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Digital Signal Processing

Design of Linear FIR filters using the Remez exchange algorithm

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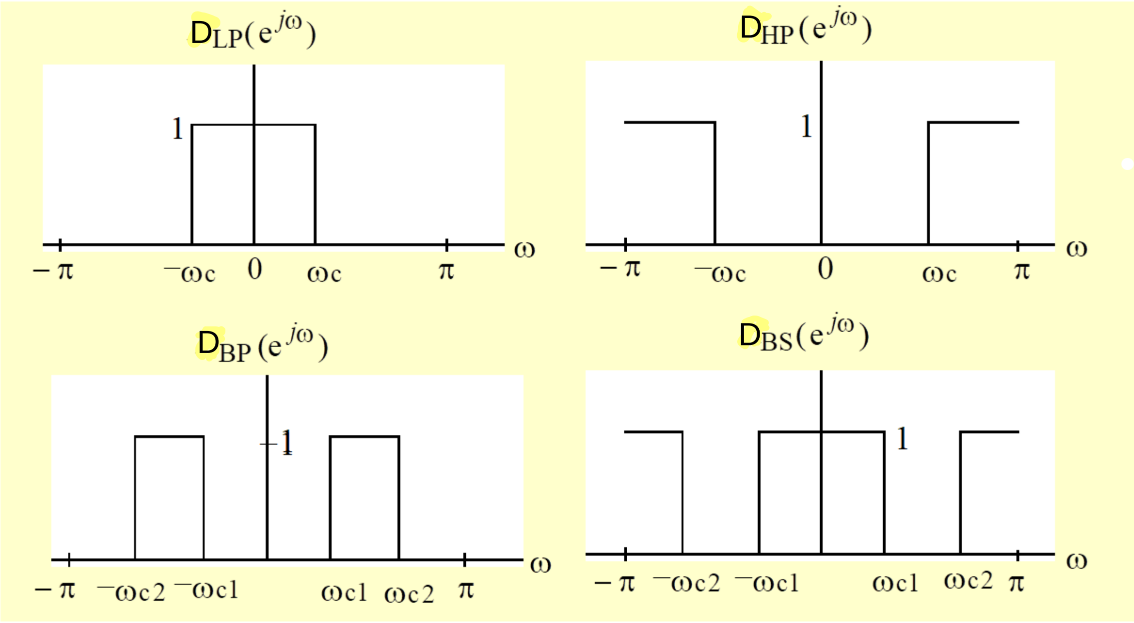
Design of linear-phase FIR filters using optimisation

- In this presentation, we consider the application of optimization techniques for the design of **linear phase** FIR filters.
- The basic idea is to minimize iteratively an error measure that is a function of the difference between the desired frequency response $D(e^{j\omega})$ and the frequency response $H(e^{j\omega})$ of the filter being designed.
- In the case of linear-phase FIR filter design, $H(e^{j\omega})$ and $D(e^{j\omega})$, i.e., real and desired are zero-phase frequency responses.



what you see in this figure is a generic diagram of the four types of ideal filters, namely, lowpass, highpass, bandpass and bandstop

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Design of linear-phase FIR filters using optimisation

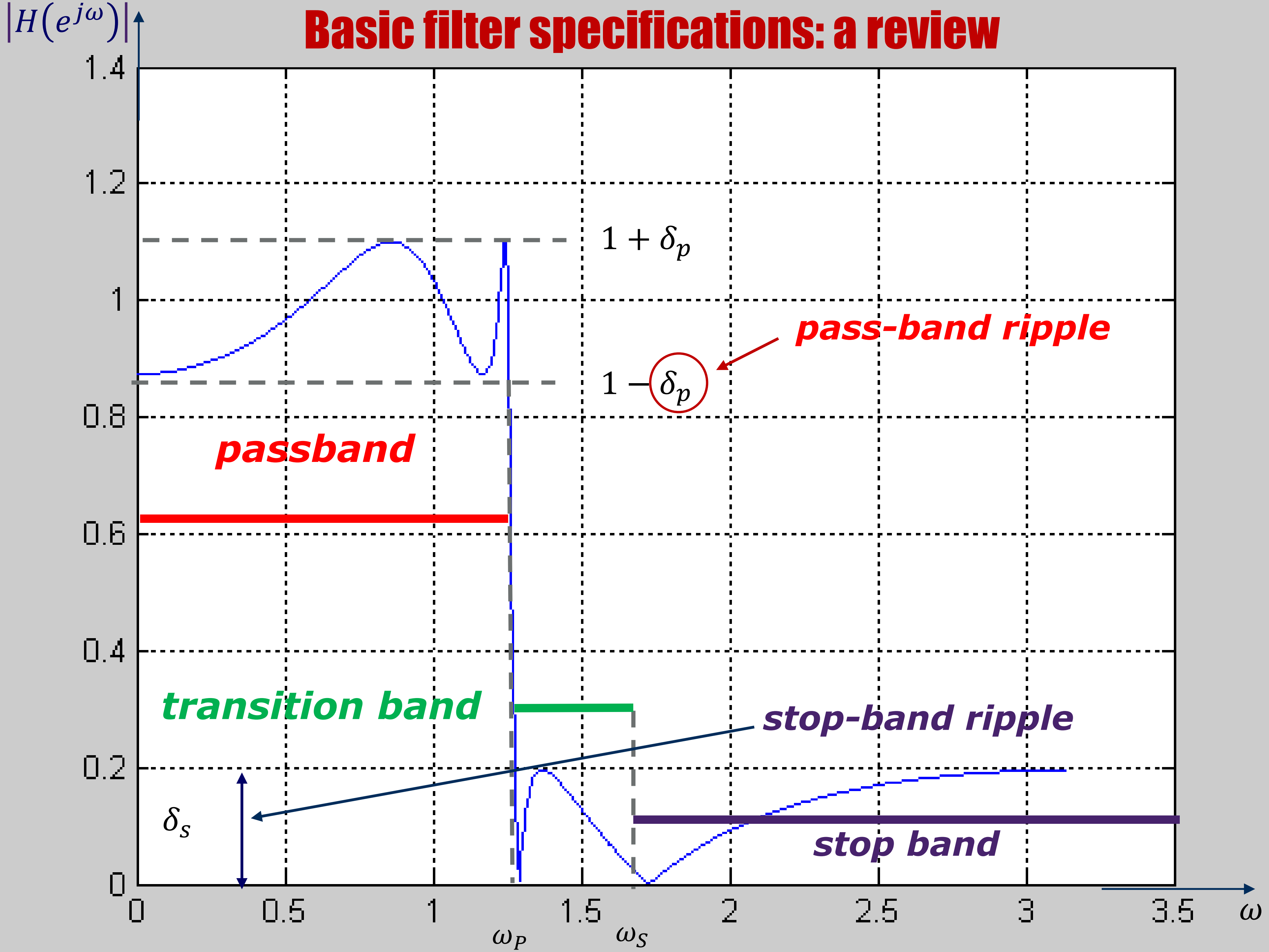
Previous part

- The windowing method is a relatively simple technique for designing linear-phase FIR filters.
- Here, a major problem, is a lack of precise control of the critical frequencies such cut-off frequencies of pass band and stop band.

This part

- The new filter design method described in this section is formulated as a so-called **Chebyshev approximation problem**. This is an optimization problem.
- It is viewed as an optimum design criterion in the sense that the maximum weighted approximation error between the desired frequency response and the actual frequency response is minimized.
- **The resulting filter designs have ripples in both the pass-band and the stop-band.**
- To describe the design procedure, let us recall the following basic filter specifications.

Basic filter specifications: a review



Design of linear-phase FIR filters using optimisation

- The design objective is to iteratively adjust the filter parameters so that the error function defined by the equation:

$$\varepsilon(\omega) = W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})]$$

is minimum according to some criterion.

  - $D(e^{j\omega})$ is the desired (ideal) amplitude response
  - $W(e^{j\omega})$ is some user-specified positive weighting function.

- The following criteria are popular:

Minimax criterion

minimize $$\max_{\omega\in R}|\varepsilon(\omega)| = \max_{\omega\in R}\left|W(e^{j\omega})[H(e^{j\omega}) - D(e^{j\omega})]\right|$$

Least squares criterion

Minimize $$\int_{\omega\in R}|\varepsilon(\omega)|^2 d\omega = \int_{\omega\in R}\left|W(e^{j\omega})\left(H(e^{j\omega}) - D(e^{j\omega})\right)\right|^2 d\omega$$

- R is a set of disjoint frequency bands in the range $0 \le \omega \le \pi$. In filtering applications, R is composed of passbands and stopbands.

basically, we don't care about what happens in the transition band

Design of linear-phase FIR filters using optimisation

- The linear phase filter that is obtained by minimizing the peak absolute value of the weighted error ε given by

$$\varepsilon = \max_{\omega \in R} |\varepsilon(\omega)| \quad \textbf{(minimax criterion)}$$

is usually called the **equiripple FIR filter**, since, after ε has been minimized, the weighted error function $\varepsilon(\omega)$ exhibits an equiripple behavior in the frequency range of interest.

- In this part we outline the **weighted-Chebyshev approximation method** advanced by Parks and McClellan (1972) for designing equiripple linear phase FIR filters.

- Therefore, this method is more commonly known as the **Parks-McClellan algorithm**.

