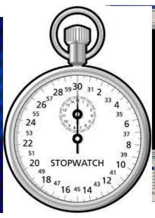
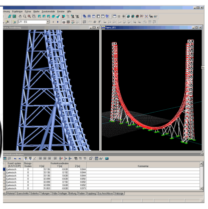
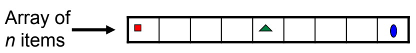
Lecture 3
Complexity Analysis





EECS 281: Data Structures & Algorithms

Metrics of Algorithm Complexity



Best-case: 1 comparison
Worst-case: *n* comparisons
Average-case: *n*/2 comparisons

Using a linear search over *n* items, how many comparisons will it take to find item *x*?

- Best-Case ■
  - Least number of comparisons required, given ideal input
  - Analysis performed over inputs of a given size
  - Example: Data is found in the first place you look
- Worst-Case ●
  - Most number of comparisons required, given hard input
  - Analysis performed over inputs of a given size
  - Example: Data is found in the last place you could possibly look
- Average-Case ▲
  - Average number of comparisons required, given any input
  - Average performed over all possible inputs of a given size

What Affects Runtime?

- The algorithm
- Implementation details
  - Skills of the programmer
- CPU Speed / Memory Speed
- Compiler (Options used)
  `g++ -g3` (for debugging, highest level of information) for 开发
  `g++ -O3` (**O**ptimization level **3** for speed) → for release
- Other programs running in parallel

- Amount of data processed (Input size)

Measuring & Using Input Size

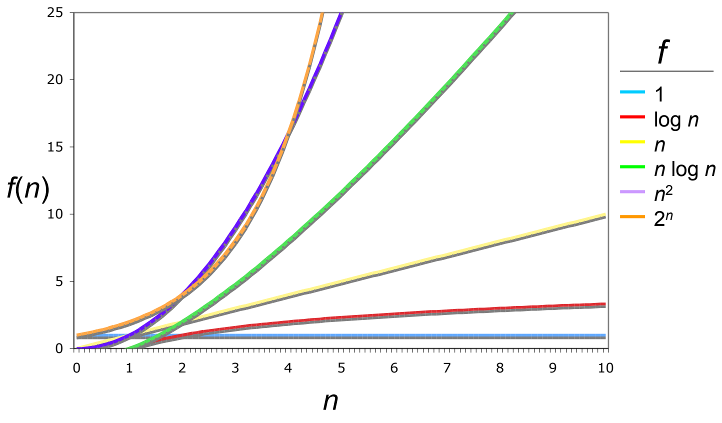
- Number of bits
  - In an `int`, a `double`? (32? 64?)
- Number of items: what counts as an item?
  - Array of integers? One integer? One digit? …
  - One string? Several strings? A char?
- Notation and terminology

– *n*	Input size
– *f(n)*	Maximum number of steps taken by an algorithm when input has size *n* ("*f* of *n*")
– ***O(f(n))***	Complexity class of *f*(*n*) ("Big-O of *f* of *n*")

Common Orders of Functions

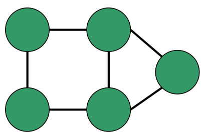
Notation	Name
***O*(1)**	Constant
***O*(log *n*)**	Logarithmic
***O*(*n*)**	Linear
***O*(*n* log *n*)**	Loglinear, Linearithmic
***O*(n^2)**	Quadratic
***O*(n^3)**, **O(n^4)**, …	Polynomial
***O*(c^n)**	Exponential
***O*(*n*!)**	Factorial
O(2^{2^n})	Doubly Exponential

Graphing *f*(*n*) Runtimes



Input Size Example

Graph *G* = (*V*, *E*):
V = 5 Vertices
E = 6 Edges



What should we measure?
- Vertices?
- Edges?
- Vertices and Edges?

