Bode plot transfer function – containing complex conjugate poles

Here is a realistic example containing three of the most common term constructions:

$$H(s) = 30\frac{s + 10}{(s^2 + 3s + 50)}$$

$$H(s) = 30\frac{10}{50}\frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1}$$

$$H(s) = 6\frac{\frac{s}{10} + 1}{\frac{s^2}{50} + \frac{3}{50}s + 1}$$

We have four terms within the Transfer Function:

1) A constant of 6
2) A zero at $s = -10$
3) Complex conjugate poles at the roots of:

$$s^2 + 3s + 50 = s^2 + 2\xi\omega_0 s + \omega_0^2 = 0$$

We’re going to compare the CE with the standard form for a damped second order system (both shown in the above equation) to get the corner frequency and the damping ratio, as follows:

Clearly we have $\omega_0 = \sqrt{50} = 7.07$ rad/s

Also we see that $3 = 2\xi\omega_0$ which leads to:

$$\xi = \frac{3}{2\omega_0} = \frac{3}{2 * 7.07} = 0.2121$$

So, it's a low frequency (LF) system which is lightly damped.
Note that the complex conjugate poles are at:

$$s = -1.5 \pm 6.9101j$$

Dealing with the magnitude first.

The constant of 6 leads to a straight line at 15.56dB. This is obtained from $20\log_{10} 6 = 15.56$ dB.

The zero is at $s = -10$ rad/s and re-applying the procedure of Lecture 11 (parts 1 & 2) we see that there will be a LF asymptote at 0 dB until the corner frequency, at which point the HF asymptote will take over, but as this term is a zero (and not a pole) then the slope will be positive, at +20 dB/decade.

Going back to the transfer function, we had this, when we replace *s* with $j\omega$:

$$H(j\omega) = 6\frac{\frac{j\omega}{10} + 1}{\frac{(j\omega)^2}{50} + \frac{3}{50}j\omega + 1}$$

In full this is:

$$H(j\omega) = \left(|6|\frac{\left|1 + \frac{j\omega}{10}\right|}{\left|\frac{(j\omega)^2}{50} + \frac{3}{50}j\omega + 1\right|}\right)\left(\angle(6) + \angle\left(1 + \frac{j\omega}{10}\right)\right.$$
$$\left. - \angle\left(\frac{(j\omega)^2}{50} + \frac{3}{50}j\omega + 1\right)\right)$$

So, we have to deal with the complex conjugate poles next.

1. Magnitude – given by:

$$|H(j\omega)| = \left| \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\xi\left(\frac{j\omega}{\omega_0}\right) + 1} \right|$$

$$= \left| \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2 + j2\xi\left(\frac{\omega}{\omega_0}\right) + 1} \right|$$

$$= \left| \frac{1}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) + j\left(2\xi\left(\frac{\omega}{\omega_0}\right)\right)} \right|$$

$$= \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_0}\right)^2}}$$

And so if we convert this into dB, as needed for the Bode plot, we get this:

$$|H(j\omega)|_{dB} = -20log_{10}\left(\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\xi\frac{\omega}{\omega_0}\right)^2}\right)$$

Note that this magnitude is preceded by a –ve sign because it's for a pole, and poles are denominator terms.

As in the case of the real pole there are three possibilities here, which we can summarise:

Case (1): $\omega \ll \omega_0$ This is the LF case again, for which the magnitude is approximated by:

$$|H(j\omega)|_{dB} = -20log_{10}(1) = 0$$

Case (2): $\omega \gg \omega_0$ This is the HF case again, and so the magnitude is approximated by:

$$|H(j\omega)|_{dB} = -20log_{10}\left(\left(\frac{\omega}{\omega_0}\right)^2\right) = -40log_{10}\left(\frac{\omega}{\omega_0}\right)$$

Case (3): $\omega \approx \omega_0$ This is a more complicated analysis because obviously a peak in the magnitude should occur near the corner frequency, effectively a form of resonance. So, if we differentiate the magnitude expression with respect to ω and then set that derivative to zero we can obtain the exact peak in the magnitude and its location.

From this differentiation we get a peak at this frequency:

$$\omega_r = \omega_0\sqrt{1 - 2\xi^2}$$

The peak magnitude for this case is obtained if we substitute ω_r for ω in the general expression for $|H(j\omega)|_{dB}$.

This gives:

$$|H(j\omega)|_{dB} = -20log_{10}\left(2\xi\sqrt{1 - \xi^2}\right)$$

The fundamental form for this is:

$$|H(j\omega)| = \frac{1}{2\xi\sqrt{1 - \xi^2}}$$

For both these expressions of magnitude we tend to take the case for low damping, for which $\sqrt{1-\xi^2} \approx 1$.

So, we draw the LF asymptote up to the corner frequency and then the HF asymptote after that, adding a peak at the corner frequency of magnitude as shown directly above.

In the case of this example we get the peak magnitude at:

$$|H(j\omega)|_{dB} = -20log_{10}\left(2\xi\sqrt{1-\xi^2}\right)$$

from which we get this:

$$-20log_{10}\left(2\xi\sqrt{1-\xi^2}\right) \approx -20log_{10}(2\xi) =$$
$$-20log_{10}(0.4242) = 7.45 \text{ dB}$$

The resonant frequency is given from the expression already obtained:

$$\omega_{r} = \omega_0\sqrt{1 - 2\xi^2}$$

$\omega_{r} = 7.07\sqrt{1 - 2 * 0.2121^2} = 6.91$ rad/s

2. Phase

The phase calculation is also different to that for the real pole and is generally done as follows.

