Exercises and Solutions

ELEC60008/70089 Control Engineering

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Latest update 02/10/2025

Exercises

Exercise 1 *Consider the matrix*

$$A = \begin{bmatrix} 7 & -6 & 2 \\ 8.8 & -7.6 & 2.8 \\ 9.6 & -7.2 & 2.6 \end{bmatrix}.$$

1. *Compute the characteristic polynomial of A and hence show that the eigenvalues of A are 1, -1 and 2.*

2. *Compute three linearly independent eigenvectors.*

3. *Find a similarity transformation L such that $\hat{A} = L^{-1}AL$ is a diagonal matrix.*

4. *Determine* $\exp(At)$ *as a function of t.*

5. *Determine* $\sin(At)$ *as a function of t.*

Solution 1

1. The characteristic polynomal is

$$\det(sI - A) = s^3 - 2s^2 - s + 2 = (s^2 - 1)(s - 2),$$

hence the claim.

2. The matrix A has three distinct eigenvalues, hence has three linearly independent eigenvectors. These are computed solving the equation

$$Av = \lambda v$$

with $\lambda = 1, -1, 2$. The eigenvector associated to $\lambda = 1$ is $v_1 = [1/2, 1, 3/2]'$; the eigenvector associated to $\lambda = -1$ is $v_2 = [1, 4/3, 0]'$; the eigenvector associated to $\lambda = 2$ is $v_3 = [1, 3/2, 2]'$. Note that, as suggested by the theory,

$$\det([v_1, v_2, v_3]) \neq 0,$$

hence the three vectors are linearly independent.

3. Let $V = [v_1, v_2, v_3]$ and
$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$
Note that
$$AV = V\hat{A},$$
hence, $L = V$ is such that $L^{-1}AL$ is a diagonal matrix.

4. By a property of the matrix exponential
$$e^{At} = e^{(V\hat{A}V^{-1})t} = Ve^{\hat{A}t}V^{-1},$$
where
$$e^{\hat{A}t} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}.$$

5. Similarly to the definition of the matrix exponential we have
$$\sin(At) = At - \frac{(At)^3}{3!} + \frac{(At)^5}{5!} - \cdots,$$
hence
$$\sin(At) = V\sin(\hat{A}t)V^{-1},$$
with
$$\sin(\hat{A}t) = \begin{bmatrix} \sin t & 0 & 0 \\ 0 & -\sin t & 0 \\ 0 & 0 & \sin 2t \end{bmatrix}.$$

Exercise 2 *Let A be a square matrix of dimension n and let $\lambda_1, \cdots \lambda_n$ denote its eigenvalues. Assume for simplicity that A has n distinct eigenvalues.*

1. *Show that the determinant of A is equal to the product of the eigenvalues of A:*
$$\det(A) = \lambda_1\,\lambda_2\,\cdots\,\lambda_n.$$

2. *Show that the trace of A is equal to the sum of the eigenvalues of A:*
$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$
($\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.)

3. *Show that if $\mathrm{tr}(A) > 0$, the system $\dot{x} = Ax$ is unstable. Is the converse true?*

4. *Show that if the system $\dot{x} = Ax$ is asymptotically stable then $\mathrm{tr}(A) < 0$.*

5. *Assume $n = 3$, $\mathrm{tr}(A) = T$ and $\det(A) = \omega^2 T$. Assume moreover that A has two purely imaginary eigenvalues. Discuss the stability of the system $\dot{x} = Ax$ as a function of T and ω and explain how to compute exp(At) as a function of t if the eigenvectors of A are known.*

Solution 2 If A has n distinct eigenvalues then there exists a nonsingular matrix L such that

$$L^{-1}AL = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \tilde{A}$$

for some $\lambda_i \in \mathbb{C}$.

1. Note that
$$\det(A) = \det(L)\ \det(L^{-1})\ \det(\tilde{A}) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

2. The characteristic polynomial of A is given by
$$\det(sI - A) = (s - a_{11})(s - a_{22}) \cdots (s - a_n) + \cdots = s^n + \alpha_{n-1}s^{n-1} + \cdots$$
with
$$\alpha_{n-1} = -a_{11} - a_{22} - \cdots - a_{nn} = -\mathrm{tr}(A).$$
Moreover
$$\det(sI - A) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n) = s^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)s^{n-1} + \cdots .$$
Hence
$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

3. If $\mathrm{tr}(A) > 0$ then there exists a λ_i with positive real part. Hence the system $\dot{x} = Ax$ is unstable. Instability of the system $\dot{x} = Ax$ does not imply $\mathrm{tr}(A) > 0$, as the sum of the real parts of the eigenvalues may be negative even if some of the real parts are positive.

4. If the system $\dot{x} = Ax$ is asymptotically stable then the real parts of all eigenvalues are negative, hence $\mathrm{tr}(A) < 0$.

5. Let $n = 3$ and, as stated,
$$\lambda_1 = -i\omega \qquad \lambda_2 = i\omega \qquad \lambda_3 = \alpha,$$
then
$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 = T \qquad \det(A) = \lambda_1\lambda_2\lambda_3 = \omega^2 T.$$

As a result, $\alpha = T$. Hence the system $\dot{x} = Ax$ is stable, not asymptotically if $T \leq 0$ and it is unstable if $T > 0$. Finally let v_1, v_2 and v_3 be the eigenvectors of A corresponding to the eigenvalues λ_1, λ_2 and λ_3. Hence

$$AV = A \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} -i\omega & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & \alpha \end{bmatrix} = V\Lambda.$$

As a result $A = V\Lambda V^{-1}$, hence

$$e^{At} = Ve^{\Lambda t}V^{-1}$$

with

$$e^{\Lambda t} = \begin{bmatrix} e^{-i\omega t} & 0 & 0 \\ 0 & e^{i\omega t} & 0 \\ 0 & 0 & e^{\alpha t} \end{bmatrix}.$$

Exercise 3 *Consider the discrete-time system $x_{k+1} = Ax_k$.*

1. *Let*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Consider the initial state

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and plot $x(k)$ on the state space for $k = 1, 2, 3, 4$. Exploiting the obtained result discuss the stability of the equilibrium $x = 0$.

2. *Let*

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Consider the initial state

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and plot $x(k)$ on the state space for $k = 1, 2, 3, 4$. Exploiting the obtained result discuss the stability of the equilibrium $x = 0$.

Solution 3

1. Note that

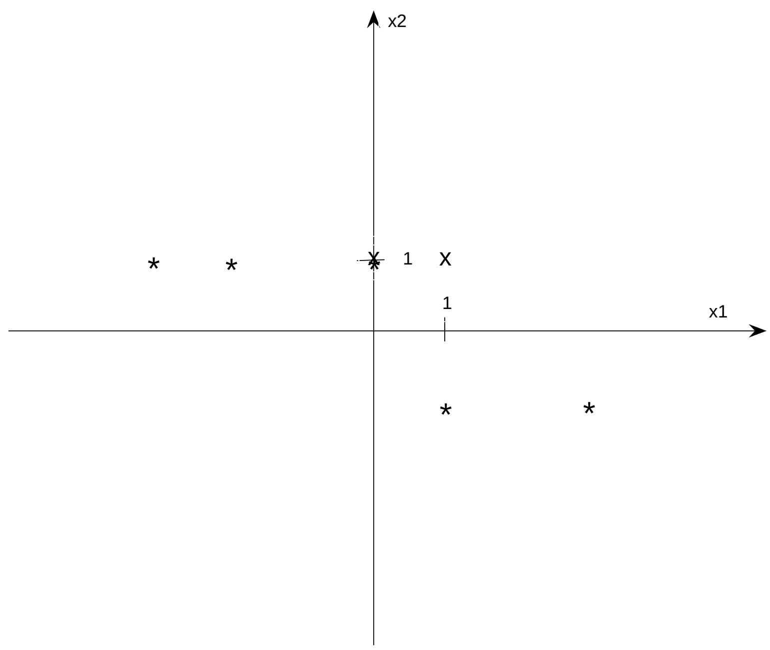
$$x(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad x(2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad x(3) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad x(4) = \begin{bmatrix} -4 \\ 1 \end{bmatrix},$$

and these are indicated in the figure with $\star$ signs. This implies that the equilibrium $x = 0$ is unstable. (Note that to decide instability of an equilibrium it is enough that one trajectory does not satisfy the "$\epsilon - \delta$" argument in the definition of stability.)

2. Note that

$$x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad x(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad x(3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad x(4) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

   and these are indicated in the figure with X signs. This trajectory is such that the "$\epsilon - \delta$" argument holds, however we cannot conclude stability of the equilibrium $x = 0$ only from properties of one trajectory.



Exercise 4 *Consider the discrete-time system*

$$x_{k+1} = Ax_k = \begin{bmatrix} 1 & 1 \\ a & -1 \end{bmatrix} x_k$$

with $a \in \mathbb{R}$.

