Problem A: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be satisfy
$$F(tx) = tF(x)$$
for all all positive real numbers t and all $x \in \mathbb{R}^n$. Assume F is differentiable at the origon. Show F is linear.

Pf Construct $r_0(h) = F(0+h) - F(0) - Df(0)h = F(h) - Df(0)h$

Then that for $t \in \mathbb{R}_{>0}$

$r_0(th) = F(th) - Df(0)(th) = tF(h) - tDf(0)h = t r_0(h)$

Claim: $\forall h \in \mathbb{R}^n$, $r_0(h) = 0$

(Pf) Suppose for contradiction that for some $h_0 \in \mathbb{R}^n$, $r_0(h_0) \neq 0$

Then $\frac{\|r_0(h_0)\|}{\|h_0\|} = c$ for some $c > 0$

By homogeneity, for any $t > 0$, $\frac{\|r_0(th_0)\|}{\|th_0\|} = \frac{t\|r_0 h_0\|}{t\|h_0\|} = \frac{\|r_0(h_0)\|}{\|h_0\|} = c$,

then for $t \to 0$, $\|th_0\| \to 0$ while $\frac{\|r_0(h_0)\|}{\|h_0\|} = c$,

contradicting that $\lim_{\|h\| \to 0} \left\| \frac{r_0(h)}{h} \right\| = 0$

This proves the claim.

So $\forall h \in \mathbb{R}^n$, $F(h) = Df(0)h$ where $Df(0) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$

Thus F is a linear transformation. $\square$

Problem B: Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ $(1 \leq i \leq m, 1 \leq j \leq n)$ exist are are bounded on A. Show that f is continuous on A.

Pf Claim $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m = \begin{pmatrix} f_1 : \mathbb{R}^n \to \mathbb{R} \\ \vdots \\ f_m : \mathbb{R}^n \to \mathbb{R} \end{pmatrix}$ is continuous

at $x_0 \in A$ iff $\forall i \in \{1, \dots, n\}$, f_i is continuous at x_0.

This directly follows from $\|f(x) - f(x_0)\|_2 = \sqrt{\sum_i (f_i(x) - f_i(x_0))^2}$

(if $\forall x \in B_\delta(x_0)$ we have $f(x) \in B_\varepsilon(f(x_0))$, then $\forall f_i$, $f_i(x) \in B_\varepsilon(f_i(x_0))$

if for all i, $\forall x \in B_\delta(x_0)$ we have $f_i(x) \in B_{\frac{\varepsilon}{\sqrt{n}}}(f_i(x_0)) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$)

There WLOG we can set $m = 1$

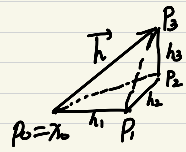
Assume $\frac{\partial f}{\partial x_j}$ $(1 \leq j \leq n)$ exist (for all $x \in A$) and bounded

WTS: f is continuous on A.

Let $\varepsilon > 0$.

Let $x = x_0 + h$ where $h \in \mathbb{R}^n$

Then $h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ for some $h_1, \dots, h_n \in \mathbb{R}$



Let $p_0 = x_0$

$p_1 = p_0 + h_1 e_1$

$\vdots$

$p_n = p_{n-1} + h_n e_n = x_0 + h$

For each $i = 1, \dots, n$, let $\varphi_i : [0, h_i] \to \mathbb{R}$

map $s \mapsto f(p_{i-1} + s e_i)$

Then $\forall s_0 \in (0, h_i)$, $\frac{d}{ds}\Big|_{s=s_0} \varphi_i(s) = \frac{d}{ds}\Big|_{s=s_0} f(p_{i-1} + s e_i) = \frac{\partial}{\partial x_i} f(p_{i-1} + s_0 e_i)$

Since all partials exist on A and bounded,

all φ_i are differentiable on $(0, h_i)$,

So by MVT, $\forall i$, $\varphi_i(h_i) - \varphi_i(0) = \left(\frac{\partial}{\partial x_i} f(p_{i-1} + s_i e_i) \right) \cdot h_i$

for some $s_i \in (0, h_i)$, we write $p_{i-1} + s_i e_i$ as q_i

Then $|f(x+h) - f(x)| = \left| \sum_{i=1}^{n} (f(p_i) - f(p_{i-1})) \right| = \left| \sum_{i=1}^{n} (\varphi_i(h_i) - \varphi_i(0)) \right|$

