Lecture 4 – Part 1

Performance and stability - Dynamics

The *Routh Hurwitz* Stability Criterion

As we now know a dynamical system is usually represented by an ordinary differential equation. In general we will have something like this:

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + a_2 \frac{d^{n-2} x}{dt^{n-2}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x = b_0 u$$

The general solution to the complementary function (for which the RHS = 0) is given by:

$$x_{CF} = Ae^{mt}$$

Substituting this into the general differential equation above leads to:

$$Ae^{mt}(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0$$

In order to get a non-trivial solution we require:

$$(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0$$

This is called the *n*-th order auxilliary equation and gives rise to *n* solutions (or roots), $m_1, m_2, \ldots, m_n$. These roots are generally complex, but since all the coefficients of the original differential equation are real they must occur in conjugate pairs, of the form:

$$\sigma \pm j\omega$$

We will come back to this.

So, the complementary function solution in detail is this:

$$x_{CF} = A_1 e^{m_1 t} + A_2 e^{m_2 t} + \cdots + A_n e^{m_n t} = \sum_{i=1}^{n} A_i e^{m_i t}$$

This is simply a summation of the *n* solutions.

In addition to the complementary function we also have the particular integral part of the solution. Given that in general the RHS of the general differential equation is $b_0 u$ then the particular integral has to resemble this form and in general turns out to be:

$$x_{PI} = \frac{b_0}{a_n} u$$

So the full solution to the general differential equation is:

$x = \sum_{i=1}^{n} A_i e^{m_i t} + \frac{b_0}{a_n} u$

We note that the n currently unknown coefficients, $A_1, A_2, \ldots, A_n$ are obtained from knowledge of the n initial conditions, that is: $x(0), \frac{dx(0)}{dt}, \frac{d^2x(0)}{dt^2}, \ldots$.

The full solution to the general differential equation (written again for clarity):

$$x = \sum_{i=1}^{n} A_i e^{m_i t} + \frac{b_0}{a_n} u$$

will be stable (i.e. will remain constant) as time increases if the exponential terms $e^{m_i t}$ decay.

So, if we first assume that the roots m_i are real then stability will be guaranteed if the roots are negative, therefore if $m_i < 0$.

If we instead assume that the roots are complex, so $m_i = \sigma_i + j\omega_i$, and from this we see that the solution will be stable if the real part is negative, i.e. if $\sigma_i < 0$.

To summarise, the dynamical system represented by the general differential equation will be stable if the real parts of the *n* roots of the auxiliary equation are negative.

We now concentrate on second order differential equations because these represent many systems that we encounter in aero-mechanical and electrical engineering.

So, we consider this differential equation now, in general:

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 u(t)$$

The ω_n^2 in the RHS is simply included as an algebraically convenient multiplier and can be easily compensated for by scaling the excitation term u(t).

In a mechanical system we have these relationships:

$2\xi\omega_n = \frac{C}{m}$ and $\omega_n^2 = \frac{k}{m}$ where the C, m, and k assume their usual meanings.

In an electrical system we have these alternative relationships:

$2\xi\omega_n = \frac{R}{L}$ and $\omega_n^2 = \frac{1}{LC}$ where R, L, and C are resistance, inductance and capacitance.

In general then, if we assume zero initial conditions then the LT of the general second order differential equation.

$$s^2X(s) + 2\xi\omega_n sX(s) + \omega_n^2 X(s) = \omega_n^2 U(s)$$

So, if we rearrange this in the usual way we get:

$$X(s) = \left[\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right] U(s)$$

The term in the square brackets is the transfer function so if we take a unit step input then we get this:

$$X(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

Now, we can factorise the denominator to get $X(s)$ in a form that allows us to see how the damping ratio can influence the roots of the denominator part.
Recall from Lecture 1 (part 2, slide 17) that the roots of the denominator of the transfer function are the *poles* and as we will see next these can be real and unequal, real and equal, or complex conjugate pairs.

We now take each in turn and summarise their meaning. To do this we take the equation above and factorise it:

$$X(s) = \frac{\omega_n^2}{s(s + \xi\omega_n + \sqrt{(\xi^2 - 1)}\omega_n)(s + \xi\omega_n - \sqrt{(\xi^2 - 1)}\omega_n)}$$

Case 1: If $\xi > 1$ the roots will be real and unequal.

Case 2: If $\xi = 1$ the roots will be real and equal.

Case 3: If $\xi < 1$ the roots will be a complex conjugate pair.

