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Digital Signal Processing
FIR Digital Filter Design using Windowing

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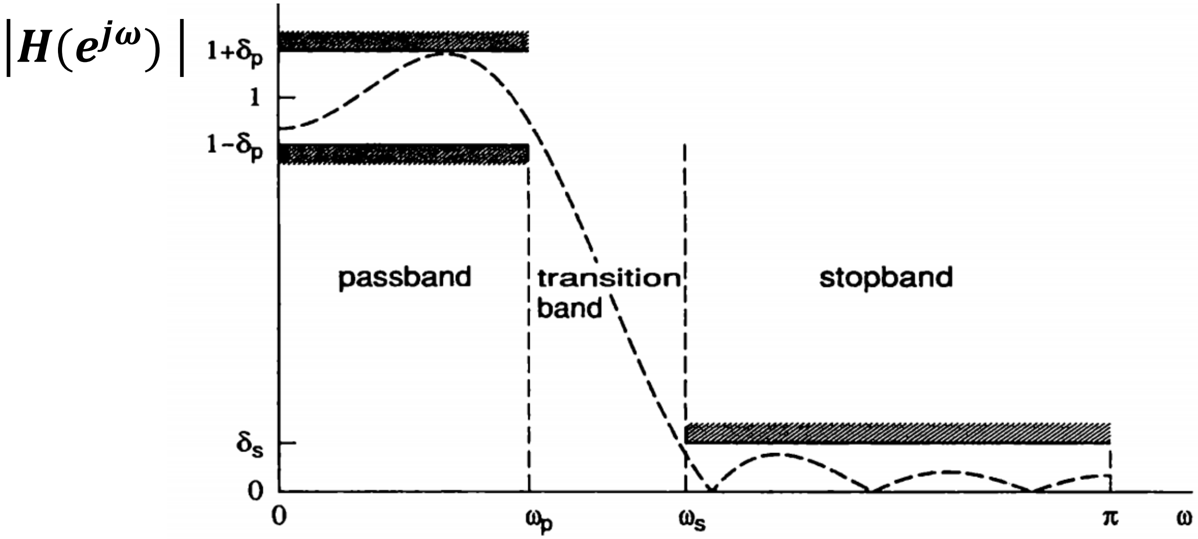
Filter design

- To design a digital filter we must specify the **desired magnitude and/or the phase response**.
- In most practical applications, the problem is the development of a **realizable approximation** to a **given ideal** magnitude response specification.
- **In this presentation we restrict our attention to the magnitude approximation problem only.**
- The phase response can be corrected by cascading our system with an allpass section.

Typical magnitude specification for a digital lowpass filter

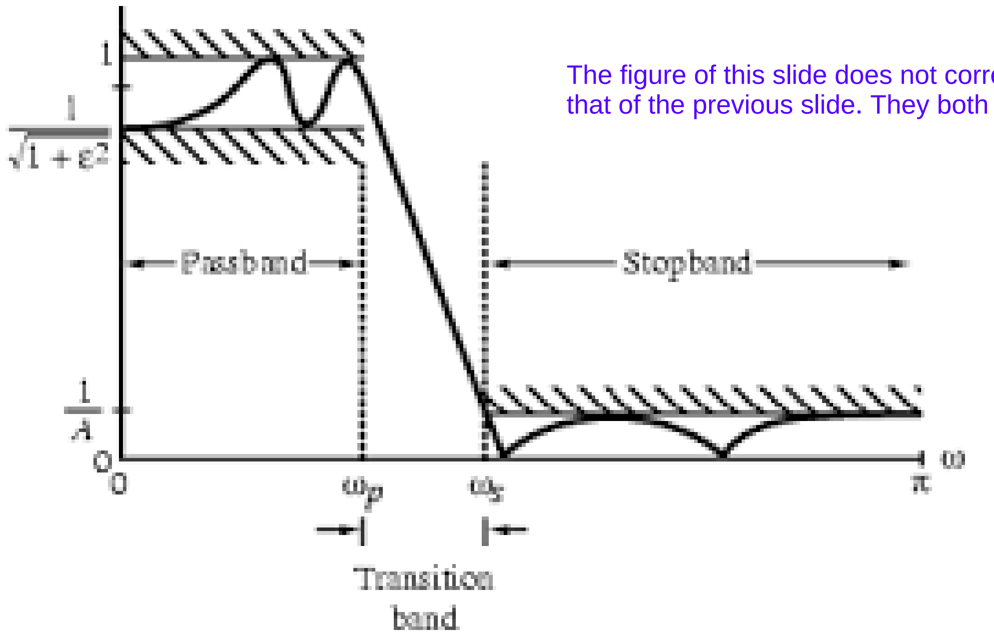
- The figure below shows an approximation to an ideal lowpass filter.
  - Passband: $0 \le \omega \le \omega_p$ (ω_p: passband edge frequency)
  - Stopband: $\omega_s \le \omega \le \pi$ (ω_s: stopband edge frequency)
  - δ_p, δ_s: peak ripple values
  - Cutoff frequency: $\omega_c = (\omega_p + \omega_s)/2$
- The loss function is defined as: $A(\omega) = -20\log_{10}\left|H\left(e^{j\omega}\right)\right|\, dB$

  The loss function is the negative of the gain in dB.
  - Peak passband ripple: $a_p = -20\log_{10}\left(1-\delta_p\right)\, dB \Rightarrow \delta_p = 1 - 10^{-a_p/20}$
  - Minimum stopband attenuation: $a_s = -20\log_{10}\delta_s\, dB \Rightarrow \delta_s = 10^{-a_s/20}$



Normalized magnitude specification for a digital lowpass filter

- The magnitude response specifications of a digital lowpass filter may alternatively be given in a normalized form.
  - Maximum passband attenuation: $a_{\max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right)\ dB$
  - For $\delta_p \ll 1$ (δ_p is defined in previous slide), it can be shown that $a_{\max} \cong -20\log_{10}(1-2\delta_p)\ dB \cong 2a_p\ dB$



The figure of this slide does not correspond to the same function as that of the previous slide. They both depict random generic functions.

