MATH 138
Honours Calculus 2

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0 Introduction

In the Calculus 2 class, we learnt about the Riemann integrals and approximations. Antiderivatives and the fundamental theorem of calculus. Change of variables, methods of integration. Applications of the integral. Improper integrals. Linear and separable differential equations and applications. Tests for convergence for series. Binomial series, functions defined as power series and Taylor series. Vector (parametric) curves in $\mathbb{R}^2$.

This course provides really good intuition for riemann integrals, which helps in upper year real analysis courses.

1 Integration

1.1 Introduction

Consider the interval $[a, b]$, If we sub-divide it into n sub-intervals we get (for example) something called an increasing sequence $P = \{t_0, t_1, t_2, \ldots, t_n\}$ a partition of the interval $[a, b]$.
The length of the i^{th} sub-interval is given by

$$\Delta t_i = t_i - t_{i-1}, i \in 1, 2, \ldots, n$$

Next let $c_i \in [t_{i-1}, t_i]$.

Definition 1.1.1 (Riemann Sum)
Given a bounded function f and partition P over the interval $[a, b]$ with $c_i \in [t_{i-1}, t_i]$ a Riemann sum of f w.r.t P is

$$S = S(f, P) = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

Note:

1. Different partitions P or different choices for the c_i will yield different values of S.
2. The value n can change from one Riemann sum to the next.

Key Ideas:

1. Shrink all Δt_i down to zero thus increasing the "resolution" of the sum. We will end up with an $\infty \cdot 0$ situation which will hopefully balance out to give a finite value, call it I.
2. If it turns out that the value of I is independent of the partition P and values c_i then we say f is integrable.
3. We denote $||P|| = \max(\Delta t_1, \Delta t_2, \ldots, \Delta t_n)$ so the previous condition can be written as $||P|| \to 0$ as $n \to \infty$.
4. If our bounded function f is integrable with value I then we write

$$I = \int_a^b f(t)dt$$

   where $f(t)$ is the integrand, dt is the variable of integration (aka dummy variable) and a, b are the limits of integration.
5. The notation $\int_a^b f(t)dt$ is called the definite integral of f from a to b.
6. Note that $\int_a^b f(t)dt = \int_a^b f(x)dx = \int_a^b f(u)du$ etc. is to be thought as $\sum_{i=1}^n f(i) = \sum_{k=1}^n f(k) = \sum_{p=1}^n f(p)$ etc.

Definition 1.1.2 (Regular n-partition)
The regular n-partition for an interval $[a, b]$ is where $\Delta t_i = \frac{b-a}{n}$ for each i. i.e., we divide $[a, b]$ into n intervals of equal width.

Eg: If we knew that $f(x) = e^x$ was integrable the one way to calculate its integral over $[1, 4]$ would be:

$$\int_1^4 e^x dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{c_i} \Delta t_i$$

where $c_i \in [t_{n-1}, t_i]$.

Assuming the regular n-partition we get,

$$\Delta t_i = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n} = \Delta t$$

We can choose c_i any way we like. One common way is to built a right-hand Riemann sum (R) by letting,

$$\begin{aligned} c_i &= a + i\Delta t \\ &= 1 + \frac{3i}{n} \end{aligned}$$

We could also make a left-hand Riemann sum (L) by choosing,

$$\begin{aligned} c_i &= a + (i-1)\Delta t \\ &= 1 + \frac{3(i-1)}{n} \end{aligned}$$

Using R, we would get

$$\begin{aligned} \int_1^4 e^x dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta t \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} e^{1+\frac{3i}{n}} \left(\frac{3}{n}\right) \\ &= \lim_{n\to\infty} \frac{3e}{n} \sum_{i=1}^{n} \left(e^{\frac{3}{n}}\right)^i \qquad \left[\text{recall } \sum_{i=1}^{n} r^i = \frac{r^{n+1}-r}{r-1}\right] \\ &= \lim_{n\to\infty} \frac{3e}{n} \left(\frac{e^{\frac{3}{n}+3} - e^{\frac{3}{n}}}{e^{\frac{3}{n}} - 1}\right) \\ &= \frac{3e(e^3-1)}{3} \\ &= e(e^3-1) \end{aligned}$$

But is $f(x) = e^x$ integrable?

