General Topology
Math 751 - PSET 3

Ryan Vaz

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[1] A complete normed (not necessarily infinite-dimesional) vector space is called a *Banach Space*

1. **Problem** Let X be a normed vector space. Recall that a norm induces a metric on X by $d(x, y) = \|x - y\|$. Let $V \subsetneq X$ be a finite-dimensional non-trivial subspace of X with basis $\{v_1, \ldots, v_N\}$.

(a) Let $\{x_n\}$ be a sequence in V which converges to the zero vector. Recall that each x_n can be represented uniquely as a linear combination.

$$x_n = a_{n,1}v_1 + \cdots + a_{n,N}v_N$$

Prove that all of the coefficients $a_{n,k}$ tend to zero.

Since V is finite-dimensional, any two norms on V are equivalent. Therefore, we can choose a norm $|\cdot|$ on V such that $|v_k| = 1$ for all k.

Let $\epsilon > 0$ be given. Since x_n converges to zero, there exists N_0 such that $|x_n - 0| = |x_n| < \epsilon$ for all $n \geq N_0$.

Now, for any $k = 1, 2, \ldots, N$, consider the sequence $a_{n,k}$.

For any fixed k, we have:

$$\begin{aligned}
|a_{n,k}| &= \left| \frac{\langle x_n, v_k \rangle}{|v_k|^2} \right| \\
&= |\langle x_n, v_k \rangle| \\
&\leq |x_n||v_k| && \text{(Cauchy-Schwarz inequality)} \\
&\leq |x_n| && \text{(since } |v_k| = 1\text{)}
\end{aligned}$$

Therefore, for all $n \geq N_0$ and for any $k = 1, 2, \ldots, N$, we have $|a_{n,k}| < \epsilon$. This shows that all the coefficients $a_{n,k}$ tend to zero as n tends to infinity, since ϵ was arbitrary.

(b) Prove that V is a closed subset of X. Hint: suppose $\{x_n\} \subset V$ and $x_n \to x \notin V$. What can you say about $\{v_1, \ldots, v_N, x\}$? Part (a) also plays a role..

Suppose $x_n \subset V$ is a convergent sequence in X with limit $x \in X$ such that $x \notin V$. We want to show that V is a closed subset of X.

x_n can be uniquely expressed as $x_n = \sum_{k=1}^{N} a_{n,k}v_k$ for some coefficients $a_{n,k}$. Since $x_n \to x$, we must have $a_{n,k} \to b_k$ for some $b_k \in \mathbb{R}$ or $\mathbb{C}$.

Now, let y be the vector in V defined by $y = \sum_{k=1}^{N} b_k v_k$. We can claim that $y = x$.

Since $x \notin V$, the vectors $v_1, \ldots, v_N, x$ are linearly independent. Therefore, the set $v_1, \ldots, v_N$ is linearly independent as well.

Now, let $z = y - x = \sum_{k=1}^{N}(b_k - a_{n,k})v_k$. Since $a_{n,k} \to b_k$, we have $z \to 0$ as $n \to \infty$.

But then part (a) of the problem statement implies that all the coefficients $b_k - a_{n,k}$ must tend to zero as well, which means $z = 0$ and hence $y = x$. Therefore, the limit of any convergent sequence in V lies in V, so V is a closed subset of X.

(c) Prove that V is nowhere dense in X.

To show that V is nowhere dense in X, we need to show that $\overline{V}$ has empty interior.

Let U be any nonempty open subset of X. We need to show that $U \setminus \overline{V} \neq \emptyset$. Since V is a proper subset of X, there exists a vector $x_0 \in X \setminus V$.

Since U is open, there exists $\epsilon > 0$ such that the ball $B(x_0, \epsilon) := x \in X : |x - x_0| < \epsilon$ is contained in U.

Now, consider the sequence $x_n \subset V$ defined by $x_n = x_0 + \frac{1}{n}(v_1 + \cdots + v_N)$. Note that $x_n \to x_0$ as $n \to \infty$.

Since $x_0 \notin V$, we must have $x_n \notin V$ for all sufficiently large n. In particular, for some $N \in \mathbb{N}$, we have $x_N \notin V$.

Since $x_N \in B(x_0, \epsilon)$, it follows that $B(x_0, \epsilon) \not\subseteq \overline{V}$. Therefore, $U \setminus \overline{V} \neq \emptyset$, as desired.

This shows that $\overline{V}$ has empty interior, so V is nowhere dense in X.

(d) Assume additionally that X is infinite-dimensional and complete [1]. Prove that any basis of X must be uncountable. Hint: let $\{v_1, v_2, v_3, \ldots\}$ be a linearly independent set, define $V_N = \text{span}\,\{v_1, \ldots, v_N\}$, and apply the BCT to show that $\{v_1, v_2, v_3, \ldots\}$ is not spanning.

Spz that X has a countable basis $v_1, v_2, v_3, \ldots$.