FIR Digital Filter Order Estimation

- Several authors have advanced formulae for estimating the minimum order of a digital filter as a function of the filter specifications.
- A rather simple formula developed by Kaiser is given by

$$M \cong \frac{-20log_{10}\left(\sqrt{\delta_p\delta_s}\right) - 13}{14.6(\omega_s - \omega_p)/2\pi}$$

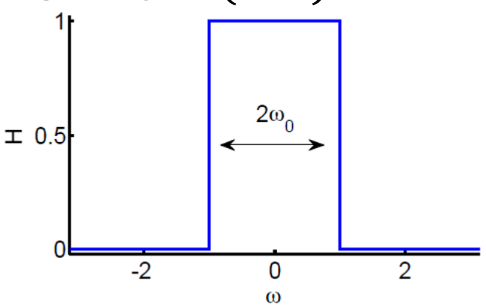
  - Peak passband ripple: $a_p = -20\log_{10}\left(1 - \delta_p\right)\, dB \Rightarrow \delta_p = 1 - 10^{-a_p/20}$
  - Minimum stopband attenuation: $a_s = -20\log_{10}\delta_s\, dB \Rightarrow \delta_s = 10^{-a_s/20}$

Ideal lowpass filter

- For any BIBO stable filter, the frequency response $H(e^{j\omega})$ is the DTFT of the impulse response $h[n]$, as follows:

$$H(e^{j\omega}) = \sum_{-\infty}^{\infty} h[n]e^{-j\omega n} \Leftrightarrow h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n} d\omega$$

- If we know $H(e^{j\omega})$ exactly, the IDTFT gives the ideal $h[n]$.
- **This could be a simple filter design method:** Specify $H(e^{j\omega})$ and then find the impulse response using IDTFT.



- Example: Design an **Ideal Lowpass Filter**.

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases} \Leftrightarrow h[n] = \frac{\sin(\omega_0 n)}{\pi n}$$

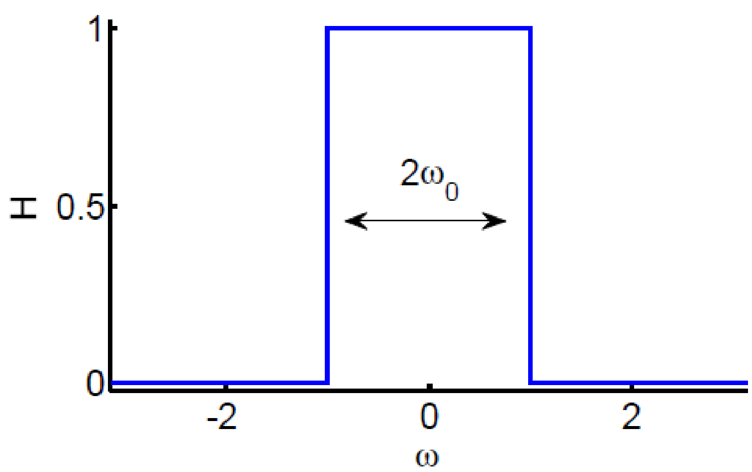
  - If the width of the filter in frequency domain is $2\omega_0$, the **width (of the main lobe) in time** is $\frac{2\pi}{\omega_0}$. Their product is always 4π.
  - Sadly, the above system is always **non-causal** and of **infinite duration**; to overcome this problem we can multiply $h[n]$ by a window.

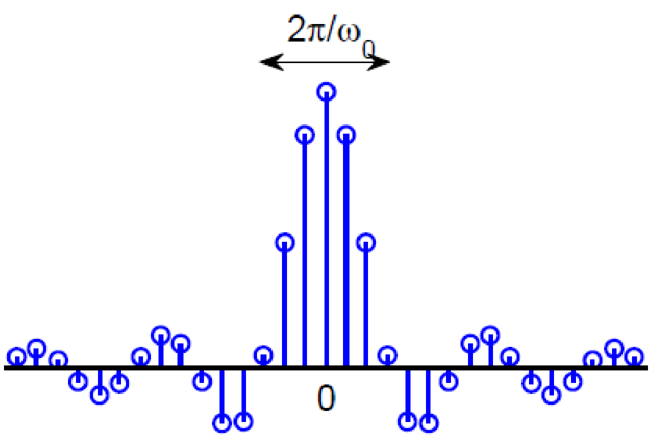
Ideal lowpass filter cont.

- For the given ideal low pass filter shown in the figure we find

$$H\left(e^{j\omega}\right) = \sum_{-\infty}^{\infty} h[n]e^{-j\omega n} \Leftrightarrow h[n] = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} 1 \cdot e^{j\omega n} d\omega = \frac{\sin(\omega_0 n)}{\pi n}$$

- We observe that $h[n]$ is always **non-causal** and of **infinite duration**; to overcome this problem we can multiply $h[n]$ by a window and shift it to the right.