The minimax criterion aims to design a filter with equiripple characteristics. In other words, it tries to make the maximum deviation (maximum ripple) between the desired response and the actual response as small as possible across all frequency points. This means that the error is distributed as evenly as possible among the passbands and stopbands, resulting in a filter with uniform ripple behavior.

Design of linear-phase FIR filters using optimisation

- The general form of the frequency response $H(e^{j\omega})$ of a causal linear-phase FIR filter of length M is given by

$$H(e^{j\omega}) = e^{-j(M-1)\omega/2} e^{j\beta} \breve{H}(\omega)$$

where $\breve{H}(\omega)$ is the **magnitude** response of $H(e^{j\omega})$ and is a **real** function of ω **(not necessarily positive)**.

(You can call $\breve{H}$ as H breve.)

- The weighted error function in this case involves the magnitude response only and is given by

$$\varepsilon(\omega) = W(\omega)\left[\breve{H}(\omega) - D(\omega)\right]$$

  - $D(e^{j\omega})$ is the desired (ideal) amplitude response
  - $W(e^{j\omega})$ is some user-specified positive weighting function.

- The **Parks-McClellan algorithm** is based on iteratively adjusting the coefficients of the magnitude response until the peak absolute value of $\varepsilon(\omega)$ is minimized using the **Remez exchange algorithm**.

Design of linear-phase FIR filters using optimisation

- If the minimum value of the peak absolute value of $\varepsilon(\omega)$ in a band $\omega_a \le \omega \le \omega_b$ is ε, then the absolute error satisfies

$$\left|\breve{H}(\omega) - D(\omega)\right| \le \frac{\varepsilon}{|W(\omega)|} = \frac{\varepsilon}{W(\omega)}, \omega_a \le \omega \le \omega_b$$

- In typical filter design applications, the desired amplitude response is given by

$$D(\omega) = \begin{cases} 1, & \text{in the passband} \\ 0, & \text{in the stopband} \end{cases}$$

- The magnitude response $\breve{H}(\omega)$ is required to satisfy the above desired response with a ripple of $\pm\delta_p$ in the passband and a ripple δ_s in the stopband.
- As a result, it is shown from the weighted error function that the weighting function can be chosen either as

$$W(\omega) = \begin{cases} 1, & \text{in the passband} \\ \delta_p/\delta_s, & \text{in the stopband} \end{cases} \text{ or } W(\omega) = \begin{cases} \delta_s/\delta_p, & \text{in the passband} \\ 1, & \text{in the stopband} \end{cases}$$

Design of linear-phase FIR filters using optimisation

- Passband:

$$\left|\breve{H}(\omega) - D(\omega)\right| \le \left|1 \pm \delta_p - 1\right| = \left|\pm\delta_p\right| = \boldsymbol{\delta_p}$$

- From previous slide we know that

$$\left|\breve{H}(\omega) - D(\omega)\right| \le \frac{\boldsymbol{\varepsilon}}{\boldsymbol{W(\omega)}}$$

- We may set $\delta_p = \frac{\varepsilon}{W(\omega)}$. For $W(\omega) = 1 \Rightarrow \varepsilon = \delta_p$.
- Stopband:

$$\left|\breve{H}(\omega) - D(\omega)\right| \le |\delta_s| = \delta_s$$

$$\delta_s = \frac{\varepsilon}{W(\omega)} = \frac{\delta_p}{W(\omega)} \Rightarrow W(\omega) = \frac{\delta_p}{\delta_s}$$

- Therefore, $W(\omega) = \begin{cases} 1, & \text{in the passband} \\ \delta_p/\delta_s, & \text{in the stopband} \end{cases}$
- Using a similar approach we can justify the choice:

$$W(\omega) = \begin{cases} \delta_s/\delta_p, & \text{in the passband} \\ 1, & \text{in the stopband} \end{cases}$$

Linear-phase FIR transfer functions

- Consider a causal FIR transfer function $H(z)$ of length M, i.e., of order $M-1$ as follows:

$$H(z) = \sum_{n=0}^{M-1} h[n]z^{-n}$$

- The above transfer function has a linear phase, if its impulse response $h[n]$ is either **symmetric**, i.e.,

$$h[n] = h[M-1-n], 0 \le n \le M-1$$

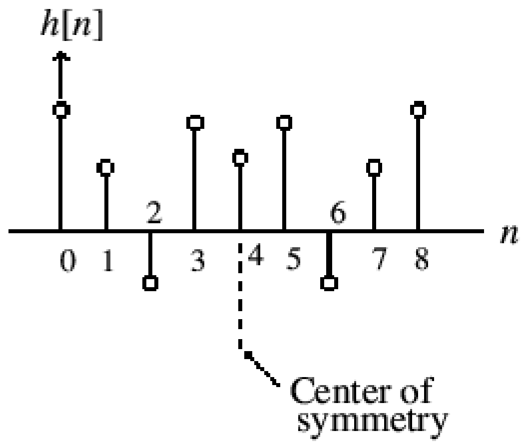
  or is **antisymmetric**, i.e.,

$$h[n] = -h[M-1-n], 0 \le n \le M-1$$

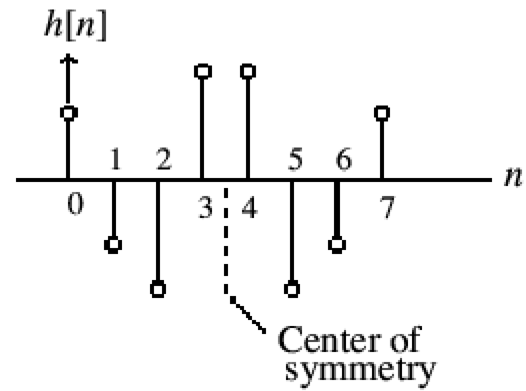
- Since the length of the impulse response can be either **even** or **odd**, we can define four types of linear-phase FIR transfer functions.
- For an antisymmetric FIR filter of odd length, the middle coefficient must be zero, i.e.,

$$h[(M-1)/2] = 0$$

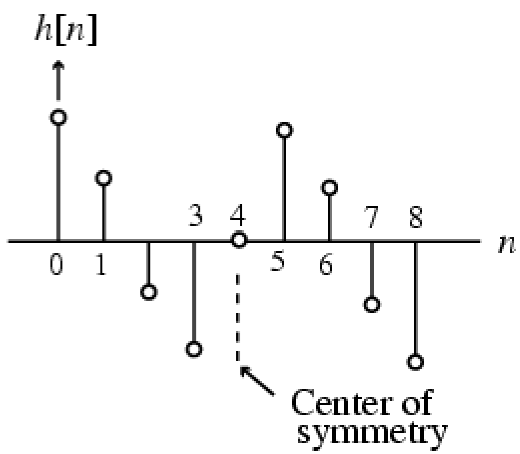
4 Types of linear-phase FIR transfer functions



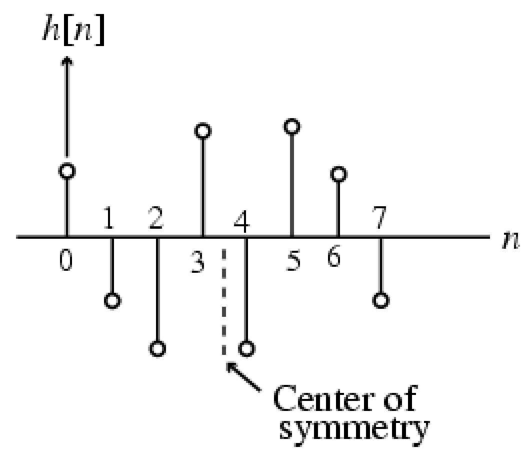
Type 1: $M = 9$



Type 2: $M = 8$



Type 3: $M = 9$



Type 4: $M = 8$

Linear-phase FIR transfer functions: MIP and AIP

- Consider a causal **symmetric** FIR transfer function $H(z)$ of length M, i.e., of order $M-1$ as follows:

$$H(z)=\sum_{n=0}^{M-1}h[n]z^{-n}$$

$H(z)=\sum_{n=0}^{M-1}h[n]z^{-n}=\sum_{r=0}^{M-1}h[M-1-r]z^{-(M-1-r)}=\sum_{r=0}^{M-1}h[r]z^{-(M-1-r)}$
$=z^{-(M-1)}\sum_{r=0}^{M-1}h[r]z^{r}\Rightarrow$

$$\boldsymbol{H(z)=z^{-(M-1)}H(z^{-1})}$$

$$\boxed{\boldsymbol{n=M-1-r}}$$

(Mirror Image Polynomial – MIP)

- Consider a causal **antisymmetric** FIR transfer function $H(z)$ of length M, i.e., of order $M-1$ as follows:

$$H(z)=\sum_{n=0}^{M-1}h[n]z^{-n}$$

$H(z)=\sum_{n=0}^{M-1}h[n]z^{-n}=\sum_{r=0}^{M-1}h[M-1-r]z^{-(M-1-r)}=-\sum_{r=0}^{M-1}h[r]z^{-(M-1-r)}\Rightarrow$

$$\boldsymbol{H(z)=-z^{-(M-1)}H(z^{-1})}$$

(Antimirror Image Polynomial – AIP)

