Digital Signal Processing

Tutorial Questions Booklet

Digital Signal Processing

Solution Sheet 1

Sampling and z-transforms

1. $Y(z) = H(z)X(z)$

2. $$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

3. $$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m)$$

4. From the definition of the z-transform

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y(n)z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(m)x(n-m)z^{-n} \\
&= \sum_{p=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h(m)x(p)z^{-m}z^{-p} \quad \text{for } p = n-m \\
&= \sum_{m=-\infty}^{\infty} h(m)z^{-m} \sum_{p=-\infty}^{\infty} x(p)z^{-p} \\
&= H(z)X(z)
\end{aligned}$$

5. Factor the denominator so that $X(z) = \dfrac{7z^2 - 5z}{(z+1)(z-3)}$ in order to find that the poles are at $z = -1$ and $z = 3$. Partial fraction expansion gives $X(z) = \dfrac{3z}{z+1} + \dfrac{4z}{z-3}$.

a) For $|z| < 1$, the ROC is inside both poles, giving two left-sided sequences.

$$x(n) = -3(-1)^n u(-n-1) - 4(3)^n u(-n-1) \quad \text{[anticausal]}$$

b) For $|z| > 3$, the ROC is outside both poles, giving right-sided sequences.

$$x(n) = 3(-1)^n u(n) + 4(3)^n u(n) \quad \text{[causal]}$$

b) For $< 1|z| < 3$, the ROC is between both poles, giving a two-sided sequence.

$$x(n) = 3(-1)^n u(n) - 4(3)^n u(-n-1) \quad \text{[noncausal]}$$

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Solution Sheet 2

Discrete Fourier Transforms

1 $$X_p[k] = \sum_{n=0}^{N-1} x_p[n] e^{-j\frac{2\pi}{N}nk} \qquad k = \ldots -1, 0, 1, \ldots$$

- find period N
- compute one period of $X_p[k]$
- extent for $-\infty < k < \infty$.

