

MATH 135

Honours Algebra

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Winter 2023[†]

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[†]Last updated: January 18, 2023

Contents

Chapter I Introduction to the Language of Mathematics

1	Sets and Mathematical Statements	1
2	Quantifiers and Nested Quantifiers	3

Chapter II Logical Analysis of Mathematical Statements

3	Truth Table	7
4	Conjunction and Disjunction	7
5	Logical Operators and Algebra	8
6	Implications	9
7	Converse and Contrapositive	10
8	If and Only if	10

Chapter III Proving Mathematical Statements

9	Proving Universally Quantified Statements	13
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I. Introduction to the Language of Mathematics

1 SETS AND MATHEMATICAL STATEMENTS

Definition. (Set) A set is a well-defined unordered (i.e., order does not matter) collection of distinct (unique) objects.

Example: Empty set = $\{\} = \emptyset$

Note:

- $\{\emptyset\} \neq \emptyset$ and $\{a, \{a, b\}\}$ is a set since $a, \{a, b\}$ are distinct objects.
- $\in \rightarrow$ "is a member of"
- $\notin \rightarrow$ "is not a member of"

Exercise 1 (True or false)

1. $\in \{A, \{A, B\}\} \rightarrow \text{false}$
2. $A \in \{A, \{A, B\}\} \rightarrow \text{true}$
3. $B \in \{A, \{A, B\}\} \rightarrow \text{false}$
4. $\{B, A\} \in \{A, \{A, B\}\} \rightarrow \text{true}$

COMMON SETS

$\mathbb{N} = \{1, 2, 3, \dots\} \rightarrow \text{Natural Numbers}$

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} \rightarrow \text{Integer Numbers}$

$\mathbb{R} = \{\text{set of real numbers}\} \rightarrow \text{Real Numbers}$

$\mathbb{P} = \{2, 3, 5, \dots\} \rightarrow \text{Prime Numbers}$

$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\} \rightarrow \text{Rational Numbers}$

Exercise 1 (True or false)

1. $\mathbb{Q} \in \mathbb{N} \rightarrow \text{false}$
2. $\sqrt{2} \in \mathbb{Q} \rightarrow \text{false}$
3. $-\infty \in \mathbb{Z} \rightarrow \text{false}$
4. $\pi \in \mathbb{R} \rightarrow \text{true}$
5. $\{\sqrt{2}, \sqrt{3}\} \subseteq \mathbb{R} \rightarrow \text{true}$

STATEMENTS

Definition. (statements) A statement is a sentence that has a definite state of being true or false. (It cannot be sometimes false or sometimes true). (i.e., it cannot be sometimes false or sometimes true)

$P(a, b, c) : a^2 + b^2 = c^2$ is NOT a statement since for some values of a, b and c the statement may be true and false for some.

Examples:

- $P(3, 4, 5) : 3^2 + 4^2 = 5^2 \rightarrow \text{true}$
- $P(1, 2, 3) : 1^2 + 2^2 = 3^2 \rightarrow \text{false}$

Exercise 1 (Which of these are statements)

1. 18 is a prime number \rightarrow statement
2. $a^3 + b^3 = c^3 \rightarrow$ not a statement
3. $\forall a, b, c \in \mathbb{R}, a^3 + b^3 = c^3 \rightarrow$ statement
4. $\exists a, b, c \in \mathbb{R}$ such that $a^3 + b^3 = c^3 \rightarrow$ statement

OPEN SENTENCE

Definition. (open sentence) An open sentence is a sentence with at least one variable that is not a statement but can become one when we give values.

$P(a, b, c) : a^3 + b^3 = c^3$ is an open sentence since if we give values for a, b and c $P(a, b, c)$ will become a statement.

Examples:

- $P(2, 3, 5) : 2^3 + 3^3 = 5^3 \rightarrow$ statement
- $\forall a, b, c \in \mathbb{R}, a^3 + b^3 = c^3 \rightarrow$ statement

Exercise 4 (Which of these are open sentences or statements)

1. $P(2, 3, 5) \rightarrow$ statement
2. $\forall a \in \mathbb{R}, P(a, b, c) \rightarrow$ open sentence
3. $P(2, 3, C) \rightarrow$ open sentence
4. $\exists a, b \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $P(a, b, c) \rightarrow$ sometimes considered as a statement or as an open sentence

NEGATION OF STATEMENTS

If P is a statement,

$$\neg P \rightarrow \text{not } P$$

Examples:

- $P \rightarrow$ There is a $x \in \mathbb{R}$ such that $x^2 = 2$
- $\neg P \rightarrow \forall x \in \mathbb{R}, x^2 \neq 2$
- $P \rightarrow \forall x \in \mathbb{R}, x^2 = 2$
- $\neg P \rightarrow \exists x \in \mathbb{R}$ such that $x^2 = 2$

$\neg(\neg P)$ is logically equivalent to P , $\neg(\neg P) = P$

2 QUANTIFIERS AND NESTED QUANTIFIERS

QUANTIFIERS

\forall = "for all" \rightarrow universal quantifier

\exists = "there exists" \rightarrow Existential quantifier

Typically, we'll have an open sentence with at least one "free" variable, x .

Examples:

- $x^2 + 2 = z^3$
- $\forall x \in \mathbb{Z}, \exists z \in \mathbb{R}, x^2 + 2 = z^3 \rightarrow$ For all $x \in \mathbb{Z}$, there exists $z \in \mathbb{R}$, such that $x^2 + 2 = z^3$

$P(x): x^2 = 2$ (open sentence)

- $\exists x \in \mathbb{N}, x^2 = 2 \rightarrow$ false
- $\exists x \in \mathbb{R}, x^2 = 2 \rightarrow$ true
- $\forall x \in \mathbb{R}, x^2 \neq 2 \rightarrow$ true
- $\forall x \in \mathbb{R}, x^2 \neq 2 \rightarrow$ false

If $Q(x): \frac{m+1}{m+2} = 5$ is open sentence, then

$$\exists m \in \mathbb{Z}, \frac{m+1}{m+2} = 5 \rightarrow \text{false}$$

We can make the above statement true by changing its domains, i.e.,

$$\exists m \in \mathbb{R}, \frac{m+1}{m+2} = 5$$

$$\exists m \in \mathbb{Q}, \frac{m+1}{m+2} = 5$$

HIDDEN QUANTIFIERS

Examples:

- 64 is a perfect square $\rightarrow \exists x \in \mathbb{Z}, x^2 = 64$ (true statement)
- $2^{2x-4} = 8$ has a integer solution $\rightarrow \exists x \in \mathbb{Z}, 2^{2x-4} = 8$ (false statement)
- The graph of $y = x^3 - 2x + 1$ has no x -intercept
 \rightarrow There is no solution in $x \in \mathbb{R}$ such that $x^3 - 2x + 1 = 0$
 \rightarrow For all $x \in \mathbb{R}, x^3 - 2x + 1 = 0$
 $\rightarrow \forall x \in \mathbb{R}, x^3 - 2x + 1 = 0$ (false statement)

NEGATION OF QUANTIFIERS

P : Everyone in this room was born in or before 2013.

$\neg P$: There exists someone in this room was born in or after 2013.

S : Set of people in this room, $Q(x) = x$ is born in or before 2013, where x is a person in the room.

- $P : \forall x \in S, Q(x)$
- $\neg P : \exists x \in S, \neg Q(x)$

Fact: If we have the statement of the form

$$P : \forall x \in S, Q(x)$$

$$\neg P : \exists x \in S, \neg Q(x)$$

Exercise 5 (Negate the statement)

1. $P : \forall x \in \mathbb{R}, |x| \geq 5$
2. $\neg P : \exists x \in \mathbb{R}, \neg(|x| \geq 5) \rightarrow \exists x \in \mathbb{R}, |x| < 5$

NESTED QUANTIFIERS

Examples: Let $Q(x, y) = x^3 - y^3 = 1$

- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1 \rightarrow \text{false}$
- $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1 \rightarrow \text{true}$
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1 \rightarrow \text{true}$
- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x^3 - y^3 = 1 \rightarrow \text{false}$

Note: Switching the order of the quantifiers in a statement makes a difference.

If we have an open sentence $Q(x, y)$,

$\exists x \in S, \forall y \in T, Q(x, y) \rightarrow$ There is an $x \in S$ such that [for all $y \in T, Q(x, y)$] is true.

NEGATING NESTED QUANTIFIERS

Examples: Let $Q(x, y, z) = x^5 + y^2 = 2^3$

$$P : \exists x \in \mathbb{Z}, \forall y \in \mathbb{Q}, \exists z \in \mathbb{R}, x^5 + y^2 = 2^3$$
$$\neg P : \forall x \in \mathbb{Z}, \exists y \in \mathbb{Q}, \forall z \in \mathbb{R}, x^5 + y^2 \neq 2^3$$

Fact:

- In order to negate a nested quantified statement, just flip \forall and \exists , and also negate the statement $P(x)$
- Also if the nested quantified statement is long, break it into shorter nested quantified statements and negate it.

