MATH 138
Honours Calculus 2

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Contents

I. Integration

1 Introduction

Consider the interval $[a, b]$, If we sub-divide it into n sub-intervals we get (for example) something called an increasing sequence $P = \{t_0, t_1, t_2, \ldots, t_n\}$ a partition of the interval $[a, b]$.
The length of the i^{th} sub-interval is given by

$$\Delta t_i = t_i - t_{i-1}, i \in 1, 2, \ldots, n$$

Next let $c_i \in [t_{i-1}, t_i]$.
Definition. (Riemann Sum) Given a bounded function f and partition P over the interval $[a, b]$ with $c_i \in [t_{i-1}, t_i]$ a Riemann sum of f w.r.t P is

$$S = S(f, P) = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

Note:

1. Different partitions P or different choices for the c_i will yield different values of S.

2. The value n can change from one Riemann sum to the next.

Key Ideas:

1. Shrink all Δt_i down to zero thus increasing the “resolution” of the sum. We will end up with an $\infty \cdot 0$ situation which will hopefully balance out to give a finite value, call it I.

2. If it turns out that the value of I is independent of the partition P and values c_i then we say f is integrable.

3. We denote $\|P\| = \max(\Delta t_1, \Delta t_2, \ldots, \Delta t_n)$ so the previous condition can be written as $\|P\| \to 0$ as $n \to \infty$.

4. If our bounded function f is integrable with value I then we write

$$I = \int_a^b f(t)dt$$

where $f(t)$ is the integrand, dt is the variable of integration (aka dummy variable) and a, b are the limits of integration.

5. The notation $\int_a^b f(t)dt$ is called the definite integral of f from a to b.

6. Note that $\int_a^b f(t)dt = \int_a^b f(x)dx = \int_a^b f(u)du$ etc. is to be thought as $\sum_{i=1}^n f(i) = \sum_{k=1}^n f(k) = \sum_{p=1}^n f(p)$ etc.

Definition. (Regular n-partition) The regular n-partition for an interval $[a,b]$ is where $\Delta t_i = \frac{b-a}{n}$ for each i. i.e., we divide $[a,b]$ into n intervals of equal width.

Eg: If we knew that $f(x) = e^x$ was integrable the one way to calculate its integral over $[1,4]$ would be:

$$\int_1^4 e^x dx = \lim_{n\to\infty} \sum_{i=1}^n e^{c_i} \Delta t_i$$

where $c_i \in [t_{n-1}, t_i]$.

Assuming the regular n-partition we get,

$$\Delta t_i = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n} = \Delta t$$

We can choose c_i any way we like. One common way is to built a right-hand Riemann sum (R) by letting,

$$\begin{aligned} c_i &= a + i\Delta t \\ &= 1 + \frac{3i}{n} \end{aligned}$$

We could also make a left-hand Riemann sum (L) by choosing,

$$\begin{aligned} c_i &= a + (i-1)\Delta t \\ &= 1 + \frac{3(i-1)}{n} \end{aligned}$$

Using R, we would get

$$\begin{aligned} \int_1^4 e^x dx &= \lim_{n\to\infty} \sum_{i=1}^n f(c_i)\Delta t \\ &= \lim_{n\to\infty} \sum_{i=1}^n e^{1+\frac{3i}{n}} \left(\frac{3}{n}\right) \\ &= \lim_{n\to\infty} \frac{3e}{n} \sum_{i=1}^n \left(e^{\frac{3}{n}}\right)^i \qquad \left[\text{recall } \sum_{i=1}^n r^i = \frac{r^{n+1}-r}{r-1}\right] \\ &= \lim_{n\to\infty} \frac{3e}{n} \left(\frac{e^{\frac{3}{n}+3} - e^{\frac{3}{n}}}{e^{\frac{3}{n}} - 1}\right) \\ &= \frac{3e(e^3-1)}{3} \\ &= e(e^3-1) \end{aligned}$$

But is $f(x) = e^x$ integrable?

> ***1.1 Theorem. (Integrability Condition)*** *If f is continuous on $[a,b]$ then f is integrable on $[a,b]$.*

Note: If f is bounded with finitely many jump discontinuities then it is also integrable.

2 Properties of Integration

2.1 Theorem. (Properties of Integrals) *If f and g are integrable over $[a,b]$ then*

a. $\int_a^b cf(t)dt = c\int_a^b f(t)dt$, *for any* $c \in \mathbb{R}$

b. $\int_a^b [f(t)+g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt$

c. If $m \leq f(t) \leq M$ *then,*

$$m(b-a) \leq \int_a^b f(t) \leq M(b-a)$$

d. $|f|$ *is integrable on* $[a,b]$, *then*

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt$$

Proof Given any partition P of $[a,b]$, note that

$$\sum_{i=1}^{n} \Delta t_i = b-a$$

Since $m \leq f(t) \leq M$ we get that

$$\sum_{i=1}^{n} m\Delta t_i \leq \sum_{i=1}^{n} f(t)\Delta t_i \leq \sum_{i=1}^{n} M\Delta t_i$$
$$m\sum_{i=1}^{n} \Delta t_i \leq \sum_{i=1}^{n} f(t)\Delta t_i \leq M\sum_{i=1}^{n} \Delta t_i \qquad \blacksquare$$

