MATH 235
Linear Algbera 2

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1.1 Vector Spaces

1.1.1 Recap from Linear Algebra I

1. The vector space $\mathbb{R}^n$: n-dimensional real vector space

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \mathbb{R} \text{ for all } i \right\}$$

2. The vector space $\mathbb{C}^n$: n-dimensional complex vector space

$$\mathbb{C}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \mathbb{C} \text{ for all } i \right\}$$

3. The vector space $\mathbb{F}^n$: n-dimensional **field**, that denotes either $\mathbb{R}$ or $\mathbb{C}$.

$$\mathbb{F}^n = \left\{ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \mathbb{F} \text{ for all } i \right\}$$

   If $\mathbb{F} = \mathbb{R}$, then $\mathbb{F}^n = \mathbb{R}^n$, and if $\mathbb{F} = \mathbb{C}$, then $\mathbb{F}^n = \mathbb{C}^n$ and the scalar α depends on the appropriate field.

1.1.2 Other Vector Spaces

4. The vector space $\mathcal{P}_n(\mathbb{F})$: The set of polynomials of degree at most n with coefficients in $\mathbb{F}$.

$$\mathcal{P}_n(\mathbb{F}) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{F} \text{ for all } i\}$$

5. The vector space $M_{m\times n}(\mathbb{F})$: m by n matrices with entries in $\mathbb{F}$

$$M_{m\times n}(\mathbb{F}) = \left\{ \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \mid a_{ij} \in \mathbb{F} \text{ for all } i, j \right\}$$

6. The vector space of real-valued continuous functions on the interval $[a, b]$:

$$\mathcal{C}([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous on } [a, b]\}$$

Definition 1.1.1 (Vector space over $\mathbb{F}$)
A vector space over $\mathbb{F}$ is a set V together with an operation $+ : V \times V \to V$ (vector addition) so that

$$\forall \vec{x}, \vec{y} \in V, \vec{x} + \vec{y} \in V$$

and an operation $\times : \mathbb{F} \times V \to V$ (scalar multiplication) so that

$$\forall s \in \mathbb{F}, \vec{x} \in V, s \cdot \vec{x} \in V$$

Definition 1.1.2 (vector space axioms)
Properties of vector spaces that closed under addition and scalar multiplication,

1. $\forall \vec{x}, \vec{y}, \vec{z} \in V$, $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$
2. There exists a vector $\vec{0} \in V$ such that, $\forall \vec{x} \in V$, $\vec{0}+\vec{x}=\vec{x}+\vec{0}=\vec{x} \quad \rightarrow$ **zero vector** of V
3. $\forall \vec{x} \in V$, there exists $-\vec{x} \in V$ such that $\vec{x}+(-\vec{x})=(-\vec{x})+\vec{x}=\vec{0} \quad \rightarrow$ **additive inverse** of $\vec{x}$
4. $\forall \vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
5. $\forall \vec{x} \in V$ and $s, t \in \mathbb{F}$, $s \cdot (t \cdot \vec{x})=(st) \cdot \vec{x}$
6. $\forall \vec{x} \in V$ and $s, t\imath\mathbb{F}$, $(s+t) \cdot \vec{x}=s \cdot \vec{x}+t \cdot \vec{x}$
7. $\forall \vec{x}, \vec{y} \in \vec{x}$ and $s \in \mathbb{F}$, $s \cdot (\vec{x}+\vec{y})=s \cdot \vec{x}+s \cdot \vec{y}$
8. $1 \cdot \vec{x}=\vec{x} \quad \rightarrow$ **multiplicative inverse** of $\vec{x}$

Proposition 1.1.1
Let V be a vector space over $\mathbb{F}$. Then,

a. The zero vector in V is **unique**. If $\vec{z} \in V$ satisfies the property that $\vec{x}+\vec{z}=\vec{x}$, $\forall \vec{x} \in V$, then it must be the case that $\vec{z}=\vec{0}$.

b. Let $\vec{x} \in V$. The additive inverse of $\vec{x}$ is **uniquely determined** by $\vec{x}$. That is, if $\vec{y}$ satisfies the property that $\vec{x}+\vec{y}=\vec{y}+\vec{x}=\vec{0}$, then $\vec{y}=-\vec{x}$

Proof
Trivial proof using axioms . . .

