omega}$ we see that $|H_M(e^{j\omega})| = 1$. This comes from the fact that $\bar{A}_M(z^{-1})$ becomes the complex conjugate of $A_M(z)$ since every term in $\bar{A}_M(z^{-1})$ is now the complex conjugate of the corresponding term in $A_M(z)$. In that case $|\bar{A}_M(z^{-1})| = |A_M(z)|$ and $|H_M(e^{j\omega})| = 1$. Therefore, $H_M(z)$ is an allpass filter.

Note that the coefficients $d[n]$ are constants and do not change with the location of z. Furthermore, the coefficients in this particular exercise are complex.

(ii) Comment on the locations of the zeros and the poles of a causal, real, stable, allpass filter's transfer function within the z −plane, with respect to the unit circle. Justify your answer. **[5]**

Answer

We have proven in the lectures that if z_0 is a pole of an allpass filter, then $1/z_0$ is a zero of the allpass filter. The poles of a causal stable transfer function must lie inside the unit circle. As a result, ALL zeros of a causal stable allpass transfer function lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle.

(b) Given an allpass filter

$$H(z) = \frac{d + ez + fz^2}{1 + bz + cz^2}$$

with poles at $\frac{1}{2}$ and $\frac{1}{3}$, find b, c, d, e, f. **[5]**

Answer

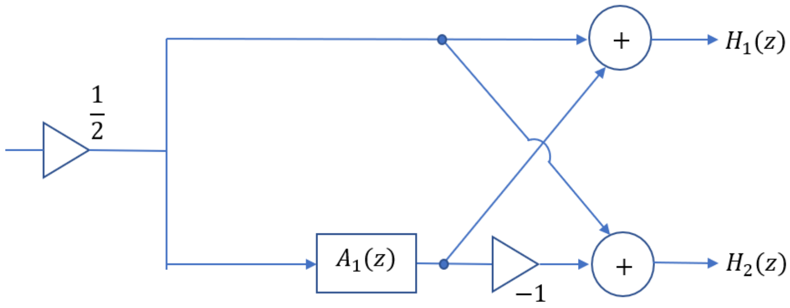
$$1 + bz + cz^2 = c\left(\frac{1}{c} + \frac{b}{c}z + z^2\right) = c\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right) = c\left(\frac{1}{6} - \frac{5}{6}z + z^2\right)$$

$c = 6, b = -5$

$$H(z) = \frac{d + ez + fz^2}{1 - 5z + 6z^2}$$

and therefore, $d = 6, e = -5, f = 1$.

(c) In the structure below, $A_1(z)$ is a first-order allpass transfer function. Find the type of filters $H_1(z)$ and $H_2(z)$ (lowpass etc.).



[5]

Answer

$H_1(z) = \frac{1}{2}(1 + A_1(z))$

$H_1(z) = \frac{1}{2}(1 + \frac{-az+1}{z-a}) = \frac{1}{2}\frac{(1-a)(z+1)}{z-a}$

$z = 1 \Longrightarrow \omega = 0, H_1(1) = 1$

$z = -1 \Longrightarrow \omega = \pi, H_1(-1) = 0$

Therefore, $H_1(z)$ is a lowpass filter.

$H_2(z) = \frac{1}{2}(1 - \frac{-az+1}{z-a}) = \frac{1}{2}\frac{(1+a)(z-1)}{z-a}$

$z = 1, H_1(1) = 0$

$z = -1, H_1(-1) = 1$

Therefore, $H_2(z)$ is a highpass filter.

