

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS

SOLUTIONS

1. (a) All vectors in V have the following shape

$$\mathbf{v}_i = \begin{bmatrix} x_1 \\ x_2 \\ 2x_2 - x_1 \end{bmatrix}$$

Therefore,

$$\mathbf{v}_i = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

This means that V has dimension two and that a basis is

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

- (b) We put the 4 vectors along the rows of the matrix \mathbf{A} and row reduce:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 & 2 \\ -1 & 8 & 6 & 5 \\ 3 & 0 & 4 & -1 \\ 5 & -4 & 3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 12 & 11 & 7 \\ 0 & -12 & -11 & -7 \\ 0 & -24 & -22 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 12 & 11 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore S has dimension 2 and a possible basis is: $[1, 4, 5, 2]^T, [0, 12, 11, 7]^T$.

- (c) We first row reduce the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 0 & 1 \\ 2 & 6 & -1 & 1 \end{bmatrix}$$

to find the dimension and the basis of its null space and range space. We have

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 1 & 3 & 0 & 1 \\ 2 & 6 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

Column one and three are linearly independent while the other two column can be obtained from one and three so range space has dimension 2 and the null space has dimension $n - \text{rank}(A) = 4 - 2 = 2$. A basis for the range space is given by the first and third column of \mathbf{A} , that is, $u_1 = (1, 1, 2)^T, u_2 = (1, 0, -1)^T$. We then find a basis for the null space by solving $\mathbf{Ax} = \mathbf{0}$ with \mathbf{A} in echelon form. We have

$$\begin{bmatrix} -3x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

So a basis for the null space is $(-3, 1, 0, 0)^T, (-1, 0, -1, 0)^T$. The range space of \mathbf{A}^T has dimension two since the null space had dimension two and a basis is, e.g., the first two rows of \mathbf{A} which are linearly independent, i.e., $(1, 3, 1, 2)^T, (1, 3, 0, 1)^T$. Finally the dimension of the null space of \mathbf{A}^T is $3 - \text{rank}(A) = 1$. We find its single basis element \mathbf{n}_1 by imposing $\langle u_1, n_1 \rangle = 0$ and $\langle u_2, n_1 \rangle = 0$ which yields a basis $n_1 = (1, -3, 1)^T$.

(d) The matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

is invertible so it doesn't affect the null space of \mathbf{A} . We find the echelon form for

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 2 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix has two pivots so it has two free columns. We find the null space by computing:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

which yield the two conditions $x_1 = -x_3 - 2x_4$ and $x_2 = 2x_3 + x_4$. Therefore the null space is given by:

$$\begin{bmatrix} -x_3 - 2x_4 \\ 2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for the null space is $(-1, 2, 1, 0)^T, (2, 1, 0, 1)^T$.

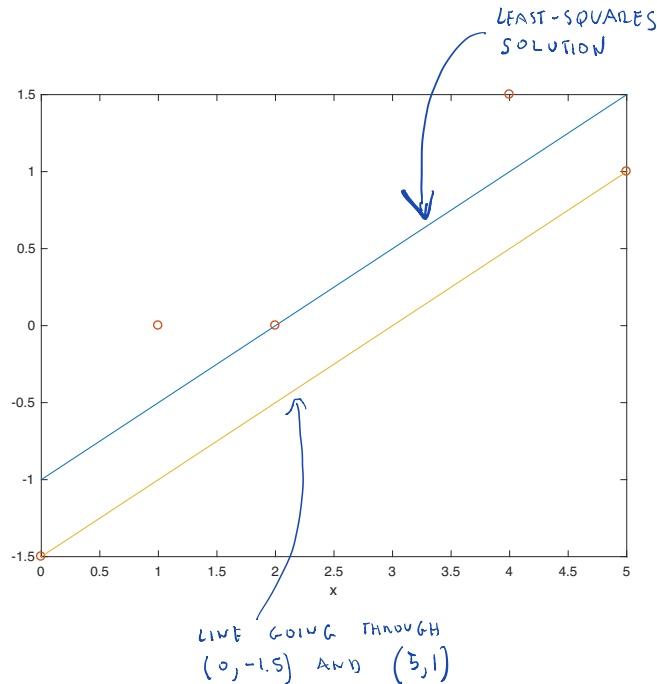
(e) The matrix \mathbf{A} is invertible if and only if its determinant is non-zero. This is a Vandermonde matrix and its determinant is $(t_2 - t_1)(t_3 - t_1)(t_3 - t_2)$. Therefore the matrix is invertible for arbitrary distinct values of t_1, t_2, t_3 . The best way to compute the determinant is by performing row reduction and then by using the determinant formula. This yields:

$$\begin{vmatrix} 1 & t_1 & t_1^2 \\ 0 & t_2 - t_1 & t_2^2 - t_1^2 \\ 0 & t_3 - t_1 & t_3^2 - t_1^2 \end{vmatrix} = (t_2 - t_1)(t_3 - t_1)(t_3 + t_1) - (t_3 - t_1)(t_2 - t_1)(t_2 + t_1) = (t_2 - t_1)(t_3 - t_1)(t_3 - t_2)$$

2. (a) We use the fact that $\text{trace}(\mathbf{ABC}) = \text{trace}(\mathbf{CAB})$, therefore

$$\text{trace}(\mathbf{A}) = \text{trace}\left(\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right) = 7.$$

- (b) i. The figure below shows the points and the lines.