$$x_p[n] = 10\sin\left(\frac{2\pi n}{3}\right), \quad N = 3$$

$n=0 \quad x_p[0] = 0$

$n=1 \quad x_p[1] = 10\sin(2\pi/3) = 8.66$

$n=2 \quad x_p[2] = 10\sin(4\pi/3) = -8.66$

$k=0 \quad X_p[0] = x_p[0].e^0 + x_p[1].e^0 + x_p[2].e^0 = 0$

$k=1 \quad X_p[1] = x_p[0].e^0 + x_p[1].e^{-j2\pi/3} + x_p[2].e^{-j4\pi/3}$

$= 0 +$

$8.66(-0.5 - j0.866) +$

$-8.66(-0.5 + j0.866)$

$= 0 - j15$

$k=2 \quad X_p[2] = x_p[0].e^0 + x_p[1].e^{-j4\pi/3} + x_p[2].e^{-j8\pi/3}$

$= 0 +$

$8.66(-0.5 + j0.866) +$

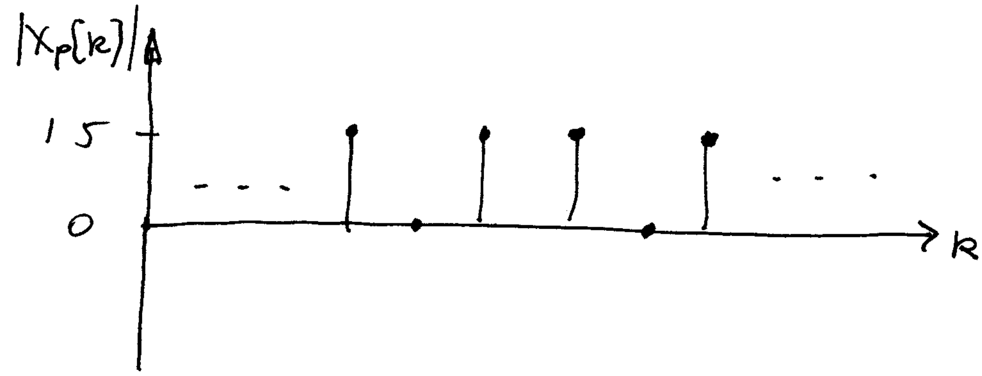
$-8.66(-0.5 - j0.866)$

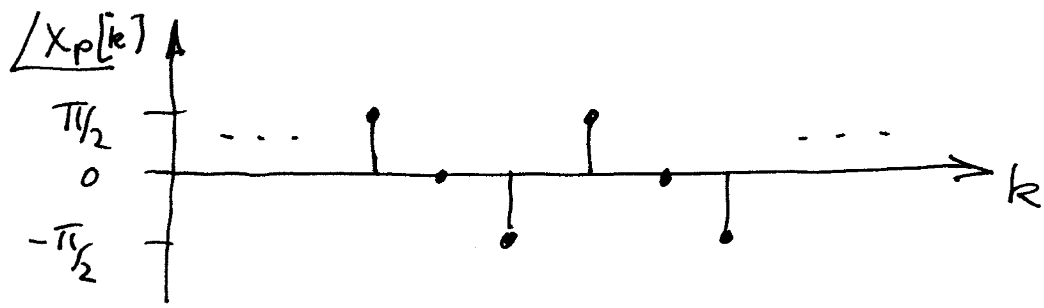
$= 0 + j15$

$$X_p[k] = [\ldots, j15, 0, -j15, j15, 0, \ldots]$$

$$|X_p[k]| = [\ldots, 15, 0, 15, 15, 0, \ldots]$$

$$\angle X_p[k] = [\ldots, \pi/2, 0, -\pi/2, \pi/2, 0, \ldots]$$





2. Method similar to Q1 but using the DFT since the sequence is aperiodic.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$$

Magnitude spectrum given by $|X[k]|$

i.e. $\left(\mathrm{Re}\{X[k]\}^2 + \mathrm{Im}\{X[k]\}^2\right)^{1/2}$

$$|X[k]| = [10.0 \quad 3.1 \quad 0.7 \quad 0.7 \quad 3.1]$$

3 a) 8 kHz, 1024 samples $\Rightarrow$ 128 ms

b) Freq. resolution $= \frac{1}{NT} = 7.8$ Hz

c) One way to view this operation is as a multiplication in the time domain by a rectangular window which is 1 for 512 points and 0 for 512 points. This will be seen in the frequency domain as sinc interpolation.

The formula $1/NT$ would yield a frequency resolution of 7.8 Hz but, in fact, every-other frequency domain point is due to the interpolation effect.

New a) 64 ms

New b) 7.8 Hz if sinc interpolation is a good fit.

4. $y[n] = [x[0], 0, x[1], 0, x[2], 0, \ldots]$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\, 2\pi/N \cdot n k} \qquad \text{where } N = 8$$

$$Y[k] = \sum_{n=0}^{2N-1} y[n]\, e^{-j\, 2\pi/2N \cdot n . k}$$

$$= \sum_{\substack{n=0 \\ n \text{ even}}}^{2N-1} x[\tfrac{n}{2}]\, e^{-j 2\pi/2N \cdot n . k} + \sum_{\substack{n=0 \\ n \text{ odd}}}^{2N-1} 0 . e^{-j\, 2\pi/2N \cdot n . k}$$

$X[k]$ is evaluated at N uniformly spaced frequencies around the unit circle

$Y[k]$ is evaluated at $2N$ uniformly spaced frequencies around the unit circle and

$$Y[k]\Big|_{k=0,1,..7} = Y[k]\Big|_{k=8,9,\ldots 15} = X[k]$$

since values of $Y[k]$ computed using $e^{-j \frac{2\pi}{16} . n . 0}$ will be the same as values computed using $e^{-j \frac{2\pi}{16} . n . 8}$ and so on. The final solution can be written $Y[mN+k] = X[k]$, $k = 0, 1, .. N-1$, $m = 0, 1$.