When in doubt, measure input size in bits

Use *V* and *E* to determine which contributes more to the total number of steps
- Big-O examples: *E* log *V*, *EV*, V^2 log *E*

From Analysis to Application

- Algorithm comparisons are independent of hardware, compilers and implementation tweaks
- Predict which algorithms will eventually be faster
  - *For large enough inputs*
  - ***O*(n^2)** time algorithms will take longer than ***O*(*n*)** algorithms
- Constants can often be ignored because they do not affect asymptotic comparisons

Q: What counts as one step in a program?

A: Primitive operations

- Variable assignment
- Arithmetic operation
- Comparison
- Array indexing or pointer reference
- Function call (not counting the data)
- Function return (not counting the data)

Runtime of 1 step is independent on input

Counting Steps: `for` Loop

- The basic form of a for-loop:
  `for (initialization; test; update)`
- The initialization is performed once (1)
- The test is performed every time the body of the loop runs, plus once for when the loop ends (*n* + 1)
- The update is performed every time the body of the loop runs (*n*)
- Total is **2*n* + 2** steps (used a lot next slide)

Counting Steps: Polynomial

```
1  int func1(int n) {                      1
2    int sum = 0;                          2  1 step
3    for (int i = 0; i < n; ++i) {         3  2n + 2 steps
4      sum += i;                           4    1 step (loop n times)
5    } // for                              5
6    return sum;                           6  1 step
7  } // func1()                            7
```

Total steps: 3*n* + 4

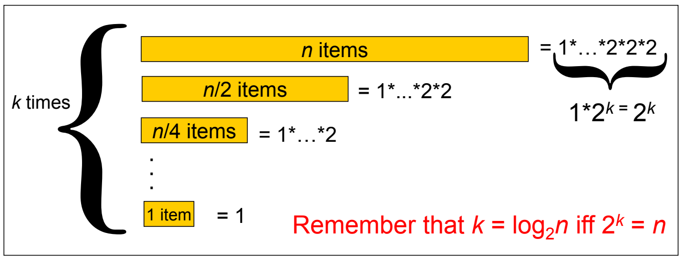
```
8  int func2(int n) {                      8
9    int sum = 0;                          9  1 step
10   for (int i = 0; i < n; ++i) {         10 2n + 2 steps
11     for (int j = 0; j < n; ++j)         11   2n + 2 steps (inside loop)
12       ++sum;                            12     1 step (inside 2 loops)
13   } // for i                            13
14   for (int k = 0; k < n; ++k) {         14 2n + 2 steps
15     --sum;                              15   1 step (inside loop)
16   } // for                              16
17   return sum;                           17 1 step
18 } // func2()                            18
```

Total steps: 3*n*^{2} + 7*n* + 6

Counting Steps: Logarithmic

```
1  int func3(int n) {                      1
2    int sum = 0;                          2  1 step
3    for (int i = n; i > 1; i /= 2)        3  (2 * ~log n) + 2 steps
4      sum += i;                           4    1 step (loop ~log n times)
5                                          5
6    return sum;                           6  1 step
7  } // func3()                            7
```

Total steps: (3 * ~log *n*) + 4 = *O*(log *n*)



Remember that $k = \log_2 n$ iff $2^k = n$

Examples of *O*(log *n*) Time

```
uint32_t logB(uint32_t n) {
  // find binary log, round up
  uint32_t r = 0;
  while (n > 1) {
    n /= 2
    r++;
  } // while
  return r;
} // logB()
```

```
int *bsearch(int *lo, int *hi, int val) {
  // find position of val between lo,hi
  while (hi >= lo) {
    int *mid = lo + (hi - lo) / 2;
    if (*mid == val)
      return mid;
    else if (*mid > val)
      hi = mid - 1;
    else
      lo = mid + 1;
  } // while
  return nullptr;
} // bsearch()
```

