Imperial College London

Digital Signal Processing

Learning through problem solving

Discrete Time Fourier Transform DTFT

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Problems

- **Problem:** Find the DTFT of a shifted discrete Dirac function $\delta[n-k]$:

  Solution:

$$\Delta(\omega) = \sum_{n=-\infty}^{\infty} \delta\,[n-k]e^{-j\omega n} = e^{-j\omega k}$$

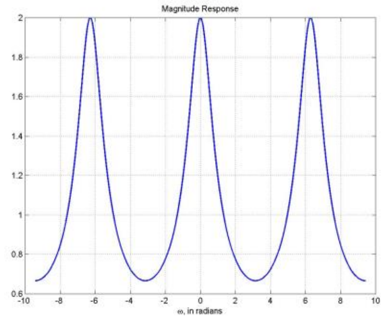
- Notice that the amplitude is 1. What does this mean?

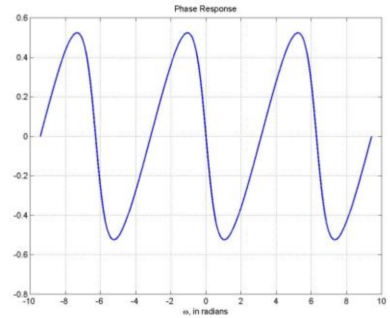
- What is the phase? Can you plot it?

- **Problem:** Find the DTFT of the causal sequence $x[n] = \alpha^n u[n], |\alpha| < 1$. Plot the magnitude and the phase for $\alpha = 0.5$.

  Solution:

  - $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty}(\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$ if $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$
  - For $\alpha = 0.5$, the magnitude and phase of $X(e^{j\omega}) = 1/(1-0.5e^{-j\omega})$ are shown below.





Problem

- **Problem:** Verify the IDTFT relationship

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Solution:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega\right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\pi\,(n-\ell)}{\pi(n-\ell)}$$

$\frac{\sin\pi(n-\ell)}{\pi(n-\ell)} = \begin{cases}1 & n=\ell\\ 0 & n\neq\ell\end{cases}$ or we can write: $\frac{\sin\pi(n-\ell)}{\pi(n-\ell)} = \delta[n-\ell]$

Hence,

$$\sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\pi\,(n-\ell)}{\pi(n-\ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] = x[n]$$

Relationship between energy and absolute summability

- The sequence $x[n] = \alpha^n u[n]$ is absolutely summable for $|\alpha| < 1$ since

$$\sum_{n=-\infty}^{\infty} |\alpha^n| u[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT converges uniformly to $\frac{1}{1-\alpha e^{-j\omega}}$. Please take time to prove this.

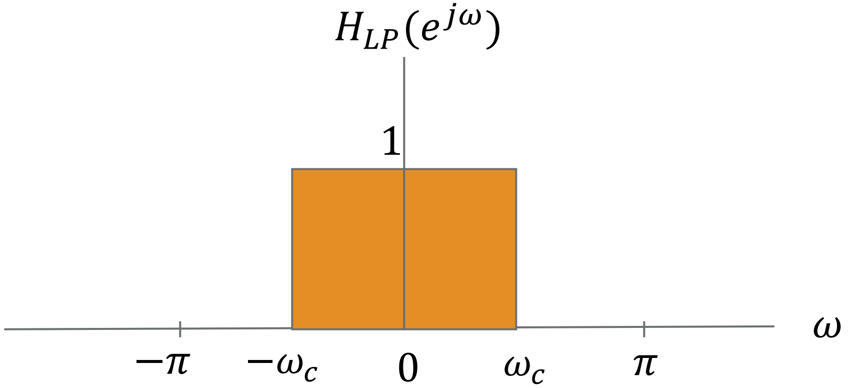
- Note that:
  - Since $\sum_{n=-\infty}^{\infty} |x[n]|^2 = (\mathbf{Energy}) \le \left(\sum_{n=-\infty}^{\infty} |x[n]|\right)^2$, an absolutely summable sequence has always finite energy.
  - However, a finite energy sequence is not necessarily absolutely summable.
  - The sequence $x[n] = \begin{cases} 1/n & n \ge 1 \\ 0 & n \le 0 \end{cases}$

    has finite energy equal to $\sum_{n=1}^{\infty} (1/n)^2 = \pi^2/6$ but is not absolutely summable.