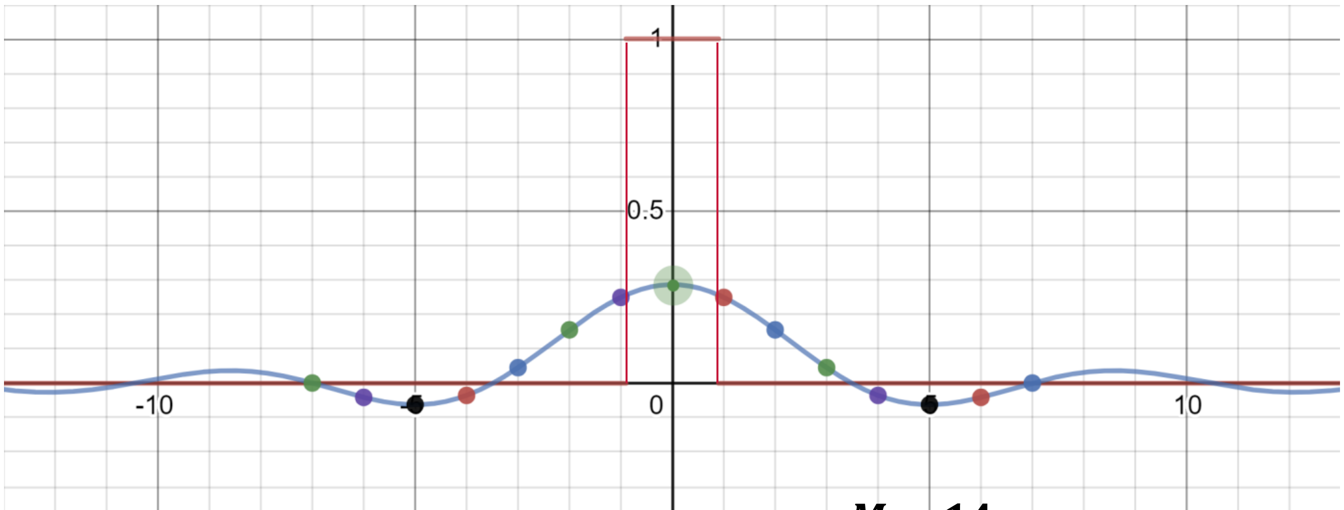
Truncated impulse response

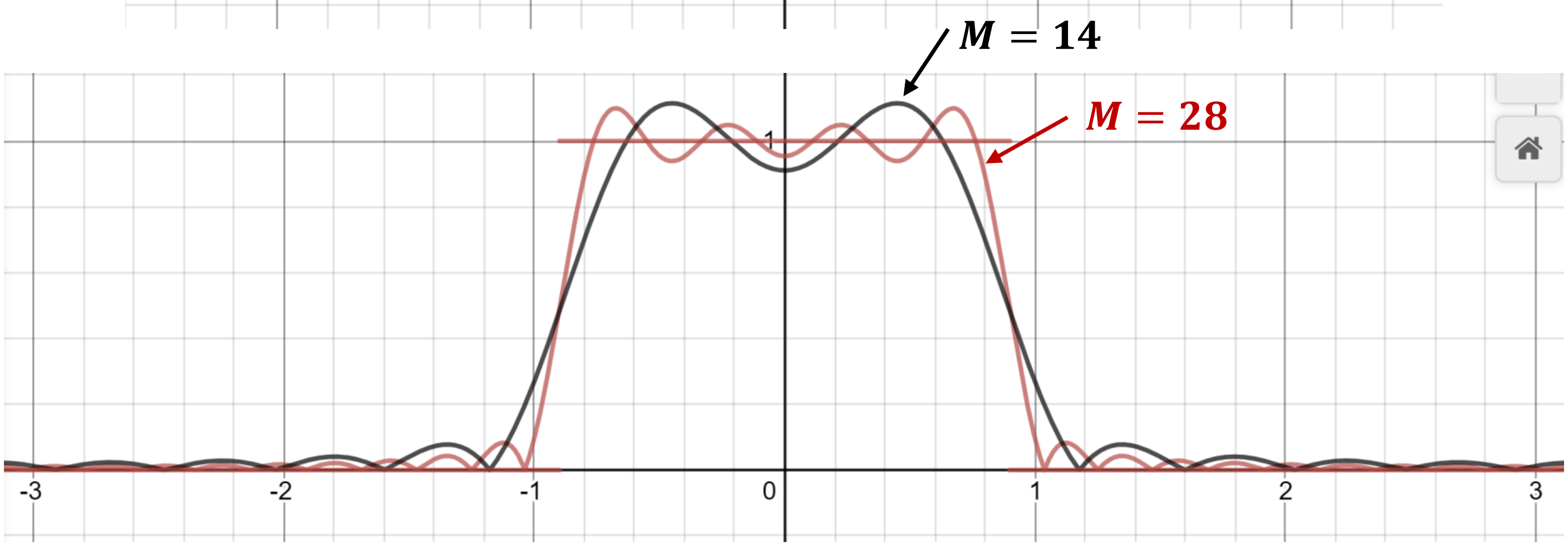
- We truncate the impulse response to $\pm\frac{M}{2}$ to make it of finite duration. Suppose we find an optimal impulse response $h_1[n]$ of length $M+1$.
- The Mean Squared Error (MSE) in the frequency domain is the average of the error due to the truncation;

  $E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\omega}\right) - H_1\left(e^{j\omega}\right)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\omega}\right) - \sum_{-M/2}^{M/2} h_1[n]e^{-j\omega n}\right|^2 d\omega$
- The error E is minimum when $h_1[n] = h[n], -\frac{M}{2} \le n \le \frac{M}{2}$.

  Proof
  - From Parseval's theorem: $E = \sum_{-M/2}^{M/2}\left|h[n] - h_1[n]\right|^2 + \sum_{|n|>M/2}\left|h[n]\right|^2$.
  - The component $\sum_{|n|>M/2}\left|h[n]\right|^2$ is always part of the error.
  - The component $E = \sum_{-M/2}^{M/2}\left|h[n] - h_1[n]\right|^2$ is zero when $h_1[n] = h[n]$.
- To make the truncated filter causal we must delay it by $M/2$. This multiplies $H\left(e^{j\omega}\right)$ by $e^{-j\frac{M}{2}\omega}$.

Ideal and truncated filter in frequency and time domain





Truncated impulse response using a rectangular window

- Truncation involves multiplying $h[n]$ by a rectangular window:

$$w_{M+1}[n] = \begin{cases} A & |n| \le M/2 \\ 0 & |n| > M/2 \end{cases}$$

  This process yields the truncated impulse response $h_{M+1}[n] = h[n] \cdot w_{M+1}[n]$. Note that we use a window of size $M+1$ where M is even.

- The DTFT of $w_{M+1}[n]$ is:

  $W_{M+1}(e^{j\omega}) = \sum_{-M/2}^{M/2} Ae^{-j\omega n} = A + \sum_{-M/2}^{-1} Ae^{-j\omega n} + \sum_{1}^{M/2} Ae^{-j\omega n}$

  $= A + \sum_{1}^{M/2} Ae^{j\omega n} + \sum_{1}^{M/2} Ae^{-j\omega n} = A + 2A\sum_{1}^{M/2} \cos\omega n.$

  This can also be written as:

  $W_{M+1}(e^{j\omega}) = \sum_{-M/2}^{M/2} Ae^{-j\omega n} = Ae^{j\frac{M}{2}\omega} \sum_{-M/2}^{M/2} e^{-j\omega n} e^{-j\frac{M}{2}\omega} = Ae^{j\frac{M}{2}\omega}$

  $\sum_{-M/2}^{M/2} e^{-j\omega(n+\frac{M}{2})} = Ae^{j\frac{M}{2}\omega} \sum_{0}^{M} e^{-j\omega r} = Ae^{j\frac{M}{2}\omega} \sum_{0}^{M} e^{-j\omega n} = Ae^{j\frac{M}{2}\omega} \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$

  $= A\frac{e^{j\frac{1}{2}\omega(M+1)} - e^{-j\frac{1}{2}\omega(M+1)}}{e^{j\frac{1}{2}\omega} - e^{-j\frac{1}{2}\omega}} = A\frac{\sin 0.5(M+1)\omega}{\sin 0.5\omega}, \ n + \frac{M}{2} = r$

  The above function is the so-called **Dirichlet Kernel**.