Proposition 1.1.2
Let V be a vector space over $\mathbb{F}$. Then

1. $0 \cdot \vec{x}=\vec{0}$, for all $\vec{x} \in V$
2. $(-1) \cdot \vec{x}=-\vec{x}$ for all $\vec{x} \in V$
3. $t \cdot \vec{0}=\vec{0}$ for all $t \in \mathbb{F}$

Proof
For exercise ...

1.2 Subspaces

Definition 1.2.1 (subspace)
Let V be a vector space over $\mathbb{F}$ and $U \subseteq V$ a subset. U is a **subspace** of V if U, endowed with the addition and scalar multiplication from V, is itself a vector space over $\mathbb{F}$.

Theorem 1.2.1 (The subspace test)
Let V be vector space over $\mathbb{F}$ and let U be a subset of V. Then U is a subspace of V if and only if the following three conditions hold.

a. U is non-empty

b. $\forall \vec{u_1}, \vec{u_2} \in U$, $\vec{u_1} + \vec{u_2} \in U$. i.e., closed under addition

c. $\forall \alpha \in \mathbb{F}$ and $\forall \vec{u} \in U$, $\alpha\vec{u} \in U$. i.e., closed under scalar multiplication

Proof
If U is a subspace, then (b) and (c) hold as part of being a definition of a subspace, and since all vector spaces have a zero, U must be non-empty.

Suppose (a), (b) and (c) hold for a subset U of V. Properties (b) and (c) imply that the addition and scalar multiplication from V restrict to addition and scalar multiplication on U. Vector space axioms 1,4,5,6,7, and 8 hold since V is a vector space. For axiom 2, since U is non-empty, choose a vector $\vec{u} \in U$ and the by previous proposition, $0\vec{u} = \vec{0}$. Property (c) then implies that $\vec{0} \in U$. Similarly, for axiom 3, let $\vec{u} \in U$. Then by proposition and property (c), $-\vec{u} = (-1)\vec{u} \in U$, hence proved.

Corollary 1.2.1.1
Let V be a vector space over $\mathbb{F}$ and suppose that U is a subspace of V. Then $\vec{0} \in U$.

Proof
For exercise ...

Definition 1.2.2 (Span)
Let $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$. Define the **span** of S by

$$\text{span}(S) = \{t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \mid t_1, \dots, t_k \in \mathbb{F}\}$$

By convention, we define the span of the empty set to be the set consisting of the zero vector

$$\text{span}(\emptyset) = \{\vec{0}\}$$

Definition 1.2.3 (Linear combination)
A vector of the form $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$ is called a **linear combination** of the vectors $\vec{v}_1, \dots, \vec{v}_k$

Proposition 1.2.2
Let $S = \{\vec{v}_1, \dots, \vec{v}_k\} \subseteq V$. Then span$(S)$ is a subspace of V.

Proof
Since, $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_k$, $\vec{0} \in \text{span}(S)$ so span(S) is non-empty. Suppose $\vec{x}, \vec{y} \in \text{span}(S)$, and let $\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$ and $y = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k$ for elements $t_1, \dots, t_k, s_1, \dots, s_k \in \mathbb{F}$. Then

$$\vec{x} + \vec{y} = (t_1 + s_1)\vec{v}_1 + \cdots + (t_k + s_k)\vec{v}_k$$

so, $\vec{x} + \vec{y} \in \text{span}(S)$. Finally, let $\vec{x} \in \text{span}(S)$ be as above, and let $\alpha \in \mathbb{F}$. Then $\alpha\vec{x} = (\alpha t_1)\vec{v}_1 + \cdots + (\alpha t_k)\vec{v}_k$ and since $\alpha t_i \in \mathbb{F}$ for all i, $\alpha\vec{x} \in \text{span}(S)$. Therefore, by the subspace test, span(S) is a subspace of V.

1.3 Bases and Dimension

Linear Independence, Spanning Sets and Bases

Definition 1.3.1 (Spanning set, Spans)
A set of vectors $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space V is a **spanning set** for V, if $\text{span}(S) = V$. We also say that S **spans** V.

