Implicit Function Thm

Let $f: A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^n$, A open, be C^r $(r \geq 1)$
$\quad (\underset{\in\mathbb{R}^k}{x}, \underset{\in\mathbb{R}^n}{y}) \mapsto \underset{\in\mathbb{R}^n}{z}$

Suppose $a \in \mathbb{R}^k$, $b \in \mathbb{R}^n$, $(a,b) \in A$ and $f(a,b) = 0 \in \mathbb{R}^n$

If $\frac{\partial f}{\partial y}(a,b)$ is non singular

$$Df = \left[\begin{array}{ccc:ccc} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_k} & \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_n} \end{array}\right]$$

$\Rightarrow \exists$ nbh $B_\varepsilon(a) \subseteq \mathbb{R}^k$
and unique cts. function $g: B_\varepsilon(a) \to \mathbb{R}^n$
s.t. $g(a) = b$ 且 $\forall x \in B_\varepsilon(a)$, we have $f(x, g(x)) = 0$

(并且 by implicit differentiation:
$\forall x \in B_\varepsilon(a)$, $\frac{\partial f}{\partial x}(x, g(x)) + \frac{\partial f}{\partial y}(x, g(x)) \cdot Dg(x) = 0$
$$Dg(x) = -\left[\frac{\partial f}{\partial y}(x, g(x))\right]^{-1} \frac{\partial f}{\partial x}(x, g(x))\Big)$$

Implicit Function Thm告诉我们:
只要 $f(z) = 0$ 且 $\frac{\partial f}{\partial y}(z)$ nonsigular
那么 locally 一定存在某个 implicit function g
使得 g 可以定义 f 在 z 局部的 level set $\{f=0\}$
by $\{f|_{B_\varepsilon(z)} = 0\} = \{(x, g(x)) \mid x \in U\}$

IFT 和 Implict FT 都告诉我们 C^r function 的"局部变化一致"的性质
IFT 告诉我们单点的导数 nonsigular 可得到局部的 invertivity,
从而得到局部 C^r diffeo
Implicit FT 告诉我们单点属于 level set (kernel)
可得到它周围的一圈由某个 implicit function (一定存在)
定义的 surrounding manifold 都在 f 的 kernel 里.

Pf of Implicit Function Thm

① Define an auxiliary function
$$F: A \subseteq \mathbb{R}^{k+n} \to \mathbb{R}^{k+n}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$$

$$DF(x,y) = \begin{pmatrix} DF_1 \\ \vdots \\ DF_{k+n} \end{pmatrix} = \begin{pmatrix} I_k \\ \hline Df \end{pmatrix} = \left(\begin{array}{c|c} I_k & 0 \\ \hline \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{array}\right)$$

$\Rightarrow \det DF(x,y) = (\det I_k)\left(\det \frac{\partial f}{\partial y}\right) = \det \frac{\partial f}{\partial y}$ ☆
by block matrices

So $\det DF(a,b) \neq 0$

$\Rightarrow$ By IFT, $\exists\, U \times V \ni (a,b)$
s.t. U is open in $\mathbb{R}^k$,
V is open in $\mathbb{R}^n$
s.t. $F|_{U\times V}$ is a local C^r diffeo
$U \times V \to W$ for some $W \subseteq \mathbb{R}^{k+n}$

Let $G: W \to \underset{\subseteq\mathbb{R}^{k+n}}{U \times V}$ be the inverse of $F|_{U\times V}$

$\Rightarrow \forall (x,y) \in U \times V$, $(x,y) = G(\underbrace{x, f(x,y)}_{\in W})$

So let $(x,z) \in W$,
$$\boxed{(x,z) = F \circ \underbrace{G(x,z)}_{=(x,y)}}$$

因而 $G = \begin{pmatrix} G_1 \\ \vdots \\ G_k \\ \vdots \\ G_{k+n} \end{pmatrix}$ 中, $G_1, \ldots, G_k$ 都是 identity function

(因为 F 的前 k 个是 identity, $F \circ G$ 的前 k 个也是 identity, 因而 G 的前 k 个必须也是 identity)

Let $h: W \to V$ 表示 G 的后 n 个 coordinate functions
$(x,z) \mapsto y$ (s.t. $f(x,y) = z$)

即 $h = \begin{pmatrix} G_{k+1} \\ \vdots \\ G_{k+n} \end{pmatrix}$

Since G is C^r (by IFT), h 也是 C^r 的.

