IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2023

# MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS

SOLUTIONS

1. (a) i. The vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ are linearly independent. Moreover, since the solution allows for an arbitrary linear combination of them, this means that they are in the null space of $\mathbf{A} \in \mathbb{R}^{3\times 3}$. So rank of $\mathbf{A}$ is one.

ii. For $a = -1$ and $b = 2$ we have

$$\mathbf{A} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}.$$

So the first column of $\mathbf{A}$ is $[2, 4, 2]^T$. For $a = 1$ and $b = -2$ we have

$$\mathbf{A} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix}.$$

So the third column of $\mathbf{A}$ is $[1, 2, 1]^T$. Finally the second column can be found by ensuring that $[1, 2, 0]^T$ is in the null space of $\mathbf{A}$. In conclusion we have:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix}$$

iii. One way to find the minimum norm solution is by finding the vector $\mathbf{x}$ that is orthogonal to both $\mathbf{n}_1 = [1, 2, 0]^T$ and $\mathbf{n}_2 = [1, 1, -1]^T$, and that solves $\mathbf{Ax} = \mathbf{b}$. So the two orthogonality constraints lead to $\mathbf{x} = [-2a, a, -a]$. Moreover, the condition $\mathbf{Ax} = \mathbf{b}$ yields $a = -2/3$ and so $\mathbf{x}_{MN} = [4/3, -2/3, 2/3]^T$.

(b) The fact that $\mathbf{Ax} = 0$ admits a non-trivial solution implies that the null space of $\mathbf{A}$ is not trivial. So $\text{rank}(\mathbf{A}) < n$ and the same applies to $\mathbf{A}^T$. This means that we can find a $\mathbf{b}$ which is outside the range of $\mathbf{A}^T$. A simple example is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and $\mathbf{b} = [1, 2]^T$. Clearly $\mathbf{b}$ is outside the range of

$$\mathbf{A}^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

So $\mathbf{A^T y} = \mathbf{b}$ is not solvable.

(c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is full rank. So we just focus on

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

to find the null space. Clearly $\mathbf{A}$ has rank 2 so, the null space has dimension 3 and a possible basis is: $\text{null}(\mathbf{A}) = \text{span}([1,0,0,0,0]^T, [1,2,-1,0,0]^T, [1,0,-1,2,-1]^T)$. A basis for the range space of $\mathbf{A}$ is given by two linearly independent columns of $\mathbf{A}$. So $\text{range}(\mathbf{A}) = \text{span}([1,1,0]^T, [3,4,1]^T)$. The range space of $\mathbf{A}^T$ is 2 and a possible basis is given by the first and third column of $\mathbf{A}^T$. Finally, the null space of $\mathbf{A}^T$ has dimension one with basis $[1,-1,1]^T$.

(d) The solution provided $\mathbf{x}_1 = [1,2,3]^T$ is not orthogonal to $\mathbf{n}$ so it is not the minimun norm solution. We find that solution by imposing that $\mathbf{x}_2$ be orthogonal to $\mathbf{n}$ and that simultaneously solve $\mathbf{A}\mathbf{x}_2 = \mathbf{b}$. This yields $\mathbf{x}_2 = [1, 14/5, 7/5]^T$.

(e) i. It is enough to impose that $\mathbf{z}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$. So $\mathbf{z} = [1,-1,1]^T$.

ii. Any vector on the plane is orthogonal to $\mathbf{z}$ so it is sufficient to find a $\mathbf{u}$ such that $\langle \mathbf{u}, \mathbf{z} \rangle \neq 0$. This leads to, for example, $\mathbf{u} = [1,1,1]^T$.

(f) We know that $\mathbf{S}$ is symmetric, this means that $\mathbf{S} = \mathbf{S}^T$. Moreover $\mathbf{S}$ is orthogonal. This means that $\mathbf{S}^{-1} = \mathbf{S}^T$. We have that $\mathbf{S}^{-1}\mathbf{S} = \mathbf{I}$ and, due to orthogonality, we then have $\mathbf{S}^T\mathbf{S} = \mathbf{I}$. We now use symmetry to be able to say that $\mathbf{S}\mathbf{S} = \mathbf{I}$, which leads to $\mathbf{S}^2 = \mathbf{I}$.