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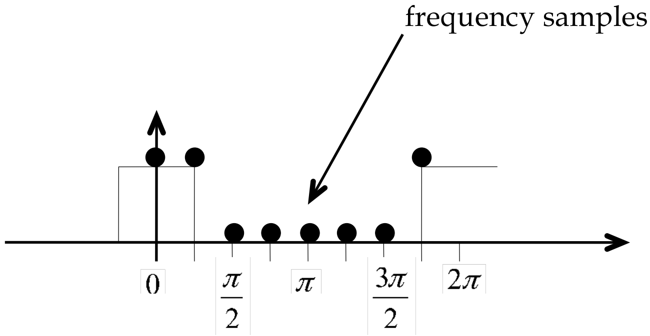
Solution Sheet 3

Digital Filters

1.



$H_s(k) = [1,1,0,0,0,0,0,1]$

$$h(n) = \frac{1}{8}\sum_{k=0}^{7} H(k)e^{j\frac{\pi}{4}nk}$$
$$= [0.375,\ \ 0.302,\ \ 0.125,\ \ -0.052,\ \ -0.125,\ \ -0.052,\ \ 0.125,\ \ 0.302]$$

$$\text{delay by } \frac{N}{2}$$
$$= [-0.125,\ \ -0.052,\ \ 0.125,\ \ 0.302,\ \ 0.375,\ \ 0.302,\ \ 0.125,\ \ -0.052]$$

In MATLAB this can be expressed as:

```
N=8;
wstep = 2*pi/N;
w = [0:wstep:(2*pi)-wstep];
H=[1 1 0 0 0 0 0 1] .* exp(-j*w*N/2);
h = ifft(H);
```

However, we can see that this solution for $h(n)$ does not satisfy the symmetry constraints required for linear phase $h(n) = h(N - 1 - n)$. To obtain exactly linear phase we would need to delay the filter's impulse response not by $N/2$ but by $(N\text{-}1)/2$ so that, in the Matlab code above, the 4th line becomes

```
H=[1 1 0 0 0 0 0 1] .* exp(-j*w*(N-1)/2);
```

and as a consequence $h(n)$ becomes complex, although with the required conjugate symmetry for complex coefficient linear-phase filters:

```
h(n) =  [0.125-0.096j  0.125-0.231j  0.125-0.231j
0.125-0.096j  0.125+0.096j  0.125+0.231j  0.125+0.231j
0.125+0.096j]
```

2. d.c. gain ($\omega = 0$)

$$\left|H(e^{j0})\right| = k.\left|\frac{1+e^{j0}}{1-ce^{j0}}\right| = \frac{2k}{1-c}$$

3. For unity gain at d.c.

$$k = \frac{1-c}{2}$$

4. $$\left|H\left(e^{j\frac{\pi}{2}}\right)\right| = \frac{1-c}{2} \cdot \left|\frac{1+e^{-j\frac{\pi}{2}}}{1-ce^{-j\frac{\pi}{2}}}\right| = \frac{1-c}{2} \cdot \left|\frac{1-j}{1+jc}\right| = \frac{1}{\sqrt{2}} \text{ for -3dB.}$$

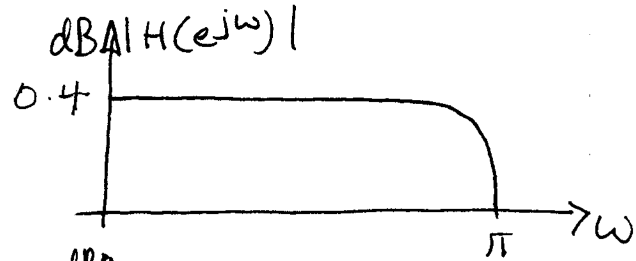
This equation is satisfied for $c = 0$.

(5) $H(z) = \dfrac{1+z^{-1}}{1-cz^{-1}} \qquad |z| > |c|$

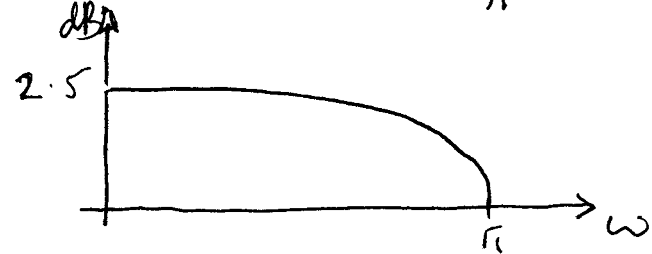
For frequency response write $z = e^{j\omega}$