(d) Consider the cascade of two causal LTI systems with impulse responses $h_1[n] = a\delta[n] + b\delta[n-1]$ and $h_2[n] = c^n u[n]$, a, b, c are real coefficients and $|c| < 1$. Determine the frequency response $H(e^{j\omega})$ of the overall system. Find all the possible sets of values of a, b, c for which the overall transfer function has an amplitude equal to 1 for all frequencies **[5]**

Answer

$$H(z) = H_1(z)H_2(z) = \frac{(a + bz^{-1})z}{z - c} = \frac{az + b}{z - c}$$

$|H(e^{j\omega})| = 1$ implies that $H(z)$ is an allpass filter.

$$|H(e^{j\omega})|^2 = \frac{a^2 + b^2 + 2ab\cos(\omega)}{1 + c^2 - 2c\cos(\omega)}$$

In order for this relationship to be satisfied for every ω the following conditions must hold:

1. $a^2 + b^2 = 1 + c^2$
2. $ab = -c$

From these two conditions we obtain:
$a^2 + b^2 = 1 + a^2b^2, a^2(1-b^2) = (1-b^2)$
If $b^2 \neq 1$ then $a^2 = 1$

- $a = 1 \Rightarrow b = -c, H(z) = \frac{z+b}{z+b} = 1$
- $a = -1 \Rightarrow b = c, H(z) = \frac{-z+b}{z-b} = -1$

If $b^2 = 1$ then

- $b = 1 \Rightarrow a = -c, H(z) = \frac{az+1}{z+a}$
- $b = -1 \Rightarrow a = c, H(z) = \frac{az-1}{z-a}$

Therefore, there are 4 solutions to the problem. Specific values for a, b, c must be found.

2. (a) Design a length-5 FIR bandpass filter with an antisymmetric impulse response $h[n]$, satisfying the following magnitude response values: $|H(e^{j0.3\pi})| = 0.3$ and $|H(e^{j0.6\pi})| = 0.8$. **[4]**

Answer

The generic form of a length-5 FIR bandpass filter with an antisymmetric impulse response $h[n]$ is given by:

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + 0e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega}$$
$$= h[0] + h[1]e^{-j\omega} - h[1]e^{-j3\omega}$$
$$- h[0]e^{-j4\omega} = e^{-j2\omega}[h[0](e^{j2\omega} - e^{-j2\omega}) + h[1](e^{j\omega} - e^{-j\omega})]$$
$$= 2je^{-j2\omega}[h[0]\sin(2\omega) + h[1]\sin(\omega)]$$

$$|H(e^{j\omega})| = 2[h[0]\sin(2\omega) + h[1]\sin(\omega)]$$

Thus,

$$|H(e^{j0.3})| = 2[h[0]\sin(0.6\pi) + h[1]\sin(0.3\pi)] = 0.3$$
$$|H(e^{j0.6})| = 2[h[0]\sin(1.2\pi) + h[1]\sin(0.6\pi)] = 0.8$$

Solving these two equations we get:

$h[0] = -0.133$ and $h[1] = 0.34$

As we see here, we have only two unknown coefficients due to the condition of anti-symmetry and for that reason we are given two pieces of information so that we are able to form two equations with two unknowns and solve them.

(b) We have shown that a real-coefficient FIR transfer function $H(z)$ with a symmetric impulse response has a linear phase response. As a result, the all-pole IIR transfer function $G(z) = \frac{1}{H(z)}$ will also have a linear-phase response. What are the practical difficulties in implementing $G(z)$? Justify your answer. **[4]**

Answer

$G(z)$ must have poles outside the unit circle making it unstable.

(c) A simple averaging filter is defined by the following input-output relationship:

$$y[n] = \frac{1}{N}(x[n] + x[n-1] + \cdots + x[n-(N-1)])$$

(i) Let $N = 4$. Determine the filter's transfer function and its zeros and poles. **[3]**

[**Hint:** You may use the relationship $\sum_{i=0}^{N-1} z^{-i} = \frac{1-z^{-N}}{1-z^{-1}}, \;\; z \neq 1$.]

