MATH 135
Honours Algebra

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Winter 2023†

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†Last updated: January 16, 2023

Contents

I. Introduction to the Language of Mathematics

1 Sets and Mathematical Statements

Definition. (Set) A set is a well-defined unordered (i.e., order does not matter) collection of distinct (unique) objects.
Example: Empty set = $\{\} = \emptyset$

Note:

- $\{\emptyset\} \neq \emptyset$ and $\{a, \{a, b\}\}$ is a set since $a, \{a, b\}$ are distinct objects.
- $\in \rightarrow$ "is a member of"
- $\notin \rightarrow$ "is not a member of"

Exercise 1 (True or false)

1. $\in \{A, \{A, B\}\} \rightarrow$ false
2. $A \in \{A, \{A, B\}\} \rightarrow$ true
3. $B \in \{A, \{A, B\}\} \rightarrow$ false
4. $\{B, A\} \in \{A, \{A, B\}\} \rightarrow$ true

Common Sets

$\mathbb{N} = \{1, 2, 3, \ldots\} \rightarrow$ Natural Numbers
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \rightarrow$ Integer Numbers
$\mathbb{R} = \{$ set of real numbers $\} \rightarrow$ Real Numbers
$\mathbb{P} = \{2, 3, 5, \ldots\} \rightarrow$ Prime Numbers
$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\} \rightarrow$ Rational Numbers

Exercise 1 (True or false)

1. $\mathbb{Q} \in \mathbb{N} \rightarrow$ false
2. $\sqrt{2} \in \mathbb{Q} \rightarrow$ false
3. $-\infty \in \mathbb{Z} \rightarrow$ false
4. $\pi \in \mathbb{R} \rightarrow$ true
5. $\{\sqrt{2}, \sqrt{3}\} \subseteq \mathbb{R} \rightarrow$ true

Statements

Definition. (statements) A statement is a sentence that has a definite state of being true or false. (It cannot be sometimes false or sometimes true). (i.e., it cannot be sometimes false or sometimes true)

$P(a,b,c) : a^2+b^2=c^2$ is NOT a statement since for some values of a,b and c the statement may be true and false for some.

Examples:

- $P(3,4,5) : 3^2+4^2=5^2 \longrightarrow$ true
- $P(1,2,3) : 1^2+2^2=3^2 \longrightarrow$ false

Exercise 1 (Which of these are statements)

1. 18 is a prime number $\longrightarrow$ statement
2. $a^3+b^3=c^3 \longrightarrow$ not a statement
3. $\forall a,b,c \in \mathbb{R}$, $a^3+b^3=c^3 \longrightarrow$ statement
4. $\exists a,b,c \in \mathbb{R}$ such that $a^3+b^3=c^3 \longrightarrow$ statement

Open Sentence

Definition. (open sentence) An open sentence is a sentence with at least one variable that is not a statement but can become one when we give values.

$P(a,b,c) : a^3+b^3=c^3$ is an open sentence since if we give values for a,b and c $P(a,b,c)$ will become a statement.

Examples:

- $P(2,3,5) : 2^3+3^3=5^3 \longrightarrow$ statement
- $\forall a,b,c \in \mathbb{R}$, $a^3+b^3=c^3 \longrightarrow$ statement

Exercise 4 (Which of these are open sentences or statements)

1. $P(2,3,5) \longrightarrow$ statement
2. $\forall a \in \mathbb{R}$, $P(a,b,c) \longrightarrow$ open sentence
3. $P(2,3,C) \longrightarrow$ open sentence
4. $\exists a,b \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $P(a,b,c) \longrightarrow$ sometimes considered as a statement or as an open sentence

NEGATION OF STATEMENTS

If P is a statement,

$$\neg P \to \text{not } P$$

Examples:

- $P \to$ There is a $x \in \mathbb{R}$ such that $x^2 = 2$
- $\neg P \to \forall x \in \mathbb{R},\, x^2 \neq 2$
- $P \to \forall x \in \mathbb{R},\, x^2 = 2$
- $\neg P \to \exists x \in \mathbb{R}$ such that $x^2 = 2$

$\neg(\neg P)$ is logically equivalent to P, $\neg(\neg P) = P$

2 QUANTIFIERS AND NESTED QUANTIFIERS

QUANTIFIERS

$\forall$ = "for all" $\to$ universal quantifier
$\exists$ = "there exists" $\to$ Existential quantifier
Typically , we'll have a open sentence with at least one "free" variable, x.

