Calculating the Root Locus

Lecture 8

Now we know something about the root locus we can start to try to use it. This requires us to be able to calculate it for a given problem.

1. First of all we see that the magnitude of each of the component vectors in the numerator and denominator is the distance of the point s from the zero or pole on the s-plane.

2. So, if the vector from the zero z_i to the point s on a pole-zero plot has a length r_i and makes an angle ϕ_i to the horizontal, and the vector from the pole p_i to the point s has a length q_i and an angle θ_i to the horizontal, and then the value of the transfer function at the point s is given by:

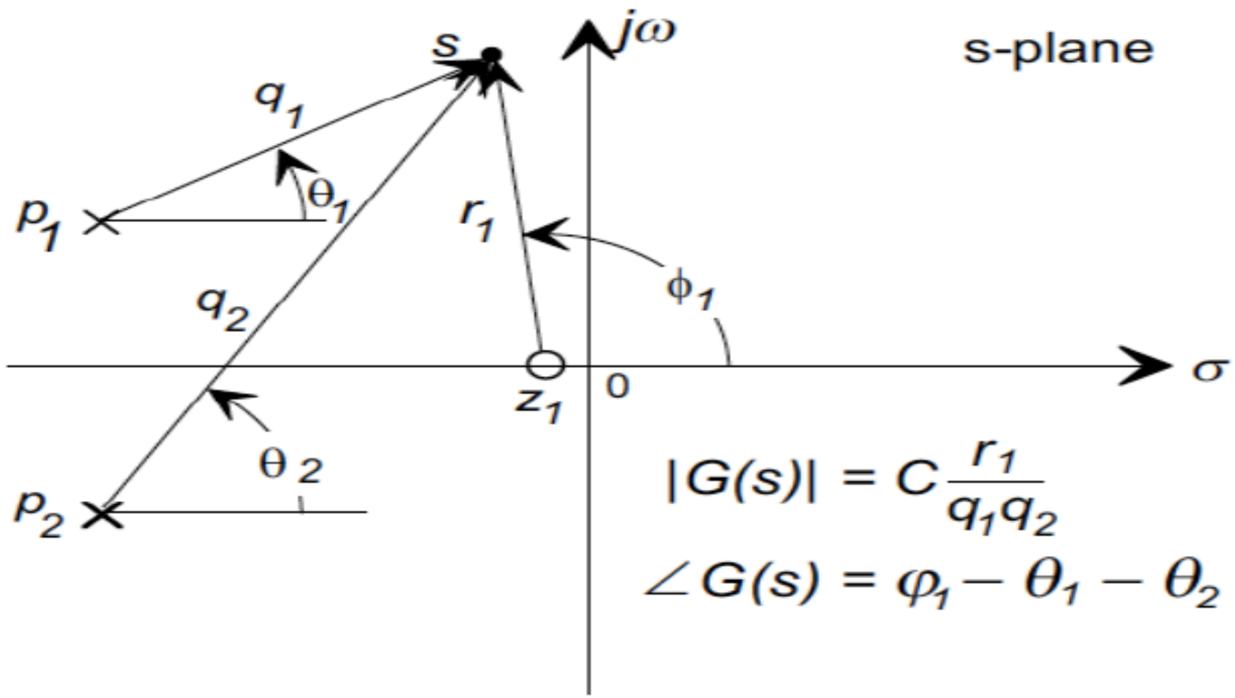
$$|G(s)| = C\frac{r_1 \ldots r_m}{q_1 \cdots q_n} \qquad \begin{matrix}(zeros)\\(poles)\end{matrix}$$

$$\angle G(s) = (\phi_1 + \cdots + \phi_m) - (\theta_1 + \cdots + \theta_n)$$

(zeros) (poles)

So,

$$\angle G(s) = \sum_{i=1}^{m} \phi_i - \sum_{i=1}^{n} \theta_i = (2n+1)\pi$$



3. Therefore the transfer function, at any value of s, can be found geometrically from the pole-zero plot, except for the overall gain factor C.

