Problem A: Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We say that a function $f : X \to Y$ is Lipschitz with constant C if for any $x, y \in X$, we have

$$d_2(f(x), f(y)) \leq C d_1(x, y).$$

(1) Show that Lipschitz maps are uniformly continuous, i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d_1(x_1, x_2) < \delta$ then $d_2(f(x_1), f(x_2)) < \epsilon$.

(2) Let $f_n : X_1 \to X_2$ be Lipschitz maps with common Lipschitz constant C. Suppose that the f_n converge uniformly to f, i.e. for all $\epsilon > 0$ there exists $N > 0$ such that for all $n > N$, and all $x \in X_1$,

$$d_2(f_n(x), f(x)) < \epsilon.$$

Is f Lipschitz? What if we only assume that the f_n are Lipschitz (without giving a common Lipshitz constant)?

(1) Pf. Suppose $f: X \to Y$ is Lipschitz.

Let $\varepsilon > 0$

By Lipschitz $\exists C$ s.t $d_2(f(x), f(y)) \leq C d_1(x, y)$

Take $\delta = \frac{\varepsilon}{C} \Rightarrow \forall x, y \in X_1$ st. $d_1(x, y) < \delta$,

we have $d_2(f(x), f(y)) \leq \frac{\varepsilon}{\delta} d_1(x, y) < \varepsilon$

$\square$

(2) Claim: If the assumptions hold true then f is Lipschitz.

Pf Let $x, y \in X_1$. Take $\varepsilon = d_1(x, y)$

So $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(f_n(x), f(x)) < \varepsilon$

and by Lipschitz, $d(f_n(x), f_n(y)) < C d(x, y)$

Then by triangular inequality of metric space,

$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y) + f(y))$

$< C d_1(x, y) + 2 d_1(x, y)$

This proves that f is Lipschitz with constant $C+2$ (can get arbitrary close to C by diff ε)

Claim 2: Without common Lipschitz condition, the proposition is false.

Consider $\left(f_n(x) = \sqrt{x + \frac{1}{n}}\right)_{n\in\mathbb{N}} \to f(x) = \sqrt{x}$, $x \in (0, +\infty)$

Obviously the convergence is uniform.

Claim $f: A \to \mathbb{R}$ is Lipschitz if and only if
(A open) it is differentiable and f' is bounded

Pf of claim: Assume f is Lipschitz with constant C

Then $\forall x \in \mathbb{R}$, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \le \lim_{h \to 0} \frac{Ch}{h} = C$, exists and is bounded

Assume f is diff'ble and f' is bounded by $M \geq 0$.

Then $\forall x, y \in A$, by FTC, $\int_x^y f' dx = f(y) - f(x) \le M(y-x)$ $\square$

$= \sum_i \int_{x_i}^{y_i} f' dx = \sum_i f(y_i) - f(x_i)$

$A = \bigcup_{i=1}^{n} (x_i, y_i)$ for some $x_1, \dots, x_n, y_1, \dots, y_n \in A$

Note: $\forall n \in \mathbb{N}$, $f_n \in C^1$

For any $n \in \mathbb{N}$, $f_n'(x) = \frac{1}{\sqrt{x + \frac{1}{n}}}$ is bounded by $\sqrt{n}$, so Lipschitz

But $f'(x) = \frac{1}{\sqrt{x}}$ is not bounded ($f' \to \infty$ when $x \to 0$), so not Lipschitz

Problem B: We say that a metric space X is connected if it cannot be written as $X = A \cup B$ where A and B are nonempty disjoint open subsets of X.

(1) Show that if $f: X \to Y$ is a continuous function between metric spaces X and Y, then $f(X)$ is connected if X is connected.

(2) Conclude that if f cts $: X \to \mathbb{R}$ and X is a connected metric space, then f admits all *intermediate* values $m \in (\inf f, \sup f)$. That is, for any such m, there exists $x_0 \in X$ such that $f(x_0) = m$.

