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Principles of the z-Transform

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The z-transform derived from the Laplace transform.

- Consider a discrete-time signal $x(t)$ sampled every T seconds.

$$x(t) = x_0\delta(t) + x_1\delta(t-T) + x_2\delta(t-2T) + x_3\delta(t-3T) + \cdots$$

- Recall that in the Laplace domain we have:

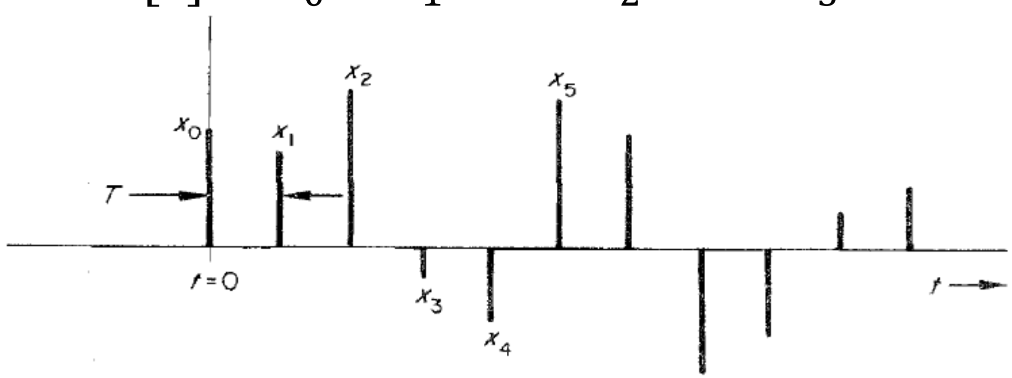
$$\mathcal{L}\{\delta(t)\} = 1$$
$$\mathcal{L}\{\delta(t-T)\} = e^{-sT}$$

- Therefore, the Laplace transform of $x(t)$ is:

$$X(s) = x_0 + x_1e^{-sT} + x_2e^{-s2T} + x_3e^{-s3T} + \cdots$$

- Now define $z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}\cos\omega T + je^{\sigma T}\sin\omega T$.
- Finally, define

$$X[z] = x_0 + x_1z^{-1} + x_2z^{-2} + x_3z^{-3} + \cdots$$



z^{-1}: the sampled period delay operator

- From the Laplace time-shift property, we know that $z = e^{sT}$ is time advance by T seconds (T is the sampling period).
- Therefore, $z^{-1} = e^{-sT}$ corresponds to one sampling period delay.
- As a result, all sampled data (and discrete-time systems) can be expressed in terms of the variable z.
- More formally, the **unilateral $z-$transform** of a causal sampled sequence:

$$x[n] = \{x[0], x[1], x[2], x[3], \dots\}$$

is given by:

$$X[z] = x_0 + x_1 z^{-1} + x_2 z^{-2} + x_3 z^{-3} + \cdots = \sum_{n=0}^{\infty} x[n] z^{-n},\ x_n = x[n]$$

- The **bilateral z –transform** for any sampled sequence is:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

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Laplace, Fourier and $z-$ transforms

	Definition	Purpose	Suitable for
Laplace transform	$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	Converts integral-differential equations to algebraic equations.	Continuous-time signal and systems analysis. Stable or unstable.
Fourier transform	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Converts finite energy signals to frequency domain representation.	Continuous-time, stable systems. Convergent signals only. Best for steady-state.
Discrete Fourier transform	$X[r\omega_0] = \sum_{n=-\infty}^{N_0-1} Tx[nT]e^{-jnr\Omega_0}$ T sampling period $\Omega_0 = \omega_0 T = 2\pi/N_0$	Converts discrete-time signals to discrete frequency domain.	Discrete time signals.
$z-$ transform	$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	Converts difference equations into algebraic equations.	Discrete-time system and signal analysis; stable or unstable.

Example: Find the z –transform of $x[n] = \gamma^n u[n]$

- Find the z –transform of the causal signal $\gamma^n u[n]$, where γ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} \gamma^n u[n] z^{-n} = \sum_{n=0}^{\infty} \gamma^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n$$

$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

- We apply the geometric progression formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, |x| < 1$$

- Therefore,

$$X[z] = \frac{1}{1-\frac{\gamma}{z}}, \left|\frac{\gamma}{z}\right| < 1$$

$$= \frac{z}{z-\gamma}, |z| > |\gamma|$$

- We notice that the z –transform exists for certain values of z. These values form the so called Region-Of-Convergence (ROC) of the transform.

