EECS 545: Machine Learning
Lecture 5. Classification 2

Honglak Lee
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Probabilistic Generative Models

Learning the Classifier

- Goal: Learn the distributions $p(C_k \mid \mathbf{x})$.
  - (a) **Discriminative** models: Directly model $p(C_k \mid \mathbf{x})$ and learn parameters from the training set.
    - Logistic regression
    - Softmax regression
  - (b) **Generative** models: Learn joint densities $p(\mathbf{x}, C_k)$ by learning $p(\mathbf{x} \mid C_k)$ and priors $p(C_k)$, and then use Bayes rule for predicting the class C_k given $\mathbf{x}$:
    - Gaussian Discriminant Analysis
    - Naive Bayes

Probabilistic Generative Models

- Bayes' theorem reduces the classification problem $p(C_k \mid \mathbf{x})$ to estimating the distribution of the data:

$$p(C_k|\mathbf{x}) = \frac{p(\mathbf{x}|C_k)p(C_k)}{p(\mathbf{x})} = \frac{p(\mathbf{x}|C_k)p(C_k)}{\sum_{k'} p(\mathbf{x}|C_{k'})p(C_{k'})}$$

- Density estimation can be decomposed into learning distributions from training data.
  - $p(C_k)$
  - $p(\mathbf{x} \mid C_k)$
- Maximum likelihood estimation for $p(\mathbf{x}, C_k)$

Probabilistic Generative Models

- For two classes, Bayes' theorem says:

$$p(C_1|\mathbf{x}) = \frac{p(\mathbf{x}|C_1)p(C_1)}{p(\mathbf{x}|C_1)p(C_1) + p(\mathbf{x}|C_2)p(C_2)}$$

- Use *log odds* (i.e., logit "score"):

$$a = \log \frac{p(C_1|\mathbf{x})}{p(C_2|\mathbf{x})} = \log \frac{p(\mathbf{x}|C_1)p(C_1)}{p(\mathbf{x}|C_2)p(C_2)}$$

- Then we can define the posterior via the *sigmoid*:

$$p(C_1|\mathbf{x}) = \frac{1}{1 + \exp(-a)} = \sigma(a)$$

Gaussian Discriminant Analysis

Gaussian Discriminant Analysis

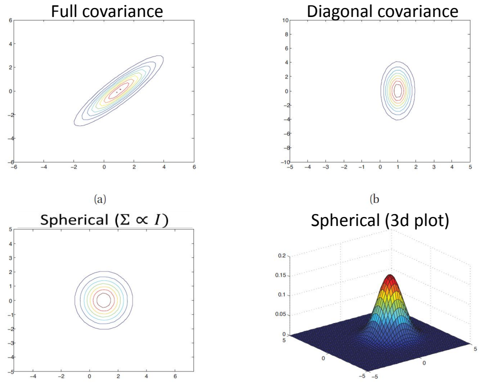
- Probability of class label
  - $p(C_k)$: Constant (e.g., Bernoulli)
- Conditional probability of data given a class
  - $p(\mathbf{x} \mid C_k)$: Gaussian distribution

$$p(\mathbf{x} \mid C_k) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2}(\mathbf{x} - \mu_k)^\top \mathbf{\Sigma}^{-1} (\mathbf{x} - \mu_k) \right\}$$

- Classification: use Bayes rule (previous slide)

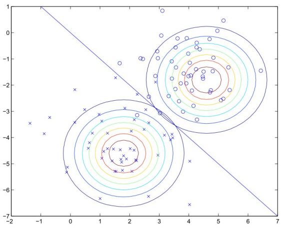
Examples of Gaussian Distributions

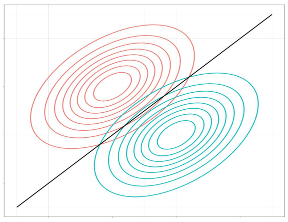
- Probability density p(x) for 2 dimensional case



Gaussian Discriminant Analysis

- Basic GDA assumes the same covariance for all classes
  - The figure below shows class-specific density and decision boundary. Note the linear decision boundary for any types of covariance matrices!





