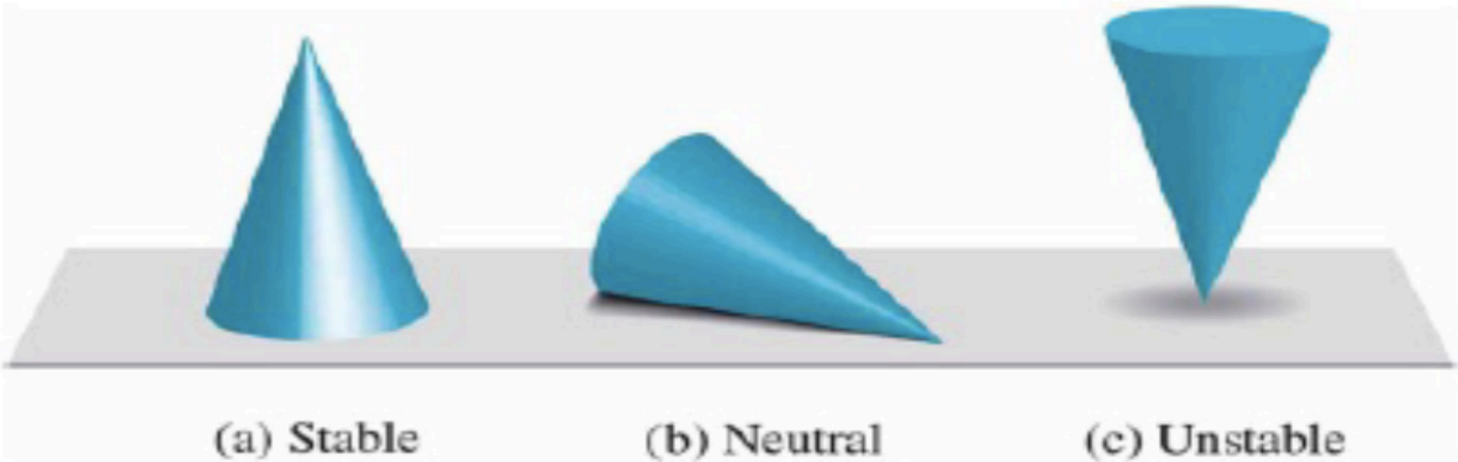
# Lecture 6
# Stability - background theory

**Stability theory** addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions.

The three cones illustrate stability in a straightforward physical way:




(a) If the cone is tipped *slightly*, it will return back to the original position (with its base resting on the ground)
(b) If the cone is pushed *slightly*, it will roll around but will remain resting on its side
(c) If the cone is displaced in any way it will tip (in fact it will simply fall when released).

Forced Spring Mass Damper system:

$$r(t) = M\ddot{y} + b\dot{y} + ky$$

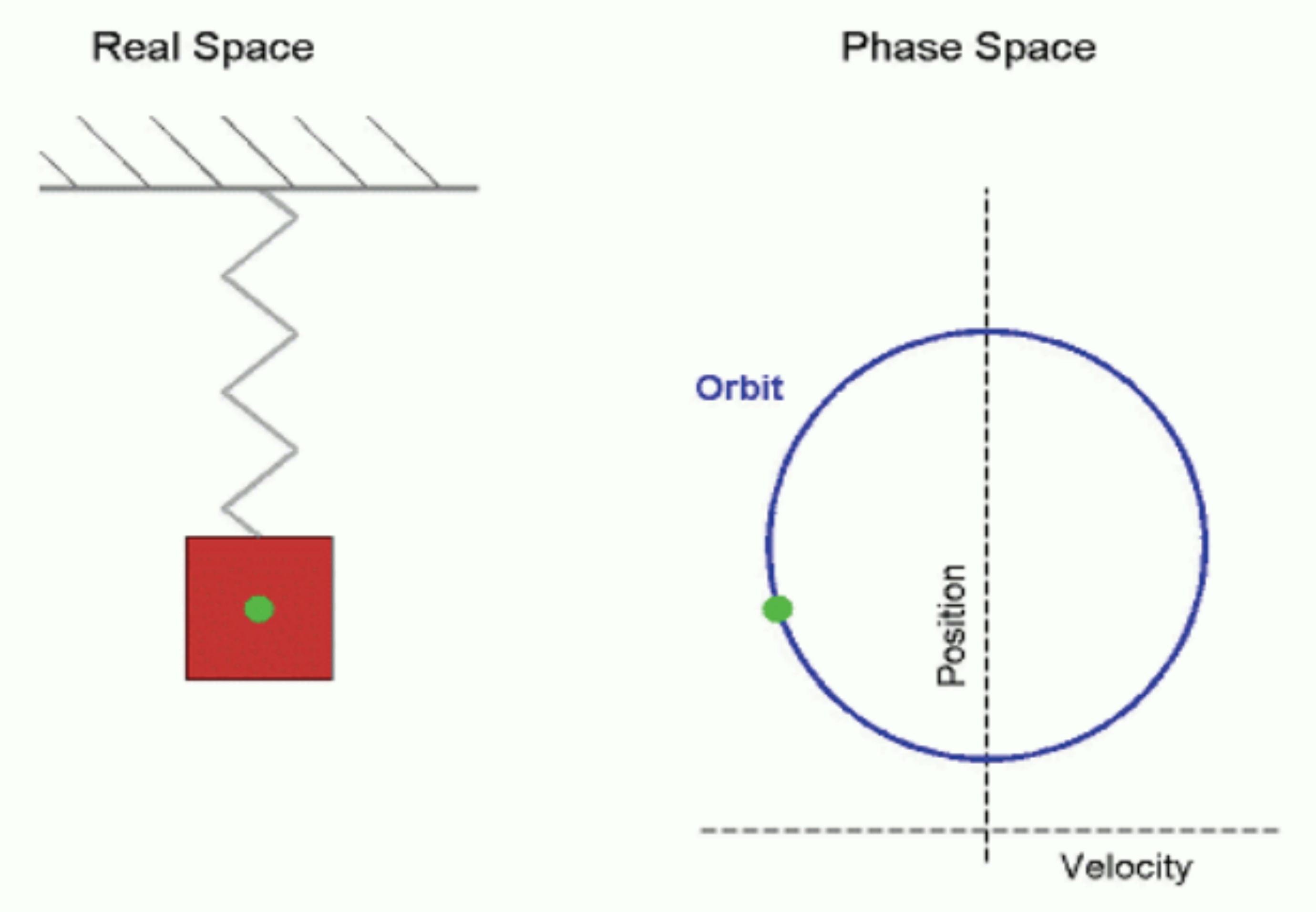
$$L\{r(t)\} = R(s) = M\left(s^2Y(s) - sy(0^-) - \frac{dy(0^-)}{dt}\right) + b\left(sY(s) - y(0^-)\right) + kY(s)$$

$$\therefore r(t) = 0,\ y(0^-) = y_0,\ \left.\frac{dy}{dt}\right|_{t=0^-} = 0$$

$$ms^2Y(s) - msy_0 + bsY(s) - by_0 + kY(s) = 0$$

$$Y(s) = \frac{(ms+b)y_0}{ms^2+bs+k} \equiv \frac{p(s)}{q(s)}$$

An orbit of a dynamical system:



## Poles & zeros

$$Y(s) = \frac{(ms+b)y_0}{ms^2+bs+k} \equiv \frac{p(s)}{q(s)}$$

Roots of numerator polynomial p(s) $\longrightarrow$ **zeros**

$$p(s) = 0 \rightarrow \lim_{s \to p_i} Y(s) = 0$$

Characteristic equation

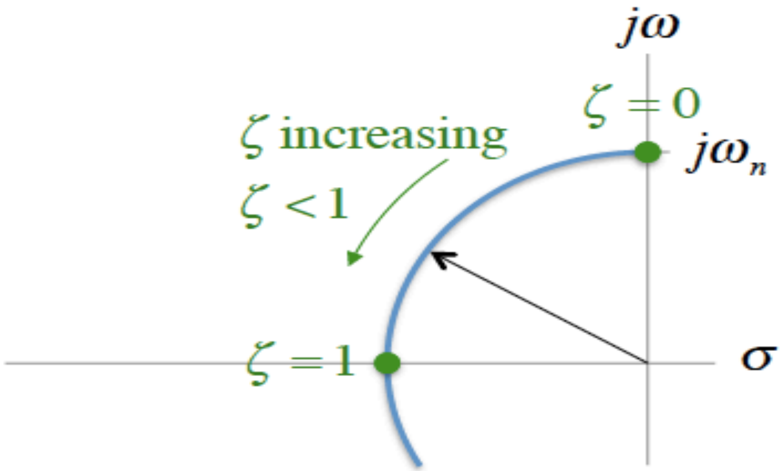
Roots of denominator polynomial q(s) $\longrightarrow$ **poles**

$$q(s) = 0 \rightarrow \lim_{s \to q_i} Y(s) = \infty$$

- The transfer function completely represents a system differential equation
- The poles and zeros effectively define the system response

Poles & zeros

Location of poles and zeros in the complex s-plane determine the response of the system – we will later develop the *Root-Locus method* as a way of finding these locations.




