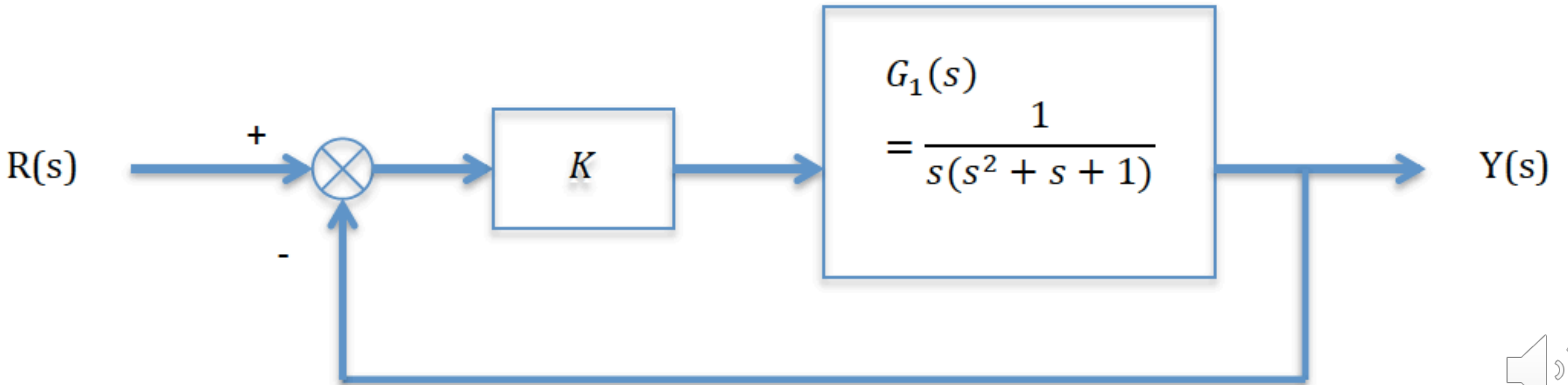
Recap of Plotting the Root Locus

Plotting the Root Locus

Finding the RL for a system already predicted to be on the threshold of stability by the Routh-Hurwitz criterion. We do this by means of a numerical example of a system with unity gain negative feedback:



The system transfer function is given by:

$$T(s) = \frac{KG_1(s)}{1 + H(s)KG_1(s)} = \frac{K}{s(s^2+s+1)\left(1 + \frac{K}{s(s^2+s+1)}\right)}$$

This is: $T(s) = \frac{K}{s(s^2+s+1)+K}$

The characteristic equation is: $s(s^2+s+1) + K = 0$

Therefore we construct the RH array for: $s^3 + s^2 + s + K = 0$

$n = 3$

The general form of the characteristic equation is:

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} = 0$$

From which we get: $a_0 = a_1 = a_2 = 1 \quad a_3 = K$

First column only

$$
\begin{array}{lllllll}
s^3: & a_0 & a_2 & \rightarrow & 1 & 1 & \rightarrow & 1 \\
s^2: & a_1 & a_3 & \rightarrow & 1 & K & \rightarrow & 1 \\
s^1: & b_1 & b_2 & \rightarrow & b_1 & b_2 & \rightarrow & 1-K \\
s^0: & c_1 & & \rightarrow & c_1 & & \rightarrow & K
\end{array}
$$

where:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{1-K}{1} = 1-K$$

$b_2 = 0$ because $a_4 = a_5 = 0$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{(1-K)K - 0}{(1-K)} = K$$

The third row suggests: $1 - K > 0$

The fourth row requires: $K > 0$

So, the criterion for stability is: $0 < K < 1$

We can test this numerically, with selected values for K.

(i) Let $K = 0.5$ and then write down the first column:

1	1
1	1
$1 - K$	0.5
K	0.5

All +ve therefore no sign changes, therefore stable.

(ii) Let $K = 1.5$ and write down the first column again:

1	1
1	1
$1 - K$	- 0.5
K	1.5

Two sign changes, therefore unstable.

(iii) Finally, let $K = 1.0$ and then write the first column for this case too:

1	1
1	1
$1 - K$	0
K	1

We have a zero in the first column.

There are two possibilities here:

(a) The element in the first column of this particular row is zero but the next element in that row is *not* zero. If this is the case then we replace the zero by a small parameter and then examine what happens to the stability prediction as that small parameter tends to zero. This is NOT the case here in this example because we have $b_2 = 0$.

