General Topology
Math 751 - PSET 1

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As in class, X denotes a metric space.

1. **Problem**

(a) Show that the discrete metric indeed defines a metric on any nonempty set X.

The discrete metric on a set X is defined as: d(x,y) = 0 if x = y, and d(x,y) = 1 if x $\neq$ y.

Knowing this, we can make some conclusions about it being a metric space for X.

- Non-Negative: For any $x, y \in X$, we have $d(x, y) \geq 0$ by definition.
- Identity: $\forall x, y \in X$, we have $d(x, y) = 0$ if and only if $x = y$.
- Triangle Inequality: For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Since the discrete metric on X follows all 3 axioms, (X, d) must be a metric space for any set X.

(b) Describe with proof the open and closed sets of a discrete metric space.

For a set to be called open, $\forall x \in S$, $\exists x \in \mathbb{R}, \epsilon$ such that the open ball with radius ϵ centered at x is contained in S

An open ball is defined as, $B_\epsilon(x) = y \in X \mid d(x, y) < \epsilon$

Since the values in a discrete metric space can only be 0 or 1, there are only two possibilities for the value of $d(x, y)$: either 0 if $x = y$ or 1 if $x \neq y$. This means that an open ball with radius ϵ can either be empty or contain only the point x itself

A set $S \subset X$ is called closed if its complement $X \setminus S$ is open. In a discrete metric space, the only open sets are the set of just a single point(x), which means that the only closed sets are the empty set $\emptyset$ and the whole space X.

This means in a discrete metric space there are only two types of sets: the sets x and the whole space X, with the empty set being the only closed set.

2. **Problem**

(a) Show that the semi-open interval $(0, 1]$ is neither open nor closed in $\mathbb{R}$ with the standard metric.

In the real numbers $\mathbb{R}$ with the standard metric, a set is considered open if for every point in the set there exists a ϵ ball around that point that is entirely contained within the set. On the other hand, a set is considered closed if its complement is open.

The interval $(0, 1]$ is not open in $\mathbb{R}$ with the standard metric because there is no ϵ ball that will be contained in (0,1].

By the same logic, the set is also not closed because it's complement $(-\infty, 0] \cup (1, \infty)$ follows the same logic using 0.

Thus, the interval (0,1] is neither open nor closed in $\mathbb{R}$ with the standard metric

(b) What about in $\mathbb{R}$ with the discrete metric?

With the discrete metric, a single point(singleton) may be both closed and open.

Since the set (0,1] can be expressed as the union of singletons, we can say it is both open and closed, and the same applies for it's complement.

Thus the set is open and closed

3. **Problem**

(a) Show that the closed ball $\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$ is a closed set.

The closed ball is defined as the set of all points y in X that are a distance of at most r from x.

We can do so by verifying that the complement of $\bar{B}(x, r)$, $\bar{B}(x, r)^c = y \in X : d(x, y) > r$ is a open set.

spz $y \in \bar{B}(x, r)^c$. Since y is farther than r away from x, we have $d(x, y) > r$. Hence, there exists a positive ϵ such that $d(x, y) > r + \epsilon$.

For any $z \in B(y, \epsilon)$, the triangle inequality implies that $d(x, z) \geq d(x, y) - d(y, z) > r + \epsilon - \epsilon = r$, so $z \in \bar{B}(x, r)^c$. (the diagram you drew in class)

Therefore, every $y \in \bar{B}(x, r)^c$ has an ϵ ball around it that is entirely contained within $\bar{B}(x, r)^c$, meaning that $\bar{B}(x, r)^c$ is an open set.

Since the complement of $\bar{B}(x, r)$ is open, it follows that it is closed.

(b) Find an example of a metric space and an open ball $B(x, r)$ whose closure $\overline{B(x, r)}$ is different from the closed ball $\bar{B}(x, r)$.

The metric space $(\mathbb{Q}, d)$, where $\mathbb{Q}$ is the set of all rational numbers and $d(x, y) = |x - y|$ for all $x, y \in \mathbb{Q}$.

4. **Problem** Let $A \subset X$. A point $x \in X$ is a limit point of A if every ball $B(x, r)$ intersects A at a point other than x.

(a) Show that every limit point of A is an adherent point of A. Find an example where the converse is false.