(assuming $\omega_n$ is held constant)

$$Y(s) = \frac{(ms+b)y_0}{ms^2+bs+k} \equiv \frac{p(s)}{q(s)}$$

$$Y(s) = \frac{(s+2\zeta\omega_n)y_0}{s^2+2\zeta\omega_n s+\omega_n{}^2} \rightarrow s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

Poles & zeros

Location of poles and zeros in the complex s-plane determine the response of the system

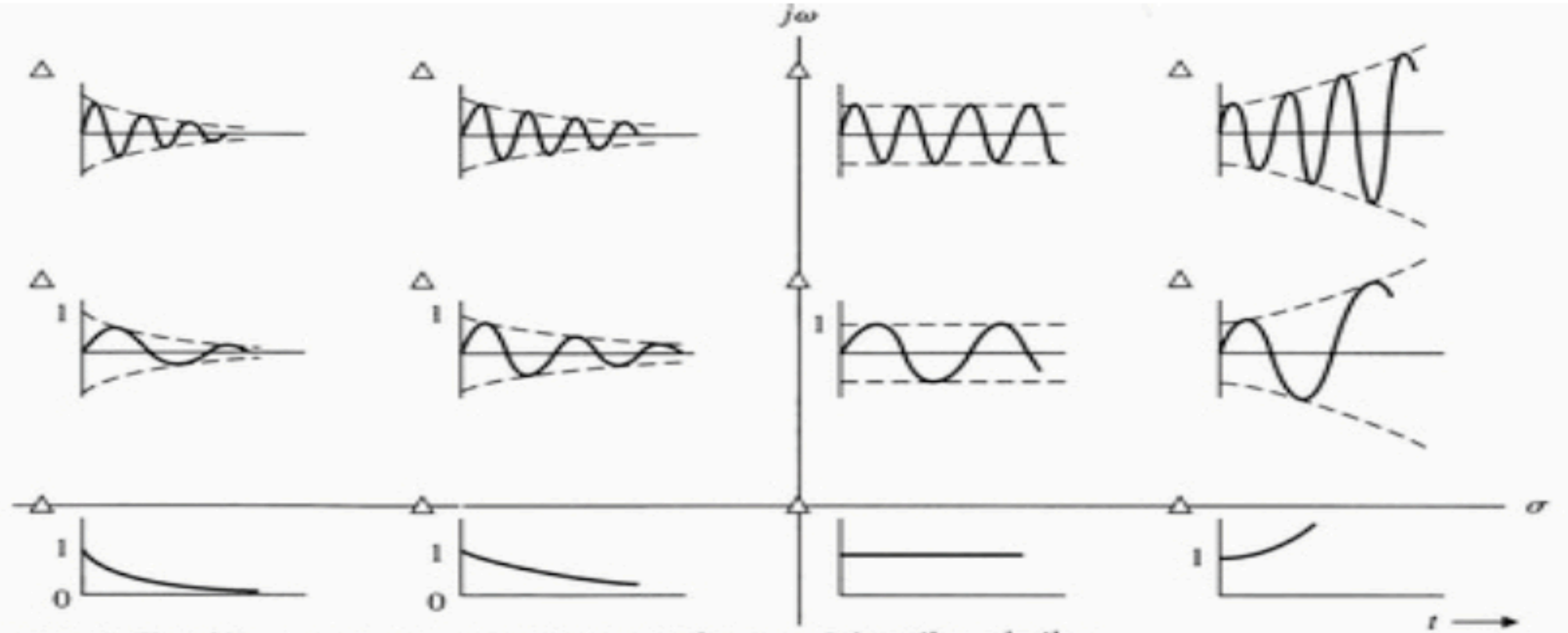


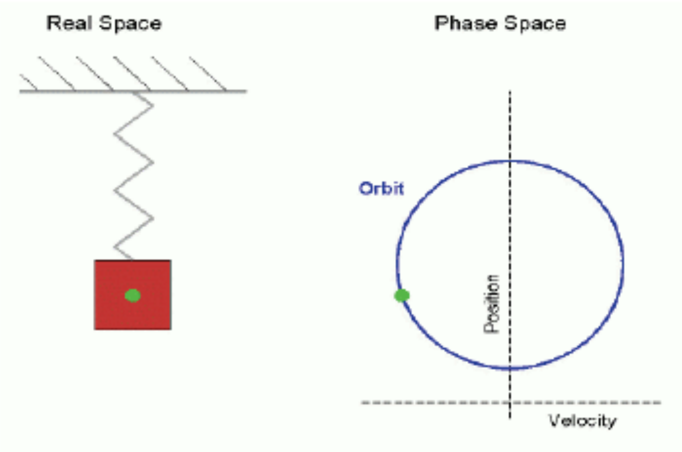

**FIGURE 5.17** Impulse response for various root locations in the $s$-plane. (The conjugate root is not shown.)

**Some important definitions for Stability, within Control Theory**

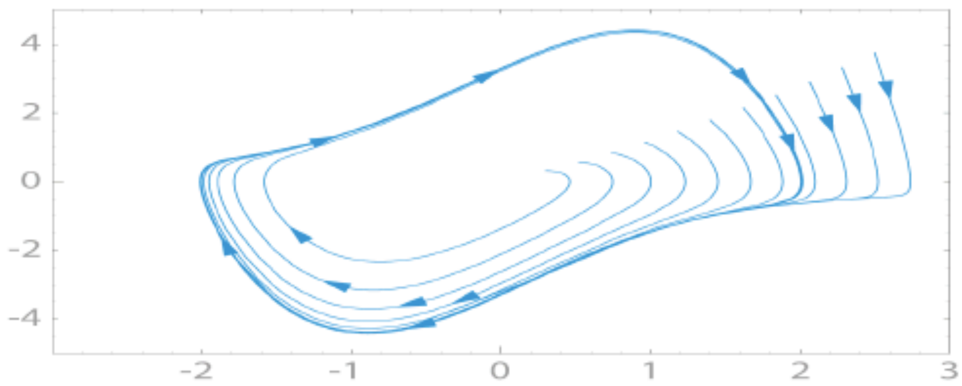
A.M.Lyapunov is considered by many to be the founder of modern stability theory ('The general problem of stability of motion', published in 1892).

The basic idea is like this – if a solution to a dynamical problem that starts out near an equilibrium point $x_e$ stays near $x_e$ forever then $x_e$ is ***Lyapunov stable***. Furthermore, and more strongly, if $x_e$ is Lyapunov stable and all solutions that start out near $x_e$ converge to $x_e$ then $x_e$ is ***asymptotically stable***.
We can extend this a bit more: if we can guarantee a minimal rate of decay then the system is also ***exponentially stable***.

We can revisit the notion of the orbit again now:




Stable limit cycle for the van der Pol oscillator



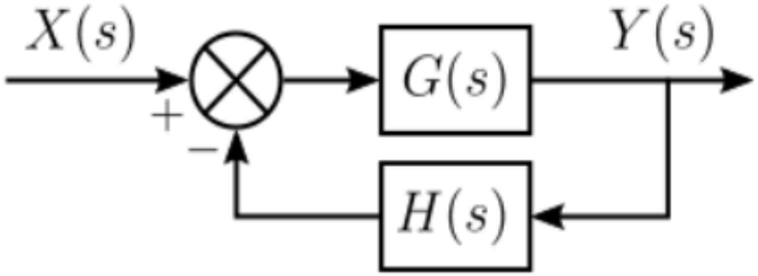
## The Root Locus Method - introduction

This is a method that is used to find how the roots of the characteristic equation move around the complex s-plane as we change a gain parameter, and what that means for the response of the system.

