MATH 138: Honours Calculus 2
Theorems

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Contents

I. Intgeration

1 Riemann Sums and Definite Integral

1.1 Theorem. (Integrability Theorem for Continuous Functions) *Let f be continuous on $[a,b]$. Then f is integrable on $[a,b]$. Moreover,*

$$\int_a^b f(t)dt = \lim_{n\to\infty} S_n$$

where,

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

is any Riemann sum associated with the regular n-partitions. In particular,

$$\int_a^b f(t)dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

and

$$\int_a^b f(t)dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

2 Properties of the Definite Integral

2.1 Theorem. (Properties of Integrals Theorem) *Assume that f and g are integrable on the interval $[a,b]$. Then:*

1. *For any $c \in \mathbb{R}$, $\int_a^b cf(t)dt = c\int_a^b f(t)dt$*
2. $\int_a^b (f+g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$
3. *If $m \leq f(t) \leq M$ for all $t \in [a,b]$, then $m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$*
4. *If $0 \leq f(t)$ for all $t \in [a,b]$, then $0 \leq \int_a^b f(t)dt$*
5. *If $g(t) \leq f(t)$ for all $t \in [a,b]$, then $\int_a^b g(t)dt \leq \int_a^b f(t)dt$*
6. *The function $|f|$ is integrable on $[a,b]$ and $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$*

2.2 Theorem. (Integrals over Subintervals Theorem) *Assume that f is integrable on an interval I containing a, b and c. Then*

$$\int_a^a f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

3 The Average Value of a Function

3.1 Theorem. (Average Value Theorem (Mean Value Theorem for Integrals)) *Assume that f is continuous on $[a.b]$. Then there exists $a \le c \le b$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$

4 The Fundamental Theorem of Calculus

4.1 Theorem. (Fundamental Theorem of Calculus (Part 1)) *Assume that f is continuous on an open interval I containing a point a. Let*

$$G(x) = \int_a^x f(t)dt$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$G'(x) = f(x)$$

Equivalently,

$$G'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

4.2 Theorem. (Extended Version of the Fundamental Theorem of Calculus) *Assume that f is continuous and that g and h are differentiable. Let,*

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(hx))g'(x)$$

4.3 Theorem. (Power Rule for Antiderivatives) *If $\alpha \neq -1$, then*

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

4.4 Theorem. (Fundamental Theorem of Calculus (Part 2)) *Assume that f is continuous and that F is any antiderivative of f, then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

5 Change of Variables

5.1 Theorem. (Change of Variables Theorem) *Assume that $g'(x)$ is continuous on $[a,b]$ and $f(u)$ is continuous on $g([a,b])$, then*

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

II. Techniques of Integration

6 Partial Fractions

6.1 Theorem. (Integration by Parts Theorem) *Assume that f and g are such that both f' and g' are continuous on an interval containing a and b. Then*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

6.2 Theorem. (Integration of Partial Fractions) *Assume that $f(x) = \frac{p(x)}{q(x)}$ admits a Type I Partial Fraction Decomposition of the form*

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_k}{x-a_k}\right]$$

Then

$$\begin{aligned}\int f(x)dx &= \frac{1}{a}\left[\int \frac{A_1}{x-a_1}dx + \int \frac{A_2}{x-a_2}dx + \cdots + \int \frac{A_k}{x-a_k}dx\right]\\ &= \frac{1}{a}[A_1\ln(|x-a_1|) + A_2\ln(|x-a_2|) + \cdots + A_k\ln(|x-a_k|)] + C\end{aligned}$$

7 Improper Integrals

7.1 Theorem. (p-Test for Type I Improper Integrals) *The improper integral*

$$\int_1^\infty \frac{1}{x^p}dx$$

converges if and only if $p > 1$. If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$$

7.2 Theorem. (Properties of Type I Improper Integrals) *Assume that $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge*

1. *$\int_a^\infty cf(x)dx$ converges for each $c \in \mathbb{R}$ and*

$$\int_a^\infty cf(x)dx = c\int_a^\infty f(x)dx$$

2. *$\int_a^\infty (f(x)+g(x))dx$ converges and*

$$\int_a^\infty (f(x)+g(x))dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$$

