MATH 135
Proof Techniques

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1 Proving Mathematical Statements

1.1 Proving Universally Quantified Statements

1.1.1 Proof Technique 1

To prove the universally quantified statement "$\forall x \in \S, P(x)$":

Choose a representative mathematical object $x \in \S$. This cannot be a specific object. It has to be a placeholder, that is, a variable, so that our argument would work for any specific member of the domain S.

Then, show that the open sentence P must be true for our representative x, using known facts about the elements of S.

1.1.2 Proof Technique 2

To disprove the universally quantified statement "$\forall x \in \S, P(x)$":

Find an element $x \in S$ for which the open sentence $P(x)$ is false. This process is called finding a **counterexample**.

1.2 Proving Existentially Quantified Statements

1.2.1 Proof Technique 3

To prove the existentially quantified statement "$\exists x \in \S, P(x)$":

Provide an explicit value of x from the domain S, and show that $P(x)$ is true for this value of x. In other words, find an element of S that satisfies property P.

1.2.2 Proof Technique 4

To disprove the existentially quantified statement "$\exists x \in S, P(x)$":

Prove the universally quantified statement "$\forall x \in S, \neg P(x)$"

1.3 Proving Implications

1.3.1 Proof Technique 5

For proving an implication:

1. To prove the implication "$A \implies B$", **assume** that the hypothesis A is true, and use this assumption to show that the conclusion B is true. The hypothesis A is what you start with. The conclusion B is where you must end up.

2. To prove the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$":
   Let x be an arbitrary element of S, assume that the hypothesis $P(x)$ is true, and use this assumption to show that the conclusion $Q(x)$ is true.

1.4 Proof by Contrapositive

1.4.1 Proof Technique 6

For proving an implication using the contrapositive:

1. To prove the implication "$A \implies B$", replace it with its contrapositive "$(\neg B) \implies (\neg A)$". Then prove this contrapositive, usually by a direct proof. That is, assume $\neg B$ is true and deduce that $\neg A$ must be true as well.

2. To prove the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$", replace it with its universally quantified contrapositive "$\forall x \in S, (\neg Q(x)) \implies (\neg P(x))$". Then prove this universally quantified contrapositive.