Taking case 1, for which the roots are real and unequal:

$$X(s) = \frac{1}{s} + \frac{A}{s + b_1} + \frac{B}{s + b_2}$$

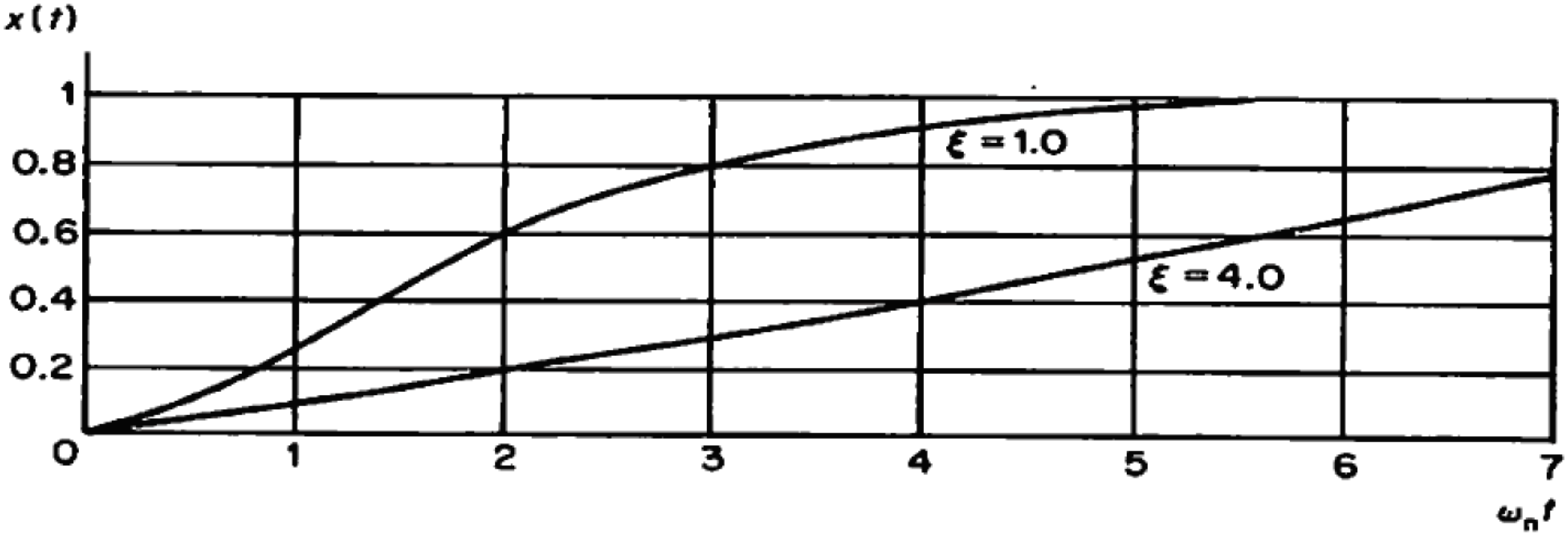
where $b_1 = \xi\omega_n + \sqrt{(\xi^2 - 1}\omega_n$ and $b_2 = \xi\omega_n - \sqrt{(\xi^2 - 1}\omega_n$

and $A = \frac{1}{2\xi\sqrt{(\xi^2-1)}+2(\xi^2-1)}$ and $B = \frac{1}{-2\xi\sqrt{(\xi^2-1)}+2(\xi^2-1)}$

Taking inverse Laplace Transforms gives:

$$x(t) = 1 + Ae^{-b_1 t} + Be^{-b_2 t}$$

If we plot this for two large (over-damped) values of ξ then we see that the system becomes progressively more unresponsive.



In the mechanical system increasing ξ means increasing the viscous damping.

For case 2 we simply have one possibility, for which $\xi = 1$ (also shown above).

