Problem A: Let B be a box in $\mathbb{R}^n$. Show that if $f, g : B \to \mathbb{R}$ are both integrable then so is the function m defined by

$$M(x) = \max(f(x), g(x)).$$

Pf Assume the hypothesis.

Denote the set of discontinuity of f, g, M by D_f, D_g, D_m respectively

Let $x \in B \setminus (D_f \cup D_g)$

Let $\varepsilon > 0$

Since f, g ctn at x, $\exists \delta_1, \delta_2$ st.

$$|f(y) - f(x)| < \varepsilon, \ \forall y \in B_{\delta_1}(x)$$

$$|g(y) - g(x)| < \varepsilon, \ \forall y \in B_{\delta_2}(x)$$

So $|M(y) - M(x)| < \varepsilon$, $\underline{\forall y \in B_{\min(\delta_1, \delta_2)}(x)}$

Therefore M is ctn at x

This proves that $B \setminus D_m \supseteq (B \setminus D_f) \cap (B \setminus D_g) = B \setminus (D_f \cup D_g)$

$\Rightarrow \underline{D_m \subseteq D_f \cup D_g}$

Since $m(D_f) = m(D_g) = 0$

$\Rightarrow m(D_f \cup D_g) \le m(D_f) + m(D_g) = 0$

$\Rightarrow m^*(D_M) \le m(D_f \cup D_g) = 0$

$\Rightarrow m(D_M) = 0$

This proves that M is Riem integrable.

Problem B: Let B be a box in $\mathbb{R}^n$. Show that if $f : B \to \mathbb{R}$ is integrable then so is $|f|$ and moreover

$$\int_B f(x)dx \le \int_B |f(x)|dx.$$

Pf Denote the set of discontinuities of $f, |f|$ by $D_f, D_{|f|}$ respectively

Claim $D_{|f|} \subseteq D_f$

Let $x_0 \in D_{|f|} \Rightarrow \exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists y \in B_\delta(x_0)$ s.t. $\big||f(x_0)| - |f(y)|\big| \ge \varepsilon$

Fix ε_0

Let $\delta > 0$

Take $y \in B_\delta(x_0)$ st. $\big||f(x_0)| - |f(y)|\big| \ge \varepsilon$

By triangular ineq., we have $|f(x_0) - f(y)| \ge \big||f(x_0)| - |f(y)|\big| \ge \varepsilon$

Since δ is arbitrary, this proves the discontinuity of f at x_0

Thus $D_{|f|} \subseteq D_f$

By linearity of Lebesgue outer measure, $m^*(D_{|f|}) \le m^*(D_f) = 0$

So $m(D_{|f|}) = 0 \Rightarrow$ $\underline{|f| \text{ Riem intble.}}$ $\square$

Let P be a partition on B

$$\Rightarrow U(f, P) = \sum_{\text{subbox } S_i} \sup_{S_i} f \cdot v(S_i)$$

$$U(|f|, P) = \sum_{\text{subbox } S_i} \sup_{S_i} |f| \cdot v(S_i)$$

By triangular ineq, $\underline{|U(f, P)| \le U(|f|, P)}$

So $\left|\int_B f\right| = \left|\overline{\int_B} f\right| = \left|\inf_P U(f, P)\right| \le \inf_P |U(|f|, P)| = \overline{\int_B} |f| = \int_B f$ $\square$

Problem C:

(1) Prove that there exists a dense subset $S \subset [0, 1]^2$ such that the intersection of S with any vertical line is at most one point, and the intersection of S with any horizontal line is at most one point.

(2) Let $f(x, y) : [0, 1]^2 \to \mathbb{R}$ be the characteristic function (aka indicator function) of S (i.e. $f(x, y) = 1$ for $(x, y) \in S$ and 0 otherwise). Show that $f(x, \cdot)$ is an integrable function of y and $f(\cdot, y)$ is an integrable function of x, but f is not integrable on $[0, 1]^2$ as a function of two variables.

