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Principles of the Inverse z - Transform

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Inverse z-transform

Welcome back to the DSP class!

- The most popular method for finding a discrete signal in time using the $z-$transform (this is the so called inverse $z-$transform) is **partial fraction expansion**.
- There are 3 additional methods to calculate the inverse $z-$transform. These are called:
  - **Inspection (for power series)**
  - **Long division**
  - **Inversion formula**

  You can revise these methods for your own interest using this presentation.
- Why are we interested?
  - To be able to find the impulse response of a discrete system if we know its transfer function.
  - Recall the difference between transfer function (Laplace and $z-$domain) and frequency response (frequency domain).

Inverse z –transform by inspection

- This method is suitable if the z –domain expression is given as a sum of powers of z. For example:

$$X(z) = 1 + 2z^{-1} + 3z^{-2}$$

- We can use the following property of the z –transform of a shifted Dirac function given below:

$$Z\{A\delta(n-m)\} = Az^{-m}$$

- By using the linearity property of the z – transform we immediately see that:

$$x[n] = Z^{-1}\{X(z)\} = Z^{-1}\{1 + 2z^{-1} + 3z^{-2}\}$$
$$= \delta(n) + 2\delta(n-1) + 3\delta(n-2)$$

- Application: there are transfer functions which do not have a denominator like the one above.
  - In filter design these are called Finite Impulse Response (FIR) filters **(I believe the term FIR is obvious)**.

Inverse z −transform by long division

- This method is suitable if the z −domain expression is given as a ratio of two polynomials **(be prepared: nasty)**. For example:

$$X(z)=\frac{0.5z^2+0.5z}{z^2-z+0.5}$$

dividend

divisor

A ratio of two polynomials in z is a standard form of a system's transfer function in many engineering applications.

We carry out the standard long division of polynomials.

Step 1: We write the division as follows:

$$z^2-z+0.5\,\big/\,\overline{0.5z^2+0.5z}$$

we take the maximum powers of dividend and divisor

We now divide $0.5z^2$ with z^2. The result is 0.5. This will be part of the final **quotient**. We write this operation as shown below:

$$\begin{array}{r} 0.5 \\ z^2-z+0.5\,\big/\,\overline{0.5z^2+0.5z} \end{array}$$

quotient

Inverse z —transform by long division cont.

Step 2:

$$z^2 - z + 0.5 \overline{\big)\, 0.5z^2 + 0.5z}^{\;0.5}$$

We multiply the quotient 0.5 with the **divisor** and subtract the result from the **dividend** as follows:

$$0.5 \cdot \left(\mathbf{z^2 - z + 0.5}\right) = 0.5z^2 - 0.5z + 0.25$$

$$\mathbf{0.5z^2 + 0.5z} - (0.5z^2 - 0.5z + 0.25) = z - 0.25$$

The new dividend is $z - 0.25$.

$$\begin{array}{r l}
 & \quad 0.5 \\
z^2 - z + 0.5 \big) & 0.5z^2 + 0.5z \\
 & -(0.5z^2 - 0.5z + 0.25) \\
\hline
 & \qquad z - 0.25
\end{array}$$

Inverse z —transform by long division cont.

Step 3:

$$
\begin{array}{r l}
 & \quad 0.5 \\
z^2 - z + 0.5 \, / & \overline{\, 0.5z^2 + 0.5z \,} \\
 & -(0.5z^2 - 0.5z + 0.25) \\
\hline
 & \qquad\qquad z + 0.25
\end{array}
$$

We now divide z with z^2 . The result is z^{-1}. We add this to the already calculated quotient which is 0.5.

$$
\begin{array}{r l}
 & \quad 0.5 + z^{-1} \\
z^2 - z + 0.5 \, / & \overline{\, 0.5z^2 + 0.5z \,} \\
 & -(0.5z^2 - 0.5z + 0.25) \\
\hline
 & \qquad\qquad z - 0.25
\end{array}
$$

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Inverse z —transform by long division cont.

Step 4:

$$\begin{array}{r l} & \quad 0.5 + z^{-1} \\ z^2 - z + 0.5 & \overline{\big)\, 0.5z^2 + 0.5z} \\ & -(0.5z^2 - 0.5z + 0.25) \\ \hline & \qquad\qquad z - 0.25 \end{array}$$

We multiply the new element of the quotient z^{-1} with the **divisor** and subtract the result from the **new dividend** as follows:

$$z^{-1} \cdot \left(\boldsymbol{z^2 - z + 0.5}\right) = z - 1 + 0.5z^{-1}$$

$$\boldsymbol{z - 0.25} - \left(z - 1 + 0.5z^{-1}\right) = 0.75 - 0.5z^{-1}$$

The new dividend is $0.75 - 0.5z^{-1}$.

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Inverse z —transform by long division cont.

Step 4 cont:

$$
\begin{array}{r l}
 & 0.5 + z^{-1} \\
z^2 - z + 0.5 \,\big/ & 0.5z^2 + 0.5z \\
 & -(0.5z^2 - 0.5z + 0.25) \\
\hline
 & z - 0.25 \\
 & -\left(z - 1 + 0.5z^{-1}\right) \\
\hline
 & 0.75 - 0.5z^{-1}
\end{array}
$$

Inverse z −transform by long division cont.

