Feed back Linearisation

As applied to the parametrically excited pendulum

Feedback Linearisation

This is a technique which can be applied to systems that require to be controlled but are governed by nonlinear ODEs which can't be directly linearised. This may be because a simple linearisation of the system might trivialise the way it works when under some form of control.
So, to rectify this it's possible to calculate a form of actuation torque that accommodates the system nonlinearities in such a way that a linear homogeneous differential equation appears in terms of the error. This is known as feedback linearisation, and it introduces two active gains which operate on the error velocity and the displacement velocity.

We start with a very general ODE of the form,

$$D\ddot{\beta} + f(\beta) - \tau = \tau_c \tag{1}$$

In this equation we have a linear inertia term, where D could contain mass moment of inertia properties, for example. The second left-hand-side term $f(\beta)$ contains nonlinearities that appear in the equation, and τ represents the applied actuation torque, and this could be a harmonic excitation acceleration or force in the case of an oscillator, or some form of applied force or torque. The right-hand-side term τ_c is an additional imposed control torque, and it is this torque term that cancels out the nonlinearities and also introduces active gains operating on the velocity and displacement errors.

Feedback linearised control works like this in principle:

We define the control torque like this:

$$\tau_c = DV + f(\beta) - \tau \tag{2}$$

where,

$$V = -k_v\dot{\beta} - k_d\beta + r$$

where,

$$r = \ddot{\beta}_{sp} + k_v\dot{\beta}_{sp} + k_d\beta_{sp}$$

So,

$$V = -k_v\dot{\beta} - k_d\beta + \ddot{\beta}_{sp} + k_v\dot{\beta}_{sp} + k_d\beta_{sp}$$

Tidying up,

$$V = k_v(\dot{\beta}_{sp} - \dot{\beta}) + k_d(\beta_{sp} - \beta) + \ddot{\beta}_{sp} \tag{3}$$

This introduces active gains k_v and k_d which operate on the errors in the velocity and the displacement respectively, and the acceleration set point is also included as a reference value.

If we now substitute equation (3) into equation (2) we end up with this:

$$\tau_c = Dk_v(\dot{\beta}_{sp} - \dot{\beta}) + Dk_d(\beta_{sp} - \beta) + D\ddot{\beta}_{sp} + f(\beta) - \tau \tag{4}$$

Note that this control torque requires feedback of the displacement and velocity signals from the response of the system. If we apply this control torque to the system governed by the general differential equation for the system that we want to control we'll end up with the following error equation:

$$(\ddot{\beta} - \ddot{\beta}_{sp}) + k_v(\dot{\beta} - \dot{\beta}_{sp}) + k_d(\beta - \beta_{sp}) = 0 \tag{5}$$

In the case of the parametrically excited pendulum we have the following:

$$D = ml^2$$

$$f(\beta) = (mgl - mla\cos(\Omega t))\sin\beta$$

$$\tau = 0$$

So, if we substitute these forms into equation (4), as required, we obtain the control actuation torque for feedback linearisation control of the parametrically excited pendulum, as follows:

$$\tau_c = ml^2\left(k_v\left(\dot{\beta}_{sp} - \dot{\beta}\right) + k_d\left(\beta_{sp} - \beta\right) + \ddot{\beta}_{sp}\right) + ml(g - a\cos(\Omega t))\sin\beta \tag{6}$$

