IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2021

Mathematics for Signals and Systems

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates:

For questions involving matrix manipulations or calculations, please provide answers showing intermediate calculations steps.

The Questions

1. (a) Let $\mathbf{A} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & -1 & -1 \end{bmatrix}$.

   i. Find the singular values of $\mathbf{A}$. [2]

   ii. Given that the two left singular vectors are $\mathbf{u_1} = [1, 0]^T$ and $\mathbf{u_2} = [0, 1]^T$, find the four right singular vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. [2]

   iii. Assume $\mathbf{b} = \begin{bmatrix} 20 \\ -4 \end{bmatrix}$. One solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x}_1 = [1, 2, 3, 4]^T$. Compute the pseudo-inverse of $\mathbf{A}$ and find the solution $\mathbf{x}_2 = \mathbf{A}^+\mathbf{b}$. Compare the two solutions and explain why they are different. [2]

   (b) Show that if $\mathbf{A}$ is nonsingular, then $\mathbf{AA}^H$ is positive definite. [2]

   (c) Determine the four fundamental subspaces of

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -2 & -8 \\ 3 & 12 \end{bmatrix}.$$

[3]

Question 1 continues on the next page

(d) Your cat has three places where it likes to hide: under your bed (location 1), under the kitchen table (location 2), and under the sofa (location 3). Your cat changes place every hour. The probability it goes from location 1 to location 2 or from 1 to 3 is the same. The probability it goes from location 3 to location 2 is 0.2 and from location 2 to location 1 is 0.6.

i. Find the matrix that describes this Markov process [2]

ii. Assuming that your cat has changed places many times and now you need to find him quickly, which of the three locations should you visit first? Justify your answer and any assumption you made. [2]

(e) Assume $\mathbf{P} \in \mathbb{R}^{N \times N}$ is a "Markov matrix". This means that the columns of $\mathbf{P}$ add to 1 and that all the entries are positive or zero. Assume that $\mathbf{P}$ has a complete set of eigenvectors, does not have repeated eigenvalues and that the largest eigenvalue is $\lambda_1 = 1$ while $|\lambda_i| < 1$ for $i = 2, 3, .., N$.

i. Show that $\mathbf{P^T}$ is a matrix with eigenvalue $\gamma = 1$ and eigenvector $\mathbf{1} = [1, 1, ..., 1]^T$. [2]

ii. Show that the matrix $\mathbf{P}^n = \mathbf{X}\mathbf{\Lambda}^n\mathbf{X}^{-1}$ converges to the rank one matrix $\mathbf{P}^{\infty} = \mathbf{u}_1\mathbf{1}^T$ when $n \to \infty$. Here, $\mathbf{X}$ is the matrix whose columns are the eigenvectors of $\mathbf{P}$, $\mathbf{\Lambda}$ is diagonal with the eigenvalues of $\mathbf{P}$ along the main diagonal and $\mathbf{u}_1$ is the eigenvector of $\mathbf{P}$ related to the eigenvalue $\lambda_1 = 1$. [3]

2. (a) Consider the sequence x_n with support $N = 2$, that is, it is assumed that x_n is zero for $n \neq 0, 1, ..., N-1$. This sequence is filtered with a filter h_n with unit impulse response

$$h_n = \begin{cases} 1 & \text{for } n = 0 \\ 1 & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We denote the filtered sequence with $y_n = h_n * x_n$ where $*$ denotes the convolution operation.

i. The linear convolution can be written in matrix/vector form as $\mathbf{y} = \mathbf{Hx}$. Write the exact expression of the matrix $\mathbf{H}$ that describes the linear convolution with the filter h_n. [3]

ii. Find a basis for the range and null spaces of $\mathbf{H}$. [3]

iii. The sequence y_n that you observe has unfortunately been corrupted by noise and is given by $\mathbf{y} = [1, 2, 0]^T$. Find the $\mathbf{x}$ that minimises $\|\mathbf{y} - \mathbf{Hx}\|^2$. [4]

iv. You are told that the correct vector $\mathbf{x}$ has entries that add to one. Find the solution $\mathbf{x}$ that minimises $\|\mathbf{y} - \mathbf{Hx}\|^2$ while also satisfying the condition that the sum of the elements in $\mathbf{x}$ is equal to one. [4]

Question 2 continues on the next page

(b) Consider now a new sequence x_n and assume that this sequence is filtered with a filter h_n with unit impulse response

$$h_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Assume that x_n is periodic with period $N = 4$.

i. Write the exact expression of the matrix $\mathbf{H}$ that describes the circulant convolution with the filter h_n. [3]

ii. Find a basis for the range and null spaces of $\mathbf{H}$. [3]

3. (a) Consider the problem of transforming a given matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$ into a given matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ using an orthogonal transformation $\mathbf{S} \in \mathbb{R}^{m\times m}$, such that $\|\mathbf{B} - \mathbf{SA}\|_F^2$ is minimised. Here $\|\cdot\|_F$ denotes the Frobenius norm. In what follows we will find a solution for $\mathbf{S}$.

i. Show that $\text{trace}(\mathbf{A}^T\mathbf{A}) = \|\mathbf{A}\|_F^2$. [1]

ii. Show that $\mathbf{A^T B}$ and $\mathbf{B^T A}$ have the same diagonal and therefore $\text{trace}(\mathbf{A}^T\mathbf{B}) = \text{trace}(\mathbf{B}^T\mathbf{A})$. [1]

iii. Then show that finding an $\mathbf{S}$ that minimises $\|\mathbf{B}-\mathbf{SA}\|_F^2$ is equivalent to finding an $\mathbf{S}$ that maximises $\text{trace}(\mathbf{A^T S^T B})$. [5]

iv. Take the SVD of $\mathbf{BA^T}$ yielding $\mathbf{BA^T} = \mathbf{U\Sigma V^T}$, then show that $\text{trace}(\mathbf{A^T S^T B}) = \text{trace}(\mathbf{Z\Sigma})$ with $\mathbf{Z} = \mathbf{V^T S^T U}$. [2]

v. Show that $\mathbf{Z}$ is an orthogonal matrix. [2]

vi. Since $\mathbf{Z}$ is orthogonal, the maximum of $\text{trace}(\mathbf{Z\Sigma})$ is reached when $\mathbf{Z} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix (proof not required). Based on the fact that $\mathbf{Z} = \mathbf{I}$, find the final expression for $\mathbf{S}$. [1]

(b) The points with coordinates $\mathbf{b_1} = [1,1]^T$, $\mathbf{b_2} = [0,1]^T$ and $\mathbf{b_3} = [0,0]^T$ are believed to be a rotation of the points: $\mathbf{a_1} = [1,0]^T$, $\mathbf{a_2} = [1,1]^T$ and $\mathbf{a_3} = [0,0]^T$.

i. What are matrices $\mathbf{A}$ and $\mathbf{B}$ that we want to map? [2]

ii. Compute matrix $\mathbf{M} = \mathbf{B}\mathbf{A}^{\mathbf{T}}$ and find the singular value decomposition of $\mathbf{M}$. [4]

iii. Hence, determine the amount of rotation described by $\mathbf{S}$ and check that $\mathbf{S}^{\mathbf{T}}\mathbf{S} = \mathbf{I}$. [2]