Examples:

- $x^2 + 2 = z^3$
- $\forall x \in \mathbb{Z}, \exists z \in \mathbb{R}, x^2 + 2 = z^3 \to$ For all $x \in \mathbb{Z}$, there exists $z \in \mathbb{R}$, such that $x^2 + 2 = z^3$

P(x): $x^2 = 2$ (open sentence)

- $\exists x \in \mathbb{N}, x^2 = 2 \to$ false
- $\exists x \in \mathbb{R}, x^2 = 2 \to$ true
- $\forall x \in \mathbb{R}, x^2 \neq 2 \to$ true
- $\forall x \in \mathbb{R}, x^2 \neq 2 \to$ false

If Q(x): $\frac{m+1}{m+2} = 5$ is open sentence, then

$$\exists m \in \mathbb{Z}, \frac{m+1}{m+2} = 5 \to \text{ false}$$

We can make the above statement true by changing its domains, i.e.,

$$\exists m \in \mathbb{R}, \frac{m+1}{m+2} = 5 \qquad\qquad \exists m \in \mathbb{Q}, \frac{m+1}{m+2} = 5$$

Hidden Quantifiers

Examples:

- 64 is a perfect square $\rightarrow \exists x \in \mathbb{Z}, x^2 = 64$ (true statement)
- $2^{2x-4} = 8$ has a integer solution $\rightarrow \exists x \in \mathbb{Z}, 2^{2x-4} = 8$ (false statement)
- The graph of $y = x^3 - 2x + 1$ has no x-intercept
  $\rightarrow$ There is no solution in $x \in \mathbb{R}$ such that $x^3 - 2x + 1 = 0$
  $\rightarrow$ For all $x \in \mathbb{R}, x^3 - 2x + 1 = 0$
  $\rightarrow \forall x \in \mathbb{R}, x^3 - 2x + 1 = 0$ (false statement)

Negation of Quantifiers

P : Everyone in this room was born in or before 2013.
$\neg P$: There exists someone in this room was born in or after 2013.
S : Set of people in this room, $Q(x) = x$ is born in or before 2013, where x is a person in the room.

- $P : \forall x \in S, Q(x)$
- $\neg P : \exists x \in S, \neg Q(x)$

Fact: If we have the statement of the form

$$P : \forall x \in S, Q(x)$$
$$\neg P : \exists x \in S, \neg Q(x)$$

Exercise 5 (Negate the statement)

1. $P : \forall x \in \mathbb{R}, |x| \geq 5$
2. $\neg P : \exists x \in \mathbb{R}, \neg(|x| \geq 5) \rightarrow \exists x \in \mathbb{R}, |x| < 5$

Nested Quantifiers

Examples: Let $Q(x, y) = x^3 - y^3 = 1$

- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1 \rightarrow$ false
- $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 \rightarrow$ true
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 \rightarrow$ true
- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 \rightarrow$ false

Note: Switching the order of the quantifiers in a statement makes a difference.
If we have an open sentence $Q(x, y)$,
$\exists x \in S, \forall y \in T, Q(x, y) \rightarrow$ There is an $x \in S$ such that [for all $y \in T, Q(x, y)$] is true.