This is true for every partition P and so we end up with

$$m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$$

2.2 Corollary. *Properties of Integration*

e. Set $m = 0$ *in (c.) to get, if* $f(t) \geq 0$ *then*

$$\int_a^b f(t)dt \geq 0$$

f. If $f(t) \geq g(t)$ *then use (e.), (b.) and (a.) to get*

$$\int_a^b f(t)dt \geq \int_a^b g(t)dt$$

it can be proved by making a new function $h(t) = f(t) - g(t)$ *(hint...)*

g. We define $\int_a^b f(t)dt = 0$. Our integration interval would be $[a,a]$ and so any Riemann sum we create would be of the form

$$\sum_{i=1}^{n} f(a)\Delta t_i = f(a)\cdot 0 = 0$$

f. For $a < b$ we have

$$\int_a^b f(t)dt = -\int_b^a f(t)dt$$

The idea here is that writing $\int_a^b$ suggests moving from a to b where t_{1-i} and t_i are points on line a to b and $\Delta t_i > 0$.
Whereas $\int_a^b$ suggests moving from b to a where $\Delta t_i < 0$.

2.3 Theorem. *Given $a,b,c \in I$ over which f is integrable then*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Note: It is not required that $a < c < b$. If $a < b < c$, we get

$$\begin{aligned}\underbrace{\int_a^b f(t)dt}_{\text{Area b/w }[a,b]} &= \int_a^c f(t)dt + \int_c^b f(t)dt \\ &= \underbrace{\int_a^c f(t)dt}_{\text{Area b/w }[a,c]} - \underbrace{\int_b^c f(t)dt}_{\text{Area b/w }[b,c]}\end{aligned}$$

3 Areas and Integrals

Note that $\int_a^b f(t)dt$ will only return the "expected" are when $f \geq 0$. Generally $\int_a^b f(t)dt$ returns the "signed" area.
That is if $f < 0$ over some interval $[c,d]$ then $\int_c^d f(t)dt$ will return the negative area between f and the x-axis.

Let the interval from $[a,b]$ have 2-regular partitions, c and d, where $a < c < d < b$. Assume $A_1, A_2, A_3 > 0$ (i.e., the normal area), where A_1 be the positive area from $[a,c]$, A_2 be the positive area from $[d,b]$ and A_3 be the negative area from $[c,d]$. then

$$\int_a^b f(t)dt = A_1 + A_2 - A_3$$

Example: $\int_0^{2\pi} \sin(t)dt = 0$, since A_1 and A_2 have equal areas, where A_1 is the positive area from $[0,\pi]$ and A_2 is the negative area from $[\pi, 2\pi]$.

4 Average Value

Recall the average of a discrete set $\{x_1, x_2, \ldots, x_n\}$ is given by

$$\frac{\sum_{i=1}^{n} x_i}{n}$$

We can define a similar concept for functions as follows:

Definition. (Average Value) The average value of a continuous function over the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)dt$$

sometimes written as $\bar{f}$.

Example: Geometrically compute f_{ave} over $[0,4]$ if $f(x) = 3x$.
Geometrically, we have

$$\int_0^4 3xdx = \frac{\text{base} \times \text{height}}{2} = \frac{4 \cdot 12}{2} = 24$$

so that $f_{ave} = \bar{f} = \frac{1}{4}\int_0^4 3xdx = \frac{24}{4} = 6$.

In this case, f_{ave} occurs halfway between 0 and 12.

However, in all cases, f_{ave} will split $f(x)$ into 2 parts of equal areas. That is the area above f_{ave} and below $f(x)$ will equal the area of both $f(x)$ and below f_{ave}.
This can be proven by shifting the x-axis to instead be f_{ave}. Let $g(x) = f(x) - f_{ave}$.
If the area above $y = f_{ave}$ is equal to the below f_{ave} then we should get $\int_a^b g(x)d(x) = 0$.
Indeed,

$$\begin{aligned}
\int_a^b g(x)dx &= \int_a^b f(x) - f_{ave}dx \\
&= \int_a^b f(x) - \int_a^b f_{ave}dx \quad [\text{Recall} f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx] \\
&= (b-a)f_{ave} - (b-a)f_{ave} \\
&= 0
\end{aligned}$$

It is always the case that for an integrable function, there is a $c \in [a, b]$ such that $f(c) = f_{ave}$?
In general ”no”.

4.1 Theorem. (Average Value Theorem (AVT)) *If f is a continuous function on $[a, b]$ then there is a $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Proof By Extreme Value Theorem (EVT), there are p (min) and q (max), where $p, q \in [a, b]$ such that

$$f(p) \leq f(x) \leq f(q)$$

By integral properties,

$$(b-a)f(p) \leq \int_a^b f(x)dx \leq (b-a)f(q)$$

$$f(p) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(q)$$

By the Intermediate value theorem, there is a $c \in [a, b]$ where $f(c)$ is the above equation.■

5 Fundamental Theorem of Calculus (FTC)

Up until now to compute $\int_a^b f(t)dt$ we had to rely on geometry or, if using the definition, we need formulas to convert $\sum_{i=1}^n f(c_i)\Delta t_i$ to an explicit expression like

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

In many cases, this is not possible.

