HW 2

DUE FRIDAY SEPTEMBER 6

def of norm:
1. $\|x\| \geq 0$, $= 0$ iff $x = 0$
2. $\forall \alpha \in \mathbb{C}$, $\|\alpha x\| = |\alpha| \|x\|$
3. $\|x+y\| \leq \|x\| + \|y\|$ $\forall x, y$

For hints see office door. But try without the hints first.

Problem A: If $\|\cdot\|$ is a norm on a vector space V, show that $d(x,y) = \|x-y\|$ defines a metric on V.

Proof Let V be a normed vector space with norm $\|\cdot\|$

Let $x, y, z \in V$

By positivity of norm: $\|x-y\| \geq 0$ and $\|x-y\| = 0$ iff $x = y$

so $d(x,y) \geq 0$ and $= 0$ iff $x = y$ ①

By homogeneity of norm: $\|-(x-y)\| = |-1| \cdot \|x-y\|$

$\Rightarrow \|y-x\| = \|x-y\|$, so $d(x,y) = d(y,x)$ ②

By triangular inequality of norm:

$\|x-y\| = \|(x-z)+(z-y)\| \geq \|x-z\| + \|z-y\|$

So $d(x,y) \geq d(x,z) + d(z,y)$ ③

Hence $d(x,y) = \|x-y\|$ defines a metric on V. □

Conclusion: norm induces metric on a vector space

Problem B: Let $T : V_1 \to V_2$ be a linear map between normed vector spaces. The norm on V_i will be denoted $\|\cdot\|_i$. Define

$$\|T\| = \sup_{v \in V_1, v \neq 0} \frac{\|Tv\|_2}{\|v\|_1} = \sup_{v \in V_1, \|v\|_1 = 1} \|Tv\|_2.$$

$\|T(v_n) - T(v)\|_2 \to 0$ as $\|v_n - v\| \to 0$

This is either a non-negative real number or infinity. The linear map is called bounded if it is not infinity. Show that T is continuous if and only if it is bounded.

Pf ① Suppose T is bounded

Let $\varepsilon > 0$

Since T is bounded we have $\|T\| = \sup_{v \in V_1, v \neq 0} \frac{\|Tv\|_2}{\|v\|_1} = c$ for some $c > 0 \in \mathbb{R}$

Thus for all $v \in V_1$, $0 \leq \|Tv\|_2 \leq c\|v\|_1$

Let $\delta = \frac{\varepsilon}{c}$.

Take $w \in V_1$ s.t. $\|w - v\| < \delta \Rightarrow c\|w-v\| < \varepsilon$

Then $\|Tw - Tv\|_2 = \|T(w-v)\|_2 \leq c\|w-v\|_1 < \varepsilon$

Hence T is continuous by the δ-ε formulation of continuity in metric space

② Suppose T is continuous

By continuity at 0, $\exists \delta > 0$ s.t. $\|Tv\|_2 < 37$ whenever $\|v\|_1 < \delta$

Take $w \in V_1$ s.t. $\|w\|_1 = 1$

Then $\|\frac{\delta}{2} w\|_1 = \frac{\delta}{2} < \delta \Rightarrow \|T(\frac{\delta}{2} w)\|_2 < 37$, $\frac{\delta}{2}\|Tw\| < 37$ $\Rightarrow \|Tw\| < \frac{74}{\delta}$

So $\sup_{v \in V_1, \|v\|_1 = 1} \|Tv\|_2 \leq \frac{74}{\delta} < \infty$

Hence T is bounded. □

Conclusion: A linear map between two normed vector spaces is continuous iff it is bounded.

Problem C: Give an example of an unbounded linear map.

Consider $T = \frac{d}{dx}\Big|_{x=0} \in \mathrm{Hom}(C^1[0,1], \mathbb{R})$ with

$\|f\|_1 = \|f\|_\infty = \sup_{[0,1]} |f|$ and $\|x\|_2 = |x|$, $\forall f \in C^1[0,1]$ and $x \in \mathbb{R}$

(I think we have already shown that T is a linear map and $\|\cdot\|_1$, $\|\cdot\|_2$ are valid norms)

