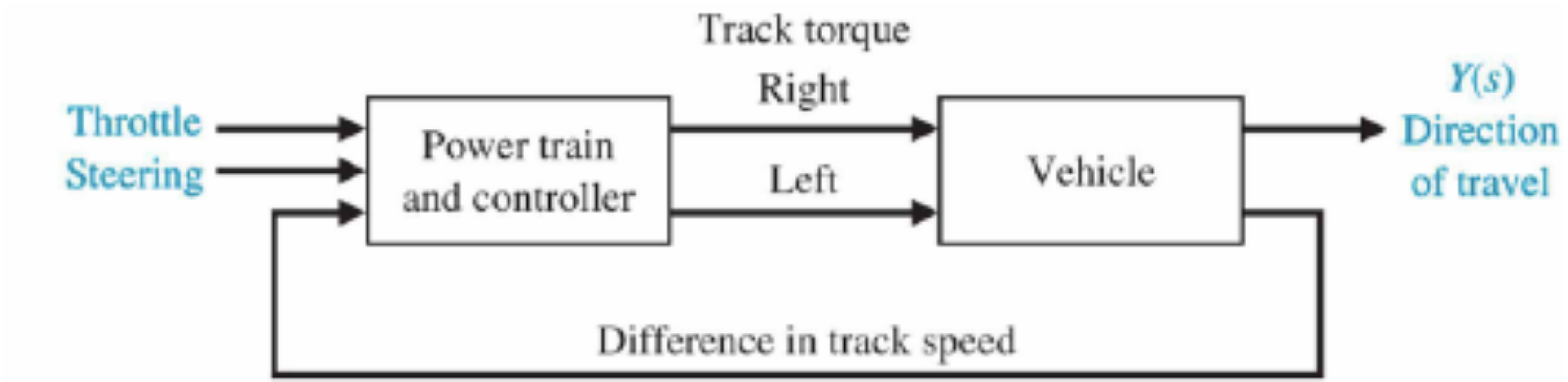
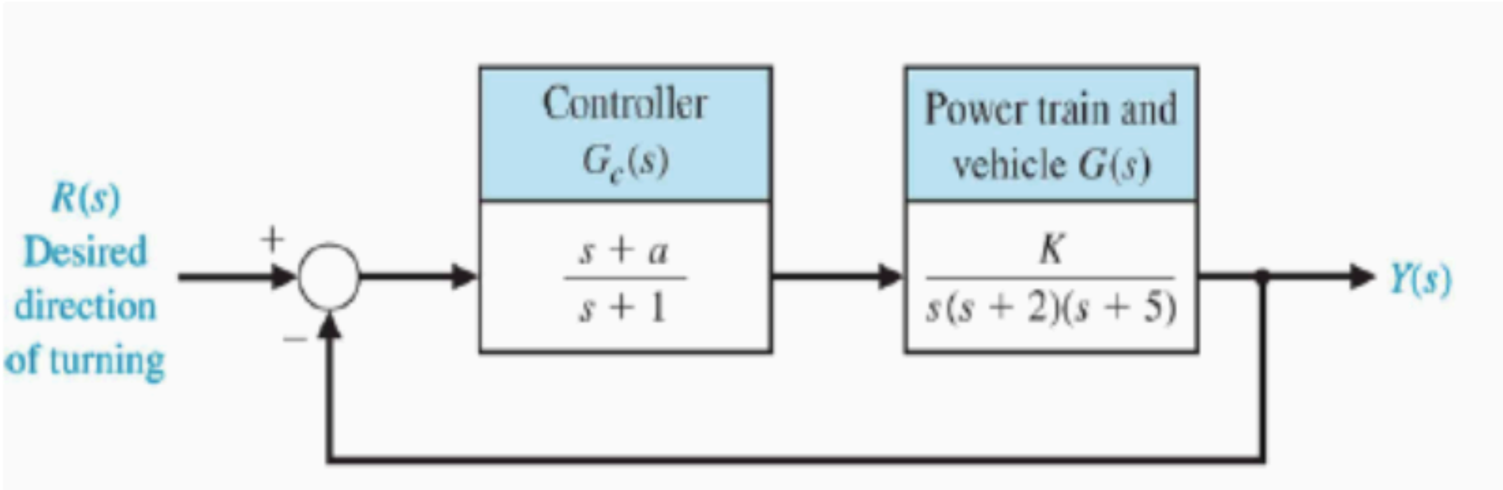
2. Stability of a tracked vehicle when turning

Tracked vehicles operate differently to steered vehicles when turning. The tracks are differentially speed controlled and when that is done the vehicle turns about a point whose precise location depends on the differential control. It is normally more or less at the centre of the vehicle if the driver has a reasonably good intuitive feel for steering this particular vehicle. In terms of the actual physical system this block diagram describes what is going on:



A mathematical model of the vehicle, with a simple controller for differential speed setting, looks like this. We want to determine a stable regime for this system, noting that this time the gains a and K relate to controller and plant, respectively.