The angle condition then states that for a point $s = \sigma + j\omega$ to be on the root locus we require that:

$$\angle G(s) = \sum_{i=1}^{m} \phi_i - \sum_{i=1}^{n} \theta_i = (2n+1)\pi$$

where m is the number of zeros and n is the number of poles, and once it has been established that s lies on the root locus then the magnitude condition can be used to determine the value of K:

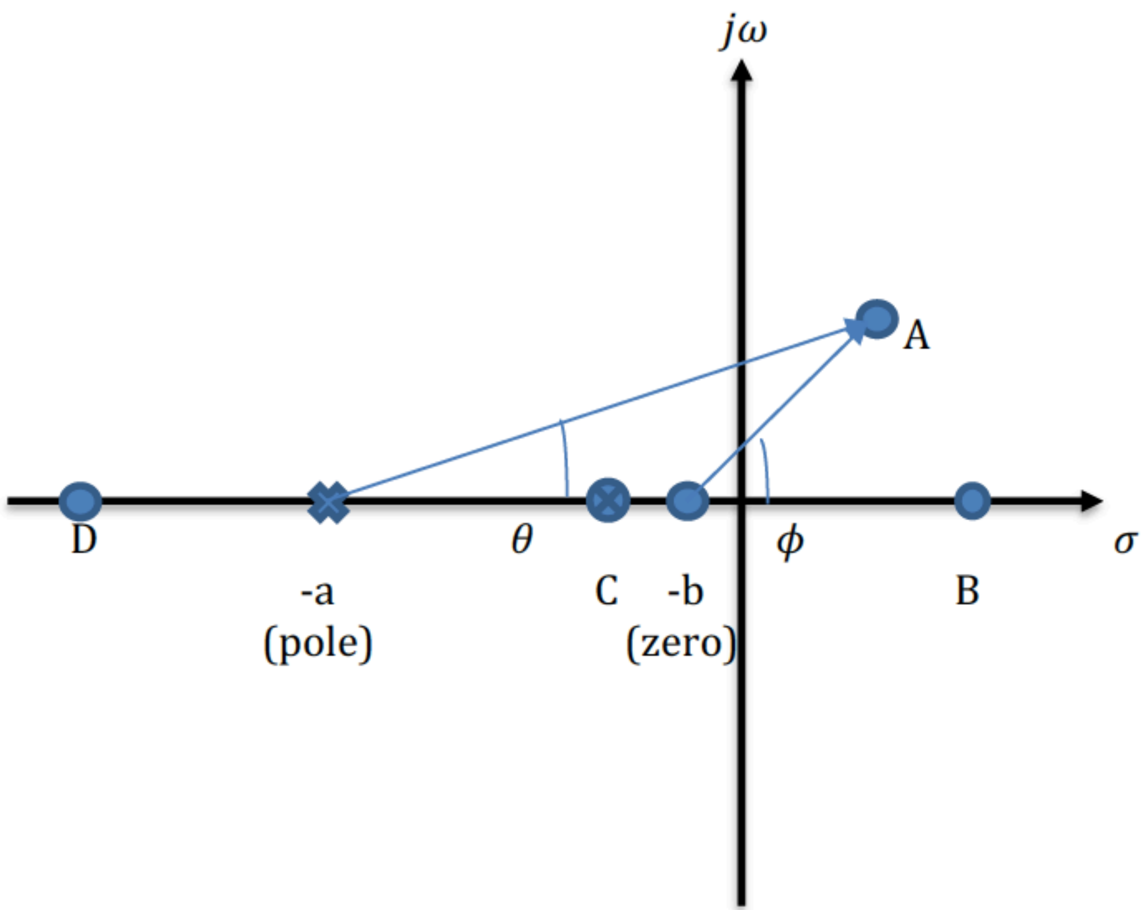
$$K = \frac{1}{|G(s)|}$$

We can now start to apply this theory to some examples.

Start with a simple OLTF:

$$G(s) = \frac{s+b}{s+a}$$

and then we can use the angle condition to see if certain points taken arbitrarily on the complex plane actually lie on the root locus. Three are on the real axis and one is complex:

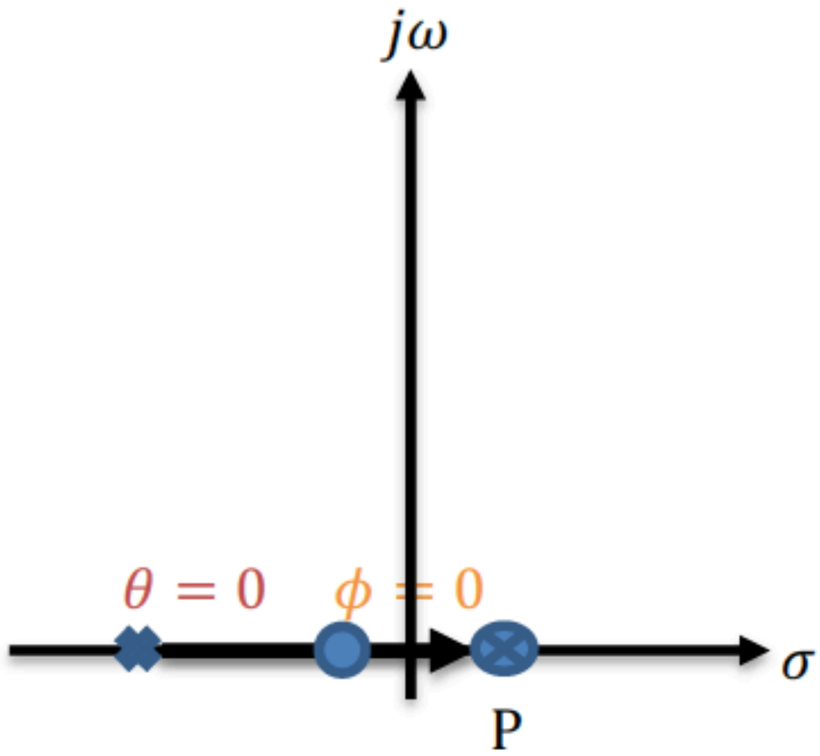


So, we start by noting that there is a zero at $-b$ and a pole at $-a$.

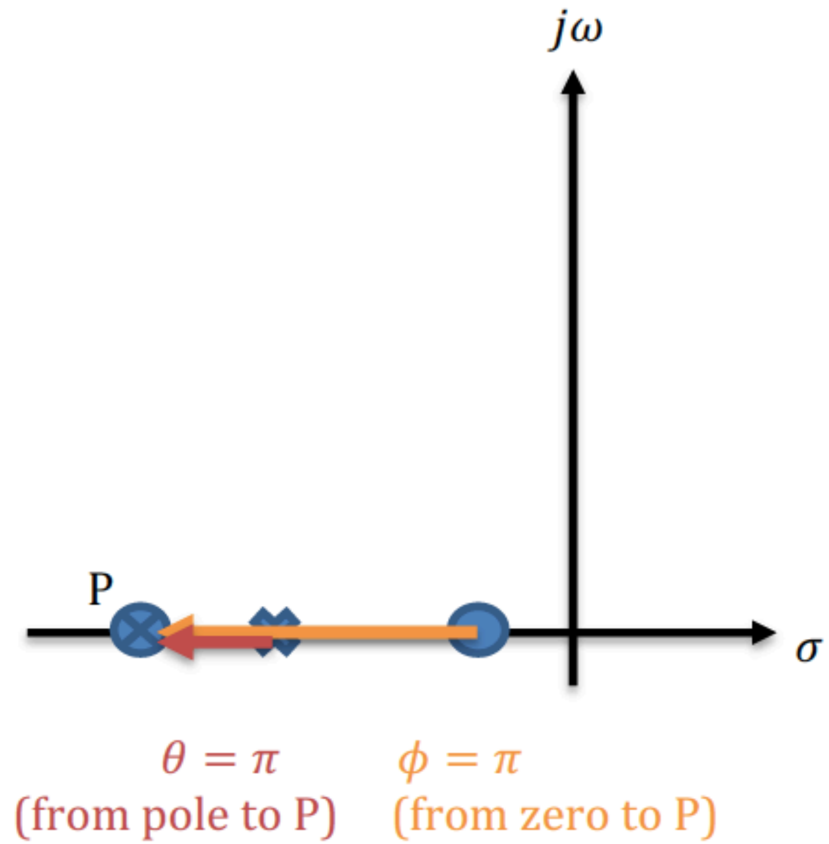
- If s lies at A then $\phi - \theta$ is some value but definitely not $(2n + 1)\pi$. **So A is not on the root locus.**
- If s lies at B then $\phi = \theta = 0$ **and so B is not on the root locus either.**
- If we now take s to be at C then ϕ goes round anticlockwise to π radians, and $\theta = 0$, so $\phi - \theta = \pi$, **so s at C is a point on the root locus.**

- Finally, if we take s to lie at D then ϕ goes round to π radians and so does θ, so $\phi - \theta = 0$, **and therefore s at D is not a point on the root locus.**

Although this is a somewhat arbitrary example it tells us something fundamentally important about points that lie on the real axis. Here are two diagrams to consider, noting that poles continue to be defined by a 'x', and zeros by a 'o'.