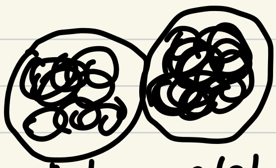
(1) Pf Suppose X is connected

Assume for contradiction that $f(X)$ is not connected

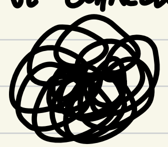
Then $\exists B_1, B_2$ open in Y s.t. $Y = B_1 \sqcup B_2$

By continuity of f, $f^{-1}(B_1)$, $f^{-1}(B_2)$ are open in X

Let $x \in X \Rightarrow f(x) \in Y \Rightarrow f(x) \in B_1$ or $f(x) \in B_2$
$\Rightarrow x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$



not connected



Connected

Then $X = f^{-1}(B_1) \sqcup f^{-1}(B_2)$ (disjoint by well-definedness of the function)
So contradicts.
This finishes the proof that $f(X)$ is connected if X is connected. $\square$

(2) Conclusion: if $f: X \to \mathbb{R}$ with X being a connected metric space, then $\mathrm{im}(f)$ is connected, i.e. f admits all intermediate value $m \in (\inf f, \sup f)$
$(\forall m \in (\inf f, \sup f), m \in \mathrm{im} f)$

Problem C: Let $f : X \to Y$ be a continuous bijective (one-to-one and onto) mapping between metric spaces X and Y.

(1) Suppose that X is compact. Show that the inverse function $f^{-1} : Y \to X$ is also continuous.
(2) Give an example to show that the requirement that X is compact is necessary.

(1) Pf Suppose X is compact
Let $B \subseteq X$ be closed $\Rightarrow$ then B is compact since it is closed subset of a compact MS.
$(f^{-1})^{-1}(B) = f(B)$

Note: $f(B)$ is compact since f ctn, and B cpt. (Lec 4)
thus closed

So $\forall$ closed $B \in X$, $(f^{-1})^{-1}(B)$ is closed in Y $\square$

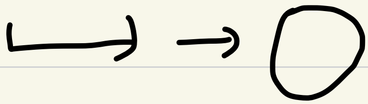
Claim C.1: For $f: X \to Y$ between topological spaces, f is ctn. iff $\forall$ close $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(Pf) Suppose $f: X \to Y$ be ctn. $C \subseteq Y$ be closed
So $Y \setminus C$ is open in $Y \Rightarrow f^{-1}(Y \setminus C)$ is open in X
$\Rightarrow f^{-1}(C) = X \setminus f^{-1}(Y \setminus C)$ is closed in X.

Suppose $\forall$ closed $C \subseteq Y$, $f^{-1}(C)$ is closed
Let $B \subseteq Y$ be open $\Rightarrow Y \setminus B$ is closed in $Y \Rightarrow f^{-1}(Y \setminus B)$ is open
$\Rightarrow f^{-1}(B) = X \setminus f^{-1}(Y \setminus B)$ is open

By claim C.1, f^{-1} is continuous. This finishes the proof. $\square$

(2) Consider $f:[0,2\pi)\longrightarrow S^1$
mapping $t\longmapsto e^{it}$



<u>Claim</u> f is continuous and bijective

For <u>bijective</u>: $e^{it}=\cos t+i\sin t$

so for distinct $t_1,t_2\in[0,2\pi)$, $e^{it_1}\neq e^{it_2}$

For <u>continuous</u>: Let $\varepsilon>0$, $t_1\in[0,2\pi)$

Take $\delta>0$ s.t. $\forall t_2\in B_\delta(t_1)$, $|\cos t_1-\cos t_2|<\frac{\varepsilon}{\sqrt{2}}$ and $|\sin t_1-\sin t_2|<\frac{\varepsilon}{\sqrt{2}}$ (can be done since sin, cos are ctn.)

Then $|e^{it_1}-e^{it_2}|=\sqrt{(\cos t_1-\cos t_2)^2+(\sin t_1-\sin t_2)^2}<\varepsilon$

But clearly f^{-1} is not ctn at $1=e^{0i}$.

Rmk: Seems that the proposition can be extended to general topological spaces since we did not use special properties of metric space during proof?

Problem D: Let f be a real valued function defined on $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$). Recall that the directional derivative $D_v f(p)$ of f at p in the direction v is vector

$$D_v f(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$$

if this limit exists.

(1) If $c \in \mathbb{R}$ and $D_v f(p)$ exists, prove that $D_{cv} f(p)$ exists and $D_{cv} f(p) = c \cdot D_v f(p)$.

(2) For $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sqrt{|xy|}$$

and $v = (1, 0)$, $v' = (0, 1)$, show that $D_v f(0, 0)$ and $D_{v'} f(0, 0)$ exist but $D_{v+v'} f(0, 0)$ does not exist.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{xy^2}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove that $D_v f(0, 0)$ exists for every $v = (a, b) \in \mathbb{R}^2$, vanishing if $v = 0$ and equal to

$$\frac{ab^2}{a^2 + b^2}$$

otherwise.

Remark 1. This formula for $D_v f(0, 0)$ is not linear in v.

Remark 2. Using polar coordinates, it is easy to see that f is continuous at $(0, 0)$.