Answer

$$Y(z) = \frac{1}{N}[1 + z^{-1} + \cdots + z^{-(N-1)}]X(z) = H(z)X(z)$$
$$H(z) = \frac{1}{N}\frac{1-z^{-N}}{1-z^{-1}} = \frac{1}{N}\frac{z^N(1-z^{-N})}{z^{N-1}z(1-z^{-1})} = \frac{1}{N}\frac{1}{z^{N-1}}\frac{z^N-1}{z-1}$$

For $N = 4$ we have $H(z) = \frac{1}{4}\frac{1}{z^3}\frac{z^4-1}{z-1}$

$z^4 - 1 = 0 \Rightarrow z^4 = 1 \Rightarrow z^2 = \pm 1$
$z^2 = 1 \Rightarrow z = \pm 1$
$z^2 = -1 \Rightarrow z = \pm j$
The zero at 1 is cancelled with the pole at 1.
Therefore, we have zeros at $-1, \pm j$ and a triple pole at 0.

(ii) Determine a general form for zeros and poles for any N. **[3]**

Answer

$$H(z) = \frac{1}{N}\frac{1}{z^{N-1}}\frac{z^N - 1}{z - 1}$$
Zeros are found from $z^N - 1 = 0 \Rightarrow z^N = 1 \Rightarrow z = e^{j\frac{2\pi k}{N}}, k = 0, \ldots, N-1$
Poles at $z = 0$ with multiplicity $N - 1$ and a pole at $z = 1$ which is cancelled by the zero at $z = 1$.
Therefore, zeros are at $z = e^{j\frac{2\pi k}{N}}, k = 1, \ldots, N-1$.

(iii) A recursive implementation of a filter refers to writing the current output as a function of the previous output and the current and past inputs. By comparing $y[n]$ and $y[n-1]$ determine a recursive implementation of the filter. **[3]**

Answer

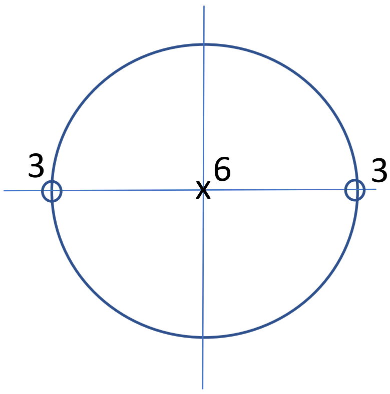
$$y[n-1] = \frac{1}{N}(x[n-1] + x[n-2] + \cdots + x[n-1-(N-1)])$$
$$y[n] = \frac{1}{N}(x[n] + x[n-1] + \cdots + x[n-(N-1)])$$
$$y[n] - y[n-1] = \frac{1}{N}(x[n] - x[n-1-(N-1)]) \Rightarrow$$
$$y[n] - y[n-1] = \frac{1}{N}(x[n] - x[n-N])$$
We observe that the current output is a function of the previous output, the current input and the input that is distanced by N samples.

(d) An FIR digital filter has the transfer function
$$H(z) = (1 - z^{-1})^3(1 + z^{-1})^3$$

(i) Sketch the pole-zero diagram of this system. **[4]**

Answer

Using polynomial expansion, we obtain
$$H(z) = \frac{(z-1)^3(z+1)^3}{z^6}$$
We see that we have 3 zeros at 1, 3 zeros at -1 and 6 poles at 0



(ii) Sketch **roughly** the amplitude response of the above filter. Would you classify this as a low-pass, high-pass, band-pass, or band-stop filter? Please briefly explain. **[4]**

Answer

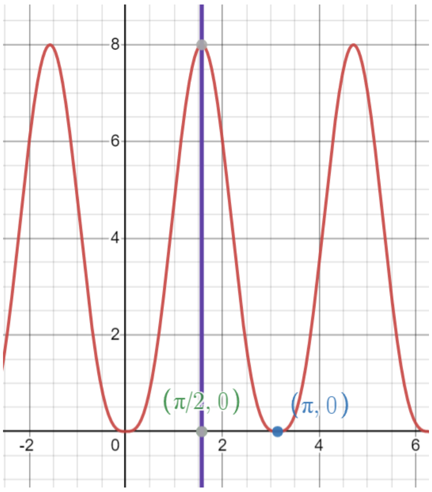
This is the Type 3 linear phase filter it can only be band pass. This answer is not complete. To prove the statement by using the expression given we see that:

$z = 1 \Rightarrow \omega = 0 \Rightarrow H(e^{j\omega}) = 0$

$z = -1 \Rightarrow \omega = \pi \Rightarrow H(e^{j\pi}) = 0$

$z = \pm j \Rightarrow \omega = \dfrac{\pi}{2} \Rightarrow H(e^{\pm j\pi/2}) = 8$

I have provided here an accurate plot of the amplitude response but in the exam paper an approximate plot based on the above results is considered as a complete answer.



3. (a) The bilinear transformation from the s −plane to the z −plane is given by

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

(i) Explain using mathematical relations, where each point $s = \sigma + j\Omega$ on the s −plane is mapped on the z −plane through the bilinear transformation. **[5]**

Answer

$$z = \frac{1 + s}{1 - s}$$

For $s = j\Omega_0$ we have that $z = \frac{1+j\Omega_0}{1-j\Omega_0}$ which has a magnitude equal to 1. This implies that a point on the imaginary axis in the s −plane is mapped onto a point on the unit circle in the z −plane where $|z| = 1$. In the general case, for $s = \sigma_0 + j\Omega_0$

$$z = \frac{1 + \sigma_0 + j\Omega_0}{1 - \sigma_0 - j\Omega_0} \Rightarrow |z|^2 = \frac{(1 + \sigma_0)^2 + {\Omega_0}^2}{(1 - \sigma_0)^2 + {\Omega_0}^2}$$

A point in the left half s −plane with $\sigma_0 < 0$ is mapped onto a point inside the unit circle in the z −plane as $|z| < 1$. Likewise, a point in the right half s −plane with $\sigma_0 > 0$ is mapped onto a point outside the unit circle in the z −plane as $|z| > 1$.

(ii) Prove that the relationship between the continuous-time angular frequency and the discrete-time angular frequency is non-linear. **[5]**

Answer

The variable s is reduced on the imaginary axis to $s = j\Omega$. The variable z is reduced on the unit circle to $z = e^{-j\omega}$. Since the bilinear transformation maps one plane to the other and vice versa and we have proven that the imaginary axis on the s plane is mapped to the unit circle on the z plane, we can write that:

$j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j\tan\left(\frac{\omega}{2}\right) \Rightarrow \Omega = \tan\left(\frac{\omega}{2}\right)$ which is a non-linear relationship.

(b) A given real-coefficient, digital IIR lowpass filter has a rational transfer function $H_L(z)$ and a cutoff frequency ω_c. The transfer function of $H_L(z)$ is transformed by replacing z by $F(z) = \frac{az+b}{cz+d}$ to a real highpass rational transfer function $H_H(z) = H_L(F(z))$ of the same order as $H_L(z)$ but different cutoff frequency. By using the constraints $H_H(e^{j\pi}) = H_L(e^{j0}) = 1$ and $H_H(e^{j0}) = H_L(e^{j\pi}) = 0$, derive relationships among the parameters a, b, c, d and explain what type of filter is $F(z)$. **[5]**

Answer

$$H_H(e^{j0}) = H_L\left(F(e^{j0})\right) = H_L(F(1)) = 0$$

Therefore, $F(1) = \frac{a+b}{c+d} = e^{j\pi} = -1 \Rightarrow a + b = -c - d$

$$H_H(e^{j\pi}) = H_L\left(F(e^{j\pi})\right) = H_L(F(-1)) = 1$$

Therefore, $F(-1) = \frac{-a+b}{-c+d} = 1 \Rightarrow -a + b = -c + d$

Therefore, $b = -c$ and $a = -d$

$$F(z) = \frac{az + b}{-bz - a} = -\frac{az + b}{bz + a}$$

$F(z)$ is an allpass filter.

