Problem A: Let $f:\mathbb{R}^3\to\mathbb{R}^2$ be of class C^1 and such that $f(1,2,3)=0$ and

$$Df(1,2,3)=\begin{bmatrix}1&2&1\\1&-1&1\end{bmatrix}.$$

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(1) Does the equation $f(x,y,z)=0$ define and implicitly a function of some of the variables in terms of the rest? If so, what variables can be expressed in terms what others? Discuss all the possibilities.

(2) Suppose there is a function $g:B\to\mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}$ such that $f(x,g(x))=0$ for $x\in B$ and $g(1)=(2,3)$. Compute $Dg(1)$.

(1) $f:\mathbb{R}^{1+2}\to\mathbb{R}^2$ is C^1

Implicit function theorem tells us that we can solve two variable in terms of the other one under certain conditions around $(1,2,3)$

$$\det\left(\frac{\partial f}{\partial(x,y)}(1,2,3)\right)=\begin{vmatrix}1&2\\1&-1\end{vmatrix}=-3\neq 0,$$

So (x,y) can be expressed in terms of z around $(1,2,3)$

$$\det\left(\frac{\partial f}{\partial(y,z)}(1,2,3)\right)=\begin{vmatrix}2&1\\-1&1\end{vmatrix}=3\neq 0$$

So (y,z) can be expressed in terms of x around $(1,2,3)$

$$\det\left(\frac{\partial f}{\partial(x,z)}(1,2,3)\right)=\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$$

(x,z) cannot be expressed in terms of y around $(1,2,3)$

(2) Let $h:B\to\mathbb{R}^3$

$x\mapsto\begin{pmatrix}x\\g(x)\end{pmatrix}$

Then $f(x,g(x))=0\Rightarrow f\circ h(x)=0$

$\Rightarrow Df(h(x))\,Dh(x)=0$ by chain rule

at $x=1$, we have ${}_2\Big[\underset{1}{\frac{\partial f}{\partial x}}\ \vdots\ \underset{2}{\frac{\partial f}{\partial(y,z)}}\Big]\Big|_{h(1)}\begin{array}{c}1\\2\end{array}\begin{bmatrix}I_1\\ \hline Dg\end{bmatrix}\Big|_{x=1}=0$

$$\frac{\partial f}{\partial x}(1,2,3)+\frac{\partial f}{\partial(y,z)}(1,2,3)\,Dg(1)=0$$

Since $\frac{\partial f}{\partial(y,z)}$ is non-sigular,

$$Dg(1)=-\left[\frac{\partial f}{\partial(y,z)}(1,2,3)\right]^{-1}\frac{\partial f}{\partial x}(1,2,3)$$

$$=-\begin{bmatrix}2&1\\-1&1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

By Cramer's rule, $\begin{bmatrix}2&1\\-1&1\end{bmatrix}^{-1}=-\frac{1}{3}\begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{matrix}1\\1\end{matrix}$

So $Dg(1)=-\frac{1}{3}\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$

Problem B: Let $f:\mathbb{R}^{k+n}\to\mathbb{R}^n$ be of class C^1 and suppose that $f(a)=0$ and $Df(a)$ has rank n. Show that if $c\in\mathbb{R}^n$ is sufficiently close to 0, then the equation $f(x)=c$ has a solution.

Pf Since $Df(a)$ has rank n,

We can divide a into $b\in\mathbb{R}^k$, $c\in\mathbb{R}^n$

s.t. $\frac{\partial f}{\partial y}(a_1,a_2)\neq 0$

where we divide variable in $\mathbb{R}^{k+n}$ into

$x\in\mathbb{R}^k$ and $y\in\mathbb{R}^n$ with reordering s.t. $\frac{\partial f}{\partial y}(a)\neq 0$

Define $g:\mathbb{R}^n\to\mathbb{R}^n$

$y\mapsto f(a_1,y)$

So $g(a_2)=f(a)=0$. $Dg(a_2)=Df(a)$

Let $h:\mathbb{R}^n\to\mathbb{R}^{k+n}$

$y\mapsto\begin{pmatrix}a_1\\y\end{pmatrix}$

By chain rule: $Dg(y)=Df(a_1,y)\,Dh(y)$

$\Rightarrow Dg(a_2)=Df(a)$

$=\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right)\Big|_a\begin{pmatrix}0\\I_n\end{pmatrix}=I_n\frac{\partial f}{\partial y}(a)$

$=\frac{\partial f}{\partial y}(a)\neq 0$

Therefore $Dg(a_2)$ is nonsingular

and since f is $C^1\Rightarrow g$ is C^1,

by IVT: $\exists$ some nbh $B_\varepsilon(a_2)\subseteq\mathbb{R}^n$ s.t.

$g|_{B_\varepsilon(a_2)}$ is invertible

So $\exists$ some nbh $V\ni f(a)=0\in\mathbb{R}^n$

and some function $\varphi:V\subseteq\mathbb{R}^n\to B_\varepsilon(a_2)\subseteq\mathbb{R}^n$

s.t. $\varphi=g|_{B_\varepsilon(a_2)}^{-1}$

$\Rightarrow\forall c\in V$, $\exists$ some $y\in\mathbb{R}^n$ s.t. $f(a_1,y)=c$

This proves that if $c\in\mathbb{R}^n$ is sufficiently close to 0, $f(x)=c$ has solution in $\mathbb{R}^{k+n}$ $\square$

Problem C: Let B be a closed box, and $f:B\to\mathbb{R}$ is a continuous function. Show that f is integrable.

Pf B is a closed box (thus also bounded in $\mathbb{R}^n$) $\Rightarrow B$ compact.

$\Rightarrow f$ is uniformly ctn.

Let $\varepsilon>0$

By uniform ctnity, $\exists\delta>0$ s.t $\forall x,y\in B$,

$|f(x)-f(y)|<\frac{\varepsilon}{V(B)}$ whenever $\|x-y\|<\delta$ ①

Set partitions $P_1,\dots,P_n$ on $B_1,\dots,B_n$ s.t. $\max_{1\le i\le n}\|P_i\|<\delta$

By ①, $\forall$ subbox B_i we have $\left|\sup_{x\in B_i}f(x)-\inf_{x\in B_i}f(x)\right|<\frac{\varepsilon}{V(B)}$

Let $P=(P_1,\dots,P_n)$,

$$|U(f,P)-L(f,P)|=\sum_{\text{subbox index } j}\left|\sup_{x\in B_j}f(x)-\inf_{x\in B_j}f(x)\right|v(B_j)$$

$$\Rightarrow\overline{\int_B}f=\underline{\int_B}f\qquad<\sum_j\frac{\varepsilon}{V(B)}V(B_j)=\varepsilon$$

This finishes the proof that f is Riemann integrable $\square$