# Lecture 1 – Part 2

# Mathematical Modelling

In this second part of the first lecture we will start to meet some of the following theory, tools, and techniques (in black font):

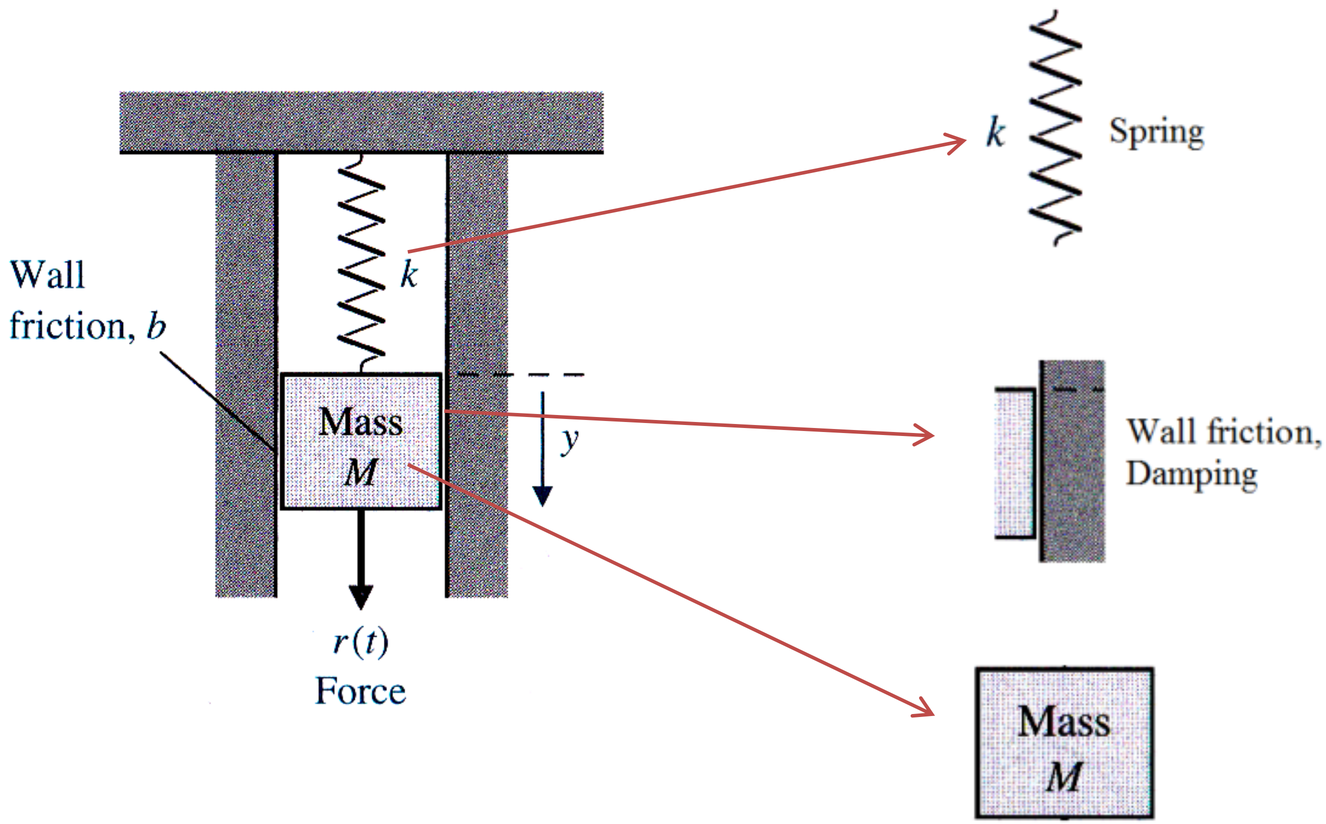
- The use of differential equations for physical system modelling,
- Linear systems and linear system approximations,
- The role of the Laplace Transform,
- The Transfer Function,
- Constructing a formal Block Diagram model,
- The use of software for control system simulation.

# The process of dynamical system modelling

- We start by defining the system and its components,
- Then we obtain the mathematical model for each component based on any assumptions given or required,
- From this we derive the differential equation(s) representing the system,
- The next step is to solve the differential equation(s),
- After that we examine the solutions and also the assumptions made.
- Finally, if necessary, we re-analyse or re-design the system.