Just for fun bonus, not to hand it: In part (1) above, show that you can pick S so its intersection with any horizontal or vertical line is *exactly* one point.

Pf let $R = \begin{pmatrix} \cos\sqrt{2}\pi & \sin\sqrt{2}\pi \\ -\sin\sqrt{2}\pi & \cos\sqrt{2}\pi \end{pmatrix}$

Note that $\mathbb{Q}^2 \cap [0,1]^2$ is dense in $[0,1]^2$

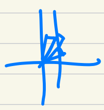
Now define $S = \{ R(a,b) \mid (a,b) \in \mathbb{Q}^2 \cap [0,1]^2 \}$

i.e $S = \operatorname{im}(R[\mathbb{Q}^2 \cap [0,1]^2])$

R is a rotation so S is still dense in $[0,1]^2$

Claim $\forall x_0 \in [0,1]$, the vertical line $x = x_0$ intersects S at most once

Pf Suppose $(x_0, y_0), (x_0, y_0') \in S$



$\Rightarrow \exists (r_1, s_1), (r_2, s_2) \in \mathbb{Q}^2 \cap [0,1]^2$ s.t.

$R(r_1, s_1) = (x_0, y_0)$, $R(r_2, s_2) = (x_0, y_0')$

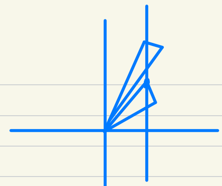
$\Rightarrow \cos\sqrt{2}\pi(r_1 - r_2) - \sin\sqrt{2}\pi(s_1 - s_2) = 0$

Note that $r_1 - r_2, s_1 - s_2 \in \mathbb{Q}$

while $\cos\sqrt{2}\pi, \sin\sqrt{2}\pi \in \mathbb{R} \setminus \mathbb{Q}$

$\Rightarrow r_1 = r_2, s_1 = s_2$

$\Rightarrow y_0 = y_0'$



This proves that S intersects any vertical line for at most one point

For similar reasoning, S intersects any horizontal line for at most one point.

(2) Since S intersects any horizontal, vertical line for at most one point,

$f(x,y) = 1$ for at most one y in $[0,1]$ for fixed x,

So $f(x, \cdot)$ is 0 a.e on $[0,1] \Rightarrow$ Riem intble

Same for $f(\cdot, y)$

For f on $[0,1]^2$, by density of S, we always have

$U(f, P) = \sum_S v(S) \cdot 1 = 1$

$L(f, P) = \sum_S v(S) \cdot 0 = 0$

so $M(f) \ne m(f)$.

So f not Riem intble on $[0,1]^2$.

Problem D: Let A be an open subset of $\mathbb{R}^2$; and let $f : A \to \mathbb{R}$ be C^2. Let Q be a box contained in A.

(1) Use Fubini's Theorem and the Fundamental Theorem of Calculus to show that
$$\int_Q \frac{\partial^2}{\partial x \partial y} f(x,y)\,dx\,dy = \int_Q \frac{\partial^2}{\partial y \partial x} f(x,y)\,dy\,dx.$$

(2) Give a new proof (other than the one we gave earlier in class) of the equality of mixed partials $\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$.

(1) Pf WLOG suppose Q is closed, then

$Q = [a_1, b_1] \times [a_2, b_2]$ for some endpoints $a_1, a_2, b_1, b_2 \in \mathbb{R}$

Since $f \in C^2(A) \Rightarrow \frac{\partial^2}{\partial x \partial y} f(x,y)$, $\frac{\partial^2}{\partial y \partial x} f(x,y)$ are ctn

thus Riem intble (their set of discontinuity has measure 0)

