1. (a) Consider the causal, real-coefficient transfer function

$$H_M(z) = \frac{\sum_{n=0}^{n=M} b[n]z^{-n}}{\sum_{n=0}^{n=M} a[n]z^{-n}}, M > 0$$

(i) Derive the conditions on the coefficients $a[n]$ and $b[n]$ to yield $|H_M(z)| = 1$, for $|z| = 1$. **[3]**

Answer

The condition $|H_M(z)| = 1$, for $|z| = 1$ implies $\left|H_M\left(e^{j\omega}\right)\right| = 1$ for all ω. That means we are looking for an allpass transfer function. An Mth order causal, real coefficient, all pass transfer function must be of the form

$$H_M(z) = \frac{\sum_{n=0}^{n=M} a[M-n]z^{-n}}{\sum_{n=0}^{n=M} a[n]z^{-n}}$$

If we denote the denominator polynomial of the allpass function as $A_M(z)$ with

$$A_M(z) = \sum_{n=0}^{n=M} a[n]z^{-n}$$

then it follows that $H_M(z)$ can be written as:

$$H_M(z) = \frac{z^{-M}A_M(z^{-1})}{A_M(z)}$$

From the above we see that:

$$H_M(z^{-1}) = \frac{z^{M}A_M(z)}{A_M(z^{-1})}$$

$$H_M(z)H_M(z^{-1}) = \frac{z^{-M}A_M(z^{-1})}{A_M(z)}\frac{z^{M}A_M(z)}{A_M(z^{-1})} = 1 \Rightarrow |H_M(z)|^2 = H_M(z)H_M(z^{-1}) = 1$$

$\Rightarrow |H_M(z)| = 1 \Rightarrow \left|H_M\left(e^{j\omega}\right)\right| = 1$

Therefore, the condition $b[n] = a[M-n]$ must hold.

(ii) Comment on the locations of the zeros of a causal, real, stable, allpass filter's transfer function within the $z-$plane, with respect to the unit circle and the poles. Justify your answer. **[3]**

Answer

The poles of a causal stable transfer function must lie inside the unit circle. As a result, all zeros of a causal stable allpass transfer function lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle.

(b) Consider a first-order, causal, real, stable, allpass transfer function $H(z)$ given as:

$$H(z) = \frac{p + z^{-1}}{1 + pz^{-1}}$$

(i) Determine the impulse response $h[n]$ for $n \geq 0$, associated with $H(z)$, if $h[n] = 0$ for $n < 0$. **[5]**

Answer

$\frac{Y(z)}{X(z)} = \frac{p+z^{-1}}{1+pz^{-1}} \Rightarrow y[n] + py[n-1] = px[n] + x[n-1], |p| < 1$
Assume that: $x[n] = \delta[n]$.
$h[0] + ph[-1] = p\delta[0] + \delta[-1] \Rightarrow h[0] = p\delta[0] \Rightarrow h[0] = p$
$h[1] + ph[0] = p\delta[1] + \delta[0] \Rightarrow h[1] + p^2 = 1 \Rightarrow h[1] = 1 - p^2$
$h[2] + ph[1] = p\delta[2] + \delta[1] \Rightarrow$
$h[2] = -ph[1] \Rightarrow \cdots$
$h[n] = -ph[n-1], n \geq 2$

(ii) Find the phase response associated with $H(z)$. **[4]**

Answer

$H(z) = \frac{p+z^{-1}}{1+pz^{-1}}, |p| < 1$
There is a single pole at $-p$.
$\angle H(e^{j\omega}) = \angle(p + e^{-j\omega}) - \angle(1 + pe^{-j\omega})$
$= \arctan\left(\frac{-\sin(\omega)}{p + \cos(\omega)}\right) - \arctan\left(\frac{-p\sin(\omega)}{1 + p\cos(\omega)}\right)$

(iii) Determine the expression for $|H(z)|^2 - 1$. **[3]**

Answer

We have proven that for an allpass filter of order M we have:

$$|H_M(z)|^2 = \prod_{i=1}^{M}\left(1 + \frac{(1-|z|^2)(1-|-p_i|^2)}{|z+p_i|^2}\right)$$

For $M = 1$ we have $|H_1(z)|^2 = |H(z)|^2 = 1 + \frac{(1-|z|^2)(1-|p|^2)}{|z+p|^2}$
Therefore, $|H(z)|^2 - 1$ is $\frac{(1-|z|^2)(1-|p|^2)}{|z-p|^2}$.

