Some examples of Taylor's Thm

ex $f(x,y) = \sin(x^2+y^2)$

compute deg 2 Taylor poly.

$$\partial^{(1,0)}f = 2x\cos(x^2+y)$$
$$\partial^{(0,1)}f = \cos(x^2+y)$$
$$\partial^{(2,0)}f = 2\cos(x^2+y) - 4x^2\sin(x^2+y)$$
$$\partial^{(1,1)}f = -2x\sin(x^2+y)$$
$$\partial^{(0,2)}f = -\sin(x^2+y)$$

So $T(x) = \cancel{f(0,0)} + \cancel{\partial^{(1,0)}f(0,0)x} + \partial^{(0,1)}f(0,0)y$
$+ \dfrac{\partial^{(2,0)}f(0,0)x^2}{2} + \cancel{\partial^{(1,1)}f(0,0)xy} + \cancel{\dfrac{\partial^{(0,2)}f(0,0)}{2}y^2}$
$= y + \frac{2x^2}{2} = y + x^2$

## The Inverse function Thm

Q: When is $f: \mathbb{R}^n \to \mathbb{R}^m$ locally invertible?

**Def** local invertibility $\to$ must have same dimension

(1) $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is locally invertible near $x_0 \in A$

if $\exists\, \delta > 0$ s.t. $f \restriction B_\delta(x_0) : B_\delta(x_0) \to \theta \subseteq \mathbb{R}^n$

is a bijection onto some open $\theta \subseteq \mathbb{R}^n$

This gives us an inverse function $g: \theta \to B_\delta(x_0)$

(2) We call $f$ a local homeomorphism near $x_0$ if

$\exists\, \delta$ s.t. $f|_{B_\delta(x_0)}$ and $g$ are continuous

Further:

(3) We say $f$ is a local diffeomorphism near $x_0$

if $f|_{B_\delta(x_0)}$, $g$ are diffble.

(4) We say $f$ is a local $C^r$ diffeomorphism near $x_0$

if $f|_{B_\delta(x_0)}$, $g$ are $C^r$ diffble

ex $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto x^3$

is locally invertible but not a local diffeo near 0.

Warm up:

if $f$ is a local diffeo at $a$

$\exists\, g$ with $g(f(x)) = x$ ($\forall x$ near $a$)

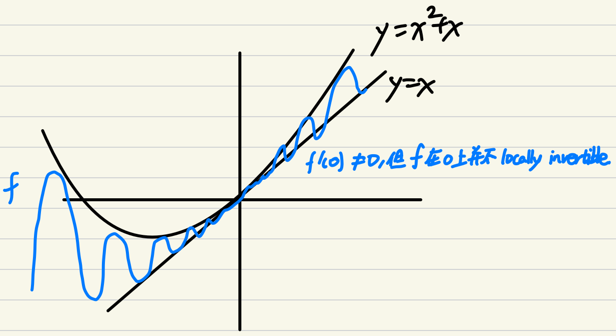
$f'(x)\, g'(f(x)) = 1 \Rightarrow f'(a) \neq 0$

If $f'(a)$ exists 且 $f'(a) \neq 0$

is it enough to imply: $f$ is locally invertible near $a$?

No

counterex




if $f'$ exists near $a$, 且 $f'(a) \neq 0$

$\Rightarrow f'(x) \neq 0$ on $(a-\varepsilon, a+\varepsilon)$

则 $\forall x, y \in B_\varepsilon(a)$, $\underbrace{f(x) - f(y)}_{} = \underbrace{f'(c)}_{\neq 0}(x-y)$ for some $c \in B_\varepsilon(a)$

($\Rightarrow f$ is locally injective near $a$)

## Inverse function Thm (IFT)

Suppose $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is $C^r$ ($r \geq 1$, $A$ open), $x_0 \in A$

Suppose $Df(x_0)$ is invertible (det $\neq 0$)

$\Rightarrow$ (1) $\exists$ open nbh $U \ni x_0$ 且 open nbh $V \ni f(x_0)$

s.t. $f(U) = V$ 且 $f|_U : U \to V$ is bij.