$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{1 - c e^{-j\omega}}$$

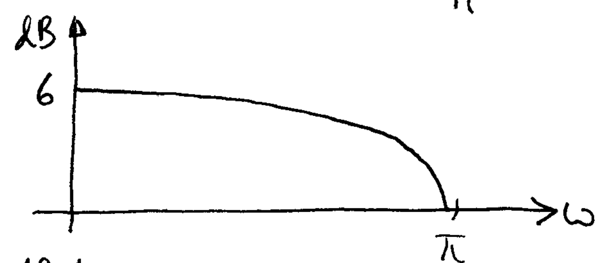
For $c = -0.9$



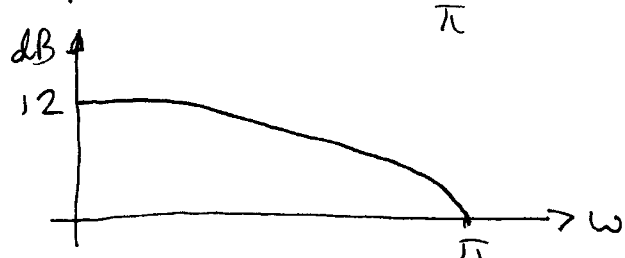
For $c = -0.5$



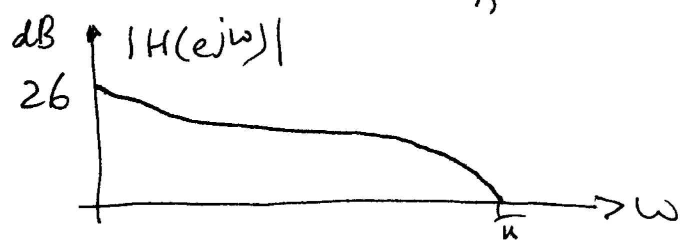
For $c = 0$



For $c = 0.5$



For $c = 0.9$



Digital Signal Processing

Solution Sheet 4

Convolution

1. $$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\, y[n-k] \right) . z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k] \;. z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] . z^{-k} . \sum_{n=-\infty}^{\infty} y[n-k] . z^{-(n-k)}$$

Now change the index of summation

$$n \longrightarrow m = n-k$$

$$\therefore W(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} . \sum_{m=-\infty}^{\infty} y[m] . z^{-m}$$

$$= X(z) . Y(z).$$

2. See page 413, Strum & Kirk.

3. Overlap Add

$x = [3, 2, 1, 4, 1, 2, 3, 2, 1]$

Choose $L + M - 1 = 4$

$M = 2$

$L = 3$

- 4 point ffts
- zero-pad x blocks by 1 zero
- zero-pad h blocks by 2 zeros

$x_1 = [3, 2, 1, 0]$

$$\boxed{\begin{aligned} h &= [1, 1, 0, 0] \\ H &= [2, 1-j, 0, 1+j] \end{aligned}}$$

$X1 = [6, 2-j2, 2, 2+j2]$

$Y1 = X1 \times H = [12, -j4, 0, j4]$

$y1 = [3, 5, 3, 1]$

$x2 = [4, 1, 2, 0]$

$X2 = [7, 2-j, 5, 2+j]$

$Y2 = X2 \times H = [14, 1-j3, 0, 1+3j]$

$y2 = [4, 5, 3, 2]$

$x3 = [3, 2, 1, 0]$ (as before)

$y3 = [3, 5, 3, 1]$

$y1$:	3	5	3	1						
$y2$:				4	5	3	2			
$y3$:							3	5	3	1
y:	3	5	3	5	5	3	5	5	3	1

