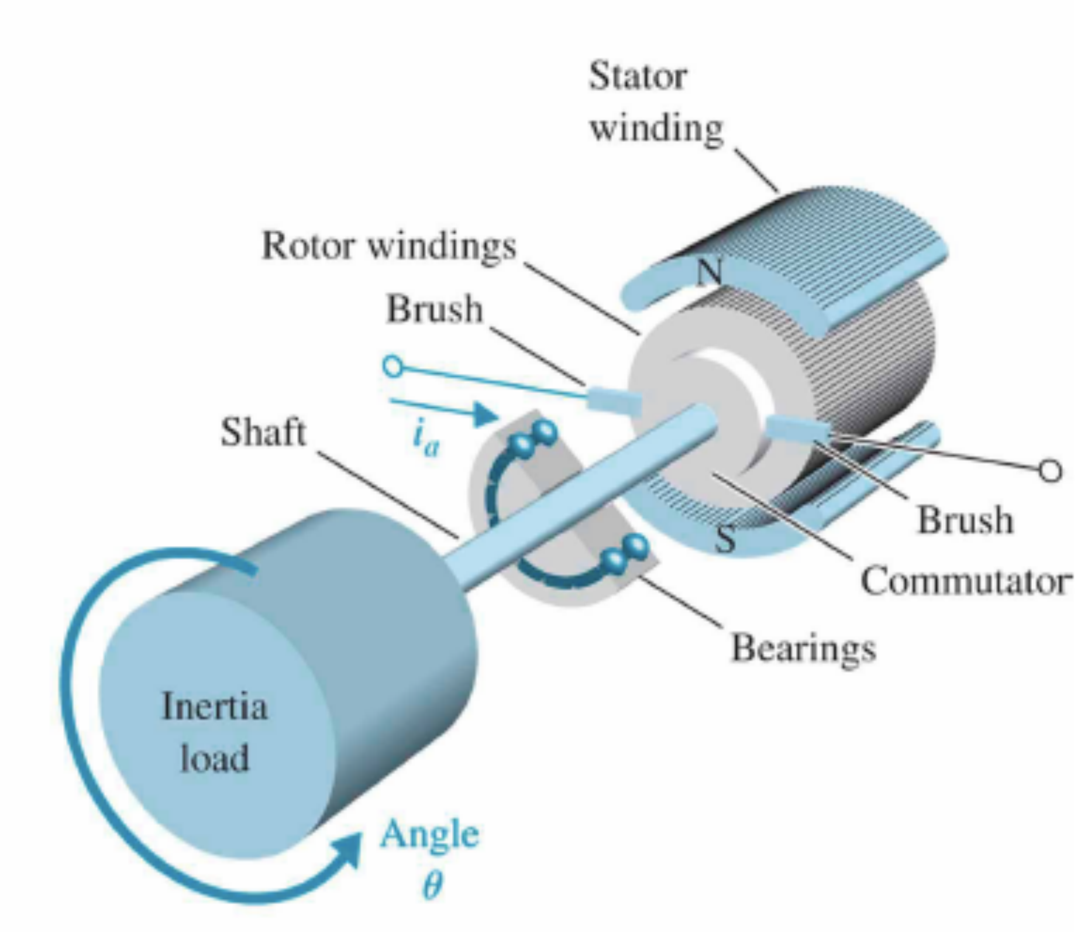
Lecture 2 – Part 2

Mathematical Modelling

Application example – DC electric motor

An important actuation device – the DC motor

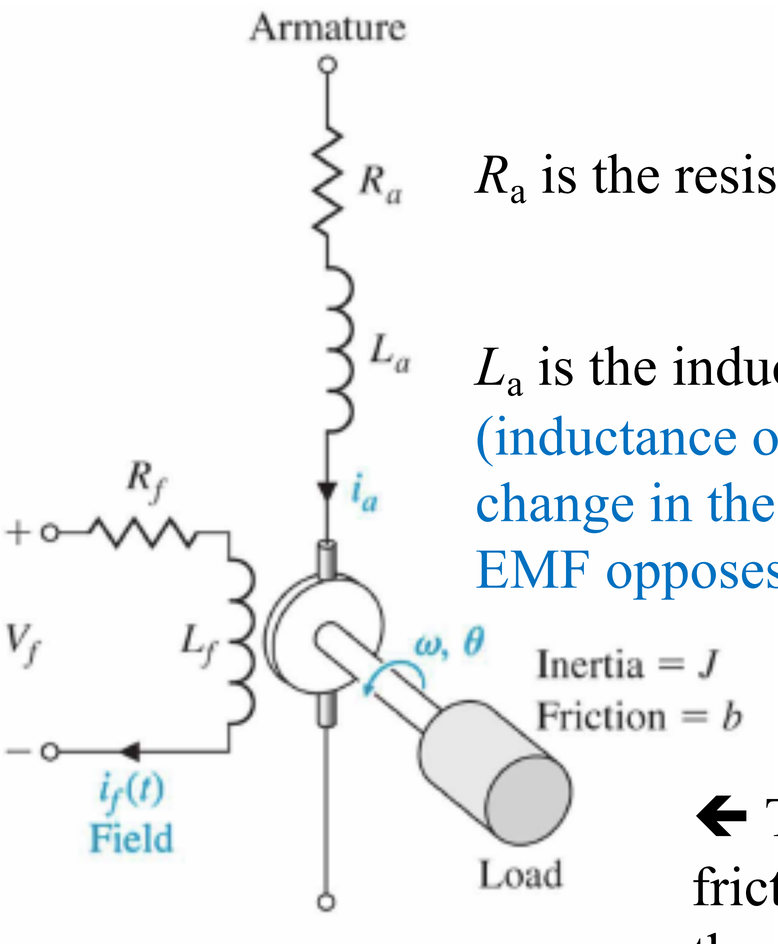


This motor converts DC current into mechanical rotation (in fact it is strictly speaking an energy convertor). Most of the torque available in the rotation is available to drive a load – an inertia of some sort. DC motors have some useful characteristics:

- high torque capability
- speed controllability
- compact and portable
- useful speed-torque output

Applications: robotic manipulator actuators, tape transports, disk drives, machine tools, servo-actuators

Assumptions when modelling DC motors – hysteresis (dependence of the state of a system on its history) and voltage drops across brushes are usually both ignored.



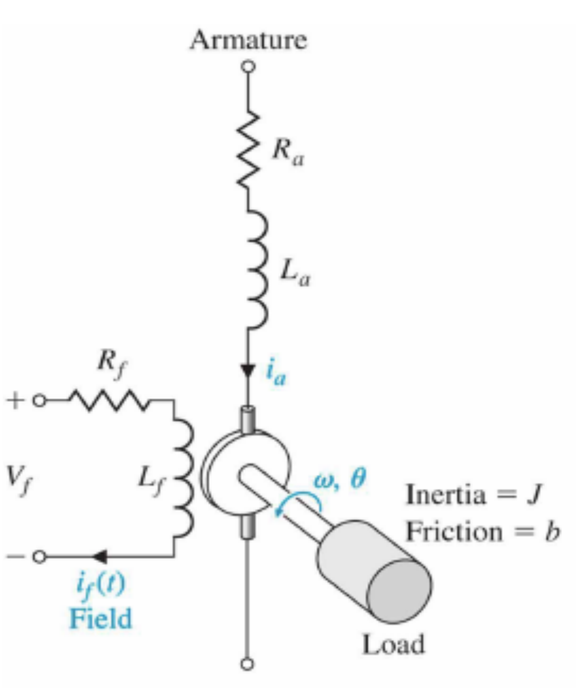
R_a is the resistance of the armature winding