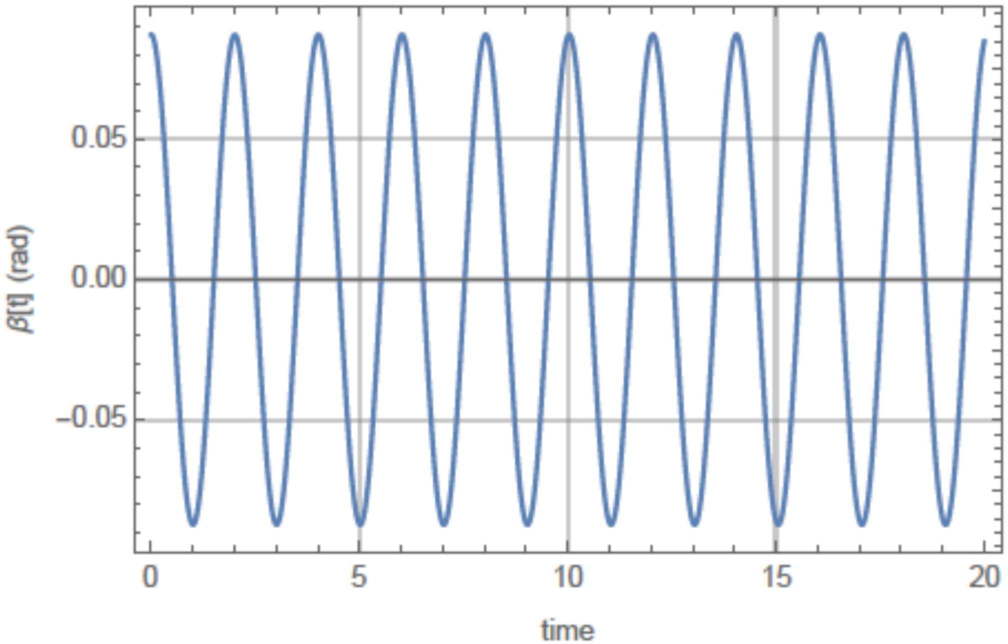
So, as in most control systems, the control torque is a function of the dynamically changing response of the system, both in terms of displacement and velocity. We can now examine this for a stable system, a metastable system, and an unstable system.

We go back to the parametrically excited pendulum analysis and using the same basic data again here we can examine the control actuation torque for the three principal cases mentioned above.

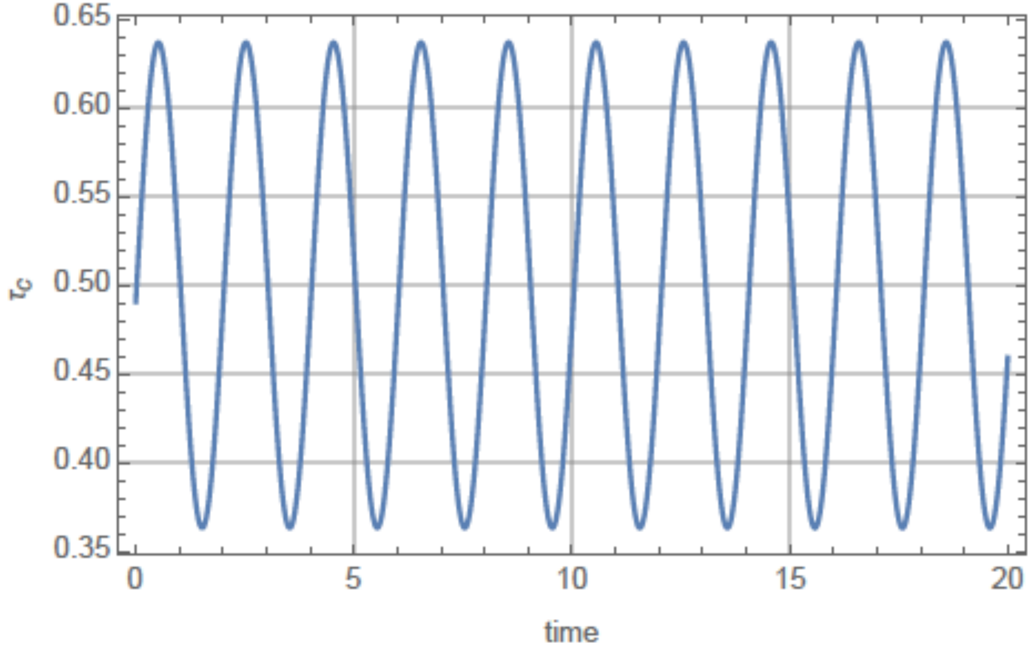
Data: $l = 1$m, $g = 9.81$m/s^2, $\epsilon = 0$ (this is a measure of 'detuning' from the perfect parametric resonance where $\Omega = 2\omega_n + \epsilon$, noting that $\omega_n = \sqrt{\frac{g}{l}}$ for the simple pendulum), $m = 0.5$ kg, $\ddot{\beta}_{sp} = 0$ rad/s^2, $\dot{\beta}_{sp} = 0$ rad/s, $\beta_{sp} = 0.1$ rad, $k_v = 1$, $k_d = 10$, integration end time $t_{end} = 20$ s.

We are only implementing a displacement set-point in this quick appraisal, and the two active gains, k_v and k_d are arbitrarily set here, with the emphasis placed on displacement control. The control torque τ_c is in the units of Nm here, so it's a real torque, and the implication is that it would be applied by a torsional actuator operating at the pivot of the pendulum.

