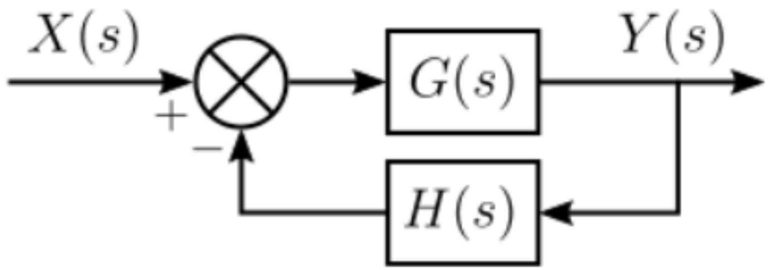
Lecture 7

Development of the theory of the Root Locus Method - 1

Recapping from Lecture 6:



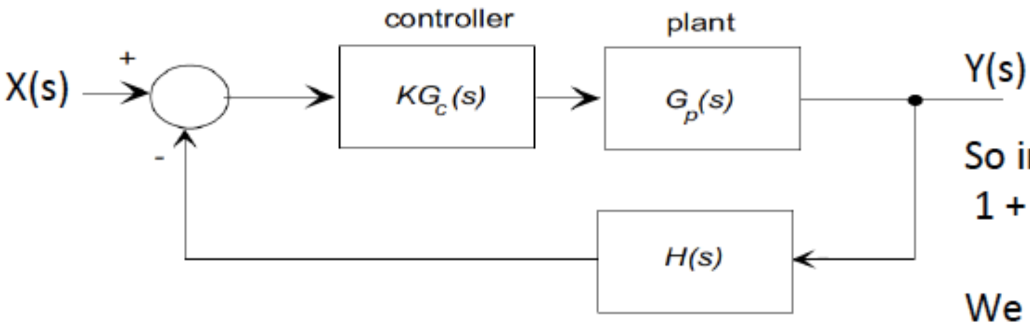
The closed loop transfer function for the basic negative feedback control system is:

$$T(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation is 1 + G(s)H(s) = 0, which can be written as 1 + F(s) = 0 so G(s)H(s) = F(s) = -1 which, because F(s) is complex, leads to:

$$|F(s)|\angle F(s) = -1$$

Making this more generalised, by taking a system where the control and plant TFs are separated, and with variable controller gain:



So in this case 1 + F(s) = 0 is actually $1 + KG_c(s)G_p(s)H(s) = 0$

We know that |F(s)|=1 and >F(s)= - 1 and this can now be written more formally in polar coordinates.

$$F(s) = |F(s)|\angle F(s) = 1 * e^{j(2n+1)\pi}, \quad n = 0,1,2,3,4, \ldots$$

where $e^{j(2n+1)\pi} = \cos\big((2n+1)\pi\big) + j sin\big((2n+1)\pi\big) = -1 + j0$

Testing this numerically:

n	(2n+1)π (radians)	cos(2n+1)π	Value of cos(2n+1)π
0	π	cos π	-1
1	3π	cos (3π)	-1
2	5π	cos (5π)	-1
3	7π	cos (7π)	-1
4	9π	cos (9π)	-1
5	11π	cos (11π)	-1

Therefore the angle criterion required to satisfy the characteristic equation is:

$\angle F(s) = (2n+1)\pi$ radians or $\angle F(s) = (2n+1)180$ degrees

The Magnitude Condition is: $|F(s)| = 1$

The Angle Condition is: $\angle F(s) = (2n+1)\pi$ → -1

These two conditions are necessary to define a point (any point) on the root locus, which is given generally by: $s = \sigma + j\omega$.

If the point of interest satisfies the angle condition then the magnitude condition is used to determine the gain K associated with that point. This has great practical consequence.

This is because if we state that $F(s) = KG_c(s)G_P(s)H(s)$ where we simplify by writing $G(s) = G_c(s)G_P(s)H(s)$, then we can state that:

$$F(s) = KG(s)$$

The magnitude condition gives us: $|F(s)| = |KG(s)| = 1$

This means that we can relate the gain K to G(s) by means of this condition:

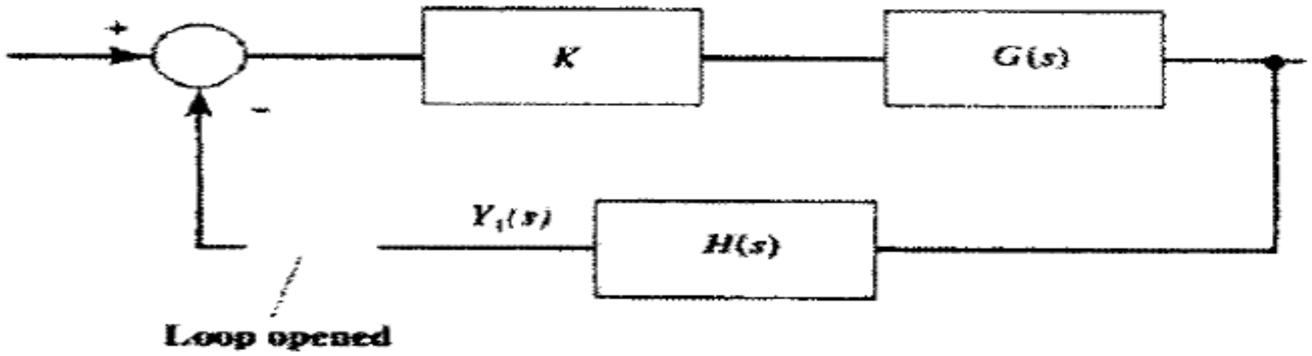
$$K = \frac{1}{|G(s)|}$$

One thing that we need to note is that the G(s) here is in fact the Open Loop Transfer Function, OLTF, (but for the closed loop system). We check this out next.

For the closed loop system the overall closed loop transfer function is given by:

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

If we open the feedback loop after the feedback block, we get this:



The feedback loop opening point is $Y_1(s)$ so we can write the OLTF for this system above as:

$$\frac{Y_1(s)}{X(s)} = KG(s)H(s)$$

So, in our main example we have

$$G(s) = G_c(s)G_P(s)H(s)$$

as the OLTF.

So, the gain which satisfies the magnitude condition $K = \frac{1}{|G(s)|}$ is the reciprocal of the magnitude of the OLTF.

Numerical examples – we can now use the above theory to do some testing of arbitrary points for a given OLTF to see if they are on that particular root locus.

Taking an OLTF of the following form:

$$G(s) = \frac{1}{(s+s)(s+4)}$$

The arbitrarily selected points are: $s = -1, \quad s = -3.5, \quad s = -3 + j5$

(a) $s = -1$ Therefore $G(s) = \frac{1}{(-1+2)(-1+4)} = \frac{1}{3}$

The fundamental condition is $KG(s) = -1$ which gave rise to the magnitude condition $|KG(s)| = 1$ and the angle condition $\angle(G(s)) = -1$

So, for $K > 0$ we get $KG(s) = \frac{K}{3}$ which is +ve, so for this point $KG(s) \neq -1$ so it is not on the root locus.

(b) $s = -3.5$ Therefore $G(s) = \frac{1}{(-3.5+2)(-3.5+4)} = \frac{1}{(-1.5)(0.5)} = \frac{-1}{0.75}$

So, $KG(s) = \frac{-K}{0.75}$ and if $K = 0.75$ then $KG(s) = -1$ which indicates that this point is on the root locus.

Exercise

Can you analyse the case for: $s = -3 + j5$?

$$s = -3 + j5$$

$$KG(s) = \frac{K}{\big((-3 + j5) + 2\big)\big((-3 + j5) + 4\big)} = \frac{K}{(-1 + j5)(1 + j5)}$$

$$= \frac{K}{-1 - j5 + j5 + j^2 25} = \frac{K}{-1 - 25} = \frac{K}{-26}$$

Therefore $KG(s) = -1$ for $K = 26$, indicating that the point defined by $s = -3 + j5$ also lies on the root locus.

Geometric Interpretation of the Transfer Function on the Complex Plane

We can evaluate a transfer function for any s where $s = \sigma + j\omega$ and we have already established that we can express the complex value of a transfer function in polar form, as a magnitude and as an angle. Using the notation of the previous general example:

$$G(s) = |G(s)|e^{j(2n+1)\pi} = |G(s)|e^{j\angle G(s)}$$

where the magnitude and angle are, respectively,

$$|G(s)| = \sqrt{Re\{G(s)\}^2 + Im\{G(s)\}^2}$$

$$\angle G(s) = \tan^{-1}\left(\frac{Im\{G(s)\}}{Re\{G(s)\}}\right)$$

(irrespective of whether the values calculated satisfy the magnitude and angle criteria).

We already know that the zeros are the roots of the numerator and the poles are the roots of the denominator, in a general transfer function. So, if we assume that the numerator and denominator of the TF can be factorised into terms $(s - z_i)$ and $(s - p_i)$ respectively then we can write the TF in this form:

$$G(s) = C\,\frac{(s - z_1)(s - z_2)\ \dots\ (s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)\ \dots\ (s - p_{n-1})(s - p_n)}$$