$\Rightarrow$ By Fubini's Thm,

$$\int_Q \frac{\partial^2}{\partial x \partial y} f(x,y)\,dx\,dy = \int_{a_2}^{b_2} \int_{a_1}^{b_1} \frac{\partial^2}{\partial x \partial y} f(x,y)\,dx\,dy$$

$$\int_{a_1}^{b_1} \frac{\partial^2}{\partial x \partial y} f(x,y)\,dx = \left[\frac{\partial f}{\partial y}(x,y)\right]_{a_1}^{b_1} = \frac{\partial f}{\partial y}(b_1, y) - \frac{\partial f}{\partial y}(a_1, y)$$

by FTC

Then $\int_Q \frac{\partial^2}{\partial x \partial y} f(x,y)\,dx\,dy = \int_{a_2}^{b_2} \frac{\partial f}{\partial y}(b_1, y) - \frac{\partial f}{\partial y}(a_1, y)\,dy$

$= f(b_1, b_2) - f(b_1, a_2) - f(a_1, b_2) + f(a_1, a_2)$ ①

by FTC

Similarly we have:

$$\int_Q \frac{\partial^2}{\partial y \partial x} f(x,y)\,dy\,dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\partial^2}{\partial y \partial x} f(x,y)\,dy\,dx$$

$$= \int_{a_1}^{b_1} \left[\frac{\partial}{\partial x} f(x,y)\right]_{a_2}^{b_2} dx = \int_{a_1}^{b_1} \left(\frac{\partial}{\partial x} f(x, b_2) - \frac{\partial}{\partial x} f(x, a_2)\right) dx$$

$= f(b_1, b_2) - f(a_1, b_2) - f(b_1, a_2) + f(a_1, a_2)$ ②

Note that ① = ② which is exactly what we want to show $\square$

(2) Let $(a,b) \in A$.

Let $\varepsilon > 0$, $Q_\varepsilon = [a, a+\varepsilon] \times [b, b+\varepsilon]$

$$\Rightarrow \int_b^{b+\varepsilon} \int_a^{a+\varepsilon} \left(\frac{\partial^2 f}{\partial x \partial y}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(x,y)\right) dx\,dy = 0$$

Since $\left(\frac{\partial^2 f}{\partial x \partial y}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(x,y)\right)$ is ctn both regard to x and y,

we can apply the integral MVT to get

$$\int_a^{a+\varepsilon} \left(\frac{\partial^2 f}{\partial x \partial y}(x,y) - \frac{\partial^2 f}{\partial y \partial x}(x,y)\right) dx = \varepsilon \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y) - \frac{\partial^2 f}{\partial y \partial x}(x_0, y)\right)$$ for some $x_0 \in [a, a+\varepsilon]$

Then get $\int_b^{b+\varepsilon} \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y) - \frac{\partial^2 f}{\partial y \partial x}(x_0, y)\right) dy = \varepsilon \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) - \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)\right)$

for some $y_0 \in [b, b+\varepsilon]$

Therefore $\varepsilon^2 \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) - \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)\right) = 0$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Problem E: Show that $f : B \to \mathbb{R}$ is Darboux integrable if and only if f is Riemann integrable.

Pf Backward direction

Suppose f is Riemann integrable

let $A \in \mathbb{R}$ be the integral

let $\varepsilon > 0$

Take $\delta > 0$ st. $\forall \|P\| < \delta$, $\left|\sum_{B_\alpha \text{ by } P} f(x_\alpha) v(B_\alpha) - A\right| < \frac{\varepsilon}{2}$

for any $x_\alpha \in B_\alpha$ for each α

Fix one arbitrary P st. $\|P\| < \delta$, let $\{B_1, \ldots, B_k\}$ be subboxes of B obtained by P

So $\forall i = 1, \ldots, k$, take arbitrary $x_i \in B_i$

we have $\left|\sum_{i=1}^n f(x_i) v(B_i) - A\right| < \frac{\varepsilon}{2}$

Since each x_i is take arbitrarily,

$$|L(f,P) - A| = \left|\sum_{i=1}^n m(B_i) v(B_i) - A\right| \leq \frac{\varepsilon}{2}$$

$$|U(f,P) - A| = \left|\sum_{i=1}^n M(B_i) v(B_i) - A\right| \leq \frac{\varepsilon}{2}$$

