IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2020

MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS (**version 1**)

SOLUTIONS

1. (a) We put the 4 vectors along the rows of the matrix $\mathbf{A}$ and row reduce:

$$\mathbf{A} = \begin{bmatrix} 2 & 6 & -4 & 8 \\ 1 & 4 & -2 & 2.5 \\ 1 & 1 & -2 & 7 \\ 3 & 5 & -6 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 6 & -4 & 8 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore S has dimension 2 and a possible basis is: $[2,6,-4,8]^T, [0,1,0,-1.5]^T$. The orthogonal basis is $\mathbf{u}_1 = [2,6,-4,8]^T$, the second vector must be orthogonal to $\mathbf{u}_1$. We denote with $\mathbf{b} = [0,1,0,-1.5]^T$, therefore, $\mathbf{u}_2 = \mathbf{b} - \mathbf{u_1}(\mathbf{u_1^T u_1})^{-1}\mathbf{u_1^T b} = [0.2, 2.6, -0.4, -2.2]^T$. The final answer is $\mathbf{q}_1 = \mathbf{u}_1/\|\mathbf{u}_1\|$ and $\mathbf{q}_2 = \mathbf{u}_2/\|\mathbf{u}_2\|$.

(b) The inverse exists if and only if the determinant is non-zero. We have

$$|\mathbf{A}| = \begin{vmatrix} -1 & 1 & 2 \\ 0 & a & b \\ -1 & 3 & 2 \end{vmatrix} = a \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} - b \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = -2b.$$

Therefore, the inverse exists if and only if $b \neq 0$, it exists for any $a \in \mathbb{C}$.

(c) In this case the null space of $\mathbf{C}$ is the same as the null space of $\mathbf{A}$ since $\mathbf{B}$ is full column rank and the null space is given by $\text{span}\{[2,0,1]^T\}$.

(d) i. The projection operator is:

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

where $\mathbf{A}$ is the matrix whose columns are $\mathbf{p}_1, \mathbf{p}_2$. Therefore $\mathbf{P}$ is

$$\mathbf{P} = \begin{bmatrix} 1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & 1/4 & -1/4 \\ -1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & -1/4 & 1/4 & 3/4 \end{bmatrix}$$

and

$$\hat{\mathbf{x}}_1 = \mathbf{P}\mathbf{x}_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

ii. The best way to solve this problem is by noticing that the projector operator $\mathbf{P}_2$ onto S is $\mathbf{I} - \mathbf{P}$ where $\mathbf{I}$ is the identity matrix. Consequently

$$\hat{\mathbf{x}}_2 = \mathbf{P}_2\mathbf{x}_2 = \begin{bmatrix} 1.75 \\ 1.25 \\ -0.75 \\ 1.25 \end{bmatrix}.$$

2. (a) i. Minimising $\|\mathbf{Ax}-\mathbf{b}\|^2$ is equivalent to minimising $\|\hat{\mathbf{b}}-\mathbf{b}\|^2$ with $\hat{\mathbf{b}}=\mathbf{A}\hat{\mathbf{x}}\in\mathcal{R}(\mathbf{A})$, the column space of $\mathbf{A}$. By the projection theorem, $\hat{\mathbf{b}}-\mathbf{b}$ is orthogonal to $\mathcal{R}(\mathbf{A})$, which means that $\hat{\mathbf{b}}-\mathbf{b}\in\mathcal{N}(\mathbf{A}^{\mathbf{H}})$, the nullspace of $\mathbf{A}^{\mathbf{H}}$. Hence,

$$\mathbf{A}^{\mathbf{H}}(\hat{\mathbf{b}}-\mathbf{b})=\mathbf{0}\Rightarrow\mathbf{A}^{\mathbf{H}}\mathbf{A}\hat{\mathbf{x}}=\mathbf{A}^{\mathbf{H}}\mathbf{b}$$

Conversely,

$$\mathbf{A}^{\mathbf{H}}\mathbf{A}\hat{\mathbf{x}}=\mathbf{A}^{\mathbf{H}}\mathbf{b}\Rightarrow\mathbf{A}^{\mathbf{H}}(\hat{\mathbf{b}}-\mathbf{b})=\mathbf{0}$$

Therefore, the vector $\hat{\mathbf{x}}$ minimises $\|\mathbf{Ax}-\mathbf{b}\|^2$ if and only if $\mathbf{A}^{\mathbf{H}}\mathbf{A}\hat{\mathbf{x}}=\mathbf{A}^{\mathbf{H}}\mathbf{b}$.

ii. No. For the minimiser to be unique, the null space of $\mathbf{A}$ has to be trivial. Full column rank guarantees a trivial null space but full row rank does not.