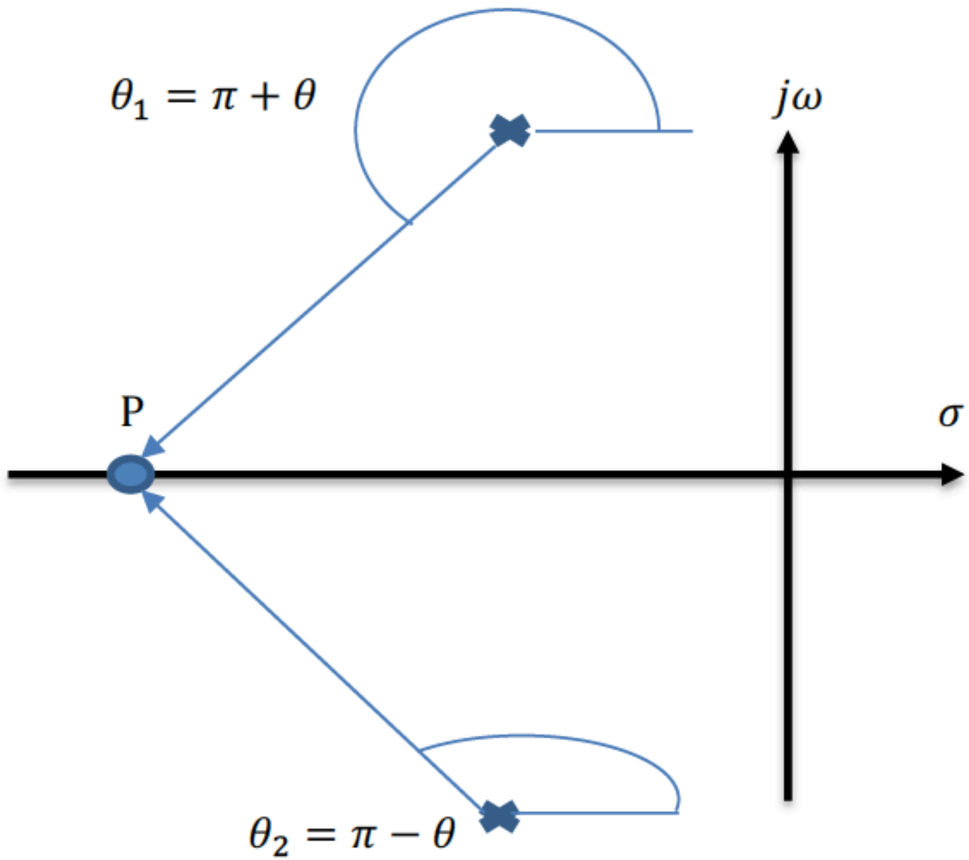


The diagram above shows a pole on the far left along the real axis and a zero just to the left of the origin. In this diagram s is at P to the right of the origin. The previous example showed that **poles and zeros that are on the real axis and lying to the left of the point of interest (denoted here by s at P) contribute zero to the angle condition.**



The diagram above here shows the pole and zero as previously but **now that they lie to the right of the point of interest (again denoted by s at P) they both contribute $\pm\pi$ to the angle condition. This is the same as point D in the previous example, and not on the root locus.**

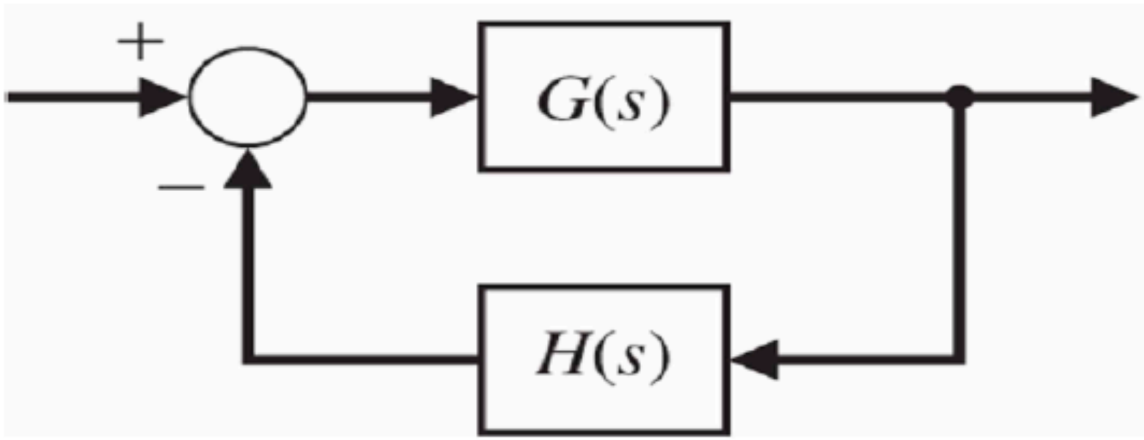
Now we look at a complex conjugate pole or zero pair (shown below as a cc pole). Note that as this is a pair then there are **two poles** here.