Algorithm Exercise

How many multiplications, if size = ***n*?**

```
// REQUIRES: in and out are arrays with size elements
// MODIFIES: out
// EFFECTS: out[i] = in[0] *...* in[i-1] * in[i+1] *...* in[size-1]
void f(int *out, const int *in, int size) {
  for (int i = 0; i < size; ++i) {
    out[i] = 1;
    for (int j = 0; j < size; ++j) {
      if (i != j)
        out[i] *= in[j];
    } // for j
  } // for i
} // f()
```

in[0]	in[1]	in[2]
3	1	2

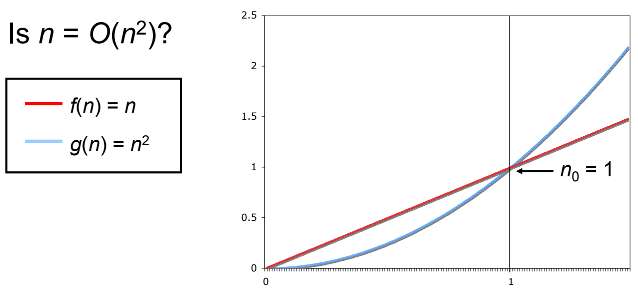
out[0]	out[1]	out[2]
2	6	3

Big-O Definition

f(*n*) = *O*(*g*(*n*)) if and only if there are constants

$\left.\begin{matrix} c > 0 \\ n_0 \geq 0 \end{matrix}\right\}$ such that $f(n) \leq c * g(n)$ whenever $n \geq n_0$

Is $n = O(n^2)$?



Big-O: Sufficient (but not necessary) Condition

If $\left[\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = d < \infty\right]$ then $f(n)$ is $O(g(n))$

$\boxed{\log_2 n = O(2n)?}$

$f(n) = \log_2 n$

$g(n) = 2n$

$\lim_{n\to\infty}\left(\frac{\log n}{2n}\right)$: $\frac{\infty}{\infty}$

$\lim_{n\to\infty}\left(\frac{1}{2n}\right)$: Use L'Hôpital's Rule

$0 = d < \infty$: $\log_2 n = O(2n)$ ✓

$\boxed{\sin\left(\frac{n}{100}\right) = O(100)?}$

$f(n) = \sin\left(\frac{n}{100}\right)$

$g(n) = 100$

$\lim_{n\to\infty}\left(\frac{\sin\left(\frac{n}{100}\right)}{100}\right)$: Condition does not hold, but it is true that $f(n) = O(g(n))$

Log Identities

Identity	Example
$\log_a(xy) = \log_a x + \log_a y$	$\log_2 (3*4)$
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_2 (4/3)$
$\log_a(x^r) = r \log_a x$	$\log_2 x^3$
$\log_a\left(\frac{1}{x}\right) = -\log_a x$	$\log_2 1/3$
$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a}$	$\log_2 1024$
$\log_a a = 1$ $\log_a 1 = 0$	

Power Identities

Identity	Example
$a^{(m+n)} = a^m \cdot a^n$	2^{2+3}
$a^{(m-n)} = \frac{a^m}{a^n}$	2^{3-2}
$(a^m)^n = a^{mn}$	$(2^2)^3$
$a^{-n} = \frac{1}{a^n}$	2^{-4}
$a^{-1} = \frac{1}{a}$ $a^0 = 1$ $a^1 = a$	

Exercise

True or False?

- $10^{100} = O(1)$ ✓
- $3n^4 + 45n^3 = O(n^4)$ ✓
- $3^n = O(2^n)$ ✗
- $2^n = O(3^n)$ ✓
- $45 \log(n) + 45n = O(\log(n))$ ✗
- $\log(n^2) = O(\log(n))$ ✓
- $[\log(n)]^2 = O(\log(n))$ ✗

Can you?