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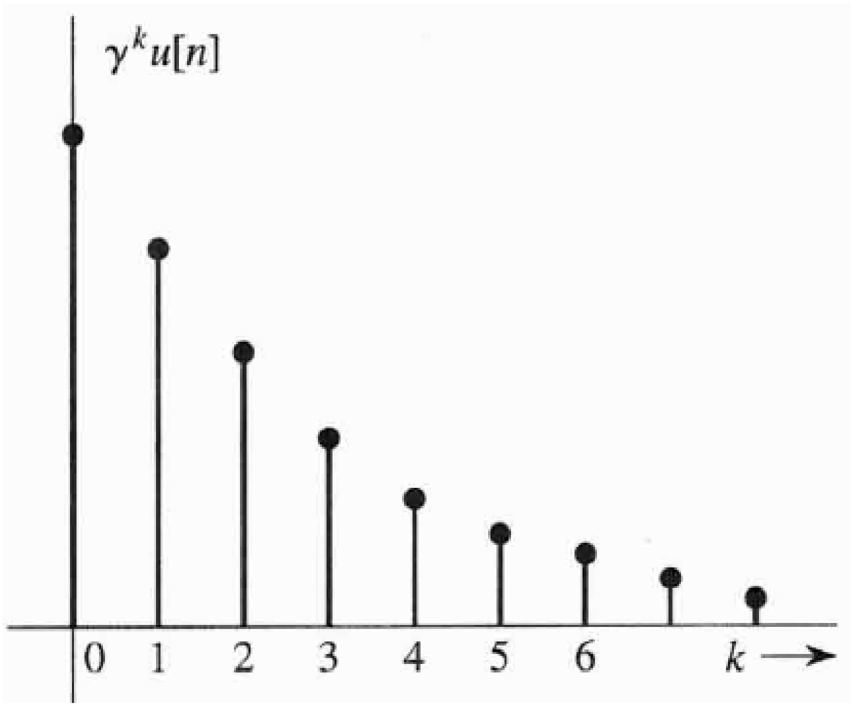
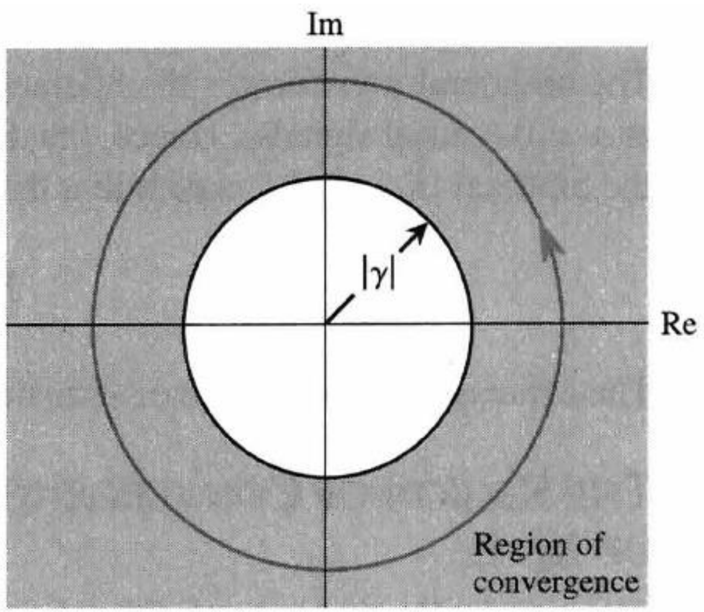
Example: Find the z −transform of $x[n] = \gamma^n u[n]$ cont.

- Observe that a simple equation in z-domain results in an infinite sequence of samples.
- The figures below depict the signal in time (left) and the ROC, shown with the shaded area, within the z −plane.