No zeros, so the ϕ_i terms disappear and instead we have the upper pole contributing $\theta_1 = \pi + \theta$ and the lower pole contributing $\theta_2 = \pi - \theta$, so the total contribution to the angle condition is 2π which is equivalent to zero.

We conclude that a point *s* lying on the real axis will <u>only</u> lie on the root locus <u>if there are an *odd number* of poles and/or zeros to its right</u> - as for point C in the previous-but-one example.

Practical Development of the theory – we start with a simple negative feedback closed loop controller, where the forward loop block $G(s)$ could be written as $K\bar{G}(s)$ leading to the usual form of the characteristic equation.



The characteristic equation is given by:

$$1 + K\bar{G}(s)H(s) = 0$$

We now take a specific numerical case and examine how the Root Locus theory can be applied to it, and in so doing generate two important rules.

$$K\bar{G}(s)H(s) = K\frac{\left(\frac{s}{2}+1\right)}{s\left(\frac{s}{4}+1\right)}$$

Therefore we have:

$$1 + K\frac{\left(\frac{s}{2}+1\right)}{s\left(\frac{s}{4}+1\right)} = 0$$

This reduces to:

$$1 + 2K\frac{(s+2)}{s(s+4)} = 0$$

Our ultimate aim is to find the locus of the roots for positive K (noting that gain can't be negative in practice) i.e. $0 < K \leq \infty$. We start by identifying the poles and zeros. There is one zero which is $s = -2$ and two poles which are $s = 0$ and $s = -4$. The zero and the poles all obviously lie on the real axis.
We start with the pole at the origin.

Then we move along the real axis to the left of the origin to a zero at -2.

The line (on the real axis) joining 0 to -2 is the first segment of the Root Locus (RL).

Then we move to the next pole at -4 and we note that the line from this pole, to the left (with no obvious end) is the second segment of the RL.

We use this by appreciating that the Root Locus theory can be used to calculate specific values of K for specific points on the RL.

We see it confirmed here that RL segments begin at a pole and end at a zero. This is a general rule.

Numerical check – informally first of all:

(1) We take $s = s_1 = -1$ (this is within the first segment of the RL, immediately to the left of the origin) and substitute this into the characteristic equation to get:

$$1 + \frac{2K(-1+2)}{(-1)(-1+4)} = 0$$

$$1 + \frac{2K}{-3} = 0, \quad K = \frac{3}{2}$$

(2) Then we take $s = s_2 = -5$ (within the second segment of the RL) and substitute to find:

$$1 + \frac{2K(-5+2)}{(-5)(-5+4)} = 0$$

$$1 + \frac{2K(-3)}{5} = 0, \qquad K = \frac{5}{6}$$

(3) Finally we take a value for s which is even further left on the second segment of the RL such that $s = s_3 = -6$ to find that:

$$1 + \frac{2K(-6+2)}{(-6)(-6+4)} = 0$$

$$1 + \frac{2K(-4)}{12} = 0, \qquad K = \frac{12}{8} = \frac{3}{2}$$

It can be seen from these calculations that $K = \frac{3}{2}$ for $s = s_1 = -1$ and $s = s_3 = -6$, therefore there are two roots on the RL for this gain value.

This example shows that the number of separate regions of the RL equals the number of poles – <u>if the number of poles is greater than the number of zeros</u> (and it usually is). This is another general rule.

We can also do these numerical checks using the magnitude condition as well. Recall that this came from the OLTF theory (given in L7, slide 4), as:

$$K = \frac{1}{|G(s)H(s)|}$$

Therefore we have:

$$K = \frac{1}{\left|\frac{(2s+4)}{s(s+4)}\right|}$$

(1) $s = s_1 = -1$

$$K = \frac{1}{\left|\frac{(-2+4)}{-1(-1+4)}\right|} = \frac{1}{\left|\frac{2}{-3}\right|} = \frac{3}{2}$$

(2) $s = s_2 = -5$

$$K = \frac{1}{\left|\frac{(-10+4)}{-5(-5+4)}\right|} = \frac{1}{\left|\frac{-6}{5}\right|} = \frac{5}{6}$$

(3) $s = s_3 = -6$

$$K = \frac{1}{\left|\frac{(-12+4)}{-6(-6+4)}\right|} = \frac{1}{\left|\frac{(-8)}{12}\right|} = \frac{3}{2}$$

Gains associated with points that can be shown to be on the RL are associated with stability of the control system.