3. *If* $f(x) \leq g(x)$ *for all* $a \leq x$*, then*

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

4. *If* $a < c < \infty$*, then* $\int_c^\infty f(x)dx$ *converges and*

$$\int_a^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx$$

7.3 Theorem. (The Monotone Convergence Theorem for Functions) *Assume that* f *is non-decreasing on* $[a,\infty)$.

1. *If* $\{f(x)|x \in [a,\infty)\}$ *is bounded above, then* $\lim_{x\to\infty} f(x)$ *exists and*

$$\lim_{x\to\infty} f(x) = L = lub(\{f(x)|x \in [a,\infty)\})$$

2. *If* $\{f(x)|x \in [a,\infty)\}$ *is not bounded above, then* $\lim_{x\to\infty} f(x) = \infty$

7.4 Theorem. (Comparison Test for Type I Improper Integrals) *Assume that* $0 \leq g(x) \leq f(x)$ *forall* $x \geq a$ *and that both* f *and* g *are continuous on* $[a,\infty)$.

1. *If* $\int_a^\infty f(x)dx$ *converges, then so does* $\int_a^\infty g(x)dx$
2. *If* $\int_a^\infty g(x)dx$ *diverges, then so does* $\int_a^\infty f(x)dx$

7.5 Theorem. (Absolute Convergence Theorem for Improper Integrals) *Let* f *be integrable on* $[a,b]$ *for all* $b > a$*. Then* $|f|$ *is also integrable on* $[a,b]$ *for all* $b > a$*. Moreover, if we assume that*

$$\int_a^\infty |f(x)|dx$$

converges, then so does

$$\int_a^\infty f(x)dx$$

In particular, if $0 \leq |f(x)| \leq g(x)$ *for all* $x \geq a$*, both* f *and* g *are integrable on* $[a,b]$ *for all* $b \geq a$*, and if* $\int_a^\infty g(x)dx$ *converges, then so does*

$$\int_a^\infty f(x)dx$$

7.6 Theorem. (p-Test for Type II Improper Integrals) *The improper integral*

$$\int_0^1 \frac{1}{x^p}dx$$

converges if and only if $p < 1$.
If $p < 1$*, then*

$$\int_0^1 \frac{1}{x^p}dx = \frac{1}{1-p}$$

III. Applications of Integration

8 Area Between Curves

8.1 Theorem. (Area Between Curves) *Let f and g be continuous on $[a,b]$. Let A be the region bounded by the graphs of f and g, the line $t = a$ and the line $t = b$. Then the area of region A is given by*

$$A = \int_a^b |g(t) - f(t)| dt$$

9 Volumes of Revolution: Disk Method

9.1 Theorem. (Volumes of Revolution: Disk Method I) *Let f be continuous on $[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$. Let W be the region bounded by the graphs of f, the x-axis and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by*

$$V = \int_a^b \pi f(x)^2 dx$$

9.2 Theorem. (Volumes of Revolution: Disk Method II) *Let f and g be continuous on $[a,b]$ with $0 \leq f(x) \leq g(x)$ for all $x \in [a,b]$. Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by*

$$V = \int_a^b \pi (g(x)^2 - f(x)^2) dx$$

9.3 Theorem. (Volumes of Revolution: The Shell Method) *Let $a \geq 0$. Let f and g be continuous on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume$V$ of the solid of revolution obtained by rotating the region W around the y-axis is given by*

$$V = \int_a^b 2\pi x (g(x) - f(x)) dx$$

10 Arc Length

10.1 Theorem. (Arc Length) *Let f be continuously differentiable on $[a,b]$. Then the arc length S of the graph of f over the interval $[a,b]$ is given by*

$$S = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

IV. Differential Equations

11 First-Order Linear Differential Equations

11.1 Theorem. (Solving First-order Linear Differential Equations) *Let f and g be continuous and let*

$$y' = f(x)y + g(x)$$

be a first-order linear differential equation. Then the solutions to this equation are of the form

$$y = \frac{\int g(x)I(x)dx}{I(x)}$$

where $I(x) = e^{-\int f(x)dx}$

12 Initial Value Problems

12.1 Theorem. (Existence and Uniqueness Theorem for FOLDE) *Assume that f and g are continuous functions on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem*

$$\begin{aligned} y' &= f(x)y + g(x) \\ y(x_0) &= y_0 \end{aligned}$$

has exactly one solution $y = \phi(x)$ on the interval I.