Linear-phase FIR transfer functions: Zero locations

- For a MIP or AIP:
  - If $z_0 = re^{j\phi}$ is a zero of $H(z)$, so is $1/z_0$.
  - Furthermore, for an FIR filter with real impulse response, the zeros occur in complex conjugate pairs. Hence, a zero at z_0 is associated with a zero at z_0^*.
  - Therefore, a **complex zero** $re^{j\phi}$ that is **not on the unit circle** is associated with a group of 4 zeros given by $z = re^{\pm j\phi}$ and $z = \frac{1}{r}e^{\pm j\phi}$.
  - For a **complex zero** $e^{j\phi}$ that is **ON the unit circle**, the reciprocal is also the complex conjugate. Hence, it is associated with a group of 2 zeros given by $z = e^{\pm j\phi}$.
  - A **real zero** $r \neq 1$ and $r \neq -1$ is associated with a group of 2 zeros given by r and $\frac{1}{r}$.
  - **A zero at ± 1 can exist on its own!**
- **The principal difference among the 4 Types of linear-phase FIR filters is with regard to the number of zeros at $z = 1$ and $z = -1$.**
  - **Note that $z = 1 = e^{j0}$ and $z = -1 = e^{j\pi}$**

Linear-phase FIR transfer functions: Zero locations

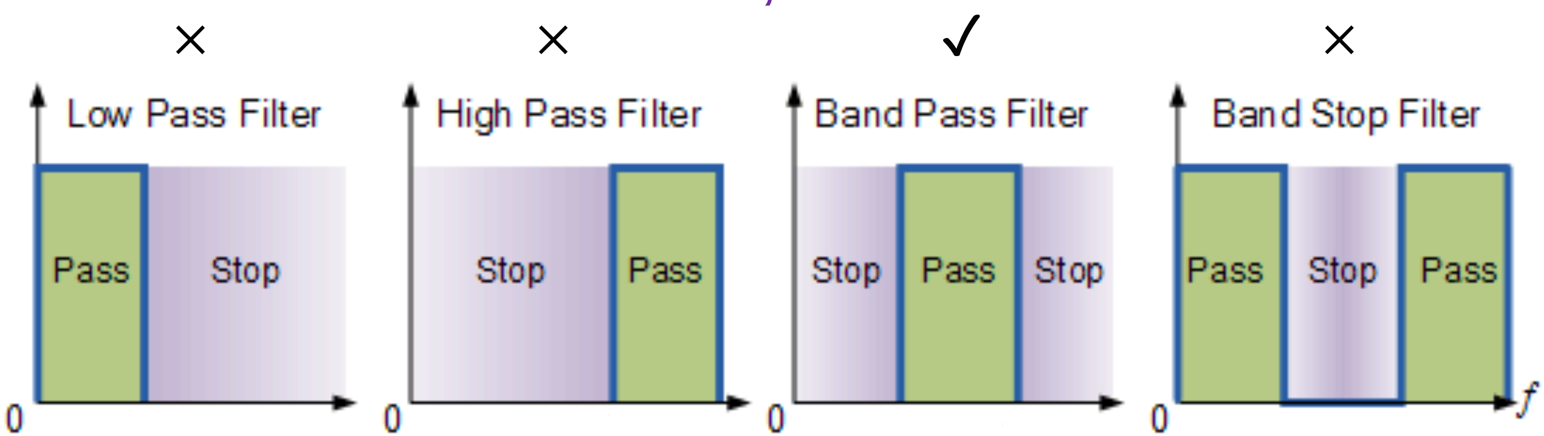
- **Type 1 filters:**
  - The length M of **Type 1** filters is odd and therefore, the order of the filter is even. This means that the number of zeros is even.
  - Therefore, **IF** the **Type 1** FIR transfer function has a zero at $z = 1$ or at $z = -1$, then because of its order being even and based on the observations of the previous slide, **there can be even number of zeros at $z = 1$ or at $z = -1$ or at both locations OR NO ZEROS at $z = 1$ and at $z = -1$.**
  - Note that if we had an odd number of zeros at $z = 1$ and an odd number of zeros at $z = -1$ the filter would be antisymmetric instead of symmetric.
    - $(z-1)(z+1) = z^2 - 1$ antisymmetric
  - A zero at $z = 1 \Rightarrow \omega = 0$ prevents us for using the filter as a lowpass or as a bandstop filter.
  - A zero at $z = -1 \Rightarrow \omega = \pi$ prevents us for using the filter as a highpass or as a bandstop filter.
  - Therefore, **Type 1** filters can be used to design any type of filters.

Linear-phase FIR transfer functions: Zero locations

- **Type 2 filters:**
  - The length M of **Type 2** filters is even and therefore, the order of the filter is odd. This means that the number of zeros is odd.
  - $H(-1) = (-1)^{-(M-1)}H(-1) = -H(-1) \Rightarrow H(-1) = 0$. **There is at least one zero at $z = -1$ ➡ TYPE 2 FILTERS CANNOT BE HIGHPASS OR BANDSTOP FILTERS**
  - **Type 2 FIR** transfer functions have on total an odd number of zeros at $z = 1$ and $z = -1$.
  - **We can show that Type 2 FIR transfer functions have an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$.** This is because having an even number of zeros at $z = -1$ and an odd number of zeros at $z = 1$ would produce an antisymmetric filter.

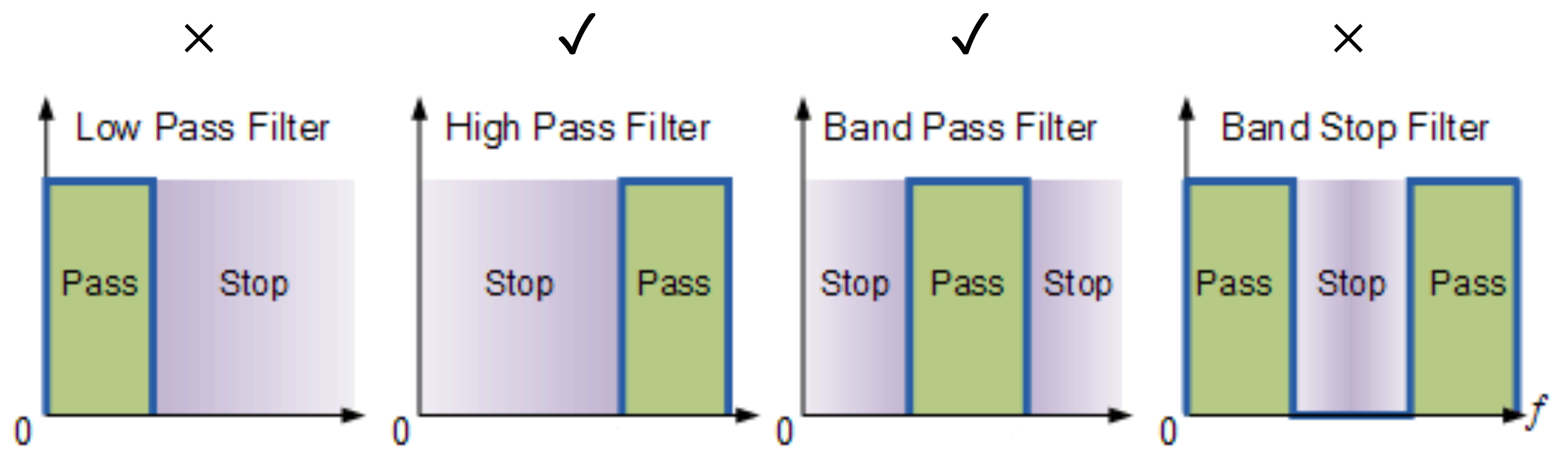
Linear-phase FIR transfer functions: Zero locations

- **Type 3 filters:**
  - The length M of **Type 3** filters is odd and therefore, the order of the filter is even. This means that the number of zeros is even.
  - $H(1) = -(1)^{-(M-1)}H(1) = -H(1) \Rightarrow H(1) = 0$
  - $H(-1) = -(-1)^{-(M-1)}H(-1) = -H(-1) \Rightarrow H(-1) = 0$
  - Therefore, **Type 3** FIR transfer functions have at least a zero at $z = 1$ and a zero at $z = -1$.
  - **We can show that Type 3 FIR transfer functions have an odd number of zeros at $z = 1$ and an odd number of zeros at $z = -1$.**
  - **TYPE 3 FILTERS CANNOT BE LP, HP or BS FILTERS!**



Linear-phase FIR transfer functions: Zero locations

- **Type 4 filters:**
  - The length M of **Type 4** filters is even and therefore, the order of the filter is odd. This means that the number of zeros is odd.
  - $H(1) = -(1)^{-(M-1)}H(1) = -H(1) \Rightarrow H(1) = 0$
  - Therefore, **Type 4** FIR transfer functions have at least a zero at $z = 1$.
  - **We can show that Type 4 FIR transfer functions have an odd number of zeros at $z = 1$ and either an even number or no zeros at $z = -1$.**
  - **TYPE 4 FILTERS CANNOT BE LP** OR BS FILTERS!



4 Types of linear-phase FIR transfer functions
Magnitude response

- The expressions for the magnitude responses of the four types of linear-phase FIR filters can be rewritten in a **similar generic fashion**.
- To develop this generic form for the magnitude response expression, we consider each of the four types of filters separately.
- These expressions can be verified easily if you choose a filter of small order.