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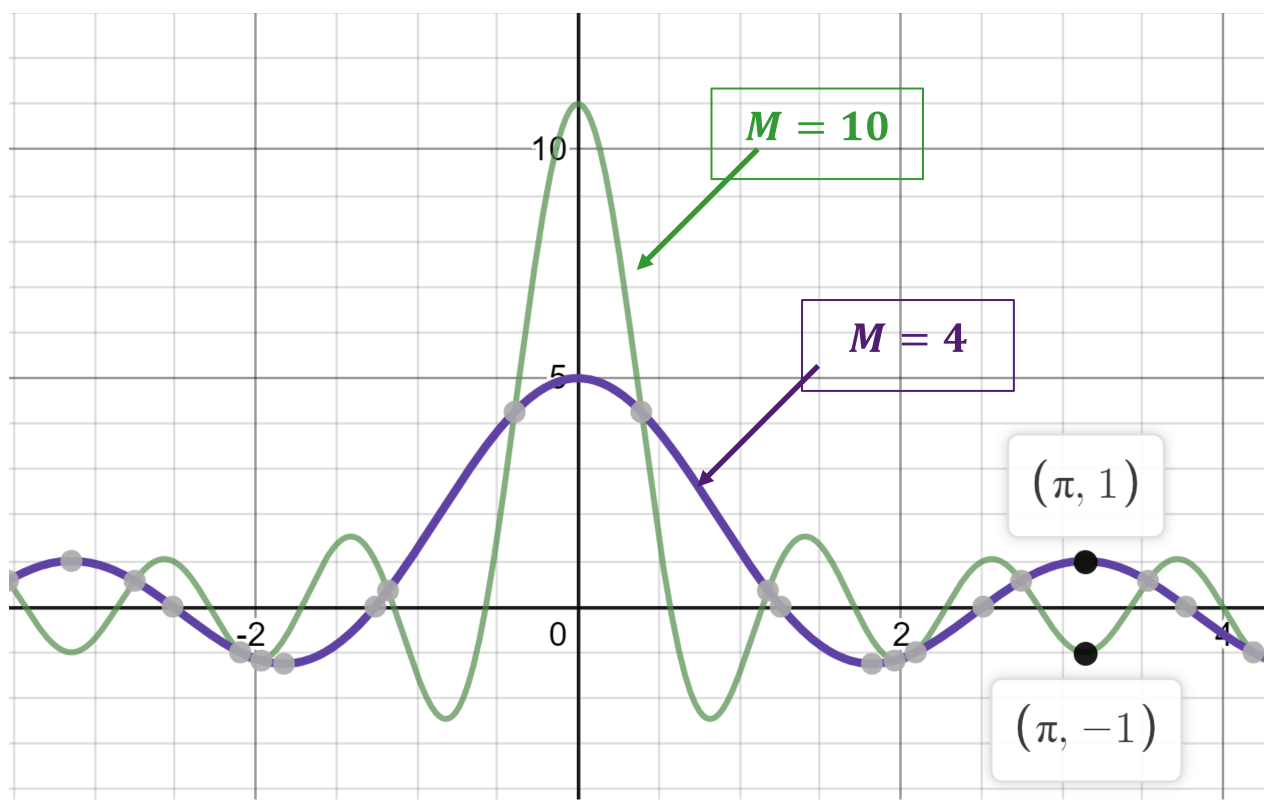
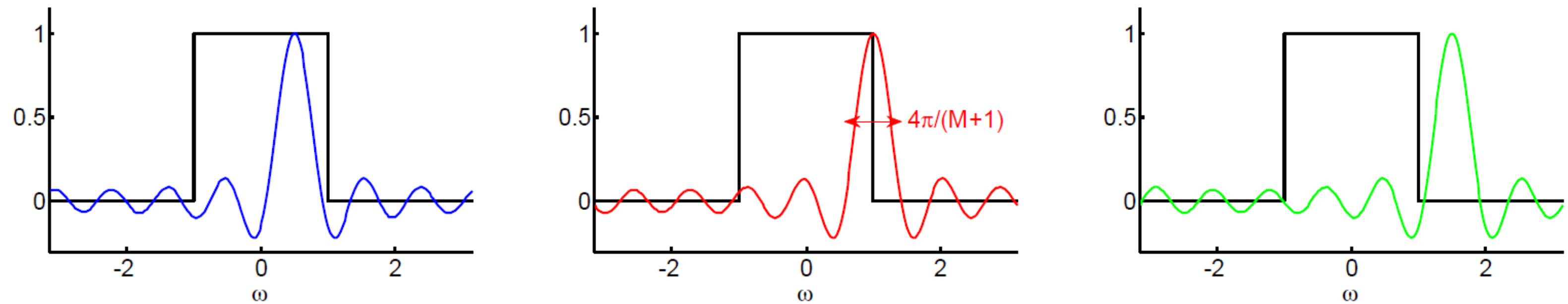
Depiction of Dirichlet kernel for different values of M

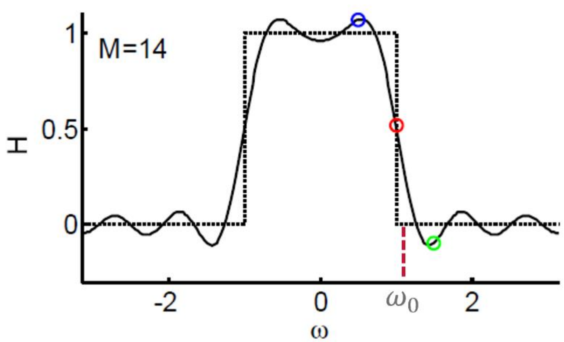


Convolution with Dirichlet kernel for $M = 14$

- Multiplication of discrete signals in time implies circular convolution in DTFT domain:

$$H_{M+1}(e^{j\omega}) = \frac{1}{2\pi} H(e^{j\omega}) \circledast W_{M+1}(e^{j\omega})$$





- The width of the main lobe of $W_{M+1}(e^{j\omega})$ is $\frac{4\pi}{M+1}$
- We want $\frac{4\pi}{M+1} \ll 2\omega_0 \Leftrightarrow M+1 \gg \frac{2\pi}{\omega_0}$
- Pass band ripple (transition from peak to peak):

$$\Delta\omega = \frac{4\pi}{M+1}$$

- Transition gradient: $\left.\frac{dH_{M+1}(e^{j\omega n})}{d\omega}\right|_{\omega=\omega_0} \approx \frac{M+1}{2\pi}$

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Alternative windows: The Hann or Hanning window

- Truncation involves multiplying $h[n]$ by the following **symmetric** window:

$$w_{M+1}[n] = 0.5\left[1 + \cos\left(\frac{2\pi n}{M}\right)\right], |n| \le M/2$$

- The DTFT of $w_{M+1}[n]$ is:

$W_{M+1}(e^{j\omega}) = \sum_{-M/2}^{M/2} w_{M+1}[n]e^{-j\omega n} = w_{M+1}[0] + \sum_{-M/2}^{-1} w_{M+1}[n]e^{-j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$

