Lecture 4 – Part 2

Performance and stability -

The *Routh Hurwitz* Stability Criterion

Absolute Stability – according to *the Routh-Hurwitz stability criterion*

We have established that a system will be stable if the real parts of the roots of the characteristic equation are negative. The problem is that calculating these roots to check this can be very time consuming.

E.J.Routh (Cambridge, UK) and A.Hurwitz (Hanover, Germany) both independently proposed a similar method for doing this which indicates the presence and number of unstable roots in a characteristic equation *but not their value*.

The Routh-Hurwitz stability criterion is a necessary and sufficient criterion for the stability of linear systems, and it is based on ordering the coefficients of the characteristic equation into an array. We then look at the way the signs change in the first column of the array, on the basis that the criterion is set up to equate this to the number of roots of the characteristic equation with positive real parts.

We start off by writing down a completely generalised characteristic equation:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

This can be written in *array form*, in such a way that the number of sign changes in the first column shows the number of roots with positive real parts.

$$\begin{array}{lllllll} s^n & : & a_0 & a_2 & a_4 & a_6 & \ldots \\ s^{n-1} & : & a_1 & a_3 & a_5 & a_7 & \ldots \\ s^{n-2} & : & b_1 & b_2 & b_3 & \ldots & \\ s^{n-3} & : & c_1 & c_2 & \ldots & & \\ s^{n-4} & : & d_1 & \ldots & & & \end{array}$$

When complete, the array consists of $n + 1$ rows.

where the following apply:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad \ldots$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \qquad \ldots$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} \qquad \ldots$$

Examples show best how to use the R-H stability criterion in practice.

1. A fourth order characteristic equation – we need to find if this represents a stable system or not.

$$s^4 + 2s^3 + 6s^2 + 7s + 5 = 0$$

We start by noting that $n = 4$, and then we compare our equation above with the general characteristic equation from which the array elements are calculated.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

So, $a_0 = 1, \quad a_2 = 6, \quad a_4 = 5, \quad a_1 = 2, \quad a_3 = 7$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{2*6-1*7}{2} = 2.5$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = \frac{2*5-1*0}{2} = 5$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{2.5*7-2*5}{2.5} = 3$$

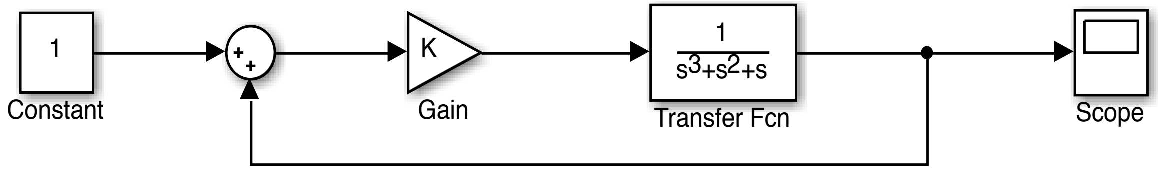
$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} = \frac{3*5-2.5*0}{3} = 5$$

So, the R-H array now looks like this:

$$\begin{array}{ll} s^n & : a_0 \;\; a_2 \;\; a_4 \;\; a_6 \;\; \ldots \\ s^{n-1} & : a_1 \;\; a_3 \;\; a_5 \;\; a_7 \;\; \ldots \\ s^{n-2} & : b_1 \;\; b_2 \;\; b_3 \;\; \ldots \\ s^{n-3} & : c_1 \;\; c_2 \;\; \ldots \\ s^{n-4} & : d_1 \;\; \ldots \end{array} \qquad\qquad \begin{array}{ll} s^n & : 1 \;\; 6 \;\; 5 \;\; \ldots \\ s^{n-1} & : 2 \;\; 7 \;\; \ldots \\ s^{n-2} & : 2.5 \;\; 5 \;\; \ldots \\ s^{n-3} & : 3 \;\; \ldots \\ s^{n-4} & : 5 \;\; \ldots \end{array}$$

We see that the first column is all positive with no sign changes, so the characteristic equation's roots all have negative real parts.

2. Use R-H to determine the range of K for which this control system is stable:



Steps:

- Work out the overall systems transfer function
- Take the denominator as the characteristic equation, and set this to zero
- Decide on the number of rows for the Routh-Hurwitz array (i.e. define *n*)
- Calculate the coefficients for the array (using the formulas in this lecture)
- Then look at the first column and decide what *K* has to be to ensure that there are no sign changes
- Also note the value of *K* necessary for instability.

The system transfer function is given by:

$$\frac{G(s)}{1+G(s)H(s)}$$

and in this case that is:

$$\frac{K}{(s^3+s^2+s)\left(1+\frac{K}{(s^3+s^2+s)}\right)}=\frac{K}{s^3+s^2+s+K}$$

Therefore the characteristic equation is this:

$$s^3+s^2+s+K=0$$

We compare this with:

$$a_0s^3+a_1s^2+a_2s+a_3=0$$

From which we can get the a_i coefficients directly, and also we see that $n = 3$.

We then establish the R-H array:

s^3: a_0 a_2
s^2: a_1 a_3
s^1: b_1
s^0: c_1

We have to calculate b_1 and c_1

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{1 * 1 - 1 * K}{1} = 1 - K$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{(1 - K)K - 1 * 0}{(1 - K)} = K$$

First column is therefore:

1
1
1 - K
K

First column is therefore:

1
1
1 - K
K

So, for stability we need $1 - K > 0$ from the third row and $K > 0$ from the fourth row, so:

$$0 < K < 1$$

for stability.

Where do we go next with stability?

In the next lecture we will discuss the conditions if a system is *Lyapunov* stable, asymptotically stable, exponentially stable, or simply unstable.

Then we will continue to analyse the stability of a system, moving on from the Routh-Hurwitz method, using *Root Locus plots*, and concluding with the *Bode stability criterion*.