MATH 235
Honours Linear Algebra 2

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Contents

I. Intgeration

1 Basis for Row Space, Null space, Left Null Space

The key lemma in understanding $\mathrm{Col}(A)$ was to show $\mathrm{Col}(PA) = P \cdot \mathrm{Col}(A)$, where P is invertible.

1.1 Lemma. *If $A \in M_{m\times n}$ and $P \in M_{n\times n}$, then $Col(AP) = Col(A)$, where P is invertible.*

Proof To prove equality, we need to show that $\mathrm{Col}(AP) \subseteq \mathrm{Col}(A)$ and $\mathrm{Col}(A) \subseteq \mathrm{Col}(AP)$. First, we will show $\mathrm{Col}(AP) \subseteq \mathrm{Col}(A)$.
So let $y \in \mathrm{Col}(AP)$, then by definition $\exists \vec{X} \in \mathbb{R}^n$ such that

$$\begin{aligned} y &= AP\vec{x} \\ &= A(P\vec{x}) \in \mathrm{Col}(A) \end{aligned}$$

Next, we will show that $\mathrm{Col}(A) \subseteq \mathrm{Col}(AP)$ (use the hypothesis that P is invertible).
Let $y \in \mathrm{Col}(A)$, so $\exists \vec{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} y &= A\vec{x} \\ &= AP(P^{-1}\vec{x}) \in \mathrm{Col}(AP) \end{aligned}$$

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Pro-tip: We just showed that for all matrix A and invertible P, then $\mathrm{Col}(AP) \subseteq \mathrm{Col}(A)$
Now choose the matrix AP^{-1} and P, so

$$\mathrm{Col}(AP^{-1} \cdot P) = \mathrm{Col}(A) \subseteq \mathrm{Col}(AP^{-1})$$

and that's true for all P invertible so choose $P^{-1} \implies \mathrm{Col}(A) \subseteq \mathrm{Col}(AP)$.

1.2 Theorem. *If $A \in M_{m\times n}$, then $Row(A) = Row(R)$, where $R = RREF(A)$*

Proof By definition, $\mathrm{Row}(A) = \mathrm{Col}(A^T)$, where $A = PR$ for some invertible P.

$$A^T = R^T P^T$$

From the previous lemma we know that

$$\mathrm{Col}(A^T) = \mathrm{Col}(R^T P^T)$$

and

$$\mathrm{Col}(A^T) = \mathrm{Col}(R^T)$$

since P^T is invertible, therefore

$$\mathrm{Row}(A) = \mathrm{Col}(A^T) - \mathrm{Col}(R^T) = \mathrm{Row}(R)$$

1.3 Corollary. *A basis for Row(A) is the non-zero rows of its RREF.*

Example: If $A \in M_{3\times 3}$

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis for Row}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Important: The Basis for Row(A) is not the row 1 and 2 of A.