IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2020

Mathematics for Signals and Systems (**version 1**)

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates:

For questions involving matrix manipulations or calculations, please provide answers showing intermediate calculations steps.

The Questions

1. (a) Consider the following subspace of $\mathbb{R}^4$:

$$S = \text{ span}\left\{ \begin{bmatrix} 2 \\ 6 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \\ 2.5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ -6 \\ 18 \end{bmatrix} \right\}.$$

i. Find a basis for S [3]

ii. Use the Gram-Schmidt method to orthogonalize your basis [3]

(b) Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & a & b \\ -1 & 3 & 2 \end{bmatrix}. \quad (1)$$

For what values of a and b does $\mathbf{A}$ have an inverse? Justify your answer. [2]

(c) Consider now the matrix $\mathbf{C} = \mathbf{B} \cdot \mathbf{A}$ where $\mathbf{A}$ is given in Eq. (1) with $a = 1$ and $b = 0$ and

$$\mathbf{B} = \begin{bmatrix} 0.5 & 1 & 0 \\ 3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find the null space of $\mathbf{C}$. [4]

Question 1 continues on the next page

(d) Let

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{p}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}.$$

i. Determine the vector $\hat{\mathbf{x}}_1 \in \text{span}\{\mathbf{p}_1, \mathbf{p}_2\}$ that minimises $\|\mathbf{x}_1 - \hat{\mathbf{x}}_1\|$. [4]

ii. Denote with S the subspace orthogonal to $V = \text{span}\{\mathbf{p}_1, \mathbf{p}_2\}$ and such that $S \bigoplus V = \mathbb{R}^4$, this means that S is the orthogonal complement to V in $\mathbb{R}^4$. Consider the vector

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}.$$

Determine the vector $\hat{\mathbf{x}}_2 \in S$ that minimises $\|\mathbf{x}_2 - \hat{\mathbf{x}}_2\|$. [4]

2. (a) Consider the problem of finding $\mathbf{x}$ that minimizes $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$.

i. Show that the vector $\hat{\mathbf{x}}$ minimises $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2$ if and only if $\mathbf{A}^{\mathbf{H}}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A}^{\mathbf{H}}\mathbf{b}$ [3]

ii. Can you claim that the minimiser $\hat{\mathbf{x}}$ is unique if $\mathbf{A}$ has a full row rank? Briefly justify your answer. [2]

(b) Assume you observe a function $f(x)$ at instants $x_i : y_i = f(x_i)$ for $i = 1, 2, 3, 4$. You would like to approximate $f(x)$ with a straight line, that is, $y_i = c_1 x_i + c_2$, from these observations.

i. Write down the problem in matrix/vector form. That is, express $\mathbf{y}$ and $\mathbf{A}$ in terms of x_i and y_i such that $\mathbf{y} = \mathbf{A}\mathbf{c} + \mathbf{e}$, where $\mathbf{c} = [c_1, c_2]^T$ and $\mathbf{e}$ is the approximation error. [2]

Consider the set of data $x_i = \{-2, -1, 0, 1\}$ $y_i = f(x_i) = \{1, 0, 0, 2\}$

ii. Find the best fit straight line to the data (x_i, y_i), that is, find least-square solution $\mathbf{c}$ that minimizes $\|\mathbf{y} - \mathbf{A}\mathbf{c}\|^2$. [3]

iii. Compute the norm of the error $\|\mathbf{y} - \mathbf{A}\mathbf{c}\|^2$ using the results from question 2(b)ii. [1]

(c) Suppose there is an epidemic in which every month a quarter of those who are well become sick and 1/8 of those who are sick recover and acquire immunity for life.

i. Find the matrix that describe this Markov process [3]

ii. Find the steady state of this process. Briefly justify your answer. [2]

Question 2 continues on the next page

(d) Consider now the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \tag{2}$$

i. Find the pseudo-inverse of $\mathbf{A}$ [3]

ii. Consider the system of linear equations $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ given in Eq. (2). Can you claim that if a solution to this system of equations exist, it is unique? Briefly justify your answer. [1]

3. (a) Consider the column vectors $\mathbf{x}_i \in \mathbb{R}^d$, with $i = 1, 2, ..., N$ and consider the Euclidian distance matrix $\mathbf{D}$, where the entry $d_{i,j}$ is defined as $d_{i,j} = \|\mathbf{x}_j - \mathbf{x}_i\|^2$.

i. Using the properties of the inner product, show that $d_{i,j} = \mathbf{x}_i^T \mathbf{x}_i + \mathbf{x}_j^T \mathbf{x}_j - 2\mathbf{x}_i^T \mathbf{x}_j$ [Hint: remember that $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$]. [2]

ii. The matrix $\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \ldots \quad \mathbf{x}_N]$ has the coordinates of point $\mathbf{x}_i \in \mathbb{R}^d$ in column i. Express the matrix $\mathbf{X}^T\mathbf{X}$ in terms of the inner products $\mathbf{x}_i^T \mathbf{x}_j$, for $i = 1, 2, ..., N$ and $j = 1, 2, ..., N$. [2]

iii. What are the dimensions of $\mathbf{X}$? Hence, what is the maximum rank of $\mathbf{X}^T\mathbf{X}$? Justify your answer. [2]

iv. Show that we can express $\mathbf{D} = \mathbf{1} diag\left(\mathbf{X}^T\mathbf{X}\right)^T - 2\mathbf{X}^T\mathbf{X} + diag\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{1}^T$, where $\mathbf{1}$ denotes the column vector of all ones and $diag(\mathbf{A})$ is the column vector of the diagonal entries of $\mathbf{A}$. Hence, prove that the rank of $\mathbf{D}$ satisfies $rank(\mathbf{D}) \leq d + 2$. [5]

(b) Consider the following points in $\mathbb{R}^2$: $\mathbf{x}_1 = [1, 1]^T$, $\mathbf{x}_2 = [2, 1]^T$, $\mathbf{x}_3 = [2, 2]^T$, $\mathbf{x}_4 = [1, 2]^T$.

i. Compute matrix $\mathbf{M}$ where $m_{i,j} = (\mathbf{x}_i - \mathbf{x}_1)^T (\mathbf{x}_j - \mathbf{x}_1)$. [2]

ii. Factorise $\mathbf{M}$ using SVD as $\mathbf{M} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T$ and explain why $\mathbf{U} = \mathbf{V}$. Hence, find $\mathbf{Y}$ such that $\mathbf{M} = \mathbf{Y}^T\mathbf{Y}$. [5]

iii. Is your solution $\mathbf{Y}$ unique? Justify your answer. [2]