IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2022

Mathematics for Signals and Systems

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates:

For questions involving matrix manipulations or calculations, please provide answers showing intermediate calculations steps.

A Hermitian matrix $\mathbf{A}$ is positive definite if and only if:

- All the eigenvalues satisfy $\lambda_i > 0$
- All the pivots (without row exchange) satisfy $d_i > 0$

The Questions

1. (a) Consider the system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{3\times3}$ and $\mathbf{b} = [9, -4.5, -4.5]^T$. The complete solution to this system is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} + a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

for arbitrary a and b.

i. Find the rank of $\mathbf{A}$ [2]

ii. Find $\mathbf{A}$ [2]

iii. Find the minimum norm solution. [2]

(a) You want to approximate a function $y(t)$ with a polynomial $p(t)$. You observe $y(t)$ at instants $t_1 = 0, t_2 = 1, t_3 = 2, t_4 = 4$ and $t_5 = 5$ and we have that $y_1 = y(t_1) = -1$, $y_2 = y(t_2) = 0.5$, $y_3 = y(t_3) = 0.5$, $y_4 = y(t_4) = 2$ and $y_5 = y(t_5) = 1.5$.

i. Assume you want to find a polynomial of degree two that satisfies $p(t_1) = y_1$, $p(t_4) = y_4$ and $p(t_5) = y_5$. Can you claim that this polynomial exists and is unique? Briefly justify your answer. [4]

ii. Find a polynomial $p_1(t) = a_1t + b_1$, that satisfies $p_1(t_1) = y_1$ and $p_1(t_4) = y_4$. [2]

iii. Find now a polynomial $p_2(t) = a_2t + b_2$ that minimises

$$e_2 = \sum_{i=1}^{5} |y_i - a_2t_i - b_2|^2.$$

[4]

iv. Can you claim that the solution you found is unique? [2]

v. Verify that $e_2 < \sum_{i=1}^{5} |y_i - a_1t_i - b_1|^2$. [2]

2. (a) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

You are told that the largest singular value of $\mathbf{A}$ is $\sigma_1 = 6$, the singular vector $\mathbf{u}_1 = [1/\sqrt{6}, 2/\sqrt{6}, 1/\sqrt{6}]^T$ and $\mathbf{v}_2 = [1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}]^T$.

i. Find all the singular values of $\mathbf{A}$ [2]

ii. Find the SVD of $\mathbf{A}$ [4]

iii. Find the least-square minimum norm solution to $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{b} = [9, 9, -9]^T.$$

[4]

(b) Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A}$ given by:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & c \end{bmatrix}.$$

i. Assume $c = 0.5$, can you guarantee that the system has always at least one solution? Justify your answer. [2]

ii. Assume now that $c = 1$ and that $\mathbf{b} = [-1, 2]^T$, find the minimum norm solution to $\mathbf{Ax} = \mathbf{b}$. [2]

(c) Try to solve the following constrained minimisation. Given the vector $\mathbf{y} = [1, 1, -1]^T$, find the three dimensional vector $\mathbf{x} = [x_1, x_2, x_3]^T$ that is closest in the least-squares sense to $\mathbf{y}$ among all three-dimensional vectors that have average value $\mu = 1$. That is find the $\mathbf{x}$ that minimises $\|\mathbf{y} - \mathbf{x}\|^2$ subject to $(x_1/3 + x_2/3 + x_3/3) = 1$. [4]

3. (a) Consider the matrix

$$\mathbf{Q} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Is $\mathbf{Q}$ positive definite and why? [2]

(b) Let $\boldsymbol{Q}$ be an $n \times n$ positive definite matrix. The directions $\boldsymbol{d}_0, \boldsymbol{d}_1, ..., \boldsymbol{d}_k \in \mathbb{R}^n$ are called $\mathbf{Q}$-conjugate if for all i, $\boldsymbol{d}_i^T \boldsymbol{Q} \boldsymbol{d}_i \neq 0$, and for all $i \neq j$, $\boldsymbol{d}_i^T \boldsymbol{Q} \boldsymbol{d}_j = 0$. If $\boldsymbol{d}_0, \boldsymbol{d}_1, ..., \boldsymbol{d}_k (k \leq n-1)$ are $\boldsymbol{Q}$-conjugate, show that they are linearly independent. [hint: you can use proof by contradiction.] [5]

(c) Find a set of $\boldsymbol{Q}$-conjugate directions $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathbb{R}^3$ for

$$\boldsymbol{Q} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

[Hint: you can set $\mathbf{d}_0 = [1, 0, 0]^T$] [3]

(d) We can use the conjugate directions to solve $\boldsymbol{Qx} = \boldsymbol{b}$. We have shown $\boldsymbol{d}_0, ..., \boldsymbol{d}_{n-1}$ form a basis for $\mathbb{R}^n$, therefore, we may express $\boldsymbol{x}$ as $\boldsymbol{x} = \sum_{i=0}^{n-1} \beta_i \boldsymbol{d}_i$, and consequently $\boldsymbol{Qx} = \sum_{i=0}^{n-1} \beta_i \boldsymbol{Q} \boldsymbol{d}_i$.

i. Write the expression for β_i in terms of $\boldsymbol{b}, \boldsymbol{Q}$ and $\boldsymbol{d}_i$. [3]

ii. Given $\mathbf{Q}$ of part (c) and assuming $\mathbf{b} = [6, 0, 2]^T$, find $\mathbf{x}$ using the conjugate directions. [3]

Question 3 continues on the next page

(e) Consider the unweighted and undirected graph shown in Fig. 3.

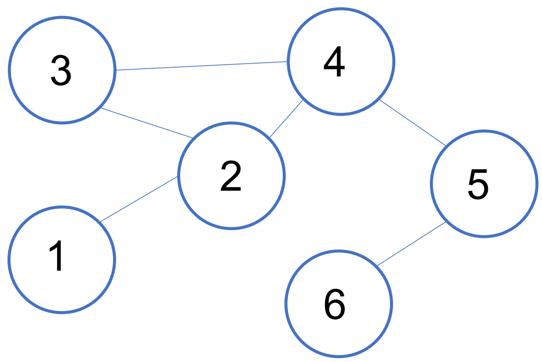

Figure 3: Graph with six nodes

i. Compute the adjacency matrix [2]

ii. Compute two iterations of the power method to estimate the most important node in the graph. [Hint: use the first iteration to estimate the degree of the nodes.] [2]