MATH 136: Linear Algebra 1
Example Sheets

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UW Example Sheet 1

1. (a) Given that $\begin{bmatrix} a \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 8 \\ b \\ 12 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ c \end{bmatrix}$, determine the values of a, b, and c.

   (b) Let $\vec{v} = \begin{bmatrix} 2+i \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3+6i \\ -3-3i \end{bmatrix}$. Determine $\vec{u} \in \mathbb{C}^2$ such that $9\vec{v} - 3i\vec{u} = -i\vec{w}$.

2. Given two unit vectors in $\mathbb{R}^3$ that are orthogonal to both $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$. Prove that these are the only two unit vectors in $\mathbb{R}^3$ that are orthogonal to both $\vec{v}$ and $\vec{w}$.

3. Let $n \in \mathbb{N}$. Prove or disprove the following statements.

   (a) $\forall \vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$ and $\vec{w} \neq \vec{0}$, $\text{proj}_{\vec{w}}(\text{perp}_{\vec{v}}(\vec{w})) = \vec{0}$.

   (b) $\forall \vec{v}, \vec{w} \in \mathbb{R}^n$ with $\vec{w} \neq \vec{0}$, $\text{proj}_{\vec{w}}(\text{perp}_{\vec{w}}(\vec{v})) = \vec{0}$.

4. Let $\vec{v}, \vec{w} \in \mathbb{R}^3$. Prove that $||\vec{v} \times \vec{w}||^2 = ||\vec{v}||^2||\vec{w}||^2 - (\vec{v} \cdot \vec{w})^2$.

5. (a) Disprove the following statement,

$$\forall c \in \mathbb{C} \text{ and all } \vec{u}, \vec{v} \in \mathbb{C}^2, \langle \vec{u}, c\vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$$

   (b) Prove the Conjugate Linearity property,

$$\forall c \in \mathbb{C} \text{ and all } \vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n, \langle \vec{u}, \vec{v} + c\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + c\langle \vec{u}, \vec{w} \rangle$$

UW Example Sheet 2

1. Let $(-1, 2)$ and $(2, -3)$ be two points on a line $\mathcal{L}$ in $\mathbb{R}^2$.
   (a) Find a vector equation for $\mathcal{L}$
   (b) find a parametric equation for $\mathcal{L}$
   (c) Determine whether the point $(3, -4)$ is on $\mathcal{L}$.

2. Prove that $\text{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \in \mathbb{R}^3$
   (**Hint**: to prove two sets are equal, you must show that they are subsets of each other.)

3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line in $\mathbb{R}^3$ with respect to vector equations:
$$\vec{\ell}_1 = \begin{bmatrix}1\\2\\5\end{bmatrix} + t\begin{bmatrix}2\\4\\-3\end{bmatrix} \text{ and } \vec{\ell}_2 = \begin{bmatrix}1\\2\\4\end{bmatrix} + t\begin{bmatrix}2\\-3\\4\end{bmatrix}, t \in \mathbb{R}$$
   Prove that $\mathcal{L}_1$ and $\mathcal{L}_2$ do not intersect.

4. The points $A = (3, -2, 1)$ and $B = (-2, 4, 3)$ lie on a plane $\mathcal{P}$.
   (a) Prove that the point $C = (-7, 10, 5)$ must also lie on $\mathcal{P}$.
   (b) Given that the line with vector equation $\vec{\ell} = t\begin{bmatrix}\frac{5}{2}\\-3\\-1\end{bmatrix}, t \in \mathbb{R}$ also lies on $\mathcal{P}$, find a scalar equation of $\mathcal{P}$.

5. (a) Find a and b such that $\begin{bmatrix}2a\\4\end{bmatrix} \notin \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\b\end{bmatrix}\right\} \in \mathbb{R}^2, \forall a, b \in \mathbb{R}$
   (b) Determine the solution set, S, to the following system of linear equations.
$$\begin{aligned} x_1 + 3x_2 + 4x_4 &= 0 \\ x_1 + 3x_2 - x_3 + 3x_4 &= 0 \end{aligned}$$
   Express S as the span of one or more vectors.

UW Example Sheet 3

1. Consider the following systems of linear equations:

$$\begin{aligned} -3x_1 - 6x_2 - 3x_3 &= 3 \\ 2x_1 + 4x_2 + 3x_3 &= -4 \\ -4x_1 - 8x_2 - 3x_3 &= 2 \end{aligned}$$

   (a) Give the coefficient matrix and the augmented matrix of this system.

   (b) Determine the reduced row echelon form of the augmented matrix.

   (c) State the rank of the coefficient matrix and the rank of the augmented matrix. State the nullity of the coefficient matrix.

   (d) Determine the solution set for the system.

2. Solve the following system of linear equations.

$$\begin{aligned} iz_1 + (1+i)z_2 &= -5 - 3i \\ 2iz_1 + (3+2i)z_2 + 2iz_3 &= -15 \\ 3z_1 + (3-3i)z_2 + z_3 &= -6 + 16i \\ 3iz_1 + (4+3i)z_2 + 2iz_3 &= -20 - 3i \end{aligned}$$

3. A system of linear equations has the following augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 2a & -4a & 6a \\ 0 & 0 & b^2 - 25 & b+5 \end{array}\right]$$

   $\forall a, b \in \mathbb{R}$. Determine all values of a and b.

