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Digital Signal Processing

Logistics of the course
Discrete Time Fourier Transform DTFT

DR TANIA STATHAKI
READER (ASSOCIATE PROFESSOR) IN SIGNAL PROCESSING
IMPERIAL COLLEGE LONDON

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Logistics of the course

Welcome to the DSP Class

- **Teacher**'s coordinates:
  - Name: **Tania Stathaki**
  - Office: **812**
  - Email: **t.stathaki@imperial.ac.uk**
  - https://scholar.google.com/citations?user=sAB5gI8AAAAJ&hl=en

- **Assessment:** 100% exam

- **Class material** is available on TEAMS

- **Textbook**

  Digital Signal Processing, A computer-Based Approach, Sanjit K. Mitra, McGraw Hill.

Digital Signal Processing in general

- **Digital signal processing** (**DSP**) is the use of digital processing, such as processing by computers or more specialized digital processors, to perform a wide variety of signal processing operations.
- The digital signals processed in this manner are a sequence of numbers that represent samples of a continuous variable in a domain such as time, space, or frequency.
- Digital signal processing and analog signal processing are subfields of signal processing.
- DSP applications include:
  - audio and speech processing,
  - sonar, radar and other sensor array processing
  - spectral density estimation
  - statistical signal processing
  - digital image processing,
  - biomedical engineering
  - seismology, among others.

- What philosophy drives the field of signal processing?
- How does it work behind the scenes?
- What language is used when we talk about signal processing?

Digital Signal Processing: This course

- Some of the topics that we will tackle in this course:
  - Expansion on previous topics such as Discrete Time Fourier Transform (DTFT), Discrete Fourier Transform and others.
  - FILTER DESIGN
  - MULTIRATE DSP
  - ...

Transition from Signals and Systems to DSP

- In this lecture we will extend our knowledge on the so-called Discrete Time Fourier Transform (DTFT).
- This is not the Fourier Transform (FT). It is not the Discrete Fourier Transform (DFT) either.
- If you are not familiar with the term DTFT that is not a problem.
- The DTFT is the frequency representation of a sampled signal. DTFT is still a continuous transform.
- More specifically, we will introduce an alternative form of DTFT in which the time and frequency variables are scaled.

  - Note that the DTFT is different from the Discrete Fourier Transform (DFT)
  - The DFT is discrete both in time and frequency
  - The DTFT is the transition between FT and DFT
    (FT: Fourier Transform)

Transition from Signals and Systems to DSP

- In Years 1 and 2 you emphasized in the Fourier Transform and the Discrete Fourier Transform.
- The Fourier Transform is a continuous-frequency transform of continuous-time signals.
  - **The Fourier transform is the building block of all subsequent frequency transforms.**
- In Year 3 DSP we will emphasize in discrete signals and systems, since, in most research areas in modern EEE, researchers work with digital computers.
- Many authors denote the Fourier Transform of a signal $x(t)$ with the function $X(\omega)$, with ω the so-called **real angular frequency**.
- **In DSP I will introduce the subscript a for analogue and the symbol Ω instead of ω for the frequency of a continuous signal.**
  - Therefore, the Fourier Transform of a signal $x(t)$ will be denoted with the function $X_a(\Omega)$.

Recall the Fourier Transform of a sampled signal (SAS)

- Consider an **analogue** signal bandlimited to B Hz with Fourier transform $X_a(\Omega)$. Let's agree that $2B < f_s$ for correct sampling.
- The sampled version of the signal $x(t)$ at a rate f_s Hz can be expressed as the multiplication of the original signal with an impulse train as follows:

(1) $$\bar{x}(t) = x(t)\delta_{T_s}(t) = \sum_n x(nT_s)\delta(t - nT_s),\ T_s = 1/f_s$$

- We can express a periodic impulse train using Fourier Series as follows:

(2) $$\delta_{T_s}(t) = \frac{1}{T_s}[1 + 2\cos\Omega_s t + 2\cos 2\Omega_s t + 2\cos 3\Omega_s t + \cdots],\ \Omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