(iv) Find for which values of z the condition $|H(z)|^2 < 1$ holds. **[3]**

Answer

$|H(z)|^2 < 1$ if $|H(z)| < 1$.
$|H(z)| < 1$ if $\frac{(1-|z|^2)(1-|p|^2)}{|z+p|^2} < 0$. Since $|p| < 1$, $1 - |z|^2 < 0$ and therefore, $|z| > 1$.

(c) What is a major drawback in designing an all-pole IIR linear-phase transfer function? Justify your answer. **[4]**

Solution

All IIR filters have either poles or both poles and zeros and must be BIBO stable. A BIBO stable filter must have its poles within the unit circle. However, in order to get linear phase, an IIR filter would need conjugate reciprocal poles outside of the unit circle, making it BIBO unstable.

2. (a) Show that an antisymmetric linear-phase Finite Impulse Response (FIR) transfer function $H(z)$ of odd length N can be expressed as follows:

$$H(z) = z^{-\left(\frac{N-1}{2}\right)}\left(\sum_{n=1}^{\frac{N-1}{2}} h\left[\frac{N-1}{2}-n\right](z^{n}-z^{-n})\right) \qquad (1)$$

[8]

Solution

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{\frac{N-3}{2}} h[n]z^{-n} + h\left[\frac{N-1}{2}\right]z^{-(\frac{N-1}{2})} + \sum_{n=\frac{N+1}{2}}^{N-1} h[n]z^{-n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h[n]z^{-n} + 0z^{-(\frac{N-1}{2})} + \sum_{n=\frac{N+1}{2}}^{N-1} h[n]z^{-n}$$

$(N-1) - n = k$

$n = \dfrac{N+1}{2} \Rightarrow k = \dfrac{N-3}{2}$

$n = N-1 \Rightarrow k = 0$

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{\frac{N-3}{2}} h[n]z^{-n} + \sum_{k=0}^{\frac{N-3}{2}} h[(N-1)-k]z^{-[(N-1)-k]}$$

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{\frac{N-3}{2}} h[n]z^{-n} - \sum_{k=0}^{\frac{N-3}{2}} h[k]z^{-[(N-1)-k]}$$

$$= z^{-(\frac{N-1}{2})}\left(\sum_{n=0}^{\frac{N-3}{2}} h[n]z^{-n+\frac{N-1}{2}} - \sum_{n=0}^{\frac{N-3}{2}} h[n]z^{n-\frac{N-1}{2}}\right)$$

$n = \dfrac{N-1}{2} - k$

$n = 0 \Rightarrow k = \dfrac{N-1}{2}$

$n = \dfrac{N-3}{2} \Rightarrow k = \dfrac{N-1}{2} - \dfrac{N-3}{2} = 1$

$$H(z) = z^{-(\frac{N-1}{2})}\left(\sum_{k=1}^{\frac{N-1}{2}} h[\frac{N-1}{2}-k]z^{k} - \sum_{k=1}^{\frac{N-1}{2}} h[\frac{N-1}{2}-k]z^{-k}\right)$$

$$= z^{-(\frac{N-1}{2})}\left(\sum_{n=1}^{\frac{N-1}{2}} h[\frac{N-1}{2}-n]z^{n} - \sum_{n=1}^{\frac{N-1}{2}} h[\frac{N-1}{2}-n]z^{-n}\right)$$

We have proven the required relationship.

