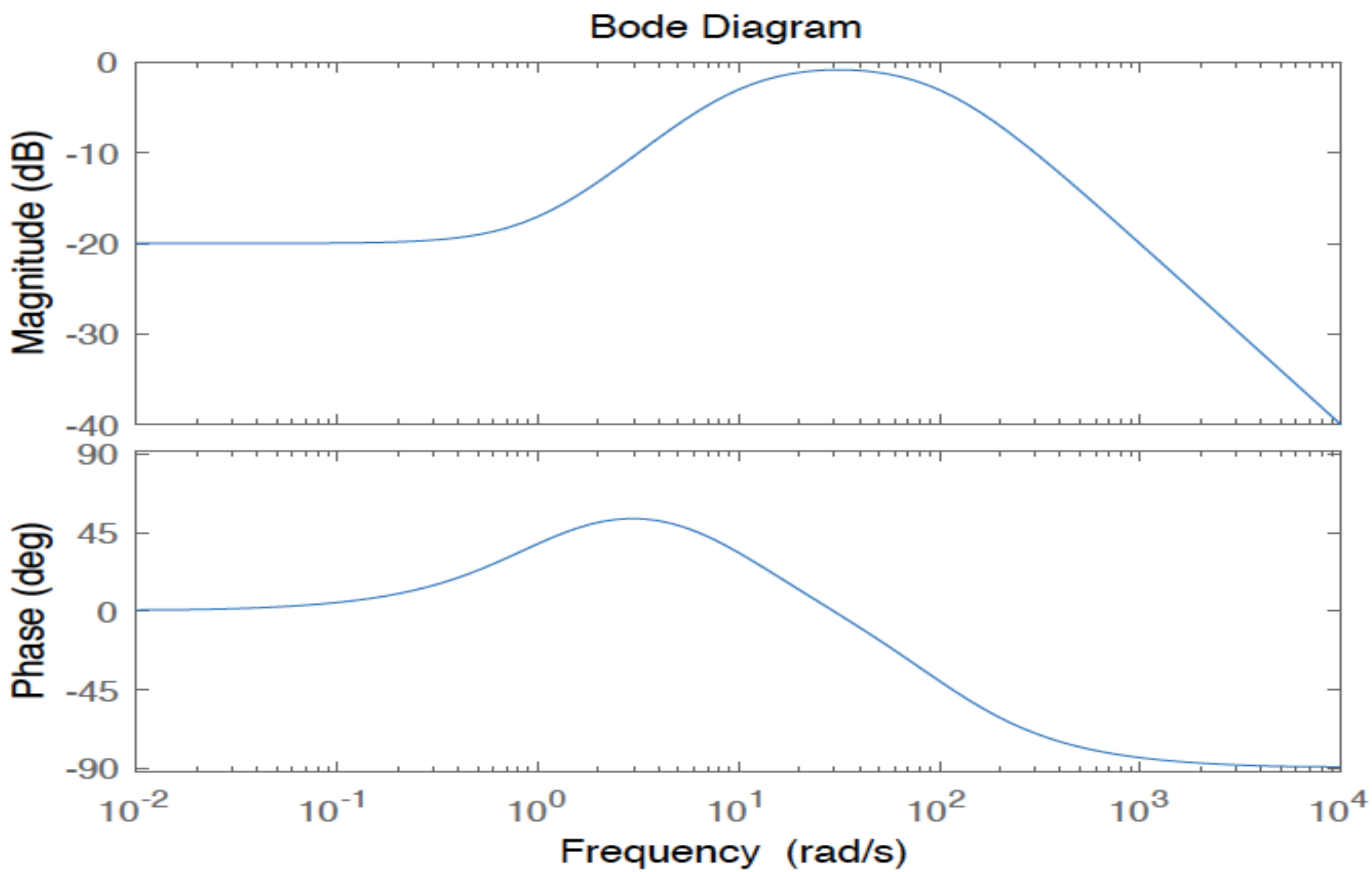
The Bode Stability Criterion – Part 1

Bode plots are a very useful way to represent the magnitude (gain) and phase of a system as a function of frequency. They can also tell us about stability. The gain and phase response of a system is referred to as its **frequency domain** behaviour. In this lecture we summarise the basic frequency domain characteristics obtainable from the Bode concept. It's easiest to examine this by taking an arbitrary transfer function, such as:

$$H(s) = \frac{100(s+1)}{(s+10)(s+100)} = \frac{100(s+1)}{(s^2+110s+1000)}$$

Then we wish to know how we can display this function. The most useful way is to display the function with two plots, the first showing the magnitude of the transfer function and the second showing its phase. One way to do this is by simply entering many values for the frequency (with s = $j\omega$), calculating the magnitude and phase at each frequency and then displaying them. If we do this we get plots like these shown below.



s=jω is chosen because that allows us to convert the Laplace transform into a Fourier transform, as is normally used for frequency domain representation. In the Laplace Transform **s** is a complex variable, but the Fourier representation just requires the imaginary component, hence σ = 0.