Please refer to the Appendix of this presentation

- Therefore, from (1) and (2)

$$\bar{x}(t) = x(t)\delta_{T_s}(t) = \{x(nT_s)\}$$

We denote the FT of $\bar{x}(t)$ with $\bar{X}(\omega)$

$$= \frac{1}{T_s}[x(t) + 2x(t)\cos\boldsymbol{\Omega_s}t + 2x(t)\cos\boldsymbol{2\Omega_s}t + 2x(t)\cos\boldsymbol{3\Omega_s}t + \cdots] \quad (3)$$

- Since the following holds: $x(t)\cos\Omega_s t \Leftrightarrow \frac{1}{2}[X_a(\Omega + \Omega_s) + X_a(\Omega - \Omega_s)]$ (4)

we have $\bar{X}(\Omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X_a(\Omega - n\Omega_s)$. (Take the FT in both sides of (3) and use (4).)

Time and frequency scaling

- If we replace $x(nT_s)$ with $x[n]$ and then ignore T_s, it is like we **divide the independent variable nT_s with T_s.**
  - In other words, we "**normalise**" (or "**scale**") the variable of **time**.
- We know from the properties of the Fourier Transform that if $x(t) \Leftrightarrow X(\Omega)$, then for any real constant a the following property holds.

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\Omega}{a}\right) \quad (1)$$

$\bar{x}(t) = x(nT_s)$

$\Rightarrow \bar{x}\left(\frac{\mathbf{1}}{\boldsymbol{T_s}} t\right) = x(\frac{\mathbf{1}}{\boldsymbol{T_s}} nT_s) = x(n)$

- In our case $a = |a| = 1/T_s = f_s$ and $\frac{1}{|a|} = T_s$.
- Therefore, the Fourier Transform of $x[n]$ is

- Combine the last equation of the previous slide with property (1)

$$T_s \bar{X}(\Omega \cdot T_s) = T_s \cdot \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X_a(\Omega \cdot T_s - n\Omega_s \cdot T_s)$$

$$= \sum_{n=-\infty}^{\infty} X_a(\Omega \cdot T_s - 2\pi n) = \sum_{n=-\infty}^{\infty} X_a(\Omega / f_s - 2\pi n)$$

[Note that $\Omega_s \cdot T_s = 2\pi f_s \cdot T_s = 2\pi$]

For more detailed analysis look at the appendix

Time and frequency scaling cont.

- The Fourier Transform of $x[n]$ is
  $T_s\bar{X}(\Omega \cdot T_s) = \sum_{n=-\infty}^{\infty} X_a(\Omega/f_s - 2\pi n)$
- We denote $\omega = \frac{\Omega}{f_s} \Leftrightarrow \omega = \Omega T_s \Leftrightarrow \boldsymbol{\Omega = \frac{\omega}{T_s}}$.
  - This time, we "**normalise**" (or "**scale**") the variable of **frequency**.
- We denote $X(\omega) = T_s\bar{X}(\Omega \cdot T_s)$.
- Therefore, $X(\omega) = \sum_{n=-\infty}^{\infty} X_a(\omega - 2\pi n)$.
- We divide all real angular frequencies Ω with f_s and we divide all real times by T_s.
  - To scale back to real-world values we must multiply all times by T_s and all frequencies and angular frequencies by $f_s = 1/T_s$.

Fourier Transform of a sampled signal and Discrete Time Fourier Transform

- The Fourier transform of the **normalised** sampled signal is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} X_a(\omega - 2\pi n)$$

- $X(\omega)$ is periodic with **fundamental period** $T_0 = 2\pi$ and **fundamental frequency** $\Omega_0 = \frac{2\pi}{T_0} = 1$; it can be represented using Fourier Series.

$$X(\omega) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\overset{=1}{\Omega_0}\omega} = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega}$$

$$D_n = \frac{1}{2\pi}\int_{2\pi} X(\omega)\, e^{-jn\omega} d\omega$$

What makes the integral on the left to not look exactly like an inverse Fourier is the range of integration. In inverse Fourier it goes from -oo to +oo. This mystery is solved in the Appendix.