So we have this situation:

First two columns

1	1
1	K
$1 - K$	0
K	

We see here that in the row of interest (this is the third row for s^1) we have two zeros, for $K = 1$, so this numerical case *does not satisfy* this first possibility.

(b) The second possibility is that we have all zeros in the row of interest, and this is indeed what we have here for $K = 1$.

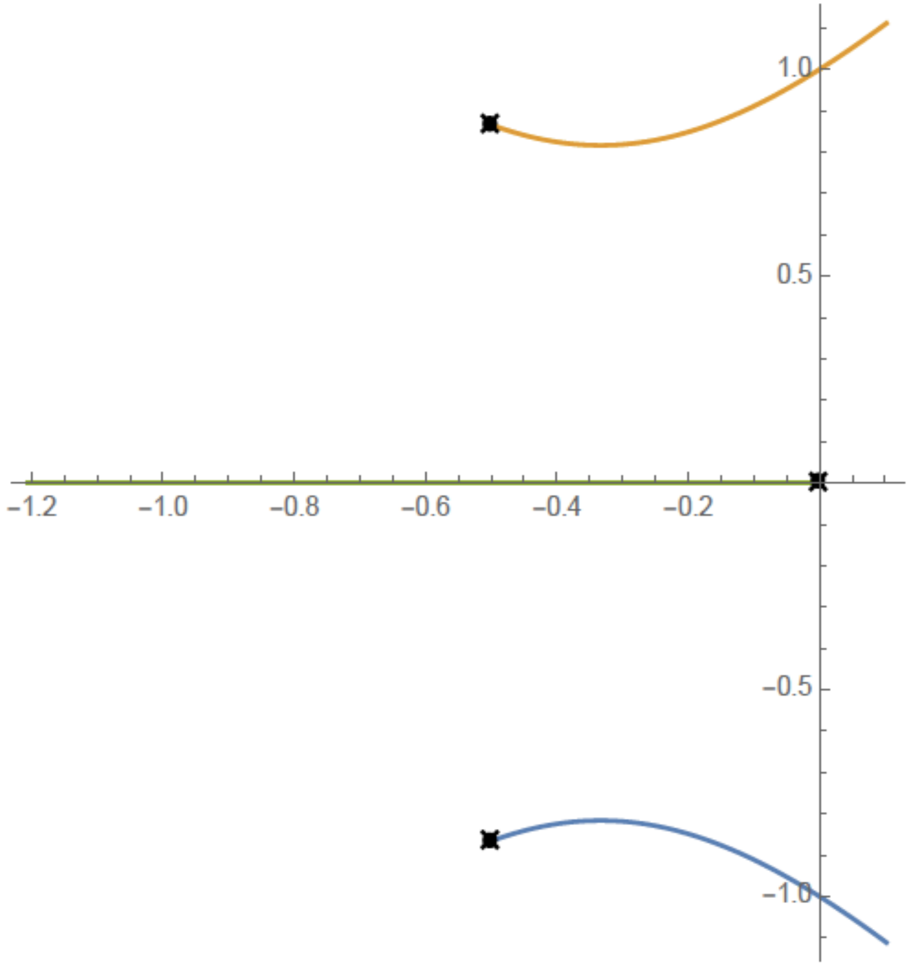
For this numerical condition we can find that the characteristic equation has *only* imaginary roots (no real parts). This indicates that the system is on the threshold of stability, and once excited it will then oscillate continuously.

We can check that this is true in this example by using *Mathematica* or MATLAB to solve the characteristic equation for $K = 1$.

This shows that the roots are $s = \pm j$.

Therefore this confirms the threshold stability condition for this example, where $K = 1$.

Here is the RL plot for this system:



$$T(s) = \frac{K}{s(s^2+s+1)+K}$$

with no zeros, a pole at $s = 0$ and two poles at:
$s = -0.5 \pm 0.866j$

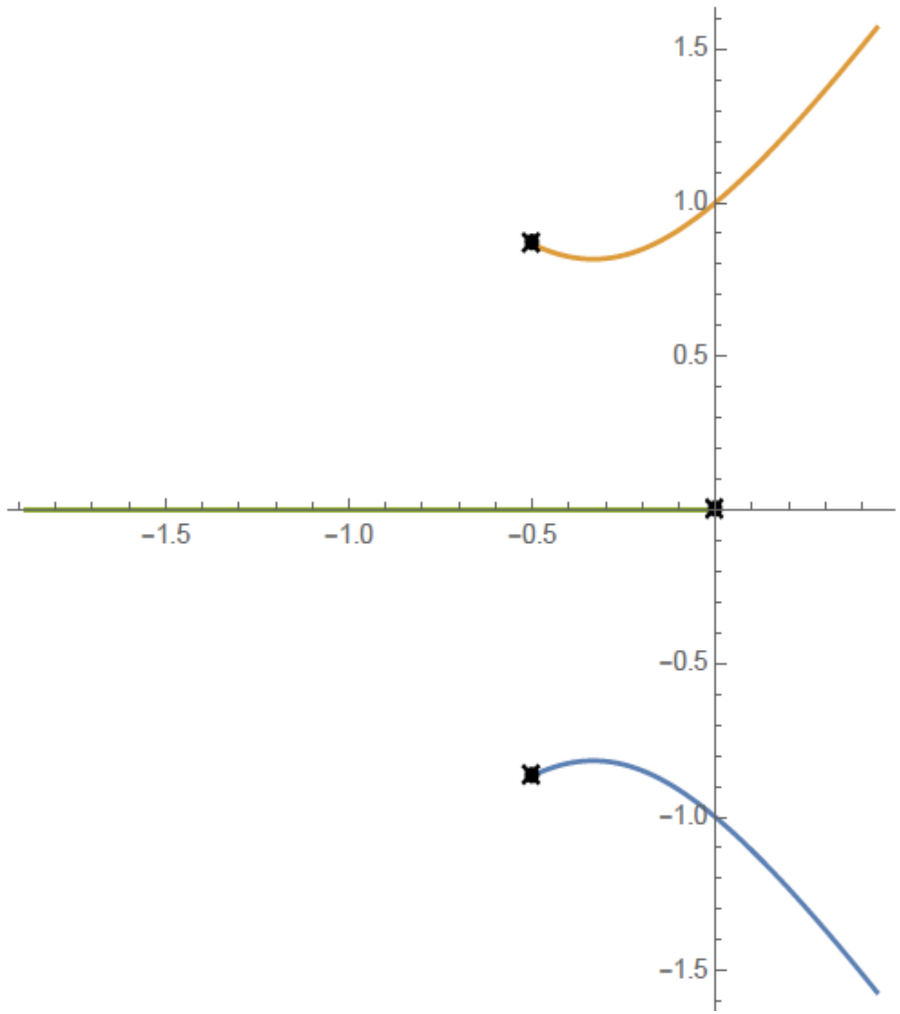
The RL is plotted by taking the characteristic equation for the above transfer function,

$$s(s^2+s+1)+K = 0$$

and substituting for K over $0 < K < 1$.

We note that there are no zeros in this case so where does the RL end?

If we plot the RL for $0 < K < 5$ we see that the RL just continues deeper within the right-hand (unstable) region of the s-plane. There are no zeros in this system to define the end of the RL segment.



We see from this example that the Routh-Hurwitz stability criterion and the Root-Locus method are very closely allied. In fact they are based on the same premise that the stability of a system is associated with roots of the characteristic equation having negative real parts.