Consider a sequence of functions $(f_n(x) = \sin\frac{nx}{n})_{n \in \mathbb{N}}$ in $C^1[0,1]$

$$\sup_{n \in \mathbb{N}} \frac{\|Tf_n(x)\|_2}{\|f_n(x)\|_1} = \sup_{n \in \mathbb{N}} \frac{\lim_{h \to 0} \frac{\frac{\sin(nh)}{n} - 0}{h}}{\left|\sup_{x \in [0,1]} \frac{\sin(nx)}{n}\right|} = \sup_{n \in \mathbb{N}} \frac{1}{\frac{1}{n}} = \sup_{n \in \mathbb{N}} n \to \infty$$

So $\sup_{f \in V_1, \|f\|_1 \neq 0} \frac{\|Tf\|_2}{\|f\|_1} \geq \sup_{n \in \mathbb{N}} \frac{\|Tf_n(x)\|_2}{\|f_n(x)\|_1} \to \infty$

Thus T is an unbounded linear map.

Problem D: Given an example of a sequence (T_i) of diagonalizable 2 by 2 real matrices whose eigenvalues stay bounded but for which $\|T_i\| \to \infty$. (Here the matrices define linear maps from $\mathbb{R}^2$ to itself, and we use the Euclidean norm on $\mathbb{R}^2$.)

Consider $(T_i)_{i \in \mathbb{N}}$ while $T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ for each $i \in \mathbb{N}$

Notice that $\forall i \in \mathbb{N}$, eigenvalue of T_i is $\lambda_1 = \lambda_2 = 1$

Consider the vector $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

Then for each $i \in \mathbb{N}$, $\frac{\|T_i V_1\|_2}{\|V_1\|_2} = \sqrt{(1+i)^2 + 1} = \sqrt{i^2 + 2i + 2} > i$

So $\|T_i\| = \sup_{V \in \mathbb{R}^2, V \neq 0} \frac{\|T_i V\|_2}{\|V\|_2} \geq \frac{\|T_i V_1\|_2}{\|V_1\|_2} > i$

Hence $\|T_i\| \to \infty$

Problem E: Show that if a subset of a metric space is totally bounded, then it is also separable (i.e. there exists a countable dense subset).

Proof Let (X, d) be a metric space with $S \subseteq X$ is totally bounded

For each $n \in \mathbb{N}$, we apply a finite cover $U_n = \{B_{\frac{1}{n}}(x_i^{(n)}) \mid i = 1, \ldots, k_n\}$ to cover S, guaranteed by totally boundedness.

We denote the centers of balls in U_n as $x_i^{(n)}$, $i = 1, \ldots, k_n$

Consider the set $\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}$

This set is countable since it is a countable union of finitely many points.

Claim: $\overline{\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}} = X$

We show that by showing that $\forall x \in X$, either $x \in \bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}$ or x is a limit point of it

Let $x \in X$

if $x \in \bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}$, it is done.

if $x \notin \bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}$, $x \in B_1(x_{i_0}^{(1)})$ for some $x_{i_0}^{(1)} \in \{x_i^{(1)} \mid i = 1, \ldots, k_n\}$

So $d(x, x_{i_0}^{(1)}) < 1$

Given $x_{i_1}^{(1)}, x_{i_2}^{(2)}, \ldots, x_{i_n}^{(n)}$, $x \in B_{\frac{1}{n+1}}(x_{i_{n+1}}^{(n+1)})$ for some $x_{i_{n+1}}^{(n+1)} \in \{x_i^{(n+1)} \mid i = 1, \ldots, k_{n+1}\}$

Then $d(x, x_{i_{n+1}}^{(n+1)}) < \frac{1}{n+1}$

Hence the sequence $(x_{i_n}^{(n)})_{n \in \mathbb{N}} \to x$ since for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x, x_{i_n}^{(n)}) < \frac{1}{N} < \varepsilon$ for all $n \geq N$

So x is a limit point of $\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}$

This finishes the proof that $\overline{\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \ldots, k_n\}} = X$

Hence this countable subset is dense in X, showing that X is separable. □

Problem F: Let X be define as infinitely many copies of $[0,1]$ with all their left endpoints glued together, with the natural metic d.