Example: Find the z −transform of $x[n] = -\gamma^n u[-n-1]$

- Find the z −transform of the anticausal signal $-\gamma^n u[-n-1]$, where γ is a constant.
- By definition:

$$X[z] = \sum_{n=-\infty}^{\infty} -\gamma^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -\gamma^n z^{-n} = -\sum_{n=1}^{\infty} \gamma^{-n} z^n = -\sum_{n=1}^{\infty} \left(\frac{z}{\gamma}\right)^n$$

$$= -\frac{z}{\gamma}\sum_{n=0}^{\infty} \left(\frac{z}{\gamma}\right)^n = -\left(\frac{z}{\gamma}\right)\left[1 + \left(\frac{z}{\gamma}\right) + \left(\frac{z}{\gamma}\right)^2 + \left(\frac{z}{\gamma}\right)^3 + \cdots\right]$$

- Therefore,

$$X[z] = -\left(\frac{z}{\gamma}\right)\frac{1}{1-\frac{z}{\gamma}}, \ \left|\frac{z}{\gamma}\right| < 1$$

$$= \frac{z}{z-\gamma}, \ |z| < |\gamma|$$

- We notice that the z −transform exists for certain values of z, which consist the complement of the ROC of the function $\gamma^n u[n]$ with respect to the z −plane.

Summary of previous examples

- We proved that the following two functions:
  - The causal function $\gamma^n u[n]$ and
  - the anti-causal function $-\gamma^n u[-n-1]$ have:
    - The same analytical expression for their z −transforms.
    - Complementary ROCs. More specifically, the union of their ROCS forms the entire z −plane.

- Observe that the ROC of $\gamma^n u[n]$ is $|z| > |\gamma|$.

- In case that $\gamma^n u[n]$ is part of a causal system's impulse response, we see that the condition $|\gamma| < 1$ must hold. This is because, since $\lim_{n\to\infty}(\gamma)^n = \infty$, for $|\gamma| > 1$, the system will be unstable in that case.

- Therefore, in causal systems, stability requires that the ROC of the system's transfer function includes the circle with radius 1 centred at origin within the z −plane. This is the so called **unit circle**.

Example: Find the z –transform of $\delta[n]$ and $u[n]$

- By definition $\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$.

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = \delta[0] z^{-0} = 1$$

- By definition $u[n] = 1$ for $n \geq 0$.

$$X[z] = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-\frac{1}{z}}, \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}, |z| > 1$$

Example: Find the z −transform of $\cos\beta n u[n]$

- We write $\cos\beta n = \frac{1}{2}\left(e^{j\beta n} + e^{-j\beta n}\right)$.
- From previous analysis we showed that:

$$\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma}, \ |z| > |\gamma|$$

- Hence,

$$e^{\pm j\beta n} u[n] \Leftrightarrow \frac{z}{z-e^{\pm j\beta}}, \ |z| > \left|e^{\pm j\beta}\right| = 1$$

- Therefore,

$$X[z] = \frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right] = \frac{z(z-\cos\beta)}{z^2-2z\cos\beta+1}, \ |z| > 1$$

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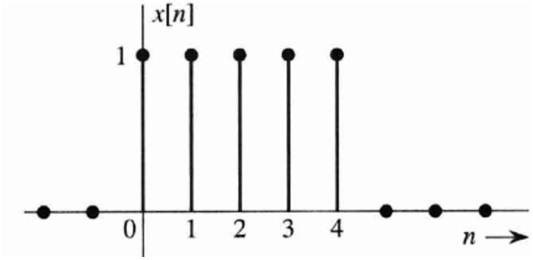
z –transform of 5 impulses

- Find the z –transform of the signal depicted in the figure.



- By definition:

$$X[z] = 1 + \frac{1}{z} + \frac{1`}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} = \sum_{k=0}^{4} (z^{-1})^k = \frac{1-(z^{-1})^5}{1-z^{-1}} = \frac{z}{z-1}(1-z^{-5})$$

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z —transform Table

No.	$x[n]$	$X[z]$
1	$\delta[n-n]$	z^{-k}
2	$u[n]$	$\frac{z}{z-1}$
3	$nu[n]$	$\frac{z}{(z-1)^2}$
4	$n^2u[n]$	$\frac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\frac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\frac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\frac{\gamma z}{(z-\gamma)^2}$

z —transform Table

No.	$x[n]$	$X[z]$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos \beta n \, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert \cos \beta)}{z^2-(2\lvert\gamma\rvert \cos \beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin \beta n \, u[n]$	$\dfrac{z\lvert\gamma\rvert \sin \beta}{z^2-(2\lvert\gamma\rvert \cos \beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{rz[z \cos \theta-\lvert\gamma\rvert \cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert \cos \beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n] \qquad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n \cos(\beta n+\theta)u[n]$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}} \qquad \beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert} \qquad \theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

Inverse z –transform

- As with other transforms, inverse z –transform is used to derive $x[n]$ from $X[z]$, and is formally defined as:

$$x[n] = \frac{1}{2\pi j} \oint X[z] z^{n-1} dz$$

- Here the symbol $\oint$ indicates an integration in counter-clockwise direction around a closed path within the complex z-plane (known as contour integral).
- Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the inverse z –transform.
- One such technique is to use the z –transform pairs Table shown in the last two slides with partial fraction expansion.