(b) By using the relation

$$U_r\left(\frac{z+z^{-1}}{2}\right)=\frac{z^{r+1}-z^{-(r+1)}}{z-z^{-1}}$$

where $U_r(x)$ is the $r-$th order **Chebyshev Polynomial of the Second Kind** in x, express $H(z)$ of (a) above in the form

$$H(z) = z^{-\left(\frac{N-1}{2}\right)}(z-z^{-1})\sum_{n=0}^{M} a[n]\left(\frac{z+z^{-1}}{2}\right)^n \qquad (2)$$

in the case where $N=9$. Determine the relation between $a[n]$ and $h[n]$ for $N=9$. You can easily determine the value of the parameter M as a function of N, by ensuring that the orders of the polynomials of equations (1) and (2) are equal.

The Chebyshev polynomials of second kind satisfy the following recursive relationship:
$U_r(x)=2xU_{r-1}(x)-U_{r-2}(x), r\geq 2$
$U_0(x)=1$, $U_1(x)=2x$

[8]

Solution

For $(9-1)/2=4$ we have

$$H(z)=z^{-4}\left(\sum_{n=1}^{4}h[4-n](z^n-z^{-n})\right)$$
$$=z^{-4}\big(h[3](z-z^{-1})+h[2](z^2-z^{-2})+h[1](z^3-z^{-3})+h[0](z^4-z^{-4})\big)$$

$U_r(x)=2xU_{r-1}(x)-U_{r-2}(x), r\geq 2$
$U_0(x)=1$, $U_1(x)=2x$, $U_2(x)=4x^2-1$
$U_3(x)=8x^3-4x$

$$H(z)=z^{-4}\big(h[3](z-z^{-1})+h[2](z^2-z^{-2})+h[1](z^3-z^{-3})+h[0](z^4-z^{-4})\big)$$

$$H(z)=z^{-4}\left(h[3](z-z^{-1})+h[2](z-z^{-1})U_1\left(\frac{z+z^{-1}}{2}\right)+h[1](z-z^{-1})U_2\left(\frac{z+z^{-1}}{2}\right)\right.$$
$$\left.+h[0](z-z^{-1})U_3\left(\frac{z+z^{-1}}{2}\right)\right)$$

$$=z^{-4}\left(h[3](z-z^{-1})+h[2](z-z^{-1})(z+z^{-1})+h[1](z-z^{-1})\left(4\left(\frac{z+z^{-1}}{2}\right)^2-1\right)\right.$$
$$\left.+h[0](z-z^{-1})\left(8\left(\frac{z+z^{-1}}{2}\right)^3-4\left(\frac{z+z^{-1}}{2}\right)\right)\right)$$

$$z^{-4}(z-z^{-1})\left(h[3]+2h[2]\left(\frac{z+z^{-1}}{2}\right)+h[1]\left(4\left(\frac{z+z^{-1}}{2}\right)^2-1\right)\right.$$
$$\left.+h[0]\left(8\left(\frac{z+z^{-1}}{2}\right)^3-4\left(\frac{z+z^{-1}}{2}\right)\right)\right)$$

$$= z^{-4}(z - z^{-1})\left((h[3] - h[1]) + (2h[2] - 4h[0])\left(\frac{z + z^{-1}}{2}\right) + 4h[1]\left(\frac{z + z^{-1}}{2}\right)^2 + 8h[0]\left(\frac{z + z^{-1}}{2}\right)^3\right)$$

We observe that $M = \frac{N-3}{2} = 3$.

(c) Develop a realization of $H(z)$ based on equation (1) above in the form of **Figure 2.1** below, where $F_1(z^{-1})$ and $F_2(z^{-1})$ are causal filters. Determine the form of $F_1(z^{-1})$ and $F_2(z^{-1})$. The triangular sign indicates multiplication with the parameter next to it.
[**Hint:** To solve this part, it is convenient to write equation (2) as a function of M only instead of N only or both M and N]. **[9]**