(1) Pf Suppose $c \in \mathbb{R}$ and $\exists D_v f(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$

then if $c = 0$, $\lim_{t\to 0} \frac{f(p+ctv) - f(p)}{t} = \lim_{t\to 0} 0 = 0 = cD_v f(p)$

if $c \neq 0$, $\frac{f(p+t(cv)) - f(p)}{t} = \frac{f(p+ctv) - f(p)}{ct} \cdot c$

so $\lim_{t\to 0} \frac{f(p+t(cv)) - f(p)}{t} = \lim_{h\to 0} \frac{f(p+hv) - f(p)}{h} \cdot c = cD_v f(p)$

Therefore for all $c \in \mathbb{R}$, $D_{cv} f(p)$ exists and is equal to $cD_v f(p)$

(2) $f: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \sqrt{|xy|}$, $v = (1,0)$, $v' = (0,1)$

$$D_v f(0,0) = \lim_{t\to 0} \frac{f\left(\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\end{pmatrix}\right) - f\begin{pmatrix}0\\0\end{pmatrix}}{t} = \lim_{t\to 0} \frac{\sqrt{0}-\sqrt{0}}{t} = 0$$

$$D_{v'} f(0,0) = \lim_{t\to 0} \frac{f\left(\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}0\\1\end{pmatrix}\right) - f\begin{pmatrix}0\\0\end{pmatrix}}{t} = \lim_{t\to 0} \frac{\sqrt{0}-\sqrt{0}}{t} = 0$$

$$D_{v+v'} f(0,0) = \lim_{t\to 0} \frac{f\left(\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}1\\1\end{pmatrix}\right) - f\begin{pmatrix}0\\0\end{pmatrix}}{t} = \lim_{t\to 0} \frac{\sqrt{t^2}}{t} = \lim_{t\to 0} \frac{|t|}{t}$$

does not exist since $\lim_{t\to 0^+} \frac{|t|}{t} = 1$, $\lim_{t\to 0^-} \frac{|t|}{t} = -1$

(3) PF

For $v=0$, $D_v f(0,0) = \lim_{t\to 0} \frac{f(0,0) - f(0,0)}{0} = 0$

For $v \neq 0$, $D_v f(0,0) = \lim_{t\to 0} \frac{f(ta,tb) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{(ta)(tb)^2}{(ta)^2+(tb)^2} - 0}{t}$

So $D_v f(0,0)$ exists for every $v \in \mathbb{R}^2$ $\quad = \lim_{t\to 0} \frac{t^3 ab^2}{t^3(a^2+b^2)} = \frac{ab^2}{a^2+b^2}$

Problem E: Give the statement of the Baire Category Theorem (from Worksheet 1). (Test yourself by seeing if you can write it down from memory!)

For complete metric space (X,d), any sequence of open dense sets in $(U_n)_{n\in\mathbb{N}}$ in X has $\bigcap_{n=1}^{\infty} U_n$ also dense in X.

Problem F: Submit a writeup of Problem B from Worksheet 2.

WS2 <u>Problem B</u>

Contradict that $m: P(\mathbb{R}^d) \to [0,\infty)$ satisfying:

(a) $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$ for countable $(E_n)_{n\in\mathbb{N}}$ where $E_n \subset \mathbb{R}^d$

(b) $m(E) = m(F)$ if E congruent to F.

(c) $m([0,1)^d) = 1$

does not exist.

By the construction: say $x \sim y$ if $x-y \in \mathbb{Q}$

Take $N \subseteq [0,1)$ s.t. N contains exactly one element of each congruent class.

$R = [0,1) \cap \mathbb{Q}$

For each $r \in R$, define $N_r = \{x+r : x \in N \cap [0,1-r)\} \cup \{x+r-1 : x \in N \cap [1-r,1)\}$

(1) $[0,1) = \bigsqcup_r N_r$

Pf Claim 1.1: $[0,1) = \bigcup_r N_r$

Denote the set of all congruent class as $con(\mathbb{R})$

let $x \in [0,1)$

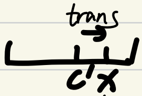
if $x \in N \Rightarrow x \in N_0 \Rightarrow$ done

if $x \notin N$, at least $x \in [c]$ for some $[c] \in con(\mathbb{R})$

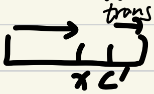
And by def of N, $\exists$ some $c' \in N$ st. $c' \in [c] \Rightarrow x - c' \in \mathbb{Q}$

Notice that $c' \in [0,1)$.

