IFT states the fact: For $f: A\subseteq\mathbb{R}^n\to\mathbb{R}^n$, if $Df(x_0)$ is non-singular

then for some $U\ni x$, specifying $y_1,\dots,y_n\in f(U)$

completely determines $x_1,\dots,x_n\in U$

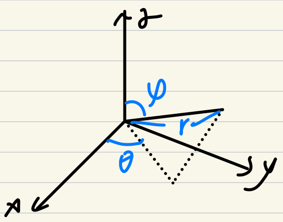
(and while U and $f(U)$ are diffeomorphic)

which means we can use $(y_1,\dots,y_n)$ instead of

$(x_1,\dots,x_n)$ as a coordinate system locally.

ex $f:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^3$

$(r,\varphi,\theta)\mapsto r(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$



$$Df(r,\varphi,\theta)=\begin{pmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta\\ \cos\varphi & -r\sin\theta & 0\end{pmatrix}$$

By Laplace expansion, taking $j=3$, $\det Df=\sum_{i=1}^{n}(-1)^{i+j}\det A_{ij}$

$$=(-1)^{1+3}(-r\sin\varphi\sin\theta)\det\begin{pmatrix}\sin\varphi\sin\theta & r\cos\varphi\sin\theta\\ \cos\varphi & -r\sin\theta\end{pmatrix}+(-1)^{2+3}(r\sin\varphi\cos\theta)\det\begin{pmatrix}\sin\varphi\cos\theta & r\cos\varphi\cos\theta\\ \cos\varphi & -r\sin\theta\end{pmatrix}$$

$=(-r\sin\varphi\sin\theta)(-r\sin\theta)-r\sin\varphi\cos\theta(-r\cos\theta)$

$=r^2\sin\varphi\sin^2\theta+r^2\sin\varphi\cos^2\theta$

$=r^2\sin\varphi$

So $\det Df(r,\varphi,\theta)\neq 0$ iff $r\neq 0$ and $\varphi\neq k\pi$

which means $\forall r\neq 0,\varphi\neq k\pi$,

$\exists U\ni(r,\varphi,\theta)$ s.t. $f(U)$ diffeomorphic U

can use $x=f_1,\ y=f_2,\ z=f_3$ as local coord.

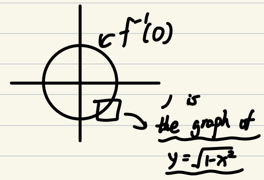
(In this case, have explicit formula but in other cases, not)

The implicit function theorem

ex $f:\mathbb{R}^2\to\mathbb{R}$

$f(x,y)=x^2+y^2-1$

$f^{-1}(0)=\{(x,y)\mid x^2+y^2=1\}$ is



Q when does an equation $f(x,y)=0$ locally define the graph of a function $y=g(x)$?

In this case we say $f(x,y)$ defines y implicitly in terms of y

For (a,b) on the unit circle, we can write $f(x,y)=0$ as

$f:(x,y)\mapsto x^2+y^2-1$

$y=g(x)$ for a small nbh of (a,b) as long as $(a,b)\neq(1,0),(-1,0)$

$y:x\mapsto\sqrt{1-x^2}$

(Note: $\frac{\partial f}{\partial y}=2y$ at $(1,0)$, $(-1,0)$)

Context for Implicit function Thm

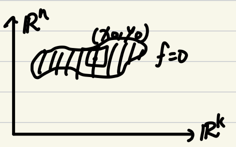
Given $f:A\subseteq\mathbb{R}^{k+n}\to\mathbb{R}^n$

A open

when can we say the level set $\{(x,y)\in\mathbb{R}^k\times\mathbb{R}^n \mid f(x,y)=0\}$

locally as the graph of a function $y=g(x)$?