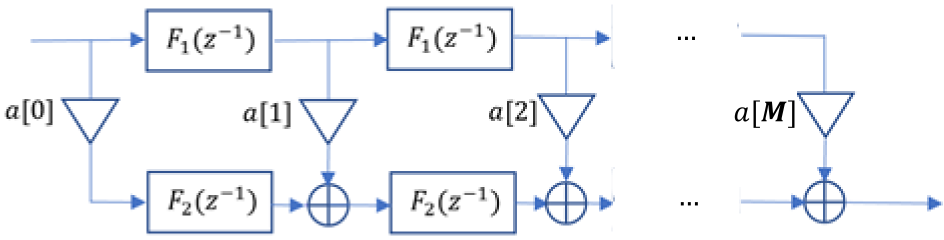


Figure 2.1

Solution

$$M = \frac{N-3}{2}$$

$$H(z) = z^{-(M+1)}(z - z^{-1})\sum_{n=0}^{M} a[n]\left(\frac{z + z^{-1}}{2}\right)^n$$

$$H(z) = a[0]F_2^M(z^{-1}) + a[1]F_1(z^{-1})F_2^{M-1}(z^{-1}) + a[2]F_1^2(z^{-1})F_2^{M-2}(z^{-1}) + \cdots + a[M]F_1^M(z^{-1})$$

$$= \sum_{n=0}^{M} a[n]F_1^n(z^{-1})F_2^{M-n}(z^{-1})$$

We immediately see that

$$F_1^n(z^{-1})F_2^{M-n}(z^{-1}) = z^{-(M+1)}(z - z^{-1})\left(\frac{z + z^{-1}}{2}\right)^n$$

$$n = 0 \Rightarrow F_2^M(z^{-1}) = z^{-(M+1)}(z - z^{-1}) = z^{-M}(1 - z^{-2}) \Rightarrow F_2(z^{-1}) = z^{-1}(1 - z^{-2})^{1/M}.$$

$$n = M \Rightarrow F_1^M(z^{-1}) = z^{-(M+1)}(z - z^{-1})\left(\frac{z + z^{-1}}{2}\right)^M \Rightarrow$$

$$F_1^M(z^{-1}) = z^{-M}(1 - z^{-2})\left(\frac{z + z^{-1}}{2}\right)^M \Rightarrow F_1(z^{-1}) = (1 - z^{-2})^{1/M}\frac{1 + z^{-2}}{2}$$

3. (a) The bilinear transformation from the s −plane to the z −plane is given by

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

(i) Illustrate, employing mathematical relationships, how the bilinear transformation maps every point $s = \sigma + j\Omega$ in the s −plane to the z −plane. **[5]**

Answer

$$z = \frac{1 + s}{1 - s}$$

For $s = j\Omega_0$ we have that $z = \frac{1+j\Omega_0}{1-j\Omega_0}$ which has a magnitude equal to 1. This implies that a point on the imaginary axis in the s −plane is mapped onto a point on the unit circle in the z −plane where $|z| = 1$. In the general case, for $s = \sigma_0 + j\Omega_0$

$$z = \frac{1 + \sigma_0 + j\Omega_0}{1 - \sigma_0 - j\Omega_0} \Rightarrow |z|^2 = \frac{(1 + \sigma_0)^2 + {\Omega_0}^2}{(1 - \sigma_0)^2 + {\Omega_0}^2}$$

A point in the left half s −plane with $\sigma_0 < 0$ is mapped onto a point inside the unit circle in the z −plane as $|z| < 1$. Likewise, a point in the right half s −plane with $\sigma_0 > 0$ is mapped onto a point outside the unit circle in the z −plane as $|z| > 1$.

(ii) Prove that the relationship between the continuous-time angular frequency and the discrete-time angular frequency is non-linear. **[5]**

Answer

The variable s is reduced on the imaginary axis to $s = j\Omega$. The variable z is reduced on the unit circle to $z = e^{-j\omega}$. Since the bilinear transformation maps one plane to the other and vice versa and we have proven that the imaginary axis on the s plane is mapped to the unit circle on the z plane, we can write that:

$j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = j\tan\left(\frac{\omega}{2}\right) \Rightarrow \Omega = \tan\left(\frac{\omega}{2}\right)$ which is a non-linear relationship.

