IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2018

MATHEMATICS FOR SIGNALS AND SYSTEMS SOLUTIONS

SOLUTIONS

1. (a) i. We apply L to both $\cos x$ and $\sin x$.

$$L(\cos x) = -\cos x + \sin x + 2\cos x = \cos x + \sin x$$

$$L(\sin x) = -\sin x - \cos x + 2\sin x = -\cos x + \sin x$$

The first corresponds to the vector $(1,1)^T$ with respect to the given basis, the second corresponds to the vector $(-1,1)^T$ Thus the matrix representing L with respect to the given basis is:

$$\mathbf{L} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

ii. To find a solution to the differential equation we just need to find a $\mathbf{x}$ such that

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

. So $\mathbf{x} = (1,-2)^T$ and the solution is $\cos x - 2\sin x$.

(b) $\mathbf{AB} = \mathbf{0}$ means that for any vector $\mathbf{x}$ we have that $\mathbf{ABx} = \mathbf{0}$. If the range space of $\mathbf{B}$ is not in the null space of $\mathbf{A}$ that means that there is a vector $\mathbf{y} = \mathbf{Bx}$ that in not in the null space of $\mathbf{A}$ therefore $(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx}) \neq \mathbf{0}$ which contradicts $\mathbf{AB} = \mathbf{0}$.

(c) We put the 4 vectors along the rows of the matrix $\mathbf{A}$ and row reduce:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 2 & -1 & 0 \\ 1 & 8 & 0 & 0 \\ 2 & 0 & 4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 6 & -3 & 1 \\ 0 & 4 & 2 & -1 \\ 0 & -8 & 8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 6 & -3 & 1 \\ 0 & 0 & 4 & -5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore S has dimension 3 and a possible basis is: $[1,4,-2,1]^T, [0,6,-3,1]^T, [0,0,4,-5/3]^T$.

(d) By inspection we see that $\mathbf{A}$ has rank=2 so range space has dimension 2 and a basis for it is given by the first and third column since they are linearly independent so $\mathbf{u}_1 = [1,2]^T, \mathbf{u}_2 = [3,-1]^T$. The null space has dimension $n - \text{rank}(\mathbf{A}) = 4 - 2 = 2$. We then find a basis for the null space by solving $\mathbf{Ax} = \mathbf{0}$ with $\mathbf{A}$ in echelon form. We have that the echelon form for $\mathbf{A}$ is

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & -7 & -3 \end{bmatrix}.$$

We then have

$$\begin{bmatrix} -3x_2 - 5/7x_4 \\ x_2 \\ -3/7x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5/7 \\ 0 \\ -3/7 \\ 1 \end{bmatrix}.$$

So a basis for the null space is $[-3,1,0,0]^T, [-5/7,0,-3/7,1]^T$.

[3/5]

The range space of $\mathbf{A^T}$ has dimension two since the null space had dimension two and a basis is given by the two rows of $\mathbf{A}$ which are linearly independent, i.e., $[1, 3, 3, 2]^T$, $[2, 6, -1, 1]^T$. Finally the dimension of the null space of $\mathbf{A^T}$ is $2 - rank(A) = 0$. [5/5]

(e) i. We first multiply $\mathbf{A}$ by $\mathbf{E_{21}}$ given by:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which yields:

$$\mathbf{E_{21}A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & 0 \\ 1 & 0 & 5 \end{bmatrix}.$$

We then multiply by:

$$\mathbf{E_{31}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

which leads to

$$\mathbf{E_{31}E_{21}A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & 0 \\ 0 & -3 & 4 \end{bmatrix}.$$

Finally we multiply by

$$\mathbf{E_{32}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/7 & 1 \end{bmatrix}$$

which yields the upper triangular matrix

$$\mathbf{U} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The corresponding lower triangular matrix is:

$$\mathbf{L} = \mathbf{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -3/7 & 1 \end{bmatrix}$$

ii. Given the LU factorization of $\mathbf{A}$, it is then easy to compute $\det(\mathbf{A})$. In fact we have that:

$$\det(\mathbf{A}) = \det(\mathbf{LU}) = \det(\mathbf{L})\det(\mathbf{U}) = -28$$

.

2. (a) i. The convolution formula is $y_n = \sum_{k=0}^{1} h_k x_{n-k}$, since $x_n = 0$ for $n \neq 0, 1, 2, 3$ we have that $y_n = 0$ for $n \neq 0, 1, 2, 3, 4$ and that we can express the non zero-entries of the filtered sequence as follows: $\mathbf{y} = \mathbf{H}\mathbf{x}$ where the matrix $\mathbf{H}$ is 'tall' with size 5×4 and is given by

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

ii. the matrix has size 5×4

iii. The four columns are clearly linearly independent, so the matrix is full column rank and the null space is trivial. A possible basis for the range space is given by the four columns of $\mathbf{H}$.

iv. When $N = 2$ then $\mathbf{H}$ is given by:

A.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

B. The least square solution is obtained by imposing: $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{y}$, which in this case yields:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Consequently, $\mathbf{x} = [2/3, 5/3]^T$.

(b) i. Due to periodicity assumption we have that $\mathbf{H}$ is circulant and is given by :

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

ii. the matrix is now square and of size 4×4

iii. By inspection we see that the rank of the matrix is 3 since the last column can be obtained with a proper linear combination of the other 3 columns. The consequence is that now the null space is not trivial and has dimension $N-3 = 1$. The first 3 columns of $\mathbf{H}$ are a possible basis of the range space of $\mathbf{H}$. We then need to find a vector $\mathbf{n}$ such that $\mathbf{H}\mathbf{n} = \mathbf{0}$. One such vector is $\mathbf{n} = [1, -1, 1, -1]^T$ and the null space of $\mathbf{H}$ is given by $\text{span}\{\mathbf{n}\}$.

iv. Since in this case the null space is non-trivial we cannot uniquely retrieve x_n from $y_n = h_n * x_n$. Assume, for example that $\hat{x}_n$ satisfies $y_n = h_n * \hat{x}_n$, so does $\hat{x}_n + n_n$, where n_n is the sequence obtained by turning $\mathbf{n}$ in a periodic sequence of period $N = 4$.