(b) i. $\mathbf{y}$ and $\mathbf{A}$ can be expressed as

$$\mathbf{y}=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix},\mathbf{A}=\begin{bmatrix}x_1&1\\x_2&1\\x_3&1\\x_4&1\end{bmatrix}.$$

The full system can hence be written as $\mathbf{y}=\mathbf{Ac}+\mathbf{e}$:

$$\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}x_1&1\\x_2&1\\x_3&1\\x_4&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}+\mathbf{e}\Rightarrow\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}c_1x_1+c_2\\c_1x_2+c_2\\c_1x_3+c_2\\c_1x_4+c_2\end{bmatrix}+\mathbf{e}.$$

ii. The solution is $\mathbf{c}=(\mathbf{A}^{\mathbf{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{H}}\mathbf{y}$. Substituting $x_i=\{-2,-1,0,1\}$ and $y_i=f(x_i)=\{1,0,0,2\}$ into the equation yields

$$\mathbf{A}^{\mathbf{H}}\mathbf{A}=\begin{bmatrix}6&-2\\-2&4\end{bmatrix}\Rightarrow(\mathbf{A}^{\mathbf{H}}\mathbf{A})^{-1}=\frac{1}{20}\begin{bmatrix}4&2\\2&6\end{bmatrix}=\begin{bmatrix}0.2&0.1\\0.1&0.3\end{bmatrix}$$

$$\mathbf{c}=\begin{bmatrix}0.2&0.1\\0.1&0.3\end{bmatrix}\begin{bmatrix}-2&-1&0&1\\1&1&1&1\end{bmatrix}\begin{bmatrix}1\\0\\0\\2\end{bmatrix}=\begin{bmatrix}0.3\\0.9\end{bmatrix}$$

iii.

$$\mathbf{Ac}=\begin{bmatrix}-2&1\\-1&1\\0&1\\1&1\end{bmatrix}\begin{bmatrix}0.3\\0.9\end{bmatrix}=\begin{bmatrix}0.3\\0.6\\0.9\\1.2\end{bmatrix}$$

$$\|\mathbf{y}-\mathbf{Ac}\|^2=0.7^2+(-0.6)^2+(-0.9)^2+0.8^2=2.3.$$

(c) i. The transition matrix which describes the process is:

$$M=\begin{bmatrix}0.75&0&0\\0.25&7/8&0\\0&1/8&1\end{bmatrix}$$

ii. This is a Markov matrix so we know that at least one eigenvalue is $\lambda_1 = 1$ and the corresponding eigenvector corresponds to the steady state. We only need to check that there is only one eigenvalue equal to one, otherwise we get more than one steady state. We know that the trace satisfies $trace(M) = \lambda_1 + \lambda_2 + \lambda_3$ and $det(M) = \lambda_1\lambda_2\lambda_3$. Since $\lambda_1 = 1$ then the other two eigenvalues are $\lambda_2 = 7/8$ and $\lambda_3 = 0.75$ and are smaller than one. The eigenvector related to λ_1 is the steady state and is equal to $[0, 0, 1]$. This is not surprising because it indicates that eventually the entire population will become immune.

(d) i. The matrix has rank one, so one singular values is $\sigma_2 = 0$ and the other is $\sigma_1 = 5$ since $\mathbf{A}$ is symmetric. The singular vectors are

$$\mathbf{U} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$$

and

$$\mathbf{V} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$$

The pseudo inverse of $\mathbf{A}$ is given by:

$$\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^T,$$

where

$$\mathbf{\Sigma}^+ = \begin{bmatrix} 1/5 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore

$$\mathbf{A}^+ = \frac{1}{25} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

ii. We cannot claim that if a solution exists it is unique since the matrix is not full column rank.