1. *Show that the system is asymptotically stable for all* $a \in (-2, 0)$ *and it is unstable for* $a < -2$ *and* $a > 0$.

2. Let $a = 0$. Discuss the stability properties of the system.

3. Let $a = -2$. Discuss the stability properties of the system.

Solution 4 The characteristic polynomial of the matrix A is

$$\det(sI - A) = (s - 1)(s + 1) - a = s^2 - 1 - a.$$

Hence the eigenvalues of A are

$$\lambda_1 = +\sqrt{1 + a} \qquad \lambda_2 = -\sqrt{1 + a}.$$

1. The system is asymptotically stable if (and only if)

$$|\lambda_1| < 1 \qquad |\lambda_2| < 1.$$

Observe that λ_1 and λ_2 are real if $a \geq -1$ and are imaginary if $a < -1$. Moreover, $|\lambda_1| = |\lambda_2|$ and this is smaller than one if (and only if) $a \in (-2, 0)$.

2. If $a = 0$ the eigenvalues are real and have modulo equal to one. Moreover they are distinct, hence they have algebraic multiplicity and multiplicity of the minimal polynomial equal to one. Therefore the system is stable, not asymptotically.

3. If $a = -2$ the eigenvalues are imaginary and have modulo equal to one. Moreover they are distinct, hence they have algebraic multiplicity and multiplicity of the minimal polynomial equal to one. Therefore the system is stable, not asymptotically.

Exercise 5 *Consider two systems $\sigma x_i = A_i x_i + B_i u$, $y_i = C_i x_i$, with $i = 1, 2$ and*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Note that the systems have the same input.

1. Show that if $x_1(0) = x_2(0) = 0$ then $y_1(t) = y_2(t)$ for all t. (This is equivalent to $C_1 A_1^k B_1 = C_2 A_2^k B_2$ for all $k \geq 0$, in discrete-time and to $C_1 e^{A_1 t} B_1 = C_2 e^{A_2 t} B_2$ for all $t \geq 0$, in continuous-time.)

2. Show that the two systems are not algebraically equivalent.

Solution 5

1. If $x_1(0) = x_2(0) = 0$ then (suppose the systems are discrete-time)

$$y_1(k) = \sum_{i=0}^{k-1} C_1 A_1^{k-1-i} B_1 u(i) \qquad\qquad y_2(k) = \sum_{i=0}^{k-1} C_2 A_2^{k-1-i} B_2 u(i)$$

   Note that $A_1 = A_2'$, $B_1 = C_2'$ and $C_1 = B_2'$. Hence

$$C_1 A_1^k B_1 = (C_1 A_1^k B_1)' = B_1'(A_1')^k C_1' = C_2 A_2^k B_2$$

   which proves that the output responses of the two systems coincide.

2. The systems are algebraically equivalent if and only if there exists a nonsingular L such that

$$A_1 = L^{-1} A_2 L \quad (\text{or } LA_1 = A_2 L) \qquad B_1 = L^{-1} B_2 \quad (\text{or } LB_1 = B_2) \qquad C_1 = C_2 L.$$

   Setting

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}$$

   yields

$$LB_1 = \begin{bmatrix} L_1 \\ L_3 \end{bmatrix} = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} = C_2 L = \begin{bmatrix} L_1 & L_2 \end{bmatrix},$$

   hence $L_1 = L_2 = L_3 = 1$. Finally, for any L_4,

$$LA_1 = \begin{bmatrix} 1 & 3 \\ 1 & 1 + 2L_4 \end{bmatrix} \neq A_2 L = \begin{bmatrix} 1 & 1 \\ 1 & 3 + 2L_4 \end{bmatrix},$$

   which shows that the two systems are not algebraically equivalent.
   It is worth noting that the system with state x_1 is not reachable but observable, whereas the system with state x_2 is reachable but not observable (it is the dual of the system with state x_1). Finally, if two systems are reachable and observable and the forced responses of the output coincide then the systems are algebraically equivalent.

Exercise 6 *The function $y = x - \log x$ has a minimum for $x = 1$. This minimum can be computed using Newton's algorithm which yields the discrete-time system*

$$x_{k+1} = 2x_k - x_k^2.$$

1. *Compute the equilibrium points of this system.*

2. *Study the stability properties of the equilibrium points.*

3. *Show that*

   (a) *if* $x_0 = 2$ *or* $x_0 = 0$ *then* $x_k = 0$*, for all* $k \geq 1$*;*

   (b) *if* $x_0 > 2$ *then* $x_{k+1} < x_k < 0$*, for all* $k \geq 1$*;*

   (c) *if* $x_0 \in (0, 2)$ *then* $x_k \in (0, 1)$*, for all* $k \geq 1$*, and* $\lim_{k \to \infty} x_k = 1$*.*

Solution 6

1. At an equilibrium point of a discrete-time system we have $x_{k+1} = x_k$. Hence, the equilibrium points of the system are the solutions of the equation

$$x = 2x - x^2.$$

   We have therefore two equilibria

$$x = 0 \qquad\qquad x = 1.$$

2. The linearization of the system around an equilibrium point x_{eq} is described by

$$(\delta_x)_{k+1} = (2 - 2x_{eq})(\delta_x)_k.$$

   Hence, for $x_{eq} = 0$ we have $(\delta_x)_{k+1} = 2(\delta_x)_k$, which is an unstable system, while for $x_{eq} = 1$ we have $(\delta_x)_{k+1} = 0$, which is a stable system. As a result, by the principle of stability in the first approximation, the equilibrium point $x = 0$ is unstable and the equilibrium point $x = 1$ is locally asymptotically stable.

3. If $x_0 = 2$, then $x_1 = 0$, and by definition of equilibrium $x_k = 0$ for all $k > 1$. If $x_0 = 0$ then, again by definition of equilibrium, $x_k = 0$, for all $k > 0$.
   Note now that the relation

$$x_{k+1} = 2x_k - x_k^2,$$

   implies $x_{k+1} < x_k$ if and only if $x_k < 0$ or $x_k > 1$.
   If $x_0 > 2$ then $x_1 < 0$, hence $x_2 < x_1 < 0$ and, in general,

$$x_{k+1} < x_k < \cdots < x_2 < x_1 < 0.$$

   This shows that, for any $x_0 > 2$ the sequence $\{x_k\}$ diverges, i.e. $\lim_{k \to \infty} x_k = -\infty$.
   If $x_0 \in (0, 2)$ then $x_1 \in (0, 1)$. Moreover, $x_1 \in (0, 1)$ implies that $x_2 \in (0, 1)$, and so on. Therefore, if $x_0 \in (0, 2)$ then $x_k \in (0, 1)$ for all $k > 0$. Note now that if $x_k \in (0, 1)$ then $x_{k+1} > x_k$. As a result, for any $x_0 \in (0, 2)$ we have

$$0 < x_1 < x_2 < \cdots < x_k < x_{k+1} < 1,$$

   which states that the sequence $\{x_k\}$ is monotonically increasing and bounded from above, hence should have a limit. However, any limit of the sequence has to be an equilibrium point of the system, which implies that, for any $x_0 \in (0, 2)$ we have that $\lim_{k \to \infty} x_k = 1$.

Exercise 7 *Consider the discrete-time system*

$$\begin{aligned} x_{1,k+1} &= \frac{1}{2}x_{1,k} + \alpha(x_{2,k})x_{2,k} \\ x_{2,k+1} &= x_{2,k} - \alpha(x_{2,k})x_{1,k} - r(x_{2,k})x_{2,k} \end{aligned}$$

with

$$r(x_{2,k}) = \frac{2}{3}(\alpha(x_{2,k}))^2,$$

and $\alpha(\cdot)$ a differentiable function.

1. *Show that if*

$$0 < |\alpha(x_{2,k})| < \frac{\sqrt{3}}{2}$$

the origin is a (locally) asymptotically stable equilibrium.

Solution 7 To study the stability of the origin we use the principle of stability in the first approximation.

1. The linearization of the system around the zero equilibrium is described by

$$\delta_x(k+1) = A\delta_x(k) = \begin{bmatrix} \frac{1}{2} & \alpha \\ -\alpha & 1 - \frac{2}{3}\alpha^2 \end{bmatrix} \delta_x(k).$$

The characteristic polynomial of the matrix A is

$$\det(sI - A) = s^2 + s\left(\frac{2}{3}\alpha^2 - \frac{3}{2}\right) + \left(\frac{2}{3}\alpha^2 + \frac{1}{2}\right) = s^2 + s\eta + (\eta + 2)$$

with $\eta = \left(\frac{2}{3}\alpha^2 - \frac{3}{2}\right)$. Note that we are interested in values of α^2 in the set $(0, 3/4)$ and hence of η in the set $(-3/2, -1)$. We need to show that, for $\eta \in (-3/2, -1)$, the roots of the characteristic polynomial above have modulo not larger than one. The simplest way to perform this test, without computing the roots, is to consider the change of variable

$$s \to \frac{1+\xi}{1-\xi},$$

and then apply Routh test to the resulting polynomial. Note in fact that is $|s| < 1$ then the real part of ξ is negative. The polynomial to study is thus

$$p(\xi) = (1+\xi)^2 + (1+\xi)(1-\xi)\eta + (1-\xi)^2(\eta+2) = 3\xi^2 + 2(-1-\eta)\xi + (3+2\eta).$$

By Routh test, this polynomial has all roots with negative real part if and only if $\eta \in (-3/2, -1)$. Hence, the matrix A has all roots with modulo smaller than one if and only if $\alpha^2 \in (0, 3/4)$. By the principle of stability in the first approximation, for all $\alpha^2 \in (0, 3/4)$ the zero equilibrium of the nonlinear system is locally asymptotically stable. Note that, if $\alpha^2 = 0$ or $\alpha^2 = 3/4$, the linearized system is stable, not asymptotically, but we cannot conclude anything on the stability properties of the zero equilibrium of the nonlinear system.