if $x < c' \Rightarrow x \in N_{c'-x}$



if $x \leq c' \Rightarrow x \in N_{1-c'+x}$



Claim 1.2 $\forall r_1, r_2 \in R$, $N_{r_1} = N_{r_2}$ or $N_{r_1} \cap N_{r_2} = \emptyset$ (i.e. N_{r_1}, N_{r_2} are disjoint)

Suppose $N_{r_1} \cap N_{r_2} \neq \emptyset$

let $x \in N_{r_1} \cap N_{r_2}$

Note : $x \in [c]$ for some congruent class $[c]$

Clearly, each N_r must and can only has one element from each congruent class, otherwise if $x_1, x_2 \in [c], x_1, x_2 \in N_r \Rightarrow$ by translation there will be two elements from $[c]$ in N.

Therefore $x \in N_{r_1}$ and $x \in N_{r_2}$ are translated from the same element from N.

$\Rightarrow x+r \equiv x+s \mod 1$

Since $r_1, r_2 \in [0,1)$, we must have $r=s$

By claim 1.1 & 1.2, $[0,1) = \bigsqcup_{r \in R} N_r$

(2) If $m: P(\mathbb{R}^d) \to [0,\infty)$ satisfying (a)(b)(c), then $m(N) = m(N_r)$ for every $r \in R$

Let $r \in R$. Denote $A = [0, 1-r)$, $B = [1-r, 1)$

$N = (N \cap A) \sqcup (N \cap B)$

Note that $N_r = (N \cap A) + r \sqcup (N \cap B) + (1-r)$

So by property (b) of the measure function we assume, we must have $m(N \cap A) = m(N \cap A + r)$

$m(N \cap B) = m(N \cap B + 1 - r)$

Then by property (a), $m(N) = m(N \cap A) + m(N \cap B)$

$= m(N \cap A + r) + m(N \cap B + 1 - r)$

$= m(N_r)$

(c) Arrive at a contradiction

Pf If $m(N)=0$, then since $\mathbb{R}$ is infinite,

$$m([0,1)) = m(\bigsqcup_r N_r) = \sum_{r\in\mathbb{R}} N_r = \sum_{r\in\mathbb{R}} N = +\infty$$

If $m(N)\neq 0$, then $m([0,1)) = \sum_{r\in\mathbb{R}} N = 0$

Thus in whatever way we define $m(N)$, the propery (c) will fail to be true. (Basically if m defines N to be measurable, then property (a)(b) contradicts (c).)

Bonus problem: A metric space (X,d) is said to be uniformly disconnected if there is $\epsilon_0 > 0$ so that no pair of distinct points $x, y \in X$ can be connected by an ϵ_0-chain, where an ϵ_0-chain connecting x and y is a sequence of points

$$x = x_0, x_1, \ldots, x_m = y$$

satisfying

$$d(x_i, x_{i+1}) \leq \epsilon_0 d(x,y).$$

(1) Show that the Cantor set is uniformly disconnected.
(2) Show that a metric space (X,d) is uniformly disconnected if and only if there is an ultrametric d' on X for which there is some $C > 1$ such that

$$d'(x,y)/C \leq d(x,y) \leq C d'(x,y).$$

An ultrametric is a metric which satisfies the following improvement of the triangule inequality:

$$d(x,z) \leq \max(d(x,y), d(y,z))$$

for all x, y, z. The discrete metric, where the distance between any pair of distinct points is 1, is an example of an ultrametric. Many other more interesting and important examples exist.

For a hint on the bonus, see office door. But try without first.

(1) Pf Consider $\varepsilon = \frac{9}{30}$

Denote the Cantor set as Cat

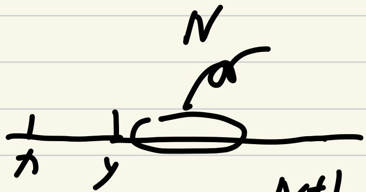
Let $x, y \in Cat$ with WLOG $y > x$

Let $n \in \mathbb{N}$

For $n=1$, we take away $(\frac{1}{3}, \frac{2}{3})$ from $[0,1]$ and get Cat_1

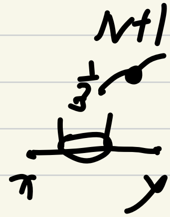
Suppose that the union of the middle parts of all disjoint intervals in $[0,1]$ we take away is I_n and we get

$$Cat_n = Cat_{n-1} \setminus I_n$$



Then $Cat = [0,1] \setminus \bigsqcup_{n \in \mathbb{N}} I_n$

Write $|I| = b-a$ if $I = [a,b]$ for some $a, b \in \mathbb{R}$



Since $x, y \in$ Cantor, $\exists$ some $N \in \mathbb{N}$ s.t.