Step 5:

$$
\begin{array}{r}
0.5 + z^{-1} \; \color{red}{\mathbf{+0.75z^{-2}}} \\
z^2 - z + 0.5 \,\big)\, \overline{0.5z^2 + 0.5z} \\
-(0.5z^2 - 0.5z + 0.25) \\
\hline
z - 0.25 \\
-(z - 1 + 0.5z^{-1}) \\
\hline
0.75 - 0.5z^{-1}
\end{array}
$$

We divide 0.75 with z^2 to obtain $0.75z^{-2}$, the new element of the quotient.

Inverse z —transform by long division cont.

The above procedure can continue. If we keep the first three terms of the division, we have:

$$X(z) = \frac{0.5z^2+0.5z}{z^2-z+0.5} \approx 0.5 + z^{-1} + 0.75\, z^{-2}$$

Therefore,

$$x[n] = Z^{-1}\{X(z)\} \approx Z^{-1}\left\{0.5 + \ z^{-1} + 0.75\, z^{-2}\right\}$$

$$\approx 0.5\delta(n) + \delta(n-1) + 0.75\delta(n-2)$$

Note that the above result is an approximation.

Inverse z –transform using the inversion formula

- As with other transforms, inverse z –transform may be used to derive $x[n]$ from $X[z]$, and is formally defined as:

$$x[n] = \frac{1}{2\pi j} \oint_C X[z] z^{n-1} dz$$

- Here the symbol $\oint$ indicates an integration in counter-clockwise direction around a closed path within the complex z-plane (known as contour integral).
- Such contour integral is difficult to evaluate but could be done using Cauchy's residue theorem:

$$x[n] = \frac{1}{2\pi j} \oint_C X[z] z^{n-1} dz =$$

$$= \sum(\text{residues of } X[z] z^{n-1} \text{at the poles inside contour } C)$$

Inverse z –transform using the inversion formula cont.

- Cauchy's residue theorem:

$$x[n] = \frac{1}{2\pi j}\oint_C X[z]z^{n-1}dz =$$
$$= \sum(\text{residues of } X[z]z^{n-1} \text{at the poles inside contour } C)$$

- The function $X[z]z^{n-1}$ must be expressed in the form

$$X[z]z^{n-1} = \frac{\phi(z)}{(z-z_0)^s}$$

This expression must be formed FOR EACH POLE in order to calculate the residue related to one specific pole zo with multiplicity s.

For each pole the expression is different!

This function has s poles at z_0.

- Then $\text{Res}\{X[z]z^{n-1} \text{ at } z = z_0\} = \frac{1}{(s-1)!}\left[\frac{d^{s-1}\phi(z)}{dz^{s-1}}\right]_{z=z_0}$

Inverse z –transform using the inversion formula cont.

- **Example:** Find the inverse z –transform:

$$X[z] = \frac{1}{1-az^{-1}} \text{ for } |z| > |a|$$

- $x[n] = \frac{1}{2\pi j}\oint_C X[z]z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{1}{1-az^{-1}}z^{n-1}dz$

$$= \frac{1}{2\pi j}\oint_C \frac{z}{z-a}z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{z-a}dz$$

  C is a circular contour of radius greater than the magnitude of the single pole.

- Comparing with the form $X[z]z^{n-1} = \frac{z^n}{z-a} = \frac{\phi(z)}{(z-z_0)^s}$ gives $s = 1$, $z_0 = a$ and $\phi(z) = z^n$.

- For $n \geq 0$ the only pole of $X[z]z^{n-1}$ is at $z = a$ which implies $s = 1$, with residue:

$$\frac{1}{(s-1)!}\left[\frac{d^{s-1}\phi(z)}{dz^{s-1}}\right]_{z=z_0} = \frac{1}{0!}\left[\frac{d^0 z^n}{dz^0}\right]_{z=a} = a^n \qquad 0!=1$$

Inverse z —transform using the inversion formula cont.

- For $n < 0$ there is at least one pole at $z = 0$.
  - For $n = -1$, $X[z]z^{n-1} = \frac{z}{z-a} z^{-1-1} = \frac{1}{z(z-a)}$, $s = 1$ and poles are at $0, a$.

    $x[-1] = \sum(\text{residues of } \frac{1}{z(z-a)} \text{ at the poles inside contour } C)$

    $= \frac{1}{(1-1)!}\left[\frac{1}{z-a}\right]_{z=0} + \frac{1}{(1-1)!}\left[\frac{1}{z}\right]_{z=a} = -\frac{1}{a} + \frac{1}{a} = 0$

  - For $n = -2$, $X[z]z^{n-1} = \frac{1}{z^2(z-a)}$,

    $s = 2$ for the pole at 0 and $s = 1$ for the pole at a

    $x[-2] = \sum(\text{residues of } \frac{1}{z^2(z-a)} \text{ at the poles inside contour } C)$

    $= \frac{1}{(2-1)!}\left[d(\frac{1}{z-a})/dz\right]_{z=0} + \frac{1}{(1-1)!}\left[\frac{1}{z^2}\right]_{z=a} = -\frac{1}{a^2} + \frac{1}{a^2} = 0$

Inverse z —transform using the inversion formula cont.

- We can easily continue and show that for $n < 0$, $x[n] = 0$.
- Therefore,

$$x[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

  or

  $x[n] = a^n u[n]$.

- Previously, we found easily that the z –transform of the function $x[n] = a^n u[n]$, is $X[z] = \frac{1}{1-az^{-1}}$ $|\mathbf{z}| > |\mathbf{a}|$
- Whether you find the forward transform easier or harder will depend on your expertise.
- **As mentioned, inverse z – transform is useful when we are interested in finding the impulse response of a discrete system from its transfer function.**