Exercise 8 *Consider the nonlinear model of a bioreactor described by the equations*

$$\begin{aligned} \dot{X} &= \mu(S)X - Xu \\ \dot{S} &= -\mu(S)X + (S_{in} - S)u \\ y &= S + kS^3, \end{aligned}$$

in which $S \geq 0$*,* $X \geq 0$*,* $S_{in} > 1$ *is a constant,* k *is a constant,* u *is an external signal and*

$$\mu(S) = 2\frac{S}{1 + 2S + S^2}.$$

1. *Suppose* u *is constant and determine all equilibrium points of the system in the following cases*
   (a) $u \in (0, 1/2)$*;*
   (b) $u = 1/2$*;*
   (c) $u > 1/2$*.*

   Sketch the position of the equilibrium points on the (X, S)*-plane as a function of* u*.*
2. *Write the linearized model of the system around the equilibrium point with both nonzero components determined in part 1.(b). Study the stability of such linearized system.*

Solution 8

1. The equilibrium points are the solutions of the equations $\dot{X} = 0$ and $\dot{S} = 0$. From the first equation we obtain $X = 0$ or $u = \mu(S)$. The first alternative yields, exploiting the second equation, $S = S_{in}$. The second alternative yields, exploiting again the second equation, $X = S_{in} - S$, where S is the solution of $u = \mu(S)$.
   Note now that the function $\mu(S)$ is always non-negative, it is zero for $S = 0$, it tends to zero as $S \to \infty$, and it has a maximum for $S = 1$ equal to $1/2$. As a result we obtain the following equilibria.
   (a) $u \in (0, 1/2)$. There are three equilibria: $(0, S_{in})$, $(S_{in} - S_1, S_1)$ and $(S_{in} - S_2, S_2)$, where S_1 and S_2 are the two solutions of $u = \mu(S)$ with $u \in (0, 1/2)$.
   (b) $u = 1/2$. There are two equilibria: $(0, S_{in})$ and $(S_{in} - 1, 1)$.
   (c) $u > 1/2$. There is one equilibrium: $(0, S_{in})$.
2. The system linearized around the equilibrium point $(S_{in} - 1, 1)$ is described by (note that $\frac{d\mu}{dS}(1) = 0$)

$$\dot{\delta}_x = A\delta_x + Bu = \begin{bmatrix} 0 & 0 \\ -1/2 & -1/2 \end{bmatrix} \delta_x + \begin{bmatrix} 1 - S_{in} \\ S_{in} - 1 \end{bmatrix} \delta_u$$

$$\delta_y = \begin{bmatrix} 0 & 1 + 3k \end{bmatrix} \delta_x.$$

   The linearized system is stable, not asymptotically, but nothing can be concluded on the stability of the equilibrium of the nonlinear system.

Exercise 9 *The (simplified and normalized) model of a patient in the presence of an infectious disease is described by the equations*

$$\dot{x} = 1 - x - Txy \qquad \dot{y} = Txy - y - ISy,$$

in which x represents the number of non-infected cells, y represents the number of infected cells, T represents the effect of the therapy and $IS \in (0, 1)$ represents the action of the immune system.

1. *Determine the two equilibrium points of the system. Show that one equilibrium corresponds to a healthy patient, i.e. the number of infected cells is zero, and one equilibrium corresponds to an ill patient, i.e. the number of infected cells is non-zero.*
2. *Write the linearized models of the system around the two equilibrium points.*
3. *Without therapy $T > 1 + IS$. Show that the equilibrium point associated to a healthy patient is unstable and the one associated to an ill patient is locally asymptotically stable.*
4. *With therapy $T < 1 + IS$. Show that the equilibrium point associated to a healthy patient is locally asymptotically stable and the one associated to an ill patient is unstable.*

Solution 9

1. The equilibrium points of the system are the solutions of the equations $\dot{x} = 0$ and $\dot{y} = 0$. From the second equation we have $y = 0$ or $x = \frac{1+IS}{T}$. From the equation $\dot{x} = 0$, the former yields $x = 1$, and the latter yields $y = \frac{T-1-IS}{T(1+IS)}$. Hence, the equilibrium $(1, 0)$ corresponds to a healty patient, and the equilibrium $(\frac{1+IS}{T}, \frac{T-1-IS}{T(1+IS)})$ corresponds to an ill patient.
2. The linearized model of the system around the first equilibrium point is

$$\begin{bmatrix} \dot{\delta}_x \\ \dot{\delta}_y \end{bmatrix} = \begin{bmatrix} -1 & -T \\ 0 & T - 1 - IS \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$

   The linearized model of the system around the second equilibrium point is

$$\begin{bmatrix} \dot{\delta}_x \\ \dot{\delta}_y \end{bmatrix} = \begin{bmatrix} -\frac{T}{1+IS} & -1 - IS \\ \frac{T-1-IS}{1+IS} & 0 \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}.$$

3. Suppose $T > 1 + IS$. The system linearized around the equilibrium associated to a healthy patient has an eigenvalue with positive real part, hence the equilibrium is unstable. On the contrary, the system linearized around the equilibrium associated to a ill patient has both eigenvalues with negative real part, hence the equilibrium is (locally) asymptotically stable.

4. Suppose $T < 1 + IS$. The system linearized around the equilibrium associated to a healthy patient has both eigenvalues with negative real part, hence the equilibrium is (locally) asymptotically stable. On the contrary, the system linearized around the equilibrium associated to a ill patient has one eigenvalue with positive real part, hence the equilibrium is unstable.

Exercise 10 *Consider the model of a simple robot arm given by*

$$\ddot{\phi} + m \sin \phi = T,$$

in which ϕ is the angle of the arm with respect to a vertical line directed upward, T is the applied torque, and m is a positive parameter.

1. *Let $T = 0$. Determine the equilibrium points of the system.*
2. *Let $T = 0$. Consider the energy of the system $E = \frac{1}{2}\dot{\phi}^2 - m \cos \phi$. Compute $\dot{E}$ and show that E remains constant for all t.*
3. *Let $T = -k\dot{\phi}$ with $k > 0$. Show that the equilibrium $(\phi, \dot{\phi}) = (0, 0)$ is locally asymptotically stable.*

Solution 10

1. The equilibrium points of the system are such that $\phi(t)$ is constant for all t, hence are such that $\dot{\phi} = \ddot{\phi} = 0$. As a result, the equilibrium points are the solutions of the equation $\sin \phi = 0$. This equation has infinitely many solutions, however, from a physical point of view, we have only two solutions $\phi = 0$ (the arm is directed upward) and $\phi = \pi$ (the arm is directed downward).
2. Note that
$$\dot{E} = \dot{\phi}\ddot{\phi} + m \sin \phi \dot{\phi}$$
and replacing the equation describing the motion of the arm yields
$$\dot{E} = T\dot{\phi}.$$
Hence, if $T = 0$ we have that $\dot{E} = 0$, hence $E(t)$ is constant for all t.
3. Setting $T = -k\dot{\phi}$ yields
$$\ddot{\phi} + k\dot{\phi} + m \sin \phi = 0.$$
To study the stability of the equilibrium $(0, 0)$ we linearize this equation around this point and we obtain
$$\ddot{\delta}_\phi + k\dot{\delta}_\phi + m\delta_\phi = 0.$$
The characteristic polynomial associated with this linear system is (Laplace transform the equation, and factor $\mathcal{L}(\phi(s))$)
$$s^2 + ks + m,$$
which has all roots with negative real part, by Routh test. As a result, the equilibrium $(0, 0)$ of the nonlinear system is locally asymptotically stable.

Exercise 11 *Consider the nonlinear system modelling the dynamics of the angular velocities of a rigid body in space (for example a satellite), described by the equations (known as Euler's equations)*

$$\begin{aligned}\dot{x}_1 &= \frac{I_2 - I_3}{I_1} x_2 x_3 \\ \dot{x}_2 &= \frac{I_3 - I_1}{I_2} x_3 x_1 \\ \dot{x}_3 &= \frac{I_1 - I_2}{I_3} x_1 x_2,\end{aligned}$$

with $I_1 \neq 0$, $I_2 \neq 0$ *and* $I_3 \neq 0$.