Theorem 1.1.1 (Integrability Condition)
If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Note: If f is bounded with finitely many jump discontinuities then it is also integrable.

1.2 Properties of Integration

Theorem 1.2.1 (Properties of Integrals)
If f and g are integrable over $[a, b]$ then

a. $\int_a^b cf(t)dt = c\int_a^b f(t)dt$, for any $c \in \mathbb{R}$

b. $\int_a^b [f(t) + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt$

c. If $m \leq f(t) \leq M$ then,

$$m(b-a) \leq \int_a^b f(t) \leq M(b-a)$$

d. $|f|$ is integrable on $[a, b]$, then

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$$

Proof
Given any partition P of $[a, b]$, note that

$$\sum_{i=1}^{n} \Delta t_i = b - a$$

Since $m \leq f(t) \leq M$ we get that

$$\sum_{i=1}^{n} m\Delta t_i \leq \sum_{i=1}^{n} f(t)\Delta t_i \leq \sum_{i=1}^{n} M\Delta t_i$$
$$m\sum_{i=1}^{n} \Delta t_i \leq \sum_{i=1}^{n} f(t)\Delta t_i \leq M\sum_{i=1}^{n} \Delta t_i$$

This is true for every partition P and so we end up with

$$m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$$

Corollary 1.2.1.1
Properties of Integration

e. Set $m = 0$ in (c.) to get, if $f(t) \geq 0$ then

$$\int_a^b f(t)dt \geq 0$$

f. If $f(t) \geq g(t)$ then use (e.), (b.) and (a.) to get

$$\int_a^b f(t)dt \geq \int_a^b g(t)dt$$

it can be proved by making a new function $h(t) = f(t) - g(t)$ (hint...)

g. We define $\int_a^b f(t)dt = 0$. Our integration interval would be $[a,a]$ and so any Riemann sum we create would be of the form

$$\sum_{i=1}^{n} f(a)\Delta t_i = f(a) \cdot 0 = 0$$

f. For $a < b$ we have

$$\int_a^b f(t)dt = -\int_b^a f(t)dt$$

The idea here is that writing $\int_a^b$ suggests moving from a to b where t_{1-i} and t_i are points on line a to b and $\Delta t_i > 0$.
Whereas $\int_a^b$ suggests moving from b to a where $\Delta t_i < 0$.

Theorem 1.2.2
Given $a, b, c \in I$ over which f is integrable then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Note: It is not required that $a < c < b$. If $a < b < c$, we get

$$\begin{aligned}\underbrace{\int_a^b f(t)dt}_{\text{Area b/w } [a,b]} &= \int_a^c f(t)dt + \int_c^b f(t)dt \\ &= \underbrace{\int_a^c f(t)dt}_{\text{Area b/w } [a,c]} - \underbrace{\int_b^c f(t)dt}_{\text{Area b/w } [b,c]}\end{aligned}$$

1.3 Areas and Integrals

Note that $\int_a^b f(t)dt$ will only return the "expected" are when $f \geq 0$. Generally $\int_a^b f(t)dt$ returns the "signed" area.
That is if $f < 0$ over some interval $[c,d]$ then $\int_c^d f(t)dt$ will return the negative area between f and the x-axis.

Let the interval from $[a,b]$ have 2-regular partitions, c and d, where $a < c < d < b$. Assume $A_1, A_2, A_3 > 0$ (i.e., the normal area), where A_1 be the positive area from $[a,c]$, A_2 be the positive area from $[d,b]$ and A_3 be the negative area from $[c,d]$. then

$$\int_a^b f(t)dt = A_1 + A_2 - A_3$$

Example: $\int_0^{2\pi} \sin(t)dt = 0$, since A_1 and A_2 have equal areas, where A_1 is the positive area from $[0,\pi]$ and A_2 is the negative area from $[\pi, 2\pi]$.