Step 2 Construct the implicit function g.
Let $B \ni a$ s.t. $B \times \{0\} \subseteq W$
$\Rightarrow \forall (x,y) \in B \times V$
$f(x,y) = 0$ iff $F(x,y) = (x,0)$
iff $(x,y) = G(x,0) = (x, h(x,0))$

Let $g: \mathbb{R}^k \to \mathbb{R}^n$
$x \mapsto h(x,0) = G(x,0) = (x,y)$

Then we have $f(x,y) = 0$ iff $y = g(x)$ for $x \in B$

Clearly g is C^r since h is C^r
(Existence □)

(And clearly: $(a,b) = G(a,0) = (a, h(a,0))$
$b = h(a,0) = g(a)$ as desired)

Step 3 Prove Uniqueness of g.

Suppose $g': B \to V$ is another ctn. function satisfying the implicit condition

Let $S = \{x \in B \mid g(x) = g'(x)\}$

WTS: $S = B$ (which implies $g = g'$)

It suffices to show that S is nonempty, both open and closed

(Since B is connected and $B \subseteq \mathbb{R}^k$, S open and closed in $B \Rightarrow S = B$) ☆

nonempty: Consider $g'(a) = g(a) \Rightarrow S$ nonempty ✓ ①

closed: Since g, g' both ctn and $S = (g-g')^{-1}(\{0\})$
$\Rightarrow S$ is closed ②

$\underset{\text{ctn}}{\uparrow}$ $\underset{\text{closed}}{\uparrow}$

open: Let $x_0 \in S \Rightarrow g'(x_0) = g(x_0) \in V$ is open
$\Rightarrow \exists$ nbh $B'_\varepsilon(x_0)$ s.t. $g'(B'_\varepsilon(x_0)) \subseteq V$

So $f(x, g'(x)) = 0 \ \forall x \in B' \subseteq B$

$F(x, g'(x)) = (x, 0) \Rightarrow (x, g'(x)) = G(x,0)$
$= (x, h(x,0)) = (x, g(x))$

So $\forall x \in B'_\varepsilon(x_0)$, we have $g'(x) = g(x)$
$\Rightarrow x \in S$

This proves that S is open ③

By ①②③ $\Rightarrow S = B \Rightarrow g' = g$

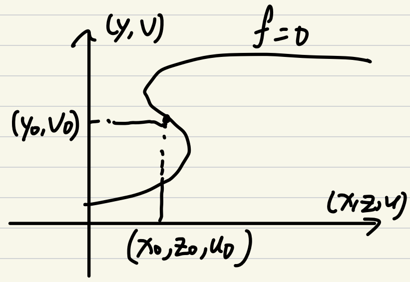
(Uniqueness □)

□

Application of Implicit Function Thm

Suppose $f: A \subseteq \mathbb{R}^5 \to \mathbb{R}^2$ is C^r
we can solve for (y, v) in terms of (x, z, u)
for $f = 0$, near a pt. $(x_0, y_0, z_0, u_0, v_0)$
if $f(x_0, y_0, z_0, u_0, v_0) = 0$ and $Df(x_0, y_0, z_0, u_0, v_0) \neq 0$
write $y = \varphi(x, z, u)$, $v = \psi(x, z, u)$

We have $\frac{\partial(\varphi, \psi)}{\partial(x,z,u)}(x_0, z_0, u_0) = -\left[\frac{\partial f}{\partial(y,v)(x_0, \dots, v_0)}\right]^{-1} \frac{\partial f}{\partial x}(x, g(x))$



<u>ex</u> $f: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x^3 - y^3 + z^3 \\ z\cos(\pi x) + \sin(\pi y) \end{pmatrix}$$

note that $f(1,1,0) = 0$

$$Df = \begin{pmatrix} 3x^2 & -3y^2 & 2z \\ -\pi z \sin \pi x & \pi \cos \pi y & \cos \pi x \end{pmatrix}$$

$$Df(1,1,0) = \begin{pmatrix} 3 & - & 0 \\ 0 & -\pi & -1 \end{pmatrix}$$

$$\frac{\partial f}{\partial(x,z)} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \text{ non-singular.}$$

Therefore, can solve for (x, z) in term of y near $y = 1$

$\exists\ \varphi, \psi: B \subseteq \mathbb{R} \to \mathbb{R}^2$
s.t. $f(\varphi(y), y, \psi(y)) = 0 \ \forall y \in B$

The sol set is a one-parameter family of sols.
(a manifold of dimension one)