Find the inverse z −transform in the case of real unique poles

- Find the inverse z −transform of $X[z] = \frac{8z-19}{(z-2)(Z-3)}$

Solution

$$\frac{X[z]}{z} = \frac{8z-19}{z(z-2)(Z-3)} = \frac{(-\frac{19}{6})}{z} + \frac{3/2}{z-2} + \frac{5/3}{z-3}$$

$$X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

By using the simple transforms that we derived previously we get:

$$x[n] = -\frac{19}{6}\delta[n] + \left[\frac{3}{2}2^n + \frac{5}{3}3^n\right]u[n]$$

Find the inverse z –transform in the case of real repeated poles

- Find the inverse z –transform of $X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

Solution

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- We use the so called **covering method** to find k and a_0

$$k = \left.\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}\right|_{z=1} = -3$$

$$a_0 = \left.\frac{(2z^2-11z+12)}{(z-1)(z-2)^3}\right|_{z=2} = -2$$

The shaded areas above indicate that they are excluded from the entire function when the specific value of z is applied.

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Find the inverse z –transform in the case of real repeated poles cont.

- Find the inverse z –transform of $X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

Solution

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{-3}{z-1} + \frac{-2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- To find a_2 we multiply both sides of the above equation with z and let $z \to \infty$.

$$0 = -3 - 0 + 0 + a_2 \Rightarrow a_2 = 3$$

- To find a_1 let $z \to 0$.

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2} \Rightarrow a_1 = -1$$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{(z-2)} \Rightarrow$$

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

Find the inverse z −transform in the case of real repeated poles cont.

$$X[z] = \frac{-3z}{z-1} - \frac{2z}{(z-2)^3} - \frac{z}{(z-2)^2} + \frac{3z}{(z-2)}$$

- We use the following properties:
  - $\gamma^n u[n] \Leftrightarrow \frac{z}{z-\gamma}$
  - $\frac{n(n-1)(n-2)\ldots(n-m+1)}{\gamma^m m!}\gamma^n u[n] \Leftrightarrow \frac{z}{(z-\gamma)^{m+1}}$

$$[-\frac{2z}{(z-2)^3} = (-2)\frac{z}{(z-2)^{2+1}} \Leftrightarrow (-2)\,\frac{n(n-1)}{2^2 2!}\gamma^n u[n] = -2\,\frac{n(n-1)}{8}\cdot 2^n u[n]$$

- Therefore,

$$x[n] = [-3\cdot 1^n - 2\,\frac{n(n-1)}{8}\cdot 2^n - \frac{n}{2}\cdot 2^n + 3\cdot 2^n]u[n]$$

$$= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right]u[n]$$

Find the inverse z –transform in the case of complex poles

- Find the inverse z –transform of $X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

Solution

$$X[z] = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

$$\frac{X[z]}{z} = \frac{(2z^2-11z+12)}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

Whenever we encounter a complex pole we need to use a special partial fraction method called **<u>quadratic factors method</u>**.

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

We multiply both sides with z and let $z \to \infty$:

$$0 = 2 + A \Rightarrow A = -2$$

Therefore,

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

Find the inverse z −transform in the case of complex poles cont.

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B we let $z = 0$:

$$\frac{-34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$$

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25} \Rightarrow X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

- We use the following property:

$r|\gamma|^n \cos(\beta n + \theta)\, u[n] \Leftrightarrow \frac{z(Az+B)}{z^2+2az+|\gamma|^2}$ with $A = -2$, $B = 16$, $a = -3$, $|\gamma| = 5$.