(c) Consider the Finite Impulse Response (FIR) filter transfer functions $G_1(z) = \frac{1}{4}(z + 2 + z^{-1})$ and $G_2(z) = \frac{1}{4}(-z + 2 - z^{-1})$. Explain the type of filters $G_1(z)$ and $G_2(z)$ (lowpass etc.), by using the following two approaches:

(i) Experimental approach. In this approach you will apply the filters to the signal $x[n] = u[n]$, with $u[n]$ the discrete unit step function and observe the effects they have on $x[n]$. **[5]**

Answer

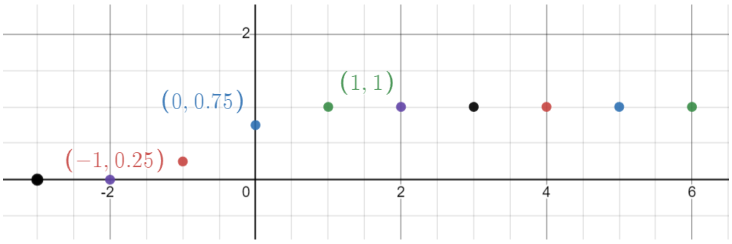
The input-output relationship of the given transfer function $G_1(z)$ is

$$y[n] = \frac{1}{4}(x[n+1] + 2x[n] + x[n-1])$$

If $x[n] = u[n]$, with $u[n]$ the discrete unit step function we have:

$$y[n] = \frac{1}{4}(u[n+1] + 2u[n] + u[n-1])$$

$$y[n] = \begin{cases} 0 & n < -1 \\ 1/4 & n = -1 \\ 3/4 & n = 0 \\ 1 & n \geq 1 \end{cases}$$



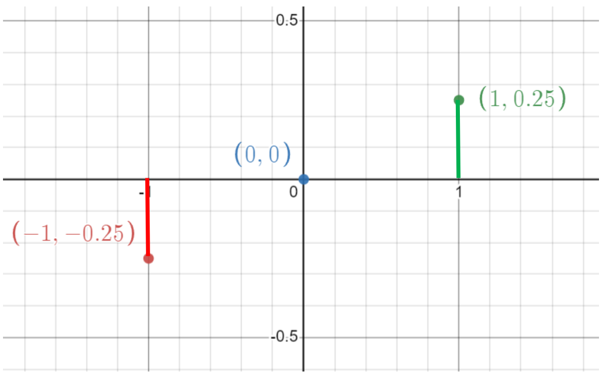
The input-output relationship of the given transfer function $G_2(z)$ is

$$y[n] = \frac{1}{4}(-x[n+1] + 2x[n] - x[n-1])$$

If $x[n] = u[n]$, with $u[n]$ the discrete unit step function we have:

$$y[n] = \frac{1}{4}(-u[n+1] + 2u[n] - u[n-1])$$

$$y[n] = \begin{cases} 0 & n < -1 \\ -1/4 & n = -1 \\ 1/4 & n = 0 \\ 0 & n \geq 1 \end{cases}$$



As we see by applying $u[n]$ as input to $G_1(z)$, $G_1(z)$ is a lowpass filter because the sharpness of $u[n]$ at $n = 0$ is destroyed by applying the filter.

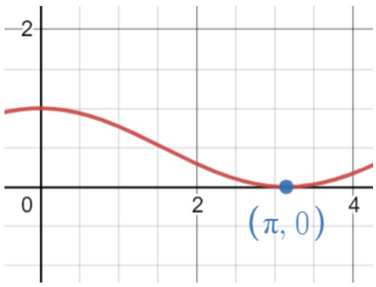
On the other hard, $G_2(z)$ is a highpass filter because we see by applying $u[n]$ as input to $G_2(z)$, the constant areas are eliminated, and we are left only with an enhanced abrupt change around $n = 0$ which is the original abrupt change point.