1.4 Average Value

Recall the average of a discrete set $\{x_1, x_2, \ldots, x_n\}$ is given by

$$\frac{\sum_{i=1}^{n} x_i}{n}$$

We can define a similar concept for functions as follows:

Definition 1.4.1 (Average Value)
The average value of a continuous function over the interval $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(t)dt$$

sometimes written as $\bar{f}$.

Example: Geometrically compute f_{ave} over $[0, 4]$ if $f(x) = 3x$.
Geometrically, we have

$$\int_0^4 3xdx = \frac{\text{base} \times \text{height}}{2} = \frac{4 \cdot 12}{2} = 24$$

so that $f_{ave} = \bar{f} = \frac{1}{4}\int_0^4 3xdx = \frac{24}{4} = 6$.

In this case, f_{ave} occurs halfway between 0 and 12.

However, in all cases, f_{ave} will split $f(x)$ into 2 parts of equal areas. That is the area above f_{ave} and below $f(x)$ will equal the area of both $f(x)$ and below f_{ave}.
This can be proven by shifting the x-axis to instead be f_{ave}. Let $g(x) = f(x) - f_{ave}$.
If the area above $y = f_{ave}$ is equal to the below f_{ave} then we should get $\int_a^b g(x)d(x) = 0$.
Indeed,

$$\begin{aligned}\int_a^b g(x)dx &= \int_a^b f(x) - f_{ave}dx \\ &= \int_a^b f(x) - \int_a^b f_{ave}dx \qquad [\text{Recall} f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx] \\ &= (b-a)f_{ave} - (b-a)f_{ave} \\ &= 0\end{aligned}$$

It is always the case that for an integrable function, there is a $c \in [a, b]$ such that $f(c) = f_{ave}$? In general "no".

Theorem 1.4.1 (Average Value Theorem (AVT))
If f is a continuous function on $[a, b]$ then there is a $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Proof
By Extreme Value Theorem (EVT), there are p (min) and q (max), where $p, q \in [a, b]$ such that

$$f(p) \leq f(x) \leq f(q)$$

By integral properties,

$$(b-a)f(p) \leq \int_a^b f(x)dx \leq (b-a)f(q)$$

$$f(p) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(q)$$

By the Intermediate value theorem, there is a $c \in [a,b]$ where $f(c)$ is the above equation.

1.5 Fundamental Theorem of Calculus (FTC)

Up until now to compute $\int_a^b f(t)dt$ we had to rely on geometry or, if using the definition, we need formulas to convert $\sum_{i=1}^n f(c_i)\Delta t_i$ to an explicit expression like

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

In many cases, this is not possible.

Let us investigate how an integral behaves on a function on x. That is,
Let $A(x) = \int_a^x f(t)dt$ for a continuous function f. Now consider $x+h$

$$A(x+h) = \int_a^{x+h} f(t)dt$$

The incremental area is given by

$$A(x+h) - A(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$$

Divide both sides by h to get

$$\frac{A(x+h)-A(x)}{h} = \frac{1}{h}\int_a^{x+h} f(t)dt$$

since f is continuous, by AVT, $\exists c \in [x, x+h]$ such that

$$\frac{A(x+h)-A(x)}{h} = f(c)$$

Also since f is continuous and c depends on h,

$$\lim_{h\to 0} f(c) = f(x)$$

Finally by definition

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h} = A'(x)$$

Thus $A'(x) = f(x)$, i.e.,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Theorem 1.5.1 (Fundamental Theorem of Calculus I (FTC I))
If f is continuous on an open interval containing x and a, then

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

That is, the derivative **cancels** the integral.

How does this help us compute $\int_a^b f(t)dt$? If we let $g(x) = \int_a^x f(t)dt$ then by FTC I, we know that $g'(x) = f(x)$ so we begin a search for a function $g(x)$ which after we take a derivative gives $f(x)$.