$= \left| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(q_i) h_i \right|$

Since all partials are bounded by some $M \in \mathbb{R}$ and

$|h_i| \leq \|h\| = \|x - x_0\|$

Thus we have:

$|f(x) - f(x_0)| \leq nM \|x - x_0\|$

This implies that f is Lipschitz on A, thus (uniformly) continuous (by hw 3). $\square$

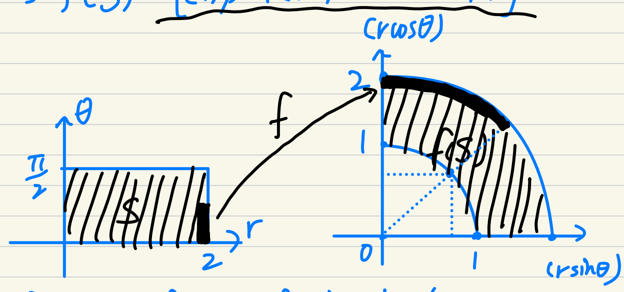
Problem C: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation:
$$f(r, \theta) = (r\cos\theta, r\sin\theta).$$

(1) Calculate Df and $\det Df$.
(2) Let $S = [1, 2] \times [0, \pi/2]$. Find $f(S)$ and sketch it.
(3) Show that f is a homeomorphism from S on $f(S)$ and compute the inverse function f^{-1}.
(4) Compute Df^{-1} and $\det Df^{-1}$.
(5) What relation can you find between Df and Df^{-1}?

(1) $Df = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$\det Df = r\cos^2\theta + r\sin^2\theta = r$

(2) $f(S) = \{(x, y) : 1 \leq x^2 + y^2 \leq 4 \text{ and } x, y \geq 0\}$



(3) Claim ①. $f : S \to f(S)$ is bijective

Pf it is surjective since we take $f(S)$ to be in place of codomain

Now prove injectivity: suppose $f(r_1, \theta_1) = f(r_2, \theta_2)$

then $r_1\cos\theta_1 = r_2\cos\theta_2$

$r_1\sin\theta_1 = r_2\sin\theta_2$

$\Rightarrow r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta = r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2 \Rightarrow r_1^2 = r_2^2 \Rightarrow r_1 = r_2$ since $r_1, r_2 > 0$

$\Rightarrow \cos\theta_1 = \cos\theta_2 \Rightarrow \theta_1 = \theta_2$ since $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$

Claim ②. f is continuous.

Since $f(r,\theta) = \begin{pmatrix} f_1(r,\theta)=r\cos\theta \\ f_2(r,\theta)=r\sin\theta\end{pmatrix}$ where f_1, f_2 are all continuous functions, f is always continuous.

Claim ③ f^{-1} is continuous

let $f(r,\theta)=\binom{x}{y} \Rightarrow x=r\cos\theta,\ y=r\sin\theta \Rightarrow x^2+y^2=r^2$

$\Rightarrow r=\sqrt{x^2+y^2}$ since $r>0$;

and $\tan\theta = \frac{y}{x} \Rightarrow \theta=\tan^{-1}(\frac{y}{x})$ since $x,y>0$, $x\in[0,\frac{\pi}{2}]$

So $f^{-1}: f(S)\to S$

sending $\binom{x}{y}\mapsto \binom{\sqrt{x^2+y^2}}{\tan^{-1}(\frac{y}{x})}$, which is continuous since f_1^{-1}, f_2^{-1} are continuous.

Claim ①②③ proves that f is a homeomorphism.

(4) $\frac{\partial}{\partial x} r = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial}{\partial y} r = \frac{y}{\sqrt{x^2+y^2}}$, $\frac{\partial}{\partial x}\theta = \frac{\partial}{\partial x}\arctan(\frac{y}{x}) = \frac{-y}{x^2+y^2}$, $\frac{\partial}{\partial y}\theta = \frac{x}{x^2+y^2}$

$\Rightarrow Df^{-1}(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r}\end{pmatrix}$, $\det Df^{-1} = \frac{x^2}{r^3}+\frac{y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$

(5) $Df\cdot Df^{-1} = \begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ \frac{\sin\theta}{-r} & \frac{\cos\theta}{r}\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} \Rightarrow (Df)(Df^{-1}) = I_2$

Problem D: Give an example of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that, at the origin, all directions derivatives exist and are zero, but F is not differentiable at the origin.