Let V_N be the span of $v_1, \ldots, v_N$. Then $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ is an increasing sequence of finite-dimensional subspaces of X. By the finite-dimensional case of the BCT, we know that X cannot be written as the union of countably many nowhere dense subsets. Since each V_N is a nowhere dense subset of X (by part c), it follows that $\bigcup_{N=1}^{\infty} V_N$ is a nowhere dense subset of X.

However, we also have $\bigcup_{N=1}^{\infty} V_N = \text{span}\, v_1, v_2, v_3, \ldots = X$ (by the assumption that $v_1, v_2, v_3, \ldots$ is a basis of X). This is a contradiction, since X is complete and cannot be the union of a nowhere dense set and a dense set.

Therefore, our assumption that X has a countable basis is false, and any basis of X must be uncountable.

2. **Problem** Let $\mathcal{C}$ denote the middle-thirds Cantor set. Show that $\partial\mathcal{C} = \mathcal{C}$.

To show that $\partial\mathcal{C} = \mathcal{C}$, we need to show that $\mathcal{C} \subseteq \partial\mathcal{C}$ and $\partial\mathcal{C} \subseteq \mathcal{C}$.

First, $\mathcal{C} \subseteq \partial\mathcal{C}$. Recall that $\mathcal{C}$ is defined as the intersection of a sequence of nested closed sets C_n, where $C_0 = [0, 1]$ and C_{n+1} is obtained by removing the middle thirds of each interval in C_n.

It follows that every point in $\mathcal{C}$ is a limit point of both $\mathcal{C}$ and its complement C^c. Therefore, $\mathcal{C} \subseteq \overline{\mathcal{C}} \cap \overline{C^c} = \partial\mathcal{C}$.

Next, let's show that $\partial\mathcal{C} \subseteq \mathcal{C}$. Suppose $x \in \partial\mathcal{C}$. Then $x \in \overline{\mathcal{C}}$ and $x \in \overline{C^c}$.

If $x \in \mathcal{C}$, then we are done. Otherwise, $x \in C^c$. Since C^c is open, there exists $\epsilon > 0$ such that the open ball $B(x, \epsilon) := y \in \mathbb{R} : |y - x| < \epsilon$ is contained in C^c.

Since $\mathcal{C}$ is a closed set, $B(x, \epsilon) \cap \mathcal{C}$ is also closed. Moreover, $B(x, \epsilon) \cap \mathcal{C} \neq \emptyset$, since x is a limit point of $\mathcal{C}$.

By the assumption that $x \in \partial\mathcal{C}$, we have $B(x, \epsilon) \cap \mathcal{C} \neq \emptyset$ and $B(x, \epsilon) \cap C^c \neq \emptyset$. This implies that $B(x, \epsilon)$ contains a point that is not in C^c, which is a contradiction.

Therefore, we must have $x \in \mathcal{C}$, and hence $\partial\mathcal{C} \subseteq \mathcal{C}$.

Combining the above arguments, we conclude that $\partial\mathcal{C} = \mathcal{C}$.

3. **Problem** This problem proves the existence of a perfect set of irrational numbers. Let $\mathcal{C}$ denote the middle-thirds Cantor set. If $a \in \mathbb{R}$, define $a+\mathcal{C} = \{a+c : c \in \mathcal{C}\}$, i.e., the translation of $\mathcal{C}$ by a.

(a) Prove that

$$\bigcup_{q\in\mathbb{Q}} q+\mathcal{C} \subsetneq \mathbb{R}$$

Assume for contradiction that $\bigcup_{q\in\mathbb{Q}} q+\mathcal{C} = \mathbb{R}$. Then, for any $x \in \mathbb{R}$, there exists $q_x \in \mathbb{Q}$ and $c_x \in \mathcal{C}$ such that $x = q_x + c_x$. Since $\mathcal{C} \subset [0,1]$, it follows that q_x must belong to the interval $[x-1, x+1]$.

Now consider the collection $q_x : x \in \mathbb{R}$. This is a countable set since it is a subset of the countable set $\mathbb{Q}$. Therefore, there exists a real number y that is not in the collection $q_x : x \in \mathbb{R}$. In particular, y does not belong to any interval of the form $[x-1, x+1]$.

Since $\mathcal{C}$ is a perfect set, there exists a sequence (c_n) of distinct points in $\mathcal{C}$ that converges to y. Since $c_n \in [y-1, y+1]$ for all n, it follows that c_n must converge to some point $c \in [y-1, y+1]$. But then $y = y+c-c$ belongs to the set $y+\mathcal{C}$, which contradicts the fact that y is not in the collection $q_x : x \in \mathbb{R}$.

Therefore, we must have $\bigcup_{q\in\mathbb{Q}} q+\mathcal{C} \subsetneq \mathbb{R}$.

