



- HW2 due on Feb. 11th (Tuesday) at 11:55 PM

Outline

- Kernel methods: Motivation
- Kernel functions
- Kernel trick
 - Converting a ML model (objective function, prediction function, etc.) expressed with **feature vectors** to **kernel functions**
 - Kernel trick for linear regression
- Constructing (valid) kernels
- Kernel regression
 - Simple application of kernel method

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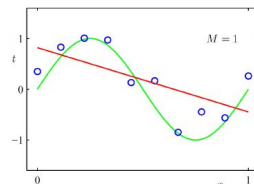
Linear regression

- Example: 1D regression
 - one input x , one output $h(x)$
- Linear model $h(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$ can only produce straight lines through origin
- Not very flexible/powerful
- How do we deal with this?

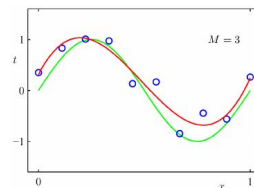
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Feature mappings

- Replace $x \rightarrow (1, x)$



- Replace $x \rightarrow (1, x, x^2, x^3)$



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Linear regression with (nonlinear) features

- Linear regression model

$$h(\mathbf{x}, \mathbf{w}) = \mathbf{w}^\top \phi(\mathbf{x}) = \sum_{j=0}^M w_j \phi_j(\mathbf{x})$$

- Least-squares with L2 regression

$$J(\mathbf{w}) = \frac{1}{2} \sum_{n=0}^N (\mathbf{w}^\top \phi(\mathbf{x}^{(n)}) - y^{(n)})^2 + \frac{\lambda}{2} \mathbf{w}^\top \mathbf{w}$$

- Closed form solution:

$$\mathbf{w} = (\Phi^\top \Phi + \lambda \mathbf{I})^{-1} \mathbf{y}$$

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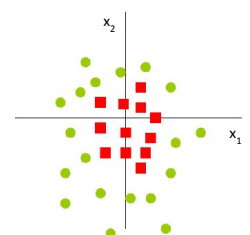
This is nice, but...

- What features to use?
- Computational complexity
 - Φ : $N \times M$ matrix
 - N : the number of examples
 - M : the number of features
 - Need to invert $\Phi^\top \Phi (M \times M)$ matrix
 - Computational complexity scales with $O(M^3)$

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Linear classifiers

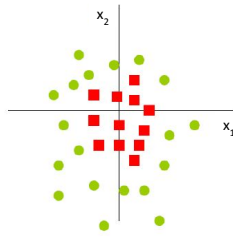
- No linear separating plane
- Linear classifiers not very flexible/powerful
- Can we do better?



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Linear classifiers with nonlinear features

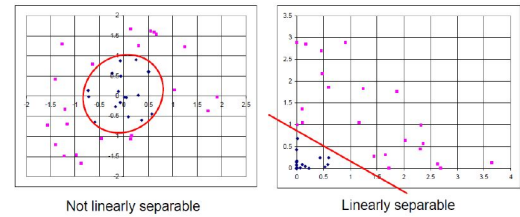
- Add distance to origin $(x_1^2 + x_2^2)^{1/2}$ as a third feature
- Data now lives on a parabolic surface in 3D.
- Linear separation in 3D feature space.
- In original feature space, decision boundary is an ellipse



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Linear classifiers with nonlinear features

- Another way: Replace $x_1 \rightarrow x_1^2$, $x_2 \rightarrow x_2^2$



- Different expansions make the problem solvable with linear methods.

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Linear classifiers with nonlinear features

- Data has been mapped to a new, higher dimensional space
- Alternative way to think about this: data still lives in original space, but the definition of distance or inner product has been changed

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Classifiers with nonlinear features

- We have been mapping each data point \mathbf{x} through a fixed non-linear mapping to get a feature vector $\phi(\mathbf{x})$
 - The feature vector extracts important properties from \mathbf{x} .
 - E.g., polynomial combinations of the original features, up to some order
 - It may make many regression/classification problems easier.
- Unfortunately, the feature vector may be high-dimensional, even infinite-dimensional.
 - Problems: computational complexity

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Kernels to the rescue (kernel trick)

- Embed data in a high dimensional space, and use simple models (linear relations) in this space.
- Use algorithms that do not need the coordinates of the embedded points, but only pairwise inner products
- Compute these inner products efficiently using a kernel