The locations of poles and zeros affect the shape of the plots, and then we can start to predict how a system behaves in the frequency domain by looking at its transfer function.

We start this off by considering the so-called proper form for the Transfer Function – this is when the lowest order terms in the numerator and denominator are both unity. This simplifies the arithmetic to come because it reduces the terms to be analysed to two types, a constant and a logarithmic term.

Here is the previous example:

$$H(s) = 100\frac{s+1}{(s+10)(s+100)}$$

$$H(s) = 100\frac{1+\frac{s}{1}}{10(1+\frac{s}{10})100(1+\frac{s}{100})}$$

$$H(s) = 0.1\frac{1+\frac{s}{1}}{(1+\frac{s}{10})(1+\frac{s}{100})}$$

We draw the magnitude and phase of the TF by using $s = j\omega$.

Before we start we note that the TF is made up of four terms, a constant of 0.1, a zero (at $s = -1$) and two poles (at $s = -10$ and $s = -100$).

We then re-write the TF, substituting for *s*, as four phasors (recall the root locus theory):

$$H(s) = 0.1\frac{1+\frac{j\omega}{1}}{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{100}\right)} =$$

$$[|0.1|\angle(0.1)]\,\frac{\left[\left|1+\frac{j\omega}{1}\right|\angle(1+\frac{j\omega}{1})\right]}{\left[\left|1+\frac{j\omega}{10}\right|\angle(1+\frac{j\omega}{10})\right]\left[\left|1+\frac{j\omega}{100}\right|\angle(1+\frac{j\omega}{100})\right]}$$

We can separate the magnitude and the angle parts out into a single phasor each, simply by using the general fact that $H(s) = H(j\omega) = |H(j\omega)|\angle H(j\omega)$.

$$H(j\omega) = \left(|0.1|\frac{\left|1+\frac{j\omega}{1}\right|}{\left|1+\frac{j\omega}{10}\right|\left|1+\frac{j\omega}{100}\right|}\right)\left(\angle(0.1) + \angle\left(1+\frac{j\omega}{1}\right) - \angle\left(1+\frac{j\omega}{10}\right) - \angle\left(1+\frac{j\omega}{100}\right)\right)$$

It is relatively easy to imagine drawing the phase because the phase terms are added, but the magnitude terms are multiplied. It would be much easier if they could be added, then we could draw each term on a graph and simply add them. There is a way to solve this problem!

We use logarithms to transform multiplication into addition:

$$X = 20log_{10}(Q)$$

noting that this converts Q into dB.

So:

$$20log_{10}(|H(j\omega)| = 20log_{10}\left(|0.1|\frac{\left|1+\frac{j\omega}{1}\right|}{\left|1+\frac{j\omega}{10}\right|\left|1+\frac{j\omega}{100}\right|}\right) =$$

$$20log_{10}(|0.1|) + 20log_{10}\left(\left|1+\frac{j\omega}{1}\right|\right) + 20log_{10}\left(\frac{1}{\left|1+\frac{j\omega}{10}\right|}\right)$$

$$+ 20log_{10}\left(\frac{1}{\left|1+\frac{j\omega}{100}\right|}\right) =$$

$$20log_{10}(|0.1|) + 20log_{10}\left(\left|1+\frac{j\omega}{1}\right|\right) - 20log_{10}\left(\left|1+\frac{j\omega}{10}\right|\right) - 20log_{10}\left(\left|1+\frac{j\omega}{100}\right|\right)$$

A quick look at the right hand side reveals just two types of term:

(1) a constant term

(2) a term of the form: $20log_{10}\left(\left|1+\frac{j\omega}{\omega_0}\right|\right)$

This is a useful feature when we come to plot the **magnitude part** of the TF against frequency later.