$$[x,y] \subseteq Cat_N \text{ but } [x,y] \subseteq Cat_{N+1}$$

Then by definition of Cantor set,

$$\left| I_{N+1} \cap [x,y] \right| \geq \frac{1}{3}(y-x)$$

And $I_{N+1} \cap [x,y] = I'_{N+1} = [b-a]$ for some $x \leq a \leq b \leq y$

Let $x = x_1, \ldots, x_m = y$ be arbitrary ε-chain in Cat, with $\varepsilon = \frac{9}{30}$, between y and x

Then $\exists$ some x_{m_1}, x_{m_2} in the chain s.t. $x_{m_1} \leq a \leq b \leq x_{m_2}$

So $d(x_{m_2}, x_{m_1}) \geq \frac{1}{3} d(x,y) > \varepsilon\, d(x,y)$

This finishes the proof that Cat is uniformly disconnected. $\square$

(2) Hint: $d'(x,y) = \inf(\gamma : \exists \text{ a } \frac{\gamma}{d(x,y)}\text{-chain from } x \text{ to } y)$

For $\Rightarrow$: (Uniformly disconnected implies $\exists$ such ultrametric d')

Pf Assume (X,d) is uniformly disconnected.

Construct $d': X \to \mathbb{R}$ sending

$$(x,y) \mapsto \inf\left(\gamma : \exists \text{ a } \frac{\gamma}{d(x,y)}\text{-chain from } x \text{ to } y\right)$$

d' is nonnegative and $d'(x,y)=0$ iff $x=y$ since X is uniformly disconnected and $d'(x,y)=d'(y,x)$ for all $x,y\in X$ since an ε-chain is an ε-chain from x to y iff it is an ε-chain from y to x.

So it suffices to show that d' satisfies the improved triangular ineq. in order to show that d' is an ultrametric

Let $x,y,z\in X$.

Suppose for contradiction that

$$d'(x,z) > d'(x,y) \text{ and } d'(x,z) > d'(y,z)$$

WLOG let $\gamma = d'(x,y) \geq d'(y,z)$

Then $\exists \frac{\gamma}{d(x,y)}$ chain $x=x_1,\dots,x_m=y$, and

$\frac{\gamma}{d(x,z)}$ chain $y=y_1,\dots,y_m=z$

Thus $x=x_1,\dots,x_m,y_1,\dots,y_m=z$ is a $\frac{\gamma}{d(x,z)}$ chain from x to z, so $d'(x,z)\leq\gamma=d'(x,y)$, reaching contradiction

① This proves that d' is an ultrametric on X induced by d

Now we want to show that $\exists C>1$ s.t.

$$\frac{d'(x,y)}{C} \leq d(x,y) \leq C\,d'(x,y) \text{ for all } x,y\in X.$$

Since (X,d) is uniformly disconnected, we can take ε_0 s.t. $\forall x,y\in X$, there is no ε_0-chain between x,y

Take $C = \frac{1}{\varepsilon_0}$

Let $x, y \in X$

WTS: $\frac{d'(x,y)}{C} \le d(x,y) \le C d'(x,y)$ and we must have $C > 1$

This part I have not finished yet.

For ($\Leftarrow$): Such ultrametric exists imply that X is uniformly disconnected.

Pf Assume $\exists$ ultrametric d' on X and $C > 1$ s.t.

$$\frac{d'(x,y)}{C} \le d(x,y) \le C d'(x,y)$$

Then let $\varepsilon = \frac{1}{2C}$

Let $x, y \in X$

Suppose for contradiction that $\exists$ an ε_0-chain

$x = x_0, x_1, \ldots, x_m = y$ s.t. $d(x_i, x_{i+1}) \le \varepsilon d(x,y)$ for each $i = 0, \ldots, m$

Then $d'(x_i, x_{i+1}) \le C d(x_i, x_{i+1}) \le C\varepsilon d(x,y) = \frac{1}{2} d(x,y)$ for each i

Then by the ultrametric inequality,

$$d'(x,y) \le \max_i d'(x_i, x_{i+1}) \le \frac{1}{2} d'(x,y)$$

$\Rightarrow d'(x,y) = 0$, reaching a contradiction

Therefore for $\varepsilon = \frac{1}{2C}$, no pair of distinct points in X can be connected by a ε-chain

$\Rightarrow X$ is uniformly disconnected

$\square$

This finishes the proof of the iff statement.