# MATH 135
# Proof Techniques

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# Contents

# 1 Proving Mathematical Statements

## 1.1 Proving Universally Quantified Statements

**Proof Technique 1.1.1**
To prove the universally quantified statement "$\forall x \in S, P(x)$":

Choose a representative mathematical object $x \in S$. This cannot be a specific object. It has to be a placeholder, that is, a variable, so that our argument would work for any specific member of the domain $S$.
Then, show that the open sentence $P$ must be true for our representative $x$, using known facts about the elements of $S$.

**Proof Technique 1.1.2**
To disprove the universally quantified statement "$\forall x \in S, P(x)$":
Find an element $x \in S$ for which the open sentence $P(x)$ is false. This process is called finding a ***counter-example***.

## 1.2 Proving Existentially Quantified Statements

**Proof Technique 1.2.1**
To prove the existentially quantified statement "$\exists x \in S, P(x)$":
Provide an explicit value of $x$ from the domain $S$, and show that $P(x)$ is true for this value of $x$. In other words, find an element of $S$ that satisfies property $P$.

**Proof Technique 1.2.2**
To disprove the existentially quantified statement "$\exists x \in S, P(x)$":
Prove the universally quantified statement "$\forall x \in S, \neg P(x)$".

## 1.3 Proving Implications

**Proof Technique 1.3.1**
For proving an implication:

1. To prove the implication "$A \implies B$", **assume** that the hypothesis $A$ is true, and use this assumption to show that the conclusion $B$ is true. The hypothesis $A$ is what you start with. The conclusion $B$ is where you must end up.

2. To prove the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$":
Let $x$ be an arbitrary element of $S$, assume that the hypothesis $P(x)$ is true, and use this assumption to show that the conclusion $Q(x)$ is true.

## 1.4 Proof by Contrapositive

**Proof Technique 1.4.1**
For proving an implication using the contrapositive:

1. To prove the implication "$A \implies B$", replace it with its contrapositive "$(\neg B) \implies (\neg A)$". Then prove this contrapositive, usually by a direct proof. That is, assume $\neg B$ is true and deduce that $\neg A$ must be true as well.
2. To prove the universally quantified implication "$\forall x \in S, P(x) \implies Q(x)$", replace it with its universally quantified contrapositive "$\forall x \in S, (\neg Q(x)) \implies (\neg P(x))$". Then prove this universally quantified contrapositive.

## 1.5 Proof by Contradiction

**Proof Technique 1.5.1**
To prove $P(x)$ is true. Assume $\neg P(x)$ is true and come to a conclusion that $\neg P(x)$ is false, which proves that $P(x)$ is true.

### 1.5.1 Proving Uniqueness

**Proof Technique 1.5.2**
To prove the statement "There is a unique element $x \in S$ such that $P(x)$ is true":

1. ("Existence"): Prove that there is at least one element $x \in S$ such that $P(x)$ is true (ie., prove the existentially quantified statement "$\exists\, x \in S,\ P(x)$").
2. ("Uniqueness"): Do either (a) or (b) below.
   (a) Assume that $P(x)$ and $P(y)$ are true for $x, y \in S$, and prove that this assumption leads to the conclusion $x = y$,
   (b) Assume that $P(x)$ and $P(y)$ are true for distinct $x, y \in S$ (so $x \neq y$), and prove that this assumption leads to a contradiction.

## 1.6 Proving If and Only if

**Proof Technique 1.6.1**
For proving an if and only if statement:

1. To prove the statement "$A \iff B$", it is equivalent to prove both the implication "$A \implies B$" and its converse "$B \implies A$".

2. To prove the universally quantified statement "$\forall\ x \in S,\ P(x) \iff Q(x)$", it is equivalent to do either (a) or (b) below.

   (a) Let $x$ be an arbitrary element of $S$, and prove both the implication "$P(x) \implies Q(x)$" and its converse "$Q(x) \implies P(x)$",

   (b) Prove both the universally quantified implication "$\forall\ x \in S,\ P(x) \implies Q(x)$" and its universally quantified converse "$\forall\ x \in S, Q(x) \implies P(x)$".

# 2 Mathematical Induction and Sets

## 2.1 Proof by Induction

**Proof Technique 2.1.1**
To prove the universally quantified statement "$\forall\ n \in \mathbb{N}, P(n)$":

1. Prove "$P(1)$".
2. Prove the universally quantified implication "$\forall\ k \in \mathbb{N},\ P(k) \implies P(k+1)$".

where, $P(1)$ is the *base case/step*, $P(k)$ is the *inductive hypothesis* and $P(k+1)$ is the *inductive conclusion* and together $P(k) \implies P(k+1)$ is the *inductive step*.

## 2.2 Proof by Strong Induction

**Proof Technique 2.2.1**
To prove the universally quantified statement "$\forall\ n \in \mathbb{N}, n \geq b, P(n)$":

1. Prove "$P(b) \wedge P(b+1) \wedge \cdots \wedge P(B)$", for some integer $B \geq b$.
2. Prove the universally quantified implication "$\forall\ k \in \mathbb{Z}, k \geq B,\ P(b) \wedge P(b+1) \wedge \cdots \wedge P(k) \implies P(k+1)$".

To implement the proof method of strong induction, what changes are needed in terms of the standard format?

- For the base case, prove all of $P(b)$, $P(b+1)$, $\ldots$, $P(B)$. When $b < B$ there is more than one case to prove, so we label them as Base cases, and refer to $b$ as the smallest base case and $B$ as the largest base case.
- For the inductive step, we assume that $k$ is an arbitrary integer where $k \geq B$. We assume the inductive hypothesis, $P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)$. That is, we assume $P(i)$, for all integers $i = b, b+1, \ldots, k$. We then prove $P(k+1)$ using the assumption $P(i)$ for all integers $i = b, b+1, \ldots, k$.

## 2.3 Proving subset

**Proof Technique 2.3.1**
To prove that $S \subseteq T$, prove the universally quantified implication:

$$\forall x \in \mathcal{U}, (x \in S) \implies (x \in T)$$

## 2.4 Proving equality

**Proof Technique 2.4.1**
To prove that $S = T$, prove $S \subseteq T$ and $T \subseteq S$ (ie., universally qunatified if and only if):

$$\forall x \in \mathcal{U}, (x \in S) \iff (x \in T)$$