(2) The inverse function $g: V \to U$ is $C^r$

且 $\forall x \in U$, $Dg(f(x)) = (Df(x))^{-1}$ locally $g(f(x)) = I_n$

显然 $D(g \circ f)(x) = Dg(f(x))\, Df(x) = I_n$

So informally under reasonable assumptions

$Df(a)$ invertible $\Rightarrow f$ is locally invertible near $a$

and the inverse function as good as $f$.

**Rmk** One interpretation of IFT is that it allows us to solve

$$\begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ \quad \vdots \\ y_n = f_n(x_1, \dots, x_n) \end{cases}$$

for $x$ in terms of $y$ locally

**ex** $f: (1,2) \times (\pi, 3\pi) \longrightarrow \mathbb{R}^2$

$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

$$Df(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det Df(r,\theta) = r\cos^2\theta + r\sin^2\theta = r \neq 0$$

**Lemma 1** Let $E$ be an invertible $n \times n$ matrix

$\forall x, y \in \mathbb{R}^n$, "quantifying" the invertibility

$$|Ex - Ey| \geq \frac{1}{\|E^{-1}\|}|x-y| > 0$$

**Pf** for $v \in \mathbb{R}^n$

$|v| = |E^{-1}(Ev)| \leq \|E^{-1}\| \cdot |Ev|$ $\leftarrow$ note: $|Cx| \leq \|C\| |x|$

So $|E(v)| \geq \dfrac{|v|}{\|E^{-1}\|}$ so we have $|x| = |E^{-1}Ex| \leq \|E^{-1}\| |Ex|$

take $v = x - y$ $\square$

**Lemma 2** If $H: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is $C^1$ and $A$ contains the line from $x$ to $y$

then $|H(x) - H(y)| \leq \max\limits_{t \in [0,1]} \left| \partial^{e_j} H_i (x + t(y-x)) \right|$

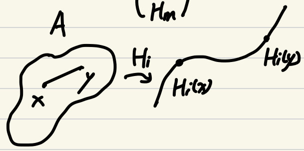
the largest term of $J$

$$H = \begin{pmatrix} H_1 \\ \vdots \\ H_m \end{pmatrix}$$

**Pf** Consider $\varphi_i(t) = H_i(x + t(y-x))$

$$\varphi_i'(t) = \left(\frac{\partial H_i}{\partial x_1}, \dots, \frac{\partial H_i}{\partial x_n}\right) \cdot (y - x)$$

$$= \sum_{j=1}^{n} \frac{\partial H_i}{\partial x_j}(y_j - x_j)$$




By MVT,

$$|H_i x - H_i y| \leq n\left(\max_j \left|\frac{\partial H_i}{\partial x_j}\right|\right)\left(\max_j |x_j - y_j|\right)$$

**Lemma 3** Suppose $f: A \subseteq \mathbb{R}$ is $C^1$ ($A$ open)

If $Df(x_0)$ is invertible

$\Rightarrow \exists\, \alpha > 0$ and open nbh $U \ni x_0$ $(U \subseteq A)$ s.t.

$|f(x) - f(y)| \geq \alpha |x - y|$ for all $x, y \in U$

(better that loc. inj. ☺)

**Pf** Set $E = Df(x_0)$

Set $H(x) = f(x) - E(x)$

Note $DH(x) = Df(x) - E$

$\Rightarrow DH(x_0) = 0$

Using Lemma 2 and ctn. of partials

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $|H(x) - H(y)| \leq \dfrac{1}{2\|E^{-1}\|}|x - y|$

$$|H(x) - H(y)| = |f(x) - f(y) - (Ex - Ey)|$$

$$\geq |E(x-y)| - |f(x) - f(y)| \quad \text{by tri. eq.}$$

So $|f(x) - f(y)| \geq |E(x-y)| - \dfrac{1}{2\|E^{-1}\|}|x - y|$

$\geq \dfrac{1}{\|E^{-1}\|}|x-y|$ by Lemma 1

$$\Longrightarrow |f(x) - f(y)| \geq \frac{1}{2\|E^{-1}\|}|x - y|$$

$\square$