Let us investigate how an integral behaves on a function on x. That is,
Let $A(x) = \int_a^x f(t)dt$ for a continuous function f. Now consider $x+h$

$$A(x+h) = \int_a^{x+h} f(t)dt$$

The incremental area is given by

$$A(x+h) - A(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$$

Divide both sides by h to get

$$\frac{A(x+h)-A(x)}{h} = \frac{1}{h}\int_a^{x+h} f(t)dt$$

since f is continuous, by AVT, $\exists c \in [x, x+h]$ such that

$$\frac{A(x+h)-A(x)}{h} = f(c)$$

Also since f is continuous and c depends on h,

$$\lim_{h\to 0} f(c) = f(x)$$

Finally by definition

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = A'(x)$$

Thus $A'(x) = f(x)$, i.e.,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

5.1 Theorem. (Fundamental Theorem of Calculus I (FTC I)) *If f is continuous on an open interval containing x and a, then*

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

That is, the derivative **cancels** the integral.

How does this help us compute $\int_a^b f(t)dt$? If we let $g(x) = \int_a^x f(t)dt$ then by FTC I, we know that $g'(x) = f(x)$ so we begin a search for a function $g(x)$ which after we take a derivative gives $f(x)$.

Example: Compute

$$\int_3^5 2t dt$$

We need a function $g(x)$ such that $g'(x) = 2x$. That is we seek an antiderivative of $2x$ (Recall from MATH 137 that given a funtion, any 2 antiderivatives of that function can differ by at most a constant.)
Let us denote $G(x) = x^2 + c$ as the family of antiderivatives of $2x$. By FTC I we have that

$$G(x) = x^2 + c = \int_3^x 2t dt$$

since G'(x) = 2x

But what is c? We know that $G(3) = 9 + c = \int_3^3 2t dt = 0$, so $c = -9$. Thus

$$G(5) = \int_3^5 2t dt = 25 - 9 = 16$$

Notice that if we instead let $g(x) = x^2 + 4$ and evaluated $g(5) - g(3)$ we should still get 16.

This leads us to:

5.2 Theorem. (Fundamental Theorem of Calculus II (FTC II)) *Let F be any antiderivative of a continuous function f. Then,*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Example: Compute

$$\int_1^4 \cos(x) dx$$

Using a Riemann sum would require a foumula for

$$\sum_{i=1}^{n} \cos(1 + \frac{3i}{n})$$

and then a limit as $n \to \infty$

Using FTC however we know that since $\frac{d}{dx}\sin(x) = \cos(x)$, we get

$$\int_1^4 \cos(x)dx = \sin(4) - \sin(1)$$

Notation: The expression $g(x)\Big|_a^b = g(b) - g(a)$

Example:

$$\begin{aligned}
\int_{-1}^{1} |x| dx &= \int_{-1}^{0} (-x)dx + \int_0^1 (x)dx \\
&= -\int_{-1}^{0} (x)dx + \int_0^1 (x)dx \\
&= \frac{-x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_0^1 \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

6 Extended Fundamental Theorem of Calculus I

We learned that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

What about $\frac{d}{dx}\int_x^a f(t)dt$ or $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt$? we solve these using the **chain rule**.

Example: Compute

$$\frac{d}{dx}\int_2^{\ln(x)} \sin(t^2)dt$$

Let $u = \ln(x)$ and let $g(u) = \int_2^u \sin(t^2)dt$.
We seek $\frac{d}{dx}[g(u)]$, by chain rule $\frac{d}{dx}[g(u)] = \frac{d}{du}[g(u)] \cdot \frac{du}{dx}$. [We can end up with $\frac{d}{dx}$ by multiplying various derivatives, i.e. $\frac{du}{dt} \cdot \frac{dt}{dp} \cdot \frac{dp}{dx}$]

Now, $\frac{du}{dx} = \frac{1}{x}$ and by FTC,

$$\frac{d}{du}[g(u)] = \frac{d}{du}\int_2^u \sin(t^2)dt = \sin(u^2)$$

so we get

$$\frac{d}{dx}[g(u)] = \sin(u^2)\left(\frac{1}{x}\right)$$

i.e.

$$\frac{d}{dx}\int_2^{\ln(x)} \sin(t^2)dt = \frac{\sin([\ln x]^2)}{x}$$

General rule: For a continuous f and differentiable at $a(x)$ and $b(x)$, we get

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$$

Tip: Even if you can't find an explicit F such that $F' = f$ you can "imagine" one exists.
Then by FTC II,

$$\begin{aligned}\int_{a(x)}^{b(x)} f(t)dt &= F(b(x)) - F(a(x))\\ \frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt &= F'(b(x))\cdot b'(x) - F'(a(x))\cdot a(x)\\ &= f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)\end{aligned}$$