II. Logical Analysis of Mathematical Statements

3 TRUTH TABLE

Let p be statement.

1. $\neg P$ = "not P "
2. $\neg P$ is true when P is false and false when P is true.

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

- \neg is a logical operator, somethin take takes a statement and creates a new statements.
- $\neg P$ is a logical expression.

Notice: P and $\neg(\neg P)$ have the same truth value So P and $\neg(\neg P)$ are logically equivalent (\equiv), i.e,

$$P \equiv \neg(\neg P)$$

4 CONJUNCTION AND DISJUNCTION

1. Conjunction (\wedge) = "and"
2. Disjunction (\vee) = "or"

We can use conjunction and disjunction to create compound statements, that are built from two or more statement using things like \vee and \wedge .

Example: A and B are statements (statement variables)

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- $A \wedge B$ and $A \vee B$ are called compound statements.

- $\vee \rightarrow$ If one of the statements are true or both of them are, then the compound statement is true.
- $\wedge \rightarrow$ Even if one of the statement is false, then the compound statement is false.

Example: $\forall x \in \mathbb{R}, (x^2 \geq 0) \wedge (\sin^2(x) + \cos^2(x) = 1)$ is a true statement because both the statement variables are true. Therefore the compound statement is true.

5 LOGICAL OPERATORS AND ALGEBRA

DE MORGAN'S LAWS

A	B	$\neg(A \wedge B)$	$\neg(A \vee B)$	$\neg A \vee \neg B$	$\neg A \wedge \neg B$
T	T	F	F	F	F
T	F	T	F	T	F
F	T	T	F	T	F
F	F	T	T	T	T

$$1. \neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$$

$$2. \neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$$

Exercise: Negate the statement

Let L be a line and P be a parabola. Statement: The point $(1, 2)$ lies on L or on P

$$(1, 2) \in L \vee (1, 2) \in P$$

Sol:

$$\begin{aligned} & \neg((1, 2) \in L \vee (1, 2) \in P) \\ & \equiv (\neg(1, 2) \in L \wedge \neg(1, 2) \in P) \\ & \equiv (1, 2) \notin L \wedge (1, 2) \notin P \end{aligned}$$

Negated statement: The point $(1, 2)$ does not lie on L and does not lie on P .

OTHER LOGICAL OPERATORS LAWS

Commutative Laws: (order does not matter)

- $P \wedge Q \equiv Q \wedge P$
- $P \vee Q \equiv Q \vee P$

Associative Laws: (parentheses does not matter)

- $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
- $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

Distributive Laws:

- $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
- $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Exercise:

1. Prove without using the truth table

$$\neg(A \wedge (\neg B \wedge C)) \equiv \neg(A \wedge C) \vee B$$

PROOF Lets consider the LHS,

$$\begin{aligned} & \neg(A \wedge (\neg B \wedge C)) \\ & \equiv \neg A \vee (\neg(B \wedge C)) && [\text{By De Morgan's Law}] \\ & \equiv \neg A \vee (\neg(\neg B) \vee \neg C) && [\text{By De Morgan's Law}] \\ & \equiv \neg A \vee (B \vee \neg C) && [\text{By Double Negation}] \\ & \equiv (\neg A \vee \neg C) \vee B && [\text{By Associative Law}] \\ & \equiv \neg(A \wedge C) \vee B && [\text{By De Morgan's Law}] \end{aligned}$$

■

2. True or false statement:

- a) $\forall x \in \emptyset, x^2 = 1 \rightarrow$ vacuously true
- b) $\exists x \in \emptyset, x^2 = 1 \rightarrow$ false

6 IMPLICATIONS

P and Q are statements \rightarrow If P , then Q

"If the statement P is true, then the statement Q is true"

Exercise:

1. If Alice is from Canada, then Alice is from North America \rightarrow True
2. If Alice is from North America, then Alice is from Canada \rightarrow False
3. If I am an animal, then I will give you \$10 \rightarrow True
4. If $x > 3$, then $x > 5 \rightarrow$ False
5. If $x > 3$, then $x \geq 1 \rightarrow$ True

Note: If P is false, then it does not matter, if Q is false or true, the implication will be **true**.