We depart from the LF asymptote (0 degrees) at:

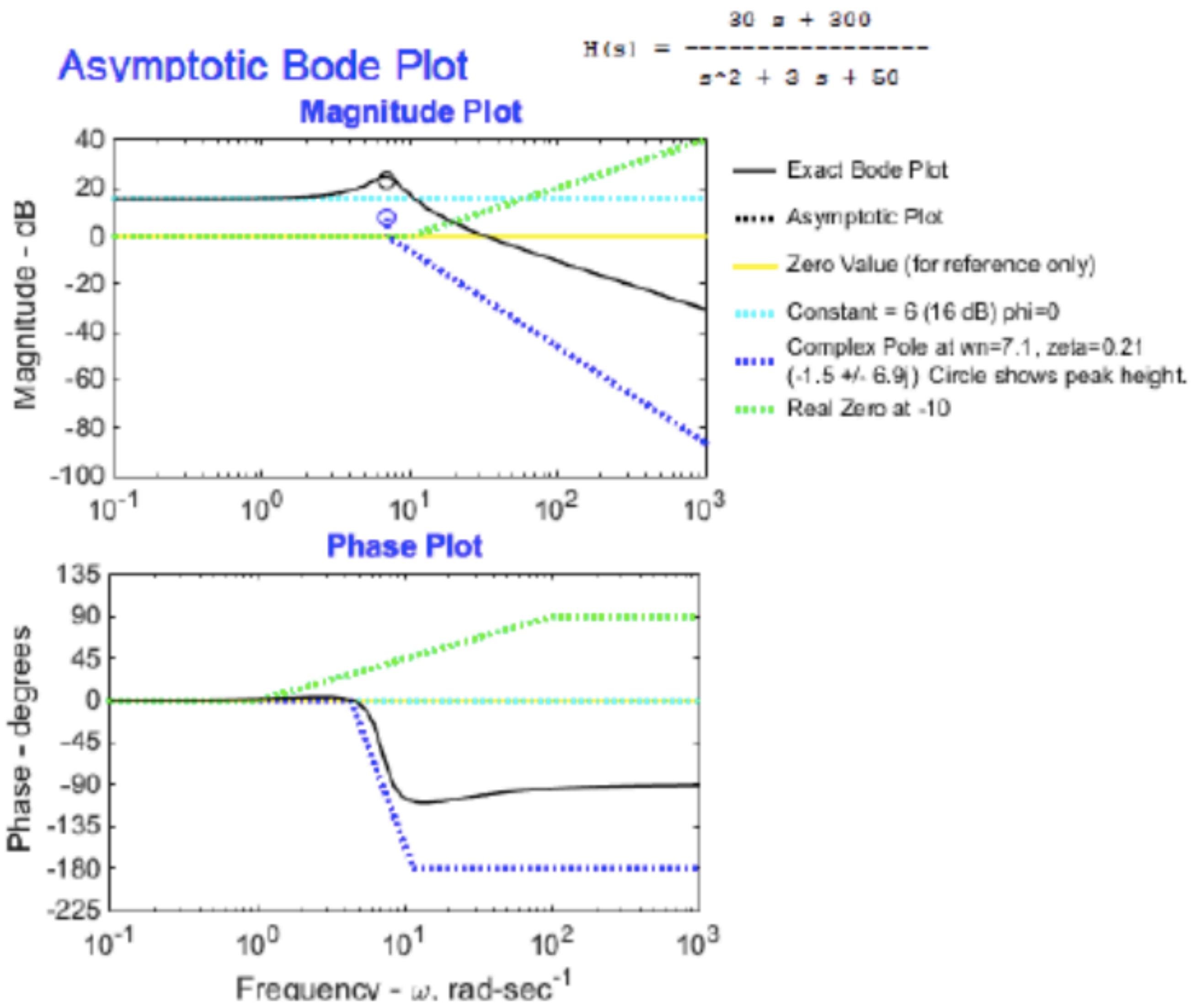
$\omega = \frac{\omega_0}{10^{\xi}} = \frac{7.07}{10^{0.2121}} = 4.33$ rad/s to the HF asymptote at:

$\omega = \omega_0 10^{\xi} = 7.07 * 10^{0.2121} = 11.52$ rad/s.

Note that there are at least four different variants on the analysis for Bode phase calculation for CC poles, but this is a simple result with reasonable accuracy.

We can now plot the magnitude and phase on a Bode plot for this CC pole transfer function.

The exact response is the black line.



Summary of the two plots

Magnitude – the blue/green line shows the constant at 15.56 dB. The green line is the zero, which is of zero magnitude until the corner frequency and then rises with a +ve slope of 20 dB/dec. The CC poles are zero magnitude until the corner frequency (7.07 rad/s) and then a –ve slope of – 40 dB/dec thereafter. There is a peak value for the CC pole magnitude at 7.45 dB at 6.91 rad/s.

Phase – straight line at 0 deg for the constant term. For the zero the phase is 0 deg up to 1/10 the corner frequency then it rises linearly to 90 deg at 10 times the corner frequency. In the case of the CC poles the phase goes from the LF asymptote (0 deg) at 4.33 rad/s to the HF asymptote at 11.52 rad/s. The HF asymptote for CC pole phase is approximately at -180 deg.

The same approaches can be used to obtain the Bode plots for a CC pair of zeros.