Negating Nested Quantifiers

Examples: Let $Q(x, y, z) = x^5 + y^2 = 2^3$

$$P : \exists x \in \mathbb{Z}, \forall y \in \mathbb{Q}, \exists z \in \mathbb{R}, x^5 + y^2 = 2^3$$
$$\neg P : \forall x \in \mathbb{Z}, \exists y \in \mathbb{Q}, \forall z \in \mathbb{R}, x^5 + y^2 \neq 2^3$$

Fact:

- In order to negate a nested quantified statement, just flip $\forall$ and $\exists$, and also negate the statement $P(x)$
- Also if the nested quantified statement is long, break it into shorter nested quantified statements and negate it.

II. Logical Analysis of Mathematical Statements

3 Truth Table

Let p be statement.

1. $\neg P$ = "not P"
2. $\neg P$ is true when P is false and false when P is true.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

- $\neg$ is a logical operator, somethin take takes a statement and creates a new statements.
- $\neg P$ is a logical expression.

Notice: P and $\neg(\neg P)$ have the same truth value So P and $\neg(\neg P)$ are logically equivalent ($\equiv$), i.e,

$$P \equiv \neg(\neg P)$$

4 Conjuction and Disjunction

1. Conjuction ($\wedge$) = "and"
2. Disjunction ($\vee$) = "or"

We can use conjunction and disjunction to create compound statements, that are built from two or more statement using things like $\vee$ and $\wedge$.

Example: A and B are statements (statement variables)

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- $A \wedge B$ and $A \vee B$ are called compound statements.

- $\vee \rightarrow$ If one of the statements are true or both of them are, then the compound statement is true.
- $\wedge \rightarrow$ Even if one of the statement is false, then the compound statement is false.

Example: $\forall x \in \mathbb{R}, (x^2 \geq 0) \wedge (\sin^2(x) + \cos^2(x) = 1)$ is a true statement because both the statement variables are true. Therefore the compound statement is true.

5 Logical Operators and Algebra

De Morgan's Laws

A	B	$\neg(A \wedge B)$	$\neg(A \vee B)$	$\neg A \vee \neg B$	$\neg A \wedge \neg B$
T	T	F	F	F	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	T	T	T	T

1. $\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$
2. $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$

Exercise: Negate the statement
Let L be a line and P be a parabola. Statement: The point $(1,2)$ lies on L or on P

$$(1,2) \in L \vee (1,2) \in P$$

Sol:

$$\begin{aligned}
&\neg((1,2) \in L \vee (1,2) \in P) \\
\equiv&(\neg(1,2) \in L \wedge \neg(1,2) \in P) \\
\equiv&(1,2) \notin L \wedge (1,2) \notin P
\end{aligned}$$

Negated statement: The point $(1,2)$ does not lie on L and does not lie on P.

Other Logical Operators Laws

Commutative Laws: (order does not matter)

- $P \wedge Q \equiv Q \wedge P$
- $P \vee Q \equiv Q \vee P$

Associative Laws: (parentheses does not matter)

- $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
- $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

Distributive Laws:

- $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Exercise:

1. Prove without using the truth table

$$\neg(A \wedge (\neg B \wedge C)) \equiv \neg(A \wedge C) \vee B$$

Proof Lets consider the LHS,

$$\begin{aligned}
&\neg(A \wedge (\neg B \wedge C)) && \\
\equiv&\neg A \vee (\neg(B \wedge C)) && \text{[By De Morgan's Law]} \\
\equiv&\neg A \vee (\neg(\neg B) \vee \neg C) && \text{[By De Morgan's Law]} \\
\equiv&\neg A \vee (B \vee \neg C) && \text{[By Double Negation]} \\
\equiv&(\neg A \vee \neg C) \vee B && \text{[By Associative Law]} \\
\equiv&\neg(A \wedge C) \vee B && \text{[By De Morgan's Law]}
\end{aligned}$$

■

2. True or false statement:
   a) $\forall x \in \emptyset, x^2 = 1 \rightarrow$ vacuously true
   b) $\exists x \in \emptyset, x^2 = 1 \rightarrow$ false