# Differential equation for simple physical systems

## Mechanical spring-mass-damper system



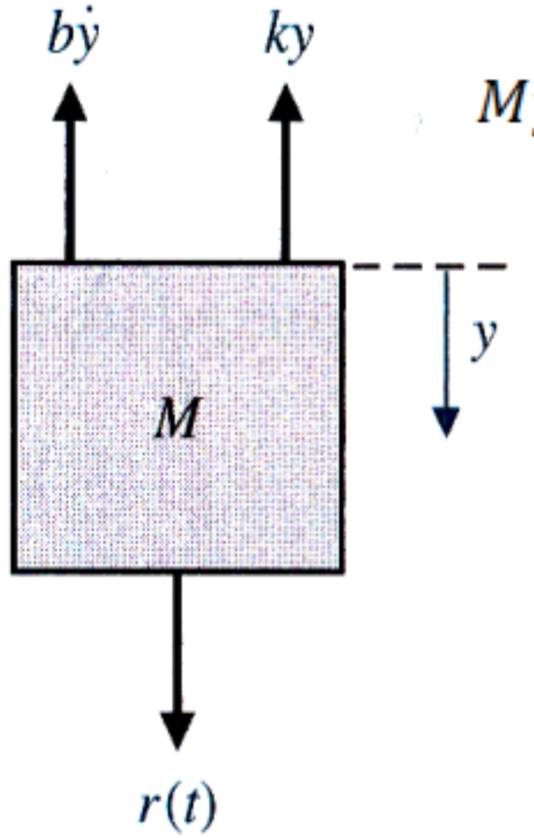

**Assumptions:**

1. Massless spring.
2. Linear spring (Hooke's Law).
3. Linear damping through viscous friction.

1. Rigid lumped mass.

We can also use appropriate electrical laws to model circuits – see next slide.

# S-M-D system

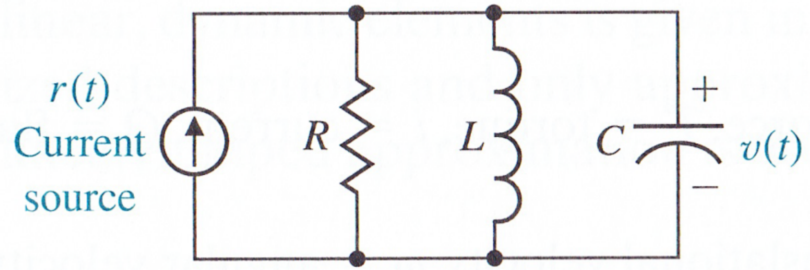



Free body diagram leads to this:

$$M\ddot{y} + b\dot{y} + ky = r(t)$$

In this mechanical system the coordinate is *y*, i.e. *y*(t).

# R-L-C system

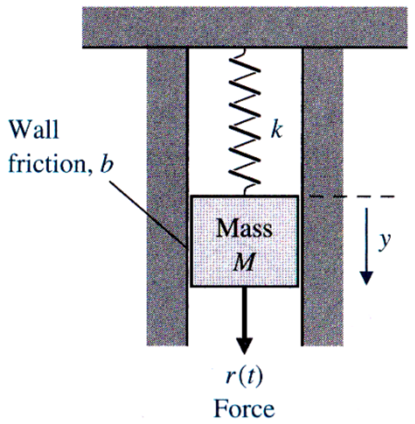


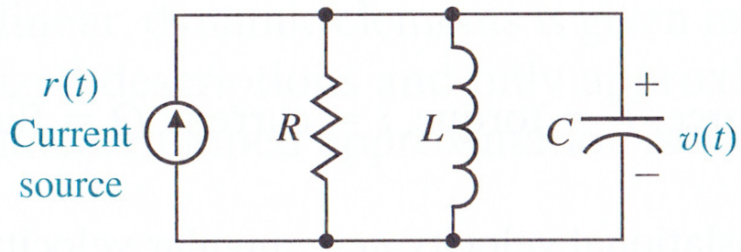

Kirchhoff's Current Law leads to this:

$$\frac{v}{R} + C\dot{v} + \frac{1}{L}\int_0^t v\,dt = r(t)$$

In this electrical system the coordinate is *v*, i.e. *v*(t).