   (a) is inconsistent

   (b) has a unique solution

   (c) has infinitely many solutions.

4. Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. Assume that $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \mathbb{R}^n$. Prove that $n \leq k$, i.e., $\mathbb{R}^n$ cannot be spanned by fewer than n vectors.

(a) Let $\vec{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{bmatrix}, \ldots, \vec{v}_k = \begin{bmatrix} v_{k1} \\ v_{k2} \\ \vdots \\ v_{kn} \end{bmatrix}$, then

$$\begin{aligned} v_{11}x_1 + v_{21}x_2 + \cdots + v_{k1}x_k &= b_1 \\ v_{12}x_1 + v_{22}x_2 + \cdots + v_{k2}x_k &= b_2 \\ &\vdots \\ v_{1n}x_1 + v_{2n}x_2 + \cdots + v_{kn}x_k &= b_n \end{aligned}$$

Prove that the system of LE's is consistent, $\forall b_1, b_2, \ldots, b_n \in \mathbb{R}$.

(b) Using part (a), prove that $n \leq k$.

5. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^2$ be distinct vectors. Prove or disprove the following statements.

   (a) If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a spanning set for $\mathbb{R}^2$, then every vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ in a unique way.

   (b) If $\{\vec{v}_1, \vec{v}_2\}$ is a spanning set of $\mathbb{R}^2$, then every vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ in a unique way.

UW Example Sheet 4

1. Consider $A = \begin{bmatrix} 2 & -1 & 3 & s \\ 0 & 2 & -3 & 1 \\ -3 & 4 & 1 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ t \\ 0 \\ -2 \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} -7 \\ 0 \\ u \end{bmatrix}$

   (a) Given that $A\vec{b} = \vec{c}$, determine the value of s, t and u.

   (b) Use your answer from part (a) to write $\vec{c}$ as a linear combination of the columns of A.

2. Consider the system of linear systems

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &= 2 \\ 2x_1 - x_2 - 3x_3 + 4x_4 &= 6 \\ x_1 - 2x_3 + x_4 &= 4 \end{aligned}$$

   The solution set of this system of equations is,

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 4 \\ 0 \\ 2 \end{bmatrix} : \forall s, t \in \mathbb{F} \right\}$$

3. Prove that, $\forall n \in \mathbb{N}$

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2^{n+1} & 2^{n+1} \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2^n & 2^n \end{bmatrix}$$

   Hint: Use proof by induction

4. (a) Let $A \in M_{n\times n}(\mathbb{F})$ be such that $A^2 = \mathcal{O}_{n\times n}$. Prove that $\mathrm{Col}(A) \subseteq \mathrm{Null}(A)$

   (b) Let $Q \in M_{n\times n}(\mathbb{R})$ be such that $Q^T Q = I_n$, and let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Prove that $\vec{u}$ is orthogonal to $\vec{v}$ if and only if $Q\vec{u}$ is orthogonal to $Q\vec{v}$.
   Hint: Consider the product $\vec{u}^T\vec{v}$ of the $1 \times n$ and $n \times 1$ matrices $\vec{u}^T$ and $\vec{v}$.

5. Let $A \in M_{m\times n}$ and let $\vec{b} \in \mathbb{R}^m$. prove that consistent system of linear equations $A\vec{x} = \vec{b}$ has a unique solution if and only if $\mathrm{Null}(A) = \{\vec{0}\}$.

UW Example Sheet 5

1. (a) Determine the kernel of the linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ x_1 + 2x_2 \\ x_1 \end{bmatrix}$$
   (b) Is T_1 one-to-one?
   (c) Determine the range of the linear transformation $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ defined by,
$$T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$$
   (d) Is T_2 onto?

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by a projection onto the line with vector equation $\vec{\ell} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$ followed by a counter-clockwise rotation about the origin by an angle of $\dfrac{\pi}{6}$.
   (a) Determine the standard matrix of T
   (b) Use the standard matrix to find the image of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ under the transformation T.

3. Prove or Disprove the following statements
   (a) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$ is linear.
   (b) Let $A, B \in M_{n\times n}(\mathbb{F})$. If A and B are invertible, then AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$

4. Let $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$ (two fixed vectors). Let $T : \mathbb{R}^n \to \mathbb{R}^2$ be defined as $T(\vec{x}) = \begin{bmatrix} \vec{a} \cdot \vec{x} \\ \vec{b} \cdot \vec{x} \end{bmatrix}$

   (a) Prove that T is linear

   (b) Find all choices of $\vec{a}$ and $\vec{b}$ such that T is onto. (show proof)

   (c) Prove that if $n \geq 3$, then for all choices of $\vec{a}$ and $\vec{b}$, T is not one-to-one.

5. Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a transformation.We say that T is invertible if there exists a transformation $S : \mathbb{F}^m \to \mathbb{F}^n$ such that

$$\forall \vec{x} \in \mathbb{F}^n, (S \circ T)(\vec{x}) = \vec{x}$$

   and

$$\forall \vec{x} \in \mathbb{F}^m, (T \circ S)(\vec{x}) = \vec{x}$$

   (a) Prove that if S exists as defined above, then S is unique.

   (b) Assume now that T is a linear transformation. Prove that if S exists as defined above, then S must be a linear transformation.