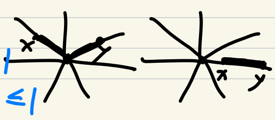
Formally, we can first define $\hat{X} = \mathbb{N} \times [0,1]$, and define an equivalence relation on $\hat{X}$ by $(i,x) \sim (j,y)$ if and only if $(i,x) = (j,y)$ or $x = y = 0$. Let X be the set of equivalence classes, and define a metric d by setting $d([(i,x)],[(j,y)])$ to be $|x| + |y|$ if $i \neq j$ and $|x-y|$ if $i = j$. You should convince yourself that this makes sense but don't have to write this up.

Prove that (X,d) is bounded but not totally bounded.

Pf Take arbitrary $[(i,x)] \in X$

if $i = 0$ then $d([(0,x)],[(0,0)]) = |x-0| \leq 1$

if $i \neq 0$ then $d([(i,x)],[(0,0)]) = |x| + 0 \leq 1$



So $X \subseteq B_1([(0,0)]) \Rightarrow (X,d)$ is bounded

To show that (X,d) is not totally bounded,

we take $\varepsilon = \frac{1}{2}$

Claim: any open ball of radius $\frac{1}{2}$ can cover at most one point of form $[(i,1)]$ where $i \in \mathbb{N}$

Suppose for contradiction that the claim does not hold,

then $\exists [(i_0,x_0)], [(i,1)], [(j,1)] \in X$ s.t. $\{[(i,1)],[(j,1)]\} \subseteq B_{\frac{1}{2}}([(i_0,x_0)])$

which would imply that $d([(i_0,x_0)],[(i,1)]), d([(i_0,x_0)],[(j,1)]) < \frac{1}{2}$

So $d([(i,1)],[(j,1)]) < d([(i_0,x_0)],[(i,1)]) + d([(i_0,x_0)],[(j,1)]) < 1$

which contradicts with $d([(i,1)],[(j,1)]) = 2$

Thus the claim is true

Hence in order to cover all points of the form $[(i,1)]$, $i \in \mathbb{N}$, we need infinitely many open balls of radius ε.

This finishes the proof that (X,d) is not ttl. bdd.

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Problem G: Let c_0 be the subspace of $\ell^\infty(\mathbb{N})$ of sequences that converge to zero, with the sup metric. Show that a subset Q of c_0 is totally bounded if and only if it is bounded and for all $\epsilon > 0$ there exists $N > 0$ such that for all $(x_n) \in Q$ and all $n \geq N$ we have $|x_n| < \epsilon$.

Pf ① Suppose Q is totally bounded

Take $\delta = 1$. By ttl bddness, we can use finitely many, say k_δ, δ-balls to cover Q.

Then $\operatorname{diam} \bigcup_{n=1}^{k_\delta} B_n \leq 2\delta k_\delta$ ✗ actually not bounded by the max of finitely many bounds.

So by taking any point $q \in Q$, $Q \subseteq B_{2\delta k_\delta}(q)$

Thus Q is bounded

Let $\varepsilon > 0$

Suppose such N does not exist, i.e. $(\forall N > 0, \exists (x_n) \in Q$ st. $\exists n \geq N, |x_n| \geq \varepsilon)$ for contradiction

Thus for each $N > 0$, we can pick such sequence to make a sequence $((x_t)^{(N)})_{N \in \mathbb{N}}$ of sequences in Q

For each term $(x_t)^{(N)}$, since it converges to 0, $\exists T \in \mathbb{N}$ st $x_t < \frac{\varepsilon}{2}$ whenever $t \geq T$

So $\forall M > T$, $d((x_t)^{(M)}, (x_t)^{(N)}) > \frac{\varepsilon}{2}$

Thus we can make a subsequence of $((x_t)^{(N)})_{N \in \mathbb{N}}$ st. $\forall N \in \mathbb{N}$ and $M \in \mathbb{N}$, $d((x_t)^{(N)}, (x_t)^{(M)}) > \frac{\varepsilon}{2}$

Thus Q can not be covered by finitely many $\frac{\varepsilon}{2}$-balls, contradicting ttl bddness.

Thus by contradiction we have proved the existence of such $N \in \mathbb{N}$.

This finishes the proof that ttl bddness implies bddness and the other condition

② Next we show that the two conditions can imply ttl bddness.

Assume the hypothesis

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ s.t. $(|x_n| < \frac{\varepsilon}{2}\ \forall (x_n) \in Q$ and $n \geq N)$

By boundedness we have $\sup_{n \in \mathbb{N}} |x_n| \leq M$ for all $(x_n) \in Q$

For the first N terms of sequences in Q, the possible range of any term of any sequence is $[-M,M]$.