A point $x \in X$ is an adherent point of A if for every $\epsilon > 0$, the ball $B(x, \epsilon)$ intersects A at some point.

spz that x is a limit point of A. Then, for every $r > 0$, the ball $B(x, r)$ intersects A at a point other than x. In particular, for every $\epsilon > 0$, the ball $B(x, \epsilon)$ intersects A at some point, so x is an adherent point of A.

For an example where the converse is false, we can use the set $A = \mathbb{Q} \cup a$ in the metric space $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$ and a is an irrational number.

In this case, a is an adherent point of A, since for every $\epsilon > 0$, the ball $B(a, \epsilon)$ contains a rational number. However, a is not a limit point of A, since every ball $B(a, r)$ with $r > 0$ contains only irrational numbers.

(b) Show that an adherent point of A which lies in A^c is in fact a limit point of A.

Suppose that x is an adherent point of A and $x \in A^c$. Then, for every $\epsilon > 0$, the ball $B(x, \epsilon)$ intersects A at some point. Since $x \in A^c$, $B(x, \epsilon_0) \subseteq A^c$. However, $B(x, \epsilon_0)$ also intersects A at some point, which means that there exists a point $y \in B(x, \epsilon_0) \cap A$.

Since $y \neq x$, it follows that y is a limit point of A that is contained in the ball $B(x, \epsilon_0)$. Hence, every ball $B(x, r)$ with $r > 0$ intersects A at a point other than x, and so x is a limit point of A.

(c) Show that the following are equivalent:

- x is a limit point of A
- every ball $B(x, r)$ contains infinitely-many points of A
- there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset A$ of distinct points converging to x.

x is a limit point of A if and only if every ball $B(x, r)$ contains at least one point of A different from x.

Suppose that every ball $B(x, r)$ contains infinitely-many points of A. Let $\epsilon > 0$ be arbitrary. Then, the ball $B(x, \epsilon)$ contains an infinite number of points of A, so there exists a sequence of distinct points such that $x_j \to x$ as $j \to \infty$.

Since the first point implies A equals B, and the second implies B equals C, it follows that A equals C, therefore all the statements above are equivalent.

5. **Problem** Let $A \subset X$. A point $x \in X$ is a boundary point of A if every ball $B(x, r)$ intersects both A and A^c. The set of all boundary points of A, denoted ∂A, is called the boundary of A.

(a) Prove that $\partial A = \bar{A} \cap \overline{A^c}$.

Let $x \in \partial A$. Then, every ball $B(x, r)$ intersects both A and A^c. Hence, x is an adherent point of both A and A^c. It follows that $x \in \bar{A}$ and $x \in \overline{A^c}$, so $x \in \bar{A} \cap \overline{A^c}$.

Conversely, let $x \in \bar{A} \cap \overline{A^c}$. Then, x is an adherent point of A and A^c. This means that for every $\epsilon > 0$, the ball $B(x, \epsilon)$ intersects both A and A^c, so x is a boundary point of A and $x \in \partial A$.

Therefore, $\partial A = \bar{A} \cap \overline{A^c}$.

(b) Prove that A is closed iff $A \supset \partial A$.

($\rightarrow$) Assume A is closed. Let $x \in \partial A$. Then, x is an adherent point of both A and A^c, so $x \in \bar{A}$ and $x \in \overline{A^c}$. Since A is closed, $x \in A$. Hence, $A \supset \partial A$.

($\leftarrow$) Assume $A \supset \partial A$. Let $x \in \bar{A} \cap A^c$. Then, x is an adherent point of A and $x \notin A$. By definition, this means that $x \in \partial A$, and thus $x \in A$, which contradicts $x \notin A$. Hence, $\bar{A} \cap A^c = \emptyset$, which implies that A is closed.

(c) Prove that A is open iff $A \cap \partial A$ is empty.

($\rightarrow$) Assume A is open. Let $x \in A \cap \partial A$. Then, $x \in A$ and x is a boundary point of A, so every ball $B(x, r)$ intersects both A and A^c. However, this contradicts the fact that $x \in A$ and A is open, so the assumption that $x \in A \cap \partial A$ is false. Hence, $A \cap \partial A = \emptyset$.

($\leftarrow$) Assume $A \cap \partial A = \emptyset$. Let $x \in A$. Then, every ball $B(x, r)$ for $r > 0$ is contained in A, since x is not a boundary point of A. Hence, A is open.