Consider $F\binom{x}{y} = \begin{pmatrix}\frac{x^2y}{x^2+y^2}\\ \frac{x^2y}{x^2+y^2}\end{pmatrix}$ for $\binom{x}{y}\ne\binom{0}{0}$ and $F\binom{0}{0}=\binom{0}{0}$

$D_{e_1}F(0,0) = \lim_{t\to0}\frac{F\binom{0+t}{0}-F\binom{0}{0}}{t} = \lim_{t\to0}(0,0)=(0,0)$

Similarly $D_{e_2}F(0,0) = \lim_{t\to0}(0,0)=(0,0)$

Since $\forall u\in\mathbb{R}^2$, u is a linear comb of e_1,e_2 and $D_uF(\vec{0})$ is linear in u, all directional derivatives exist and are 0 at origin:

$D_uf\binom{0}{0} = D_{u_1e_1+u_2e_2}f\binom{0}{0} = u_1D_{e_1}f\binom{0}{0}+u_2D_{e_2}f\binom{0}{0} = \binom{0}{0}$

The Jacobian matrix $J_f(0) = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$$\lim_{\|(x,y)\|\to0}\frac{f(x,y)-f(0,0)-J_f(0)\binom{x}{y}}{\sqrt{x^2+y^2}} = \lim_{\|(x,y)\|\to0}\begin{pmatrix}\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\\ \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}}\end{pmatrix}$$

Consider the sequence $((x_n,y_n)=(\frac{1}{n},\frac{1}{n}))_{n\in\mathbb{N}}$

this sequence converge to 0 by norm

But $\lim_{n\to\infty}\frac{x_n^2y_n}{(x_n^2+y_n^2)^{\frac{3}{2}}} = \lim_{n\to\infty}\frac{1}{n^3}\Big/\frac{2\sqrt{2}}{n^3} = \frac{\sqrt{2}}{4}\ne0$

Hence the Jacobian matrix is not the derivative of f at 0 which suffices to indicate that f is not differentiable at 0 □

Problem E: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting $f(0) = 0$ and

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}.$$

(1) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at 0.
(2) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x,y)\neq 0$.
(3) Show that $f\in C^1(\mathbb{R}^2)$.
(4) Show that

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

exist everywhere on $\mathbb{R}^2$, but they are not equal at $(x,y)=0$.

(1) $\frac{\partial f}{\partial x}(0) = \lim_{t\to0}\frac{f\binom{t}{0}-f\binom{0}{0}}{t} = \lim_{t\to0}\frac{0}{t} = 0$

$\frac{\partial f}{\partial y}(0) = \lim_{t\to0}\frac{f\binom{0}{t}-f\binom{0}{0}}{t} = \lim_{t\to0}\frac{0}{t} = 0$

(2) Claim product and quotient rule of differentiation holds for partial derivatives.

Pf $\frac{\partial}{\partial x}(u(x,y)v(x,y)) = \lim_{t\to0}\frac{u(x+t,y)v(x+t,y)-u(x,y)v(x,y)}{t}$

$= \lim_{t\to0}\frac{u(x+t,y)(v(x+t,y)-v(x,y))}{t} + \frac{v(x,y)(u(x+t,y)-u(x,y))}{h}$

$= u(x,y)\frac{\partial v(x,y)}{\partial x}+v(x,y)\frac{\partial u(x,y)}{\partial x}$

Quotient rule follows from the product rule.

Now we use the rules for calculation:

for $\binom{x}{y}\ne0$, $\frac{\partial}{\partial x}f(x,y) = \frac{x^2-y^2}{x^2+y^2}\frac{\partial}{\partial x}(xy) + xy\frac{\partial}{\partial x}\left(\frac{x^2-y^2}{x^2+y^2}\right)$

where $\frac{\partial}{\partial x}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \frac{(x^2+y^2)2x-(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$

$\Rightarrow \frac{\partial}{\partial x}f(x,y) = y\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^3}{(x^2+y^2)^2}$

$\frac{\partial}{\partial y}f(x,y) = \frac{x^2-y^2}{x^2+y^2}\frac{\partial(xy)}{\partial y}+xy\frac{\partial}{\partial y}\left(\frac{x^2-y^2}{x^2+y^2}\right)$