# Time responses of simple physical systems




Standard methods for the solution of differential and integro-differential equations can be used to get the time domain responses of both equations.

Is there anything that unifies these two system models? Yes, there is if we re-write the mechanical displacement coordinate $y$ as velocity $V$ (upper case $V$ now distinguishes velocity from voltage). So, the S-M-D model now goes from

$$M\ddot{y} + b\dot{y} + ky = r(t) \qquad \text{to} \qquad M\dot{V} + bV + k\int_0^t V\,dt = r(t)$$

because $\quad V = \dot{y}$

# System responses

We now have similarly structured mathematical models for the two systems, as follows:

Mechanical S-M-D system

$$M\dot{V} + bV + k\int_0^t V dt = r(t)$$

Electrical R-L-C system

$$\frac{v}{R} + C\dot{v} + \frac{1}{L}\int_0^t v dt = r(t)$$

We recall that the mechanical S-M-D system is represented now by velocity $V$, and the electrical R-L-C system is represented by voltage $v$. Both are functions of time, of course.

Time domain responses are given by the following (transforming back from $V$ to $y$ in the mechanical S-M-D system:

$$y = K_1 e^{-\alpha_1 t} \sin\left(\beta_1 + \theta_1\right)$$

$$v = K_2 e^{-\alpha_2 t} \sin\left(\beta_2 + \theta_2\right)$$

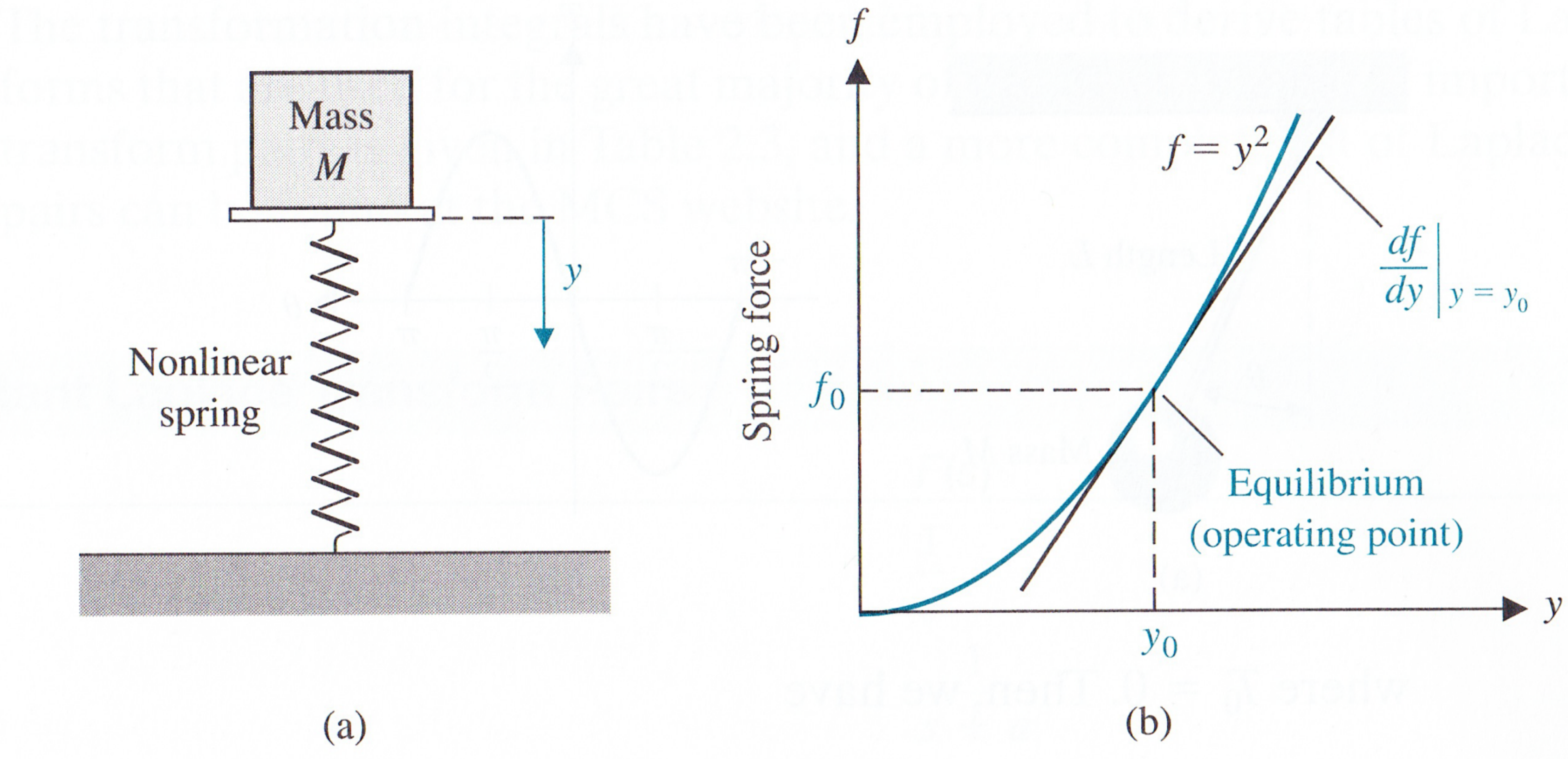
Certain conditions apply for each system (initial conditions, each under-damped, and constant current).