We start with the stable case for which we could take $a = 0.01\mathrm{m/s^2}$.



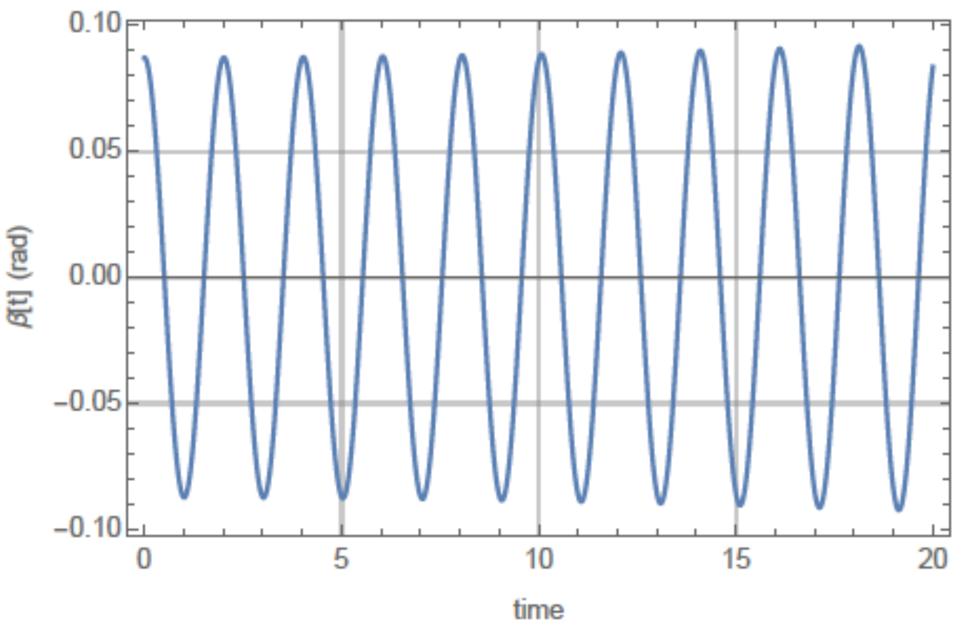
Response β against time, stable at ICs.

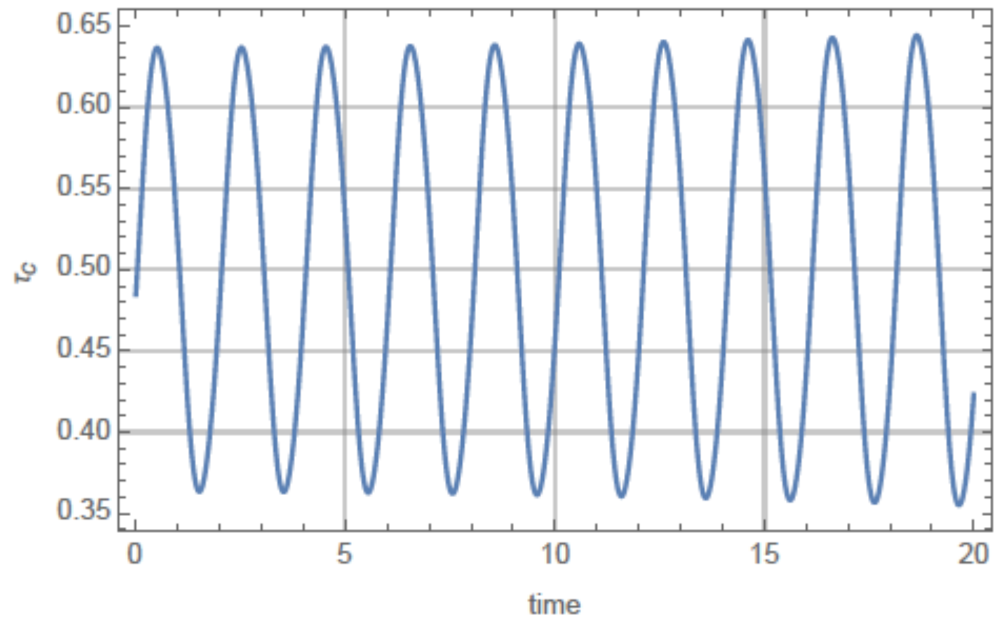


Control actuator torque τ_c against time.

