MATH 138
Theorems

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Contents

1 Intgeration

1.1 Riemann Sums and Definite Integral

Theorem 1.1.1 (Integrability Theorem for Continuous Functions)
Let f be continuous on $[a,b]$. Then f is integrable on $[a,b]$. Moreover,

$$\int_a^b f(t)dt = \lim_{n\to\infty} S_n$$

where,

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

is any Riemann sum associated with the regular n-partitions. In particular,

$$\int_a^b f(t)dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

and

$$\int_a^b f(t)dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

1.2 Properties of the Definite Integral

Theorem 1.2.1 (Properties of Integrals Theorem)
Assume that f and g are integrable on the interval $[a,b]$. Then:

1. For any $c \in \mathbb{R}$, $\int_a^b cf(t)dt = c\int_a^b f(t)dt$
2. $\int_a^b (f+g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$
3. If $m \leq f(t) \leq M$ for all $t \in [a,b]$, then $m(b-a) \leq \int_a^b f(t)dt \leq M(b-a)$
4. If $0 \leq f(t)$ for all $t \in [a,b]$, then $0 \leq \int_a^b f(t)dt$
5. If $g(t) \leq f(t)$ for all $t \in [a,b]$, then $\int_a^b g(t)dt \leq \int_a^b f(t)dt$
6. The function $|f|$ is integrable on $[a,b]$ and $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

Theorem 1.2.2 (Integrals over Subintervals Theorem)
Assume that f is integrable on an interval I containing a, b and c. Then

$$\int_a^a f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

1.3 The Average Value of a Function

Theorem 1.3.1 (Average Value Theorem (Mean Value Theorem for Integrals))
Assume that f is continuous on $[a, b]$. Then there exists $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

1.4 The Fundamental Theorem of Calculus

Theorem 1.4.1 (Fundamental Theorem of Calculus (Part 1))
Assume that f is continuous on an open interval I containing a point a. Let

$$G(x) = \int_a^x f(t)dt$$

Then $G(x)$ is differentiable at each $x \in I$ and

$$G'(x) = f(x)$$

Equivalently,

$$G'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Theorem 1.4.2 (Extended Version of the Fundamental Theorem of Calculus)
Assume that f is continuous and that g and h are differentiable. Let,

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(hx))g'(x)$$

Theorem 1.4.3 (Power Rule for Antiderivatives)
If $\alpha \neq -1$, then

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

Theorem 1.4.4 (Fundamental Theorem of Calculus (Part 2))
Assume that f is continuous and that F is any antiderivative of f, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

1.5 Change of Variables

Theorem 1.5.1 (Change of Variables Theorem)
Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$, then

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

2 Techniques of Integration

2.1 Partial Fractions

Theorem 2.1.1 (Integration by Parts Theorem)
Assume that f and g are such that both f' and g' are continuous on an interval containing a and b. Then

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

Theorem 2.1.2 (Integration of Partial Fractions)
Assume that $f(x) = \frac{p(x)}{q(x)}$ admits a Type I Partial Fraction Decomposition of the form

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_k}{x-a_k}\right]$$

Then

$$\begin{aligned}\int f(x)dx &= \frac{1}{a}\left[\int \frac{A_1}{x-a_1}dx + \int \frac{A_2}{x-a_2}dx + \cdots + \int \frac{A_k}{x-a_k}dx\right] \\ &= \frac{1}{a}[A_1 \ln(|x-a_1|) + A_2 \ln(|x-a_2|) + \cdots + A_k \ln(|x-a_k|)] + C\end{aligned}$$

2.2 Improper Integrals

Theorem 2.2.1 (p-Test for Type I Improper Integrals)
The improper integral

$$\int_1^\infty \frac{1}{x^p}dx$$

converges if and only if $p > 1$. If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$$

Theorem 2.2.2 (Properties of Type I Improper Integrals)
Assume that $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge

1. $\int_a^\infty cf(x)dx$ converges for each $c \in \mathbb{R}$ and
$$\int_a^\infty cf(x)dx = c\int_a^\infty f(x)dx$$
2. $\int_a^\infty (f(x)+g(x))dx$ converges and
$$\int_a^\infty (f(x)+g(x))dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$$
3. If $f(x) \leq g(x)$ for all $a \leq x$, then
$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$
4. If $a < c < \infty$, then $\int_c^\infty f(x)dx$ converges and
$$\int_a^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx$$

Theorem 2.2.3 (The Monotone Convergence Theorem for Functions)
Assume that f is non-decreasing on $[a, \infty]$.