6 Implications

P and Q are statements $\rightarrow$ If P, then Q
"If the statement P is true, then the statement Q is true"

Exercise:

1. If Alice is from Canada, then Alice is from North America $\rightarrow$ True
2. If Alice if from North America, then Alice is from Canada $\rightarrow$ False
3. If I am an animal, then I will give you \$10 $\rightarrow$ True
4. If $x > 3$, then $x > 5 \rightarrow$ False
5. If $x > 3$, then $x \geq 1 \rightarrow$ True

Note: If P is false, then it does not matter, if Q is false or true, the implication will be **true**.
Implication Law: $(P \implies Q) \equiv (\neg P \vee B)$

Exercise:

1. Prove using the implication law

$$\forall x \in \mathbb{R}, (x > 2) \implies (x^2 > 1)$$

Proof Let $A(x) = x > 2$ and $B(x) = x^2 > 1$. Then for $(A(x) \implies B(x))$ to be true. We need $B(x)$ to be true or $A(x)$ to be false. Notice $B(x)$ is true for $x \in (1, \infty) \cup (-\infty, -1)$ and $A(x)$ is false for $x \in (-\infty, 2]$. So $B(x)$ is true or $A(x)$ is false, holds for $x \in (1, \infty) \cup (-\infty, -1) \cup (-\infty, 2] = \mathbb{R}$.
So, $\forall x \in \mathbb{R}, B(x) \vee \neg A(x) \equiv \forall x \in \mathbb{R}, A(x) \implies B(x)$ ■

2. Let $\mathbb{P}$ be a set of prime numbers. Prove that

$$\forall p \in \mathbb{P}, (p > 2) \implies (P + 1) \text{ is even.}$$

Proof If the hypothesis is true, then p is a prime greater than 2 and since p it prime it cannot be a multiple of 2, so by definition of even and odd, p is odd. so $p+1$ is even. Thus if the hypothesis is true, the conclusion is true. so $\forall p \in \mathbb{P}, (p > 2) \implies (P + 1)$ is even. ■

Negating Implication

What is $\neg(A \implies B)$?

$$\begin{aligned}
\neg(A \implies B) &\equiv \neg(B \vee (\neg A)) && [\text{we know } (A \implies B) \equiv (B \vee (\neg A))] \\
&\equiv \neg B \wedge \neg(\neg A) && [\text{By De Morgan's Law}] \\
&\equiv \neg B \wedge A && [\text{By Double Negation}]
\end{aligned}$$

Exercise: If $\mathbb{P}$ are the set of prime numbers, then ”There is at most one prime number less than 3”.

$$\forall x \in \mathbb{P}, \forall y \in \mathbb{P}, ((x < 3) \wedge (y < 3) \implies (x = y))$$

7 Converse and Contrapositive

Definition. (Converse) The implication $B \implies A$ is called the **converse** of $A \implies B$

Note: A common mistake is to think that the implication $A \implies A$ and its converse $B \implies A$ are logically equivalent. They are not!

Definition. (contrapositive) The implication $(\neg B) \implies (\neg A)$ is called the **contrapositive** of $A \implies B$.

Contrpositive equivalence Law: An Implication is logically equivalent to its contrapositive.

$$(P \implies Q) \equiv ((\neg B) \implies (\neg A))$$

Implication Law:

$$(A \implies B) \equiv ((\neg A) \vee B) \qquad \neg(A \implies B) \equiv (A \wedge (\neg B))$$

8 If and Only if

Definition. (if and only if (iff)) The truth value for “A if and only if B”, written symbolically as $A \iff B$ is true when A and B have the same truth values, and is false when they have opposite truth values.

More Laws:

1. $(A \iff B) \equiv ((A \implies B) \land (B \implies A))$
2. $\Big(\forall x \in X, P(x) \iff Q(x)\Big) \equiv \Big((\forall x \in X, P(x) \implies Q(x)) \land (\forall x \in X, Q(x) \implies P(x))\Big)$