Example: Compute

$$\int_3^5 2tdt$$

We need a function $g(x)$ such that $g'(x) = 2x$. That is we seek an antiderivative of $2x$ (Recall from MATH 137 that given a function, any 2 antiderivatives of that function can differ by at most a constant.)
Let us denote $G(x) = x^2 + c$ as the family of antiderivatives of $2x$. By FTC I we have that

$$G(x) = x^2 + c = \int_3^x 2tdt$$

since G'(x) = 2x

But what is c? We know that $G(3) = 9 + c = \int_3^3 2tdt = 0$, so $c = -9$. Thus

$$G(5) = \int_3^5 2tdt = 25 - 9 = 16$$

Notice that if we instead let $g(x) = x^2 + 4$ and evaluated $g(5) - g(3)$ we should still get 16.

This leads us to:

Theorem 1.5.2 (Fundamental Theorem of Calculus II (FTC II))
Let F be any antiderivative of a continuous function f. Then,

$$\int_a^b f(t)dt = F(b) - F(a)$$

Example: Compute

$$\int_1^4 \cos(x)dx$$

Using a Riemann sum would require a formula for

$$\sum_{i=1}^{n} \cos\left(1 + \frac{3i}{n}\right)$$

and then a limit as $n \to \infty$

Using FTC however we know that since $\frac{d}{dx}\sin(x) = \cos(x)$, we get

$$\int_1^4 \cos(x)dx = \sin(4) - \sin(1)$$

Notation: The expression $g(x)\Big|_a^b = g(b) - g(a)$

Example:

$$\begin{aligned}
\int_{-1}^{1}|x|dx &= \int_{-1}^{0}(-x)dx + \int_{0}^{1}(x)dx \\
&= -\int_{-1}^{0}(x)dx + \int_{0}^{1}(x)dx \\
&= \frac{-x^2}{2}\bigg|_{-1}^{0} + \frac{x^2}{2}\bigg|_{0}^{1} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

1.5.1 Extended Fundamental Theorem of Calculus I

We learned that

$$\frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)$$

What about $\frac{d}{dx}\int_{x}^{a} f(t)dt$ or $\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt$? we solve these using the **chain rule**.

Example: Compute

$$\frac{d}{dx}\int_{2}^{\ln(x)} \sin\left(t^2\right)dt$$

Let $u = \ln(x)$ and let $g(u) = \int_{2}^{u}\sin\left(t^2\right)dt$.
We seek $\frac{d}{dx}[g(u)]$, by chain rule $\frac{d}{dx}[g(u)] = \frac{d}{du}[g(u)] \cdot \frac{du}{dx}$. [We can end up with $\frac{d}{dx}$ by multiplying various derivatives, i.e. $\frac{du}{dt} \cdot \frac{dt}{dp} \cdot \frac{dp}{dx}$]

Now, $\frac{du}{dx} = \frac{1}{x}$ and by FTC,

$$\frac{d}{du}[g(u)] = \frac{d}{du}\int_{2}^{u}\sin\left(t^2\right)dt = \sin\left(u^2\right)$$

so we get

$$\frac{d}{dx}[g(u)] = \sin\left(u^2\right)\left(\frac{1}{x}\right)$$

i.e.

$$\frac{d}{dx}\int_{2}^{\ln(x)}\sin\left(t^2\right)dt = \frac{\sin\left([\ln x]^2\right)}{x}$$

General rule: For a continuous f and differentiable at $a(x)$ and $b(x)$, we get

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)$$

Tip: Even if you can't find an explicit F such that $F' = f$ you can "imagine" one exists.
Then by FTC II,

$$\begin{aligned}
\int_{a(x)}^{b(x)} f(t)dt &= F(b(x)) - F(a(x)) \\
\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt &= F'(b(x)) \cdot b'(x) - F'(a(x)) \cdot a(x) \\
&= f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)
\end{aligned}$$

1.6 Indefinite Integrals

So far we have defined $\int_a^b f(t)dt$ as the definite integral of f over $[a,b]$. It evaluates to a number. We call $\int f(x)dx$ the indefinite integral of $f(x)$. It is a function.