4 Types of linear-phase FIR transfer functions
Magnitude response of Type 1

- For **Type 1 linear-phase FIR filter** **(odd length and symmetric)** of length M, the frequency response can be written using the notation

$$H(e^{j\omega}) = \breve{H}(\omega)e^{-j\omega(M-1)/2}$$

$$\breve{H}(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h[n]\cos\left(\omega(\frac{M-1}{2} - n)\right)$$

M is the length of filter which is odd. The order of the filter is $M-1$ (even).

- Furthermore, for the same filter of length M the magnitude response can be rewritten in the form

$$\breve{H}(\omega) = \sum_{k=0}^{(M-1)/2} a[k]\,\mathbf{cos(\omega k)}$$

$$a[0] = h\left[\frac{M-1}{2}\right], \quad a[k] = 2h\left[\frac{M-1}{2} - k\right], \quad 1 \le k \le \frac{M-1}{2}$$

- From now on my goal will be to have weighted combinations of $\mathbf{cos(\omega k)}$ terms.

4 Types of linear-phase FIR transfer functions
Magnitude response of Type 2

- For **Type 2 linear-phase FIR filter** **(even length and symmetric)** of length M, the frequency response can be written using the notation

$$H\left(e^{j\omega}\right)=\breve{H}(\omega)e^{-j\omega(M-1)/2}$$

$$\breve{H}(\omega)=2\sum_{n=0}^{\frac{M}{2}-1}h[n]\cos\left(\omega(\frac{M-1}{2}-n)\right)$$

- For the same filter, the magnitude response can be rewritten in the form

$$\breve{H}(\omega)=\sum_{\boldsymbol{k=1}}^{M/2}b[k]\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$$

$$b[k]=2h\left[\frac{M}{2}-k\right],\quad \mathbf{1}\leq k\leq\frac{M}{2}$$

- However, recall that from now on our goal will be to have weighted combinations of $\boldsymbol{\cos(\omega k)}$ terms. So we are still not happy!

4 Types of Linear-Phase FIR Transfer Functions
Magnitude Response of Type 2 cont.

- We managed to wrote

$$\breve{H}(\omega) = \sum_{k=1}^{M/2} b[k] \cos\left(\omega\left(k - \frac{1}{2}\right)\right)$$

$$b[k] = 2h\left[\frac{M}{2} - k\right], \quad 1 \le k \le \frac{M}{2}$$

- The above can also be expressed in the form:

$$\breve{H}(\omega) = \cos\left(\frac{\omega}{2}\right) \sum_{k=0}^{\frac{M}{2}-1} \tilde{b}[k] \cos(\omega k)$$

where

$$\tilde{b}\left[\frac{M}{2} - 1\right] = 2b\left[\frac{M}{2}\right]$$

$$\tilde{b}[k-1] = 2b[k] - \tilde{b}[k], \quad 2 \le k \le \frac{M}{2} - 1$$

$$\tilde{b}[0] = b[1] - \frac{1}{2}\tilde{b}[1]$$

- **Use $\cos(a)\cos(b) = \frac{\cos(a+b)+\cos(a-b)}{2}$ to move between the two relationships.**

4 Types of linear-phase FIR transfer functions
Phase response of Type 1 and Type 2

- The phase characteristics of the symmetric filters for both M odd and M even (Type 1 and Type 2) are:

$$\Theta(\omega) = \begin{cases} -\omega(\frac{M-1}{2}) & \breve{H}(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & \breve{H}(\omega) < 0 \end{cases}$$

- Observe that the phase is indeed a linear function.

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4 Types of linear-phase FIR transfer functions
Magnitude response of Type 3

- For the **Type 3 linear-phase FIR filter** **(odd length and antisymmetric)** of length M, the frequency response can be written using the notation

$$H\left(e^{j\omega}\right) = \breve{H}(\omega) e^{-j\left(\frac{\omega(M-1)}{2} - \frac{\pi}{2}\right)}$$

$$\breve{H}(\omega) = 2 \sum_{n=0}^{(M-3)/2} h[n] \sin\left(\omega\left(\frac{M-1}{2} - n\right)\right)$$

- The magnitude response can be rewritten using the form

$$\breve{H}(\omega) = \sum_{k=1}^{(M-1)/2} c[k] \sin(\omega k)$$

$$c[k] = 2h\left[\frac{M-1}{2} - k\right], \quad 1 \le k \le (M-1)/2$$

4 Types of linear-phase FIR transfer functions
Magnitude response of Type 3 cont.

- The previous can also be expressed in the form:

$$\breve{H}(\omega) = \sin(\omega) \sum_{k=0}^{(M-3)/2} \tilde{c}[k] \cos(\omega k)$$

Where

$$\tilde{c}\left[\frac{M-3}{2}\right] = 2c\left[\frac{M-1}{2}\right]$$

$$\tilde{c}\left[\frac{M-5}{2}\right] = 2c\left[\frac{M-3}{2}\right]$$

$$\tilde{c}[k] = 2c[k+1] + \tilde{c}[k+2], 1 \le k \le \frac{M-7}{2}$$

$$\tilde{c}[0] = c[1] + \frac{1}{2}\tilde{c}[2]$$

The red line exists if $\frac{M-7}{2} \ge 1 \Rightarrow M-7 \ge 2 \Rightarrow M \ge 9$

- **Use $\sin(a)\cos(b) = \frac{\sin(a+b)+\sin(a-b)}{2}$ to move between the two previous relationships.**

4 Types of linear-phase FIR transfer functions
Magnitude response of Type 4

- For the **Type 4 linear-phase FIR filter** **(even length and antisymmetric)**, the frequency response can be written using the notation

$$H\left(e^{j\omega}\right) = \breve{H}(\omega) e^{-j\left(\frac{\omega(M-1)}{2}-\frac{\pi}{2}\right)}$$

$$\breve{H}(\omega) = 2\sum_{n=0}^{\frac{M}{2}-1} h[n]\sin\left(\omega\left(\frac{M-1}{2}-n\right)\right)$$

- For the same filter, the magnitude response can be rewritten using the form

$$\breve{H}(\omega) = \sum_{k=1}^{M/2} d[k]\sin\left(\omega\left(k-\frac{1}{2}\right)\right)$$

$$d[k] = 2h\left[\frac{M}{2}-k\right],\quad 1 \le k \le \frac{M}{2}$$

4 Types of linear-phase FIR transfer functions
Magnitude response of Type 4 cont.

- We wrote

$$\breve{H}(\omega) = \sum_{k=1}^{M/2} d[k] \sin\left(\omega\left(k - \frac{1}{2}\right)\right)$$

$$d[k] = 2h\left[\frac{M}{2} - k\right], \quad 1 \le k \le \frac{M}{2}$$

- The above can also be expressed in the form:

$$\breve{H}(\omega) = \sin\left(\frac{\omega}{2}\right) \sum_{\boldsymbol{k=0}}^{\boldsymbol{\frac{M}{2}-1}} \tilde{d}[k] \cos(\omega k)$$

Where

$$\tilde{d}\left[\boldsymbol{\frac{M}{2} - 1}\right] = 2d\left[\frac{M}{2}\right]$$

$$\tilde{d}[k-1] = \tilde{d}[k] + 2d[k] \; 2 \le k \le \frac{M}{2} - 1$$

$$\tilde{d}[\mathbf{0}] = \frac{1}{2}\tilde{d}[1] + d[1]$$

4 Types of linear-phase FIR transfer functions
Phase response of Type 3 and Type 4

- The phase characteristics of the antisymmetric filters for both M odd and M even (Type 3 and Type 4) are:

$$\Theta(\omega) = \begin{cases} \pi/2 - \omega(\frac{M-1}{2}) & \breve{H}(\omega) > 0 \\ 3\pi/2 - \omega\left(\frac{M-1}{2}\right) & \breve{H}(\omega) < 0 \end{cases}$$

- We observe again that the phase responses are linear.

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Magnitude response of linear-phase FIR filters: Generic form.

Type 1: Odd Length M Symmetric	$\breve{H}(\omega) = Q(\omega)A(\omega)$, $\uparrow = 1$	$\breve{H}(\omega) = \sum_{k=0}^{L=(M-1)/2} \tilde{a}[k]\cos(\omega k)$ $\tilde{a}[k] = a[k]$
Type 2: Even Length M Symmetric	$\breve{H}(\omega) = Q(\omega)A(\omega)$	$\breve{H}(\omega) = \cos\left(\frac{\omega}{2}\right) \sum_{k=0}^{L=\frac{M}{2}-1} \tilde{b}[k]\cos(\omega k)$
Type 3: Odd Length M Antisymmetric	$\breve{H}(\omega) = Q(\omega)A(\omega)$	$\breve{H}(\omega) = \sin(\omega) \sum_{k=0}^{L=(M-3)/2} \tilde{c}[k]\cos(\omega k)$
Type 4: Even Length M Antisymmetric	$\breve{H}(\omega) = Q(\omega)A(\omega)$	$\breve{H}(\omega) = \sin\left(\frac{\omega}{2}\right) \sum_{k=0}^{L=\frac{M}{2}-1} \tilde{d}[k]\cos(\omega k)$

Linear-phase FIR filter design by optimization

- The magnitude response for all 4 types of linear-phase FIR filters can be expressed as

$$\breve{H}(\omega) = Q(\omega)A(\omega)$$

- Before, we wrote the weighted error function as

$$\varepsilon(\omega) = W(\omega)\left[\breve{H}(\omega) - D(\omega)\right]$$

- The modified form of the weighted error function is now

$$\varepsilon(\omega) = W(\omega)[Q(\omega)A(\omega) - D(\omega)] = W(\omega)Q(\omega)\left[A(\omega) - \frac{D(\omega)}{Q(\omega)}\right]$$

$$\boldsymbol{\varepsilon(\omega) = \widetilde{W}(\omega)[A(\omega) - \widetilde{D}(\omega)]}$$

where

$$\widetilde{W}(\omega) = W(\omega)Q(\omega)$$

$$\widetilde{D}(\omega) = \frac{D(\omega)}{Q(\omega)}$$

Optimization problem formulation

- **Problem formulation**

  Determine the samples $\{\tilde{a}[k]\}$ which minimize the peak absolute value of

  $$\varepsilon(\omega) = \widetilde{W}(\omega)[\sum_{k=0}^{L} \tilde{a}[k]\cos(\omega k) - \widetilde{D}(\omega)]$$

  over the specified frequency bands $\omega \in R$.