Look at the Appendix

We prove that $D_n = x[-n]$, i.e., the Inverse FT of $X(\omega)$ evaluated at $-n$.

- Many authors use $X(e^{j\omega})$ instead of $X(\omega)$. **We will use $X(e^{j\omega})$ too!**
- Therefore, $X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[-n]\, e^{jn\omega} \Rightarrow \boldsymbol{X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega}}$.

- The above relationship is the **Discrete Time Fourier Transform (DTFT)**.
  - As mentioned, it is periodic with period 2π.
  - It is the continuous frequency representation of a discrete signal.
  - Note that we have **discrete time – continuous frequency**.

The two continuous-frequency transforms

To summarize, there are two useful representations of signals in continuous frequency domain.

- **Continuous-Time Fourier Transform (CTFT) or Fourier Transform (FT)**
  - For continuous aperiodic signals. Continuous time and **continuous frequency**.
- **Discrete Time Fourier Transform (DTFT)**
  - For discrete aperiodic signals. Discrete time and **continuous frequency**.

	Forward Transform	Inverse Transform
CTFT	$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$ Ω: “real” frequency	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega$
DTFT	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $\omega = \Omega T_s$: "normalised" angular frequency	$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$

Discrete Time Fourier Transform (DTFT): expression

- The **discrete-time Fourier transform (DTFT)** $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- In general $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega})$$

where $X_{\mathrm{re}}(e^{j\omega})$ and $X_{\mathrm{im}}(e^{j\omega})$ are the real and imaginary parts of $X(e^{j\omega})$ and are real functions of ω.

- $X(e^{j\omega})$ can alternatively be expressed **with polar coordinates** as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

where $|X(e^{j\omega})|$ and $\theta(\omega)$ are the **amplitude** and **phase** of $X(e^{j\omega})$ and are also real functions of ω.

Discrete Time Fourier Transform (DTFT): phase

- For a real sequence $x[n]$, $\left|X(e^{j\omega})\right|$ and $X_{\mathrm{re}}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{\mathrm{im}}(e^{j\omega})$ are odd functions of ω.
- Note that for any integer k
$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j(\theta(\omega)+2\pi k)} = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$
- The above property indicates that the phase function $\theta(\omega)$ cannot be uniquely specified for the DTFT.
- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:
$$-\pi \leq \theta(\omega) < \pi$$
called the **principal values**.

Discrete Time Fourier Transform (DTFT): phase unwrapping

- The DTFTs of some sequences exhibit discontinuities of 2π in their phase responses.
- An alternate type of phase function that is a continuous function of ω is often used.
- It is derived from the original phase function by removing the discontinuities of 2π.
- The process of removing the discontinuities is called **phase unwrapping.**
- Sometimes the continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$.
- The DTFT is the z –transform evaluated at the point $e^{j\omega}$.
  - Recall that $X(z) = \sum_{-\infty}^{\infty} x[n]\, z^{-n}$.
  - The DTFT converges if the ROC of the related z –transform includes $|z| = 1$.

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Does DTFT exist (converge)?

- When we say that the **Discrete-Time Fourier Transform (DTFT)** of a sequence **converges**, we are referring to the mathematical condition where the infinite summation that defines the DTFT yields a finite and well-defined result for all angular frequencies.

- An infinite series of the form $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$ may or may not converge.

- A sequence $x[n]$ is **absolutely summable** if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. We observe that:
$|X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]| \left|e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$.
Therefore, we can say that in the case of **absolute summability** the DTFT **always** converges!

Convergence of the DTFT

- Let $X_K(e^{j\omega}) = \sum_{n=-K}^{n=K} x[n]\, e^{-j\omega n}$ be a truncated version of DTFT. This is an approximation of the DTFT with a finite number of terms.
- When we refer to the **"type of convergence"** of the DTFT, we are specifically discussing **how** the finite sum that defines the approximation $X_K(e^{j\omega})$ of the DTFT approaches a finite value as the number of terms in the approximation increases, and whether this convergence occurs in a consistent manner across different frequencies.
- There are several types of convergence in mathematics. In relation to the DTFT I will mention briefly, in the next two slides, two types: **uniform** and **mean square** convergence.
- In general, there are more types such as Pointwise, Absolute, Almost Everywhere (a.e.), L^p, Cesàro Summability, Distributional, Weak and others.