$g:B\subseteq\mathbb{R}^k\to\mathbb{R}^n$



For understanding: $f:A\subseteq\mathbb{R}^k\times\mathbb{R}^n\to\mathbb{R}^n$

level set: $\{(x,y)\mid f(x,y)=0\}$

we have $k+n$ dims taking n dims to be 0

$\Rightarrow$ k dims left should be

ex when f is linear

$f:\mathbb{R}^k\times\mathbb{R}^m\to\mathbb{R}^n$

f is a $n\times(k+m)$ matrix

$f=(A|B)$

$f(x,y)=(A|B)\binom{x}{y}=Ax+By$

$f^{-1}(0)=\{By=-Ax\}$

To solve for y, sufficient that B is invertible (with $n=m$)

$y=-B^{-1}Ax$

Let $f:A\subseteq\mathbb{R}^n\to\mathbb{R}^m$ be diff'ble with components $f_1,\dots,f_m:\mathbb{R}^n\to\mathbb{R}$

If $(x_1,\dots,x_n)=(y,z)$ with $y\in\mathbb{R}^k$, $z\in\mathbb{R}^{n-k}$

Define:

$\frac{\partial f}{\partial y}=\frac{\partial(f_1,\dots,f_m)}{\partial(y_1,\dots,y_n)}$ to be the $m\times k$ matrix whose cols are $\frac{\partial f}{\partial y_i}$ $(1\le i\le k)$

$$\frac{\partial f}{\partial y}=\begin{array}{c|c} & k\\ \hline m & \begin{bmatrix}\frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_k}\end{bmatrix}\end{array}$$

Similarly define $\frac{\partial f}{\partial z}$

$$\begin{array}{c|c} & n\\ \hline m & \begin{bmatrix}\frac{\partial f}{\partial z_1} & \cdots & \frac{\partial f}{\partial z_{n-k}}\end{bmatrix}\end{array}=\frac{\partial f}{\partial z}$$

The implicit function Thm roughly says that

if $f:A\subseteq\mathbb{R}^k\times\mathbb{R}^n\to\mathbb{R}$,

the level set $\{(x,y)\in\mathbb{R}^k\times\mathbb{R}^n \mid f=0\}$ is the graph

of a function near (a,b) if $\frac{\partial f}{\partial y}$ is non-singular

$n\times n$ matrix

Warm up

Thm Implicit differentiation

Let $A \subseteq \mathbb{R}^{k+n}$ open, $f: A \to \mathbb{R}^n$ differentiable

(we write $f(x,y)$ for $x \in \mathbb{R}^k, y \in \mathbb{R}^n$)

Suppose $\exists g: \underset{(\text{open})}{B} \subseteq \mathbb{R}^k \to \mathbb{R}^n$ s.t. $\forall x \in B, f(x, g(x)) = 0 \in \mathbb{R}^n$

$$\Rightarrow \forall x \in B, \ \underbrace{\frac{\partial f}{\partial x}(x, g(x))}_{\substack{n \times k \\ \text{matrix}}} + \underbrace{\frac{\partial f}{\partial y}(x, g(x))}_{\substack{n \times n \\ \text{matrix}}} \cdot \underbrace{Dg(x)}_{\substack{n \times k \\ \text{matrix}}} = 0 \qquad f: A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$$

$\mathbb{R}^k$ $\mathbb{R}^n$

In particular, if $\frac{\partial f}{\partial y}(x, g(x))$ is invertible.

$$\Rightarrow \text{then } Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \frac{\partial f}{\partial x}(x, g(x))$$

Pf Let $h: B \subseteq \mathbb{R}^k \to \mathbb{R}^{k+n}$

map $\underset{\in \mathbb{R}^k}{x} \mapsto \begin{pmatrix} x \in \mathbb{R}^k \\ g(x) \in \mathbb{R}^n \end{pmatrix}$

So $\forall x \in B$, $\boxed{f \circ h(x) = f(x, g(x)) = 0}$

The chain rule gives $Df(h(x))\, Dh(x) = 0$

$$\text{Note} \quad Dh = \begin{pmatrix} \nabla h_1 \\ \nabla h_2 \\ \vdots \\ \nabla h_{k+n} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \hline & \nabla g_1 & \\ & \vdots & \\ & \nabla g_n & \end{pmatrix} = \begin{pmatrix} I_k \\ \hline Dg \end{pmatrix}$$

$$Df(x) = \left(\frac{\partial f}{\partial x_1} \middle| \cdots \middle| \frac{\partial f}{\partial y_n} \right) = \left(\frac{\partial f}{\partial x} \middle| \frac{\partial f}{\partial y} \right)$$

$$Df \circ h(x) = Df(h(x))\, Dh(x) = \left(\frac{\partial f}{\partial x} \middle| \frac{\partial f}{\partial y} \right)_{h(x)} \begin{pmatrix} I \\ Dg \end{pmatrix}_x = \frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x))\, Dg(x) \qquad \square$$