- After $\tilde{a}[k]$ has been determined, construct the impulse response $h[n]$.

- Solution is obtained via the so-called **Alternation Theorem**.

- The optimal solution has equiripple behavior.

- **Parks and McClellan** used the **Remez** algorithm to develop a procedure for designing linear phase FIR digital filters.

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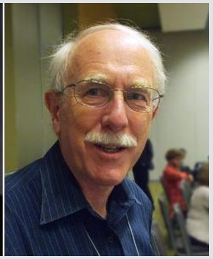
The Parks-McClellan algorithm

Definition of Alternation Theorem



- **Problem formulation**

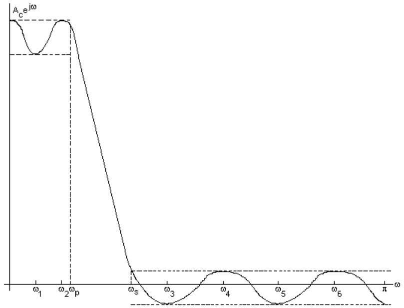
  Determine the samples $\{\tilde{a}[k]\}$ which minimise the peak absolute value of

$$\varepsilon(\omega) = \widetilde{W}(\omega)[\sum_{k=0}^{L} \tilde{a}[k]\cos(\omega k) - \widetilde{D}(\omega)]$$

- Parks and McClellan solved the above problem (1972) applying the following theorem from the theory of **Chebyshev Approximation**.

Alternation Theorem: The amplitude function $A(\omega)$ is the best unique approximation of the desired amplitude response obtained by minimizing the peak absolute value ε of $\varepsilon(\omega)$, **if and only if** there exist **at least** $L+2$ extremal angular frequencies $\omega_0, \omega_1, \ldots, \omega_{L+1}$, in a closed subset R of the frequency range $0 \leq \omega \leq \pi$ such that $\omega_0 < \omega_1 < \cdots < \omega_L < \omega_{L+1}$ and $\varepsilon(\omega_i) = -\varepsilon(\omega_{i+1})$, with $|\varepsilon(\omega_i)| = \varepsilon$ for all i in the range $0 \leq i \leq L+1$.

The solution to the problem of minimizing the peak absolute value of the error is called the best weighted Chebyshev approximation to D(). Because it minimizes the maximum value of the error, it is also called the minimax solution. The Remez algorithm for computing the best Chebyshev solution uses the alternation theorem.



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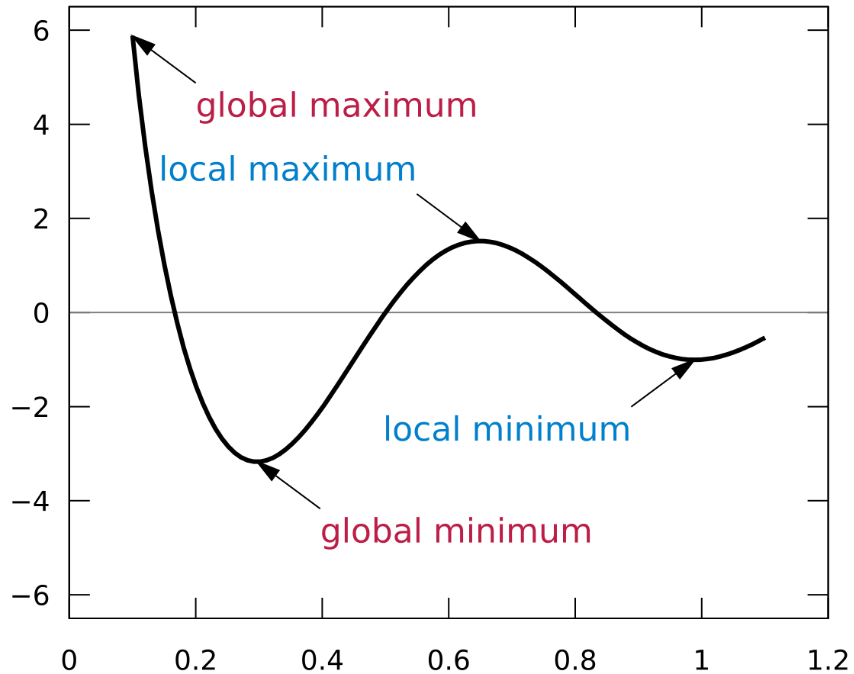
Extrema of a function

- What are **extrema** of functions? An **extremum** (plural extrema) is a point of a function at which it has the highest (maximum) or lowest (minimum) value. A global maximum or minimum is the highest or lowest value of the entire function, whereas a local maximum or minimum is the highest or lowest value in its neighbourhood.



The Parks-McClellan algorithm

Reformulation of magnitude response using Chebyshev polynomials

- Let us examine the behaviour of the magnitude response for a Type 1 equiripple lowpass FIR filter whose approximation error $\varepsilon(\omega)$ satisfies the condition of the alternation theorem. For Type 1 filters $Q(\omega) = 1$.
- The peaks of $\varepsilon(\omega)$ are at $\omega = \omega_i$, $0 \leq i \leq L+1$, where

$$\frac{d\varepsilon(\omega)}{d\omega} = 0$$

- Since within the passband and the stopband, $\widetilde{W}(\omega)$ and $\widetilde{D}(\omega)$ are piecewise constant $(\boldsymbol{\varepsilon(\omega) = \widetilde{W}(\omega)[A(\omega) - \widetilde{D}(\omega)]})$ we see that

$$\left.\frac{d\varepsilon(\omega)}{d\omega}\right|_{\omega=\omega_i} = 0 \Rightarrow \left.\frac{dA(\omega)}{d\omega}\right|_{\omega=\omega_i} = 0$$

or, in other words, the amplitude response $A(\omega)$ also has peaks at $\omega = \omega_i$.
- We use the relation $\cos(\omega \boldsymbol{k}) = T_{\boldsymbol{k}}(\cos\omega)$ where $T_k(x)$ is the kth order Chebyshev polynomial of the first kind, defined through the recurrence relation:

$$T_0(x) = 1, T_1(x) = x, T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$$

The amplitude response $A(\omega)$ can be expressed as a power series in $\cos\omega$.

$$A(\omega) = \sum_{k=0}^{L} a[k]\cos(\omega k) = \sum_{k=0}^{L} a[k]\left[\sum_{n=0}^{k} \beta_{nk}\,(\cos\omega)^n\right] = \sum_{k=0}^{L} a'[k](\cos\omega)^k$$

Chebyshev polynomial revision

- Chebyshev polynomials of 1st kind:

  $T_0(x) = 1$
  $T_1(x) = x$
  $T_2(x) = 2x^2 - 1$
  $T_3(x) = 4x^3 - 3x$
  $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

  We know that
  $\cos\mathbf{2}\omega = 2\cos^{\mathbf{2}}\omega - 1 = T_2(\cos\omega)$
  $\cos\mathbf{3}\omega = 4\cos^{\mathbf{3}}\omega - 3\cos\omega = T_3(\cos\omega)$

  It is proven that
  $\cos k\omega = T_k(\cos\omega)$

  The amplitude response $A(\omega)$ can be expressed as a power series in $\cos\omega$.

  polynomial

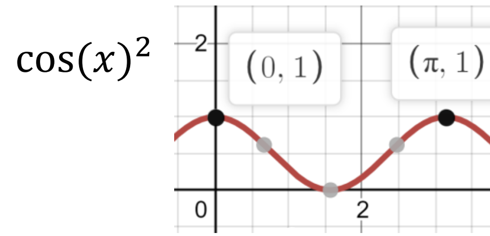
$$A(\omega) = \sum_{k=0}^{L} a'[k](\cos\omega)^k$$

The Parks-McClellan algorithm

Counting the extrema

- The amplitude response $A(\omega)$ can be expressed as a power series in $\cos\omega$

$$A(\omega) = \sum_{k=0}^{L} a'[k](\cos\omega)^k$$



- It is an L^{th} order polynomial in $\cos\omega$.
- As a result $A(\omega)$ can have at most $\boldsymbol{L-1}$ local minima and maxima in R.
- Moreover, at the band-edge frequencies $\boldsymbol{\omega = \omega_p}$ and $\boldsymbol{\omega = \omega_s}$, $|\varepsilon(\omega)|$ is maximum and therefore, $A(\omega)$ has extrema in these angular frequencies.
- In addition $A(\omega)$ **usually** has extrema at $\boldsymbol{\omega = 0}$ and $\boldsymbol{\omega = \pi}$.
- Therefore, there are, at most $L-1+4 = L+3$ extremal frequencies of $\varepsilon(\omega)$.
- We can generalize and say that in the case of a linear phase FIR filter with K specified band-edge frequencies, designed using the Remez exchange algorithm, there can be at most $L-1+2+K = L+K+1$ extremal frequencies.
- To arrive at the optimum solution we need to solve the set of $L+2$ equations:

$$\widetilde{W}(\omega_i)[A(\omega_i) - \widetilde{D}(\omega_i)] = (-1)^i\varepsilon,\ 0 \le i \le L+1$$

for the unknowns $\tilde{a}(i)$ and ε, provided the $L+2$ extremal angular frequencies are known.