Note that this numerical simulation is of the open loop system, so β is therefore not controlled here. The simulation just shows what the computed control actuator torque would be against time, for the stable case.

Then we move to the metastable case for which we find that $a \approx 0.15$ m/s².



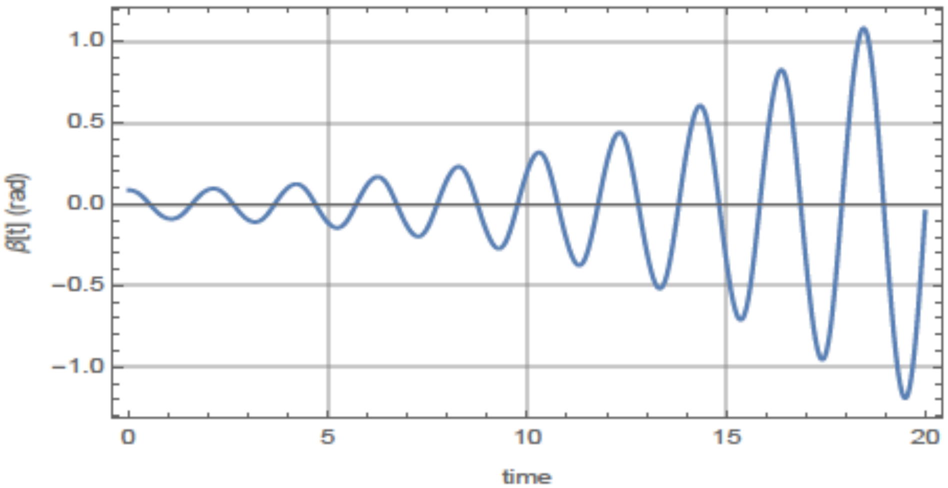


Response β against time, metastable at ICs.

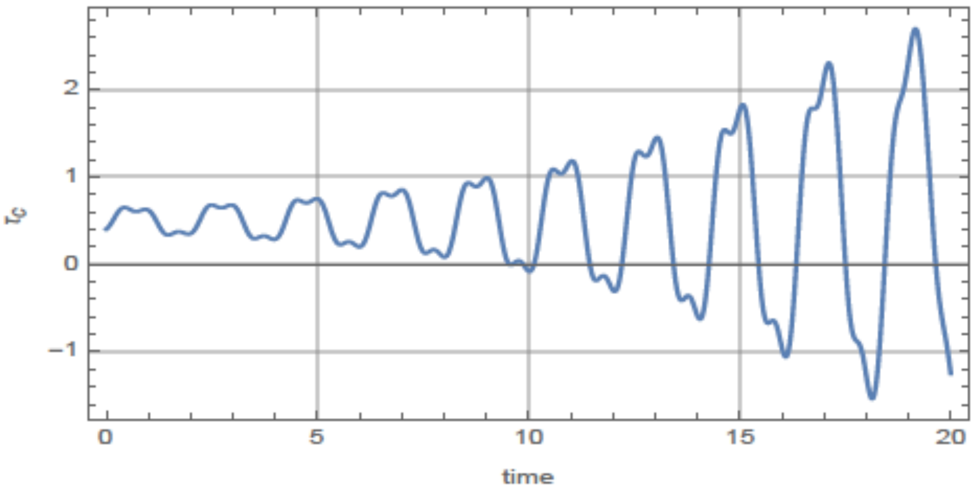
Control actuator torque τ_c against time.

Here we see that the response of the pendulum is just very slightly beginning to grow. It's maybe just over the metastability limit at this value of a. The control actuator torque mirrors this behaviour, very slightly growing in time. It's relevant to note that there's a phase lag between τ_c and β, so if the control was implemented it would, as one might expect, lag the response slightly.

Finally, we look at the unstable case, for which we can use $a = 2$ m/s².



β against time, unstable and growing from ICs.



Control actuator torque τ_c against time.

Again we see a phase lag, but the control torque now appears to be significantly different in form to the response. *If implemented as a control torque τ_c would bring β down close to the set-point $\beta_{sp} = 0.1$ rad, in time, and in the closed loop it would reduce very significantly as β reduces.*

We can also see how capable this nonlinear controller could be if we were to increase the excitation acceleration magnitude up to $a = 5\ \mathrm{m/s^2}$, for example. This is shown directly below. Note the very large displacement of the pendulum, and the vigorous control torque generated.