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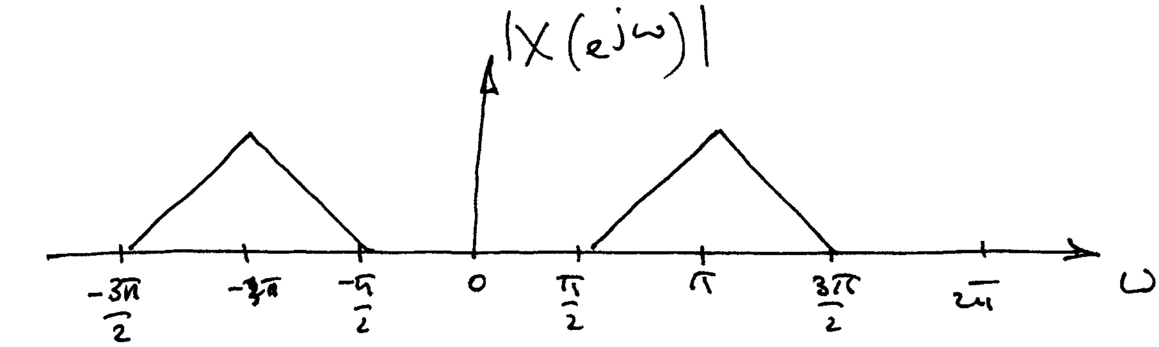
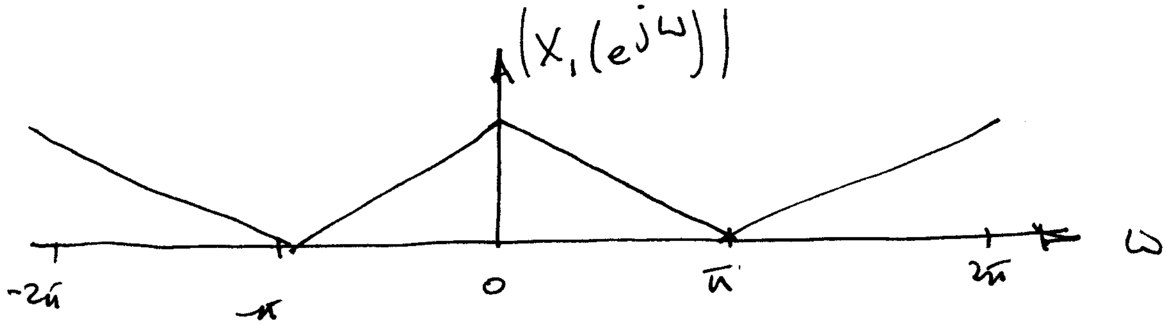
Solution Sheet 5