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Kernel functions

- A kernel function $k(\mathbf{x}, \mathbf{x}')$ is intended to represent the similarity between \mathbf{x} and \mathbf{x}' .
- A popular way to express similarity is as the inner product of feature vectors: $k(\mathbf{x}, \mathbf{x}') = \phi(\mathbf{x})^\top \phi(\mathbf{x}')$
- We *define* a kernel function $k(\mathbf{x}, \mathbf{x}')$ as one that can be expressed as an inner product, but we may not need to compute it that way.
- This definition immediately leads to symmetricity of kernels:

$$k(\mathbf{x}, \mathbf{x}') = k(\mathbf{x}', \mathbf{x})$$

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Example: 2D input data

- Inner product between two vectors (x_1, x_2) and (z_1, z_2)

$$k(\mathbf{x}, \mathbf{z}) = \mathbf{x}^\top \mathbf{z} = x_1 z_1 + x_2 z_2$$

- Let's replace this by its square

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^2 = x_1^2 z_1^2 + x_2^2 z_2^2 + 2x_1 x_2 z_1 z_2$$

- This is the same as inner product between

$$(x_1^2, \sqrt{2}x_1 x_2, x_2^2) \text{ and } (z_1^2, \sqrt{2}z_1 z_2, z_2^2)$$

- Or between

$$(x_1^2, x_1 x_2, x_2^2) \text{ and } (z_1^2, z_1 z_2, z_2^2)$$

- Note: solution is not unique.

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Example: 2D input data

- Consider higher-order polynomial of degree p :

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^p = \left(\sum_{j=1}^M x_j z_j \right)^p = \sum_{(j_1, j_2, \dots, j_M), \sum_k j_k = p} \binom{p}{j_1, j_2, \dots, j_M} (x_1 z_1)^{j_1} (x_2 z_2)^{j_2} \dots (x_M z_M)^{j_M}$$

- Feature mapping:

$$\phi(\mathbf{x}) = \left[\dots, \sqrt{\binom{p}{j_1, j_2, \dots, j_M}} x_1^{j_1} x_2^{j_2} \dots x_M^{j_M}, \dots \right]^\top$$

- All monomials of degree p

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Example: 2D input data

- Inhomogeneous polynomial up to degree p :

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z} + c)^p = \left(\sum_{j=1}^M x_j z_j + c \right)^p, \quad c > 0$$

- Feature mapping:
 - All monomials of degree $\leq p$

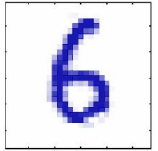
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Example: handwritten digits images

- Take the pixel values and compute

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z} + 1)^p$$

- Here \mathbf{x} is $28 \times 28 = 784$ dimensional



- You need to compute the inner product in the space of all monomials up to degree p .
- For $\dim(\mathbf{x})=784$ and $p=4$ a 16-billion dimensional space!

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Kernel trick

- Kernels allow you to achieve a high-dimensional feature space which is desirable for better separability for classes, i.e., classification performance.
- Crucially, we don't have to compute the high-dimensional feature explicitly, the inner product of the features are computed directly via the kernel function.
- Many algorithms can be expressed completely in terms of kernels $k(\mathbf{x}, \mathbf{x}')$, rather than other operations on \mathbf{x} .
- In this case, you can replace one kernel with another, and get a new algorithm that works over a different domain.

$$k(\mathbf{x}, \mathbf{z}) = (\underbrace{\mathbf{x}^\top \mathbf{z}}_{784 \text{ dim}} + 1)^4 = \underbrace{\phi(\mathbf{x})^\top \phi(\mathbf{z})}_{16 \text{ billion dim}}$$

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Kernel trick

- The kernel trick represents the problem formulation and its solutions entirely in terms of kernels (this is called "**dual** representation").
- The elements of the Gram matrix $K = \Phi \Phi^\top$

$$K_{nm} = \phi(\mathbf{x}^{(n)})^\top \phi(\mathbf{x}^{(m)}) = k(\mathbf{x}^{(n)}, \mathbf{x}^{(m)})$$

- These represent the pairwise similarities among all the observed feature vectors.
 - Assumption: we may be able to compute the kernels more efficiently than the feature vectors.

