The Bode Stability Criterion – Part 2

In this lecture we continue the theoretical development of the Bode plot, from Lecture 11 Part 1

Summarising the **phase part** of the TF:

$$\angle H(s) = \angle(0.1) + \angle\left(1 + \frac{j\omega}{1}\right) - \angle\left(1 + \frac{j\omega}{10}\right) - \angle\left(1 + \frac{j\omega}{100}\right)$$

Again, just two types of term, a constant term and a term of the form,

$\left(1 + \frac{j\omega}{\omega_0}\right)$.

In both the magnitude and phase cases plotting the constant part is trivial but the other term in each case takes a little more work.

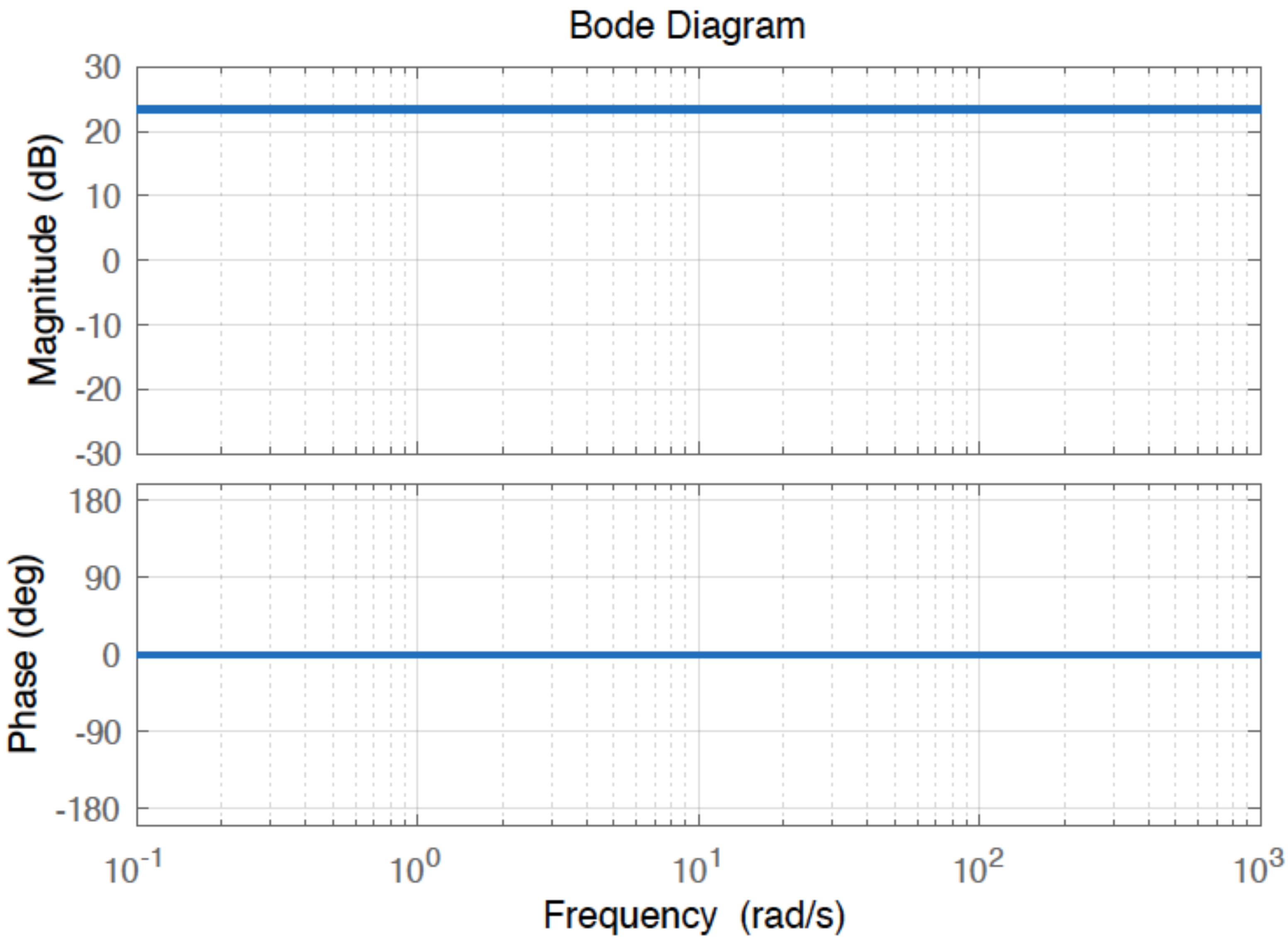
If we take the constant part of the magnitude first then we have this:

$$20log_{10}(|0.1|) = -20 \text{ dB}$$

So, it is simply a straight line against frequency.

In the case of phase the phase value is zero for a positive constant and $-\pi$ for a negative constant.

