Applying the Gain Margin and the Phase Margin

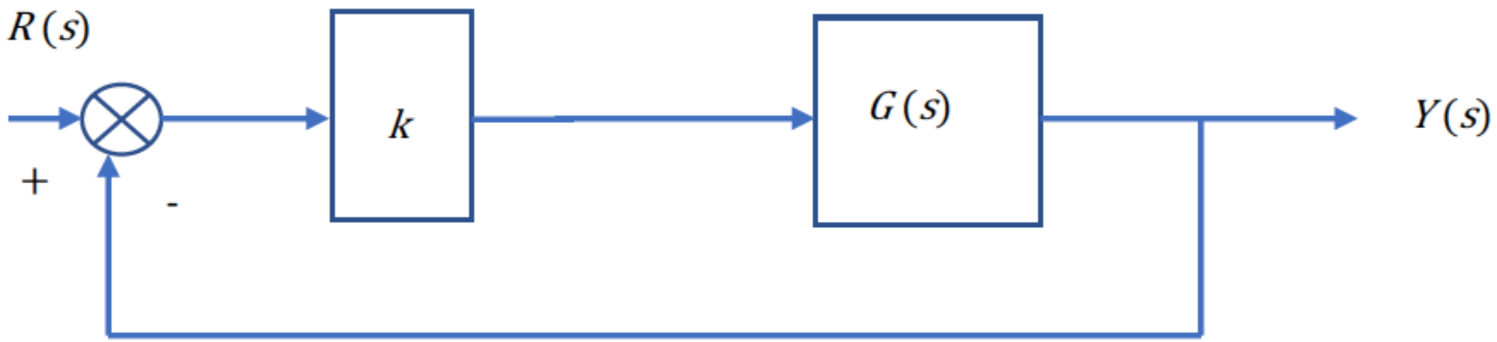
In this lecture we apply the concept of gain margin and phase margin to assess the stability of a system.

Summary: The Bode criterion comprises a pair of plots: (a) Log Magnitude against Log frequency and (b) Angle (phase) against Log frequency:

$20\,log_{10}|G(j\omega)|$ against $log_{10}\omega$ (magnitude)

$\angle G(j\omega)$ against $log_{10}\omega$ (phase)

Now we can examine this in the usual context of the closed loop system with negative feedback, where $H(s) = 1$:



$$T(j\omega) = \frac{kG(j\omega)}{1 + kG(j\omega)}$$

In the simplest case of $k = 1$ (and re-writing s as $j\omega$) then the magnitude of the transfer function can be written as,

$$|T(j\omega)| = \frac{|G(j\omega)|}{|1 + G(j\omega)|}$$

In Bode notation this becomes,

$$20log_{10}|T(j\omega)| = 20log_{10}|G(j\omega)| - 20log_{10}|1 + G(j\omega)|$$

From previous work involving the Characteristic Equation we are always interested in $1 + G(j\omega) \to 0$, so it is the behaviour of the second term on the RHS above that particularly interests us, for the condition when $1 + G(j\omega) \to 0$. This means that,

$$\lim_{1+G(j\omega)\to 0} -20log_{10}|1 + G(j\omega)| = \infty$$

This represents an instability, in the context of Bode, for the case where $1 + G(j\omega) = 0$.

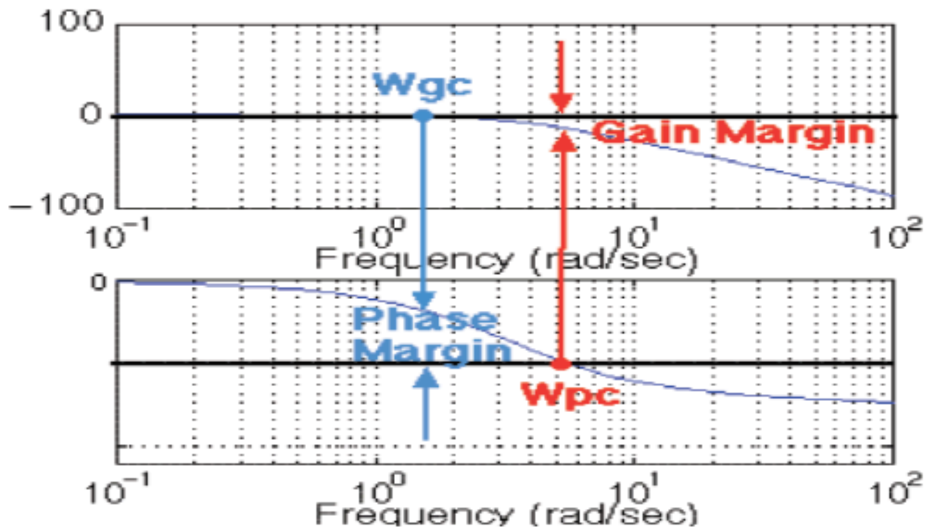
We know from our root locus work (L7 notes) that for this to happen we need two underlying conditions:

$$|G(j\omega)| = 1$$

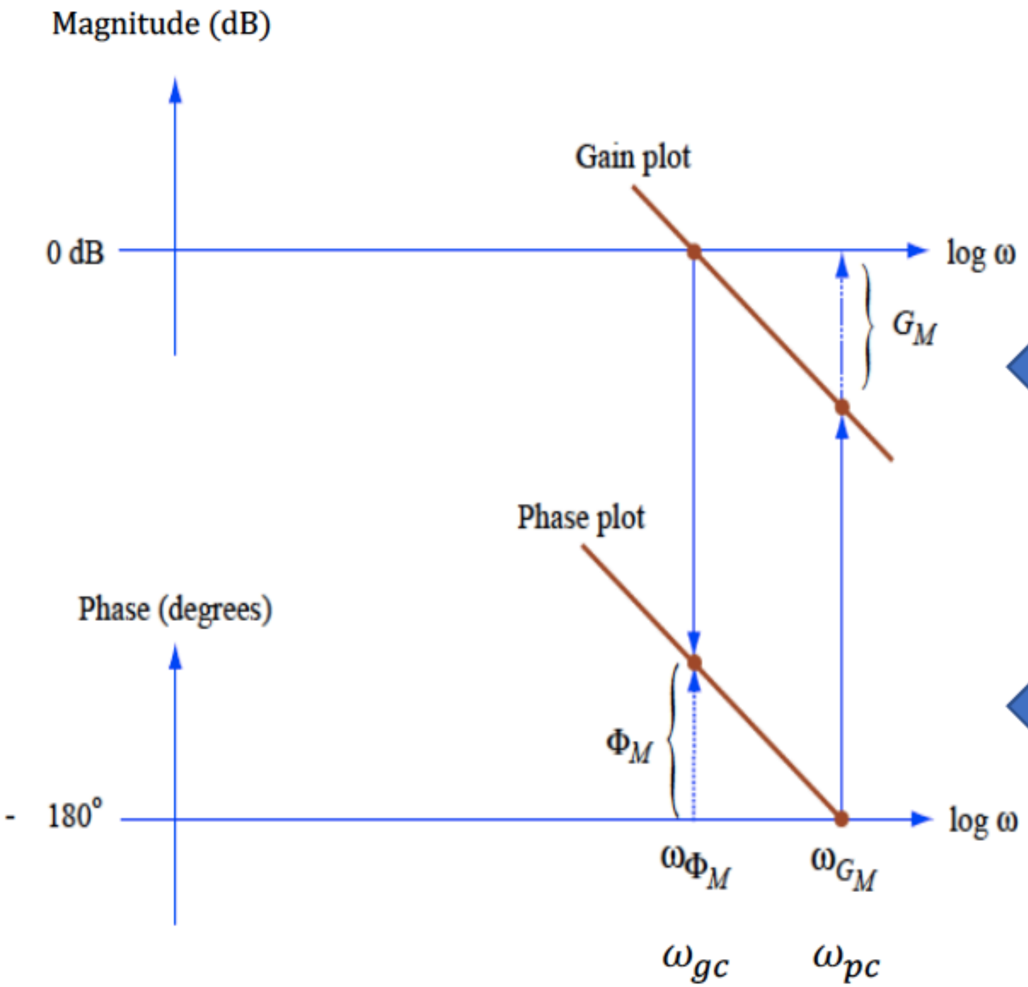
$$\angle G(j\omega) = -180°$$

In the Bode plot the *stability margin* is the general measure of how far we are from these two conditions above.

To establish the stability margin we need to find the frequency which relates to the first condition, $|G(j\omega)| = 1$. This is the *gain crossover frequency* ω_{gc}. The frequency that relates to the second condition $\angle G(j\omega) = -180°$ is the *phase crossover frequency* ω_{pc} . We define the *phase margin* on the phase plot, at ω_{gc} , and the *gain margin* on the gain plot, at ω_{pc}.