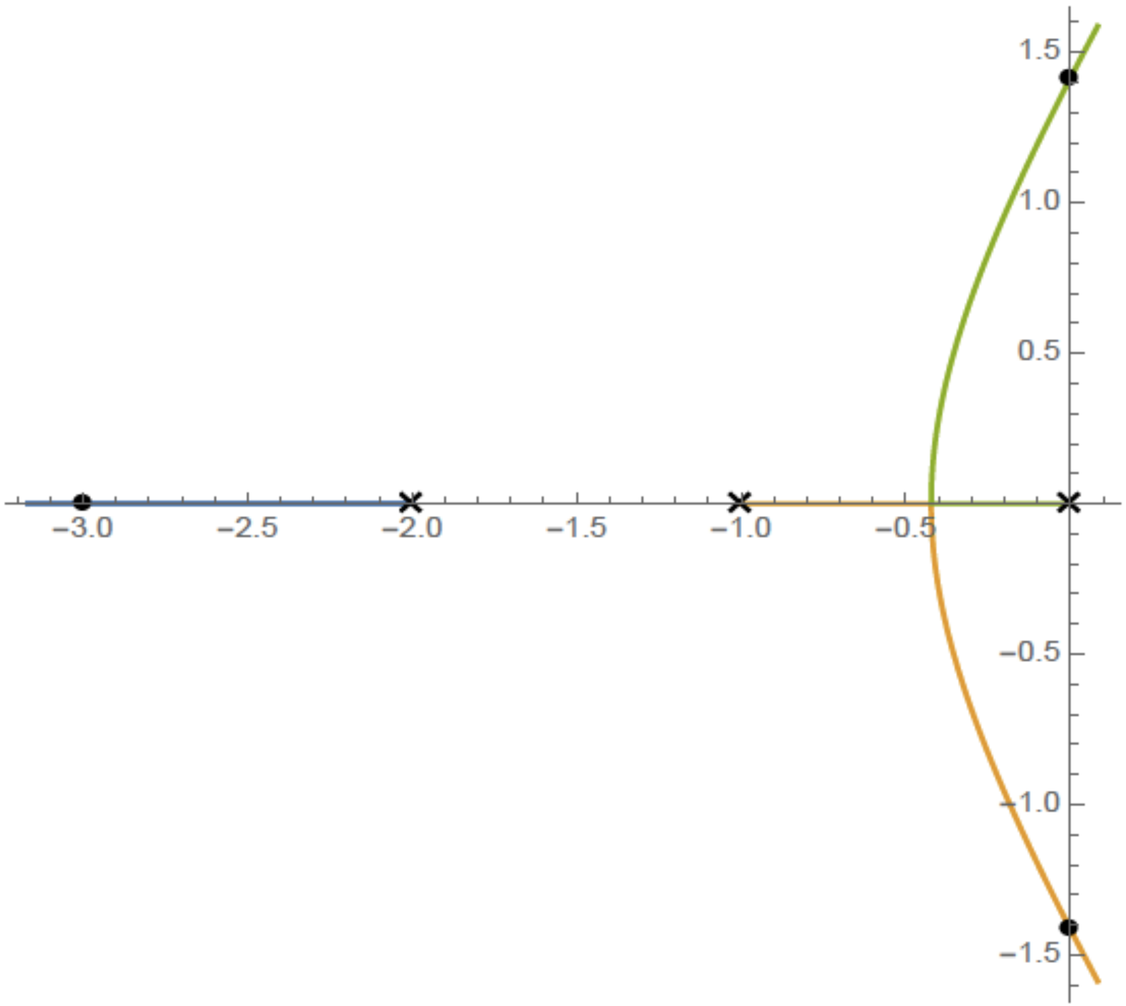
Here is another example:

$$KG_1(s) = \frac{K}{s(s+1)(s+2)}$$

The characteristic equation works out to be,

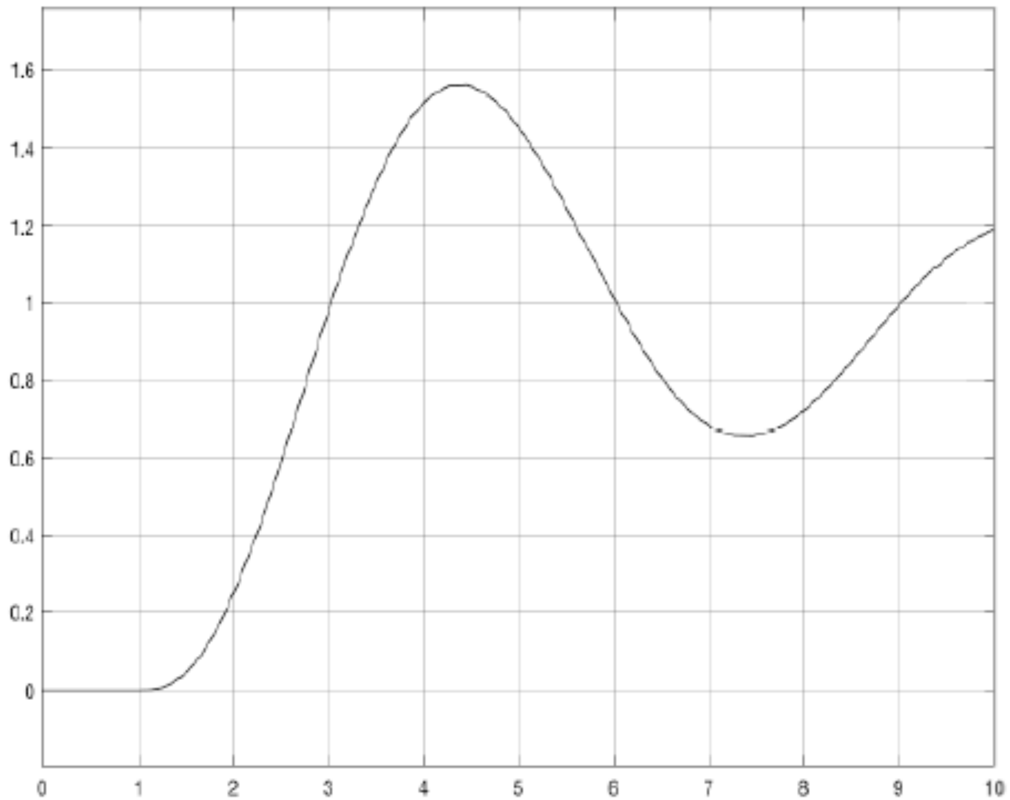
$$s(s+1)(s+2) + K = s^3 + 3s^2 + 2s + K = 0$$