(ii) Mathematical approach. In this approach you must find the frequency response of the two filters. **[5]**

Answer

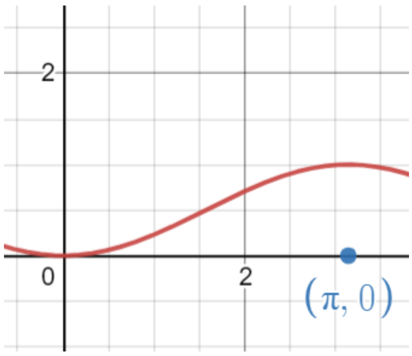
$G_1(z) = \frac{1}{4}(z + 2 + z^{-1}) = \frac{1}{4}(2 + 2\cos(\omega)) = \frac{1}{2}(1 + \cos(\omega))$

$|G_1(e^{j\omega})| = \frac{1}{2}|1 + \cos(\omega)|$. We observe that $|G_1(e^{j\omega})|$ decreases as the frequency approaches π.



$G_2(z) = \frac{1}{4}(-z + 2 - z^{-1}) = \frac{1}{4}(2 - 2\cos(\omega)) = \frac{1}{2}(1 - \cos(\omega))$

$|G_2(e^{j\omega})| = \frac{1}{2}|1 - \cos(\omega)|$



We observe that $|G_2(e^{j\omega})|$ increases as the frequency approaches π.

4. (a) Consider the multirate structure of **Figure 1** below, where $H_0(z), H_1(z)$ and $H_2(z)$ are ideal, zero phase, real-coefficient lowpass, bandpass and highpass filters respectively, with frequency responses as follows:

$$H_0(e^{j\omega}) = u(\omega) - u(\omega - \frac{2\pi}{3})$$
$$H_1(e^{j\omega}) = u(\omega - \frac{\pi}{3}) - u(\omega - \frac{2\pi}{3})$$
$$H_2(e^{j\omega}) = u(\omega - \frac{\pi}{3}) - u(\omega - \pi)$$

The function $u(\omega)$ is the well-known unit step function, defined as

$$u(\omega) = \begin{cases} 1 & \omega \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

If the input is a real sequence with a Discrete Time Fourier Transform

$$X(e^{j\omega}) = (-\frac{3\omega^2}{\pi} + \frac{\pi}{3})\left(u(\omega) - u\left(\omega - \frac{\pi}{3}\right)\right)$$

sketch the Discrete Time Fourier Transform of the outputs $y_0[n], y_1[n]$ and $y_2[n]$.
Notice that all frequency domain representations above are provided only within the range $[0, \pi]$. **[15]**

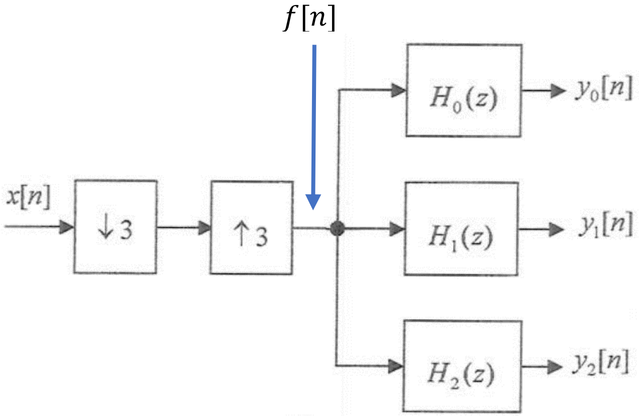
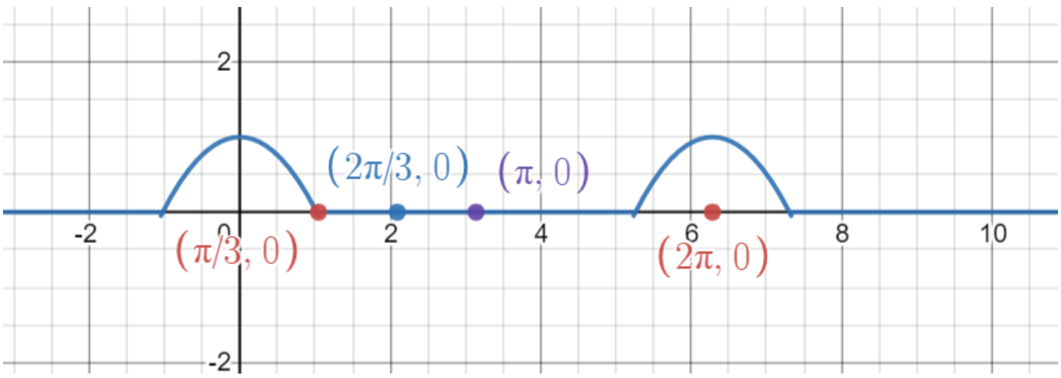