3. (a) i. The norm of the vector can be expressed as an inner product as follows:

$$\begin{aligned}\|\mathbf{x}_j - \mathbf{x}_i\|^2 &= \langle \mathbf{x}_j - \mathbf{x}_i, \mathbf{x}_j - \mathbf{x}_i \rangle = \left(\mathbf{x}_j - \mathbf{x}_i\right)^T \left(\mathbf{x}_j - \mathbf{x}_i\right) \\ &= \mathbf{x}_j^T \mathbf{x}_j - \mathbf{x}_j^T \mathbf{x}_i - \mathbf{x}_i^T \mathbf{x}_j + \mathbf{x}_i^T \mathbf{x}_i \\ &\overset{(a)}{=} \mathbf{x}_j^T \mathbf{x}_j - 2\mathbf{x}_i^T \mathbf{x}_j + \mathbf{x}_i^T \mathbf{x}_i,\end{aligned}$$

where (a) follows from the fact that inner products of real vectors are commutative.

ii. We can express matrix $\mathbf{X}^T\mathbf{X}$ as follows:

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_N\end{bmatrix}^T \begin{bmatrix}\mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_N\end{bmatrix} = \begin{bmatrix}\mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_2 & \dots & \mathbf{x}_1^T\mathbf{x}_N \\ \mathbf{x}_2^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \dots & \mathbf{x}_2^T\mathbf{x}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_N^T\mathbf{x}_1 & \mathbf{x}_N^T\mathbf{x}_2 & \dots & \mathbf{x}_N^T\mathbf{x}_N\end{bmatrix}.$$

iii. The dimensions of $\mathbf{X}$ are $d \times N$, which shows that the rank of $\mathbf{X}$ cannot exceed d. Since the rank of a matrix product $\mathbf{AB}$ cannot exceed the rank of the individual matrices $\mathbf{A}$ and $\mathbf{B}$ respectively, this shows that the rank of $\mathbf{X}^T\mathbf{X}$ cannot exceed d.

iv. We begin by expading $\mathbf{1}diag\left(\mathbf{X}^T\mathbf{X}\right)^T$ as:

$$\mathbf{1}diag\left(\mathbf{X}^T\mathbf{X}\right)^T = \begin{bmatrix}1 \\ 1 \\ \vdots \\ 1\end{bmatrix} \begin{bmatrix}\mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \dots & \mathbf{x}_N^T\mathbf{x}_N\end{bmatrix} = \begin{bmatrix}\mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \dots & \mathbf{x}_N^T\mathbf{x}_N \\ \mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \dots & \mathbf{x}_N^T\mathbf{x}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_2^T\mathbf{x}_2 & \dots & \mathbf{x}_N^T\mathbf{x}_N\end{bmatrix}.$$

The matrix $diag\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{1}^T$ is the transpose of $\mathbf{1}diag\left(\mathbf{X}^T\mathbf{X}\right)^T$:

$$diag\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{1}^T = \begin{bmatrix}\mathbf{x}_1^T\mathbf{x}_1 & \mathbf{x}_1^T\mathbf{x}_1 & \dots & \mathbf{x}_1^T\mathbf{x}_1 \\ \mathbf{x}_2^T\mathbf{x}_2 & \mathbf{x}_2^T\mathbf{x}_2 & \dots & \mathbf{x}_2^T\mathbf{x}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_N^T\mathbf{x}_N & \mathbf{x}_N^T\mathbf{x}_N & \dots & \mathbf{x}_N^T\mathbf{x}_N\end{bmatrix}.$$

We already have an expression for $\mathbf{X}^T\mathbf{X}$ from part ii, and hence using $\mathbf{D} := (d_{ij}) = \mathbf{x}_j^T\mathbf{x}_j - 2\mathbf{x}_i^T\mathbf{x}_j + \mathbf{x}_i^T\mathbf{x}_i$, it follows immediately that $\mathbf{D} = \mathbf{1}diag\left(\mathbf{X}^T\mathbf{X}\right)^T - 2\mathbf{X}^T\mathbf{X} + diag\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{1}^T$.