$= 1 + \sum_{1}^{M/2} w_{M+1}[-n]e^{j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$

$= 1 + \sum_{1}^{M/2} w_{M+1}[n]e^{j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$

$= 1 + 2\sum_{1}^{M/2} 0.5\left[1 + \cos\left(\frac{2\pi n}{M}\right)\right]\cos\omega n = 1 + \sum_{1}^{M/2}\left[1 + \cos\left(\frac{2\pi n}{M}\right)\right]\cos\omega n.$

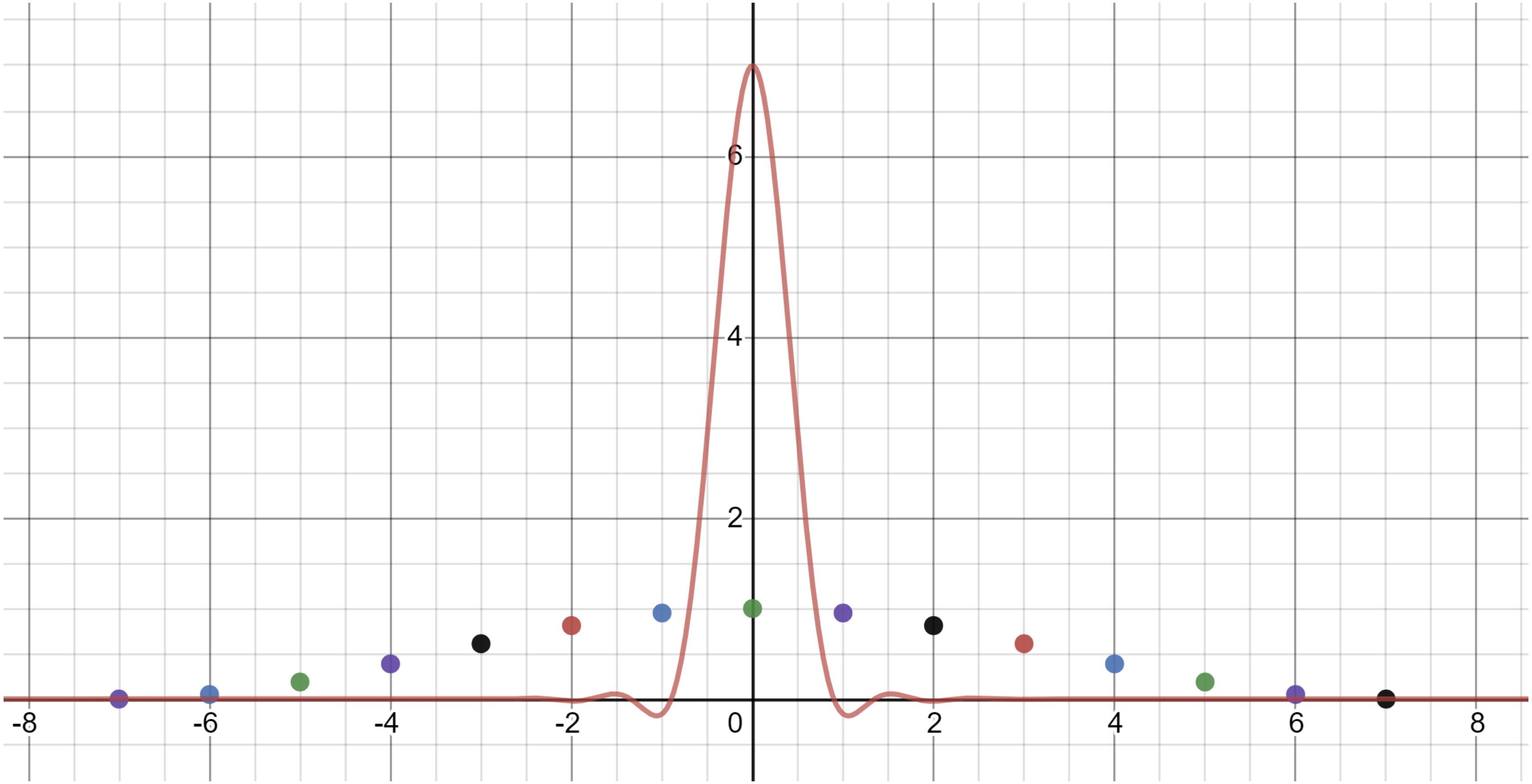
- It is named after the Austrian metereologist Julius von Hann (1839-1921).

- Sometimes it is referred to as Hanning, presumably because researchers confuse it with the Hamming window! **This name is wrong.**

- It is also known as **raised cosine** because it is an elevated cosine function**.**

The Hann or Hanning window

- What you see below is the Hann window with 15 $(M+1=15)$ coefficients (dotted signal) and its DTFT (**red** curve).
- Observe that the end point of the Hann window always touches zero.



The Hamming window

- It is similar to the Hann window. It is defined as follows:

$$w_{M+1}[n] = \frac{25}{46} + \frac{21}{46}\cos\left(\frac{2\pi n}{M}\right), |n| \le \frac{M}{2}$$

$\left(\mathbf{\frac{25}{46} + \frac{21}{46} = 1}\right)$

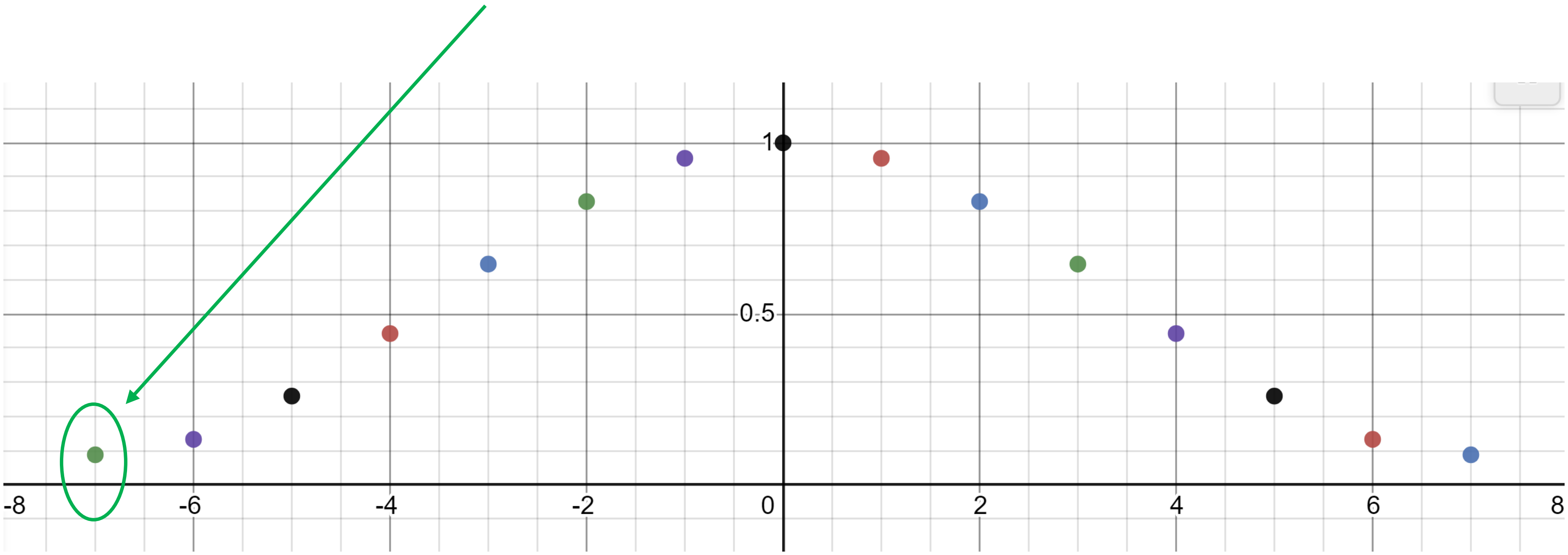
- The DTFT of $w_{M+1}[n]$ is:

$W_{M+1}\left(e^{j\omega}\right) = \sum_{-M/2}^{M/2} w_{M+1}[n]e^{-j\omega n} = w_{M+1}[0] + \sum_{-M/2}^{-1} w_{M+1}[n]e^{-j\omega n} +$
$\sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n} = 1 + \sum_{1}^{M/2} w_{M+1}[-n]e^{j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$
$= 1 + \sum_{1}^{M/2} w_{M+1}[n]e^{j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$
$= 1 + 2\sum_{1}^{M/2}\left[\frac{25}{46} + \frac{21}{46}\cos\left(\frac{2\pi n}{M}\right)\right]\cos\omega n = 1 + \sum_{1}^{M/2}\left[\frac{50}{46} + \frac{42}{46}\cos\left(\frac{2\pi n}{M}\right)\right]\cos\omega n.$

- It is named after the American mathematician Richard Hamming (1915-1998).

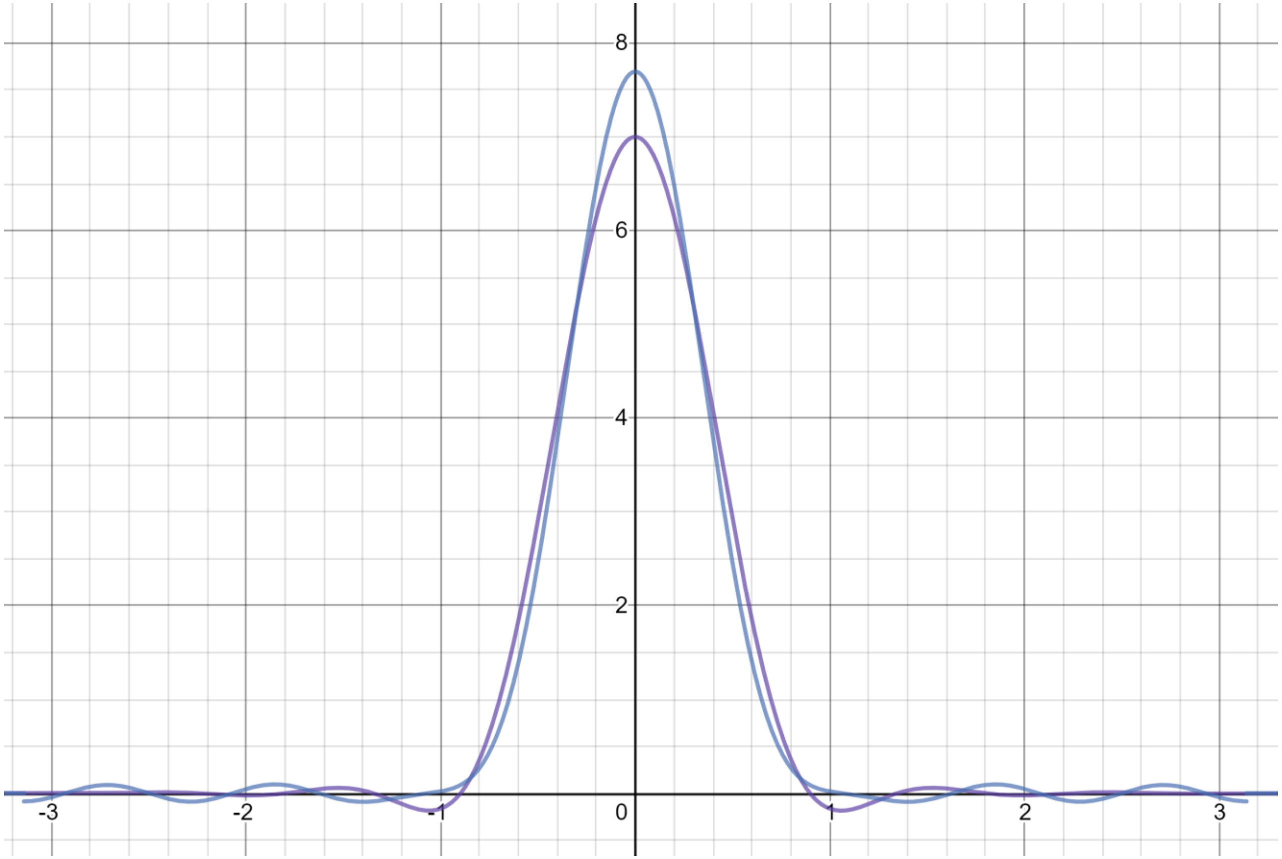
Comparison between Hann and Hamming windows

- What you see below is the Hamming window with 15 coefficients (dotted signal). Observe that, in contrast with the Hann window, the end points of the Hamming window do not touch zero.