Prediction: Class-Conditional Densities

- Suppose we model $p(\mathrm{x} \mid C_k)$ as Gaussians with the <u>same covariance</u> matrix.

$$p(\mathbf{x} \mid C_k) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\mathbf{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\mu_k)^\top \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu_k)\right\}$$

- This gives us $p(C_1 \mid \mathbf{x}) = \sigma(\mathbf{w}^\top \mathbf{x} + w_0)$
  - where $\mathbf{w} = \mathbf{\Sigma}^{-1}(\mu_1 - \mu_2)$

  and $w_0 = -\frac{1}{2}\mu_1^\top \mathbf{\Sigma}^{-1}\mu_1 + \frac{1}{2}\mu_2^\top \mathbf{\Sigma}^{-1}\mu_2 + \log \frac{p(C_1)}{p(C_2)}$

Derivation

$$\begin{aligned}
P(x, C_1) &= P(x \mid C_1)\, P(C_1) \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu_1)^\top \Sigma^{-1}(x-\mu_1)\right\} P(C_1) \\
P(x, C_2) &= P(x \mid C_2)\, P(C_2) \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu_2)^\top \Sigma^{-1}(x-\mu_2)\right\} P(C_2)
\end{aligned}$$

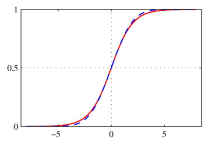
$$\log \frac{P(C_1 \mid \mathbf{x})}{P(C_2 \mid \mathbf{x})} = \log \frac{P(C_1 \mid \mathbf{x})}{1 - P(C_1 \mid \mathbf{x})}$$ "Log-odds"

$$\begin{aligned}
&= \log \frac{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mu_1)^\top \Sigma^{-1}(\mathbf{x}-\mu_1)\right\}}{\exp\left\{-\frac{1}{2}(\mathbf{x}-\mu_2)^\top \Sigma^{-1}(\mathbf{x}-\mu_2)\right\}} + \log \frac{P(C_1)}{P(C_2)} \\
&= \left\{-\frac{1}{2}(\mathbf{x}-\mu_1)^\top \Sigma^{-1}(\mathbf{x}-\mu_1)\right\} - \left\{-\frac{1}{2}(\mathbf{x}-\mu_2)^\top \Sigma^{-1}(\mathbf{x}-\mu_2)\right\} + \log \frac{P(C_1)}{P(C_2)} \\
&= (\mu_1-\mu_2)^\top \Sigma^{-1}\mathbf{x} - \frac{1}{2}\mu_1^\top \Sigma^{-1}\mu_1 + \frac{1}{2}\mu_2^\top \Sigma^{-1}\mu_2 + \log \frac{P(C_1)}{P(C_2)} \\
&= \left(\Sigma^{-1}(\mu_1-\mu_2)\right)^\top \mathbf{x} + w_0
\end{aligned}$$

where $w_0 = -\frac{1}{2}\mu_1^\top \mathbf{\Sigma}^{-1}\mu_1 + \frac{1}{2}\mu_2^\top \mathbf{\Sigma}^{-1}\mu_2 + \log \frac{p(C_1)}{p(C_2)}$

Prediction: Class-Conditional Densities for shared covariances

- $p(C_k \mid \mathbf{x})$ is a sigmoid function:



$$\sigma(a) = \frac{1}{1+\exp(-a)}$$

  - with log-odds (*logit* function):

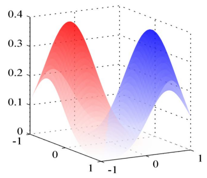
$$a = \log\left(\frac{\sigma}{1-\sigma}\right) = \left(\mathbf{\Sigma}^{-1}(\mu_1-\mu_2)\right)^\top \mathbf{x} + w_0$$