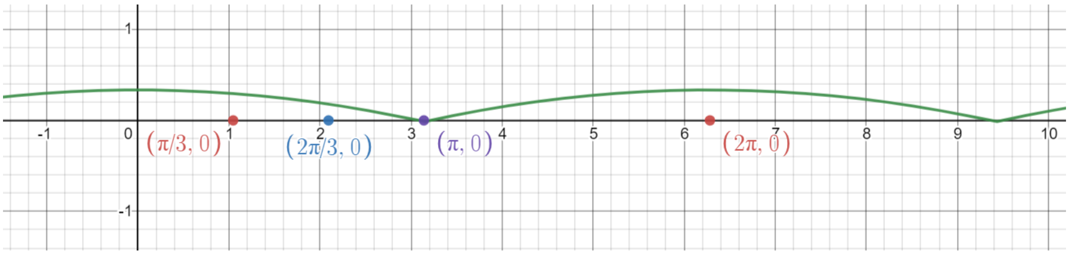
Figure 1

Answer

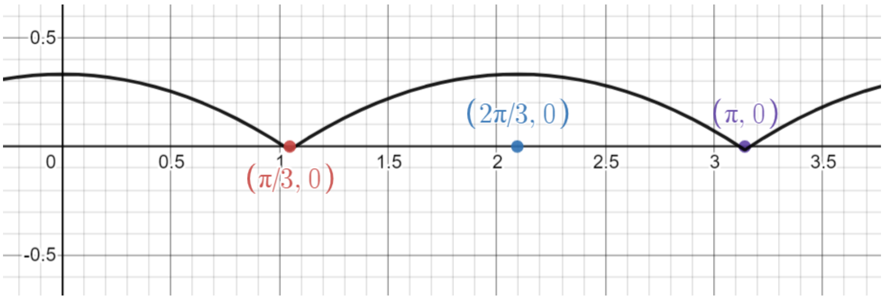
The DTFT of the input is shown below:



After decimation by 3:



After interpolation by 3 we obtain the DTFT of the signal $f[n]$ shown below.



The output of the three filters is obvious.

(b) Develop an expression for the output $y[n]$ as a function of the input $x[n]$ for the multirate structure of Figure 3 below. **[10]**

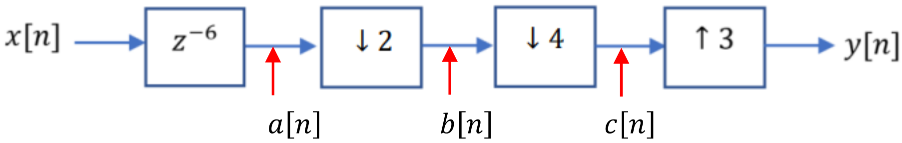


Figure 2

Answer

$a[n] = x[n-6]$
$b[n] = a[2n]$
$c[n] = b[4n] = a[8n] = x[8n-6]$

$$y[n] = \begin{cases} x[\frac{8n}{3} - 6] & n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$$

It is important that the answer specifies clearly that the relationship $y[n] = x[\frac{8n}{3} - 6]$ is only valid if n is a multiple of 3. If this is not specified, then the signal given is different.