Comparison between Hann and Hamming windows

- What you see below is DTFT of the 15-coefficient Hann window (**purple**) and the DTFT of the 15-coefficient Hamming window (**blue**).
- **Observe that the Hamming window "cancels" the first sidelobe of the Hann window.**



The 3-term Blackman window

- This is a popular symmetric window with 3 terms. It is defined as follows:

$$w_{M+1}[n] = 0.42 + 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right), |n| \le M/2$$

$(\mathbf{0.42 + 0.5 + 0.08 = 1})$

- The DTFT of $w_{M+1}[n]$ is:

$W_{M+1}(e^{j\omega}) = \sum_{-M/2}^{M/2} w_{M+1}[n]e^{-j\omega n} = w_{M+1}[0] + \sum_{-M/2}^{-1} w_{M+1}[n]e^{-j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$

$= 1 + \sum_{1}^{M/2} w_{M+1}[-n]e^{j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$

$= 1 + \sum_{1}^{M/2} w_{M+1}[n]e^{j\omega n} + \sum_{1}^{M/2} w_{M+1}[n]e^{-j\omega n}$

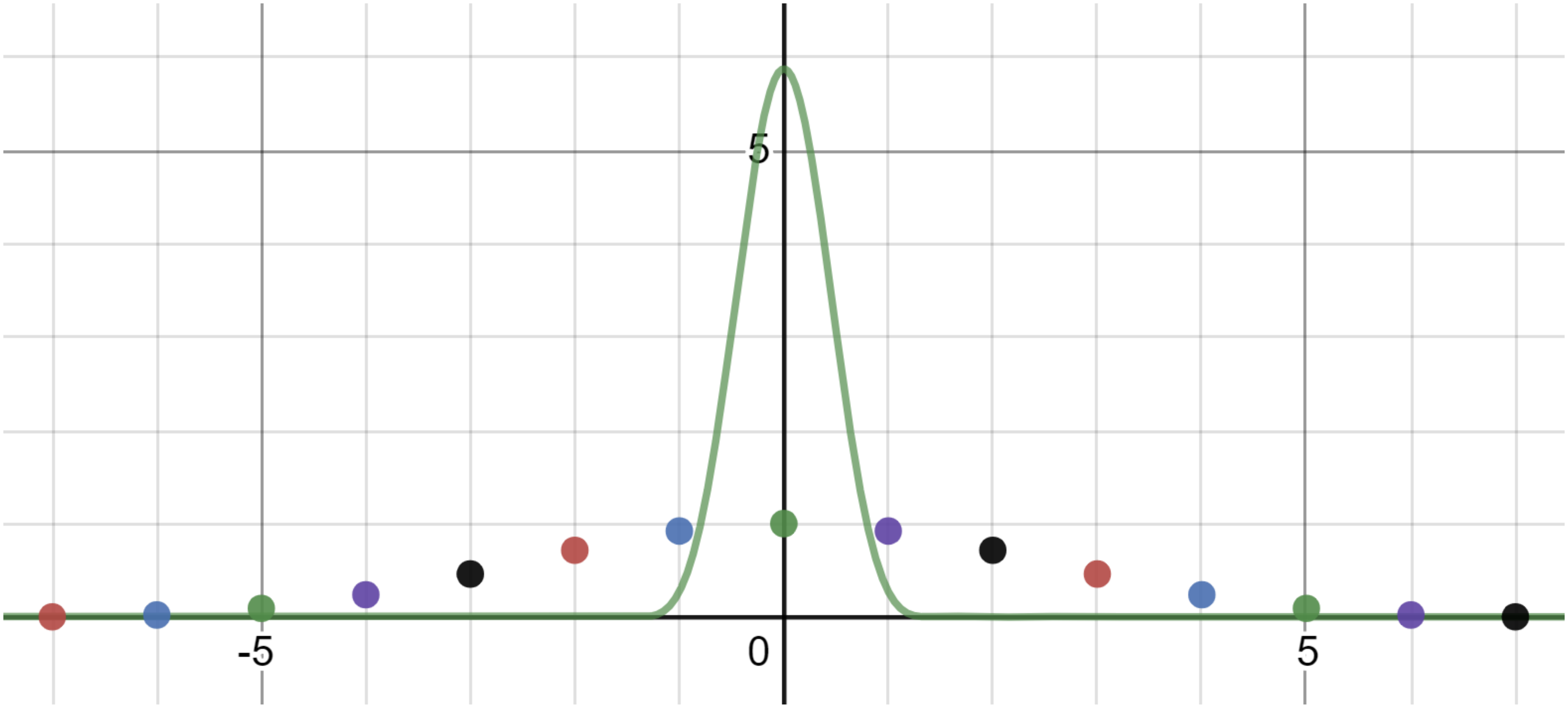
$$= 1 + 2\sum_{1}^{M/2}\left[0.42 + 0.5\cos\left(\frac{2\pi n}{M}\right) + 0.08\cos\left(\frac{4\pi n}{M}\right)\right]\cos\omega n$$

$= 1 + \sum_{1}^{M/2}\left[0.84 + \cos\left(\frac{2\pi n}{M}\right) + 0.16\cos\left(\frac{4\pi n}{M}\right)\right]\cos\omega n.$

- The term Blackman is "unqualified".

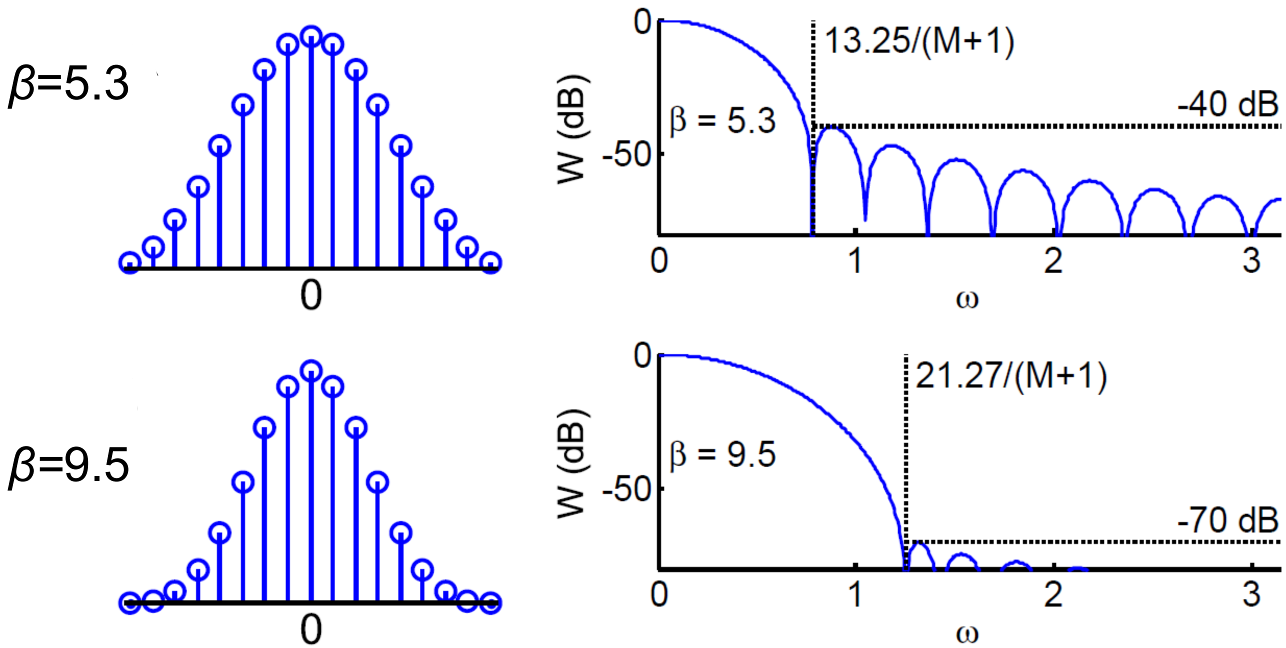
The Blackman window

- What you see below is the Blackman window with 15 coefficients (dotted signal) and its DTFT (**green** curve).