We also notice that the matrix $\mathbf{1}diag\left(\mathbf{X}^T\mathbf{X}\right)^T$ has repeated rows, and hence it is a rank-one matrix. Similarly, its transpose, $diag\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{1}^T$ also has rank one.

Moreover, in part iii, we showed that $rank(\mathbf{X}^T\mathbf{X}) \leq d$.

We can then use the property of the rank of a sum of matrices. Since we know that $rank(\mathbf{A} + \mathbf{B}) \leq rank(\mathbf{A}) + rank(\mathbf{B})$, we deduce that $rank(\mathbf{D}) \leq rank(\mathbf{1}diag\left(\mathbf{X}^T\mathbf{X}\right)^T) + rank(\mathbf{X}^T\mathbf{X}) + rank(diag\left(\mathbf{X}^T\mathbf{X}\right)\mathbf{1}^T) \leq 1 + d + 1 = d + 2$.

(b) i. Using the formula given for $\mathbf{M}$, we get:

$$\mathbf{M} = \begin{bmatrix}0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1\end{bmatrix}.$$

ii. Using singular value decomposition, we want to express $\mathbf{M} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T$.
We first need to find the eigenvalues of $\mathbf{M}^T\mathbf{M}$, where:

$$\mathbf{M}^T\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix}.$$

The eigenvalues λ of $\mathbf{M}^T\mathbf{M}$ can be found by setting the determinant of $\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}$ to zero. This gives the equation $\lambda^2\left(-\lambda^2 + 10\lambda - 9\right) = 0$, which has the repeated root $\lambda_1 = \lambda_2 = 0$ and two other roots, $\lambda_3 = 9$ and $\lambda_4 = 1$ respectively.
Hence, matrix $\mathbf{\Lambda}$ in the SVD factorisation of $\mathbf{M}$ is:

$$\mathbf{\Lambda} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

The eigenvectors of $\mathbf{M}^T\mathbf{M}$ will then be the columns of $\mathbf{U}$. For each eigevalue λ_i we find the corresponding eigenvector $\mathbf{u}_i$ by solving $\mathbf{M}^T\mathbf{M}\mathbf{u}_i = \lambda_i\mathbf{u}_i$.
Let $\mathbf{u_1} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$. Then, to find the two eigenvectors corresponding to the repeated eigenvalue $\lambda_1 = \lambda_2 = 0$ we have that:

$$\mathbf{M}^T\mathbf{M}\mathbf{u_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can solve the system of equations using any method (e.g. Gaussian elimination with back substitution) above to obtain:

$$\begin{cases} x_3 = -x_2 \\ x_4 = x_2 \end{cases},$$

which gives us the general form of eigenvector:

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

We can then obtain the two eigenvectors corresponding to the repeated eigenvalue:

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T,$$

and

$$\mathbf{u}_2 = \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}^T,$$

which we can normalise to obtain:

$$\tilde{\mathbf{u}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\tilde{\mathbf{u}}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix}^T.$$

Similarly, the eigenvector corresponding to $\lambda = 9$ is:

$$\tilde{\mathbf{u}}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 1 & 2 & 1 \end{bmatrix}^T,$$

and the eigenvector corresponding to $\lambda = 1$ is:

$$\tilde{\mathbf{u}}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T.$$

We can then form matrix $\mathbf{U}$ as:

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

Since $\mathbf{M}$ is symmetric, $\mathbf{V} = \mathbf{U}$, which means we can factorise $\mathbf{M}$ as;

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

iii. Since we can write $\mathbf{M} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T$, we can set $\mathbf{Y} = \sqrt{\boldsymbol{\Lambda}}\mathbf{U}^T$ to get $\mathbf{M} = \mathbf{Y}^T\mathbf{Y}$. If we replace $\mathbf{Y}$ with $\mathbf{QY}$, where $\mathbf{Q}$ is an arbitrary orthogonal matrix, we get $\tilde{\mathbf{M}} = \mathbf{Y}^T\mathbf{Q}^T\mathbf{QY} = \mathbf{Y}^T\mathbf{Y} = \mathbf{M}$. This shows that the solution is not unique. In fact, there is an infinite number of solutions $\mathbf{Y}$ that satisfy $\mathbf{M} = \mathbf{Y}^T\mathbf{Y}$.