So $|U(f,P) - L(f,P)| \leq |L(f,P) - A| + |U(f,P) - A| \leq \varepsilon$

Since ε is arbitrary and we always have $U(f,P) \geq L(f,P)$

this proves that $\inf_P U(f,P) = \sup_P L(f,P)$ where P indexes over all partitions of B

Thus f is Darboux integrable

Forward direction

Lemma Let $B' \subseteq B$ be a box

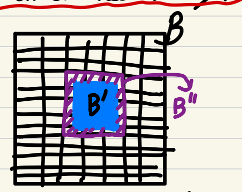
$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. any partition P on B s.t. $\|P\| < \delta$,

we have $\left(\sum_{S \cap B' \neq \emptyset} v(S)\right) - v(B') \leq \varepsilon$,

where S indexes on subboxes of B by P

Pf of Lemma

Fix $\varepsilon > 0$. Consider $\delta = \frac{\sqrt[n]{\varepsilon}}{2}$



Let P be arbitrary partition s.t. $\|P\| < \delta$

$B' = [a_1, b_1] \times \ldots \times [a_n, b_n]$

Let $G = \{S_1, S_2, S_3, \ldots, S_k\}$ be set of subboxes of B by P that have non empty intersection with B'

write each $S_i := [\alpha_1^{(i)}, \beta_1^{(i)}] \times \ldots \times [\alpha_n^{(i)}, \beta_n^{(i)}]$

Let $\alpha_j = \min\{\alpha_j^{(i)} \mid 1 \leq i \leq k\}$

$\beta_j = \max\{\beta_j^{(i)} \mid 1 \leq i \leq k\}$ for each $j = 1, \ldots, n$

Since $\|P\| < \delta \Rightarrow \forall j = 1, \ldots, n$, $a_j - \alpha_j \leq \delta$ and $b_j - \beta_j \leq \delta$

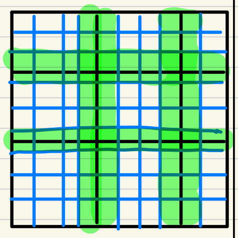
$(\alpha_j \leq a_j,\ \beta_j \geq b_j\ \forall j)$

Consider $B'' = [\alpha_1, \beta_1] \times \ldots \times [\alpha_n, \beta_n] \supseteq B'$

$\Rightarrow B' \subseteq \bigcup_G S \subseteq B''$ so $v(B') \leq \sum_G v(S) \leq v(B')$

since subboxes in G are almost disjoint and $v(B'') - v(B') \le (2\delta)^n = (\sqrt[n]{\varepsilon})^n \le \varepsilon$

So $\sum_{S\in G} v(S) - v(B') < \varepsilon$ □ Lemma



Now we have come to

Pf of Darboux integrable ⇒ Riem integrable

Suppose f is Darboux integrable

So $\inf_P U(f,P) = \sup_P L(f,P) = A \in \mathbb{R}$

Let $\varepsilon > 0$

Let P_ε be a partition with subboxes collection $G = \{S_1, \dots, S_k\}$

s.t. $\left|\sum_{B_i\in G} m(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$ and $\left|\sum_{B_i\in G} M(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$

For each subbox B_i by P_ε, by Lemma, we can pick $\delta_i > 0$ that

for any partition P on B s.t. $\|P\| < \delta_i$ we have $\left(\sum_{S\cap B_i \neq \emptyset} v(S)\right) - v(B_i) < \frac{\varepsilon}{2k(\sup_B f - \inf_B f)}$

where S indexes over subboxes of B by P

Take $\delta = \min\{\delta_1, \dots, \delta_k\}$ and fix it

Let P_{new} be arbitrary partition of B s.t. $\|P_{new}\| < \delta$

Let G_{new} be subboxes of B by P_{new}

Let $G_\partial \subseteq G_{new}$ be the collection of subboxes by P_{new} not completely contained in some subboxes by P_ε