The Parks-McClellan algorithm

Matrix problem formulation

- To arrive at the optimum solution we need to solve the set of $L+2$ equations:

$$\widetilde{W}(\omega_i)[A(\omega_i) - \widetilde{D}(\omega_i)] = (-1)^i \varepsilon,\ 0 \le i \le L+1$$

$$A(\omega_i) - \frac{1}{\widetilde{W}(\omega_i)}(-1)^i \varepsilon = \widetilde{D}(\omega_i), 0 \le i \le L+1 \quad (1)$$

for the unknowns $\tilde{a}(i)$ and ε, provided the $L+2$ extremal angular frequencies are known.

- The above is rewritten in matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_0) & \dots & \cos(L\omega_0) & -1/\widetilde{W}(\omega_0) \\ 1 & \cos(\omega_1) & \dots & \cos(L\omega_1) & \text{-(-1)\^{}1=}1/\widetilde{W}(\omega_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega_L) & \dots & \cos(L\omega_L) & -(-1)^L/\widetilde{W}(\omega_L) \\ 1 & \cos(\omega_{L+1}) & \dots & \cos(L\omega_{L+1}) & -(-1)^{L+1}/\widetilde{W}(\omega_{L+1}) \end{bmatrix} \begin{bmatrix} \tilde{a}[0] \\ \tilde{a}[1] \\ \vdots \\ \tilde{a}[L] \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \widetilde{D}(\omega_0) \\ \widetilde{D}(\omega_1) \\ \vdots \\ \widetilde{D}(\omega_L) \\ \widetilde{D}(\omega_{L+1}) \end{bmatrix}$$

- The **Remez Exchange Algorithm** is used to solve the above.

If the set of extremal points in the alternation theorem were known in advance, then the solution could be found by solving the system of equations (1). The system (1) represents an interpolation problem, which in matrix form becomes the formulation above.

The Parks-McClellan algorithm
Using the Remez Exchange Algorithm

- The **Remez Exchange Algorithm**, a highly efficient iterative procedure, is used to determine the locations of the extremal frequencies and consists of the following steps at each iteration stage.
- **Step 1:** A set of initial values for the extremal frequencies are either chosen or are available from the completion of the previous iteration. **These values are initially guessed!**
- **Step 2:** Solving the system of equations we obtain

$$\varepsilon = \frac{c_0\tilde{D}(\omega_0) + c_1\tilde{D}(\omega_1) + \cdots + c_{L+1}\tilde{D}(\omega_{L+1})}{\frac{c_0}{\tilde{W}(\omega_0)} - \frac{c_1}{\tilde{W}(\omega_1)} + \cdots + \frac{(-1)^{L+1}c_{L+1}}{\tilde{W}(\omega_{L+1})}}$$

$$c_n = \prod_{\substack{i=0 \\ i\neq n}}^{L+1} \frac{1}{\cos(\omega_n) - \cos(\omega_i)}$$

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The Parks-McClellan algorithm

Using the Remez Exchange Algorithm cont.

- **Step 3:** The values of the amplitude response $A(\omega)$ at $\omega = \omega_i$ are then computed using

$$A(\omega_i) = \frac{(-1)^i \varepsilon}{\widetilde{W}(\omega_i)} + \widetilde{D}(\omega_i),\ 0 \le i \le L+1$$

- **Step 4:** The polynomial $A(\omega)$ is determined by interpolating the above values at the $L+2$ extremal frequencies using the **Lagrange interpolation formula**:

$$A(\omega) = \sum_{i=0}^{L+1} A(\omega_i)\, P_i(\cos\omega)$$

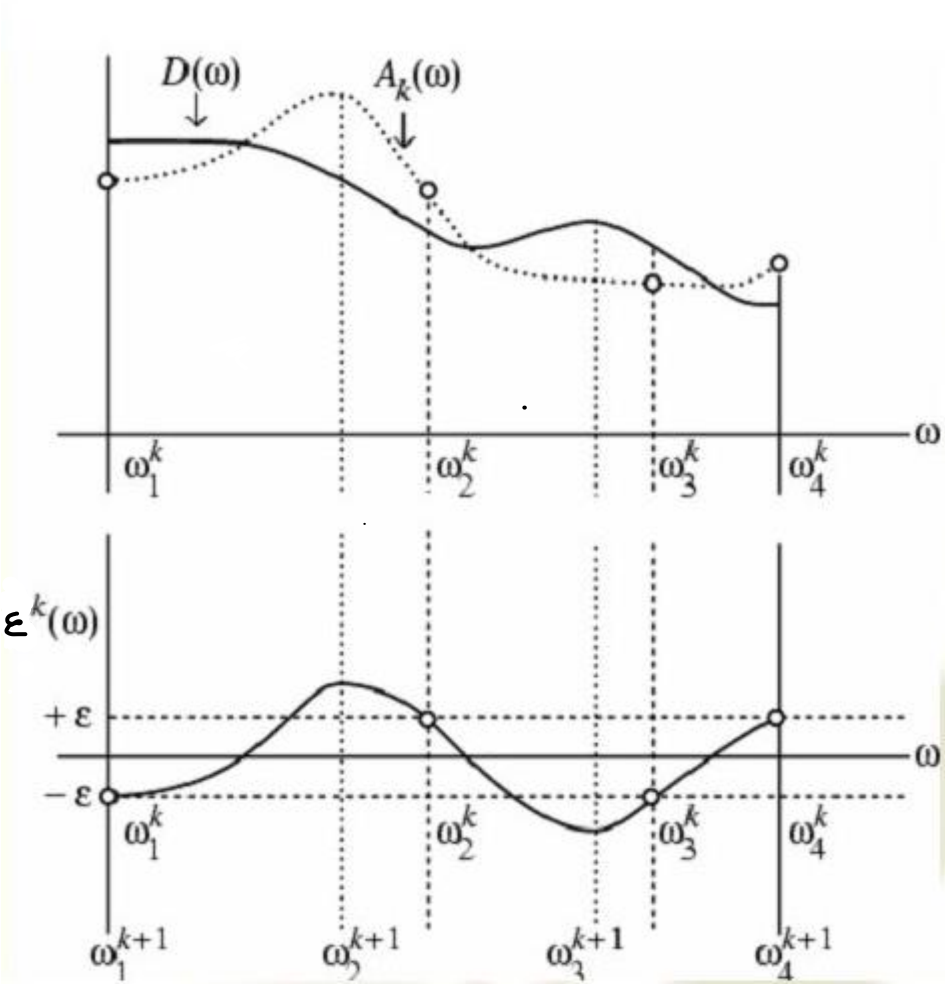
where $P_i(\cos\omega) = \prod_{\substack{l=0 \\ l \ne i}}^{L+1} \left(\frac{\cos\omega - \cos\omega_l}{\cos\omega_i - \cos\omega_l}\right),\ 0 \le i \le L+1$

- **Step 5:** The new weighted error function $\varepsilon(\omega)$ is computed at a dense set $S (S \ge L)$ of frequencies. In practice, $S = 16L$ is adequate. Determine the $L+2$ new extremal frequencies from the values of $\varepsilon(\omega)$ evaluated at the dense set of frequencies.
- **Step 6:** If the peak values ε are equal in magnitude, the algorithm has converged. Otherwise, we go bask to Step 2.

The Parks-McClellan algorithm

Illustration of iterations

- Plots of the desired response $D(\omega)$, the amplitude response $A_k(\omega)$ and the error $\varepsilon^k(\omega)$ at the end of the kth iteration. The locations of the new extremal frequencies are given by ω_i^{k+1}.
- The iteration process is stopped after the difference between the value of the peak error ε calculated at any stage and that at the previous stage is below a present threshold value, such as 10^{-6}.
- In practice the process converges after very few iterations.