$$r = \sqrt{\frac{A^2|\gamma|^2+B^2-2AaB}{|\gamma|^2-a^2}} = \sqrt{\frac{4\cdot25+256-2\cdot(-2)\cdot(-3)\cdot16}{25-9}} = 3.2,\ \beta = \cos^{-1}\frac{-a}{|\gamma|} = 0.927rad,$$

$$\theta = \tan^{-1}\frac{Aa-B}{A\sqrt{|\gamma|^2-a^2}} = -2.246rad.$$

Therefore, $x[n] = [2 + 3.2\cos(0.927n - 2.246)]u[n]$

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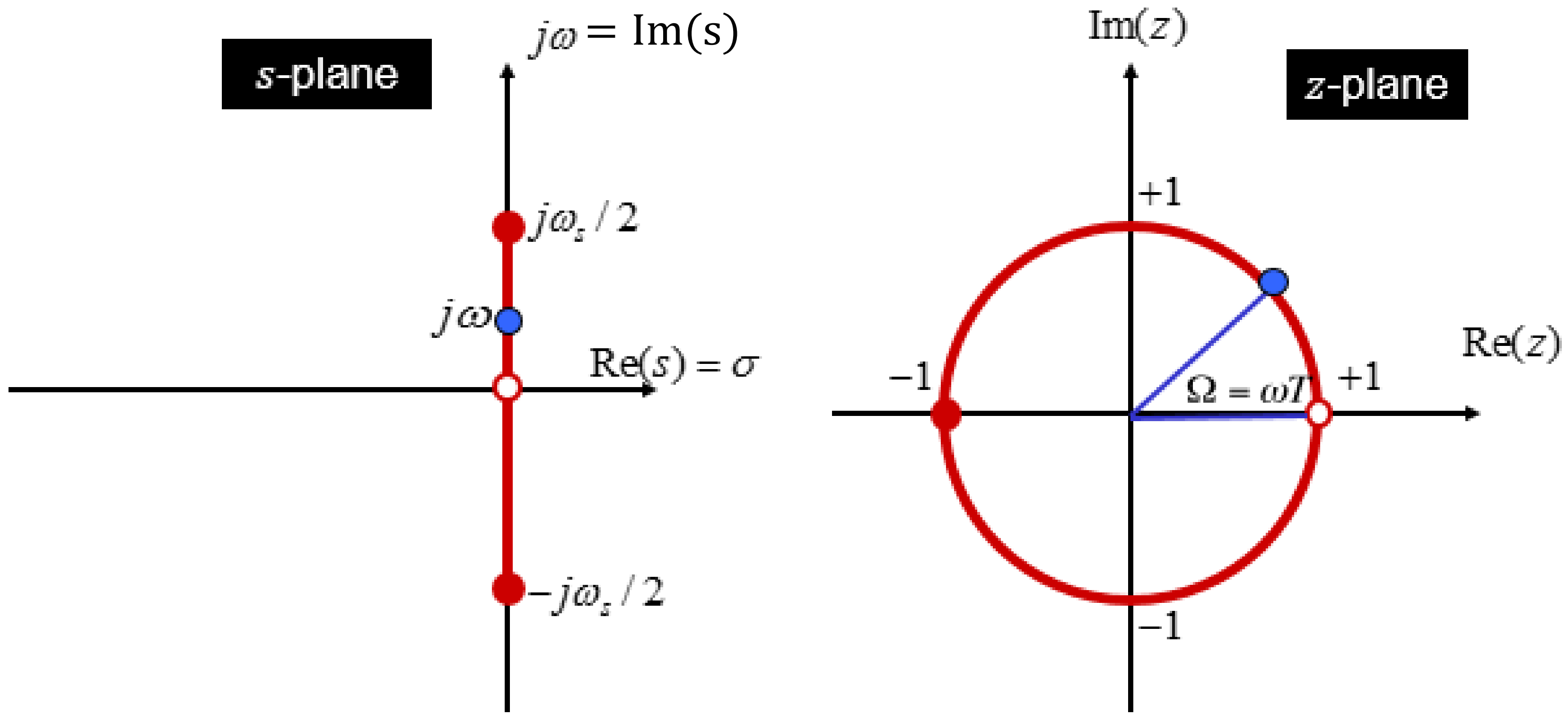
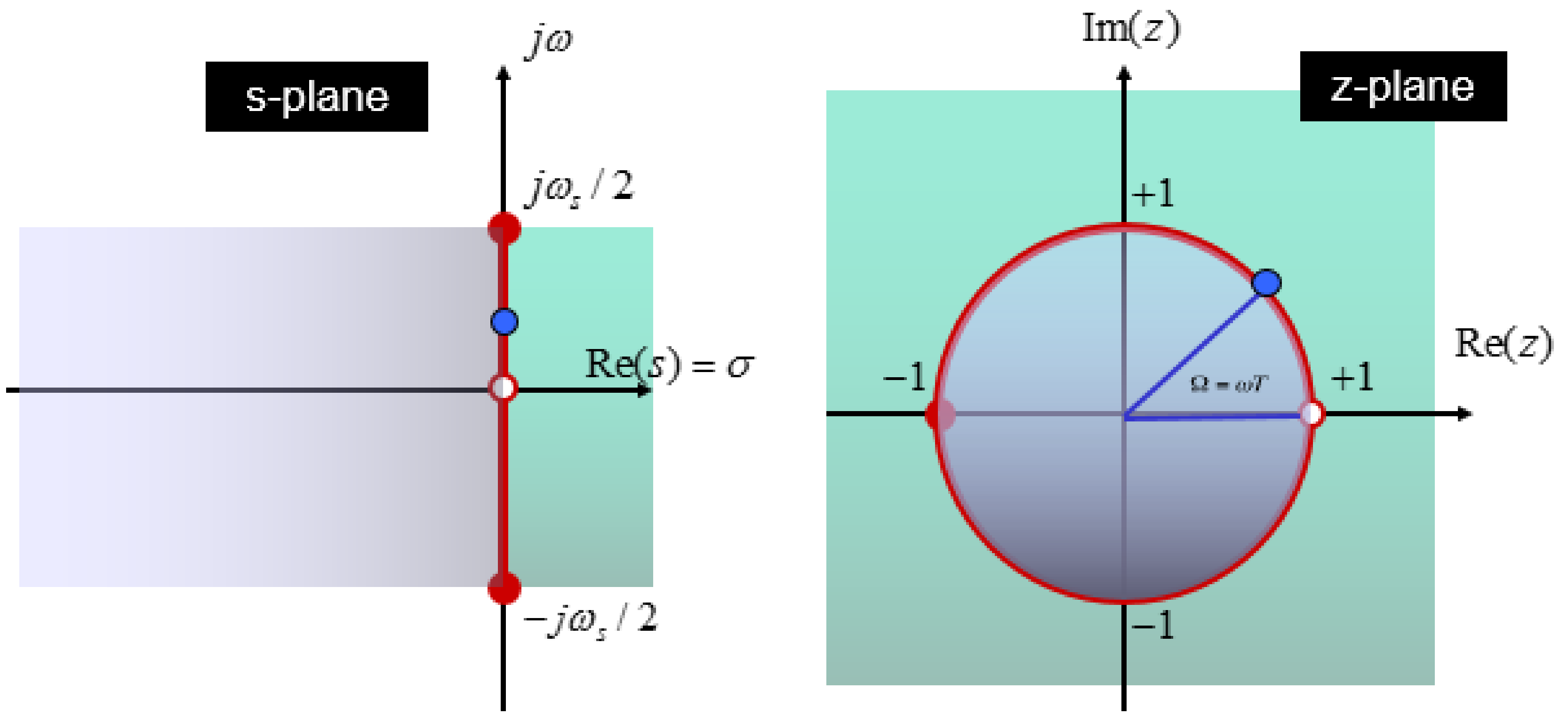
Mapping from s $-$plane to z $-$plane