Definition 1.3.2 (Linearly independent and dependent)
A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in a vector space V is **linearly independent** if the only solution to the equation

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$$

is $t_1 = \cdots = t_k = 0$. The set is **linearly dependent** otherwise.

By convention, the empty set $\emptyset$ is linearly independent.

Definition 1.3.3 (Basis)
A **basis** for a vector space V is a linearly independent subset that spans V.

Theorem 1.3.1
Every vector space has a basis.

Dimension

For $\mathbb{F}^n$, we will define the dimension of a vector space V to be the number of vectors in a basis for V .

Lemma 1.3.2
Let V be a vector space over $\mathbb{F}$ and suppose that $V = \text{span}(\{\vec{v}_1, \dots, \vec{v}_n\})$. If $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a linearly independent set in V, then $k \leq n$

Proof
Since $\text{span}(\{\vec{v}_1, \dots, \vec{v}_n\}) = V$, we have

$$\begin{aligned}\vec{u}_1 &= a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n \\ &\vdots \\ \vec{u}_k &= a_{k1}\vec{v}_1 + \cdots + a_{kn}\vec{v}_k\end{aligned}$$

where $a_{ij} \in \mathbb{F}$, for all i and j. We will now aim to show that if $k > n$, then there is a solution to $t_1\vec{u}_1 + \cdots + t_k\vec{u}_k = \vec{0}$, where not all the t_i are 0. We have

$$\begin{aligned}t\vec{u}_1 + \cdots + t_k\vec{u}_k &= t_1(a_{11}\vec{v}_1 + \cdots + a_{1n}\vec{v}_n) + \cdots + t_k(a_{k1}\vec{v}_1 + \cdots + a_{kn}\vec{v}_k) \\ &= (a_{11}t_1 + \cdots + a_{k1}t_k)\vec{v}_1 + \cdots + (a_{1n}t_1 + \cdots + a_{kn}t_k)\vec{v}_n\end{aligned}$$

Now, if $k > n$ the system of linear equations

$$\begin{aligned} a_{11}t_1 + \cdots + a_{k1}t_k &= 0 \\ &\vdots \\ a_{1n}t_1 + \cdots + a_{kn}t_k &= 0 \end{aligned}$$

has a solution where not all the t_i are 0. Consider such a solution. We then have,

$$\begin{aligned} \vec{0} &= 0\vec{v}_1 + \cdots + 0\vec{v}_n \\ &= (a_{11}t_1 + \cdots + a_{k1}t_k)\vec{v}_1 + \cdots + (a_{1n}t_1 + \cdots + a_{kn}t_K)\vec{v}_n \\ &= t_1\vec{u}_1 + \cdots + t_k\vec{u}_k \end{aligned}$$

contradicting the assumption that $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is linearly independent. So $k \leq n$.

Theorem 1.3.3
Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\mathcal{C} = \{\vec{u}_1, \ldots, \vec{u}_k\}$ are both bases of a vector space V. Then $k = n$.

Proof
Since $\mathcal{B}$ spans V and $\mathcal{C}$ is linearly independent, $k \leq n$. However, since $\mathcal{C}$ spans V and $\mathcal{B}$ is linearly independent, $n \leq k$. Thus, $k = n$.

Definition 1.3.4 (Dimension)
The **dimension** of a vector space V, denoted by $\dim(V)$, is the size of any basis for V.

- $\dim(\{\vec{0}\}) = 0$ since by convention $\emptyset$ is a basis for $\{\vec{0}\}$.
- $\dim(\mathbb{F}^n) = n$ since the standard basis has size n.
- $\dim(\mathcal{P}_n(\mathbb{F})) = n + 1$ since the standard basis has a size $n + 1$.
- $\dim(M_{m\times n}(\mathbb{F})) = mn$ since the standard basis has size nm.

If there is no finite basis for a vector space V, then V is infinite-dimensional vector space.

Theorem 1.3.4
Let V be an n-dimensional vector space over $\mathbb{F}$. Then

a. A set of more than n vectors in V must be linearly dependent.

b. A set of fewer than n vectors in V cannot span V.

c. A set with exactly n vectors in V is a spanning set for V if and only if its linearly independent.

Theorem 1.3.5
Let V be a finite-dimensional vector space over $\mathbb{F}$ and let W be a subspace of V. Then $\dim(W) \leq \dim(V)$ with equality if and only if $W = V$.