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Uniform Convergence of the DTFT

- Let $X_K(e^{j\omega}) = \sum_{n=-K}^{n=K} x[n]\, e^{-j\omega n}$ be a truncated version of DTFT.
- For the so-called **uniform convergence** of $X(e^{j\omega})$ we require:

$$\lim_{K\to\infty} \left|X(e^{j\omega}) - X_K(e^{j\omega})\right| = 0$$

- We observe that the absolute error $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ in the DTFT approximation gradually decreases toward zero as the number of terms used in the approximation increases.
- Uniform convergence implies that the sequence of functions $X_K(e^{j\omega})$ converges to the limiting function $X(e^{j\omega})$ at the same rate across all frequencies. In other words, $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ depends on K and not ω.
- **Observation:** When a sequence is absolutely summable, the DTFT converges; however, the convergence is not necessarily uniform.

Mean Square Convergence of the DTFT

- A **square-summable** sequence satisfies the condition:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

- In this case, the so-called **mean square convergence** of $X(e^{j\omega})$ holds:

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega = 0$$

- We observe that the energy (integral/sum - absolute value - square) of the error function in the DTFT approximation gradually decreases toward zero as the number of terms used in the approximation increases.

- Uniform convergence is a stronger condition than mean square convergence. Therefore, unifrm convergence implies mean square convergence.

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Common DTFT pairs

$x[n]$	$X(e^{j\omega})$
$\delta[n]$	1
$x[n] = 1$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\alpha^n u[n], (\|\alpha\| < 1)$	$\frac{1}{1 - \alpha e^{-j\omega}}$

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DTFT properties (listed without proof)

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

DTFT properties (listed without proof)

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

$x[n]$: A complex sequence

Note: $X_{\mathrm{cs}}(e^{j\omega})$ and $X_{\mathrm{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\mathrm{cs}}[n]$ and $x_{\mathrm{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

DTFT properties (listed without proof)

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$ $x_{od}[n]$	$X_{re}(e^{j\omega})$ $jX_{im}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$ $X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

$x[n]$: A real sequence

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

APPENDIX: Fourier Series of a Train of Impulses (Slide 7)

- We construct the periodic signal $\delta_{T_s}(t) = \sum_n \delta(t - nT_s)$
- The periodic signal $\delta_{T_s}(t)$ is called an **impulse train**.
- This periodic signal can be expressed using Fourier series.

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\Omega_s t},\ \Omega_s = 2\pi f_s = \frac{2\pi}{T_s}$$

$$c_n = \frac{1}{T_s}\int_{T_s} \delta_{T_s}(t)\, e^{-jn\Omega_s t} dt = \frac{1}{T_s}\, e^{-jn\Omega_s 0} = \frac{1}{T_s}$$

$$\delta_{T_s}(t) = \frac{1}{T_s}\sum_{n=-\infty}^{n=\infty} e^{jn\Omega_s t} = \frac{1}{T_s}\sum_{n=-\infty}^{n=-1} e^{jn\Omega_s t} + \frac{1}{T_s}\sum_{n=+1}^{n=\infty} e^{jn\Omega_s t} + \frac{1}{T_s}\, e^{j0\Omega_s t}$$

$$= \frac{1}{T_s}\sum_{n=+1}^{n=\infty} e^{-jn\Omega_s t} + \frac{1}{T_s}\sum_{n=+1}^{n=\infty} e^{jn\Omega_s t} + \frac{1}{T_s}\, e^{j0\Omega_s t}$$

$$= \frac{1}{T_s}\sum_{n=1}^{n=\infty} 2\cos(n\Omega_s t) + \frac{1}{T_s} = \frac{1}{T_s}(1 + 2\cos(\Omega_s t) + 2\cos(2\Omega_s t) + \cdots)$$