Discrete Time Fourier Transform: convergence

- Consider the DTFT:

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$



- The inverse DTFT is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin \omega_c n}{\pi n},\ -\infty < n < \infty$$

- The energy of $h_{LP}[n]$ is given by $E_h = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_{LP}(e^{j\omega})\right|^2 d\omega = \frac{\omega_c}{\pi}$.
- $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable.

Problem (cont from previous slide)

- **Problem:** Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin\omega_c n}{\pi n}, \ -\infty < n < \infty$$

Solution

Here,

$$\sum_{n=-\infty}^{\infty} \left|h_{LP}[n]\right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega})\right|^2 d\omega$$

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Therefore,

$$\sum_{n=-\infty}^{\infty} \left|h_{LP}[n]\right|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

Hence, $h_{LP}[n]$ is a finite energy sequency.

Discrete Time Fourier Transform: convergence cont.

- The infinite sum below

$$\sum_{n=-K}^{K} h_{LP}[n]e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

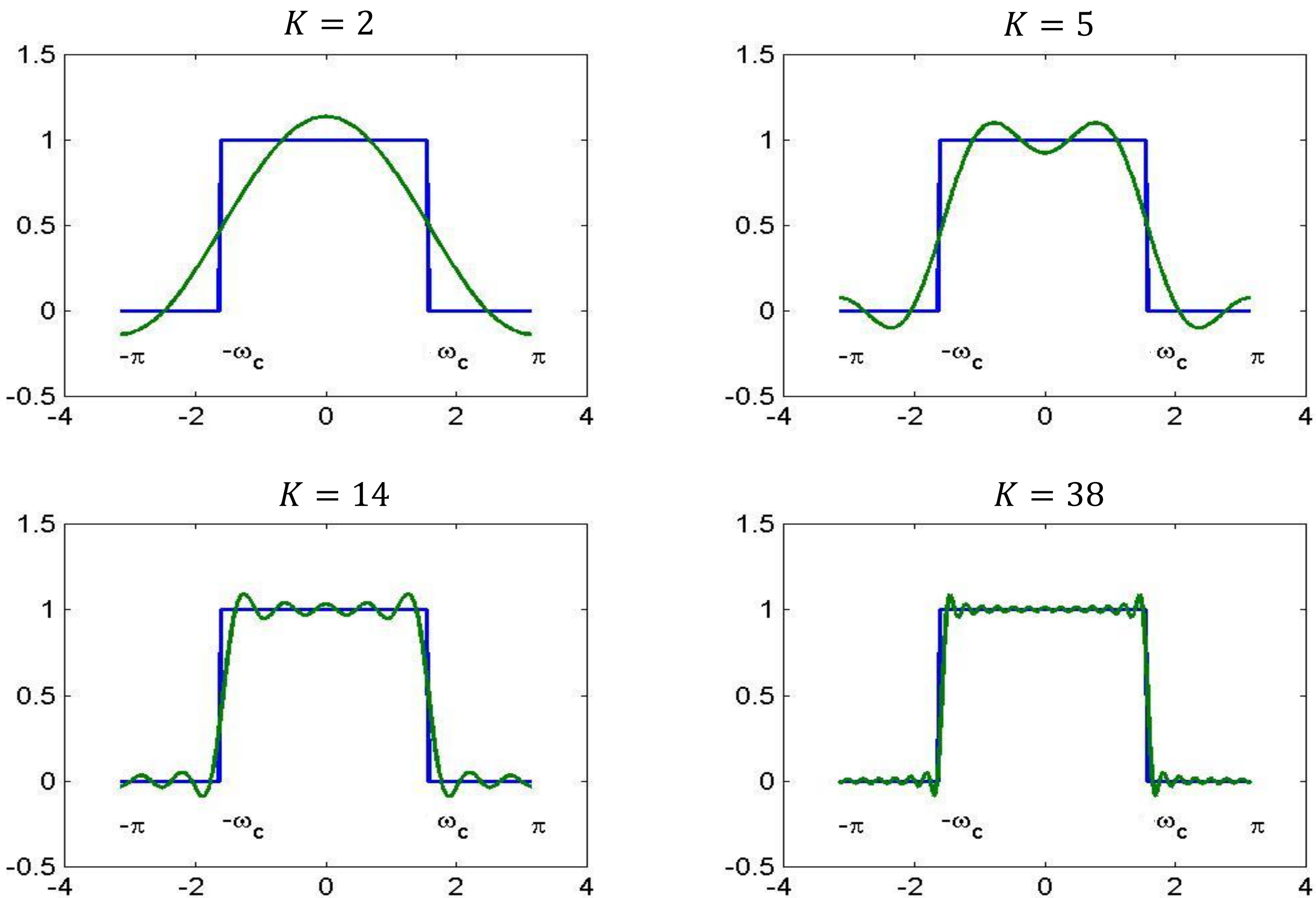
for all values of ω but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense.

- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of K as shown next.

Discrete Time Fourier Transform: convergence cont.



Discrete Time Fourier Transform: convergence cont.

- As it can be seen from these plots, independently of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the points $\omega = \pm\,\omega_c$.
- The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K.
- As K goes to infinity, we can prove that the condition

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega})\right|^2 d\omega = 0$$

  holds, indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$.
- The oscillatory behavior observed in $H_{LP,K}(e^{j\omega})$ is known as the **Gibbs phenomenon**.

Problem

- **Problem:** Consider the complex exponential sequence $x[n] = e^{j\omega_o n}$, ω_o real. Show that its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and $-\pi \le \omega_o \le \pi$.

Solution

In that case, it is easier to verify the above by taking IDTFT of $X(e^{j\omega})$:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - (\omega_o - 2\pi k))e^{j\omega n}d\omega$$

The impulses are located at ω_o, $\omega_o - 2\pi$, $\omega_o + 2\pi$, $\omega_o - 4\pi$, $\omega_o + 4\pi$, etc.

Since the integration is from $-\pi$ to π and because $-\pi \le \omega_o \le \pi$ we are only left with one term inside the integration as follows:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_o)e^{j\omega n}d\omega = \int_{-\pi}^{\pi}\delta(\omega - \omega_o)e^{j\omega n}d\omega = e^{j\omega_o n}$$

Problem

- **Problem:** Determine the DTFT of the sequence

$$y[n] = (n+1)\alpha^n u[n], |\alpha| < 1$$

Solution

Let $x[n] = \alpha^n u[n], |\alpha| < 1$. We can, therefore, write

$$y[n] = nx[n] + x[n]$$

The DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

Using the differentiation property of the DTFT (look at Tables provided), we observe that the DTFT of $nx[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$$

Next, using the linearity property of the DTFT given in previous tables we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

Problem

- **Problem:** Determine the DTFT of the sequence $v[n]$ defined by
$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

Solution

It is straightforward to show that the DTFT of $\delta[n]$ is 1.

Using the time-shifting property of the DTFT given in previous Tables, we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$.

Using the linearity property we obtain the frequency-domain representation of
$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$ as
$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$
Solving the above equation for $V(e^{j\omega})$ we get
$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\varepsilon_g = \sum_{n=-\infty}^{\infty} \left|g[n]\right|^2$$

- From Parseval's Theorem we know that

$$\varepsilon_g = \sum_{n=-\infty}^{\infty} \left|g[n]\right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|G(e^{j\omega})\right|^2 d\omega$$

- The quantity $S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$ is called the **energy density spectrum**.
- The area under the function $\left|G(e^{j\omega})\right|^2$ in the range $-\pi \leq \omega \leq \pi$ divided by 2π is the energy of the sequence.