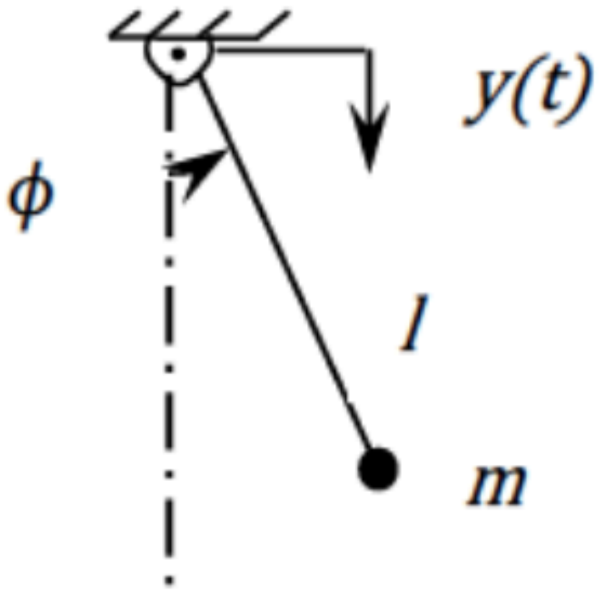
Stability of a simple pendulum under parametric excitation

The behaviour of the vertically cantilevered beam in the video can be fairly closely approximated by a simple vertical pendulum. It's easier to consider a hanging down configuration because this is unconditionally stable at rest. The behaviour of this pendulum is similar to that of the beam in the video.

Parametric excitation, for both the glass epoxy beam in the video and the pendulum example here, is when the system is driven by an input which modulates a system parameter, such as stiffness. In that case it is stiffness modulation. The system is also likely to be *nonlinear*. In this case the nonlinearity arises because when the deflection gets large, $sin\,\phi \neq \phi$, so the usual small angle approximation no longer applies.



The excitation is a base motion $y(t)$ and the pendulum motion is defined by the angular co-ordinate $\phi(t)$. The length of the pendulum is l and the mass of the bob is m, with the mass of the pendulum wire considered to be negligible. The equation of motion for this configuration shows that the gravitational restoring force is modulated by the excitation.

We can use Lagrange's equation to derive the equation of motion for the pendulum by setting up the kinetic energy and the (gravitational) potential energy. This leads to the equation of motion for a parametrically excited damped pendulum, which is:

$$ml^2\ddot{\phi} + c\dot{\phi} + \ mgl \ sin\phi - ml(y_0\Omega^2) \ cos \ (\Omega t). sin\phi = 0$$

where we take $\ddot{y}(t)$ as the acceleration of the displacement $y(t)$.

So, if the displacement is harmonic, and given by,

$$y(t) = y_0 cos(\Omega t)$$

then the acceleration associated with this is, as shown above, is:

$$\ddot{y}(t) = -(y_0\Omega^2) \ cos(\Omega t).$$

So, the peak acceleration is given by $y_0\Omega^2$. We can replace $y_0\Omega^2$ with a to simplify the notation. We can also replace ϕ with β to remind ourselves that we'll consider an undamped system here because this gives the same stability thresholding effects, without the added complication of further phenomena due to damping.

Our goal is now to find some time domain responses for this undamped parametrically excited pendulum, for different excitation amplitudes.

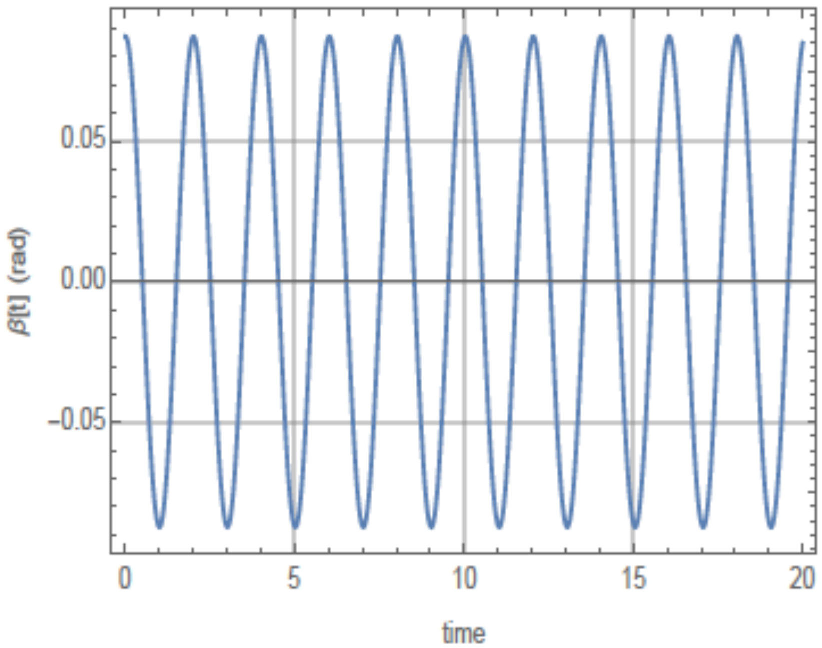
The slightly simplified equation of motion for an undamped system is therefore:

$$\ddot{\beta} + \left(\frac{g - a\,cos(\Omega t)}{l}\right) sin\beta = 0$$

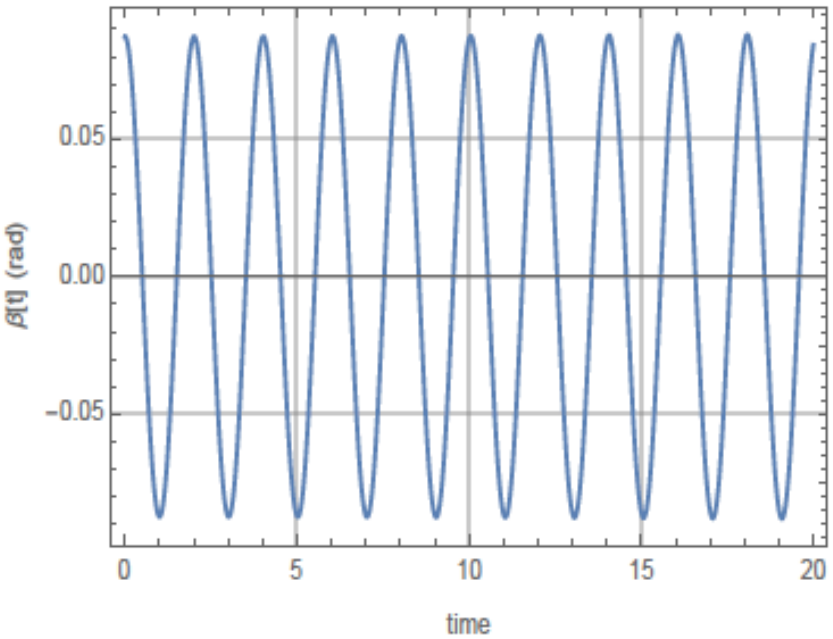
This is a nonlinear ordinary differential equation and it can't be solved by conventional linear ODE theory. There are several analytical techniques but they are not relevant to this course, so here we go for an accurate numerical integration procedure, using the powerful NDSolve package in *Mathematica*.

We select initial conditions: $\beta(0) = 5°$ (or 0.0872 rad) and $\dot{\beta}(0) = 0°/s$ (or 0 rad/s). The displacement condition is an arbitrary choice but a small deflection is a good choice as it accentuates the effect of instability when it appears. It simulated mathematically the small 'push' given to the beam in the video.

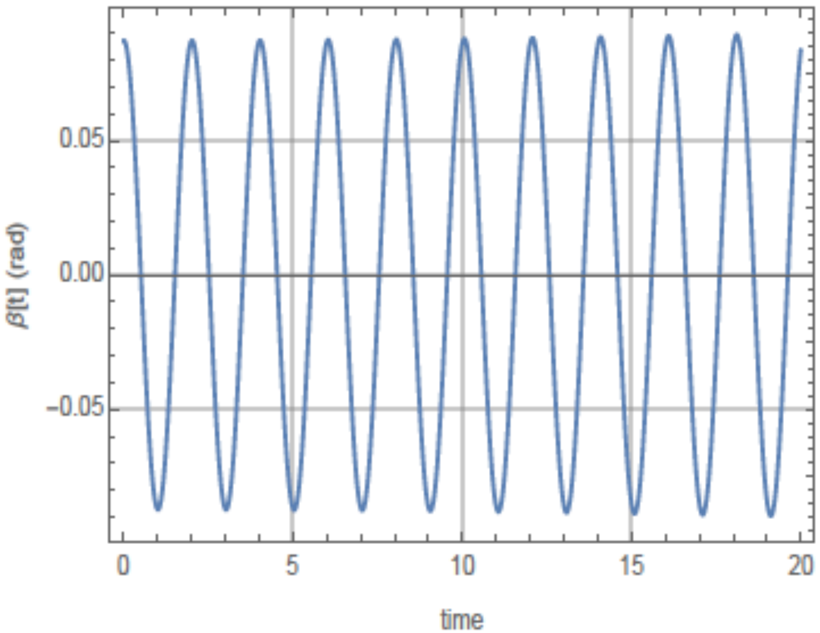
The plots below show the solutions to the equation (for β) plotted against time.



