## Recall

extended Integral (此定义并非 most general)

If $A \subseteq \mathbb{R}^n$ open, $f: A \to \mathbb{R}$ ctn

- if $f \geq 0$, define $\int_A f := \sup_{\substack{D \in \mathcal{J}_c \\ D \subseteq A}} \int_D f$
- else, define $\int_A f := \int_A f^+ - \int_A f^-$

Temporary problem: For $A$ bdd. $f: A \to \mathbb{R}$ ctn & bdd.

We have 2 defs of $\int_A f$

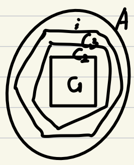
Need to show they are equal (ordinary) (extended)

Before that: will write them as $\text{ord} \int_A f$, $\text{ext} \int_A f$

Recall any open $A \subseteq \mathbb{R}^d$

$\exists$ cpt $\{C_n\}_{n=1}^{\infty} \subseteq \mathcal{J}_c$ s.t.

$\forall n$, $C_n \subseteq C_{n+1}$ and $A = \bigcup_{N=1}^{\infty} C_N$




## Thm Criterion of ext. intble

If $A \subseteq \mathbb{R}^n$ open

$f: A \to \mathbb{R}$ ctn.

$\{C_n\} \subseteq \mathcal{J}_c$ defined above.

Then:

$\text{ext} \int_A f$ exists $\iff \{\int_{C_N} |f|\}_{N=1}^{\infty}$ is bdd

$\Updownarrow$

$\{\int_{C_N} f_+\}_{N=1}^{\infty}$, $\{\int_{C_N} f_-\}_{N=1}^{\infty}$ bdd

In this sense,

$$\int_A f = \lim_{N \to \infty} \int_{C_N} f$$

In particular, $f$ is intble on $A$ iff $|f|$ is

Pf Case1 $f \geq 0$. Note $(\int_{C_N} f)_{N=1}^{\infty}$ is non-strict increasing

so it conv. iff bdd

Suppose that $f$ intble on $A$

By ctn & $C_N \in \mathcal{J}_c \Rightarrow \int_{C_N} f \; \exists$

Also, $\int_{C_N} f \leq \sup_{\substack{D \in \mathcal{J}_c \\ D \subseteq A}} \int_D f = \text{ext} \int_A f$

So $(\int_{C_N} f)_{N=1}^{\infty}$ is bdd. thus conv. (and increasing) And $\boxed{\lim_{N \to \infty} \int_{C_N} f \leq \int_A f}$ ①

Conversely, if $\lim_{N \to \infty} \int_{C_N} f \; \exists$

$\Rightarrow M = \sup_N \int_{C_N} f < \infty$

Since increasing, $M = \lim_{N \to \infty} \int_{C_N} f$

Take any $D \subseteq A$ & $D \in \mathcal{J}_c$

Note that $D \subseteq A = \bigcup_{N=1}^{\infty} C_N^{\circ}$

By cptness, $\exists N_0$ st. $D \subseteq \bigcup_{N=1}^{N_0} C_N^{\circ} = C_{N_0}^{\circ} \subseteq C_{N_0}$

So $\int_D f \leq \int_{C_{N_0}} f \leq M$

So $\int_A f \; \exists$ and $\boxed{\int_A f \leq M = \lim_{N \to \infty} \int_{C_N} f}$ ②

By ①②, $\int_A f = \lim_{N \to \infty} \int_{C_N} f$

Case2 General $f: A \to \mathbb{R}$ ctn

$f = f_+ - f_-$ intble $\iff f_+, f_-$ intble

$\overset{\text{By case1}}{\iff} \int_{C_N} f^+, \int_{C_N} f^-$ bdd.

$\iff \int_{C_N} f_+ + f_-$ bdd. $\iff \int_{C_N} |f|$ bdd $\overset{\text{By case1}}{\iff} \int_A |f| \; \exists$

So $\int_{C_N} f_+ \to \int_A f_+$, $\int_{C_N} f_- \to \int_A f_-$

So $\int_{C_N} f = \int_{C_N} f_+ - \int_{C_N} f_- \to \int_A f_+ - \int_A f_- = \int_A f$ □

## Thm Let $A$ bdd, open in $\mathbb{R}^n$

$f: A \to \mathbb{R}$ bdd, ctn

$\Rightarrow$ (a) $\text{ext} \int_A f \; \exists$

(b) if $\text{ord} \int_A f \; \exists \Rightarrow \text{ord} \int_A f = \text{ext} \int_A f$

对于 bdd open A, extended integral must exists (且如果intble 则 $= \text{ord} \int_A f$)

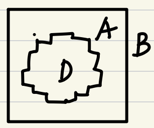
Pf (a) Let $|f| \leq M$ on $A$

Let $D \subseteq A$ and $D \in \mathcal{J}_c$ $\Rightarrow$ $\leq \int_D M = M m_J(D)$

$$\int_D f_{\pm} \leq M \int_D 1 = M m_J(D)$$

So $\text{ext} \int_A f \; \exists$.

(b) Case1 $f \geq 0$




Let $B \supseteq A$ be a box

So $\text{ord} \int_A f = \int_B f_A$ where $f_A$ is the extension of $f$ by 0

Let $D \subseteq A$ & $D \in \mathcal{J}_c$

$\Rightarrow \int_D f = \int_D f_A \leq \int_B f_A = \text{ord} \int_A f$

Taking sup over all $D \Rightarrow \text{ext} \int_A f \leq \text{ord} \int_A f$

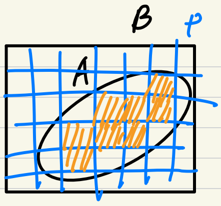
To show the reverse inequality, Let $P$ be any partition on $B$

Let $D = \bigcup_{R_i \subseteq A} R_i$ Let $R_1, \ldots, R_m$ be subboxes

($D$: union of subboxes completely contained in $A$)

Note

$$L(f_A,P) = \sum_{i=1}^{m} m_{R_i}(f_A)\, v(R_i)$$




$$= \sum_{R_i \subseteq A} m_{R_i}(f_A)\, v(R_i)$$

$$= \sum_{R_i \subseteq A} \int_{R_i} \underline{m_{R_i}(f_A)} \leftarrow \text{const function}$$

$$\leq \sum_{R_i \subseteq A} \int_{R_i} f_A = \int_D f \leq \text{ext}\int_A f$$

Taking the sup over $P$

$$\text{ord}\int_A f \leq \text{ext}\int_A f$$

Therefore $\text{ord}\int_A f = \text{ext}\int_A f$

(b) <u>Case 2</u> $f = f_+ - f_-$

Since $\text{ord}\int_A f \;\exists \implies$ so does $\text{ord}\int_A f_+$, $\text{ord}\int_A f_-$

Note $\text{ord}\int_A f = \text{ord}\int_A f_+ - \text{ord}\int_A f_-$

$$= \text{ext}\int_A f_+ - \text{ext}\int_A f_- = \text{ext}\int_A f$$

$\square$

<u>Corollary</u> Let $S \subseteq \mathbb{R}^n$ be bdd (not necessarily open) and $f: S \to \mathbb{R}$ bdd, ctn

If $\text{ord}\int_S f \;\exists \implies \text{ord}\int_S f = \text{ext}\int_{S^\circ} f$

<u>Pf</u> hw 12: $\text{ord}\int_S f \;\exists \implies \text{ord}\int_{S^\circ} f \;\exists$ &

$$\text{ord}\int_S f = \text{ord}\int_{S^\circ} f$$

$\square$