# Linear systems

In general, a system becomes nonlinear as its responses get bigger. The spring in the mass-spring-damper system is only linear for a limited range of displacement – see below.

A necessary condition for linearity is the *Principle of Superposition*.

Continuing with the spring as an example:



# Some general theory for linearisation

A linear system could be subjected to an excitation $x_1$(t) which then leads to a response $y_1$(t). If it is linear then another excitation $x_2$(t) will generate another response $y_2$(t), and furthermore *it is necessary* that these can be superpositioned so that an excitation ($x_1$(t) + $x_2$(t)) leads to a response ($y_1$(t) + $y_2$(t)).

Another requirement is that the linear system scaling remains constant.

So, input $x$(t) scales to output $y$(t) through a constant multiplier β.

This is called the Property of Homogeneity, and is a requirement for a linear system.

Conclusion: A system is linear if and only if the properties of superposition and homogeneity are both satisfied.

$y = x^2$ does not satisfy superposition and $y = \mathrm{mx} + c$ does not satisfy homogeneity. Neither system is linear.

The required relationship between excitation and response for a linear system can be written as:

$$y(t) = g\big(x(t)\big)$$

(where g indicates that y(t) is a function of x(t))

and we note that the normal operating point is given by $x_0$ , for example. As $g$ is continuous then the Taylor series expansion about the operating point is given by:

$$y = g(x) = g(x_0) + \frac{dg}{dx}\Big|_{x=x_0} \frac{(x - x_0)}{1!} + \frac{d^2g}{dx^2}\Big|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots$$

The slope at the operating point is $\frac{dg}{dx}\big|_{x=x_0}$ is a good approximation to the curve over the small range $(x - x_0)$, this being the deviation from the operating point.

So, to a reasonable approximation we get:

$$y = g(x_0) + \frac{dg}{dx}|_{x=x_0}(x - x_0) = y_0 + m(x - x_0)$$

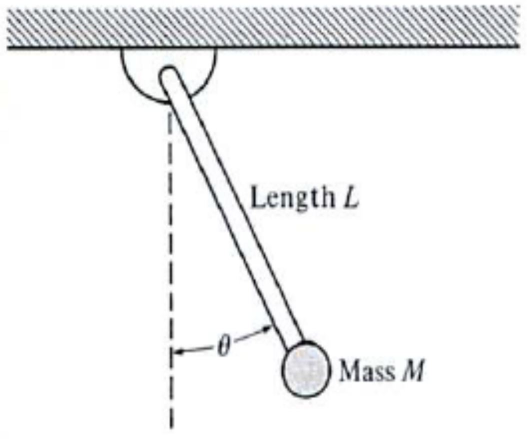
where $m$ is the slope at the operating point. From this we get:

$$(y - y_0) = m(x - x_0) \qquad \text{or} \qquad \Delta y = m\,\Delta x$$

This linear approximation is as accurate as the assumption of small signals is applicable to a specific problem.

## Linearisation example – pendulum oscillator

(Refer to Dynamics Tutorial Sheet 1, Question 1 for the model of this system)




This is exactly the same problem as in Q1 on Dynamics Tutorial sheet 1, but using different notation.