1. *Determine the equilibrium points of the system.*
2. *Consider the equilibrium* $\tilde{x} = (\alpha, 0, 0)$ *with* $\alpha \neq 0$. *Compute the linearized model around the equilibrium* $\tilde{x}$, *and study its stability as a function of* I_1, I_2 *and* I_3. *Discuss the stability properties of the equilibrium* $\tilde{x}$ *of the nonlinear system.*

Solution 11

1. The equilibrium points of the system are the solutions of the equations $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$. From the equation $\dot{x}_1 = 0$ we have $x_2 = 0$ or $x_3 = 0$. Replacing $x_2 = 0$ in the second and third equations yields the constraint $x_3 x_1 = 0$. As a result, we have the two families of equilibrium points described by

   $$(0, 0, \star) \qquad\qquad (\star, 0, 0),$$

   where $\star$ is any real number. Finally, from $x_3 = 0$ we have another family of equilibrium points described by

   $$(0, \star, 0).$$

   All the above equilibria describe steady rotations around an axis of symmetry (these are sometimes called, in mechanics, relative equilibria).

2. The system linearized around the given equilibrium is described by

   $$\dot{\delta}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{I_3 - I_1}{I_2}\alpha \\ 0 & \frac{I_1 - I_2}{I_3}\alpha & 0 \end{bmatrix} \delta_x.$$

   As a result, if

   $$\frac{I_3 - I_1}{I_2}\frac{I_1 - I_2}{I_3} < 0$$

   the linearized system is stable, but no conclusions can be drawn on the stability properties of the given equilibrium of the nonlinear system. If

   $$\frac{I_3 - I_1}{I_2}\frac{I_1 - I_2}{I_3} > 0$$

   the linearized system is unstable, hence the given equilibrium of the nonlinear systems is unstable.

Exercise 12 *A continuous-time system described by the equations*

$$\dot{x} = Ax + Bu \qquad y = Cx$$

is passive if there exists a matrix $P = P' > 0$ *such that*

$$A'P + PA \leq 0 \qquad PB = C'.$$

Consider the continuous-time system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

1. *Show that the system is passive.*
2. *Let* $u = -Ky$. *Write the equations of the closed-loop system and show that the system is asymptotically stable for all* $K > 0$.
3. *The zeros of a system with* $x \in I\!R^n$, $u \in I\!R$ *and* $y \in I\!R$, *are the complex numbers* $\bar{s}$ *such that*
$$\text{rank} \begin{bmatrix} \bar{s}I - A & B \\ C & 0 \end{bmatrix} < n + 1.$$
Show that the considered system has a zero at $\bar{s} = 0$.

Finally, show that a SISO passive system does not have zeros $\bar{s}$ *with positive real part.*

Solution 12

1. Let
$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$
and note that
$$PB = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = C'$$
and
$$A'P + PA = 0,$$
hence the system is passive.
2. The closed-loop system is described by the equations
$$\dot{x} = (A - KBC)x = \begin{bmatrix} 0 & 1 \\ -2 & -K \end{bmatrix} x = A_{cl}x.$$
The characteristic polynomial of the matrix A_{cl} is given by
$$\det(sI - A_{cl}) = s^2 + Ks + 2,$$
hence, by Routh test, its roots have negative real part if and only if $K > 0$, hence the closed-loop system is asymptotically stable for all $K > 0$.

3. Note that
$$\begin{bmatrix} \bar{s}I - A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} \bar{s} & -1 & 0 \\ -2 & \bar{s} & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
Setting $\bar{s} = 0$ yields the matrix
$$\begin{bmatrix} 0 & -1 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
The determinant of this matrix is equal to zero, hence the matrix has rank smaller than three, which proves that the system has a zero for $\bar{s} = 0$.
(This definition of zeros for single-input, single-output systems is consistent with the definition based on the transfer function of the system. For example, for the considered system, the transfer function is
$$W(s) = C(sI - A)^{-1}B = \frac{s}{s^2 + 2},$$
hence the system has a zero at zero.)

To prove that a passive, single-input, single-output system does not have zeros with positive real part we proceed as follows. Suppose that the system has a zero at some $\bar{s}$, hence
$$\text{rank} \begin{bmatrix} \bar{s}I - A & B \\ C & 0 \end{bmatrix} < n + 1.$$
This implies that there exist a nonzero (complex) vector v and a scalar w such that
$$(\bar{s}I - A)v + Bw = 0 \qquad\qquad Cv = 0 \ \ (\text{or} \ \ v^\star C' = 0),$$
where $v^\star$ is the adjoint of v. Then, left multiplying by P,
$$0 = (\bar{s}P - PA)v + PBw = (\bar{s}P - PA)v + C'w,$$
hence, left multiplying by $v^\star$,
$$0 = v^\star \big((\bar{s}P - PA)v + C'w\big) = v^\star(\bar{s}P - PA)v.$$
As a result
$$\bar{s}v^\star Pv = v^\star PAv$$
and
$$\bar{s}^\star v^\star Pv = v^\star A'Pv.$$
Adding these two last equations yields
$$(\bar{s} + \bar{s}^\star)v^\star Pv = v^\star(A'P + PA)v.$$
Finally, note that $v^\star Pv > 0$, $v^\star(A'P + PA)v \leq 0$ and $(\bar{s} + \bar{s}^\star)$ is equal to twice the real part of $\bar{s}$, which has to be non-positive, as claimed.

Exercise 13 *Consider the discrete-time system $x(k+1) = Ax(k) + Bu(k)$. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

1. *Compute the reachability matrix R.*
2. *Determine if the system is reachable and compute the set of reachable states.*
3. *Determine all states x_I such that $x(0) = x_I$ and $x(1) = 0$.*

Solution 13

1. The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 2 & -2 & -2 \end{bmatrix}.$$

2. The first two columns of the reachability matrix are linearly independent, and $\det(R) = 0$, hence the system is not reachable. The set of reachable states is two dimensional and it is described by the linear combination of the first two columns of the reachable matrix.

3. We have to determine all states which are controllable in one step. Instead of using the definition of controllable states in one step, we perform a direct calculation. Let

$$x(0) = x_I = \begin{bmatrix} x_{I,1} \\ x_{I,2} \\ x_{I,3} \end{bmatrix}$$

and note that

$$x(1) = Ax(0) + Bu(0) = \begin{bmatrix} x_{I,2} + u(0) \\ -x_{I,1} - u(0) \\ 2(x_{I,2} + u(0)) \end{bmatrix}.$$

The condition $x(1) = 0$ implies $x_{I,1} = -u(0)$, $x_{I,2} = -u(0)$, hence all states that can be controlled to zero in one step are given by

$$x_I = \begin{bmatrix} -u(0) \\ -u(0) \\ x_{I,3} \end{bmatrix},$$

and this is a two dimensional set. Note that this implies that the considered system has an eigenvalue at zero.

Exercise 14 *Consider the discrete-time system $x(k+1) = Ax(k) + Bu(k)$. Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

1. *Compute the reachable subspaces in one step, two steps and three steps.*
2. *Using Hautus test determine the unreachable modes.*
3. *Show that the system is controllable in two steps.*

Solution 14 The reachability matrix is

$$R = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and it has rank equal to two.

1. The set of reachable states in one step is

$$\mathcal{R}_1 = \text{span} B = \text{span} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The set of reachable states in two steps is

$$\mathcal{R}_2 = \text{span}[B, AB] = \text{span} \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The set of reachable states in three steps is

$$\mathcal{R}_3 = \text{span}[B, AB, A^2B] = \mathcal{R}_2.$$

2. The reachability pencil is

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{ccc|c} s & -1 & 0 & 1 \\ 1 & s & 0 & -1 \\ 0 & 0 & s & 0 \end{array}\right].$$

This matrix has rank three for all $s \neq 0$, hence the unreachable mode is $s = 0$.

3. The system is controllable, since the unreachable modes are at $s = 0$. To show that it is controllable in two steps note that

$$A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence

$$\mathrm{Im}A^2 \subseteq \mathcal{R}_2,$$

which proves the claim.

Exercise 15 *Consider the continuous-time system $\dot{x} = Ax + Bu$. Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Compute the set of states that can be reached from the state

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Solution 15 Note that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \begin{bmatrix} 0 \\ e^t \end{bmatrix} + \begin{bmatrix} \int_0^t e^{t-\tau}u(\tau)d\tau \\ 0 \end{bmatrix}.$$

Note that, by a proper selection of $u(\tau)$ in the interval $[0, t)$ it is possible to *assign* $\int_0^t e^{t-\tau}u(\tau)d\tau$. Therefore, the states that can be reached at time t from x_0 are described by

$$x(t) = \begin{bmatrix} 0 \\ e^t \end{bmatrix} + \lambda B,$$

with $\lambda \in I\!R$.

