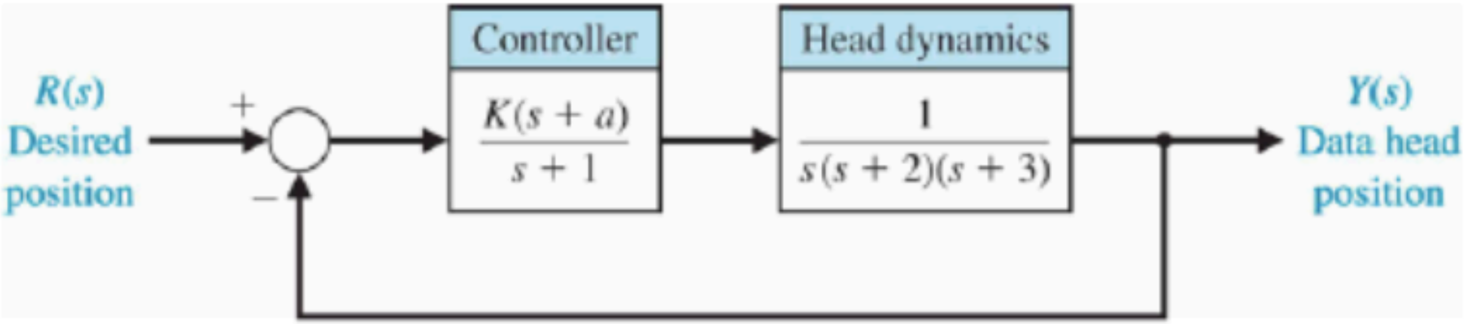
Further notes on the Routh-Hurwitz Stability Criterion

2 examples

1. Stability of a disk-drive controller

Disk-drives require fast and accurate motion of the data head above the spinning disk and so the controller must be highly responsive. We want to understand how to guarantee a stable control for the disk drive head for a typical system as shown below, where K and a are internal controller gains:



The overall system transfer function is given by:

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K(s+a)}{s(s+1)(s+2)(s+3)}}{\left(1+\frac{K(s+a)}{s(s+1)(s+2)(s+3)}\right)}$$

So, we take $1 + G(s)H(s)$ and write that as the left hand side of the characteristic equation to get:

$$1 + \frac{K(s+a)}{s(s+1)(s+2)(s+3)} = 0$$

Therefore we can rationalise this to:

$$\frac{s(s+1)(s+2)(s+3) + K(s+a)}{s(s+1)(s+2)(s+3)} = 0$$

So obviously we just need to consider:

$$s(s+1)(s+2)(s+3) + K(s+a) = 0$$

Expanding this out gives:

$$s^4 + 6s^3 + 11s^2 + (K+6)s + Ka = 0$$

From this we can build up the Routh-Hurwitz array.

We start by noting that in this example $n = 4$, and then we can compare our disk-drive characteristic equation (above) with the general characteristic equation from which the array elements are calculated.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n = 0$$

So, $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = K + 6, \quad a_4 = Ka$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{6*11 - 1*(K+6)}{6} = \frac{(60-K)}{6}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = \frac{6*Ka - 1*0}{6} = Ka$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{\frac{(60-K)}{6}*(K+6) - 6*Ka}{\frac{(60-K)}{6}} = (K+6) - \frac{6Ka}{\frac{(60-K)}{6}} = (K+6) - \frac{36Ka}{(60-K)}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} = \frac{\left[(K+6) - \frac{36Ka}{(60-K)}\right]*Ka - \frac{(60-K)}{6}*0}{\left[(K+6) - \frac{36Ka}{(60-K)}\right]} = Ka$$

So, the R-H array now looks like this:

$$\begin{array}{ll} s^n & : \; a_0 \;\; a_2 \;\; a_4 \;\; a_6 \;\; \ldots \\ s^{n-1} & : \; a_1 \;\; a_3 \;\; a_5 \;\; a_7 \;\; \ldots \\ s^{n-2} & : \; b_1 \;\; b_2 \;\; b_3 \;\; \ldots \\ s^{n-3} & : \; c_1 \;\; c_2 \;\; \ldots \\ s^{n-4} & : \; d_1 \;\; \ldots \end{array}$$

$$\begin{array}{ll} s^4 & : \; 1 \;\; 11 \;\; Ka \;\; \ldots \\ s^3 & : \; 6 \;\; (K+6) \;\; \ldots \\ s^2 & : \; \frac{(60-K)}{6} \;\; Ka \;\; \ldots \\ s^1 & : \; \left[(K+6) - \frac{36Ka}{(60-K)}\right] \;\; \ldots \\ s^0 & : \; Ka \;\; \ldots \end{array}$$

The first column contains these terms:

1

6

$\frac{(60-K)}{6}$

$\left[(K+6) - \frac{36Ka}{(60-K)}\right]$

Ka

The third element b_1 requires that $K < 60$ and the fifth element requires that $a > 0$, and the fourth element sets the relationship between K and a.

To find the most general relationship possible, whilst also satisfying the Routh-Hurwitz stability criterion that there are no sign changes in the first column, we set $c_1 = 0$. This means that we have:

$$(K+6) - \frac{36Ka}{(60-K)} = 0$$

Therefore we can write:

$$(K-60)(K+6) + 36Ka = 0$$

From which we obtain:

$$a \leq \frac{(60-K)(K+6)}{36K}$$

noting the previous condition for a being positive.

So, for example, if we take a practical plant value of $K = 40$ then we get $a \leq 0.64$.