So, we know that the equation of motion for this system is given by:

$$ML^2\ddot{\theta} + MgLsin\theta = 0$$

where the *restoring torque* part of the equation is:

$$T_{rest} = MgLsin\theta$$

The equilibrium condition for the mass is

$$\theta_0 = 0°$$

The first derivative evaluated at equilibrium gives us the linear approximation:

$$T_{rest} = \theta_0 + MgL\frac{d\ sin\theta}{d\theta}|_{\theta=\theta_0}(\theta - \theta_0)$$

So, $$T_{rest} = 0 + MgLcos\ 0^{\circ}\ (\theta - \theta_0) = MgL\theta$$

We already knew this from informal use of the MacLaurin expansion for $sin\theta$ for small $\theta$.

This approximation is in fact reasonably okay for angles up to 30° (response of the linearised model is within 2% of the nonlinear model's value even at that deflection of the pendulum).

# The Laplace Transform

Laplace transformation involves a *domain change* so that the problem can be converted from one using differential equations into one which is algebraic.

This defines the process of Laplace Transformation:

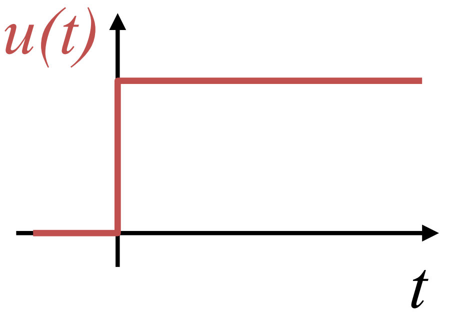
$$F(s) = \int_0^\infty f(t)e^{-st}dt = \mathbf{L}\{f(t)\}$$

Here are some useful Laplace Transforms:

| $f(t)$ | $F(s)$ |
|---|---|
| Step function, $u(t)$ | $1/s$ |
| $e^{-at}$ | $1/(s+a)$ |
| $\sin(\omega t)$ | $\omega / (s^2 + \omega^2)$ |

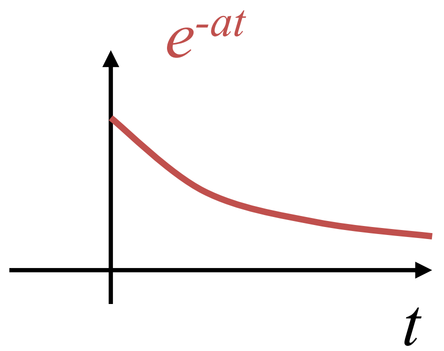
It is very useful to remember that signals that are physically possible <u>always</u> have a Laplace Transform!

Example: find the Laplace Transform of the step function *u(t)*




$$\mathbf{L}\{u(t)\} = \int_0^\infty e^{-st}dt = \left[\frac{-e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Example: find the Laplace Transform of the exponential decay $e^{-at}$




$$\mathbf{L}\{e^{-at}\} = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt$$

$$\mathbf{L}\{e^{-at}\} = \left[\frac{-e^{-(s+a)t}}{(s+a)}\right]_0^\infty = \frac{1}{(s+a)}$$

## Some common Laplace Transform pairs

| $f(t)$ | $F(s)$ |
|---|---|
| Step function, $u(t)$ | $\frac{1}{s}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $e^{-at}f(t)$ | $F(s+a)$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$ | $s^k F(s) - s^{k-1} f(0^+) - s^{k-2} f'(0^+) - \cdots - f^{(k-1)}(0^+)$ |
| $\int_{-\infty}^{t} f(t)dt$ | $\frac{F(s)}{s} + \frac{\int_{-\infty}^{0} f\,dt}{s}$ |
| Impulse function $\delta(t)$ | $1$ |

Once the Laplace Transformation of the problem has been done, and the algebraic problem that then arises has been solved, the result has to be 'inverse transformed' back into the original domain. We can directly use the pairs in the table opposite, as long as they are in recognisable form. Partial fractions are frequently used to achieve that in practice. Examples to follow.

# The Inverse Laplace Transform and Laplace Transforms for derivatives

The inverse LT is given by the following:

$$f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} F(s)\, e^{st} ds$$

where $\alpha$ is some real number representing the numerical limit of the function. We rarely need to use this explicitly for reasons already given.