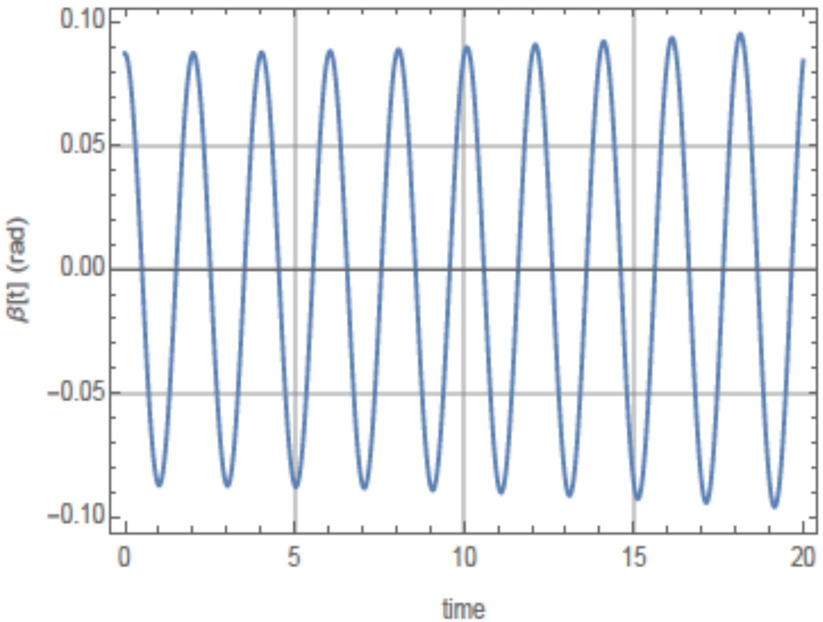
Stable response at $\beta(0)$ for a $= 0.01$



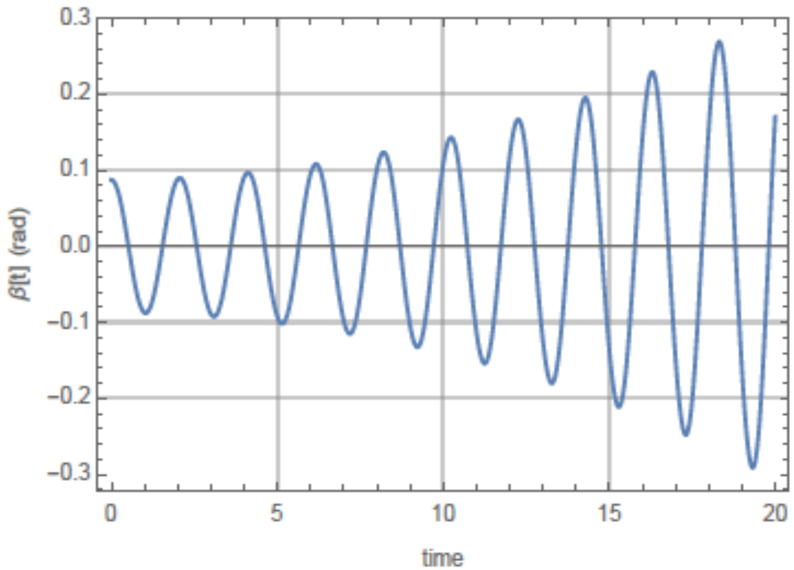
Stable response at $\beta(0)$ for a $= 0.05$

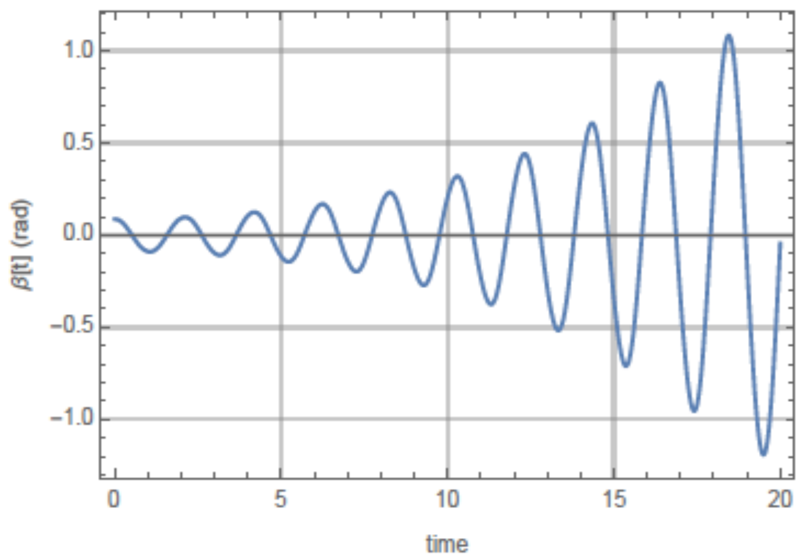


Stable response at $\beta(0)$ for a = 0.10



Just above stability threshold: unstable for a = 0.20





Well above stability threshold: unstable for $a = 1.00$

Far above stability threshold: unstable for $a = 2.00$

The main point to note here is that as soon as the magnitude of the excitation acceleration crosses the threshold, the response of the system very quickly builds up. In a linear parametrically excited system it would build up in an unbounded manner, and would appear to tend to infinity. In practice this can never happen because ALL real systems are nonlinear in one way or another, and the nonlinearity always bounds the response to a finite level. This level may still be so high that failure follows, and that's commonplace. In control terms the general objective with such a system might be either to control the magnitude of the excitation acceleration, its frequency, or both, so that the threshold into instability is never crossed.