2. (a) i. The matrix $\mathbf{A}$ has rank one, so the column space has dimension one and a possible basis is $\mathbf{c} = [1, 4]^T$. Consequently,

$$\mathbf{P}_C = \mathbf{c}(\mathbf{c}^T\mathbf{c})^{-1}\mathbf{c}^T = \frac{1}{17}\begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}.$$

ii. Similarly, the row space has dimension one and a possible basis is $\mathbf{r} = [1, 3, 1]^T$. Consequently,

$$\mathbf{P}_R = \mathbf{r}(\mathbf{r}^T\mathbf{r})^{-1}\mathbf{r}^T = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

iii. If $\hat{\mathbf{x}}$ is the projection then, due to the orthogonality principle, the error vector $\mathbf{x} - \hat{\mathbf{x}}$ must be orthogonal to $\mathbf{r}$, namely, $\langle \mathbf{x} - \hat{\mathbf{x}}, \mathbf{r} \rangle = 0$ and it is easy to see that in this case the $\langle \mathbf{x} - \hat{\mathbf{x}}, \mathbf{r} \rangle$ indeed equals zero and so $\hat{\mathbf{x}}$ is the orthogonal projection.

(b) i. In this case the range space has dimension one, but $\mathbf{b} \in \mathbb{R}^2$ so we cannot guarantee that we always have a solution. For example $\mathbf{b} = [1, 1]^T$ does not have a solution.

ii. In this case $\mathbf{b}$ is first projected in the column space of $\mathbf{A}$ and so the problem always has a solution. Given that our $\mathbf{A}$ is rank deficient we always have an infinite number of solutions.

iii. $\mathbf{P}_C\mathbf{b} = [1, 4]^T/17$. So a possible solution is $\mathbf{x} = [1/17, 0, 0]^T$.

iv. This is a projection matrix since $(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - 2\mathbf{P} + \mathbf{P}^2) = \mathbf{I} - \mathbf{P}$, where in the last equality we used the fact that $\mathbf{P}^2 = \mathbf{P}$ since $\mathbf{P}$ is a projection. Moreover, $(\mathbf{I} - \mathbf{P})^T = \mathbf{I}^T - \mathbf{P}^T = \mathbf{I} - \mathbf{P}$ since $\mathbf{P}^T = \mathbf{P}$. Therefore the matrix $\mathbf{I} - \mathbf{P}$ is idempotent and self-adjoint.

3. (a) i. We apply $L$ to $1, x$ and $x^2$.

$$L(1) = -1$$
$$L(x) = 1 - x$$
$$L(x^2) = 2x - x^2$$

The first corresponds to the vector $[-1, 0, 0]^T$ with respect to the given basis, the second corresponds to the vector $[1, -1, 0]^T$ and the third to $[0, 2, -1]^T$ Thus the matrix representing $L$ with respect to the given basis is:

$$\mathbf{L} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

ii. To find a solution to the differential equation we need to find a $\mathbf{f}$ such that

$$\mathbf{Lf} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

The inverse of $\mathbf{L}$ is given by:

$$\mathbf{L^{-1}} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

So $\mathbf{f} = [-6, -3, -2]^T$ and the solution is $f(x) = -56 - 3x - 2x^2$.

(b) i. The vectors span a three dimensional space and a possible selection of basis vectors is $[0, 1, 0, 0]^T$,$[1, 0, 1, 0]^T$,$[2, 0, 0, 1]^T$

ii. The vectors cover a three dimensional space and the basis vectors are $[-1, 1, 0, 0]^T$, $[0, 0, 1, 0]^T$, $[0, 0, 0, 1]^T$.

(c) i. We need all entries to be non-negative and not greater than one and the columns to sum up to one. Consequently $a = 0.5$ and $b = 0$ So:

$$\mathbf{Q} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.5 & 0.7 & 0.5 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

ii. The steady state corresponds to the eigenvector related to the eigenvalue $\lambda = 1$. In this case, $\mathbf{u}_1 = [1, 5/3, 0]$.

iii. At the steady state, the first town will have reached a population of 30,000 habitant, the second town 50,000 and the last one will have none left.

(d) The matrix has two non-zero singular values. The easiest way to find them is to compute the eigenvalues of

$$\mathbf{AA}^T = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

We then have $\sigma_1 = \sqrt{5}$ and $\sigma_2 = \sqrt{2}$.