Proof
Since any basis for W can be extended to a basis for V, the inequality $\dim(W) \leq \dim(V)$ follows.

Suppose now that $\dim(W) = \dim(V)$. Then according to previous theorem (c) part, a basis $\mathcal{B}$ for W will automatically be a basis for V, since it is a linearly independent set of size $\dim(V)$. It follows that $V = \text{span}(\mathcal{B}) = W$. Conversely, if $W = V$, then $\dim(W) = \dim(V)$.

Obtaining Bases

1. **Extending a linearly independent subset.** Given a linearly independent subset $\{\vec{v}_1, \ldots, \vec{v}_k\} \in V$. If it is a spannings set, then its a basis. If not, choose a vector $\{\vec{v}_{k+1}\}$ not in the span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$. Then $\{\vec{v}_1, \ldots, \vec{v}_{k+1}\}$ must be linearly independent. If this new spans, then it's a basis. If not, then repeat. This process must eventually stop since our vector space is finite-dimensional, and you will be left with a basis containing $\{\vec{v}_1, \ldots, \vec{v}_k\}$.

2. **Reducing an arbitrary finite spanning set.** Given a finite spanning set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for a vector space V, and assume that it doesn't contain $\vec{0}$. If it is linearly independent, it is a basis. If not, say v_i as a linear combination of the others. Now $\text{span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) = \text{span}(\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\})$, so $\{\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_k\}$ spans the vector space. If this new set is linearly independent, then it is a basis. If not, repeat to remove another vector. This process must eventually stop since we started with finitely many vectors in our spanning set. The final product will be a basis made up entirely out of vectors from our original spanning set.

Coordinates w.r.t a basis

Lemma 1.3.6
Let V be a vector space, let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a subset of V, and let $U = \text{span}(S)$. Then every vector in U can be expressed in a unique way as a linear combination of the vectors in S if and only if S is linearly independent.

Proof
Suppose every vector in U is expressed uniquely as a linear combination of the vectors in S. Then there is only one way to write

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

which is $t_1 = \cdots = t_k = 0$, so S is linearly independent. Conversely, suppose S is linearly independent and

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k$$

Rearranging we have $(t_1 - s_1)\vec{v}_1 + \cdots + (t_k - s_k)\vec{v}_k = \vec{0}$. Since S is linearly independent, this can only be true if $t_i = s_i$ for all i, hence proved.

Theorem 1.3.7 (Unique representation theorem)
Let V be a vector space and let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of V. Then for all $\vec{v} \in V$, there exist an unique scalar $x_1, \ldots, x_n \in \mathbb{F}$ such that

$$\vec{v} = x_1\vec{v} + \cdots + x_n\vec{v}_n$$

Definition 1.3.5 (Ordered basis)
Let V be a vector space over $\mathbb{F}$. An **ordered basis** for V is a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ for V together with a fixed ordering.

A basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ gives rise to $n!$ ordered bases, one for each possible permutation of the vectors in the basis.

Definition 1.3.6 (Coordinate vector)
Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an ordered basis for a vector space V. If $\vec{x} \in V$ is written as

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

the the coordinate vector of $\vec{x}$ with repsect to $\mathcal{B}$ is

$$[\vec{x}]_{\mathcal{B}} = (x_1, \ldots, x_n)$$

Once we have chosen a basis for V, every vector can now be represented as a column vector. Column vectors, as we know, come with their own addition and scalar multiplication.

Theorem 1.3.8
Let V be a vector space over $\mathbb{F}$ with ordered basis $\mathcal{B}$. Then

$$[\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}} = [\vec{x} + \vec{y}]_{\mathcal{B}} \qquad \text{and} \qquad t[\vec{x}]_{\mathcal{B}} = [t\vec{x}]_{\mathcal{B}}$$

for all $\vec{x}, \vec{y} \in V$ and all $t \in \mathbb{F}$.

Proof
This is just a matter of using the definition to determine $[\vec{x}]_{\mathcal{B}}$, $[\vec{y}]_{\mathcal{B}}$, $[\vec{x} + \vec{y}]_{\mathcal{B}}$ and $[t\vec{x}]_{\mathcal{B}}$