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Kernel substitution

- To use the kernel trick, we must formulate (training and test) algorithms purely in terms of inner products between data points
- We *cannot* access the coordinates of points in the high-dimensional feature space
- This seems a huge limitation, but it turns out that quite a lot can be done

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Example: distance

- Distance between samples can be expressed in inner products:

$$\begin{aligned} \|\phi(\mathbf{x}) - \phi(\mathbf{z})\|^2 &= \langle \phi(\mathbf{x}) - \phi(\mathbf{z}), \phi(\mathbf{x}) - \phi(\mathbf{z}) \rangle \\ &= \langle \phi(\mathbf{x}), \phi(\mathbf{x}) \rangle + \langle \phi(\mathbf{z}), \phi(\mathbf{z}) \rangle - 2\langle \phi(\mathbf{x}), \phi(\mathbf{z}) \rangle \\ &= \kappa(\mathbf{x}, \mathbf{x}) + \kappa(\mathbf{z}, \mathbf{z}) - 2\kappa(\mathbf{x}, \mathbf{z}) \end{aligned}$$

- So nothing stops you from doing k -nearest neighbor searches in high dimensional spaces

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Example: mean

- Can you determine the mean of data in the mapped feature space through kernel operations only?
 - A: No, you cannot compute any point explicitly

$$\phi_s = \frac{1}{N} \sum_{i=1}^N \phi(\mathbf{x}^{(i)})$$

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Example: distance to the mean

- Mean of data points given by: $\phi_s = \frac{1}{N} \sum_{i=1}^N \phi(\mathbf{x}^{(i)})$
 - cannot** be computed with kernel functions only

- Distance to mean:

- can** be computed with kernel functions only

$$\begin{aligned} \|\phi(\mathbf{x}) - \phi_s\|^2 &= \langle \phi(\mathbf{x}), \phi(\mathbf{x}) \rangle + \langle \phi_s, \phi_s \rangle - 2\langle \phi(\mathbf{x}), \phi_s \rangle \\ &= \kappa(\mathbf{x}, \mathbf{x}) + \frac{1}{N^2} \sum_{i,j=1}^N \kappa(\mathbf{x}^{(i)}, \mathbf{x}^{(j)}) - \frac{2}{N} \sum_{i=1}^N \kappa(\mathbf{x}, \mathbf{x}^{(i)}) \end{aligned}$$

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Kernel trick for linear regression

- Recall regression problems with error function

$$J(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^N \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \}^2 + \frac{\lambda}{2} \mathbf{w}^\top \mathbf{w}$$

- $J(\mathbf{w})$ is minimized at

$$\mathbf{w}_{ML} = (\lambda \mathbf{I} + \Phi^\top \Phi)^{-1} \Phi^\top \mathbf{y}$$

- Recall the $N \times M$ design matrix that is central to this solution.
- We can approach the solution a different way

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Recap: the design matrix

- The design matrix is an $N \times M$ matrix, applying
 - the M basis functions (M : number of columns)
 - to N data points (N : number of rows)

$$\Phi = \begin{pmatrix} \phi_0(\mathbf{x}^{(1)}) & \phi_1(\mathbf{x}^{(1)}) & \cdots & \phi_{M-1}(\mathbf{x}^{(1)}) \\ \phi_0(\mathbf{x}^{(2)}) & \phi_1(\mathbf{x}^{(2)}) & \cdots & \phi_{M-1}(\mathbf{x}^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\mathbf{x}^{(N)}) & \phi_1(\mathbf{x}^{(N)}) & \cdots & \phi_{M-1}(\mathbf{x}^{(N)}) \end{pmatrix}$$

$$\Phi \mathbf{w} \approx \mathbf{y}$$

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The gram matrix

- For regression, a key term is the $M \times M$ matrix

$$\Phi^\top \Phi \quad \text{"covariance"}$$

- Here, we will use the $N \times N$ Gram matrix

$$\mathbf{K} = \Phi \Phi^\top \quad \text{"pairwise similarity"}$$

- Note that $K_{nm} = \phi(\mathbf{x}^{(n)})^\top \phi(\mathbf{x}^{(m)}) = k(\mathbf{x}^{(n)}, \mathbf{x}^{(m)})$
 - the pairwise similarities of all the data points in the training set
- Note that kernel methods use only \mathbf{K} , not Φ