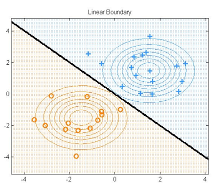
where $w_0 = -\frac{1}{2}\mu_1^\top \mathbf{\Sigma}^{-1}\mu_1 + \frac{1}{2}\mu_2^\top \mathbf{\Sigma}^{-1}\mu_2 + \log \frac{p(C_1)}{p(C_2)}$

- Generalizes to *normalized exponential*, or *softmax* : $p_i = \frac{\exp(q_i)}{\sum_j \exp(q_j)}$

Prediction: Linear Decision Boundaries

- At decision boundary, we have $p(C_1 \mid \mathrm{x}) = p(C_2 \mid \mathrm{x})$
- With the same covariance matrices, the boundary $p(C_1 \mid \mathrm{x}) = p(C_2 \mid \mathrm{x})$ is linear.
  - Different class priors $p(C_1)$, $p(C_2)$ just shift it around.





Likelihood function of generative models

- The likelihood of Data $\{(\mathbf{x}^{(n)}, y^{(n)})\}$

$$P(D|\mathbf{w}) = \prod_{i=1}^{N} P(\mathbf{x}^{(i)}, y^{(i)}|\mathbf{w}) \longrightarrow P(\mathbf{X}, \mathbf{y}|\mathbf{w})$$

Compact notation: This is called joint likelihood.

Decomposition of the joint probability

$$= \prod_{i=1}^{N} P(\mathbf{x}^{(i)}|y^{(i)}, \mathbf{w}) P(y^{(i)}|\mathbf{w})$$

Learning parameters via maximum likelihood

- Given training data $\{(\mathbf{x}^{(1)}, y^{(1)}), \cdots, (\mathbf{x}^{(N)}, y^{(N)})\}$ and a generative model ("shared covariance")

$$p(y) = \phi^y (1-\phi)^{1-y}$$

$$p(\mathbf{x}|y=0) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu_0)^\top \Sigma^{-1}(\mathbf{x}-\mu_0)\right)$$

$$p(\mathbf{x}|y=1) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu_1)^\top \Sigma^{-1}(\mathbf{x}-\mu_1)\right)$$

Learning via maximum likelihood

- Maximum likelihood estimation (HW2):

$$\phi = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 1\}$$

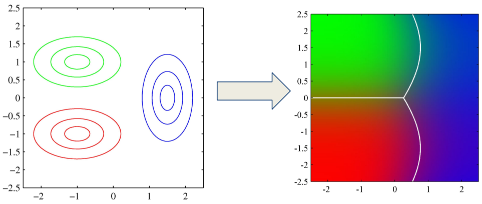
$$\mu_0 = \frac{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\}\mathbf{x}^{(i)}}{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 0\}}$$

$$\mu_1 = \frac{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 1\}\mathbf{x}^{(i)}}{\sum_{i=1}^{N} \mathbb{I}\{y^{(i)} = 1\}}$$

$$\Sigma = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{x}^{(i)} - \mu_{y^{(i)}})(\mathbf{x}^{(i)} - \mu_{y^{(i)}})^\top$$

Different Covariance

- Decision boundaries between some classes can be quadratic when they have **different** covariances.



Comparison between GDA and Logistic regression (or softmax regression)

- Logistic regression:
  - For an M-dimensional feature space, this model has M parameters to fit.
- Gaussian Discriminative Analysis
  - $2M$ parameters for the means of $p(\mathbf{x} \mid C_1)$ and $p(\mathbf{x} \mid C_2)$
  - $M(M+1)/2$ parameters for the shared covariance matrix
- Logistic regression has less parameters and is more flexible about data distribution.
- GDA has a stronger modeling assumption, and works well when the distribution follows the assumption.