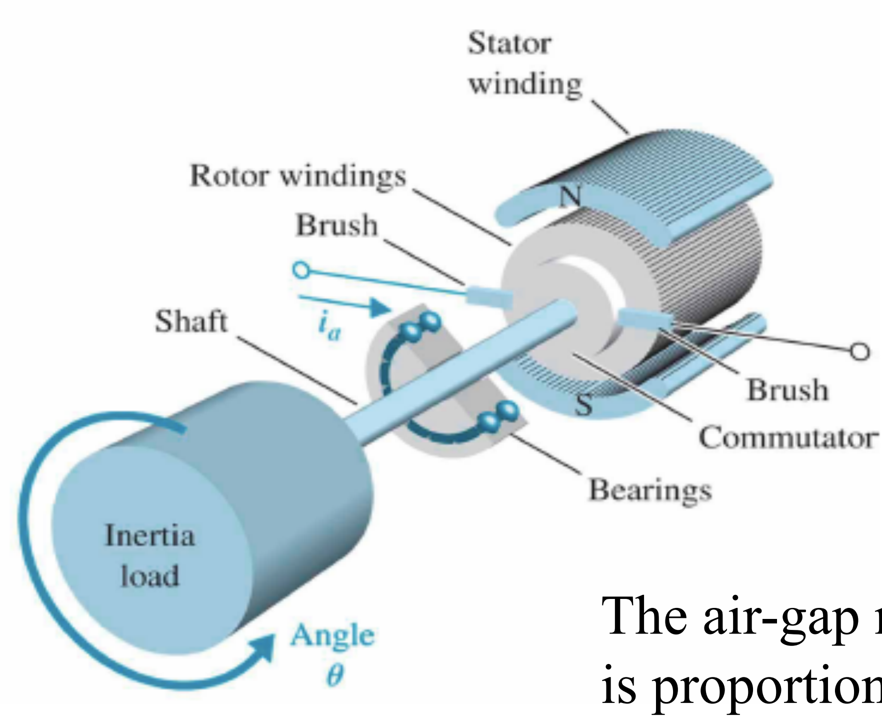
L_a is the inductance of the armature winding (inductance of a coil causes an EMF due to a change in the current flowing through it, this EMF opposes current flow)

← The mechanical load can be inertial and frictional in nature, and is external, connected to the motor shaft by a coupling.

↑

The magnetic field in the stator can be obtained from a permanent magnet, or, as shown here, from a field coil winding. The field coil DC motor is more versatile and controllable so we will work on that variant from here on.





← Note that there is an air gap between the rotor and the stator.

The air-gap magnetic flux is proportional to the field current.

So, the air-gap magnetic flux is given by

$$\phi = K_f i_f$$

where i_f is the current in the field winding (Amperes, A).

DC motor torque is linearly proportional both to air-gap magnetic flux and armature current, i_a. Both the field and armature currents are necessarily time variant, so they are designated by $i_f(t)$ and $i_a(t)$.

Therefore,

$$T_m = K_1 \phi i_a(t) = K_1 K_f i_f(t) i_a(t)$$

Repeating the motor torque equation:

$$T_m = K_1 \phi i_a(t) = K_1 K_f i_f(t) i_a(t)$$

we see that we can only vary one current at a time for a linear element for which we can obtain a transfer function, so we can use the field current as the input, for example, and then take the LT:

$$T_m(s) = (K_1 K_f I_a) I_f(s)$$

noting that the armature current in the time domain, $i_a(t)$, becomes denoted by a constant I_a in the s domain.

We can aggregate the constants to get:

$$T_m(s) = K_m I_f(s)$$

where K_m is a motor constant, specific to the electromechanical design of the motor itself.

Now we have to relate field current to field voltage because it is the voltage applied to the winding that is usually used to control the current flow in the wire of the winding. Given that coils possess inductance as well as resistance we apply both Ohm's law and also the fundamental relationship between inductance, voltage, and current, in order to get the LT. These are brought together by the *Kirchhoff voltage equation*, which we can apply directly in LT form.

The Kirchhoff voltage equation for the field coil – this being an inductor (the formal name for a coil or a winding) is:

$$V_f(s) = (R_f + L_f s) I_f(s)$$

Also we need to be able to represent the actual situation with the motor torque. We can divide it into two parts, the part associated with the external load, and the part associated with any disturbances acting on the motor. A disturbance could be wind loading on a motor driven antenna for example.

So, we can write:

$$T_m(s) = T_L(s) + T_d(s)$$

Since the mechanical load on our motor is both inertial and frictional we can state it as:

$$T_L(s) = Js^2\theta(s) + fs\theta(s)$$

Why? Because we have an inertia term and a damping (or friction) term, written in LT form.

From this we get:

$$\theta(s) = \frac{T_L(s)}{(Js^2 + fs)} = \frac{T_L(s)}{s(Js + f)} = \frac{T_m(s) - T_d(s)}{s(Js + f)}$$

We take the motor-load combination on its own now and then insert the disturbance later, within the block diagram.