1. If $\{f(x)|x \in [a,\infty]\}$ is bounded above, then $\lim_{x\to\infty} f(x)$ exists and
$$\lim_{x\to\infty} f(x) = L = lub(\{f(x)|x \in [a,\infty)\})$$
2. If $\{f(x)|x \in [a,\infty]\}$ is not bounded above, then $\lim_{x\to\infty} f(x) = \infty$

Theorem 2.2.4 (Comparison Test for Type I Improper Integrals)
Assume that $0 \leq g(x) \leq f(x)$ forall $x \geq a$ and that both f and g are continuous on $[a, \infty)$.

1. If $\int_a^\infty f(x)dx$ converges, then so does $\int_a^\infty g(x)dx$
2. If $\int_a^\infty g(x)dx$ diverges, then so does $\int_a^\infty f(x)dx$

Theorem 2.2.5 (Absolute Convergence Theorem for Improper Integrals)
Let f be integrable on $[a, b]$ for all $b > a$. Then $|f|$ is also integrable on $[a, b]$ for all $b > a$. Moreover, if we assume that

$$\int_a^\infty |f(x)|dx$$

converges, then so does

$$\int_a^\infty f(x)dx$$

In particular, if $0 \leq |f(x)| \leq g(x)$ for all $x \geq a$, both f and g are integrable on $[a, b]$ for all $b \geq a$, and if $\int_a^\infty g(x)dx$ converges, then so does

$$\int_a^\infty f(x)dx$$

Theorem 2.2.6 (p-Test for Type II Improper Integrals)
The improper integral

$$\int_0^1 \frac{1}{x^p}dx$$

converges if and only if $p < 1$.
If $p < 1$, then

$$\int_0^1 \frac{1}{x^p}dx = \frac{1}{1-p}$$

3 Applications of Integration

3.1 Area Between Curves

Theorem 3.1.1 (Area Between Curves)
Let f and g be continuous on $[a, b]$. Let A be the region bounded by the graphs of f and g, the line $t = a$ and the line $t = b$. Then the area of region A is given by

$$A = \int_a^b |g(t) - f(t)|dt$$

3.2 Volumes of Revolution: Disk Method

Theorem 3.2.1 (Volumes of Revolution: Disk Method I)
Let f be continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of f, the x-axis and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi f(x)^2 dx$$

Theorem 3.2.2 (Volumes of Revolution: Disk Method II)
Let f and g be continuous on $[a, b]$ with $0 \leq f(x) \leq g(x)$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of f and g, and the lines $x = a$ and $x = b$. Then the volume V of the solid of revolution obtained by rotating the region W around the x-axis is given by

$$V = \int_a^b \pi(g(x)^2 - f(x)^2)dx$$

Theorem 3.2.3 (Volumes of Revolution: The Shell Method)
Let $a \geq 0$. Let f and g be continuous on $[a, b]$ with f$(x) \leq g(x)$ for all $x \in [a, b]$. Let W be the region bounded by the graphs of$f$ and g, and the lines $x = a$ and $x = b$. Then the volume$V$ of the solid of revolution obtained by rotating the region W around the y-axis is given by

$$V = \int_a^b 2\pi x(g(x) - f(x))dx$$

3.3 Arc Length

Theorem 3.3.1 (Arc Length)
Let f be continuously differentiable on $[a, b]$. Then the arc length S of the graph of f over the interval $[a, b]$ is given by

$$S = \int_a^b \sqrt{1 + (f'(x))^2}dx$$

4 Differential Equations

4.1 First-Order Linear Differential Equations

Theorem 4.1.1 (Solving First-order Linear Differential Equations)
Let f and g be continuous and let

$$y' = f(x)y + g(x)$$

be a first-order linear differential equation. Then the solutions to this equation are of the form

$$y = \frac{\int g(x)I(x)dx}{I(x)}$$

where $I(x) = e^{-\int f(x)dx}$

4.2 Initial Value Problems

Theorem 4.2.1 (Existence and Uniqueness Theorem for FOLDE)
Assume that f and g are continuous functions on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem

$$\begin{aligned} y' &= f(x)y + g(x) \\ y(x_0) &= y_0 \end{aligned}$$

has exactly one solution $y = \phi(x)$ on the interval I.

5 Numerical Series

6 Power Series