Here is a Bode plot for a constant value of 15:



Bode Diagram
Magnitude (dB)
30
20
10
0
-10
-20
-30
Phase (deg)
180
90
0
-90
-180
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

Looking at the other sort of term next; this can be a real pole term, a real zero term, a term representing a complex conjugate pair of poles, and a term representing a complex conjugate pair of zeros. We will examine real pole terms only, for conciseness, bearing in mind that the rules for construction are the same for the other variants, although the results are more complicated.

In general, a real pole term looks like:

$$H(s) = \frac{1}{1 + \frac{s}{\omega_0}} = \frac{1}{1 + j\frac{\omega}{\omega_0}}$$

The quantity ω_0 has several names. It is the 'break frequency' the 'corner frequency' and the '3 dB frequency'.

So**, the magnitude** of this pole term is this:

$$|H(j\omega)| = \left|\frac{1}{1 + j\frac{\omega}{\omega_0}}\right| = \frac{1}{\sqrt{1^2 + \left(\frac{\omega}{\omega_0}\right)^2}}$$

So, in dB this is:

$$|H(j\omega)|_{dB} = 20log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2}}\right)$$

There are three important cases for the value of the frequency.

Case (1): $\omega \ll \omega_0$ This is the low frequency case, so:

$$\sqrt{1 + \left(\frac{\omega}{\omega_0}\right)^2} \approx 1$$

so

$$|H(j\omega)|_{dB} = 20log_{10}\left(\frac{1}{1}\right) = 0$$

This is the blue line in the diagram (later) and is called the ‘low frequency asymptote’.

Case (2): $\omega \gg \omega_0$ This is the high frequency case, and so here we have:

$$\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2} \approx \sqrt{\left(\frac{\omega}{\omega_0}\right)^2} \approx \frac{\omega}{\omega_0}$$

So,

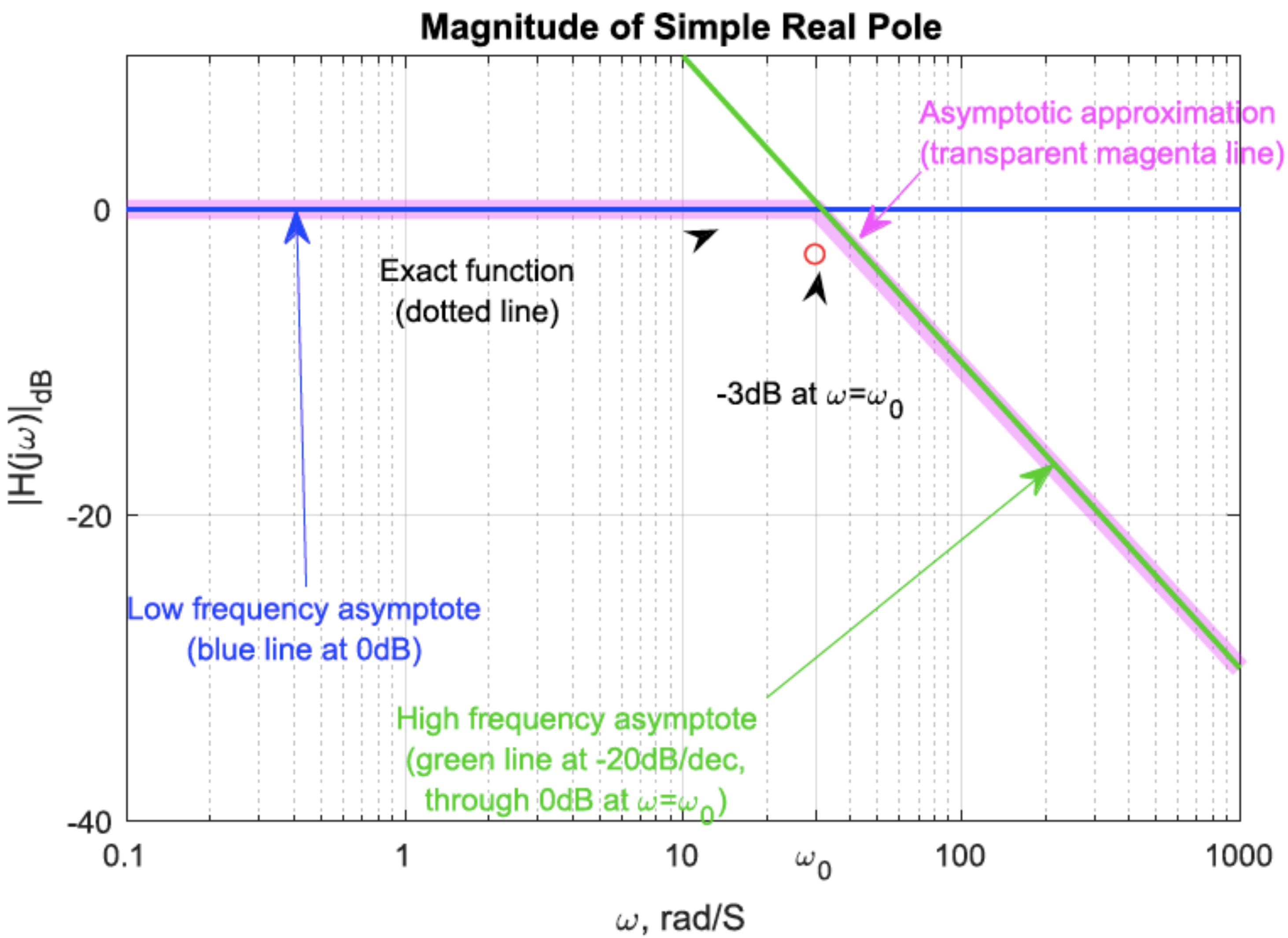
$$|H(j\omega)|_{dB} = 20log_{10}\left(\frac{\omega_0}{\omega}\right)$$