The Laplace Transformation of the derivative (taking the time domain as the base domain from which we transform to the $s$ domain, but it could equally be another physically real domain) is given by:

$$L\left[\frac{dx}{dt}\right] = sX(s) - x(0^+)$$

$$L\left[\frac{d^2x}{dt^2}\right] = s^2X(s) - sx(0^+) - \frac{dx(0^+)}{dt}$$

We now have enough information to attempt to solve a second order linear differential equation such as the S-M-D system earlier in these notes:

$$M\ddot{y} + b\dot{y} + ky = r(t)$$

$$M\ddot{y} + b\dot{y} + ky = r(t)$$

We start by taking the LTs of each term in the equation, like this:

$$M\left(s^2Y(s) - sy(0^+) - \frac{dy(0^+)}{dt}\right) + b(sY(s) - y(0^+)) + kY(s) = R(s)$$

Now, for a free vibration context we have $r(t) = 0$

Initial conditions are included here: $y(0^+) = y_0$ and $\frac{dy(0^+)}{dt} = \dot{y}_0$ (often zero)

So, we get a simplified form now:

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0$$

Now we can simply solve for $Y(s)$ to get:

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}$$

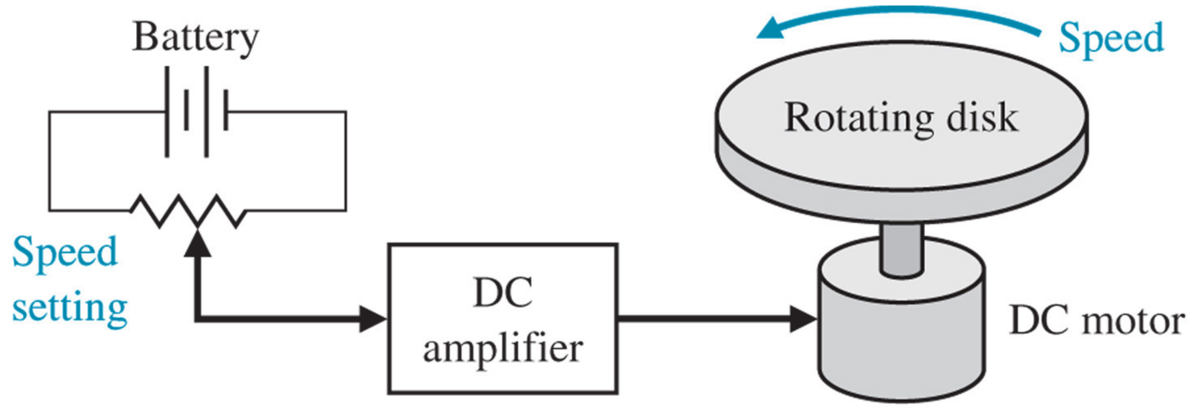
**Exercise**: Take k/M = 2 and b/M = 3 and try to show that:

$$Y(s) = \frac{(s + 3)y_0}{(s + 1)(s + 2)}$$

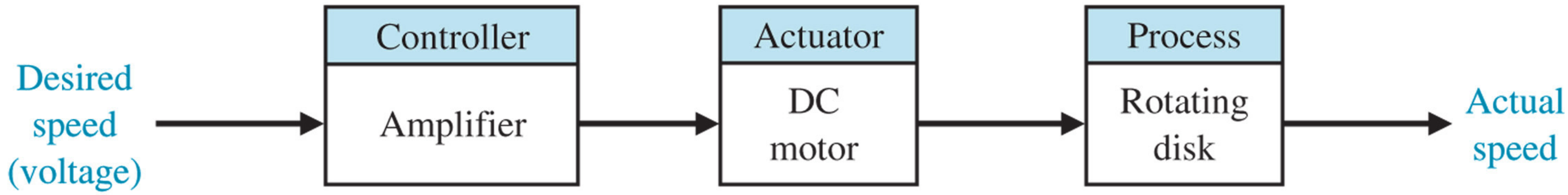
$$\therefore \quad Y(s) = \frac{C_1}{s + 1} + \frac{C_2}{s + 2}$$