For each $B_\alpha \in G_{new}$, take arbitrary $x_\alpha \in B_\alpha$

We have $\sum_{G_{new}} f(x_\alpha)v(B_\alpha)$

$$= \sum_{G_\partial} f(x_\alpha)v(B_\alpha) + \sum_{G_{new}\setminus G_\partial} f(x_\alpha)v(B_\alpha)$$

This is between $\sum_{G_\partial} \inf_B f\, v(B_\alpha) + \sum_{G_{new}\setminus G_\partial} m(B_\alpha)v(B_\alpha)$

and $\sum_{G_\partial} \sup_B f\, v(B_\alpha) + \sum_{G_{new}\setminus G_\partial} M(B_\alpha)v(B_\alpha)$

which is within $\left[\sum_{B_i\in G} m(B_i)v(B_i) - \frac{\varepsilon}{2},\ \sum_{B_i\in G} M(B_i)v(B_i) + \frac{\varepsilon}{2}\right]$

by our assumption of P_{new} through lemma

Since we also have

$\left|\sum_{B_i\in G} m(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$ and $\left|\sum_{B_i\in G} M(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$,

we obtain that

$\left|\sum_{G_{new}} f(x_\alpha)v(B_\alpha) - A\right| < \varepsilon$ by triangular ineq

Since P_{new} s.t. $\|P_{new}\| < \delta$ is taken arbitrarily, this shows the Riemann's condition

Then we finishes the proof of Darboux intble ⇒ Riem intble

since ε is arbitrary. □

Problem F: Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2 and given a n by n matrix A, define a new function by $g(x) = f(Ax)$. Calculate $Dg(0)$ in terms of $Df(0)$ and A. Calculate the Hessian of g at 0 in terms of the Hessian of f at 0.

By chain rule, $Dg(x) = D(f\circ A)(x)$

$$= Df(Ax)\underbrace{DA(x)}_{=A}$$

So $Dg(0) = Df(0)A$

$$H_g(x) = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2}(x) & \frac{\partial^2 g}{\partial x_1\partial x_2}(x) & \cdots & \frac{\partial^2 g}{\partial x_1\partial x_n}(x) \\ \frac{\partial^2 g}{\partial x_2\partial x_1}(x) & & \ddots & \vdots \\ \vdots & & & \\ \frac{\partial^2 g}{\partial x_n\partial x_1}(x) & \cdots & & \frac{\partial^2 g}{\partial x_n^2}(x) \end{pmatrix} = \begin{pmatrix} D\frac{\partial g}{\partial x_1}(x) \\ D\frac{\partial g}{\partial x_2}(x) \\ \vdots \\ D\frac{\partial g}{\partial x_n}(x) \end{pmatrix}$$

Let $y(x) = Ax \ \forall x \in \mathbb{R}^n$

Denote the Jacobian matrix of f by $J_f = \left(\frac{\partial f}{\partial y_1} \cdots \frac{\partial f}{\partial y_n}\right)$

the Jacobian matrix of y by $J_y = \left(\frac{\partial y}{\partial x_1} \cdots \frac{\partial y}{\partial x_n}\right)$

Note that $\frac{\partial g}{\partial x_i} = (J_f)(A)_{col\ i} = \sum_{j=1}^n A_{ji}\frac{\partial f}{\partial y_j}$

So $D\frac{\partial g}{\partial x_i}(0) = \left(\frac{\partial}{\partial x_1}\left(\frac{\partial g}{\partial x_i}(0)\right) \cdots \frac{\partial}{\partial x_n}\left(\frac{\partial g}{\partial x_i}(0)\right)\right)$

$$= \left(\sum_{j=1}^n A_{ji}\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial y_j}(0)\right) \cdots \sum_{j=1}^n A_{ji}\frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial y_j}(0)\right)\right)$$

$$= \left(\sum_{j=1}^n\sum_{k=1}^n A_{ji}\frac{\partial^2 f}{\partial y_k\partial y_j}(0)\overset{=A_{1k}}{\frac{\partial y_1}{\partial x_k}(0)} \cdots \sum_{j=1}^n\sum_{k=1}^n A_{ji}\frac{\partial^2 f}{\partial y_k\partial y_j}(0)\overset{=A_{nk}}{\frac{\partial y_n}{\partial x_k}(0)}\right)$$

since $\frac{\partial f}{\partial x_n} = \sum_{k=1}^n \frac{\partial f}{\partial y_k}\frac{\partial y_k}{\partial x_n}$ implied by chain rule.