$= x\frac{x^2-y^2}{x^2+y^2}+x\frac{-4x^2y^2}{(x^2+y^2)^2}$

(3) Since we have shown that for all $(x,y)\in\mathbb{R}^2$, all partials at (x,y) exist, it suffices to show that $\forall(x,y)\in\mathbb{R}^2$, all partials are continuous, in order to show that $f\in C^1(\mathbb{R}^2)$

And since any directional derivative $D_uf(x_0)$ is linear in u, it suffices to show that $\forall(x,y)\in\mathbb{R}^2$, $\frac{\partial}{\partial x}f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous.

let $(x,y)\in\mathbb{R}^2$

Since $\frac{\partial}{\partial x}f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are rational functions (thus ctn.) except at $x=0$, we only need to show that $\frac{\partial}{\partial x}f(x,y), \frac{\partial}{\partial y}f(x,y)$ are ctn. at $x=0$.

$$\lim_{(x,y)\to(0,0)}\frac{\partial}{\partial x}f(x,y) = \lim_{(x,y)\to(0,0)} y\overbrace{\frac{x^2-y^2}{x^2+y^2}}^{\in[-1,1]}+y\frac{4x^2y^2}{x^4+2x^2y^2+y^4}\in(0,2]$$

the expression is bounded by $|3y|$, so its limit when $(x,y)\to0$ is 0.

Similarly, $$\lim_{(x,y)\to(0,0)}\frac{\partial}{\partial y}f(x,y) = \lim_{(x,y)\to(0,0)} x\underbrace{\frac{x^2-y^2}{x^2+y^2}}_{\in[-1,1]}+x\frac{-4x^2y^2}{x^4+2x^2y^2+y^4}\in[-2,0)$$

the expression is bounded by $|3x|$, so its limit when $(x,y)\to0$ is 0.

Notice that $\mathbb{R}^2$ has no isolated pt., so f is ctn. at origin and thus ctn. since it is rational elsewhere

(4) Let $(x,y)\in\mathbb{R}^2$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial x}\left(x\frac{x^2-y^2}{x^2+y^2}+x\frac{-4x^2y^2}{(x^2+y^2)^2}\right) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial y}\left(y\frac{x^2-y^2}{x^2+y^2}+\frac{4x^2y^3}{(x^2+y^2)^2}\right) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

So $\frac{\partial^2}{\partial x \partial y}$ and $\frac{\partial^2}{\partial y \partial x}$ exists everywhere and equal except on the origin

On the origin: $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(0,0)\right) = \lim_{t\to 0}\frac{\frac{\partial}{\partial y}f(t,0) - \frac{\partial}{\partial y}f(0,0)}{t} = \lim_{t\to 0}\frac{t-0}{t} = 1$

but $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(0,0)\right) = \lim_{t\to 0}\frac{\frac{\partial}{\partial x}(0,t) - \frac{\partial}{\partial x}(0,0)}{t} = \lim_{t\to 0}\frac{-t-0}{t} = -1$

Bonus: Recall that an ultametric space is a metric space where one has the following stronger than usual form of the triangle inquality:

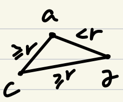
$$d(x,z) \leq \max(d(x,y), d(y,z)).$$

(1) Show that, in an ultrametric space, open balls are closed.
(2) Show that, in an ultrametric space, if two balls intersect, one of the two must be contained in the other.
(3) Show that, in an ultrametric space, every point of a ball is the center of the ball. That is, if $y \in B_r(x)$, then $B_r(x) = B_r(y)$.
(4) Let G be a connected weighted undirected graph. (The weighting is the assignment of a positive number to each edge). Let $V(G)$ be the set of vertices.
Given a path in the graph (a sequence of adjacent edges), define the length of the path to be the largest weight of an edge crossed by the path.
Given $v, w \in V(G)$, define $d(v,w)$ to be the smallest length of a path from v to w.
Show that d is an ultrametric on $V(G)$.
(5) Show that any finite ultrametric arises as in the previous part.

Just for fun (don't hand in): Imagine you have an electric car, and you live in a country that provides free charging stations, and you're not in a hurry. Why might you end up thinking about an ultrametric?