Due to FTC, $\int f(x)dx$ represents the family of antiderivatives of $f(x)$. eg: $\int 4x+3dx = 2x^2+3x+C$

1.6.1 Basic antiderivatives

To help calculate $\int_a^b f(t)dt$, it will be handy to know the following

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln(a)} + C, \qquad a > 0$ and $a \neq 1$
- $\int \sin(x)dx = -\cos(x) + C$
- $\int \cos(x)dx = \sin(x) + C$
- $\int \sec^2(x)dx = \tan(x) + C$
- $\int \frac{1}{1+x^2} dx = \arctan(x) + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$

Example: Find

$$\int 2x + \sin(x) - 1dx$$

By linearity of integration we get

$$\int 2xdx + \int \sin(x)dx - \int 1dx = x^2 - \cos(x) - x + C$$

Generally, finding antiderivatives is quite hard and sometimes impossible,
eg: e^{x^2}, $\sqrt{1+x^3}$, $\ln(\ln(x))$, $\sin(x^2)$.

When they do exists, finding them is much more of an art form compared to finding derivatives. We will introduce a few techniques that will help.

1.7 Substitution Rule

First note that by FTC,

$$\int g'(x)dx = \int \frac{d}{dx}[g(x)]dx = g(x) + C$$

Sometimes you will see this written as

$$\int d[g(x)] = g(x) + C$$

Consider $\int \sin^3 x \cdot \cos x dx$. Notice that $\cos x$ is the derivative of $\sin(x)$. In fact if we let $u = \sin(x)$, then $\frac{du}{dx} = \cos(x)$, and our integrals becomes

$$\int \sin^3 x \cdot \cos x dx = \int u^3 \frac{du}{dx} dx$$

Note that the derivative is NOT on the full integrand.

Recall however that from the chain rule

$$\frac{d}{dx}[u(x)^4] = 4u^3 \cdot \frac{du}{dx}$$

This suggests that

$$u^3 \cdot \frac{du}{dx} = \frac{d}{dx}\left[\frac{u(x)^4}{4}\right]$$

so our integral becomes,

$$\begin{aligned}\int u^3 \cdot \frac{du}{dx} dx &= \int \frac{d}{dx}\left[\frac{u(x)^4}{4}\right] dx \\ &= \frac{u(x)^4}{4} + C \qquad\qquad \text{(by FTC)}\end{aligned}$$

Notice that if we take the above integral and "cancel" the dx. We get $\int u^3 du$ which also yields $\frac{u^4}{4} + C$.

When we identify $\int f(g(x)) \cdot g'(x) dx$, we can let $u = g(x)$ to get

$$\int f(u) \cdot \frac{du}{dx} dx = \int f(u) du$$

In practice this lets us manipulate du and dx.

For example in $\int \sin^3 x \cdot \cos x dx$, let $u = \sin x$ so that $\frac{du}{dx} = \cos x$ or $dx = \frac{du}{\cos x}$ which gives

$$\int u^3 \cdot \cos x \frac{du}{\cos x} = \int u^3 du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

check that the derivative of $\frac{\sin^4 x}{4} + C$ yields to $\sin^3 x \cdot \cos x$. **Example:**

$$\int 5xe^{x^2} dx$$

let $u = x^2$, then $du = 2xdx \implies dx = \frac{du}{2x}$ which gives

$$\begin{aligned}\int 5xe^u \frac{du}{2x} &= \int \frac{5}{2} e^u du \\ &= \frac{5}{2} e^u + C \\ &= \frac{5}{2} e^{x^2} + C\end{aligned}$$

1.7.1 How to choose u?

There is no set rule for this. Often times you want to look for a function and its derivative appearing in the integrand.
eg:

- $\sin^3 x \cdot \cos x \to u = \sin x,\ u' = \cos x$
- $5xe^{x^2} \to u = x^2,\ u' = 2x$
- $\frac{\ln x}{x} \to u = \ln x,\ u' = \frac{1}{x}$

but this will not always work.