So we can mesh N of $[-M,M]$ into $\lceil \frac{4M}{\varepsilon} \rceil^N$ intervals with each one of $\frac{\varepsilon}{2}$ length.

$[a_0 \lceil \frac{4M}{\varepsilon} \rceil^N, (a_0+1)\lceil \frac{4M}{\varepsilon} \rceil^N]$

for each small interval, if $\exists$ some term of some sequence in Q whose position is N_0 and value lies in the interval, $(x_n)^{(t)}$

pick one such sequence and add $B_{\frac{\varepsilon}{2}}((x_n)^{(t)})$ to covering

if no such sequence that has such term in that interval, then continue.

Since there are only finitely many small intervals, the covering is finite.

For any $(y_n) \in Q$, the first N terms lies in the range of some $B_{\frac{\varepsilon}{2}}((x_n)^{(t)})$ in covering.

Take that $(x_n)^{(t)}$, $d((y_n), (x_n)^{(t)}) = \sup |y_n - x_n^{(t)}| \leq \varepsilon$

since if $\sup |y_n - x_n^{(t)}| = \max_{1 \leq n \leq N} |y_n - x_n^{(t)}| \Rightarrow \sup |y_n - x_n^{(t)}| \leq \frac{\varepsilon}{2}$

if not, then $\sup |y_n - x_n^{(t)}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

as $|y_n|, |x_n|$ when $n \geq N$ are bounded by $\frac{\varepsilon}{2}$. □

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Bonus problem: A map $f : X \to Y$ between metric spaces is called an isometric embedding if

$$d(f(x_1), f(x_2)) = d(x_1, x_2)$$

for all $x_1, x_2 \in X$. If such a map exists we say X embedds isometrically in Y.

Show that every seperable metric space embedds isometrically into $\ell^\infty(\mathbb{N})$.

Let X be a seperable metric space,

Take a countable dense subset $E \subseteq X$ and enumerate it as $(p_n)_{n \in \mathbb{N}}$

This induces a seq. in $\ell^\infty(X)$: $(d(x,p_n))_{n \in \mathbb{N}}$ for each $x \in X$

Then we construct $f: X \to \ell^\infty(\mathbb{N})$ by $x \mapsto (d_x(x,p_n))_{n \in \mathbb{N}}$

Let $x, y \in X \Rightarrow d_\infty(f(x), f(y)) = \sup_{n \in \mathbb{N}} |d_x(x,p_n) - d_x(y,p_n)|$

By triangular inequality, $|d_x(x,p_n) - d_x(y,p_n)| \leq d_x(x,y)$ for all $n \in \mathbb{N}$

And since E is dense in X, x is a subsequential limit of (p_n)

So $\sup_{n \in \mathbb{N}} |d_x(x,p_n) - d_x(y,p_n)| = |d_x(x,y) - 0| = d_x(x,y)$

Therefore f is an isometric embedding between X and $\ell^\infty(\mathbb{N})$

9/17 Fix. the sequence $(d_x(x,p_n))_{n \in \mathbb{N}}$ can be unbounded in X, causing it not in $\ell^\infty(\mathbb{N})$

but we can pick an arbitrary term in (p_n), name it as p_0 then fix it

And we induce another sequence $(d_x(x,p_n) - d_x(x_0,p_n))_{n \in \mathbb{N}}$ which is bounded ensured by triangular inequality:

$\forall n$, $|d_x(x,p_n) - d_x(x_0,p_n)| \leq d(x,x_0)$

So $\forall x \in X$, $(d_x(x,p_n) - d_x(x_0,p_n))_{n \in \mathbb{N}} \in \ell^\infty$

Then we construct $f_{\text{modified}}: X \to \ell^\infty(\mathbb{N})$

mapping $x \mapsto (d_x(x,p_n) - d_x(x_0,p_n))_{n \in \mathbb{N}}$

So $\forall x, y \in X$, $d_\infty(f_{\text{modified}}(x), f_{\text{modified}}(y))$

$= \sup_{n \in \mathbb{N}} |d_x(x,p_n) - d_x(y,p_n)|$

$= d_x(x,y)$ as shown above.

This completes the proof.

□