(b) A given real-coefficient, digital IIR lowpass filter has a rational transfer function $H_L(z)$ and a cutoff frequency ω_c. The transfer function of $H_L(z)$ is transformed by replacing z by $F(z) = \frac{az+b}{cz+d}$ to a real highpass rational transfer function $H_H(z) = H_L(F(z))$ of the same order as $H_L(z)$ but different cutoff frequency. By using the constraints $H_H(e^{j\pi}) = H_L(e^{j0}) = 1$ and $H_H(e^{j0}) = H_L(e^{j\pi}) = 0$, derive relationships among the parameters a, b, c, d and explain what type of filter is $F(z)$. **[5]**

Answer

$H_H(e^{j0}) = H_L\left(F(e^{j0})\right) = H_L(F(1)) = 0$

Therefore, $F(1) = \frac{a+b}{c+d} = e^{j\pi} = -1 \Rightarrow a + b = -c - d$

$H_H(e^{j\pi}) = H_L\left(F(e^{j\pi})\right) = H_L(F(-1)) = 1$

Therefore, $F(-1) = \frac{-a+b}{-c+d} = 1 \Rightarrow -a + b = -c + d$

Therefore, $b = -c$ and $a = -d$

$$F(z) = \frac{az + b}{-bz - a} = -\frac{az + b}{bz + a}$$

$F(z)$ is an allpass filter.

(c) Consider the two LTI causal digital filters with impulse responses given by:

$$h_A[n] = 0.5\delta[n] - \delta[n-1] + 0.5\delta[n-2]$$
$$h_B[n] = 0.25\delta[n] + 0.5\delta[n-1] + 0.25\delta[n-2]$$

Explain the type of filters $h_A[n]$ and $h_B[n]$ (lowpass etc.), by using the following two approaches:

(i) Practical approach. In this approach you will apply the filters to the input signal $x[n] = u[n]$, with $u[n]$ the discrete unit step function and observe the effects they have on $x[n]$. **[5]**

Answer

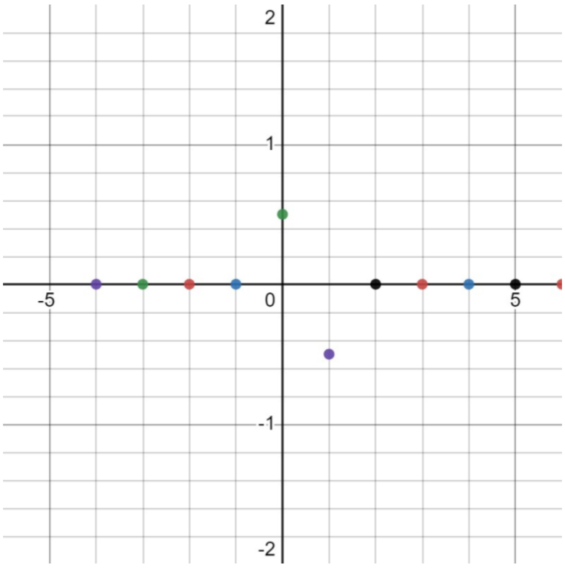
The input-output relationship of the given transfer function $H_A(z)$ is

$$y_A[n] = 0.5x[n] - x[n-1] + 0.5x[n-2]$$

If $x[n] = u[n]$, with $u[n]$ the discrete unit step function we have:

$$y_A[n] = \frac{1}{2}(u[n] - 2u[n-1] + u[n-2])$$

$$y_A[n] = \begin{cases} 0 & n \leq -1 \\ \frac{1}{2} & n = 0 \\ -\frac{1}{2} & n = 1 \\ 0 & n > 1 \end{cases}$$