- Since $z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}e^{j\omega T}$ where $T = \frac{2\pi}{\omega_s}$, we can map the s $-$plane to the z $-$plane as below.
- For $\sigma = 0$, $s = j\omega$ and $z = e^{j\omega T}$. Therefore, the imaginary axis of the s $-$plane is mapped to the unit circle on the z $-$plane.



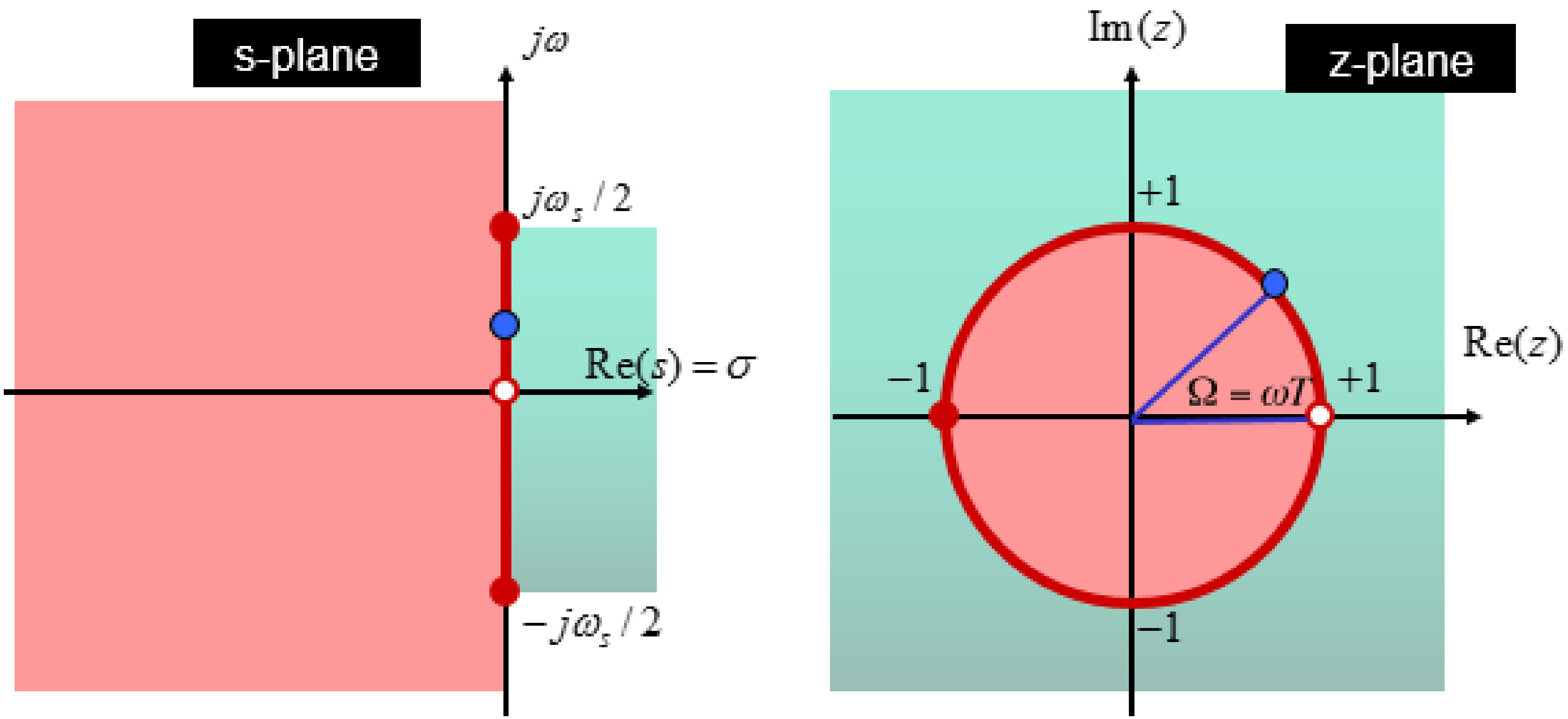
Mapping from s −plane to z −plane cont.

- For $\sigma < 0$, $|z| = e^{\sigma T} < 1$. Therefore, the left half of the s −plane is mapped to the inner part of the unit circle on the z −plane (turquoise areas).
- Note that we normally use Cartesian coordinates for the s − plane $(s = \sigma + j\omega)$ and polar coordinates for the z −plane $(z = re^{j\omega})$.



Mapping from s –plane to z –plane cont.

- For $\sigma > 0$, $|z| = e^{\sigma T} > 1$. Therefore, the right half of the $s-$plane is mapped to the outer part of the unit circle on the z –plane (pink areas).



Find the inverse z −transform in the case of complex poles

- Using the results of today's Lecture and also Lecture 9 on stability of causal continuous-time systems and the mapping from the s −plane to the z −plane, we can easily conclude that:

  - A discrete-time LTI system is stable if and only if the ROC of its system function $H(z)$ includes the unit circle, $|z| = 1$.

  - A causal discrete-time LTI system with rational z −transform $H(z)$ is stable if and only if all of the poles of $H(z)$ lie inside the unit circle – i.e., they must all have magnitude smaller than 1. This statement is based on the result of Slide 5.

Example: homework

- Consider a LTI system with input $x[n]$ and output $y[n]$ related with the difference equation:

$$y[n-2] - \frac{5}{2}y[n-1] + y[n] = x[n]$$

  Determine the impulse response and its z –transform in the following three cases:
  - The system is causal.
  - The system is stable.
  - The system is neither stable nor causal.