General Topology
Math 751 - PSET 2

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As in class, X denotes a metric space.

1. **Problem**

(a) Prove that $\text{int}(A^c) = \text{cl}(A)^c$. This is very useful to have in mind. Moving the complement sign out of the parentheses changes "int" to "cl".

To prove every point in $\text{int}(A^c)$ is in $\text{cl}(A)^c$:

Let $x \in \text{int}(A^c)$. This means that there exists an open set U such that $x \in U$ and $U \subseteq A^c$. We need to show that $x \in \text{cl}(A)^c$. Suppose for contradiction that $x \in \text{cl}(A)$. Then, every open set containing x intersects A. Furthermore, U intersects A, which contradicts the fact that $U \subseteq A^c$. Therefore, $x \in \text{cl}(A)^c$.

Now for the reverse direction:

Let $x \in \text{cl}(A)^c$. We need to show that $x \in \text{int}(A^c)$. Suppose for contradiction that this is not true. This means for every positive integer n, there exists a point $a_n \in A$ such that $a_n \in B(x, 1/n)$, where $B(x, 1/n)$ is the open ball centered at x with radius $1/n$. This also means that the sequence (a_n) converges to x. However, since $x \in \text{cl}(A)^c$, x cannot be a limit point of A. Therefore, there exists an open set V containing x such that $V \cap A = \emptyset$. But this contradicts the fact that $a_n \in B(x, 1/n)$ and $a_n \in A$ for every n. Therefore, $x \in \text{int}(A^c)$.

Since we were able to show both directions, it holds that $\text{int}(A^c) = \text{cl}(A)^c$.

2. **Problem** Let $X = [0, 1] \subset \mathbb{R}$ with the usual metric. Let $A \subset [0, 1]$ be the set of numbers whose decimal expansion does not contain the number 4 (for those numbers with two expansions, use the infinite one).

(a) Prove that A is nowhere dense.

To show that A is nowhere dense, we need to show that the closure of A has empty interior. We can do this by showing that every nonempty open set in $[0, 1]$ contains a point that is not in A or a limit point of A.

Any point x in $[0, 1]$ that does not belong to A has a decimal expansion that contains the digit 4. We can then construct a sequence in A that converges to x by replacing the first occurrence of 4 in the decimal expansion of x with a 3 followed by an infinite sequence of 9's as $\lim_{x \to .4^-} = .3\overline{99}$. Thus, x is a limit point of A.

Since every point in $[0, 1]$ is either in A or a limit point of A, the closure of A is $[0, 1]$. Therefore, the interior of the closure of A is empty, which means that A is nowhere dense.

(b) Prove that every point of A is a limit point of A. (Refer to HW 1 for the definition of limit point.)

To show that every point of A is a limit point of A, we need to show that for every $x \in A$, there exists a sequence (a_n) in A such that $a_n \neq x$ for all n and $a_n \to x$ as $n \to \infty$.

Since x belongs to $[0, 1]$, we can write its decimal expansion as $x = 0.d_1 d_2 d_3 \ldots$, where each d_n is an integer between 0 and 9. Since $x \in A$, we know that $d_n \neq 4$ for all n.

Consider the number $y = 0.4d_2d_3\ldots$. Note that $y \in A$ because its decimal expansion does not contain the digit 4. Also note that y differs from x only in the first decimal place.

Now, let (a_n) be the sequence given by $a_n = y + 1/n$. Note that each a_n belongs to A because it can be obtained from y by changing only a finite number of decimal places. Also note that a_n converges to x, because

$$a_n - x = \frac{4 - d_1}{10} + \sum_{k=2}^{\infty} \frac{d_k}{10^k} - \frac{1}{n} \to 0$$

as $n \to \infty$. Therefore, x is a limit point of A, and the proof is complete.

(c) Is A open? Closed? Justify.

A is neither open nor closed in $[0, 1]$.

To see that A is not open, consider any point $x \in A$. We need to show that there exists no open interval around x contained entirely in A. Let d_n denote the nth digit in the decimal expansion of x. Since $x \in A$, we have $d_n \neq 4$ for all n. Let y be the number obtained by replacing d_1 with 4 and all subsequent digits with 0:

$$y = 0.4\underbrace{0\ldots0}_{n-1 \text{ zeros}}.$$

Note that y belongs to A because its decimal expansion does not contain the digit 4. Moreover, y lies in the open interval $(x - 10^{-n}, x + 10^{-n})$, but y does not belong to A. Therefore, A cannot be open.

To see that A is not closed, note that its complement A^c is the set of numbers whose decimal expansion contains the digit 4. For example, $0.4 \in A^c$. But A^c is not open, because any open interval around 0.4 contains points that do not belong to A^c. Therefore, A is not closed.

3. **Problem** Define the diameter of a subset $E \subset X$ to be

$$\text{diam}(E) = \sup\{d(x, y) : x, y \in E\}.$$

Let X be a complete metric space and let

$$E_1 \supset E_2 \supset E_3 \supset \cdots$$

be a sequence of nested nonempty closed subsets with $\lim_{n\to\infty} \text{diam}\,(E_n) = 0$. Prove that $\cap_{n=1}^{\infty} E_n$ contains exactly one point. (This is a generalization of the lemma used in the proof of the BCT.)

