Proof Methods

Proving Universally Quantified Statements

To prove the universally quantified statement "$\forall x \in S, P(x)$":
Choose a representative mathematical object $x \in S$. This cannot be a specific object. It has to be a placeholder, that is, a variable, so that our argument would work for any specific member of the domain S.
Then, show that the open sentence P must be true for our representative x, using known facts about the elements of S.

Proof Methods

Disproving UQS (Proof by counter-example

To prove the universally quantified statement "$\forall x \in S, P(x)$":
Find an element $x \in S$ for which the open sentence $P(x)$ is false. This process is called finding a **counter-example**.

Proving Existentially Quantified Statements

To prove the existentially quantified statement "$\exists x \in S$", $P(x)$:
Provide an explicit value of x from the domain S, and show that $P(x)$ is true for this value of x. In other words, find an element of S that satisfies property of S.

Proof Methods

Disproving EQS

To disprove the existentially quantified statement "$\exists x \in S$", $P(x)$:
Prove that **universally** quantified statement "$\forall x \in S, \neg P(x)$".

Proof by Contradiction

Let A be a statement. Note that either A or $\neg A$ must be false, so the compound statement $A \wedge (\neg A)$ is always **false**. The statement "$A \wedge (\neg A)$ is true" is called a contradiction.

Proof Methods

Proving Implications

- To prove the implication "$A \implies B$", **assume** that the hypothesis A is true, and use this assumption to show that the conclusion B is true. The hypothesis A is what you start with. The conclusion B is where you must end up.
- To prove the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$": Let x be an arbitrary element of S, assume that the hypothesis $P(x)$ is true, and use this assumption to show that the conclusion $Q(x)$ is true.

Proof Methods

Proof by Contrapositive

- To prove the implication "$A \implies B$", replace it with its contrapositive "$(\neg B) \implies (\neg A)$". Then prove this contrapositive, usually by a direct proof. That is, assume $\neg B$ is true and deduce that $\neg A$ must be true as well.
- To prove the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$", replace it with its universally quantified contrapositive "$\forall x \in S, (\neg Q(x)) \implies (\neg P(x))$". Then prove this universally quantified contrapositive.

Proof Methods

Proving Uniqueness

To prove the statement "There is a **unique** element $x \in S$ such that $P(x)$ is true":

- ("Existence"): Prove that there is at least one element $x \in S$ such that $P(x)$ is true (i.e., prove the existentially quantified statement "$\exists x \in S, P(x)$").
- ("Uniqueness"): Do either of the below.
  - Assume that $P(x)$ and $P(y)$ are true for $x, y \in S$, , and prove that this assumption leads to the conclusion $x = y$,
  - Assume that $P(x)$ and $P(y)$ are true for distinct $x, y \in S$ (so $x \neq y$), and prove that and prove that this assumption leads to a contradiction.

Proof Methods

Proving If and Only If Statements

For proving an if and only if statement:

- To prove the statement $A \longleftrightarrow B$, it is equivalent to prove both the implication "$A \implies B$" and its converse "$B \implies A$".
- To prove the universally quantified statement "$\forall x \in S, P(x) \longleftrightarrow Q(x)$", it is equivalent to do either of the below.
  - Let x be an arbitrary element of S, and prove both the implication "$P(x) \implies Q(x)$" and its converse "$Q(x) \implies P(x)$".
  - Prove both the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$" and its universally quantified converse "$\forall x \in S, Q(x) \implies P(x)$".

Proof Methods

Proof by Mathematical Induction

To prove the universally quantified statement "For all $n \in \mathbb{N}, P(n)$":

- Prove "$P(1)$"
- Prove the universally quantified implication "For all $k \in \mathbb{N}$, if $P(k)$, then $P(k+1)$."

Proof by Strong Induction

To prove the universally quantified statement "For all integers $n \geq b, P(n)$":

- Prove "$P(b) \wedge P(b+1) \wedge \cdots \wedge P(B)$", for some integers $B \geq b$.
- Prove the universally quantified implication "For all $k \geq B$, if $P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)$, then $P(k+1)$."

Proof Methods

Prove $S \subseteq T$

To prove that $S \subseteq T$, prove the universally quantified implication:

$$\forall x \in \mathcal{U}, (x \in S) \implies (x \in T)$$