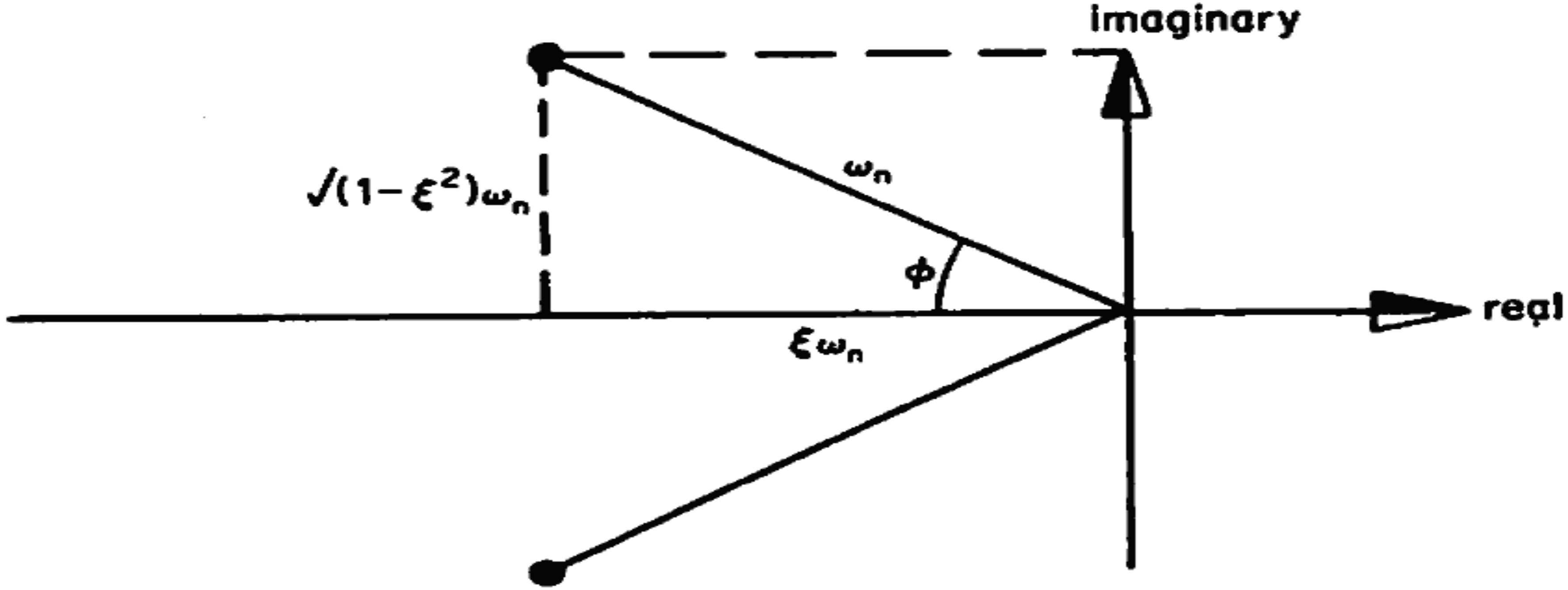
For case 3 we have a complex conjugate pair of roots. To clarify this we repeat the equation for X(s) for the unit step function:

$$X(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

and for this case the roots of the bracketed denominator term are:

$$-\xi\omega_n \pm j\sqrt{(1 - \xi^2}\,\omega_n$$

These can be plotted on the complex plane and look like this:



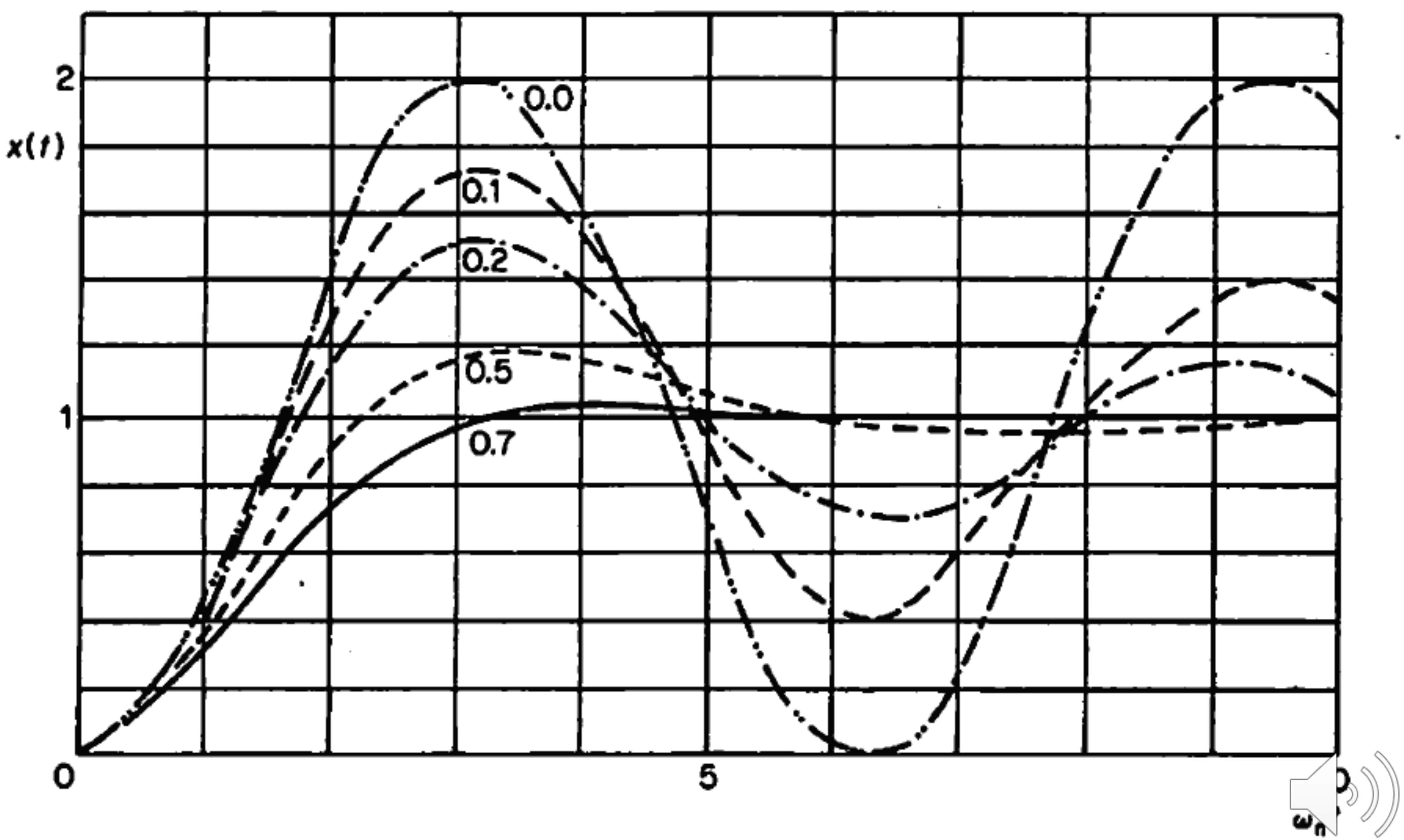
It can also be shown that $cos\phi = \xi$, the damping ratio.

After some considerable algebra the inverse Laplace Transform of X(s) turns out to be:

$$X(t) = 1 - \frac{1}{\sqrt{(1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n \sqrt{(1-\xi^2} t + \phi\right)$$

where $\phi = arctan\frac{\sqrt{(1-\xi^2)}}{\xi}$

This response can be shown for values of $0 \leq \xi \leq 1$. We see that the damping ratio dominates the response whilst ω_n merely affects the time scale.



When $\xi = 0$ we get a sinusoidal response of natural frequency ω_n. As ξ increases the curves become less oscillatory and the natural frequency is more accurately expressed by $\omega_d = \omega_n\sqrt{1-\xi^2}$. This is called the *damped natural frequency*.

We start by recalling the spring-mass-damper system:

We take the spring mass damper system again, with zero ICs, and write down the Laplace Transformed equation of motion, as follows:

$$Ms^2Y(s) + bsY(s) + kY(s) = R(s)$$

The Transfer Function is therefore:

$$\text{output/input} = G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2+bs+k}$$

This comes from the algebraic structure of the Laplace Transformed equation of motion. The input is R(s) and the output is the response (in the Laplace domain) which is Y(s).

Then, we showed that the transient part of the solution (represented by the complementary function) and the steady-state part of the solution (represented by the particular integral) could be superpositioned (remember we are dealing with linear systems), and so we got this:

The full solution to the general differential equation (written again for clarity):