Therefore $(H_g(0))_{im} = \sum_{j=1}^n\sum_{k=1}^n A_{ji}(H_f(0))_{jk}A_{km}$

$$= (A)_{row\ i}(H_f(0))(A)_{col\ m}$$

This shows that $H_g(0) = A^T H_f(0) A$

Problem G: Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2, give a formula for the degree two Taylor polynomial of f at 0 in terms of derivative and Hessian of f at 0.

$$f(x) = \sum_{|\alpha| \le n} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha + R_{a,k}(x)$$

$$T_{a,2}(x) = \sum_{|\alpha| \le 2} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha$$

$$= f(0) + \sum_{|\alpha|=1} \partial^\alpha f(0)\, x^\alpha + \sum_{|\alpha|=2} \frac{\partial^\alpha f(0)}{2} x^\alpha$$

$$= f(0) + \underbrace{\sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) x_i}_{= Df(0)x} + \frac{1}{2} \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(\sum_{k=1}^n \frac{\partial f}{\partial x_k}(0) x_k \right) x_j$$

$$= f(0) + Df(0)\,x + \underbrace{\frac{1}{2} \sum_{j=1}^n x_j \left(\sum_{k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(0) x_k \right)}_{= x^T H_f(0) x \text{ by } C^2} \quad (\text{by } C^2)$$

$$= f(0) + Df(0)\,x + \frac{1}{2} x^T H_f(0) x$$

Bonus: Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous on the irrationals and continuous on the rationals. (Hint: There is a short solution. You're welcome to use IBL results if you want to.)

Recall in topological sense: a G_δ set means a countable intersection of open sets.

Claim 1 $\forall f: X \to Y$ between metric spaces, $C_f = \{x \mid f \text{ ctn at } x\}$ is G_δ.

Pf of claim 1: exactly hw 9 problem A $\square$

Claim 2 Corollary of Baire Category Thm: a complete metric space can not be a countable union of nowhere denses sets in it.

Pf of claim 2 Let K be a complete metric space. Assume the contrary: Let (N_n) be a seq of nowhere dense sets and $X = \bigcup_n N_n$

$\Rightarrow \{O_n = X \setminus \overline{N_n}\}_{n \in \mathbb{N}}$ is dense and open in X

But $\bigcap_n O_n = \emptyset$, contradicts BCT. $\square$

Claim 3 $\mathbb{Q}$ is not G_δ.

Pf of Claim 3 Suppose for contradiction that $\mathbb{Q}$ is G_δ

Then $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$ for some countable collection of open sets $\{U_n\}$

$\Rightarrow \mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcap_{n=1}^\infty U_n = \bigcup_{n=1}^\infty (\mathbb{R} \setminus U_n)$ is a countable union of closed sets.

Note that each $\mathbb{R} \setminus U_n$ is nowhere set as its interior is empty (otherwise it contains an open set, which in $\mathbb{R}$ means that it contains open interval)

And $\mathbb{Q}$ as a countable set, is a ctbl union of singleton sets which are also nowhere dense

$\Rightarrow \mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ is a ctbl union of nowhere dense sets which contradicts with the Baire Category Thm $\square$

By claim 1, 3 we proved that $\nexists f: \mathbb{R} \to \mathbb{R}$ s.t. $D_f = \{\mathbb{R} \setminus \mathbb{Q}\}$ as that implies $\mathbb{Q}$ being G_δ. $\square$