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Kernel trick for linear regression

- Objective: $J(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^N \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \}^2 + \frac{\lambda}{2} \mathbf{w}^\top \mathbf{w}$

- Another way to minimize $J(\mathbf{w})$ is to set $\nabla_{\mathbf{w}} J(\mathbf{w}) = 0$

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \sum_{n=1}^N \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \} \phi(x^{(n)}) + \lambda \mathbf{w} = 0$$

$$\Rightarrow \mathbf{w} = -\frac{1}{\lambda} \sum_{n=1}^N \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \} \phi(x^{(n)}) = \sum_{n=1}^N a_n \phi(x^{(n)}) = \Phi^\top \mathbf{a}$$

- where $a_n = -\frac{1}{\lambda} \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \}$

- Let \mathbf{a} be the **dual** parameter, instead of \mathbf{w} .
- Transform $J(\mathbf{w})$ to $J(\mathbf{a})$ by substituting

$$\mathbf{w} = \Phi^\top \mathbf{a}$$

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Kernel trick for linear regression

- Objective function $J(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^N \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \}^2 + \frac{\lambda}{2} \mathbf{w}^\top \mathbf{w}$
- Substitute $\mathbf{w} = \Phi^\top \mathbf{a}$

$$J(\mathbf{a}) = \frac{1}{2} \mathbf{a}^\top \Phi \Phi^\top \Phi \Phi^\top \mathbf{a} - \mathbf{a}^\top \Phi \Phi^\top \mathbf{y} + \frac{1}{2} \mathbf{y}^\top \mathbf{y} + \frac{\lambda}{2} \mathbf{a}^\top \Phi \Phi^\top \mathbf{a}$$

$$= \frac{1}{2} \mathbf{a}^\top \mathbf{K} \mathbf{K} \mathbf{a} - \mathbf{a}^\top \mathbf{K} \mathbf{y} + \frac{1}{2} \mathbf{y}^\top \mathbf{y} + \frac{\lambda}{2} \mathbf{a}^\top \mathbf{K} \mathbf{a} \quad \text{K} = \Phi \Phi^\top$$

- Setting the gradient w.r.t. \mathbf{a} to zero:

$$\nabla_{\mathbf{a}} J(\mathbf{a}) = \mathbf{K} \mathbf{K} \mathbf{a} - \mathbf{K} \mathbf{y} + \lambda \mathbf{K} \mathbf{a} = 0$$

$$(\mathbf{K} + \lambda \mathbf{I}) \mathbf{a} = \mathbf{K} \mathbf{y} \quad \text{simplify (K is invertible)}$$

- Solution (closed form):

$$\mathbf{a} = (\mathbf{K} + \lambda \mathbf{I}_N)^{-1} \mathbf{y}$$

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Kernel trick for linear regression

- Objective function $J(\mathbf{w}) = \frac{1}{2} \sum_{n=1}^N \{ \mathbf{w}^\top \phi(x^{(n)}) - y^{(n)} \}^2 + \frac{\lambda}{2} \mathbf{w}^\top \mathbf{w}$
- Substitute $\mathbf{w} = \Phi^\top \mathbf{a}$

$$J(\mathbf{a}) = \frac{1}{2} \mathbf{a}^\top \Phi \Phi^\top \Phi \Phi^\top \mathbf{a} - \mathbf{a}^\top \Phi \Phi^\top \mathbf{y} + \frac{1}{2} \mathbf{y}^\top \mathbf{y} + \frac{\lambda}{2} \mathbf{a}^\top \Phi \Phi^\top \mathbf{a}$$

$$= \frac{1}{2} \mathbf{a}^\top \mathbf{K} \mathbf{K} \mathbf{a} - \mathbf{a}^\top \mathbf{K} \mathbf{y} + \frac{1}{2} \mathbf{y}^\top \mathbf{y} + \frac{\lambda}{2} \mathbf{a}^\top \mathbf{K} \mathbf{a} \quad \text{K} = \Phi \Phi^\top$$

- Solution (closed form): $\mathbf{a} = (\mathbf{K} + \lambda \mathbf{I}_N)^{-1} \mathbf{y}$