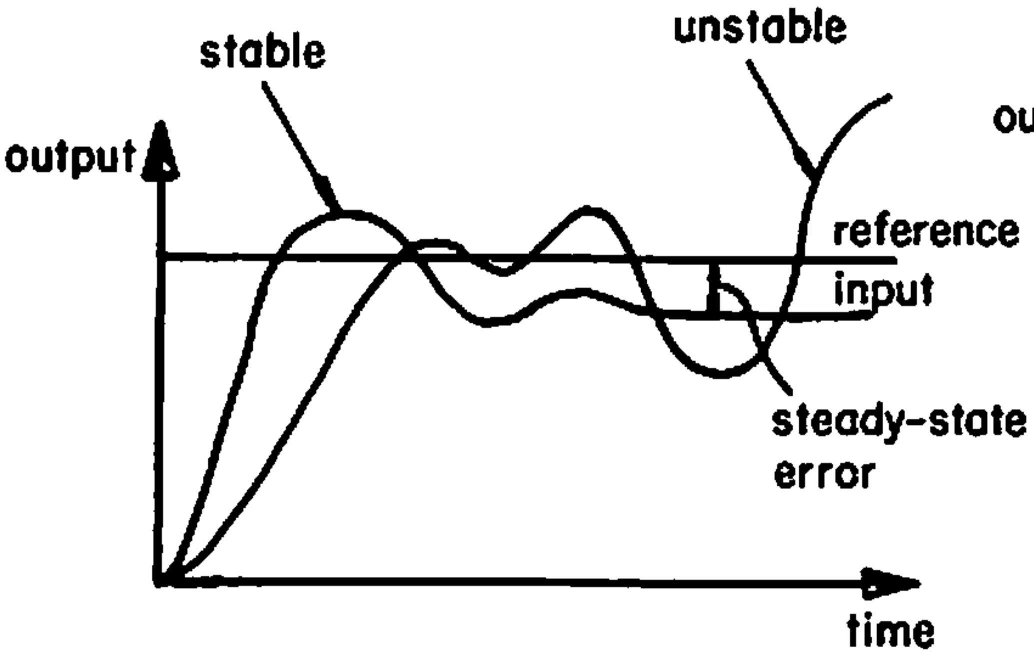
$$x = \sum_{i=1}^{n} A_i e^{m_i t} + \frac{b_0}{a_n} u$$

will be stable (i.e. will remain constant) as time increases if the exponential terms $e^{m_i t}$ decay.

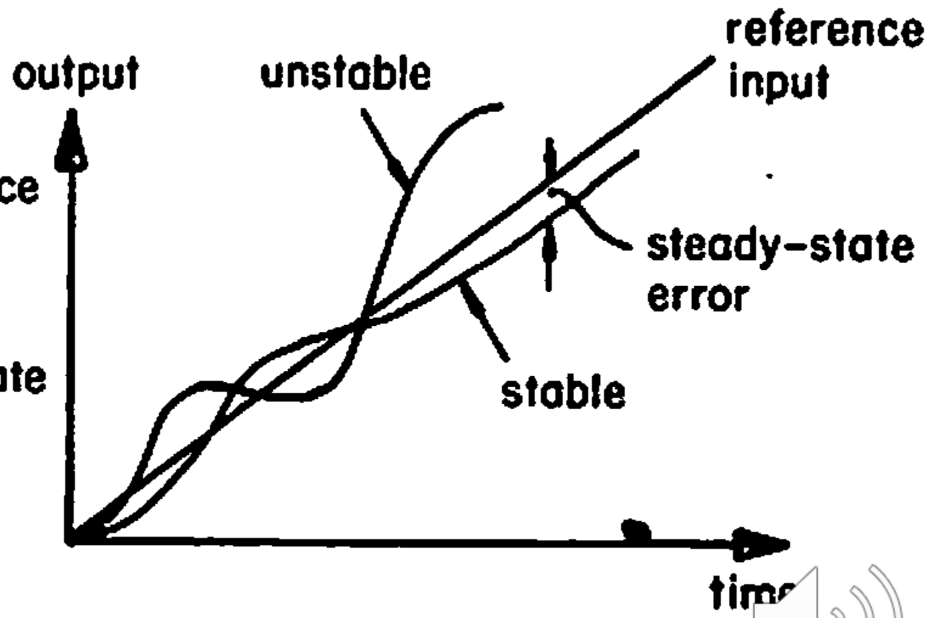
Then, we showed that if the real parts of the roots of the auxilliary equation are negative then the condition above is fulfilled. In practice we apply this to the roots of the characteristic equation, an example of this being the denominator in the Transfer Function G(s) shown in the previous slide for the SMD system. The roots of the denominator are called the poles.

Four important practical considerations to start off with:

1. The system should be absolutely stable. So, when it is excited it should settle down to some steady value and not exhibit continuous and growing oscillations (see Figs below).
2. The system should be accurate in the steady-state, so as $t \rightarrow \infty$ then the output should be equal to the input, or nearly so.
3. The system should exhibit a satisfactory transient response.
4. The system should be insensitive to changes in system parameters, and disturbances.



Step Input



Ramp Input