Important points:

1. The relative stability and transient performance of a closed-loop control system are directly related to the location of the closed-loop roots of the characteristic equation in the s-plane.
2. It's frequently necessary to adjust one or more system parameters in order to obtain suitable root locations.
3. The Root Locus Method (W.R.Evans, 1948) is a graphical method for drawing the locus of roots in the s-plane as a parameter is varied.
4. It can be useful when used in conjunction with methods such as the Routh-Hurwitz criterion, for example.
5. It can also be used to design for the damping ratio and natural frequency in terms of a selected gain.

We start by taking a simple negative feedback system and looking at its closed-loop transfer function.




$$T(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Remember that the roots of the numerator, generally defined by *p*(s), are the zeros and the roots of the denominator, generally defines by *q*(s), are the poles.
So, the closed-loop poles are the roots of the characteristic equation *q*(s), which in this case is 1 + *G*(s)*H*(s) = 0.

So, the roots of the characteristic equation may be found whenever *G*(s)*H*(s) = -1.
The product *G*(s)*H*(s) can be written as *F*(s), so *F*(s) = -1.

Since we are operating on the complex s-plane then *F*(s) = -1 can be written as:

$$|F(s)|\angle F(s) = -1$$

The first part is the modulus (equal to 1) and the second part is the argument. We proceed to examine both in an example, in the next lecture.