- Prediction for any arbitrary input \mathbf{x} :

$$h(\mathbf{x}) = \mathbf{w}^\top \phi(\mathbf{x}) = \phi(\mathbf{x})^\top \mathbf{w} = \phi(\mathbf{x})^\top \Phi^\top \mathbf{a} = (\Phi \phi(\mathbf{x}))^\top \mathbf{a} = \sum_{n=1}^N a_n k(\mathbf{x}^{(n)}, \mathbf{x})$$

$$= k(\mathbf{x})^\top (\mathbf{K} + \lambda \mathbf{I}_N)^{-1} \mathbf{y}$$

definition: $k(\mathbf{x}) = [k(\mathbf{x}^{(1)}, \mathbf{x}), \dots, k(\mathbf{x}^{(N)}, \mathbf{x})]^\top$

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Kernel trick for linear regression

- Transform $J(\mathbf{w})$ to $J(\mathbf{a})$ by using $\mathbf{w} = \Phi^\top \mathbf{a}$ and the Gram matrix $\mathbf{K} = \Phi \Phi^\top$

- Find \mathbf{a} to minimize $J(\mathbf{a})$: $\mathbf{a} = (\mathbf{K} + \lambda \mathbf{I}_N)^{-1} \mathbf{y}$

- For predictions (for query point/test example \mathbf{x}):

$$h(\mathbf{x}) = \phi(\mathbf{x})^\top \mathbf{w} = \phi(\mathbf{x})^\top \Phi^\top \mathbf{a} = k(\mathbf{x})^\top (\mathbf{K} + \lambda \mathbf{I}_N)^{-1} \mathbf{y}$$

- where $k(\mathbf{x}) = [k(\mathbf{x}^{(1)}, \mathbf{x}), \dots, k(\mathbf{x}^{(N)}, \mathbf{x})]^\top$

- This method is called **Kernel Ridge Regression**.

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Primal versus Dual

- Primal: $\mathbf{w} = (\Phi^\top \Phi + \lambda \mathbf{I}_M)^{-1} \Phi^\top \mathbf{y}$
- Dual: $\mathbf{a} = (\mathbf{K} + \lambda \mathbf{I}_N)^{-1} \mathbf{y}$
- Primal: invert M by M matrix ($M = \dim$ feature space), \mathbf{w} vector of length M
 - cheaper because usually $N > M$, but you need to explicitly construct features.
- Dual: invert N by N matrix ($N = \text{number of data points}$)
 - can use the kernel trick (embded into very high dimensional feature space)
 - Use kernels $k(\mathbf{x}, \mathbf{x}')$ to represent similarity.
 - Kernels can be defined over vectors, images, sequences, graphs, text, etc.

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Constructing valid kernels

- One can do *kernel engineering* to create kernels for particular purposes, expressing different kinds of similarity.
- How do we verify that a kernel is valid?
- Three methods (for verification):
 1. Direct construction with feature vectors
 2. Mercer Theorem
 3. Composition of kernels with pre-defined rules

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Constructing valid kernels: Method 1 Explicit Construction by defining feature vectors

- Method 1: One way is to define the feature space mapping $\phi(\mathbf{x})$ and show that the kernel function represents the inner product of feature vectors:

$$k(\mathbf{x}, \mathbf{x}') = \phi(\mathbf{x})^\top \phi(\mathbf{x}') = \sum_{i=1}^M \phi_i(\mathbf{x}) \phi_i(\mathbf{x}')$$

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Constructing valid kernels: Method 1 Explicit Construction by defining feature vectors

- Suppose we define a kernel function directly, such as

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^2$$

- In 2D, we can explicitly identify the feature map

$$\phi(\mathbf{x}) = (x_1^2, \sqrt{2}x_1x_2, x_2^2)$$

– such that

$$k(\mathbf{x}, \mathbf{z}) = \phi(\mathbf{x})^\top \phi(\mathbf{z})$$

- Explicit feature mappings can be very complex.
 - Kernels help us avoid that complexity.