Exercise 16 *Consider the discrete-time system $x(k+1) = Ax(k)$, $y(k) = Cx(k)$. Let*

$$A = \begin{bmatrix} 0 & -4 & 0 \\ 1 & 4 & 0 \\ 0 & -4 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Determine if the system is observable and compute the unobservable subspace.

Solution 16 The observability matrix is

$$O = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 0 \\ 4 & 12 & 0 \end{bmatrix}.$$

This matrix has rank two, hence the system is not observable. The unobservable subspace $\ker O$ is spanned by the vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

this means that it is not possible to obtain information on the third component of the state from measurements of the output.

Exercise 17 *Consider the continuous-time system*

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx$$

and its Euler discrete-time approximate model

$$x_{k+1} = x_k + T(Ax_k + Bu_k) \qquad\qquad y_k = Cx_k$$

where $T > 0$ is the sampling time.

Show that the continuous-time system is observable if and only if the Euler discrete-time approximate model is observable.

Solution 17 By observability of the continuous-time system we have that

$$\text{rank}\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = n,$$

for all $s \in \mathbb{C}$. Consider now the observability pencil of the Euler model, namely

$$\left[\begin{array}{c} sI - (I + TA) \\ \hline C \end{array}\right].$$

Note now that, for all $s \in \mathbb{C}$,

$$n = \text{rank}\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \text{rank}\left[\begin{array}{c} \dfrac{s-1}{T}I - A \\ \hline C \end{array}\right] = \text{rank}\left[\begin{array}{c} zI - A \\ \hline C \end{array}\right],$$

where $z = \frac{s-1}{T}$. Hence, observability of the continuous-time system implies, and it is implied by, observability of the Euler discrete-time approximate model.

Exercise 18 *Consider the system $\sigma x = Ax$, $y = Cx$, with*

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \qquad\qquad C = \left[\begin{array}{cc} I & 0 \end{array}\right],$$

and A_{ij} matrices of appropriate dimensions. Show that the system is observable if and only if the (simpler) system $\sigma\xi = A_{22}\xi$ with output $\eta = A_{12}\xi$ is observable.

Solution 18 *Observability of the system $\sigma x = Ax$, $y = Cx$ implies, and is implied by,*

$$\text{rank}\left[\begin{array}{cc} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \\ \hline I & 0 \end{array}\right] = n$$

for all $s \in \mathbb{C}$. Suppose that $y(t) \in \mathbb{R}^p$, then

$$\text{rank}\left[\begin{array}{cc} sI - A_{11} & -A_{12} \\ -A_{21} & sI - A_{22} \\ \hline I & 0 \end{array}\right] = p + \text{rank}\left[\begin{array}{c} -A_{12} \\ sI - A_{22} \end{array}\right].$$

Note now that the matrix

$$\begin{bmatrix} -A_{12} \\ sI - A_{22} \end{bmatrix}$$

has full rank for all $s \in \mathbb{C}$ *if and only if the system* $\sigma\xi = A_{22}\xi$ *with output* $\eta = A_{12}\xi$ *is observable, and this proves the claim.*

Exercise 19 *Consider the continuous-time system*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 3 & -1+\epsilon \\ 1 & 2-\epsilon \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 \end{bmatrix} x. \end{aligned}$$

1. *Show that the system is reachable for all* $\epsilon \neq 1$.
2. *Let* $\epsilon = 1$. *Write the system in the canonical form for non-reachable systems, i.e. determine coordinates* $\hat{x}$, *defined by*

   $$x = L\hat{x}$$

   for some matrix L, *such that, in the* $\hat{x}$ *coordinates, the system is described by equations of the form*

   $$\dot{\hat{x}} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \hat{x} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u,$$

   with $\tilde{A}_{11}$ *and* $\tilde{B}_1$ *such that* $\dot{\hat{x}}_1 = \tilde{A}_{11}\hat{x}_1 + \tilde{B}_1 u$ *is reachable. (Compute explicitly* L, $\tilde{A}_{11}$, $\tilde{A}_{12}$, $\tilde{A}_{22}$ *and* $\tilde{B}_1$.*)*
3. *Show that the system is observable for all* $\epsilon \neq 1/2$.
4. *Let* $\epsilon = 1/2$. *Determine, using Hautus test, the unobservable modes.*

Solution 19

1. The reachability matrix is

$$R = \begin{bmatrix} 0 & \epsilon - 1 \\ 1 & 2 - \epsilon \end{bmatrix}.$$

If $\epsilon \neq 1$ then $\det(R) \neq 0$, hence the system is reachable.

2. If $\epsilon = 1$ the system is not reachable. Let L be constructed setting the first column equal to the first column of the reachable matrix (which is nonzero) and selecting the second column to render L non-singular, for example

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(Any matrix L of the form

$$L = \begin{bmatrix} 0 & L_{12} \\ 1 & L_{21} \end{bmatrix},$$

with $L_{12} \neq 0$, can be used.)
Note now that (recall that $\epsilon = 1$ and note that $L^{-1} = L$)

$$\begin{aligned} \dot{\hat{x}} &= L^{-1} \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} L\hat{x} + L^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} -1 & 1 \end{bmatrix} L\hat{x} = \begin{bmatrix} 1 & -1 \end{bmatrix} \hat{x}. \end{aligned}$$

Hence $\tilde{A}_{11} = 1$, $\tilde{A}_{12} = 1$, $\tilde{A}_{22} = 3$, $\tilde{B}_1 = 1$ and the system $\dot{\hat{x}}_1 = \tilde{A}_{11}\hat{x}_1 + \tilde{B}_1 u = \hat{x}_1 + u$ is reachable. (Note that the system is not stabilizable, and the unreachable mode is $s = 3$.)

3. The observability matrix is

$$O = \begin{bmatrix} -1 & 1 \\ -2 & 3 - 2\epsilon \end{bmatrix}.$$

Note that $\det(O) = 2\epsilon - 1$. Therefore the system is observable if $\epsilon \neq 1/2$.

4. The observability pencil, for $\epsilon = 1/2$, is

$$\left[\begin{array}{cc} s - 3 & 1/2 \\ -1 & s - 3/2 \\ \hline -1 & 1 \end{array}\right].$$

As the system is not observable we know that the observability pencil loses rank, i.e. it has rank equal to one, for some s. To compute such an s consider the submatrix

$$\left[\begin{array}{cc} -1 & s - 3/2 \\ \hline 1 & -1 \end{array}\right].$$

This has rank equal to one for $s = 5/2$, which is therefore the unobservable mode.

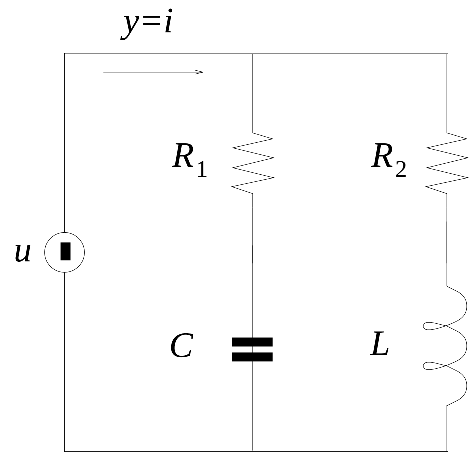


Figure A: The electrical network for Exercise 20.

Exercise 20 *Consider the linear electrical network in Figure A. Let u be the driving voltage.*

1. *Using Kirchhoff's laws, or otherwise, express the dynamics of the circuit in the standard state-space form*

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

Take x_1 to be the voltage across the capacitor, x_2 to be the current through the inductor and the output to be the current supplied by the generator.

2. *Derive a condition on the parameters R_1, R_2, C and L under which the pair (A, B) is controllable.*

3. *Derive a condition on the parameters R_1, R_2, C and L under which the pair (A, C) is observable.*

4. *Assume $R_1R_2C = L$ and $R_1 \neq R_2$. Derive the Kalman canonical form for the system.*

5. *Assume $R_1R_2C = L$ and $R_1 \neq R_2$. Define the controllable subspace and the unobservable subspace. Illustrate these subspaces as lines in R^2.*

Solution 20 Let x_1 denote the voltage across C and x_2 the current through L.