This generates the following Root Locus, for which the system destabilises towards the higher end of the range $3 < K < 8$, as can be seen when we play the simulation (to be done in class).

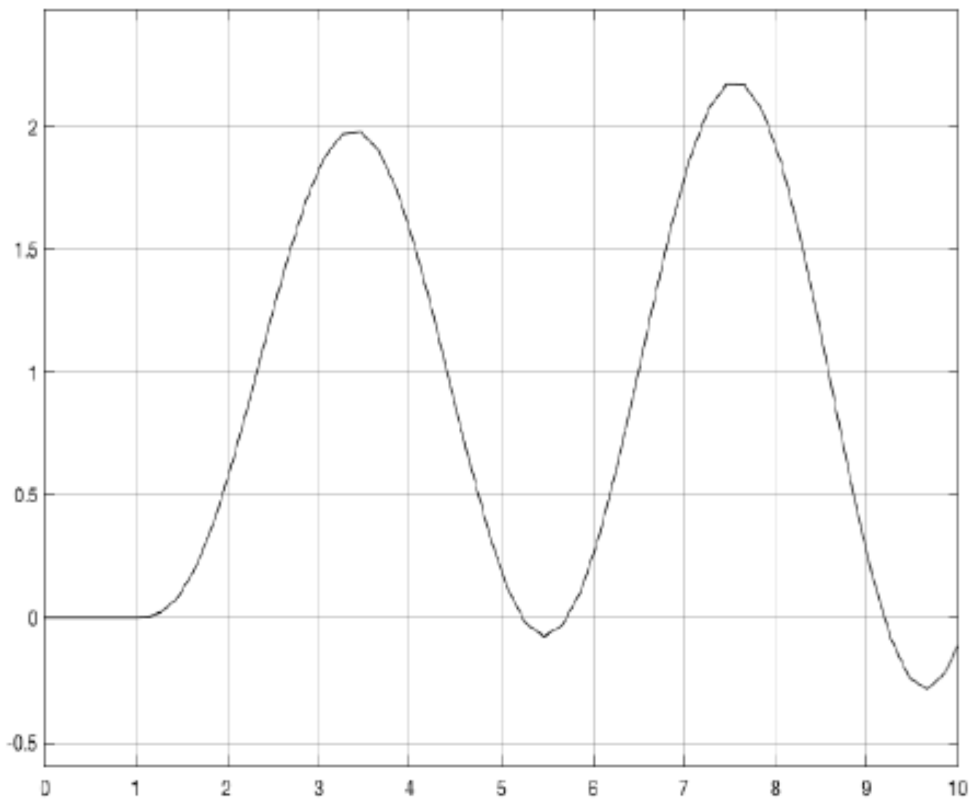


We can see the poles at $s = 0$, -1, and -2. There are no zeros once again. The black dots are not zeros they are merely points on the RL segments that relate to K. As shown the dots are for K=6.

Therefore the numerical simulation suggests that the system is on the threshold of stability at $K = 6$.



Simulink output for the time response of this unity gain controller based on K = 3. The time response is oscillatory but tending towards a stable value of unity, the value of the step function set-point.



Simulink output for this unity gain controller based on K = 7. The time response is oscillatory and growing with time, therefore obviously unstable.