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Constructing valid kernels: Method 2 Mercer Theorem

- A simpler way to test without having to construct $\phi(\mathbf{x})$
- Use the necessary and sufficient condition (Mercer Theorem) that for a function $k(\mathbf{x}, \mathbf{x}')$ to be a inner product (valid) kernel:
 - the Gram matrix \mathbf{K} , whose elements are given by $k(\mathbf{x}^{(n)}, \mathbf{x}^{(m)})$, should be positive semidefinite for all possible choices of the data set $\{\mathbf{x}^{(n)}\}$
 - i.e., \mathbf{K} is positive semidefinite:

$$\mathbf{a}^\top \mathbf{K} \mathbf{a} = \sum_{i=1}^N \sum_{j=1}^N a_i K_{i,j} a_j \geq 0 \quad \forall \mathbf{a} \in \mathbb{R}^N$$

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Constructing valid kernels: Method 3 Using Pre-Defined Rules

- There are a number of axioms that help us construct new, more complex kernels, from simpler known kernels.
- For example,

$$k(\mathbf{x}, \mathbf{x}') = f(\mathbf{x}) k_1(\mathbf{x}, \mathbf{x}') f(\mathbf{x}')$$

$$k(\mathbf{x}, \mathbf{x}') = \exp(k_1(\mathbf{x}, \mathbf{x}'))$$

$$k(\mathbf{x}, \mathbf{x}') = \exp(-\|\mathbf{x} - \mathbf{x}'\|^2 / 2\sigma^2)$$

– **Prove** that these are valid kernels (homework)

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Constructing valid kernels: Method 3 Using Pre-Defined Rules

Given valid kernels $k_1(\mathbf{x}, \mathbf{x}')$ and $k_2(\mathbf{x}, \mathbf{x}')$, the following new kernels will also be valid:

$$k(\mathbf{x}, \mathbf{x}') = c k_1(\mathbf{x}, \mathbf{x}') \quad (6.13)$$

$$k(\mathbf{x}, \mathbf{x}') = f(\mathbf{x}) k_1(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') \quad (6.14)$$

$$k(\mathbf{x}, \mathbf{x}') = q(k_1(\mathbf{x}, \mathbf{x}')) \quad (6.15)$$

$$k(\mathbf{x}, \mathbf{x}') = \exp(k_1(\mathbf{x}, \mathbf{x}')) \quad (6.16)$$

$$k(\mathbf{x}, \mathbf{x}') = k_1(\mathbf{x}, \mathbf{x}') + k_2(\mathbf{x}, \mathbf{x}') \quad (6.17)$$

$$k(\mathbf{x}, \mathbf{x}') = k_1(\mathbf{x}, \mathbf{x}') k_2(\mathbf{x}, \mathbf{x}') \quad (6.18)$$

$$k(\mathbf{x}, \mathbf{x}') = k_3(\phi(\mathbf{x}), \phi(\mathbf{x}')) \quad (6.19)$$

$$k(\mathbf{x}, \mathbf{x}') = \mathbf{x}^\top \mathbf{A} \mathbf{x}' \quad (6.20)$$

$$k(\mathbf{x}, \mathbf{x}') = k_a(\mathbf{x}_a, \mathbf{x}'_a) + k_b(\mathbf{x}_b, \mathbf{x}'_b) \quad (6.21)$$

$$k(\mathbf{x}, \mathbf{x}') = k_a(\mathbf{x}_a, \mathbf{x}'_a) k_b(\mathbf{x}_b, \mathbf{x}'_b) \quad (6.22)$$

where $c > 0$ is a constant, $f(\cdot)$ is any function, $q(\cdot)$ is a polynomial with nonnegative coefficients, $\phi(\mathbf{x})$ is a function from \mathbf{x} to \mathbb{R}^M , $k_3(\cdot, \cdot)$ is a valid kernel in \mathbb{R}^M , \mathbf{A} is a symmetric positive semidefinite matrix, \mathbf{x}_a and \mathbf{x}_b are variables (not necessarily disjoint) with $\mathbf{x} = (\mathbf{x}_a, \mathbf{x}_b)$, and k_a and k_b are valid kernel functions over their respective spaces.