Find $f(n)$ and $g(n)$, such that $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$

$f(x) = \sin x$

$g(x) = \cos x$

Big-O, Big-Theta, and Big-Omega

	Big-O (O)	Big-Theta (Θ)	Big-Omega (Ω)
Defines	Asymptotic upper bound	Asymptotic tight bound	Asymptotic lower bound
Definition	$f(n) = O(g(n))$ if and only if there exists an integer n_0 and a real number c such that for all $n \geq n_0$, $f(n) \leq c \cdot g(n)$	$f(n) = \Theta(g(n))$ if and only if there exists an integer n_0 and real constants c_1 and c_2 such that for all $n \geq n_0$: $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$	$f(n) = \Omega(g(n))$ if and only if there exists an integer n_0 and a real number c such that for all $n \geq n_0$, $f(n) \geq c \cdot g(n)$
Mathematical Definition	$\exists n_0 \in Z, \exists c \in R:$ $\forall n \geq n_0, f(n) \leq c \cdot g(n)$	$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n))$	$\exists n_0 \in Z, \exists c \in R:$ $\forall n \geq n_0, f(n) \geq c \cdot g(n)$
$f_1(n)=2n+1$	$O(n)$ or $O(n^2)$ or $O(n^3)$…	$\Theta(n)$	$\Omega(n)$ or $\Omega(1)$
$f_2(n)=n^2+n+5$	$O(n^2)$ or $O(n^3)$ …	$\Theta(n^2)$	$\Omega(n^2)$ or $\Omega(n)$ or $\Omega(1)$

Amortized Complexity

- A type of worst-case complexity
- Used when the work/time profile is "spiky" (sometimes it is very expensive, but most times it is a small expense)
- Analysis performed over a sequence of operations covering of a given range
  - The sequence selected includes expensive and cheap operations

Amortized Complexity

- Considers the average cost of one operation over a sequence of operations
  - Best/Worst/Average-case only consider operations independently
  - Different from average-case complexity!
- Average case complexity is ONE event that is representative of all possible events
- Amortized complexity is an average of a sequence of events that ALL occur
- Key to understanding expandable arrays and STL vectors, priority queues, and hash tables

Cell Phone Bill Example

- Pay $100 once per month, each call and text has no added cost (unlimited plan)
- If you make 1000 calls/texts during the month, each one effectively costs $0.10
  - This is the "sequence of events that all occur"
- The rate at which money leaves your pocket is very "spiky"
- But each call or text appears to have basically a constant cost: the *amortized* cost per text is ***O*(1)**

Common Amortized Complexity

Analyze the asymptotic runtime complexity of the `push` operation for a stack implemented using an array/vector

Method	Implementation
`push(object)`	1. If needed, allocate a bigger array and copy data 2. Add new element at `top_ptr`, increment `top_ptr`

Exercise

Analyze the asymptotic runtime complexity of the `push` operation for a stack implemented using an array/vector

Amortized ***Θ*(1)**

Assume vector is filled with ***n*** elements

Double vector size (1 + ***Θ*(*n*)**) steps

Θ*(1)** push ***n times until full

Amortized cost: $\frac{(1 + \Theta(n)) + n * \Theta(1)}{n \text{ push operations}} = \Theta(1)$

Container Growth Options

1. Constant Growth
   - When container fills, increase size by *c*
   - Amortized cost: $\frac{(1 + \Theta(n)) + c * \Theta(1)}{c \text{ push operations}} = \Theta(n)$
   - Amortized linear
2. Linear Growth θ
   - When container fills, increase size by *n*
   - Amortized cost: $\frac{(1 + \Theta(n)) + n * \Theta(1)}{n \text{ push operations}} = \Theta(1)$
   - Amortized constant

Exercise

- You have *n* billiard balls. All have equal weight, except for one which is heavier. Find the heavy ball using only a balance.
- Describe an ***O*(*n*²)** algorithm → -个-个 balance
- Describe an ***O*(*n*)** algorithm →
- Describe an ***O*(log *n*)** algorithm → 二分
- Describe another ***O*(log *n*)** algorithm

Two *O*(log *n*) solutions

$\frac{\log_2 n}{\log_3 n} = \frac{\ln 2}{\ln 3} = c$

- Two groups: $\log_2(n) = \boldsymbol{O(\log_3 n)}$ ✓
- Three groups: $\log_3(n) = \boldsymbol{O(\log_2 n)}$ ✓
- True or False? Why?