APPENDIX: Transition from $\overline{x}(t)$ to $x[n]$ (Slide 8)

- $\bar{x}(t) = x(t)\ \delta_{T_s}(t)$
- $\bar{x}(t)$ is a continuous-time (CT) signal. It only has non-zero values at multiples of T_s. Therefore, we can denote it as a sequence $\{x(nT_s)\}$ **having in mind that in times which are not multiples of T_s the signal is zero.**

  (A continuous-time (CT) signal is a function, that is defined for all time t contained in some interval on the real line. For historical reasons, CT signals are often called analog signals.) The following time-frequency pairs hold:

  $\bar{x}(t) \Leftrightarrow \bar{X}(\Omega)$ or $\{x(nT_s)\} \Leftrightarrow \bar{X}(\Omega)$

  We know the property $x(\boldsymbol{a}t) \Leftrightarrow \frac{1}{|\boldsymbol{a}|} X\left(\frac{\Omega}{\boldsymbol{a}}\right)$. We apply above and we get:

$$\left\{x\left(\frac{\mathbf{1}}{\boldsymbol{T_s}} nT_s\right)\right\} \Leftrightarrow \frac{1}{\frac{\mathbf{1}}{\boldsymbol{T_s}}} \bar{X}\left(\frac{\Omega}{\frac{\mathbf{1}}{\boldsymbol{T_s}}}\right)$$

$$\{x(n)\} \Leftrightarrow T_s\ \bar{X}(\Omega T_s)$$

APPENDIX: Elaborate on Slide 10

- The Fourier transform of the normalised sampled signal is given by

$$X(\omega) = \sum_{n=-\infty}^{n=\infty} X_a(\omega - 2\pi n)$$

- $X(\omega)$ is periodic with fundamental period $T_0 = 2\pi$ and fundamental frequency $\Omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$; it can be represented with Fourier Series as

$$X(\omega) = \sum_{n=-\infty}^{n=\infty} D_n\, e^{jn\Omega_0\omega} = \sum_{n=-\infty}^{n=\infty} D_n\, e^{jn\omega},\ D_n = \frac{1}{2\pi}\int_{\mathbf{2\pi}} X(\omega)e^{-jn\omega}d\omega$$

We can choose the area of integration as: $D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{-jn\omega}d\omega$.

- From $X(\omega) = \sum_{n=-\infty}^{n=\infty} X_a(\omega - 2\pi n)$ we see that if we restrict $X(\omega)$ within the interval $[-\pi, \pi]$ we are left with the term $X_a(\omega)$ only as follows:

$$D_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{-jn\omega}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)e^{-jn\omega}d\omega$$

- Since $\omega = \frac{\Omega}{f_s}$ we can write

$$D_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a\left(\frac{\Omega}{f_s}\right)e^{-jn\frac{\Omega}{f_s}}\mathbf{d}\left(\frac{\mathbf{\Omega}}{\boldsymbol{f_s}}\right) = \frac{\mathbf{1}}{\boldsymbol{f_s}}\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a\left(\frac{\Omega}{f_s}\right)e^{-jn\frac{\Omega}{f_s}}\mathbf{d\Omega}$$

APPENDIX: Elaborate on Slide 10 cont.

- We see that the Fourier Series coefficients

$$D_n = \frac{1}{f_s}\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\frac{\Omega}{f_s})e^{-jn\frac{\Omega}{f_s}}d\Omega$$

look like the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega t}d\Omega$$

- More specifically, if we use the property

$x(\boldsymbol{a}t) \Leftrightarrow \frac{1}{|\boldsymbol{a}|}X\left(\frac{\Omega}{\boldsymbol{a}}\right)$ we can write $x(f_s t) = \frac{1}{f_s}\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\frac{\Omega}{f_s})e^{j\Omega t}d\Omega$

we see that D_n is the same function as $x(f_s t)$ evaluated at $-\frac{n}{f_s}$.

Therefore,

$$D_n = x\left(-f_s\frac{n}{f_s}\right) = x(-n) = x[-n]$$