So, just for the motor-load combination, we get:

$\theta(s) = \frac{T_m(s)}{s(Js+f)}$

We have already established that for a field controlled motor we have:

$$T_m(s) = K_m I_f(s)$$

and we recall that Kirchhoff gave us:

$$V_f(s) = (R_f + L_f s) I_f(s)$$

So, $\quad T_m(s) = \frac{K_m V_f(s)}{(R_f + L_f s)}$

Therefore:

$$\theta(s) = \frac{K_m V_f(s)}{s(Js + f)(L_f s + R_f)}$$

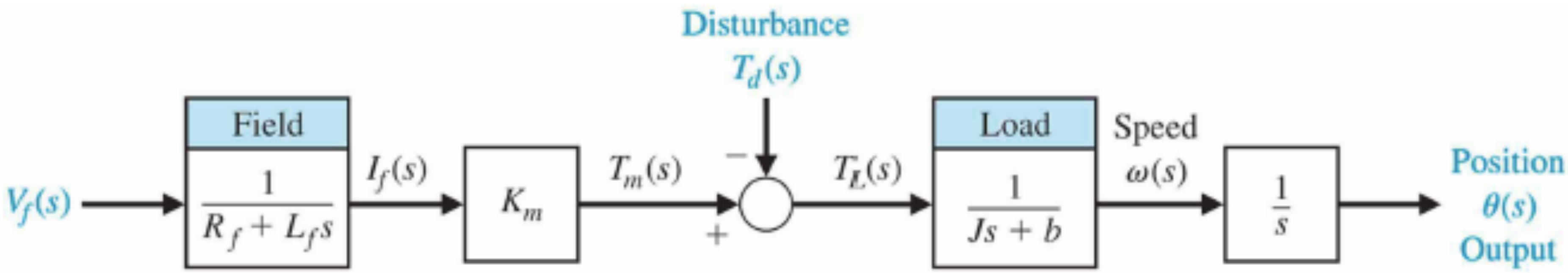
So the transfer function of the motor-load combination is given by:

$$\frac{\theta(s)}{V_f(s)} = G(s) = \frac{K_m}{s(Js+f)(L_f s+R)}$$

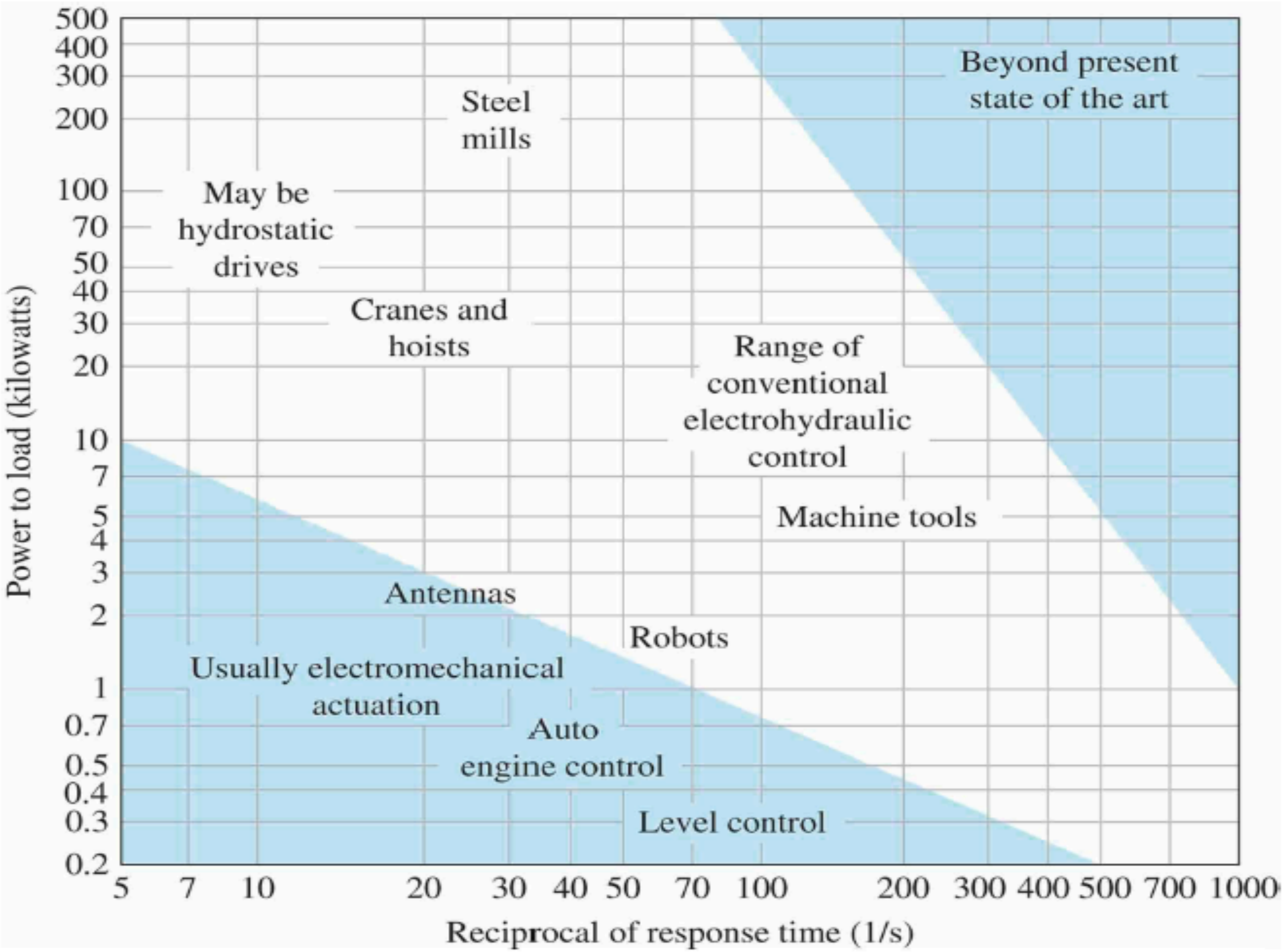
It is very important to note that this is the overall TF for the whole motor-load system, thinking of that whole system as one block.

$$V_f(s) \rightarrow \boxed{\frac{K_m}{s(Js+f)(L_f s+R)}} \rightarrow \theta(s)$$

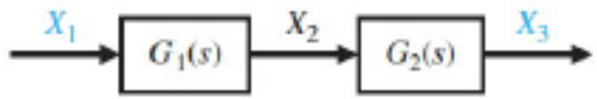
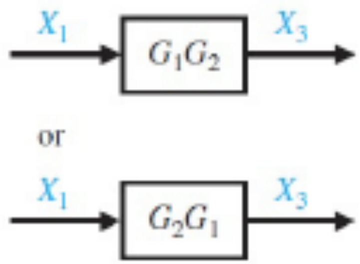
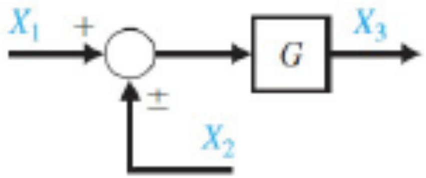
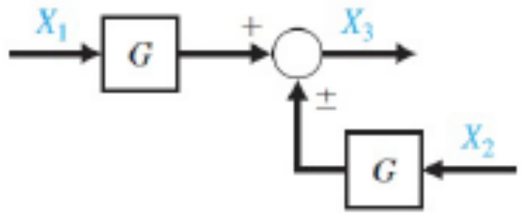
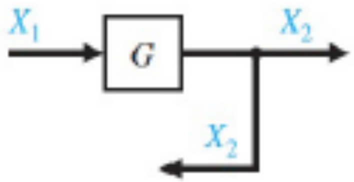