Start by showing that the characteristic equation for the system is:

$$s(s+1)(s+2)(s+5)+K(s+a)=0$$

working:

$$G_{combined}(s) = G_c(s)G(s) = \frac{(s+a)K}{s(s+1)(s+2)(s+5)}$$

We require to construct $1 + G_{combined}(s)H(s) = 0$ which will give us the characteristic equation, so we get:

$$1 + \frac{(s+a)K}{s(s+1)(s+2)(s+5)} = 0$$

Therefore:

$$\frac{s(s+1)(s+2)(s+5) + (s+a)K}{s(s+1)(s+2)(s+5)} = 0$$

From which we only need to set the numerator to zero to satisfy the characteristic equation:

$$s(s+1)(s+2)(s+5) + (s+a)K = 0$$

Then we expand this to get:

$$s^4 + 8s^3 + 17s^2 + (K + 10)s + Ka = 0$$

Then we can try to establish the Routh-Hurwitz array.

Yet again we have a case where $n = 4$, and so we compare our tracked-vehicle characteristic equation (above) with the general characteristic equation from which the array elements can be calculated.

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_{n-1}s + a_n = 0$$

So, $a_0 = 1, \quad a_1 = 8, \quad a_2 = 17, \quad a_3 = K + 10, \quad a_4 = Ka$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{8*17 - 1*(K+10)}{8} = \frac{(126-K)}{8}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = \frac{8*Ka - 1*0}{8} = Ka$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{\frac{(126-K)}{8}*(K+10) - 8*Ka}{\frac{(126-K)}{8}} = (K+10) - \frac{8Ka}{\frac{(126-K)}{8}} = (K+10) - \frac{64Ka}{(126-K)}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} = \frac{\left[(K+10) - \frac{64Ka}{(126-K)}\right]*Ka - \frac{(126-K)}{8}*0}{\left[(K+10) - \frac{64Ka}{(126-K)}\right]} = Ka$$

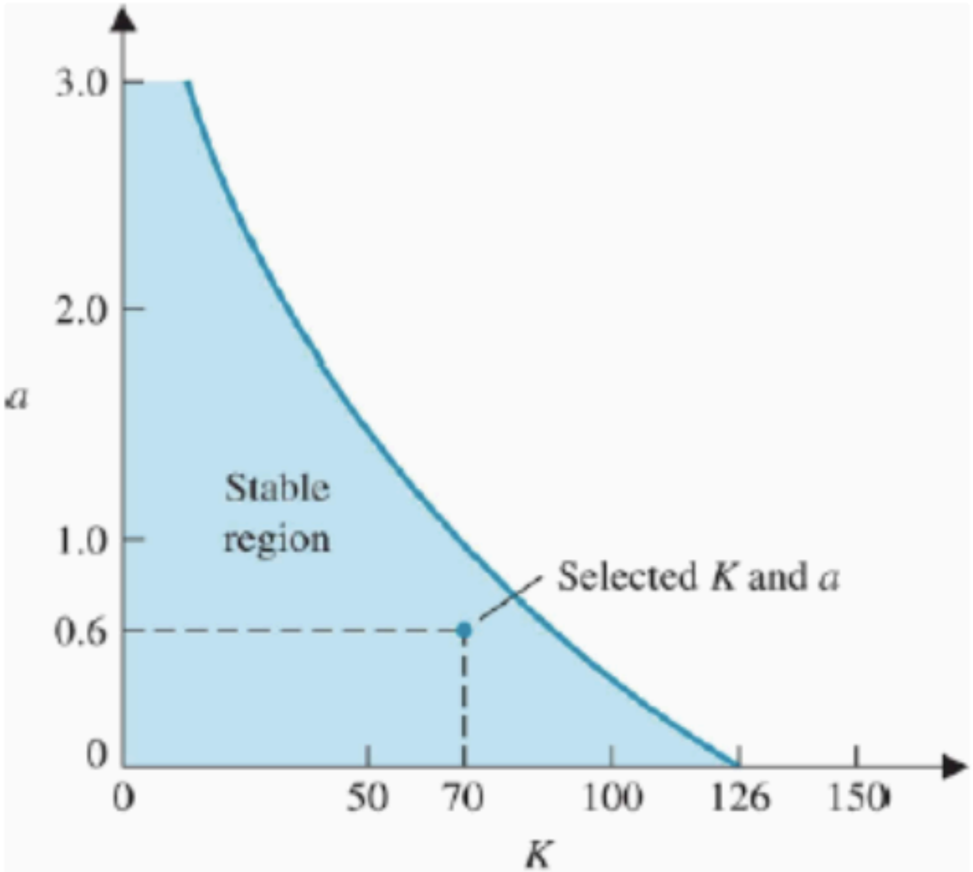
So, the R-H array now looks like this:

$s^n \ : a_0 \ a_2 \ a_4 \ a_6 \ \ldots$	$s^4: \ 1 \ \ 17 \ \ Ka \ \ \ldots$
$s^{n-1}: a_1 \ a_3 \ a_5 \ a_7 \ \ldots$	$s^3: \ 8 \ \ (K+10) \ \ \ldots$
$s^{n-2}: b_1 \ b_2 \ b_3 \ \ldots$	$s^2: \ \frac{(126-K)}{8} \ \ Ka \ \ldots$
$s^{n-3}: c_1 \ c_2 \ \ldots$	$s^1: \ \left[(K+10) - \frac{64Ka}{(126-K)}\right] \ \ldots$
$s^{n-4}: d_1 \ \ldots$	$s^0: \ Ka \ \ldots$

After this we try to establish the necessary conditions for stability according to the R-H criterion.

We see that the third element requires $K < 126$. Then we note that the fifth element requires $Ka > 0$, and finally that the third element requires:
$(K + 10)(126 - K) - 64Ka > 0.$ fourth

These conditions can be summarised in a plot of a as a function of K, as follows.



Returning to the controller block diagram we see that the overall system transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{G_{combined}(s)}{1 + G_{combined}(s)H(s)} = \frac{\dfrac{(s+a)K}{s(s+1)(s+2)(s+5)}}{1 + \dfrac{(s+a)K}{s(s+1)(s+2)(s+5)}}$$

If we take a ramp input for which we have $r(t) = At$ for $t > 0$ then we have $R(s) = \frac{A}{s^2}$.

Therefore the steady-state tracking error is:

$$E(s) = R(s)[1 - T(s)]$$

After this we can try to use the final value theorem to establish the steady-state tracking error:

$$E(s) = \frac{A}{s^2}\left[1 - \frac{\dfrac{(s+a)K}{s(s+1)(s+2)(s+5)}}{1+\dfrac{(s+a)K}{s(s+1)(s+2)(s+5)}}\right]$$

Therefore:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{sA}{s^2}\left[1 - \frac{\dfrac{(s+a)K}{s(s+1)(s+2)(s+5)}}{1+\dfrac{(s+a)K}{s(s+1)(s+2)(s+5)}}\right]$$

If we take the quantity in the [ ] first and process that for $s \to 0$ then we get this:

$$\left[1-\frac{\frac{Ka}{10s}}{1+\frac{Ka}{10s}}\right]=\left[\frac{1+\frac{Ka}{10s}-\frac{Ka}{10s}}{1+\frac{Ka}{10s}}\right]=\frac{1}{1+\frac{Ka}{10s}}$$

Therefore we now have:

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{sA}{s^2}\left[\frac{1}{1+\frac{Ka}{10s}}\right]\to\frac{A}{s+\frac{Ka}{10}}$$

(noting that this is still in the limit as $s \to 0$).

The result is this:

$$e_{ss}=\frac{10A}{Ka}$$

A common ploy in controller design is to designate a relationship between the steady-state tracking error and the magnitude of the commend signal. So, a practical (and relatively undemanding) value might be that the steady-state error for the ramp command should be less than or equal to around 25% of the magnitude of the command.

So, this means that $e_{ss} \leq 0.25A$. We can easily show from the above result that when $e_{ss} = 0.238A$ then $Ka = 42$.

This is a neat and tidy value to work with for Ka and satisfies the requirement because 23.8% is less than 25%.
Looking back at the Figure for a as a function of K we see that the point shown $(K, a) = (70, 0.6)$ is perfect – and it is within the region defined as stable in the sense of Routh-Hurwitz.

We could, however, also get this from points outwith the stable region as well, but would obviously not opt to do that if we wanted to be sure of stability.