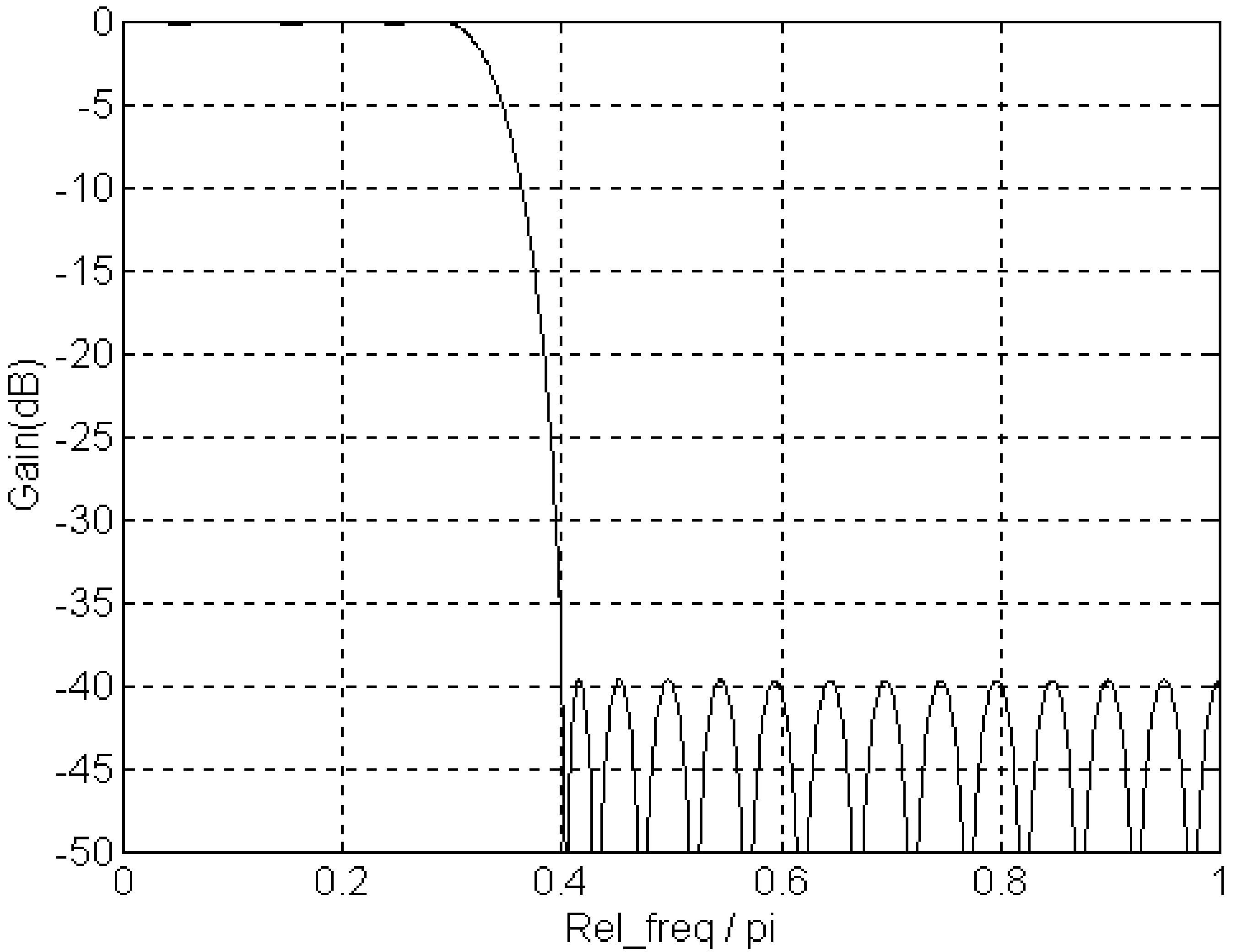
Example of the Remez exchange algorithm

- Better than windowing technique, but more complicated.
- Available in MATLAB.
- Design 40th order FIR lowpass filter whose gain is unity (0 dB) in range 0 to 0.3π radians/sample & zero in range 0.4 π to π.
- The 41 coefficients will be found in array ‘a’.
- Produces equiripple gain-responses where peaks of stop-band ripples are equal rather than decreasing with increasing frequency.
- Highest peak in stop-band lower than that of an FIR filter of same order designed by a windowing technique to have the same cut-off frequency.

a = remez(n,f,d) returns row vector a containing the n+1 coefficients of the order n FIR filter whose frequency-amplitude characteristics match those given by vectors f and d (desired).

```
a = remez (40, [0, 0.3,  0.4,1],[1, 1,  0, 0] );
[h,w] = freqz (a,1,1000);
plot([0:999]/1000,20*log10(abs(h)),'k');
axis([0,1,-50,0]);
grid on;
xlabel('Rel_freq / pi');
ylabel('Gain(dB)');
```

[h,w] = freqz(a,b,n) returns the n-point frequency response vector h and the corresponding angular frequency vector w for the digital filter transfer function coefficients stored in a and b.



Gain of 40^{th} order FIR lowpass filter designed by “ Remez ”

Design exercise 1

1. (i) Determine the weighting function $W(\omega)$ that is used to design a symmetric linear phase FIR **lowpass** filter using the Parks-McClellan method to meet the following specifications: $\omega_p = 0.45\pi, \omega_s = 0.6\ \pi$, $\delta_p = 0.2043$ and $\delta_s = 0.0454$. Assume that $F_T = 2Hz$. Sampling frequency

   (ii) Design the linear phase FIR filter described above using the function **remez** and plot its magnitude response.

Solution

(i) $W(\omega) = \begin{cases} 1, & \text{in the passband} \\ \delta_p/\delta_s, & \text{in the stopband} \end{cases}$ **or** $W(\omega) = \begin{cases} 1, & 0 \le \omega \le 0.45\pi \\ 4.5, & 0.6\ \pi \le \omega \le \pi \end{cases}$

(ii) $\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} \Rightarrow F_p = \frac{\omega_p F_T}{2\pi} = 0.45Hz$ Recall: Topic 1a Revision DSP DTFT

$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} \Rightarrow F_s = \frac{\omega_s F_T}{2\pi} = 0.6\,Hz$

$a_p = -20\log_{10}(1-\delta_p)dB = 1.985dB$ peak passband ripple

$a_s = -20\log_{10}\delta_s dB = 26.858dB$ minimum stopband attenuation

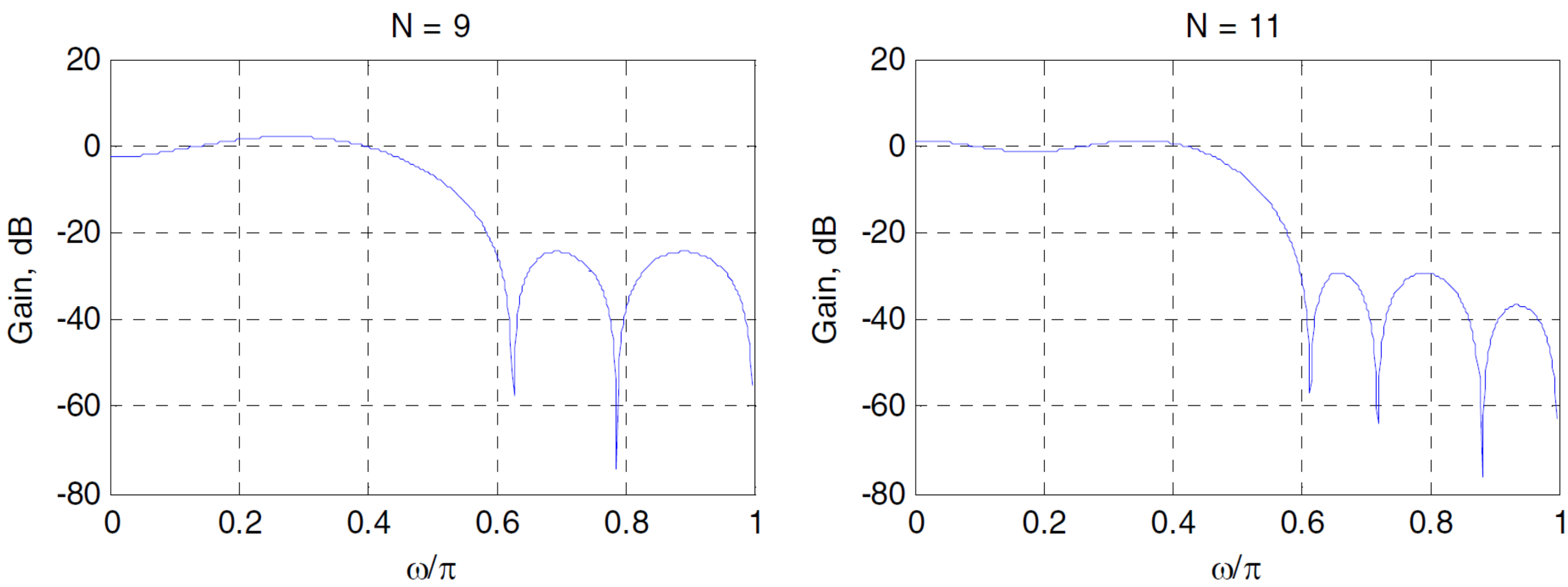
Design exercise 1 cont.

Using **remezord**, we get $N = 9$. The specifications of the filter are hardly met in the passpand. We increase N to 11 to meet the specifications.

```
Ft = 2;Fp = 0.45;Fs = 0.6;
ds = 0.0454;dp = 0.2043;
F = [Fp Fs];A = [1 0];DEV = [dp ds];
[N,Fo,Ao,W] = remezord(F,A,DEV,Ft);
b = remez(N,Fo,Ao,W);
[H,w] = freqz(b,1,512);
figure(1);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 9');
%axis([0 0.45 -3 3]);
N = 11;
b = remez(N,Fo, Ao, W);
[H,w] = freqz(b,1,512);
figure(2);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi');ylabel(Gain, dB);title('N = 11');
```

Design exercise 1 cont.

