15-463 Notes on Linearization of Rendered Images

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We are given the following optimization problem

$$\min_{g, L_{ij}} \sum_{i,j} \sum_{k} \left\{ w\left(I_{ij}^{k}\right) \left[g\left(I_{ij}^{k}\right) - \log\left(L_{ij}\right) - \log\left(t^{k}\right) \right] \right\}^{2} + \lambda \sum_{z=0}^{255} \left\{ w(z) \nabla^{2} g(z) \right\}^{2} \tag{1}$$

We want to formulate this equation in order to solve it in a least squares sense. We are given that $v = [g; log(L_{ij})]$, so essentially, we want to formulate A and b using terms from Equation (2) such that:

$$A \begin{bmatrix} g \\ log(L_{ij}) \end{bmatrix} = b \tag{2}$$

where g is a 256×1 column vector that represents a 1-1 mapping (we'll cover dimensions of A and b shortly).

Let's take a closer look at the two terms in Equation (1), and consider their relationships to v. Expanding the first and second terms respectively, we have

$$w(I_{ij}^{k})g(I_{ij}^{k}) - w(I_{ij}^{k})log(L_{ij}) - w(I_{ij}^{k})log(t^{k}) \tag{3}$$

$$\lambda w(z)g(z+1) - 2\lambda w(z)g(z) + \lambda w(z)g(z-1) \tag{4}$$

Notice that the first two terms of Equation(3) have factors of g and $log(L_{ij})$, as well as all of the terms in Equation (4). Noting that g and $log(L_{ij})$ are the terms in our v vector, this gives us some important information regarding A: **A is a matrix that contains weights (w_{ij}).**

Now, consider the absolute minimum value of Equation (1). When $Ax - b = 0$, x is an exact solution to $Ax = b$. Therefore, we want to find some g and so that Equation(1)'s value can be as close to 0 as possible.

We can rewrite these expressions as equations, under the assumption that they evaluate to 0:

$$w(I_{ij}^k)g(I_{ij}^k) - w(I_{ij}^k)log(L_{ij}) = w(I_{ij}^k)log(t^k) \tag{5}$$

$$\lambda w(z)g(z+1) - 2\lambda w(z)g(z) + \lambda w(z)g(z-1) \tag{6}$$

For convenience, let us consider Equation(3) and Equation(4) separately.

For Equation (3), we notice that while the first two terms have factors of either g or $log(L_{ij})$, the third term is independent of either g or $log(L_{ij})$. Motivated by our previous observations that A is a matrix that contains weights, and we assume the expression to evaluate to 0 in order to calculate the closest g, we can conclude that **b contains $\mathbf{log(t^k)}$**.

For Equation (4), all of the terms contain g. Therefore we can additionally conclude that **A contains the smoothing terms (Equation(6))**.

Now let's actually construct A and b. The number of columns of A should be pretty straightforward by rules of matrix multiplication. We know that the $log(L_{ij})$ component of v is of dimension $3 * S$, where $S = image\_width * image\_height$ and 3 denotes the 3 RGB color channels. So A = (?, 256 + 3S).

Now, in order to count the number of rows in A, we must consider the number of constraints on g and $log(L_j)$. From the first term in Equation(1), there are at least $3S * K$ constraints, where K is the number of images in the stack. See $g(I_{ij}^k)$ for motivation. From the second term in Equation(1), there are an additional 256 constraints. And finally, we add one more constraint to set the middle value of $g = 0$.

So the dimensions of A are $(3S + 256 + 1, 256 + 3S)$, and the dimensions of b are $(3S + 256 + 1, 1)$.

Hints: Remember that the domain of Z, the number of values a pixel can take, is finite!