This is the green line in the diagram and is called the 'high frequency asymptote'. It's a straight line with a slope of -20 dB/decade (a decade is 1-10, 10-100, 100-1000 etc rad/s).

Case (3): $\omega = \omega_0$ This is the *corner frequency*, where we get:

$$|H(j\omega)|_{dB} = 20log_{10}\left(\frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^2}}\right) = 20log_{10}\left(\frac{1}{\sqrt{2}}\right) \approx -3\text{ dB}$$

The corner frequency is shown as a red circle on the diagram.



There is an asymptotic approximation shown on the diagram as the magenta line, and the maximum error is at the corner frequency and is approximately 3 dB.

We can examine the pole term for phase too.

The phase of a single real pole term is:

$$\angle H(j\omega) = \left(\frac{1}{1 + j\frac{\omega}{\omega_0}}\right) = -\angle\left(1 + j\frac{\omega}{\omega_0}\right) = -arctan\left(\frac{\omega}{\omega_0}\right)$$

We can examine the three cases once again.

Case (1): $\omega \ll \omega_0$ This is the low frequency case, so:

$\angle H(j\omega) \approx -arctan\left(\frac{\omega}{\omega_0}\right) = 0$ rad

This is the low frequency asymptote again, in blue in the diagram.

Case (2): $\omega \gg \omega_0$ This is the high frequency case for which we get:

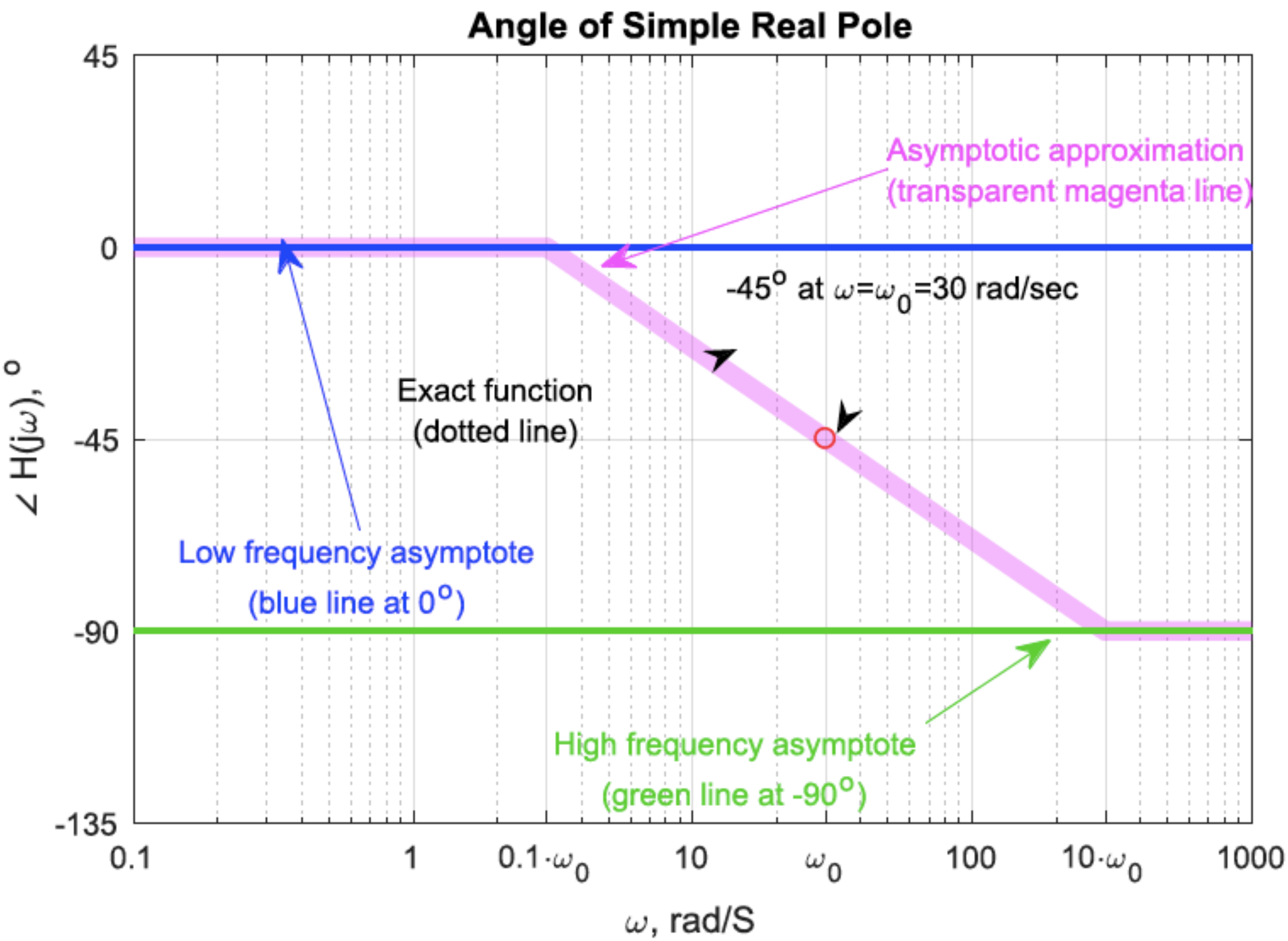
$$\angle H(j\omega) \approx -arctan\left(\frac{\omega}{\omega_0}\right) = -\arctan(\infty) = -\frac{\pi}{2}$$ rad or -90 deg