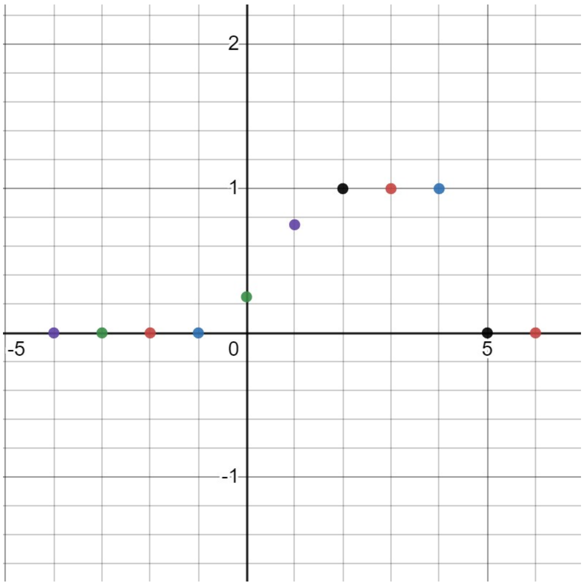
The input-output relationship of the given transfer function $H_B(z)$ is

$$y_B[n] = 0.25x[n] + 0.5x[n-1] + 0.25x[n-2]$$

If $x[n] = u[n]$, with $u[n]$ the discrete unit step function we have:

$$y_B[n] = 0.25u[n] + 0.5u[n-1] + 0.25u[n-2]$$

$$y_A[n] = \begin{cases} 0 & n \leq -1 \\ \frac{1}{4} & n = 0 \\ \frac{3}{4} & n = 1 \\ 1 & n > 1 \end{cases}$$



As we see by applying $u[n]$ as input, $H_B(z)$ is a lowpass filter because the sharpness of $u[n]$ at $n = 0$ is destroyed by applying the filter.

On the other hard, $H_A(z)$ is a highpass filter because we see by applying $u[n]$ as input, the constant areas are eliminated, and we are left only with an enhanced abrupt change around $n = 0$ which is the original abrupt change point.

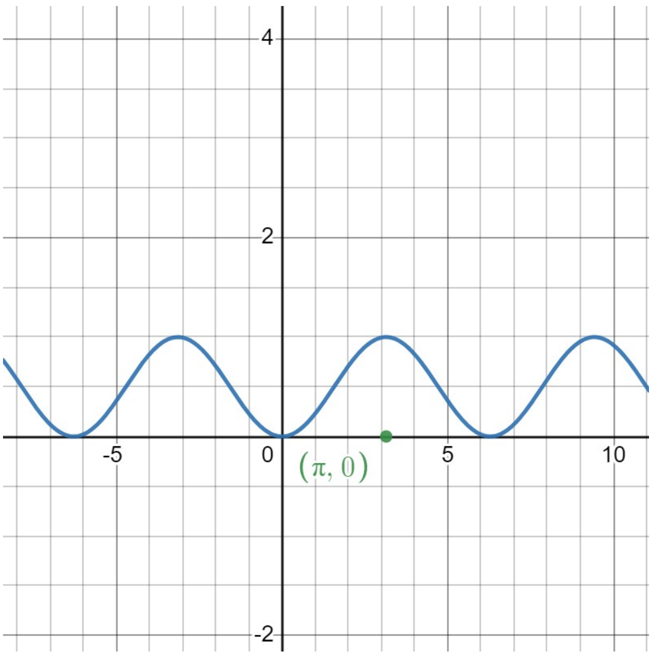
(ii) Mathematical approach. In this approach you must find and sketch the amplitude response of the two filters. **[5]**

Answer

$H_A(z) = 0.25 - 0.5z^{-1} + 0.25z^{-2} = z^{-1}(0.25z - 0.5 + 0.25z^{-1})$

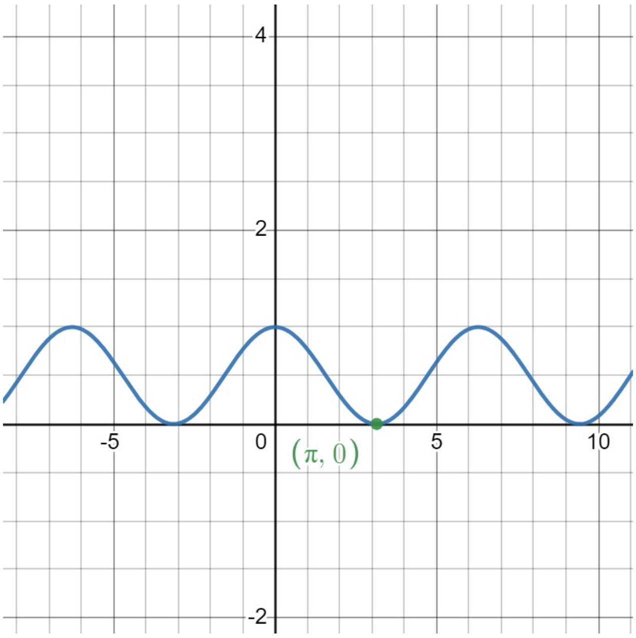
$H_A(e^{j\omega}) = e^{-j\omega}(-0.5 + 0.5\cos(\omega))$