- The gain responses for $N = 9$ and $N = 11$ are shown below.



Design exercise 2

2. (i) Determine the weighting function $W(\omega)$ that is used to design a symmetric linear phase FIR **highpass** filter using the Parks-McClellan method to meet the following specifications: $\omega_p = 0.7\pi, \omega_s = 0.55\pi$, $\delta_p = 0.03808$ and $\delta_s = 0.0112$. Assume that $F_T = 2Hz$.

   (ii) Design the linear phase FIR filter described above using the function **remez** and plot its magnitude response.

Solution

(i) $W(\omega) = \begin{cases} 1, & \text{in the passband} \\ \delta_p/\delta_s, & \text{in the stopband} \end{cases}$ **or** $W(\omega) = \begin{cases} 3.4, & 0 \le \omega \le 0.55\pi \\ 1, & 0.7\pi \le \omega \le \pi \end{cases}$

(ii) $\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} \Rightarrow F_p = \frac{\omega_p F_T}{2\pi} = 0.7Hz$

$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} \Rightarrow F_s = \frac{\omega_s F_T}{2\pi} = 0.55Hz$

$a_p = -20\log_{10}(1 - \delta_p)dB = 0.3372dB$

$a_s = -20\log_{10}\delta_s dB = 39.016dB$

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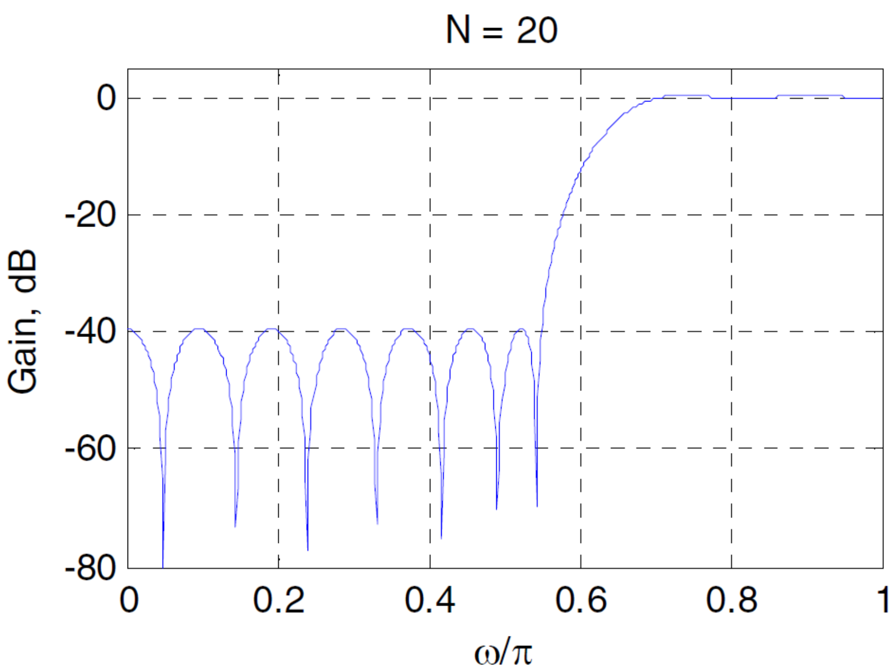
Design exercise 2 cont.

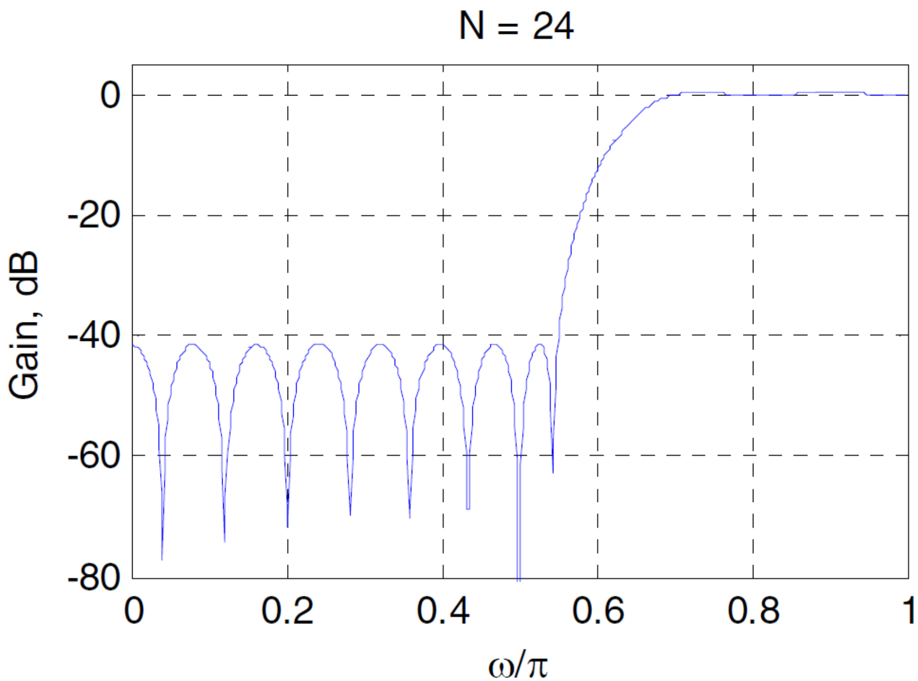
- Using **remezord**, we obtain $N = 20$. The specifications of the filter are hardly met in the stoppand. The are met with $N = 23$.
- However, for odd orders, the frequency response at the Nyquist frequency is 0. If you specify an odd-valued N, **remez** increments it by 1 $(N = 23)$.

```
Ft = 2;Fp = 0.7;Fs = 0.55;
ds = 0.0112;dp = 0.03808;
F = [Fs Fp];A = [0 1];DEV = [ds dp];
[N,Fo,Ao,W] = remezord(F,A,DEV,Ft);
b = remez(N,Fo,Ao,W);
[H,w] = freqz(b,1,512);
figure(1);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 20');
N = 23;
b = remez(N,Fo,Ao,W);
[H,w] = freqz(b,1,512);
figure(2);
plot(w/pi,20*log10(abs(H)));
xlabel('\omega/\pi');ylabel('Gain,dB ');title('N = 23');
```

Design exercise 2 cont.

- The gain responses for $N = 20$ and $N = 24$ are shown below.





Design exercise 3

3. (i) Determine the weighting function $W(\omega)$ that is used to design a linear phase FIR **bandpass** filter using the Parks-McClellan method to meet the following specifications: $\omega_{p_1} = 0.55\pi, \omega_{p_2} = 0.7\pi,\ \omega_{s_1} = 0.44\pi,\ \omega_{s_2} = 0.82\pi,\ \delta_p = 0.01$ and $\delta_{s_1} = 0.007, \delta_{s_2} = 0.002$, where $\delta_{s_1}, \delta_{s_2}$ are the ripples in the lower and upper stopbands, respectively. Assume that $F_T = 2Hz$.

   (ii) Design the linear phase FIR filter described above using the function **remez** and plot its magnitude response.

Solution

(i) $W(\omega) = \begin{cases} \delta_p/\delta_{s_1} & 0 \le \omega \le \omega_{s_1} \\ 1 & \omega_{p_1} \le \omega \le \omega_{p_2} \\ \delta_p/\delta_{s_2} & \omega_{s_2} \le \omega \le \pi \end{cases}$ **or** $W(\omega) = \begin{cases} 1.43 & 0 \le \omega \le 0.44\pi \\ 1 & 0.55\pi \le \omega \le 0.7\pi \\ 5 & 0.82\pi \le \omega \le \pi \end{cases}$

(ii) $a_p = -20\log_{10}(1 - \delta_p)dB = 0.087dB$

$a_{s_1} = -20\log_{10}\delta_{s_1}dB = 43dB$

$a_{s_2} = -20\log_{10}\delta_{s_2}dB = 54dB$

Design exercise 3 cont.

```
Ft = 2;Fp1 = 0.55;Fp2 = 0.7;Fs1 = 0.44;Fs2 = 0.82;
ds1 = 0.007;ds2 = 0.002;dp = 0.01;
F = [Fs1 Fp1 Fp2 Fs2];A = [0 1 0];DEV = [ds1 dp ds2];
[N,Fo,Ao,W] = remezord(F,A,DEV,Ft);
b = remez(N,Fo,Ao,W);
[H, w] = freqz(b, 1, 512);
figure(1);
plot(w/pi,20*log10(abs(H)));grid;
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 39');
axis([0 1 -80 10]);
N = 41;
b = remez(N, Fo, Ao, W);
[H, w] = freqz(b, 1, 512);
figure(2);
plot(w/pi, 20*log10(abs(H)));
xlabel('\omega/\pi');ylabel('Gain, dB');title('N = 41');
axis([0 1 -80 10]);
```

Design exercise 3 cont.

- Using **remezord**, we estimate the filter length to be $N = 39$. However, the minimum stopband attenuation specifications are not met in both stopbands, so we increase N to 41.