Multirate Signal Processing

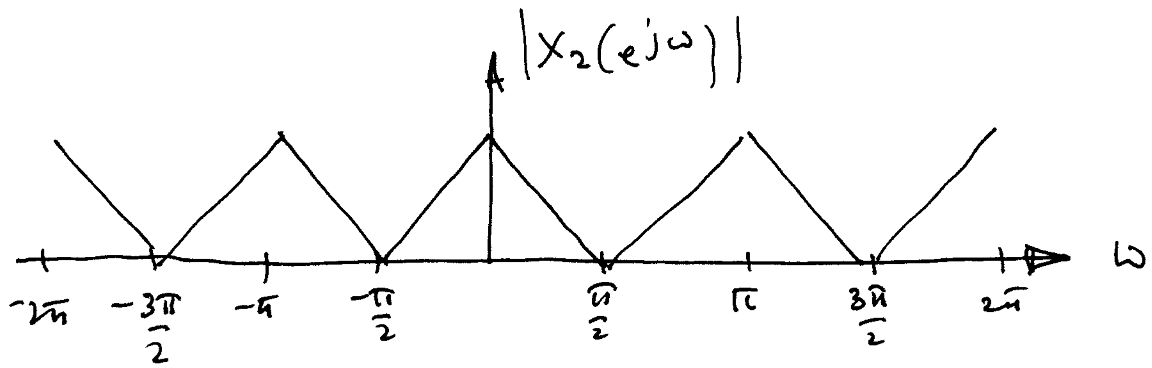
1



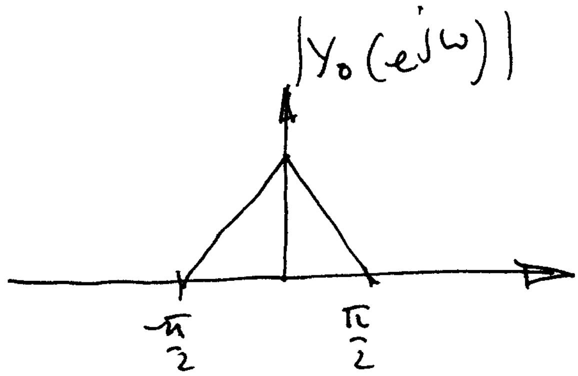
|X(ejω)|
−3π/2
−π/2
0
π/2
π
3π/2
2π
ω



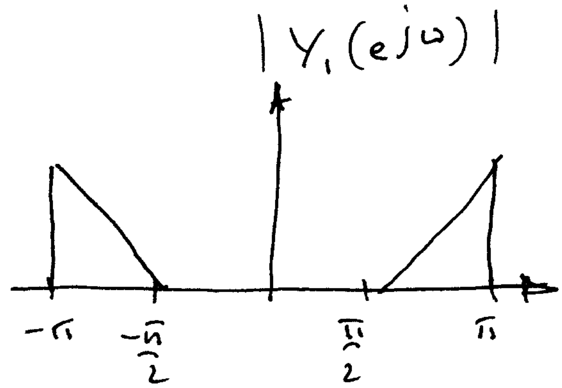
|X1(ejω)|
−2π
−π
0
π
2π
ω



|X2(ejω)|
−2π
−3π/2
−π
−π/2
π/2
π
3π/2
2π
ω

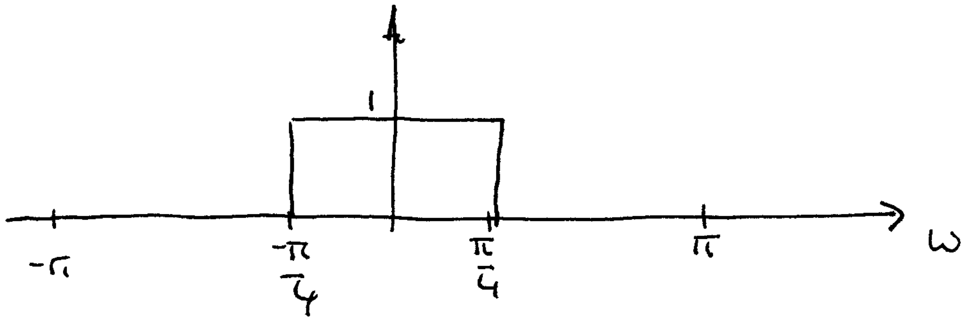


|Y0(ejω)|
−π/2
π/2



|Y1(ejω)|
−π
−π/2
π/2
π

2. $x[n] \rightarrow \boxed{\uparrow 3} \rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow 4} \rightarrow x'[n]$



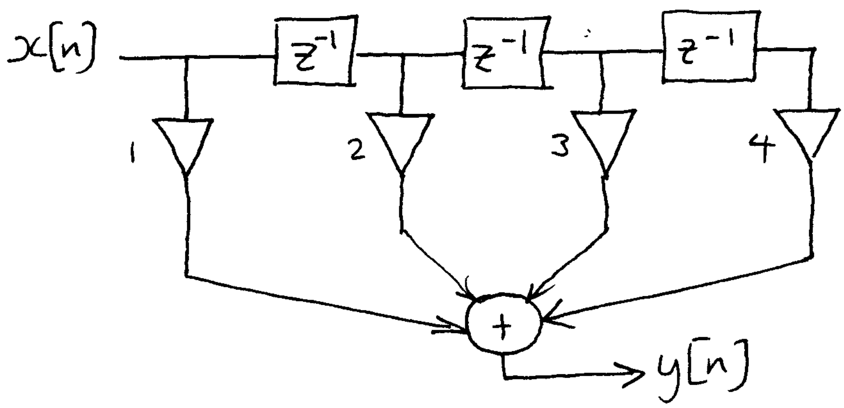
3 $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

We can write $H(z) = E_0(z^2) + z^{-1} E_1(z^2)$

for $E_0(z) = 1 + 3z^{-1}$

and $E_1(z) = 2 + 4z^{-1}$

Direct Implementation



Polyphase Implementation