$|H_A(e^{j\omega})| = |-0.5 + 0.5\cos(\omega)|$. We observe that $|H_A(e^{j\omega})|$ increases as the frequency approaches π.



$H_B(z) = 0.25 + 0.5z^{-1} + 0.25z^{-2} = z^{-1}(0.25z + 0.5 + 0.25z^{-1})$

$H_B(e^{j\omega}) = e^{-j\omega}(0.5 + 0.5\cos(\omega))$
$|H_B(e^{j\omega})| = 0.5|1 + \cos(\omega)|$. We observe that $|H_B(e^{j\omega})|$ decreases as the frequency approaches π.



4. (a) Consider the multirate structure of **Figure 4.1** below, where $H_0(z), H_1(z)$ and $H_2(z)$ are ideal, zero phase, real coefficient lowpass, bandpass and highpass filters respectively, with frequency responses as follows:

$$H_0(e^{j\omega}) = u(\omega) - u(\omega - \frac{2\pi}{3})$$
$$H_1(e^{j\omega}) = u(\omega - \frac{\pi}{3}) - u(\omega - \frac{2\pi}{3})$$
$$H_2(e^{j\omega}) = u(\omega - \frac{\pi}{3}) - u(\omega - \pi)$$

The function $u(\omega)$ is the unit step function, defined as

$$u(\omega) = \begin{cases} 1 & \omega \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

If the input is a real sequence with a Discrete Time Fourier Transform

$$X(e^{j\omega}) = (-\frac{3\omega^2}{\pi} + \frac{\pi}{3})\left(u(\omega) - u\left(\omega - \frac{\pi}{3}\right)\right)$$

sketch the Discrete Time Fourier Transform of the outputs $y_0[n], y_1[n]$ and $y_2[n]$.
It's important to note that all frequency domain representations mentioned are limited to the range $[0, \pi]$. **[15]**

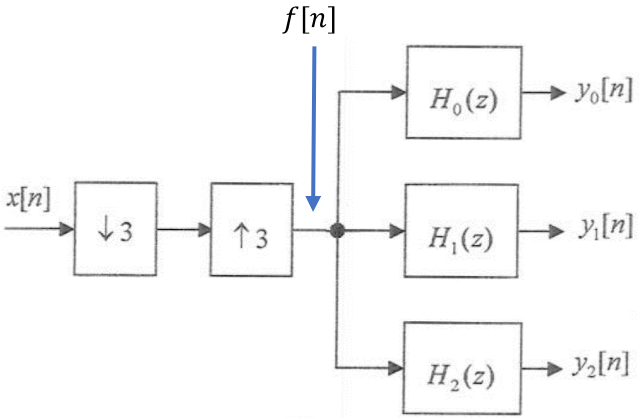
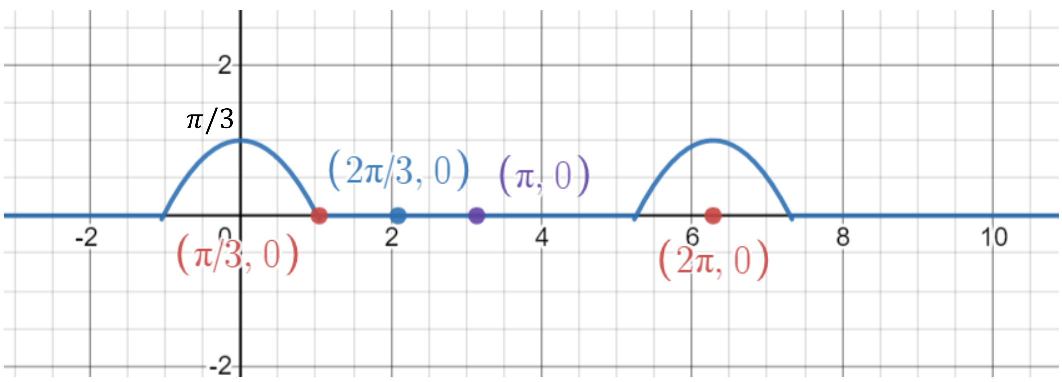