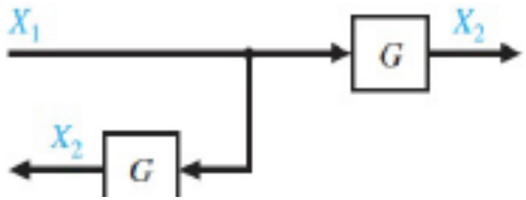
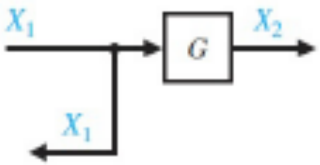
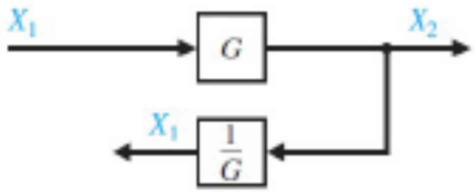
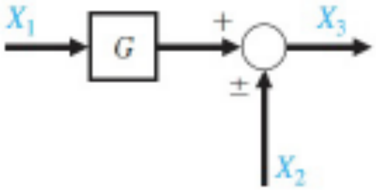
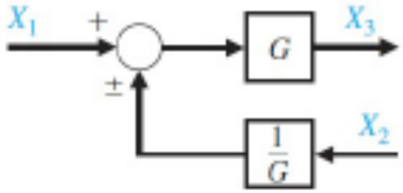
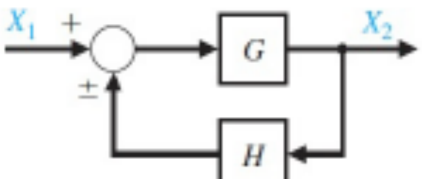
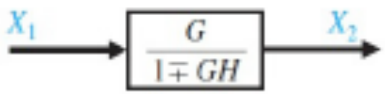
We can break that unified block diagram down into a component part block diagram, as shown below, and also add in the disturbing torque. We have to go back to the various equations in these notes to do this.



