Lecture 2 – Part 1

Mathematical Modelling

Laplace Transforms and Transfer Functions

1. Laplace Transform example – spring mass damper dynamics.

2. Finding Transfer Functions for linear systems.

3. Applying Laplace Transforms and Transfer Function theory to a real machine.

$$M\ddot{y} + b\dot{y} + ky = r(t)$$

We start by taking the LTs of each term in the equation, like this:

$$M\left(s^2Y(s) - sy(0^+) - \frac{dy(0^+)}{dt}\right) + b(sY(s) - y(0^+)) + kY(s) = R(s)$$

Now, for a free vibration context we have $r(t) = 0$

Initial conditions are included here: $y(0^+) = y_0$ and $\frac{dy(0^+)}{dt} = \dot{y}_0$ (often zero)

So, we get a simplified form now:

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0$$

Now we can simply solve for $Y(s)$ to get:

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}$$

Exercise: Take k/M = 2 and b/M = 3 and try to show that:

$$Y(s) = \frac{(s+3)y_0}{(s+1)(s+2)}$$

$$\therefore \quad Y(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2}$$

Note that q(s), when set to zero, is called the characteristic equation – the roots of this are called **poles.** The roots of p(s) are called **zeros. Poles and zeros are critical frequencies.**

Solution:

We have the following:

$\frac{k}{M} = 2$ and $\frac{b}{M} = 3$ so $k = 2M$ and $b = 3M$

Therefore,

$$Y(s) = \frac{(Ms+3M)y_0}{(Ms^2+3Ms+2M)} = \frac{(s+3)y_0}{(s^2+3s+2)} = \frac{(s+3)y_0}{(s+1)(s+2)} = \frac{p(s)}{q(s)}$$

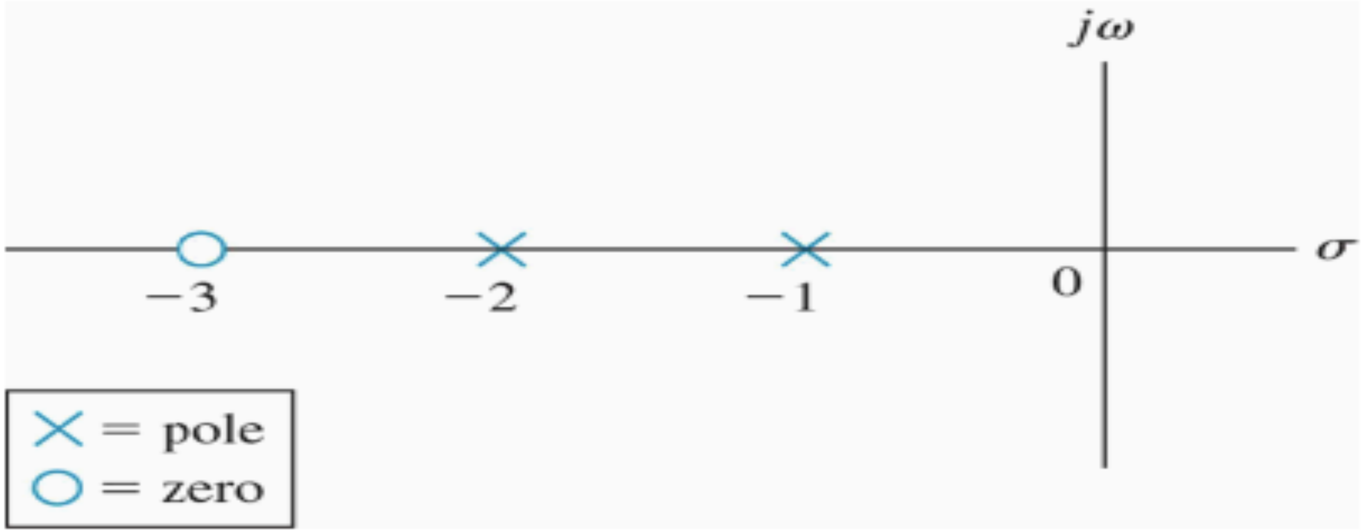
Expanding this by partial fractions leads to:

$$Y(s) = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)}$$

We recall that the roots of p(s) are the *zeros* of the system and the roots of q(s) are the *poles*. Poles and zeros are critical frequencies.

At the poles the function Y(s) becomes infinite.

At the zeros the function Y(s) becomes zero.



Pole-zero mapping for $Y(s)$ on the complex frequency s-plane

$p(s) = (s + 3)y_0$ The zero is $s = -3$

$q(s) = (s + 1)(s + 2)$ The poles are $s = -1, -2$

They are all real so they all lie on the σ axis.

Going back to $Y(s)$:

$$Y(s) = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)}$$

The quantities C_1 and C_2 are called the *residues*. We evaluate the residues at the respective poles next.

$$Y(s) = \frac{(s+3)y_0}{(s+1)(s+2)} = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)}$$

Therefore:

$$\frac{(s+3)y_0}{(s+1)(s+2)} = \frac{C_1(s+2)}{(s+1)(s+2)} + \frac{C_2(s+1)}{(s+1)(s+2)}$$

So:

$$(s+3)y_0 = C_1(s+2) + C_2(s+1)$$

Setting s equal to the poles, in turn:

$s = s_1 = -1$ for $y_0 = 1$

$$(-1+3) = C_1(-1+2) + C_2(-1+1)$$

$$2 = C_1$$

$s = s_2 = -2$ for $y_0 = 1$

$$(-2+3) = C_1(-2+2) + C_2(-2+1)$$

$$1 = -C_2$$

So, $C_1 = 2$ and $C_2 = -1$

$$Y(s) = \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

Therefore:

$$y(t) = L^{-1}\left[\frac{2}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

Giving:

$$y(t) = 2e^{-t} - 1e^{-2t} \quad \text{for} \quad y_0 = 1$$

Noting that this is the transient response because we started with $r(t) = 0$.

We also note that the solution is based on certain numerical interrelationships between mass *M*, damping *b*, and stiffness *k*, and that the displacement IC is unity, and the resulting response is decaying but not oscillatory.

More Laplace Transform examples to follow as we start to build controllers.

The Transfer Function

Definition – The TF of a linear system is the ratio of the LT of the output variable to the LT of the input variable, with all ICs assumed to be zero.

Rules:

1. A TF can only be found for a linear stationary system.
2. It is an input-output description of the behaviour of the system so it doesn't contain information about internal structure of the system.
3. Notation generally used for the TF is $G(\mathrm{s})$.
4. It is defined as $G(\mathrm{s})$ = output/input = $Y(\mathrm{s})/R(\mathrm{s})$ if $R(\mathrm{s})$ is the input or *excitation* function.

We take the spring mass damper system again, with zero ICs, and write down the Laplace Transformed equation of motion, as follows:

$$Ms^2Y(s) + bsY(s) + kY(s) = R(s)$$

The Transfer Function is therefore:

$$\text{output/input} = G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2+bs+k}$$

This comes from the algebraic structure of the Laplace Transformed equation of motion. The input is R(s) and the output is the response (in the Laplace domain) which is Y(s).