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Most popular kernels

- Simple Polynomial Kernel (terms of degree 2)

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z})^2$$

- Generalized Polynomial kernel - degree M

$$k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^\top \mathbf{z} + c)^M, \quad c > 0$$

- Gaussian Kernels

$$k(\mathbf{x}, \mathbf{z}) = \exp\left(-\frac{\|\mathbf{x} - \mathbf{z}\|^2}{2\sigma^2}\right)$$

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Gaussian kernel

- Not related to Gaussian pdf
- Translation invariant (depends only on distance between points)
- Corresponds to an infinitely dimensional space! (PRML ex6.11)

$$k(\mathbf{x}, \mathbf{z}) = \exp\left(-\frac{\|\mathbf{x} - \mathbf{z}\|^2}{2\sigma^2}\right)$$

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Kernel regression

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Kernel regression

- Recall k-nearest neighbor regression:

$$k(\mathbf{x}) = \frac{1}{k} \sum_{(\mathbf{x}', y') \in \text{KNN}(\mathbf{x})} y'$$

- Kernel regression:

$$h(\mathbf{x}) = \frac{\sum_i k(\mathbf{x}, \mathbf{x}^{(i)}) y^{(i)}}{\sum_j k(\mathbf{x}, \mathbf{x}^{(j)})} = \frac{1}{Z} \sum_i k(\mathbf{x}, \mathbf{x}^{(i)}) y^{(i)}$$

$$\text{where } Z = \sum_j k(\mathbf{x}, \mathbf{x}^{(j)})$$

- Weighted average of training responses where weight is proportional to the similarity with the corresponding feature.

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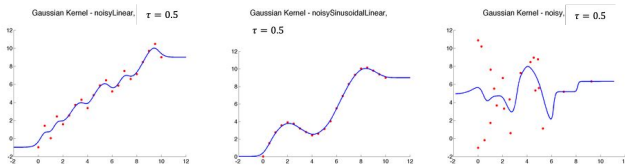
Kernel regression

- Can use different kinds of kernels as they capture similarity between features differently.

- Popular: Gaussian kernel with width τ :

$$k(\mathbf{x}, \mathbf{x}') = \exp\left(-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{2\tau^2}\right)$$

- Examples



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Kernels for classification

- We can just as easily use kernels for classification as well.
- Assume $y_i \in \{-1, +1\}$, return output as weighted majority:

$$h(\mathbf{x}) = \text{sign}\left(\sum_{i=1}^N k(\mathbf{x}, \mathbf{x}^{(i)}) y^{(i)}\right)$$

- Compare it to k-nearest neighbor classification:

$$h(\mathbf{x}) = \text{sign}\left(\frac{1}{k} \sum_{(\mathbf{x}', y') \in \text{KNN}(\mathbf{x})} y'\right)$$

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Locally-weighted Linear Regression vs. Kernel regression

- Suppose we want to predict y given a query \mathbf{x} .
- Locally-weighted linear regression

- 1. Fit \mathbf{w} to minimize $\sum_i r^{(i)} (y^{(i)} - \mathbf{w}^\top \phi(\mathbf{x}^{(i)}))^2 \leftarrow \mathbf{w} = (\Phi^\top R \Phi)^{-1} \Phi^\top R \mathbf{y}$
- 2. Output $\mathbf{w}^\top \phi(\mathbf{x}^{(i)})$
- Standard choice: $r^{(i)} = \exp\left(-\frac{\|\mathbf{x}^{(i)} - \mathbf{x}\|^2}{2\tau^2}\right)$

R is a diagonal matrix with $R_{i,i} = \frac{1}{2} r^{(i)}$

τ : "kernel width"

- Kernel regression (Using Gaussian kernel)

- output: $\frac{\sum_i \kappa(\mathbf{x}, \mathbf{x}^{(i)}) y^{(i)}}{\sum_i \kappa(\mathbf{x}, \mathbf{x}^{(i)})}$
- where $\kappa(\mathbf{x}, \mathbf{x}^{(i)}) = \exp\left(-\frac{\|\mathbf{x}^{(i)} - \mathbf{x}\|^2}{2\tau^2}\right)$

More generally, any distance metric (other than L2 or Euclidean distance) can be used.
Also, more general types of kernel function can be used.

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Any feedback (about lecture, slide, homework, project, etc.)?

(via anonymous google form: <https://forms.gle/fpYmiBtG9Me5qbP37>)



Change Log of lecture slides:

<https://docs.google.com/document/d/e/2PACX-1vSSIHjklvK7rKFSR1-5GYXvBCEW8UPtpSfCR9AR6M17K9ZQEmxf7waWaW7kLDxusthsF8WICyZl-/pub>

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