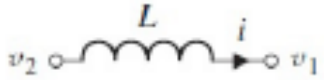
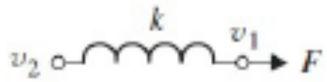
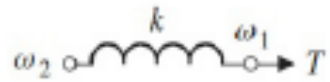
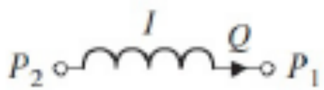
The best way to think about such a block diagram is that each TF represents a physical part of the system, or a necessary mathematical construct needed to go from one thing to another (i.e. the 1/s block that converts speed to position). Note that this is open loop, requiring the Kirchhoff voltage equation, the motor constant, and the load model to be correct. If so then all is fine.

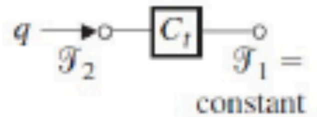
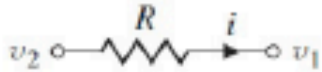
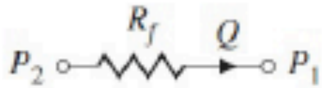
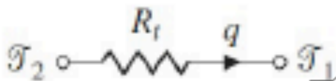


Relationship between load power and response time for different actuators

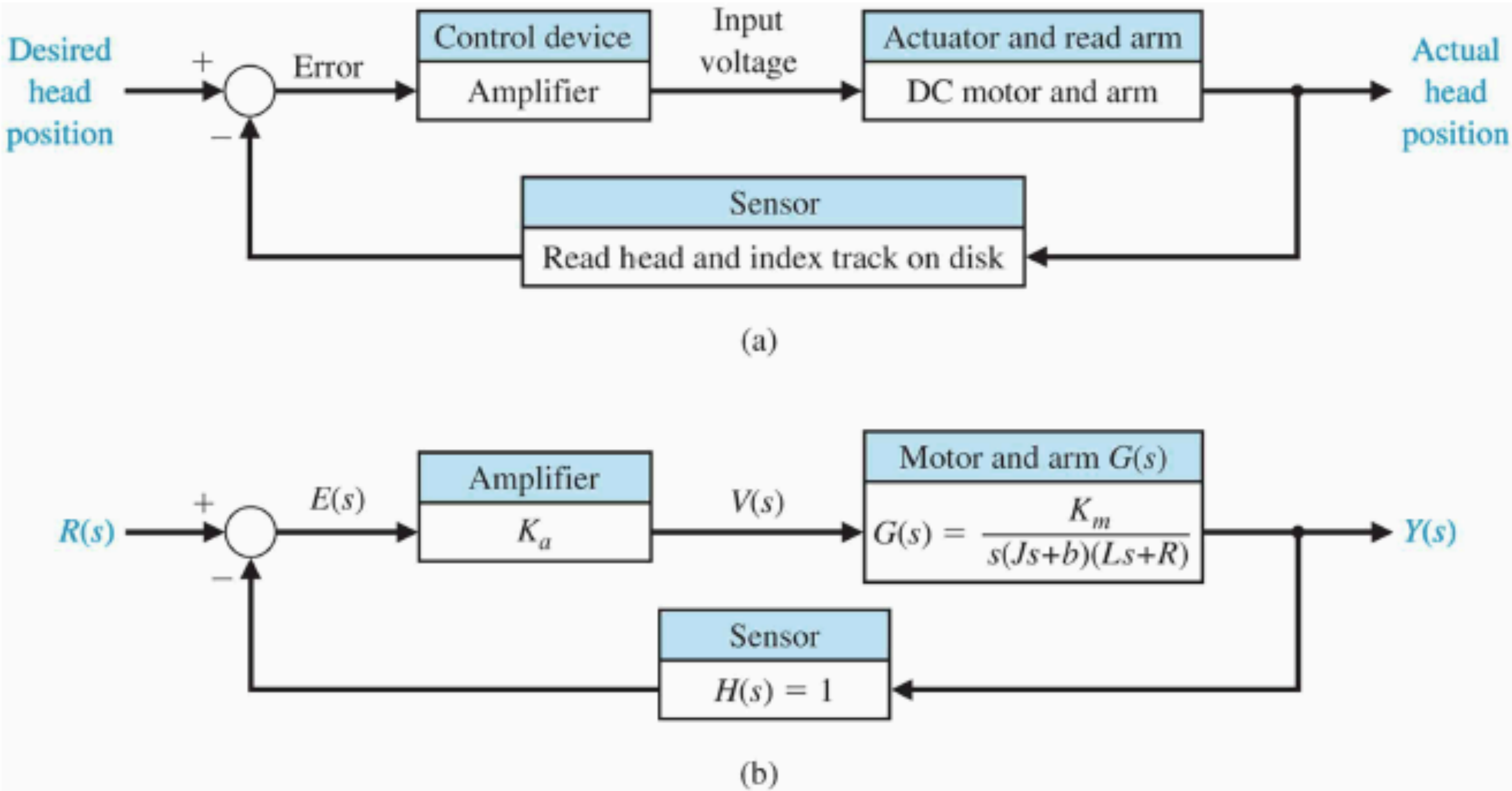
Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

Electromechanical systems – differential equations for elements in terms of energy and power

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\dfrac{di}{dt}$	$E = \dfrac{1}{2}Li^2$	
	Translational spring	$v_{21} = \dfrac{1}{k}\dfrac{dF}{dt}$	$E = \dfrac{1}{2}\dfrac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \dfrac{1}{k}\dfrac{dT}{dt}$	$E = \dfrac{1}{2}\dfrac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\dfrac{dQ}{dt}$	$E = \dfrac{1}{2}IQ^2$	

Capacitive storage	Electrical capacitance	$i = C\dfrac{dv_{21}}{dt}$	$E = \dfrac{1}{2}Cv_{21}^2$	
	Translational mass	$F = M\dfrac{dv_2}{dt}$	$E = \dfrac{1}{2}Mv_2^2$	
	Rotational mass	$T = J\dfrac{d\omega_2}{dt}$	$E = \dfrac{1}{2}J\omega_2^2$	
	Fluid capacitance	$Q = C_f\dfrac{dP_{21}}{dt}$	$E = \dfrac{1}{2}C_f P_{21}^2$	
	Thermal capacitance	$q = C_t\dfrac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \dfrac{1}{R}v_{21}$	$\mathcal{P} = \dfrac{1}{R}v_{21}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}^2$	
	Fluid resistance	$Q = \dfrac{1}{R_f}P_{21}$	$\mathcal{P} = \dfrac{1}{R_f}P_{21}^2$	
	Thermal resistance	$q = \dfrac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \dfrac{1}{R_t}\mathcal{T}_{21}$	

Block diagram for a disk drive read system



Upper diagram is in terms of technological functionality required, and the lower diagram is the mathematical representation of that technological functionality