Example:

$$\int x^3\sqrt{x^2-4}dx$$

It seems like since $\frac{d}{dx}[x^3] = 3x^2$ we should let $u = x^3$. If we try we end up with $du = 3x^2dx$.

$$\begin{aligned}\int x^3\sqrt{x^2-4}dx = \int u\sqrt{x^2-4}dx &= \int u\sqrt{x^2-4}\frac{du}{3x^2}\\ &= \frac{1}{3}\int \frac{u}{x^2}\sqrt{x^2-4}du\end{aligned}$$

Now since $u = x^3$, $u^{\frac{2}{3}} = x^2$, which gives

$$\frac{1}{3}\int \frac{u}{u^{\frac{2}{3}}}\sqrt{u^{\frac{2}{3}}-4}du = \frac{1}{3}\int \frac{u}{u^{\frac{1}{3}}}\sqrt{u^{\frac{2}{3}}-4}du\ldots$$

Ideally we would end up at one of our basic antiderivative formulas like $\int x^n dx$ or $\int \sin(x)dx$, so we should try something else.
There will be lots of **trial and error**.

1.7.2 Trigonometric Substitutions

Recall that

$$\begin{aligned}1 - \sin^2\theta &= \cos^2\theta\\ \tan^2\theta + 1 &= \sec^2\theta\\ \sec^2\theta - 1 &= \tan^2\theta\end{aligned}$$

When an integrand involves an expression such as $a^2 - x^2, x^2 + a^2$ or $x^2 - a^2$ then, based on the formulas above we sometimes make the following substitutions.

$$\begin{aligned}&\text{for } a^2 - x^2 \qquad \text{try } x = a\sin\theta, \quad \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\\ &\text{for } a^2 + x^2 \qquad \text{try } x = a\tan\theta, \quad \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\\ &\text{for } x^2 - a^2 \qquad \text{try } x = a\sec\theta, \quad \theta \in \left[-\pi, \frac{-\pi}{2}\right) \cup \left[0, \frac{\pi}{2}\right)\end{aligned}$$

Usually we only need to do this for things like $(a^2 - x^2)^p$, $(a^2 + x^2)^p$ and $(x^2 - a^2)^p$ when p is a fraction.

Example: Find

$$\int \frac{x}{\sqrt{9+x^2}}dx \qquad \text{(we could do } u = 9 + x^2 \text{ but let's say we didn't notice)}$$

Let $x = 3\tan\theta$ with $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$dx = 3\sec^2\theta d\theta$ so we get,

$$\begin{aligned}
\int \frac{x}{\sqrt{9+x^2}}dx &= \int \frac{3\tan\theta}{\sqrt{9+9\tan^2\theta}} \cdot 3\sec^2\theta d\theta \\
&= 3\int \frac{\tan\theta\sec^2\theta}{\sqrt{1+\tan^2\theta}}d\theta \\
&= 3\int \frac{\tan\theta\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta \\
&= 3\int \frac{\tan\theta\sec^2\theta}{|\sec\theta|}d\theta \\
&= 3\int \frac{\tan\theta\sec^2\theta}{\sec\theta}d\theta \qquad \text{since } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\
&= 3\int \sec\theta\tan\theta d\theta \\
&= 3\sec\theta + C
\end{aligned}$$

1.7.3 What is $\sec\theta$ in terms of x?

Since

$$x = 3\tan\theta \implies \frac{x}{3} = \tan\theta$$

so,

$$\sec\theta = \frac{\sqrt{x^2+9}}{3}$$

We get a final answer of

$$\begin{aligned}
&= 3\left(\frac{\sqrt{x^2+9}}{3}\right) + C \\
&= \sqrt{x^2+9} + C
\end{aligned}$$

Example: Find

$$\int \frac{x^2}{\sqrt{1-x^2}}dx$$

Let $x = \sin\theta$