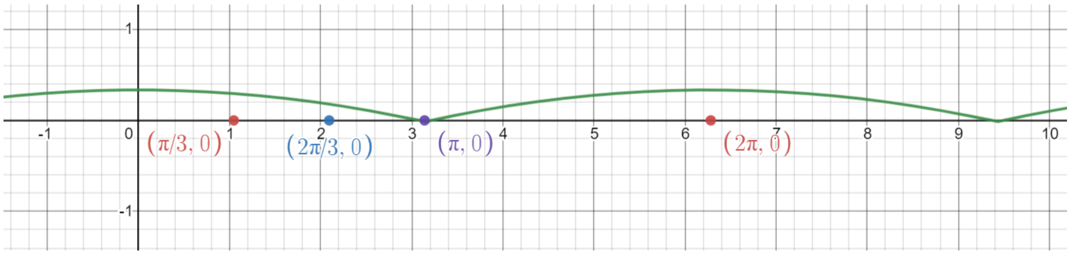
Figure 4.1

Answer

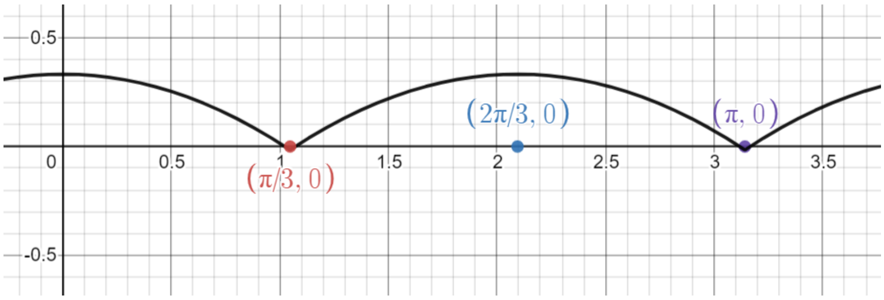
The DTFT of the input is shown below:



After decimation by 3:



After interpolation by 3 we obtain the DTFT of the signal $f[n]$ shown below.



The output of the three filters is obvious.

(b) Analyse the structure of **Figure 4.2** below and determine its input-output relations. **[10]**

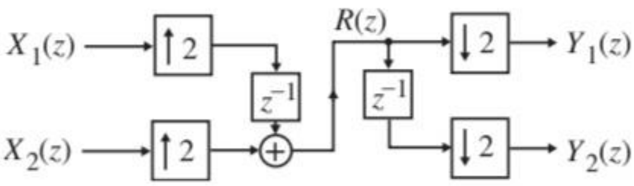


Figure 4.2

Answer

Analysis yields $R(z) = z^{-1}X_1(z^2) + X_2(z^2)$, $Y_1(z) = R(z^{1/2}) + R(-z^{1/2})$
$= [z^{-1/2}X_1(z) + X_2(z)] + [-z^{-1/2}X_1(z) + X_2(z)] = 2X_2(z)$,
$Y_2(z) = z^{-1/2}R(z^{1/2}) - z^{-1/2}R(-z^{1/2})$
$= [z^{-1}X_1(z) + z^{-1/2}X_2(z)] + [z^{-1}X_1(z) - z^{-1/2}X_2(z)] = 2z^{-1}X_1(z)$. Thus, the output $y_1[n]$ is a scaled replica of the input $x_2[n]$ while the output $y_2[n]$ is a scaled replica of the delayed input $x_1[n-1]$.