(1) Let (X,d) be an ultra-metric space.
Suppose $B = \{x : d(x,c) < r\}$ is an open ball in X centered at $c \in X$
let $z \in X \setminus B \Rightarrow d(z,c) \geq r$

Consider $B_r(z)$: let $a \in B_r(z)$, then $d(a,z) < r$
By ultrametric, $d(z,c) \leq \max\{d(a,c), d(a,z)\}$

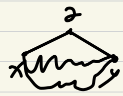


And since $d(z,c) \geq r \Rightarrow \max\{d(c,a), d(a,z)\} \geq r$
We already know that $d(a,z) < r$
Therefore $d(c,a) \geq r \Rightarrow a \in X \setminus B_r(c)$
$\Rightarrow B_r(z) \subseteq X \setminus B_r(c)$
Since z is arbitrary, this proves that $X \setminus B_r(c)$ is open
$\Rightarrow B_r(c)$ is closed
Then we can conclude that every open ball is also closed in X.

(2) Let (X,d) be an ultrametric space
Let $B_r(x), B_s(y) \subseteq X$ be two open balls with $B_r(x) \cap B_s(y) \neq \emptyset$
We only need to consider the case when $x \neq y$ since if if $x = y$ then one ball must contain the other one.
WLOG suppose $r \leq s$.
let $a \in B_r(x)$, $z \in B_r(s) \cap B_r(y)$
$\Rightarrow d(z,x) < s$, $d(z,y) < r$
$\Rightarrow d(a,y) \leq \max\{d(a,z), d(z,y)\} < r \Rightarrow a \in B_s(y)$
$\Rightarrow B_r(x) \subseteq B_s(y)$ □

(3) This directly follows from (2):
let $y \in B_r(x) \Rightarrow B_r(y) \cap B_r(x) \neq \emptyset$
$\Rightarrow B_r(y) \subseteq B_r(x)$ and $B_r(x) \subseteq B_r(y)$
$\Rightarrow B_r(y) = B_r(x)$

(4) Positivity follows from the definition of the graph and $\forall x,y \in V(G)$, $d(x,y) = d(y,x)$ since the graph is undirected (every path commutes)



So it suffices to show d is an ultrametric by showing the ultra-triangular property

Let $x,y,z \in V(G)$ st. there is at least one path from x to z and z to y and x to y
We write the weight of an edge e as $w(e)$ and the smallest weight of an edge cross a path p as $L(p)$

Case 1: the smallest-length path between x,y, say P_{xy}, goes through z.
Then $P_{xy} = P_{xz} \cup P_{zy}$ where P_{xz} is a path between x,z and P_{zy} is a path between z,y
Then $d(x,y) = L(P_{xy}) = \max\{L(P_{xz}), L(P_{zy})\}$
and $d(x,z) = \min\{L(P) : \text{path through } x,z\}$
$d(z,y) = \min\{L(P) : \text{path through } z,y\}$
so $L(P_{xz}) \leq d(x,z)$, $L(P_{zy}) \leq d(z,y)$
Thus $d(x,y) = L(P_{xy}) \leq \max\{d(x,z), d(z,y)\}$

Case 2: the smallest-length path between x,y, say P_{xy}, does not go through z.
Take path P_{xz}, P_{zy} s.t. $L(P_{xz}) = d(x,z)$, $L(P_{zy}) = d(z,y)$
Then let $P_{xy}' = P_{xz} \cup P_{zy}$, we have $L(P_{xy})' = \max\{L(P_{xz}), L(P_{zy})\}$
Since $d(x,y) = L(P_{xy}) \Rightarrow L(P_{xy}) \geq L(P_{xy}')$
$\Rightarrow d(x,y) = L(P_{xy}) > \max\{L(P_{xz}), L(P_{zy})\}$
$= \max\{d(x,z), d(z,y)\}$
In both case the ultra-triangular ineq. holds true
This finishes the proof that d is an ultrametric on $V(G)$

(5) Let (X, d_u) be a finite ultrametric field with $\#|X| = c$
WTS: we can construct a graph $G = (X, E(G))$ endowed with metric d_g in (4), s.t (X, d_u) is isometrically embedded into (G, d_g)

Construction: for each $v,w \in X$, add an edge $e(v,w)$ to $E(G)$ with $w(e(v,w)) = d_u(e,w)$
Then the graph will be a complete c-graph
By d_u, $\forall x \in X$, $d(v,w) \leq \max\{d(v,x), d(x,w)\}$
So every path P through v,w has $L(P) \geq w(e(v,w))$
So $d_g(v,w) = w(e(v,w)) = d_u(v,w)$