1. Kirchhoff's laws yield

$$u = x_1 + R_1C_1\dot{x}_1 \qquad u = R_2x_2 + l\dot{x}_2$$

and

$$y = i = x_2 + \frac{u - x_1}{R_1}.$$

As a result,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix} u$$

and

$$y = Cx + Du = \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} x + \frac{1}{R_1} u.$$

2. The reachability matrix is

$$R = \begin{bmatrix} \frac{1}{R_1 C} & -\frac{1}{R_1^2 C^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{bmatrix}$$

and

$$\det(R) = \frac{1}{R_1 C L} \left(\frac{1}{R_1 C} - \frac{R_2}{L} \right).$$

Hence the system is reachable and controllable if, and only if,

$$R_1 R_2 C \neq L.$$

3. The observability matrix is

$$O = \begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2 C} & -\frac{R_2}{L} \end{bmatrix}$$

and

$$\det(O) = \frac{1}{R_1} \left(\frac{R_2}{L} - \frac{1}{R_1 C} \right).$$

Hence the system is observable if, and only if,

$$R_1 R_2 C \neq L.$$

4. If $R_1 R_2 C = L$ then the reachable subspace is

$$\mathcal{R} = \text{span} \begin{bmatrix} R_2 \\ 1 \end{bmatrix}$$

and the unobservable subspace is

$$\ker \mathcal{O} = \text{span} \begin{bmatrix} R_1 \\ 1 \end{bmatrix}.$$

Note that, as $R_1 \neq R_2$ these two subspaces are independent. Let

$$L = \begin{bmatrix} R_2 & R_1 \\ 1 & 1 \end{bmatrix}$$

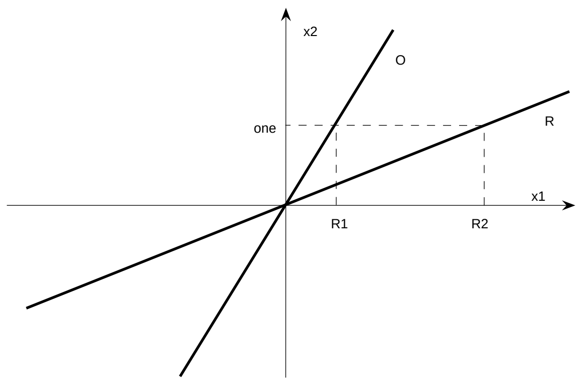
and note that the matrices of the system in Kalman canonical form are

$$\tilde{A} = L^{-1}AL = \begin{bmatrix} -\frac{R_2}{L} & 0 \\ 0 & -\frac{R_2}{L} \end{bmatrix} \qquad \tilde{B} = L^{-1}B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

and

$$\tilde{C} = CL = \begin{bmatrix} 1 - \frac{R_2}{R_1} & 0 \end{bmatrix}.$$

5. The subspaces are indicated in the figure.



Exercise 21 *Consider the electrical network depicted in Figure B.*

1. *Using Kirchoff's laws, or otherwise, write a state space description of the system.*
2. *Let $R_1 = R_2 = R$ and $C_1 = C_2 = C$. Compute the controllability and observability matrices and their ranks.*
3. *Let $R_1 = R_2 = R$ and $C_1 = C_2 = C$. Compute the Kalman canonical form of the system.*

Solution 21 Let x_j be the voltage across the capacitor C_j, considered positive "from left to right", and i_j the current through the capacitor C_j, considered positive "from left to right". The input current i is positive in the upward direction, and the output voltage is the voltage, positive "upward" between the two open terminals.

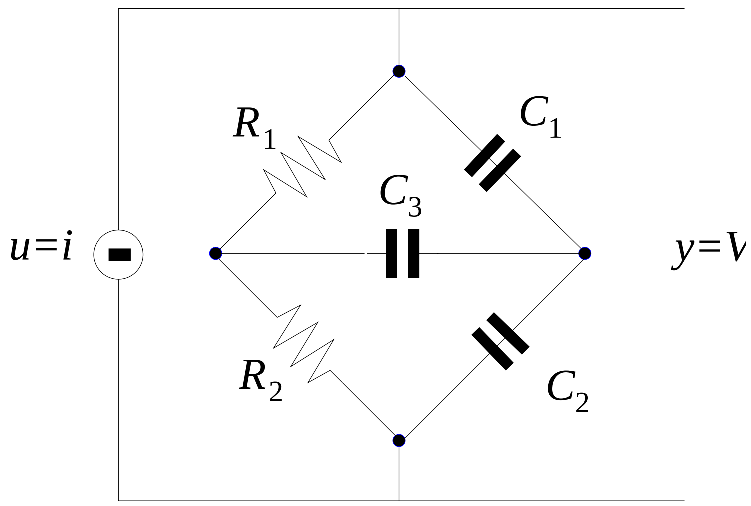


Figure B: The electrical network for Exercise 21.

1. Using the above conventions we have

$$i_1 = C_1\dot{v}_1 \qquad i_2 = C_2\dot{v}_2 \qquad i_3 = C_3\dot{v}_3$$

$$i_1 + i_2 + i_3 = 0 \qquad v_3 - v_2 = R_2(i - i_1 - i_3) \qquad v_1 - v_3 = R_1(i - i_1).$$

As a result

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_1\dot{v}_1 \\ C_2\dot{v}_2 \\ C_3\dot{v}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} \\ \frac{1}{R_1} & 0 & -\frac{1}{R_1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

yielding

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1R_1} & 0 & \frac{1}{C_1R_1} \\ 0 & -\frac{1}{C_2R_2} & \frac{1}{C_2R_2} \\ \frac{1}{C_3R_2} & \frac{1}{C_3R_2} & -\frac{1}{C_3R_1} - \frac{1}{C_3R_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} \\ -\frac{1}{C_2} \\ 0 \end{bmatrix} i.$$

Finally

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

2. Setting $R_1 = R_2 = R$, $C_1 = C_2 = C$, $RC = 1/\alpha$, $RC_3 = 1/\beta$, and $u = i/C$ yields

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & \alpha \\ 0 & -\alpha & \alpha \\ \beta & \beta & -2\beta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u.$$

The reachability matrix is

$$R = \begin{bmatrix} 1 & -\alpha & \alpha^2 \\ -1 & \alpha & -\alpha^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

R has rank one, hence the system is not reachable and not controllable. The observability matrix is

$$O = \begin{bmatrix} 1 & -1 & 0 \\ -\alpha & \alpha & 0 \\ \alpha^2 & -\alpha^2 & 0 \end{bmatrix}.$$

O has rank one, hence the system is not observable.

3. The reachability subspace is

$$\mathcal{R} = \text{span} B = \text{span} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The unobservable subspace is

$$\ker \mathcal{O} = \text{span} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that

$$\mathcal{X}_1 = \mathcal{R} \cap \ker \mathcal{O} = \emptyset \qquad \mathcal{X}_2 = \mathcal{R} \qquad \mathcal{X}_3 = \ker \mathcal{O} \qquad \mathcal{X}_4 = \emptyset,$$

hence $x = L\hat{x}$, with

$$L = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, Kalman canonical form is given by the equations

$$\dot{\hat{x}} = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha & \alpha \\ 0 & 2\beta & -2\beta \end{bmatrix} \hat{x} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 0 & 0 \end{bmatrix} \hat{x}.$$

Exercise 22 *The linearized model of an orbiting satellite about a circular orbit of radius $r_0 > 0$ and angular velocity $\omega_0 \neq 0$ is described by the equations*

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 & 0 & 0 & 2r_0\omega_0^2 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega_0/r_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/r_0 \end{bmatrix} u$$

$$y = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x.$$

The output components are variations in radius and angle of the orbit and the input components are radial and tangential forces.

1. *Show that the system is controllable.*
2. *Design a state feedback control law*

$$u = Kx + Gv = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} x + \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \end{bmatrix} v$$

such that

- *the matrix $A+BK$ has all eigenvalues equal to -1 and it is block diagonal, i.e.*

$$A + BK = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$$

with $F_i \in I\!R^{2\times 2}$;

- *the closed-loop system has unity DC gain, i.e.*

$$-C(A+BK)^{-1}BG = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution 22

1. Consider the following submatrix of the reachability matrix

$$\tilde{R} = [B \;\; AB] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2\omega_0^2 \\ 0 & 0 & 0 & 1/r_0 \\ 0 & 1/r_0 & -2\omega_0/r_0 & 0 \end{bmatrix}$$

and note that its determinant is $-1/r_0^2 \neq 0$. Hence the system is reachable and controllable.

2. Consider the closed-loop system

$$\dot{x} = (A+BK)x + BGv$$

and note that

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega_0^2 + k_{11} & k_{12} & k_{13} & 2r_0\omega_0^2 + k_{14} \\ 0 & 0 & 0 & 1 \\ k_{21}/r_0 & -2\omega_0/r_0 + k_{22}/r_0 & k_{23}/r_0 & k_{24}/r_0 \end{bmatrix}.$$

Hence, selecting

$$k_{11} = -3\omega_0^2 - 1 \quad k_{12} = -2 \quad k_{13} = 0 \quad k_{14} = -2r_0\omega_0^2$$

$$k_{21} = 0 \quad k_{22} = 2\omega_0 \quad k_{23} = -r_0 \quad k_{24} = -2r_0$$

yields

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

which shows that the first condition has been achieved.