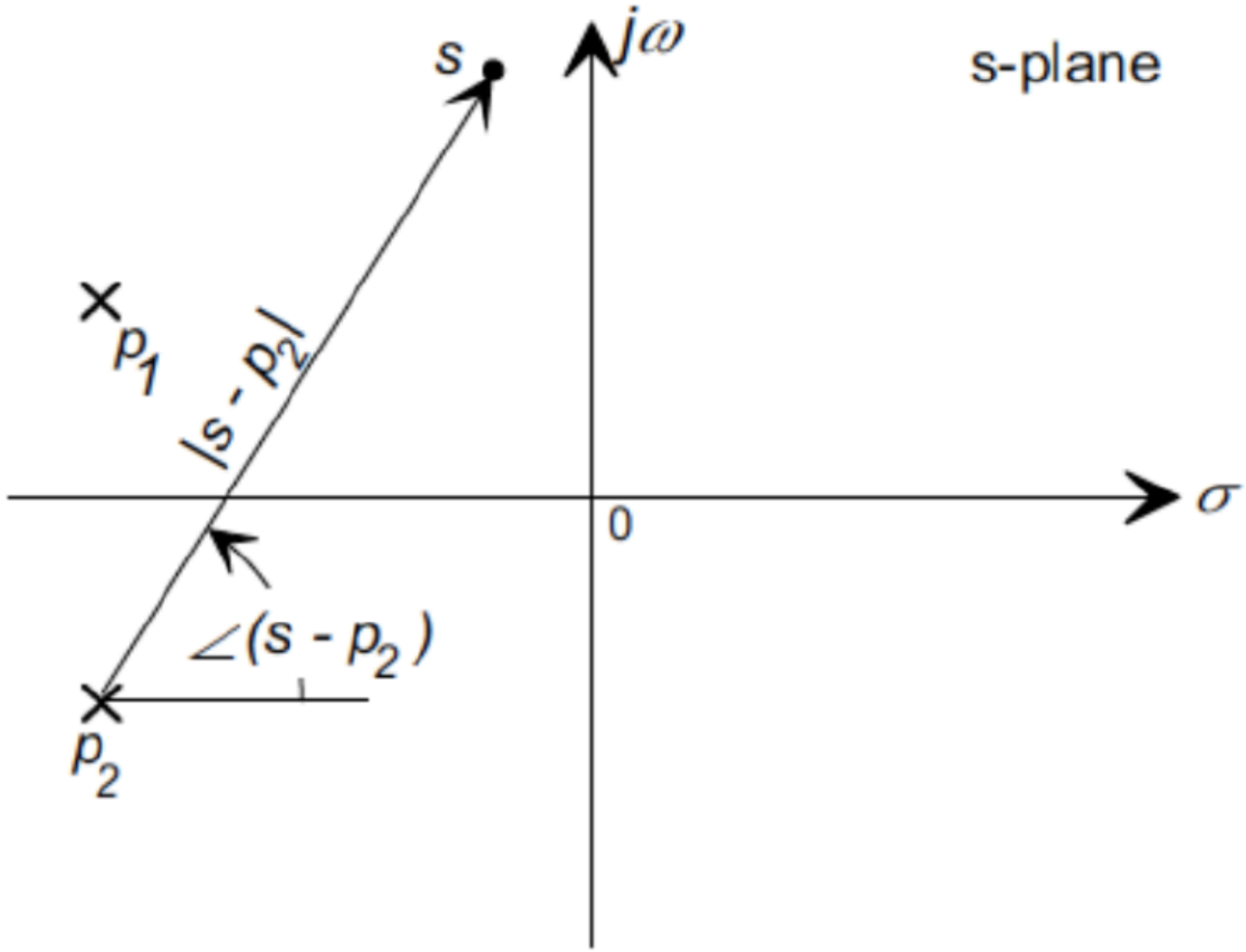
where C is a constant and each of the factors in the numerator and denominator is a complex quantity.

Each factor in the numerator and denominator is a complex quantity and can be geometrically interpreted as a vector in the complex plane, originating from the zero z_i or the pole p_i and directed to the point s at which the function is to be evaluated. Each of these vectors can be written in polar form in terms of a magnitude and an angle. For example, for a pole $p_i = \sigma_i + j\omega_i$, the magnitude and angle of the vector to the point $s = \sigma + j\omega$ are:

$$|s - p_i| = \sqrt{(\sigma - \sigma_i)^2 + (\omega - \omega_i)^2}$$

$$\angle(s - p_i) = \tan^{-1}\left(\frac{\omega - \omega_i}{\sigma - \sigma_i}\right)$$

This is how this looks geometrically.



We can generalise this geometrical interpretation by recalling a bit of complex algebra:

- the magnitude of the product of two complex quantities is the product of the individual magnitudes, so:

$$|ab| = |a||b| \text{ and also } \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

- the angle of the product of two complex quantities is the sum of the component angles, so:

$$\angle(ab) = \angle a + \angle b \text{ and also } \angle\left(\frac{a}{b}\right) = \angle a - \angle b$$

The magnitude and angle of the complete transfer function may then be written as:

$$|G(s)| = C\,\frac{\prod_{i=1}^{m}|s - z_i|}{\prod_{i=1}^{n}|s - p_i|}$$

$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} (\angle(s - p_i)$$