(b) Let $y \in \mathbb{R} - \bigcup_{q\in\mathbb{Q}} q+\mathcal{C}$. Prove that $y+\mathcal{C}$ contains no rational numbers and is perfect (you may assume that $\mathcal{C}$ is perfect)..

To show that $y+\mathcal{C}$ is perfect, we need to show that it is closed, has no isolated points, and contains no intervals.

First, we show that $y+\mathcal{C}$ is closed. Let $(y+c_n)$ be a sequence in $y+\mathcal{C}$ that converges to some $z \in \mathbb{R}$. Since $\mathcal{C}$ is closed, the sequence (c_n) converges to some $c \in \mathcal{C}$. Then, $(y+c_n)$ converges to $y+c$, which belongs to $y+\mathcal{C}$ since $\mathcal{C}$ is closed. Therefore, $y+\mathcal{C}$ is closed.

Next, we show that $y+\mathcal{C}$ has no isolated points. Let $z = y+c \in y+\mathcal{C}$ for some $c \in \mathcal{C}$. Since $\mathcal{C}$ has no isolated points, there exists a sequence (c_n) of distinct points in $\mathcal{C}$ that converges to c. Then, $(y+c_n)$ is a sequence in $y+\mathcal{C}$ that converges to z, and since z was arbitrary, it follows that $y+\mathcal{C}$ has no isolated points.

Finally, we show that $y+\mathcal{C}$ contains no intervals. spz that I is an interval contained in $y+\mathcal{C}$. Then, I has nonempty intersection with $\mathcal{C}$, since otherwise I would be contained in the set y. Let $c \in \mathcal{C}$ be a point in I, and let (c_n) be a sequence of distinct points in $\mathcal{C}$ that converges to c. Then, $(y+c_n)$ is a sequence in $y+\mathcal{C}$ that converges to $y+c$, which belongs to I since I is closed. But then I contains infinitely many points of the form $y+c_n$, which contradicts the fact that $y+\mathcal{C}$ has no isolated points. Therefore, $y+\mathcal{C}$ contains no intervals.

Since $y+\mathcal{C}$ is closed, has no isolated points, and contains no intervals, it must be a perfect set.

4. **Problem** Let (X,d) be a metric space. Prove that every sequence in X has a convergent subsequence if and only if every infinite subset of X has a limit point in X.

First, let's assume that every infinite subset of X has a limit point in X. We want to show that every sequence in X has a convergent subsequence.

Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ has infinitely many distinct terms, then we can take a subsequence of distinct terms, which is clearly convergent.

Otherwise, there are only finitely many distinct terms in the sequence, say $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ for some $k \in \mathbb{N}$. Since there are infinitely many terms in the sequence, there must be some n_i that appears infinitely many times in the sequence. We can therefore construct a subsequence $\{x_{n_k}\}$ such that all its terms are equal to x_{n_i}. This subsequence is clearly convergent, and we are done.

Next, let's assume that every sequence in X has a convergent subsequence. We want to show that every infinite subset of X has a limit point in X. spz that there exists an infinite subset $A \subseteq X$ such that A has no limit point in X. Then, for each $x \in A$, there exists some $r_x > 0$ such that $B(x, r_x) \cap A$ contains at most finitely many points of A

Let $\{x_n\}$ be a sequence in A. Since A is infinite, we can choose distinct terms $x_{n_1}, x_{n_2}, \ldots$ in $\{x_n\}$ such that $x_{n_k} \in B(x_{n_{k-1}}, r_{x_{n_{k-1}}}) \cap A$ for all $k \geq 2$.

This sequence $\{x_{n_k}\}$ is a cauchy sequence, since $d(x_{n_k}, x_{n_{k-1}}) < r_{x_{n_{k-1}}}$ for all $k \geq 2$. Since X is a metric space, we know that every cauchy sequence in X has a convergent subsequence. But this contradicts our assumption that A has no limit point in X. Therefore, every infinite subset of X must have a limit point in X. Conversely, spz that every infinite subset of X has a limit point in X, but there exists a sequence $\{x_n\}$ in X that has no convergent subsequence. Then, for each $x \in X$, we can choose an open ball $B(x, r_x)$ such that $B(x, r_x) \cap \{x_n\}$ contains at most finitely many points of $\{x_n\}$

Now, assume that every infinite subset of X has a limit point in X. Let $\{x_n\}$ be any sequence in X. If $\{x_n\}$ has infinitely many distinct terms, then we can extract a subsequence that is a copy of an infinite subset of X. By our assumption, this infinite subset has a limit point x in X. Then by definition, for any $\epsilon > 0$, there exists infinitely many terms of the subsequence within the ϵ-neighborhood of x. Thus, this subsequence converges to x.

If $\{x_n\}$ has finitely many distinct terms, then some term x_k appears infinitely often. In this case, the constant subsequence $\{x_k, x_k, x_k, \ldots\}$ clearly converges to x_k.