To achieve the second condition note that

$$-C(A + BK)^{-1}BG = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22}/r_0 \end{bmatrix}.$$

Hence, it suffices to select

$$g_{11} = 1 \qquad g_{22} = r_0.$$

Exercise 23 *Consider the continuous-time system $\dot{x} = Ax + Bu$. Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find a matrix K such that $\sigma(A + BK) = \{-1, -2\}$. Solve the problem in two ways:

1. *using the general theory discussed in the lectures;*
2. *using a direct computation, i.e. without computing the reachability matrix of the system.*

Solution 23

1. The general theory states that the state feedback is given by

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} R^{-1}p(A),$$

where R is the reachability matrix and $p(s)$ is the desired closed-loop characteristic polynomial, in this case $p(s) = (s + 1)(s + 2) = s^2 + 3s + 2$. As a result,

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \left(\frac{1}{4} \begin{bmatrix} 7 & -3 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} 12 & 16 \\ 24 & 36 \end{bmatrix} = -\begin{bmatrix} 3 & 5 \end{bmatrix},$$

yielding

$$A + BK = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix},$$

which has eigenvalues equal to -1 and -2, as requested.

2. Let
$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$
and note that
$$A + BK = \begin{bmatrix} 1+K_1 & 2+K_2 \\ 3+K_1 & 4+K_2 \end{bmatrix}.$$
The characteristic polynomial of $A+BK$ is
$$\det(sI-(A+BK)) = s^2 + s(-5-K_1-K_2) + (-2K_2+2K_1-2),$$
and this should be equal to $p(s) = s^2+3s+2$. As a result, K_1 and K_2 should be such that
$$-5-K_1-K_2 = 3 \qquad -2K_2+2K_1-2 = 2,$$
which yields $K_1 = -3$ and $K_2 = -5$. Note that, because the system has only one input and it is reachable, the state feedback assigning the eigenvalues is unique!

Exercise 24 *Consider the continuous-time system*
$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & -1 \end{bmatrix} x. \end{aligned}$$

1. *Show that the system is controllable and observable.*
2. *Assume zero initial state. Compute the response of the system when u is a unity step applied at $t=0$.*
3. *Design a state feedback control law*
$$u = Kx + Gr$$
such that the closed-loop system has two eigenvalues at -3.

Solution 24

1. The reachability matrix is
$$R = \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix},$$
which is full rank, hence the system is reachable and controllable.
The observability matrix is
$$O = \begin{bmatrix} 3 & -1 \\ 0 & -5 \end{bmatrix},$$
which is full rank, hence the system is observable.

2. We have to compute the forced response of the output of the system. Note that $\sigma(A) = \{i\sqrt{5}, -i\sqrt{5}\}$, hence the forced response of the output of the system has the form
$$y(t) = y_0 + y_1 \sin(\sqrt{5}t + \phi).$$
Note now that (recall that $u = 1$, for $t \geq 0$, and that $x(0) = 0$)
$$y(0) = \left[\begin{array}{cc} 3 & -1 \end{array}\right] x(0) = 0$$
$$\dot{y}(0) = \left[\begin{array}{cc} 0 & -5 \end{array}\right] x(0) + 4u = 4$$
$$\ddot{y}(0) = \left[\begin{array}{cc} -15 & 5 \end{array}\right] x(0) + 5u + 4\dot{u} = 5.$$
Therefore, we have to determine y_0, y_1 and ϕ from the equations
$$y_0 + y_1 \sin\phi = 0 \qquad \sqrt{5}y_1 \cos\phi = 4 \qquad -5y_1 \sin\phi = 5.$$
This yields
$$y_1 = \sqrt{\frac{21}{5}} \qquad \phi = \arctan\left(-\frac{\sqrt{5}}{4}\right) \qquad y_0 = \sqrt{\frac{21}{5}}.$$

3. Let
$$K = \left[\begin{array}{cc} K_1 & K_2 \end{array}\right]$$
and note that
$$A + BK = \left[\begin{array}{cc} 1 + K_1 & -2 + K_2 \\ 3 + K_1 & -1 - K_2 \end{array}\right].$$
The characteristic polynomial of $A + BK$ is
$$\det(sI - (A + BK)) = s^2 + s(-K_1 + K_2) + (-3K_1 + 5 - 4K_2),$$
and this should be equal to $p(s) = (s + 3)^2 = s^2 + 6s + 9$. As a result, K_1 and K_2 should be such that
$$-K_1 + K_2 = 6 \qquad -3K_1 + 5 - 4K_2 = 9,$$
which yields $K_1 = -4$ and $K_2 = 2$.

Exercise 25 *Consider the continuous-time system*
$$\begin{aligned} \dot{x} &= \left[\begin{array}{cc} 3 & -1 + \epsilon \\ 1 & 2 - \epsilon \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] u \\ y &= \left[\begin{array}{cc} -1 & 1 \end{array}\right] x. \end{aligned}$$

1. *Show that the system is controllable for any $\epsilon \neq 1$. Study the stabilisability of the system for $\epsilon = 1$.*

2. *Show that the system is observable for any $\epsilon \neq 1/2$. Study the detectability of the system for $\epsilon = 1/2$.*

3. *Assume $\epsilon = 0$. Design a state feedback control law*

$$u = Kx + Gr$$

such that the closed-loop system has two eigenvalues equal to -2.

4. *Show that the state feedback control law designed above stabilizes the system for any $\epsilon \in (-4, 1/7)$.*

Solution 25

1. The reachability matrix is

$$R = \begin{bmatrix} 0 & \epsilon - 1 \\ 1 & 2 - \epsilon \end{bmatrix},$$

and $\det(R) = 1 - \epsilon$. As a result the system is reachable and controllable for $\epsilon \neq 1$. Let $\epsilon = 1$ and consider the reachability pencil

$$\left[\begin{array}{c|c} sI - A & B \end{array}\right] = \left[\begin{array}{cc|c} s-3 & 0 & 0 \\ -1 & s-1 & 1 \end{array}\right],$$

which has rank equal to one for $s = 3$. The system is therefore not stabilizable. Note that it is possible to obtain this conclusion without computing the reachability pencil. In fact, for $\epsilon = 1$ the eigenvalues of A are $\{3, 1\}$, hence if there is an unreachable mode this is associated to a value of s with positive real part.

2. The observability matrix is

$$O = \begin{bmatrix} -1 & 1 \\ -2 & -2\epsilon + 3 \end{bmatrix},$$

and $\det(O) = 2\epsilon - 1$. As a result the system is observable for $\epsilon \neq 1/2$. Let $\epsilon = 1/2$ and consider the observability pencil

$$\left[\begin{array}{c} sI - A \\ \hline C \end{array}\right] = \left[\begin{array}{cc} s-3 & 1/2 \\ -1 & s-3/2 \\ \hline -1 & 1 \end{array}\right].$$

Because the system is not observable, this matrix has to have rank equal to one for some s. To find such an s consider the submatrix

$$\left[\begin{array}{cc} -1 & s-3/2 \\ \hline -1 & 1 \end{array}\right].$$

Its determinant is $s - 5/2$, hence the unobservable mode is $s = 5/2$ and the system is not detectable. Note that it is possible to obtain this conclusion without computing the observability pencil. In fact, for $\epsilon = 1/2$ the eigenvalues of A are $\{5/2, 2\}$, hence if there is an unobservable mode this is associated to a value of s with positive real part.

3. If $\epsilon = 0$ we have
$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}.$$
Setting $K = [K_1,\ K_2]$ yields
$$\det(sI - (A + BK)) = s^2 + s(-5 - K_2) + (K_1 + 3K_2 + 7),$$
which should be equal to $(s+2)^2$. This is achieved setting
$$K_1 = 24 \qquad\qquad K_2 = -9.$$

4. Consider now the matrix
$$A + BK = \begin{bmatrix} 3 & \epsilon - 1 \\ 25 & -7 - \epsilon \end{bmatrix}.$$
Its characteristic polynomial is
$$\det(sI - (A + BK)) = s^2 + s(4 + \epsilon) + (4 - 28\epsilon),$$
which has both roots with negative real part (by Routh test) if and only if $\epsilon \in (-4, 1/7)$.

Exercise 26 *Consider the continuous-time system $\dot{x} = Ax + Bu$, $y = Cx$, with*
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$
Design an asymptotic observer having three eigenvalues at -10.