This is the high frequency asymptote again, and in green on the diagram.

Case (3): $\omega = \omega_0$ This is the corner frequency, where we get:

$$\angle H(j\omega) \approx -arctan\left(\frac{\omega}{\omega_0}\right) = -\arctan(1) = -\frac{\pi}{4}$$ rad or -45 deg

This is the red circle on the diagram.



Angle of Simple Real Pole
Asymptotic approximation
(transparent magenta line)
-45° at ω=ω0=30 rad/sec
Exact function
(dotted line)
Low frequency asymptote
(blue line at 0°)
High frequency asymptote
(green line at -90°)
45
0
-45
-90
-135
0.1
1
0.1·ω0
10
ω0
100
10·ω0
1000
ω, rad/S
∠ H(jω), °

The piecewise linear approximation that we made quite easily for the magnitude case is less easy for phase because the high and low frequency asymptotes don't intersect. The rule that we follow is this:

Follow the low frequency asymptote until one tenth the corner frequency then decrease linearly to meet the high frequency asymptote at ten times the corner frequency. This is *an ad* hoc rule that is universally followed.

This shows how to draw a Bode diagram for just one element (a real pole). The rules for the others (a real zero term, a term representing a complex conjugate pair of poles, and a term representing a complex conjugate pair of zeros) are arrived at in the same way but give obviously different results.

Bode plots can be used to show the magnitude and phase plots and these determine the phasor representation of the transfer function at any frequency. The frequency response of a system is presented as two graphs: one showing magnitude and one showing phase. The phasor representation of the transfer

function can then be easily determined at any frequency. The magnitude of the output is the magnitude of the phasor representation of the transfer function (at a given frequency) multiplied by the magnitude of the input. The phase of the output is the phase of the transfer function added to the phase of the input.

The full list of terms that we can examine for a closed loop transfer function through Bode are: a *constant*, a *real pole* (examined in full here), a *real zero*, a *pole at the origin*, a *zero at the origin*, a *complex conjugate pair of poles*, and a *complex conjugate pair of zeros*. These are known as the seven possible constituent parts of the TF.

Example: Conversion of the TF to the proper form, then identify the constituent parts, or forms:

Find the proper form for:

$$H(s) = -100\frac{s}{s^3 + 12s^2 + 21s + 10}$$

Solution: $H(s) = -100\frac{s}{(s+1)^2(s+10)} = -10\frac{s}{(s+1)^2\left(\frac{s}{10}+1\right)}$

The TF here has 4 constituent parts:

A constant of -10

A pole at $s = -10$

A doubly repeated pole at $s = -1$

A zero at the origin ($s = 0$).

Bode plots relate to stability through the *gain* and *phase margins*.

The Gain Margin indicates absolute stability, and the degree to which a system will oscillate, without limit, given any disturbance.

The Phase Margin indicates relative stability, the tendency to oscillate during its damped response to an input change such as a step function.

Both terms are electrical in origin, relating to the notion of gain for a negative feedback amplifier.

They can be precisely defined for any system for which the Bode plot can be calculated.