Note that $q$(s), when set to zero, is called the characteristic equation – the roots of this are called **poles.** The roots of $p$(s) are called **zeros. Poles and zeros are critical frequencies.**

# Open loop control of the speed of a motor




(a)

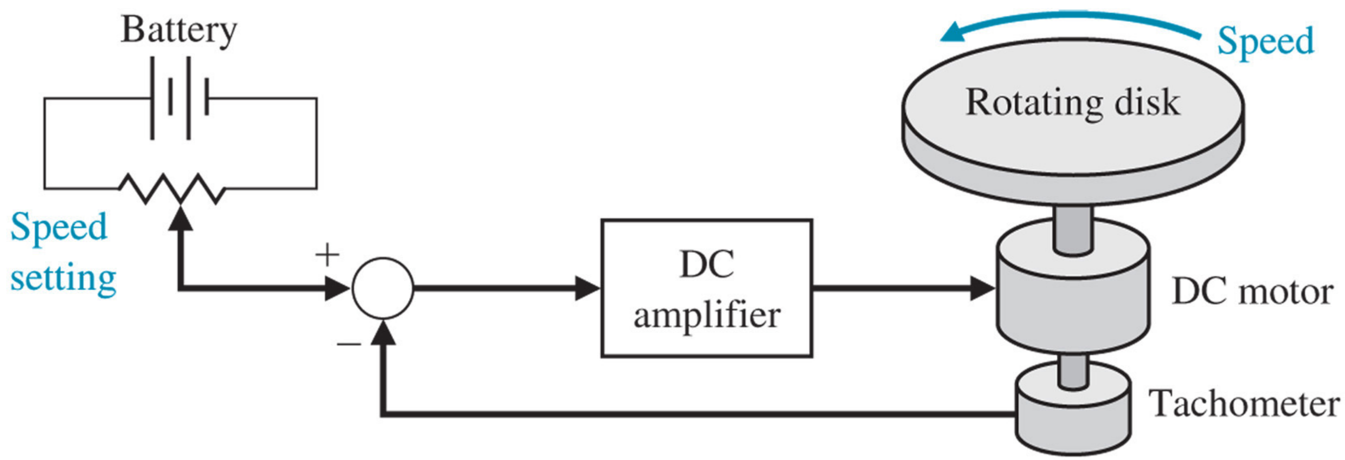



(b)

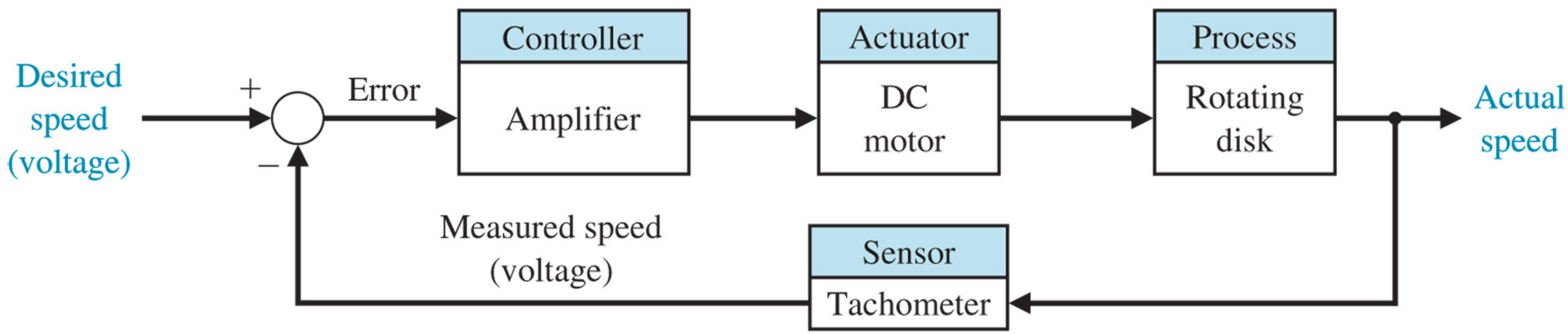


**Class Exercise: Can you now construct a closed loop control block diagram which includes feedback?**

# Closed loop control of the speed of a motor




(a)




(b)

**Exercise**: Can you construct a block diagram of a driver-based control system which will maintain the forward speed of a car?

**Solution**: driver-based control system for speed control of a car