The gain and phase margins (formally defined as G_M and Φ_M) together define the stability margin for the system. Looking a little closer at this now, to relate the different notations:



The gain margin is the difference between 0dB and the gain at the frequency where the phase is - 180°

The phase margin is the difference between the system phase and -180° at the frequency where the gain is 1, i.e. at 0dB on the gain or magnitude plot.

So, in very many practical cases it is sufficient to say that increasing the closed loop gain by an amount equal to or more than the gain margin (i.e. setting $k \geq G_M$ dB) will de-stabilise the system.

Good examples are in the Matlab R2019b documentation, at: https://www.mathworks.com/help/control/ref/margin.html

Here is one:

For this example, create a continuous transfer function.

```
sys = tf(1,[1 2 1 0])
```

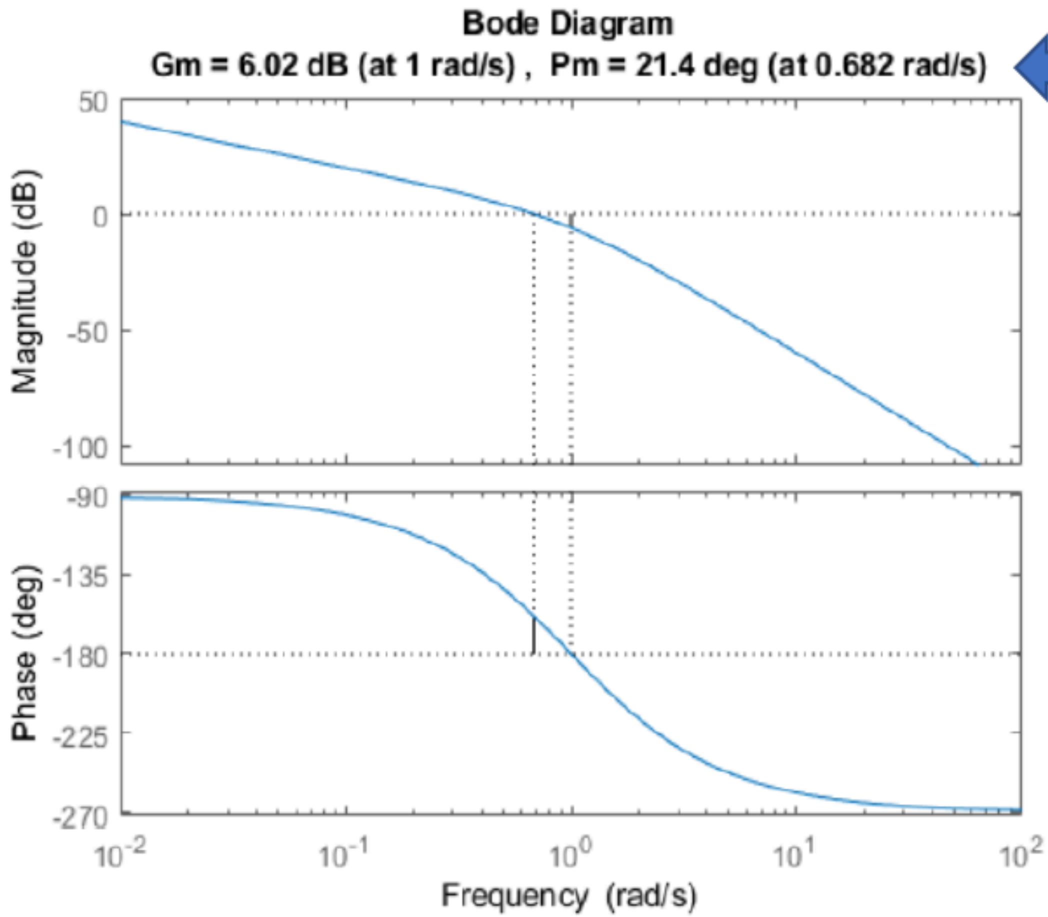
```
sys =

          1
  ---------------
  s^3 + 2 s^2 + s

Continuous-time transfer function.
```

Display the gain and phase margins graphically.

```
margin(sys)
```



We do not want to exceed either of these if we want to remain stable!

The gain margin (6.02 dB) and phase margin (21.4 deg), displayed in the title, are marked with solid vertical lines. The dashed vertical lines indicate the locations of `Wcg`, the frequency where the gain margin is measured, and `Wcp`, the frequency where the phase margin is measured.