The Kaiser window

- Another popular window defined as $w_{M+1}[n] = \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)}$, $I_0(x) = \sum_{k=0}^{\infty}\left(\frac{(x/2)^k}{k!}\right)^2$
- β is a non-negative real number that determines the shape of the window.
- $I_0(x)$ is the so-called **zeroth order modified Bessel function**.



Comparison of windows

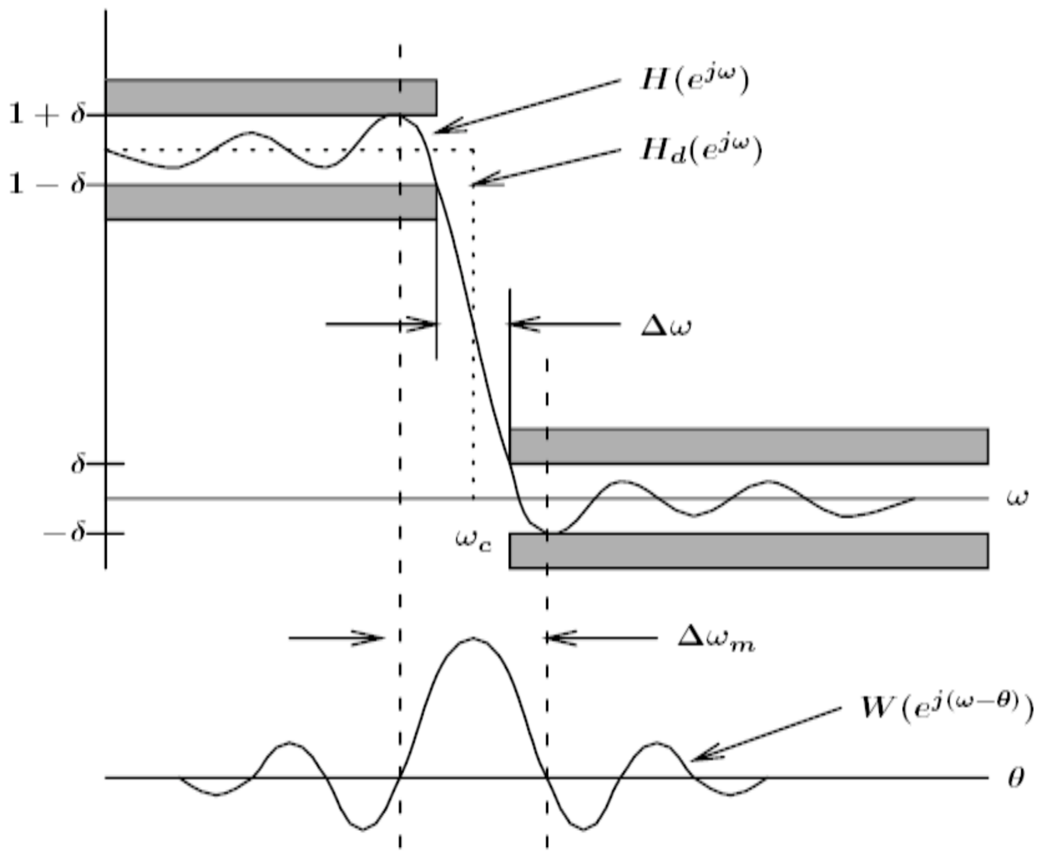
Type of window	Main lobe width	Relative sidelobe level: Difference in dB between the amplitudes of the largest sidelobe and the main lobe	Minimum stopband attenuation $a_s = -20\log_{10}\delta_s\ dB$	Transition bandwidth
Rectangular	$\frac{4\pi}{M+1}$	13.3db	20.9db	$\frac{1.84\pi}{M}$
Hann	$\frac{8\pi}{M}$	31.5db	43.9db	$\frac{6.22\pi}{M}$
Hamming	$\frac{8\pi}{M}$	42.7db	54.5db	$\frac{6.64\pi}{M}$
Blackman	$\frac{12\pi}{M}$	58.1db	75.3db	$\frac{11.12\pi}{M}$

Design exercise

- **Problem:** We want to design a lowpass filter with the following specifications:
  - passband edge $\omega_p = 0.3\pi$ and stopband edge $\omega_s = 0.5\pi$
  - minimum stopband attenuation $a_s = 40dB$.

  Investigate which of the filters presented in this lecture can achieve the above specifications and find their corresponding order.
- **Solution:**
  - From the Table of the previous slide we see that the desired **minimum stopband attenuation** a_s of $40dB$ can be achieved using Hann, Hamming and Blackman windows.
  - Using the last column of the Table we can find the order of the corresponding filter as a function of the **transition bandwidth** $\Delta\omega = \omega_s - \omega_p = 0.2\pi$.
    - Hann: $\frac{6.22\pi}{M} = 0.2\pi \Rightarrow M = 31.1 \cong 32$, Final length $M + 1 = 33$.
    - Hamming: $\frac{6.64\pi}{M} = 0.2\pi \Rightarrow M = 33.2 \cong 34$, Final length $M + 1 = 35$.
    - Blackman: $\frac{11.12\pi}{M} = 0.2\pi \Rightarrow M = 55.6 \cong 56$, Final length $M + 1 = 57$.
  - We choose the Hann since it satisfies the given specifications with the smallest length.
  - **Note that I round to the next EVEN integer and add one.**
  - **Note that due to the fact that the designed filter is the convolution between the DTFT of the window and the DTFT of the ideal filter, the filter specifications are explicitly related to the DTFT of the window employed (look next slide).**

Design exercise



Relation among the frequency response of an ideal lowpass filter, a typical window, and the windowed filter.