IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EXAMINATIONS 2018

Mathematics for Signals and Systems

There are THREE questions in this paper. Answer ALL questions. All questions carry equal marks.

Time allowed 3 hours.

Special Information for the Invigilators: none

Information for Candidates: none

The Questions

1. (a) Let V be the vector space consisting of all functions of the form

$$f(x) = \alpha \cos x + \beta \sin x.$$

Consider the linear transformation

$$L = \frac{d^2 f}{dx^2} - \frac{df}{dx} + 2f(x).$$

i. Find the matrix representing the linear mapping L with respect to the basis $\{\cos x, \sin x\}$ [2]

ii. Use the answer from part i. to find one solution to the following differential equation:

$$\frac{d^2 f}{dx^2} - \frac{df}{dx} + 2f(x) = 3\cos x - \sin x$$

[2]

(b) Show that if $\mathbf{AB} = \mathbf{0}$ then the range space of $\mathbf{B}$ is in the null space of $\mathbf{A}$. [2]

(c) Consider the following subspace of $\mathbb{R}^4$:

$$S = \text{ span}\left\{ \begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \\ -1 \end{bmatrix} \right\}.$$

Find a basis for S. [3]

Question 1 continues on the next page

(d) Find the dimension and construct a basis for the four subspaces associated with

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & -1 & 1 \end{bmatrix}.$$

[5]

(e) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

i. Find the LU factorization of $\mathbf{A}$ [3]

ii. Find the determinant of $\mathbf{A}$ [3]

2. (a) The sequence x_n has support $N = 4$, that is, it is assumed to be zero for $n \neq 0, 1, ..., N-1$. This sequence is filtered with a filter h_n with unit impulse response

$$h_n = \begin{cases} 1 & \text{for } n = 0 \text{ and } n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2a}$$

We denote the filtered sequence with $y_n = h_n * x_n$ where $*$ denotes the convolution operation.

i. The linear convolution can be written in matrix/vector form as $\mathbf{y} = \mathbf{H}\mathbf{x}$. Write the exact expression of the matrix $\mathbf{H}$ that describes the linear convolution with the filter h_n [3]

ii. What are the dimensions of $\mathbf{H}$? [1]

iii. Find a basis for the range and null spaces of $\mathbf{H}$ [3]

iv. Assume now that $N = 2$, i.e., x_n is assumed to be zero for $n \neq 0, 1$.

A. Write the exact expression of the matrix $\mathbf{H}$ that describes the linear convolution with the filter h_n as in Eq. (2a). [1]

B. You observe $\mathbf{y} = [1, 2, 2]^T$ which is a noisy version of $y_n = h_n * x_n$, that is, $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{e}$ for some noise $\mathbf{e}$. This means that you cannot recover $\mathbf{x}$ exactly. Find the least-squares solution to $\mathbf{y} = \mathbf{H}\mathbf{x}$, with $\mathbf{H}$ as in Eq. (2a) and with $\mathbf{y} = [1, 2, 2]^T$. [3]

Question 2 continues on the next page

(b) Assume now that x_n is periodic with period $N = 4$.

i. Write the exact expression of the matrix $\mathbf{H}$ that describes the circulant convolution with the filter h_n as in Eq. (2a). [3]

ii. What are the dimensions of $\mathbf{H}$? [1]

iii. Find a basis for the range and null spaces of $\mathbf{H}$. [3]

iv. Given that $y_n = h_n * x_n$ where now $*$ denotes the circulant convolution, can you always retrieve x_n from y_n? Justify your answer. [2]

3. (a) Bob and Alice play cards everyday and everyday Alice loses 1/10 of her total amount of money to Bob. At the same time Bob loses 1/4 of his total amount of money in favour of Alice.

   i. Assume that Alice had £25 at the beginning and Bob had £10, compute the total amount of money that Alice has after 2 days. [5]

   ii. Assume now that they can play an infinite number of times and that they both had £10 at the beginning, compute the total amount of money that Bob has after an *infinite* number of days. [3]

(b) Show that the eigenvectors related to two different eigenvalues of a symmetric matrix are orthogonal. [4]

(c) Consider the systems of linear equations $\mathbf{y} = \mathbf{A}\mathbf{x}$ with $\mathbf{A}$ given by:

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & \alpha \end{bmatrix}.$$

   i. Find a real value of α that guarantees that the system has always at least one solution. Justify your answer. [2]

   ii. Given the α you found and assuming that $\mathbf{y} = [-2, 1]^T$, find the minimum norm solution to $\mathbf{y} = \mathbf{A}\mathbf{x}$. [2]

Question 3 continues on the next page

(d) Let $\mathbf{y}$ be the two-dimensional vector $\mathbf{y} = [0.5, 1]^T$ and let the scalar $\alpha = 1$. Find the two dimensional vector $\mathbf{x} = [x_1, x_2]^T$ that is closest in the least-squares sense to $\mathbf{y}$ among all two-dimensional vectors that have average value $\alpha = 1$. That is find the $\mathbf{x}$ that minimizes $\|\mathbf{y} - \mathbf{x}\|^2$ subject to $(x_1/2 + x_2/2) = 1$. [4]