Implication Law: $(P \implies Q) \equiv (\neg P \vee Q)$

Exercise:

1. Prove using the implication law

$$\forall x \in \mathbb{R}, (x > 2) \implies (x^2 > 1)$$

PROOF Let $A(x) = x > 2$ and $B(x) = x^2 > 1$. Then for $(A(x) \implies B(x))$ to be true. We need $B(x)$ to be true or $A(x)$ to be false. Notice $B(x)$ is true for $x \in (1, \infty) \cup (-\infty, -1)$ and $A(x)$ is false for $x \in (-\infty, 2]$. So $B(x)$ is true or $A(x)$ is false, holds for $x \in (1, \infty) \cup (-\infty, -1) \cup (-\infty, 2] = \mathbb{R}$.

So, $\forall x \in \mathbb{R}, B(x) \vee \neg A(x) \equiv \forall x \in \mathbb{R}, A(x) \implies B(x)$ ■

2. Let \mathbb{P} be a set of prime numbers. Prove that

$$\forall p \in \mathbb{P}, (p > 2) \implies (P + 1) \text{ is even.}$$

PROOF If the hypothesis is true, then p is a prime greater than 2 and since p is prime it cannot be a multiple of 2, so by definition of even and odd, p is odd. so $p + 1$ is even. Thus if the hypothesis is true, the conclusion is true. so $\forall p \in \mathbb{P}, (p > 2) \implies (P + 1) \text{ is even.}$ ■

NEGATING IMPLICATION

What is $\neg(A \implies B)$?

$$\begin{aligned} \neg(A \implies B) &\equiv \neg(B \vee (\neg A)) \quad [\text{we know } (A \implies B) \equiv (B \vee (\neg A))] \\ &\equiv \neg B \wedge \neg(\neg A) \quad [\text{By De Morgan's Law}] \\ &\equiv \neg B \wedge A \quad [\text{By Double Negation}] \end{aligned}$$

Exercise: If \mathbb{P} are the set of prime numbers, then "There is at most one prime number less than 3".

$$\forall x \in \mathbb{P}, \forall y \in \mathbb{P}, ((x < 3) \wedge (y < 3) \implies (x = y))$$

7 CONVERSE AND CONTRAPOSITIVE

Definition. (Converse) The implication $B \implies A$ is called the **converse** of $A \implies B$

Note: A common mistake is to think that the implication $A \implies B$ and its converse $B \implies A$ are logically equivalent. They are not!

Definition. (contrapositive) The implication $(\neg B) \implies (\neg A)$ is called the **contrapositive** of $A \implies B$.

Contrapositive equivalence Law: An Implication is logically equivalent to its contrapositive.

$$(P \implies Q) \equiv ((\neg B) \implies (\neg A))$$

Implication Law:

$$(A \implies B) \equiv ((\neg A) \vee B) \quad \neg(A \implies B) \equiv (A \wedge (\neg B))$$

8 IF AND ONLY IF

Definition. (if and only if (iff)) The truth value for "A if and only if B", written symbolically as $A \iff B$ is true when A and B have the same truth values, and is false when they have opposite truth values.

More Laws:

1. $(A \iff B) \equiv ((A \implies B) \wedge (B \implies A))$
2. $(\forall x \in X, P(x) \iff Q(x)) \equiv ((\forall x \in X, P(x) \implies Q(x)) \wedge (\forall x \in X, Q(x) \implies P(x)))$

III. Proving Mathematical Statements

9 PROVING UNIVERSALLY QUANTIFIED STATEMENTS

Example:

1. Prove that $\forall \theta \in \mathbb{R}, \sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

PROOF Let θ be an arbitrary real number.

We used the identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$, with $a = \theta$ and $b = \theta$ to obtain.

$$\sin(3\theta) = \sin(\theta + 2\theta) = \sin\theta \cos(2\theta) + \cos\theta \sin(2\theta) \quad (9.1)$$

Now we substitute the double angle identities,

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \sin(2\theta) &= 2\sin\theta \cos\theta\end{aligned}$$

into the right hand side of 9.1, and we get

$$\sin(3\theta) = \sin\theta(\cos^2\theta - \sin^2\theta) + \cos\theta(2\sin\theta \cos\theta) \quad (9.2)$$

we substitute $\cos^2\theta = 1 - \sin^2\theta$, into the RHS of (9.2), we get

$$\begin{aligned}\sin 3\theta &= \sin(1 - \sin^2\theta - \sin^2\theta) + 2\sin\theta(1 - \sin^2\theta) \\ &= \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

so we've shown that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta, \forall \theta \in \mathbb{R}$, as desired. ■