3. (a) i. We can treat this as a Markov process. If we denote with a_0 the initial amount of money of Alice and b_0 the initial money of Bob at the end of the first day their saving will be given by:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.25 \\ 0.1 & 0.75 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}.$$

If we denote with

$$\mathbf{A} = \begin{bmatrix} 0.9 & 0.25 \\ 0.1 & 0.75 \end{bmatrix},$$

we have that the total savings of Alice (and Bob) in pounds after two days is:

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \mathbf{A}^2 \begin{bmatrix} 25 \\ 10 \end{bmatrix}.$$

[2/5]

To find, $\mathbf{A}^2$, we might diagonalize $\mathbf{A}$ and so we need to find its eigenvalues and eigenvectors. We know that one eigenvalue of a Markov matrix is always $\lambda_1 = 1$ and so using the fact that $1.65 = \text{trace}(\mathbf{A}) = \lambda_1 + \lambda_2$, we obtain $\lambda_2 = 0.65$. The two eigenvectors are $\mathbf{u}_1 = [5/2, 1]^T$ and $\mathbf{u}_2 = [1, -1]^T$. [4/5]

So we have:

$$\mathbf{A^2} = \begin{bmatrix} 5/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1^2 & 0 \\ 0 & (0.65)^2 \end{bmatrix} \begin{bmatrix} 2/7 & 2/7 \\ 2/7 & -5/7 \end{bmatrix}$$

and

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \mathbf{A}^2 \begin{bmatrix} 25 \\ 10 \end{bmatrix} = \begin{bmatrix} 25 \\ 10 \end{bmatrix}$$

So the total amount of money of Alice does not change. This is not surprising because $[25, 10]^T$ is actually the eigenvector related to $\lambda_1 = 1$ and represents the steady state solution. In fact each day Alice loses 1/10 of her money, that is, £2.5 but gains 1/4 of Bob's money which is also £2.5 so the total amount of money that Bob and Alice have at the end of each day is not changing. [5/5]

ii. We can compute this easily by realizing that $(0.65)^k$ for large k goes to zero. [1/3]

Therefore

$$\begin{bmatrix} a_\infty \\ b_\infty \end{bmatrix} = \begin{bmatrix} 5/2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/7 & 2/7 \\ 2/7 & -5/7 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 100/7 \\ 40/7 \end{bmatrix}.$$

So Bob has at the end £5.7143. [3/3]

(b) Symmetric matrix means that $\mathbf{A} = \mathbf{A^T}$. [1/4]

Suppose that $\mathbf{Ax} = \lambda_1 \mathbf{x}$ and that $\mathbf{Ay} = \lambda_2 \mathbf{y}$ with $\lambda_1 \neq \lambda_2$, then $\lambda_1 \mathbf{x^T y} = (\mathbf{Ax})^T \mathbf{y} = \mathbf{x}^T \mathbf{A}^T \mathbf{y} = \mathbf{x}^T \mathbf{Ay} = \lambda_2 \mathbf{x^T y}$.

[3/4]

The conditions: $\lambda_1 \mathbf{x^T y} = \lambda_2 \mathbf{x^T y}$ and $\lambda_1 \neq \lambda_2$ implies that $\mathbf{x^T y} = 0$ and so $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

[4/4]

(c) We have

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & \alpha \end{bmatrix}$$

i. $\mathbf{y} \in \mathbb{R}^2$, so we are guaranteed that a solution always exist if range($\mathbf{A}$) $= \mathbb{R}^2$. This condition is satisfied when $\mathbf{A}$ is full raw rank which happens for any $\alpha \neq 0.5$.

ii. The minimum norm solution is given by:

$$\mathbf{x}_{MN} = \mathbf{A^H}(\mathbf{AA^H})^{-1}\mathbf{y}$$

So in our case, assuming $\alpha \neq 0.5$, we obtain:

$$\mathbf{x}_{MN} = \frac{1}{5(1+\alpha^2) - (2+\alpha^2)} \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & \alpha \end{bmatrix} \begin{bmatrix} 1+\alpha^2 & -2-\alpha \\ -2-\alpha & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

As expected we note that for $\alpha = 0.5$ the determinant of $\mathbf{AA^H}$ is zero and so the inverse cannot be found, but for all other values of α the above system has a solution. The simplest case is $\alpha = 0$ which yields the solution: $\mathbf{x}_{MN} = [1, 0, -4]^T$.

(d) We solve this problem from first principles. First of all we can write the constraint as follows $\mathbf{Ax} = 1$ with $\mathbf{A} = [1/2 \; 1/2]$ The matrix has null space given by span($[1, -1]^T$). We then find all the possible solutions to this linear system which are given by $[1, 1]^T + c\mathbf{n}$ with $\mathbf{n} = [1, -1]^T$.

[2/4]

We finally find the constant c that minimizes $\|\mathbf{y} - \mathbf{x}\|^2$. So we need to find c that minimizes $(y_1 - x_1 - c)^2 + (y_2 - x_2 + c)^2 = (1/2 + c)^2 + c^2 = 2c^2 + c + 1/2$. We take the derivative with respect to c and equal to zero which yields $c = -1/4$ and finally $\mathbf{x} = [3/4, 5/4]^T$.

[4/4]