Since X is complete, any Cauchy sequence in X converges to a limit point in X. Suppose there are two distinct points $x, y \in \bigcap_{n=1}^{\infty} E_n$. Then, by the definition of the diameter, we have $d(x, y) \leq \operatorname{diam}(E_n)$ for any $n \in \mathbb{N}$, and taking the limit as $n \to \infty$, we obtain $d(x, y) = 0$. This implies that $x = y$, and hence $\bigcap_{n=1}^{\infty} E_n$ contains at most one point.

To show that it contains at least one point, we construct a sequence (x_n) in X such that $x_n \in E_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Since E_1 is closed and nonempty, we can choose $x_1 \in E_1$. Suppose we have constructed $x_1, \dots, x_n$ such that $x_k \in E_k$ for all $k \in 1, \dots, n$. Since E_{n+1} is closed and nonempty, we can choose $x_{n+1} \in E_{n+1}$ such that $d(x_{n+1}, x_n) < \frac{1}{2^{n+1}}$. By the triangle inequality, we have

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} d(x_{k+1}, x_k) = \frac{1}{2^n},$$

for any $m > n$. This implies that (x_n) is a Cauchy sequence in X. Since X is complete, there exists a limit point $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since E_n is closed for all n, we have $x \in E_n$ for all n, and hence $x \in \bigcap_{n=1}^{\infty} E_n$. Therefore, $\bigcap_{n=1}^{\infty} E_n$ contains at least one point.

Thus we have shown that $\bigcap_{n=1}^{\infty} E_n$ contains exactly one point.

4. **Problem** Define the distance between two subsets $A, B \subset X$ to be

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$$

The distance between a point x and a subset A, denoted $d(x, A)$, is simply $d(\{x\}, A)$ (i.e., treat the point as a singleton). Fix an arbitrary set A and define for each positive integer n the set $U_n = \{x \in X : d(x, A) < 1/n\}$.

(a) Prove that the U_n are open.

To prove that U_n is open, we need to show that every point in U_n has a neighborhood contained in U_n.

Let $x \in U_n$, then $d(x, A) < \frac{1}{n}$, which means there exists a point $a \in A$ such that $d(x, a) < \frac{1}{n}$. We will show that the open ball $B(x, r)$ with radius $r = \frac{1}{n} - d(x, a)$ is contained in U_n. Let $y \in B(x, r)$, then we have $d(y, x) < r$, which implies

$$\begin{aligned} d(y, A) &\leq d(y, x) + d(x, A) \\ &< r + \frac{1}{n} - d(x, a) \\ &= \frac{1}{n} \quad (\text{since } r = \frac{1}{n} - d(x, a)). \end{aligned}$$

Therefore, $y \in U_n$, and since y was an arbitrary point in $B(x, r)$, we have shown that $B(x, r) \subseteq U_n$. Hence, U_n is open for every positive integer n.

(b) Prove that $\bar{A} = \cap_{n=1}^{\infty} U_n$. In particular, every closed set is the countable intersection of open sets.

First, we need to show that $\bar{A} \subseteq \cap_{n=1}^{\infty} U_n$.

Let $x \in \bar{A}$. Then, for any $n \in \mathbb{N}$, the open ball $B(x, 1/n)$ contains a point a_n in A. This means $d(x, a_n) < 1/n$, and therefore $x \in U_n$ for every n. Thus, $x \in \cap_{n=1}^{\infty} U_n$.

Next, we need to show that $\cap_{n=1}^{\infty} U_n \subseteq \bar{A}$.

Let $x \in \cap_{n=1}^{\infty} U_n$. Then, for any $n \in \mathbb{N}$, we have $d(x, A) < 1/n$. This means there exists a point $a_n \in A$ such that $d(x, a_n) < 1/n$. Therefore, for any $\epsilon > 0$, we can choose n such that $1/n < \epsilon$, and we have $d(x, a_n) < \epsilon$. This shows that x is a limit point of A, and hence $x \in \bar{A}$.

Therefore, every closed set is the countable intersection of open sets, since the closure of any set is closed, and we can express it as the intersection of open sets.

(c) Prove that every open set is the countable union of closed sets.

Let U be an open set in a metric space (X, d). For each $n \in \mathbb{N}$, let F_n be the set of all points in X whose distance from U^c is at most $1/n$.

We claim that each F_n is a closed set. To see this, let (x_k) be a sequence in F_n converging to some point $x \in X$. We need to show that $x \in F_n$. Since $x_k \in F_n$, we have $d(x_k, U^c) \leq 1/n$ for all $k \in \mathbb{N}$. By the inverse of the triangle inequality, we have

$$|d(x, U^c) - d(x_k, U^c)| \leq d(x, x_k)$$

Thus, $d(x, U^c) \leq d(x_k, U^c) + d(x, x_k) \leq 1/n + d(x, x_k)$ for all $k \in \mathbb{N}$. Since $x_k \to x$, we have $d(x, x_k) \to 0$ as $k \to \infty$. It follows that $d(x, U^c) \leq 1/n$ and hence $x \in F_n$. Thus, F_n is closed.

Next, we claim that $U = \cup_{n=1}^{\infty} F_n$. To see this, let $x \in U$. Since U is open, there exists some $r > 0$ such that $B(x, r) \subset U$. By the Archimedean property of real numbers, we can choose some $n \in \mathbb{N}$ such that $1/n < r$. Then for any $y \in B(x, r)$, we have $d(y, U^c) \geq r > 1/n$, which implies $y \notin F_n$. Thus, $B(x, r) \subset U \cap F_n$, which implies $x \in F_n$. Therefore, $U \subset \cup_{n=1}^{\infty} F_n$.