Solution 26 *An asymptotic observer is described by*
$$\dot{\xi} = A\xi + L(C\xi - y) + Bu = (A + LC)\xi - Ly + Bu$$
for some $L = [L_1\ L_2\ L_3]'$, where ξ is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. Note that
$$A + LC = \begin{bmatrix} L_1 & 1 & 0 \\ L_2 & 0 & 1 \\ L_3 & -5 & -6 \end{bmatrix},$$
and its characteristic polynomial is
$$s^3 + s^2(6 - L_1) + s(5 - 6L_1 - L_2) + (-L_3 - 5L_1 - 6L_2).$$
This should be equal to
$$(s + 10)^3 = s^3 + 30s^2 + 300s + 1000.$$
As a result,
$$L_1 = -24 \qquad L_2 = -151 \qquad L_3 = 26.$$

Exercise 27 *Consider the continuous-time system $\dot{x} = Ax$, $y = Cx$. Let*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

1. *Show, using Hautus test, that the system is observable for all α.*
2. *Design an asymptotic observer for the system. Select the output injection gain L such that the matrix $A + LC$ has two eigenvalues equal to -3.*
3. *Suppose that one can measure $y(t)$ and a delayed copy of $y(t)$ given by $y(t - \tau)$, with $\tau > 0$. Assume (for simplicity) that $\alpha \neq 0$.*
   For $t \geq \tau$, express the vector

   $$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix}$$

   from $x(0)$.
   Show that the relation determined above can be used, for any $\tau > 0$, to compute $x(0)$ as a function of $Y(t)$, where $t \geq \tau$. Argue that the above result can be used to determine $x(t)$ from $Y(t)$, for $t \geq \tau$, exactly.

Solution 27

1. The observability pencil is

   $$\left[\begin{array}{cc} s & -1 \\ 0 & s+\alpha \\ \hline 1 & 0 \end{array}\right],$$

   which has rank two for any s and any α. Hence the system is observable.

2. An asymptotic observer is described by

   $$\dot{\xi} = A\xi + L(C\xi - y) = (A + LC)\xi - Ly$$

   for some $L = [L_1 \ L_2]'$, where ξ is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. Note that

   $$A + LC = \begin{bmatrix} L_1 & 1 \\ L_2 & -\alpha \end{bmatrix}$$

   and its characteristic polynomial is

   $$s^2 + s(\alpha - L_1) - \alpha L_1 - L_2.$$

   This should be equal to $(s + 3)^2 = s^2 + 6s + 9$, yielding

   $$L_1 = \alpha - 6 \qquad L_2 = -9 + (6 - \alpha)\alpha.$$

3. Note that
$$y(t) = Ce^{At}x(0)$$
and replacing t with $t-\tau$ one has
$$y(t-\tau) = Ce^{A(t-\tau)}x(0).$$
Then, for $t \geq \tau$,
$$Y(t) = \begin{bmatrix} y(t) \\ y(t-\tau) \end{bmatrix} = \begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} e^{At}x(0).$$
For the given A and C (using $\alpha \neq 0$) we have
$$\begin{bmatrix} C \\ Ce^{-A\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{\alpha\tau}-1}{\alpha} \end{bmatrix}$$
which is invertible for all $\alpha \neq 0$ and all $\tau > 0$. Hence
$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 0 \\ 1 & -\frac{e^{\alpha\tau}-1}{\alpha} \end{bmatrix}^{-1} Y(t).$$
The above relation implies that, for all $t \geq \tau$, it is possible to obtain exactly $x(t)$.

Exercise 28 *Consider the discrete-time system*
$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 3 & -1 \end{bmatrix} x_k. \end{aligned}$$

1. *Show that the system is observable.*
2. *Design an asymptotic observer, with state $\hat{x}_k$, such that $e_k = x_k - \hat{x}_k = 0$ for all $k \geq N$. Determine the smallest value of N for which the above condition can be satisfied.*
3. *Let*
$$u_k = K\hat{x}_k + v_k$$
with $K = [3/4, 3/4]$. Write the equations of the closed-loop system, with state $[x_k, \hat{x}_k]$, input v_k and output y_k, and determine the eigenvalues of this system.

Solution 28

1. The observability matrix is
$$O = \begin{bmatrix} 3 & -1 \\ 0 & -5 \end{bmatrix},$$
which has rank equal to two. The system is therefore observable.

2. An asymptotic observer is described by

$$\xi_{k+1} = A\xi_k + L(C\xi_k - y_k) + Bu_k = (A + LC)\xi_k - Ly_k + Bu_k$$

for some $L = [L_1\ L_2]'$, where ξ_k is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. To obtain a dead-beat observer L should be such that both eigenvalues of $A + LC$ are zero. Note that

$$A + LC = \begin{bmatrix} 1 + 3L_1 & -2 - L_1 \\ 3 + 3L_2 & -1 - L_2 \end{bmatrix},$$

and

$$\det(sI - (A + LC)) = s^2 + s(L_2 - 3L_1) + 5(1 + L_2).$$

Hence,

$$L_1 = -1/3 \qquad\qquad L_2 = -1.$$

With this selection of L we have $(A + LC)^2 = 0$, hence $N = 2$. To prove that the smallest N for which the considered condition holds is $N = 2$ it is enough to observe that there is no selection of L such that $(A + LC)^1 = 0$.

3. By the separation principle the eigenvalues of the closed-loop system are the eigenvalues of the observer and of the matrix

$$A + BK = \begin{bmatrix} 7/4 & -5/4 \\ 9/4 & -7/4 \end{bmatrix}.$$

Hence, the eigenvalues of the closed-loop system are $\{1/2, -1/2, 0, 0\}$.

Exercise 29 *Consider the simplified model of a ship described by the equation*

$$\begin{aligned} M\ddot{\theta} + d\dot{\theta} + c\alpha &= w \\ \dot{\alpha} + \alpha &= u \end{aligned}$$

where θ denotes the heading angle error (the angle between the ship's heading and the desired heading), α denotes the rudder angle, w denotes a disturbance due to wind, and u is the control input. M and c are positive parameters, and d is a non-negative parameter.

1. *Write the equation of the system, with state $(\theta, \dot{\theta}, \alpha)$, input (w, u) and output θ in standard state space form.*

2. *Let $w = 0$. Show that the system is controllable.*

3. *Show that the system is observable.*

4. *Let $w = 0$, $M = 1$, $c = 1$ and $d = 0$. Design an output feedback controller applying the separation principle. In particular, select the state feedback gain K such that the matrix $A + BK$ has three eigenvalues equal to -1 and the output injection gain L such that the matrix $A + LC$ has three eigenvalues equal to -3.*

Solution 29

1. The description of the system in standard state space form is (set $x = (\theta, \dot{\theta}, \alpha)'$)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -d/M & -c/M \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1/M & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x.$$

2. The reachability matrix is

$$R = \begin{bmatrix} 0 & 0 & -c/M \\ 0 & -c/M & c/M(d/M+1) \\ 1 & -1 & 1 \end{bmatrix}$$

and this has full rank for all positive c and M. The system is reachable and controllable.

3. The observability matrix is

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -d/M & -c/M \end{bmatrix}$$

and this has full rank for all positive c and M. The system is observable.

4. Let $K = [K_1\ K_2\ K_3]$ and note that

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ K_1 & K_2 & -1+K_3 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^3+(1-K_3)s^2+(K_2)s+(K_1)$. Hence the selection

$$K_1 = 1 \qquad K_2 = 3 \qquad K_3 = -2$$

is such that all eigenvalues of $A + BK$ are equal to -1. Let $L = [L_1\ L_2\ L_3]'$ and note that

$$A + LC = \begin{bmatrix} L_1 & 1 & 0 \\ L_2 & 0 & -1 \\ L_3 & 0 & -1 \end{bmatrix},$$

and that the characteristic polynomial of this matrix is $s^3 + (1 - L_1)s^2 + (-L_1 - L_2)s + (-L_2 + L_3)$. Hence the selection

$$L_1 = -8 \qquad L_2 = -19 \qquad L_3 = 8$$

is such that all eigenvalues of $A + LC$ are equal to -3. Finally, the controller is $\dot{\xi} = (A + BK + LC)\xi - Ly$, $u = K\xi$.

Exercise 30 *Consider the continuous-time system*

$$\dot{x} = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 3 & -1 \end{bmatrix} x$$

1. *Design an asymptotic observer with a double pole at* -6.

2. *Suppose* x_0 *is the observer state evaluated in part 1. Let*

$$u = Kx_0 + Gr$$

with $K = [-4, 2]$. *Compute the eigenvalues of the closed-loop system.*

Solution 30

1. An asymptotic observer is described by

$$\dot{\xi} = A\xi + L(C\xi - y) + Bu = (A + LC)\xi - Ly + Bu$$

for some $L = [L_1 \ L_2]'$, where ξ is the asymptotic estimate of x provided the matrix $A + LC$ has all eigenvalues with negative real part. Note that

$$A + LC = \begin{bmatrix} 1 + 3L_1 & -2 - L_1 \\ 3 + 3L_2 & -1 - L_2 \end{bmatrix},$$

and its characteristic polynomial is

$$s^2 + s(L_2 - 3L_1) + 5(1 + L_2).$$

This should be equal to

$$(s + 6)^2 = s^2 + 12s + 36.$$

As a result,

$$L_1 = -29/15 \qquad L_2 = 31/5.$$

2. By the separation principle the eigenvalues of the closed-loop system are the eigenvalues of the observer and of the matrix

$$A + BK = \begin{bmatrix} -3 & 0 \\ 7 & -3 \end{bmatrix}.$$

Hence, the eigenvalues of the closed-loop system are $\{-3, -3, -6, -6\}$.