- ii. We are after coefficient a, b such that $\sum_i |y_i - (ax_i + b)|^2$ is minimized. We build the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 1 \\ 4 & 1 \\ 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The least-squares solution is given by $\mathbf{A}^T \mathbf{A} \mathbf{c} = \mathbf{A}^T \mathbf{y}$ where $\mathbf{c} = [a, b]^T$ and $\mathbf{y} = [1, 1.5, 0, 0, -1.5]^T$. We have that:

$$\mathbf{A}^T \mathbf{A} = \begin{bmatrix} 46 & 12 \\ 12 & 5 \end{bmatrix}.$$

and

$$(\mathbf{A}^T \mathbf{A})^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} 5 & -12 \\ -12 & 46 \end{bmatrix}.$$

Consequently

$$\mathbf{c} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y} = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}.$$

- iii. $e_1 = 1$.
- iv. $a_2 = 0.5, b_2 = -1.5$ and $e_2 = 2.5 > e_1$.
- v. Eq (1) in the exam paper can be written in matrix/vector form as follows:
 $\|\mathbf{W}\mathbf{y} - \mathbf{W}\mathbf{A}\mathbf{c}_3\|^2$ where $\mathbf{c}_3 = [a_3, b_3]^T$, $\mathbf{W} = \text{diag}(1, 1, 1, 2, 2)$ and the other matrices have the same meaning as before. This is the ordinary least-squares problem with \mathbf{A} replace with $\mathbf{W}\mathbf{A}$ and \mathbf{y} replaced with $\mathbf{W}\mathbf{y}$. Therefore the solution is given by: $(\mathbf{W}\mathbf{A})^T \mathbf{W}\mathbf{A}\mathbf{c}_3 = (\mathbf{W}\mathbf{A})^T \mathbf{W}\mathbf{y}$. Since \mathbf{W} is diagonal we have: $\mathbf{A}^T \mathbf{W}^2 \mathbf{A}\mathbf{c}_3 = \mathbf{A}^T \mathbf{W}^2 \mathbf{y}$ which yields: $\mathbf{c}_3 = [0.5525, -1.0717]^T$
- (c) The matrix $\mathbf{A}^H \mathbf{A}$ is square by construction. We only need to show that its null space is trivial, i.e., it contains only the zero vector. Consider $\mathbf{A}^H \mathbf{A}\mathbf{x}$ and assume $\mathbf{A}^H \mathbf{A}\mathbf{x} = \mathbf{0}$, we want to show that this happens only when $\mathbf{x} = \mathbf{0}$. We have that $\mathbf{A}^H \mathbf{A}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^H \mathbf{A}^H \mathbf{A}\mathbf{x} = 0 \Rightarrow \|\mathbf{A}\mathbf{x}\|^2 = 0$. By definition of the norm the last equality is satisfied only when $\mathbf{A}\mathbf{x} = \mathbf{0}$ which in turns happens only when $\mathbf{x} = \mathbf{0}$ since the columns of \mathbf{A} are linearly independent.

3. (a) The matrix \mathbf{P} is self-adjoint ($\mathbf{P}^T = \mathbf{P}$) but is not idempotent: $\mathbf{P}^2 \neq \mathbf{P}$, therefore this is not a projection matrix.
- (b) i. This is a projection matrix since $(\mathbf{I} - \mathbf{P})^2 = (\mathbf{I} - 2\mathbf{P} + \mathbf{P}^2) = \mathbf{I} - \mathbf{P}$, where in the last equality we used the fact that $\mathbf{P}^2 = \mathbf{P}$ since \mathbf{P} is a projection. Moreover, $(\mathbf{I} - \mathbf{P})^T = \mathbf{I}^T - \mathbf{P}^T = \mathbf{I} - \mathbf{P}$ since $\mathbf{P}^T = \mathbf{P}$. Therefore the matrix $\mathbf{I} - \mathbf{P}$ is idempotent and self-adjoint.
- ii. Denote with V the subspace \mathbf{P} projects onto. Because of idempotency we know that if $\mathbf{x} \in V$ then $\mathbf{P}\mathbf{x} = \mathbf{x}$. We are now looking for vectors \mathbf{x} such that $(\mathbf{I} - \mathbf{P})\mathbf{x} = \mathbf{0}$; which means that $\mathbf{x} - \mathbf{P}\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{P}\mathbf{x}$. Therefore $\mathcal{N}(\mathbf{I} - \mathbf{P}) = V$.
- (c) i. Since the matrix is not full row rank we cannot claim that it has at least one solution. It may not have any solution. In this specific case since $\text{rank}(\mathbf{A}) = 1$ then the range space of \mathbf{A} has dimension one and so we cannot reach out every 2-dimensional vectors.
- ii. We look for the least-squares minimum-norm solution which is obtained by multiplying \mathbf{y} with the pseudo inverse of \mathbf{A} . We therefore first find the SVD of \mathbf{A} which is given by:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The pseudo-inverse is then given by:

$$\mathbf{A}^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/10 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/20 & -\sqrt{2}/20 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the solution is: $\mathbf{x} = \mathbf{A}^+\mathbf{y} = [\sqrt{2}/20, 0, 0]^T$.

- (d) We need to find the eigenvalues of either $\mathbf{A}^T\mathbf{A}$ or $\mathbf{A}\mathbf{A}^T$. We opt for $\mathbf{A}^T\mathbf{A}$ and the eigenvalues are $\lambda_1 = 10, \lambda_2 = 4$. Consequently, the singular values are $\sigma_1 = \sqrt{10}, \sigma_2 = 2$ and $\sigma_3 = 0$.
- (e) The right answer is (iii) since the matrix in (ii) is not rank one. Moreover while it is evident that (iii) is better than (i) we also note that the singular values of \mathbf{E} are $\sigma_1 = 2$ and $\sigma_2 = 1$ and the singular vectors are the canonical basis. Consequently the best rank one approximation, $\sigma_1 \mathbf{u}\mathbf{v}_1^H$, corresponds to the matrix in (iii).