Now we get the other direction of Lebesgue's

Thm Characterization of Riem integrability

Let $B \subseteq \mathbb{R}^d$ be bdd, $f: B \to \mathbb{R}$ be bdd.

$\Rightarrow$ f is Riem intble iff $D_f = \{x \mid f \text{ discontinuous at } x\}$ has $m(D_f) = 0$

Def For $f: B \to \mathbb{R}$, B a box

we define the oscilation of f at $x \in B$ as

$$\operatorname{osc}_\delta g(x) = \sup_{x_1, x_2 \in B \cap B_\delta(x)} [g(x_1) - g(x_2)]$$

$$\operatorname{osc} g(x) = \inf_{\delta > 0} \operatorname{osc}_\delta g(x)$$

即在x处 g 的变化幅度

exercise Show the following properties of oscilation function

(1) $\boxed{\operatorname{osc}_\delta f(x) = \sup_{B \cap B_\delta(x)} f - \inf_{B \cap B_\delta(x)} f}$ $\circledast$

$\boxed{\operatorname{osc} f(x) = \inf_{\delta > 0} \operatorname{osc}_\delta f(x)}$

(2) $\delta_1 < \delta_2 \Rightarrow \operatorname{osc}_{\delta_1} f(x) \leq \operatorname{osc}_{\delta_2} f(x)$

$\left(\text{So } \operatorname{osc} f(x) = \inf_{\delta > 0} \operatorname{osc}_\delta f(x) = \lim_{\delta \to 0^+} \operatorname{osc}_\delta g(x) \right)$

(3) f is ctn at x iff $\operatorname{osc} f(x) = 0$ $\circledast$

ex $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$\Rightarrow \forall x \in \mathbb{R}, \forall \delta, \operatorname{osc}_\delta f(x) = 1$

$\Rightarrow \forall x \in \mathbb{R}, \operatorname{osc} f(x) = 1$

Then we prove the other direction of Lebesgue's characterization of Riemann integrability:

$f: B \to \mathbb{R}$ Riem intble $\Rightarrow m(D_f) = 0$

Pf $D_f = \{x \mid \operatorname{osc} f(x) > 0\}$

Define $D_m = \{x \in B \mid \operatorname{osc} f(x) \geq \frac{1}{m}\}$

So $D_f = \bigcup_{m=1}^{\infty} D_m$

We know countable union of measure 0 sets still has measure 0.

So it suffices to show that for each $m \in \mathbb{N}$, $m(D_m) = 0$

Let $m \in \mathbb{N}$. WTS: $m(D_m) = 0$

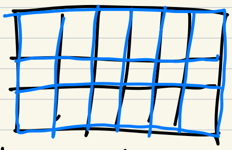
Let $\varepsilon > 0$.

Since f is Riem intble, we can find P s.t.

$U(f,P) - L(f,P) < \frac{\varepsilon}{2m}$

Write $D_m' = \{x \in D_m \mid x \in \partial S \text{ for subbox } S \text{ determined by } P\}$

$\Rightarrow D_m' = \bigcup_{\text{subbox } S} \partial S$



$\bigcup_S \partial S$

Since the boundary of a box always has measure 0, (IBL)

$m(D_m') = 0$

So we can cover D_m' by ctbly many boxes with total volume $< \frac{\varepsilon}{2}$

It remains to cover $D_m \setminus D_m'$ by ctbly many boxes of total volume $< \frac{\varepsilon}{2}$

Note: $\forall x \in D_m \setminus D_m'$ is inside a subbox S determined by P

while $\forall S$, $\sup_S f - \inf_S f \geq \frac{1}{m}$ (Since $\operatorname{osc} f(x) \geq \frac{1}{m}$)

So $U(f,P) - L(f,P) \geq \sum_{\substack{S \text{ with} \\ S \cap D_m \setminus D_m' \neq \emptyset}} \frac{1}{m} v(S)$

Since $U(f,P) - L(f,P) < \frac{\varepsilon}{2m}$

$$\Rightarrow \sum_{\substack{S \text{ with} \\ S \cap (D_m \setminus D_m') \neq \emptyset}} v(S) < \frac{\varepsilon}{2}$$

So $D_m \setminus D_m'$ can be covered by boxes of total vol $< \frac{\varepsilon}{2}$

Then in conclusion, $\forall m, \forall \varepsilon > 0$, D_m can be covered by boxes of total vol $< \varepsilon$ $\square$

Rmk 看似这个 characterization 表明 Riem integral 对仅在某个零测集上的改变很友好，但实则不然：

改变一个零测集上的值可能会改变整个函数的 continuity，（甚至仅仅是可数集） 从而改变 Riem integrability

ex $f: [0,1] \to \mathbb{R}$

$f(x) = 0$: $D_f = \emptyset$, Riem intble

$f(x) = \begin{cases} 0, & x \in \mathbb{R} \setminus \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$: $D_f = [0,1]$, not Riem intble

Notation

almost everywhere: this term means that this behavior of the function happens everywhere except on a set of measure 0

Thm Let $B \subseteq \mathbb{R}^n$ be a box. $f: B \to \mathbb{R}$ be Rien intble (Rmk: not true without f being Rien intble)

(a) If f is 0 almost everywhere (i.e. $m(\{x \mid f(x) \neq 0\}) = 0$)

$\Rightarrow \int_B f = 0$

(b) If $f \geq 0$ and $\int_B f = 0$

$\Rightarrow f$ is 0 almost everywhere

Pf (a) Let $E = \{x \mid f(x) \neq 0\}$ so $m(E) = 0$

Let P be a partition & S a subbox

Since $m(E) = 0 \Rightarrow S$ cannot be included in E

$\Rightarrow \exists x \in S$ with $f(x) = 0$

So $\inf_S f \leq 0$, $\sup_S f \geq 0$

$\Rightarrow L(f,P) \leq 0$, $U(f,P) \geq 0$

$\underline{\int_B} f \leq 0$, $\overline{\int_B} f \geq 0$

Since $\int_B f$ exists $\Rightarrow \int_B f = \underline{\int_B} f = \overline{\int_B} f = 0$

(b) Suppose $f \geq 0$ & $\int_B f = 0$

Claim if f is ctn at x then $f(x) = 0$

Pf of claim Otherwise if $f(x) \neq 0$ and f ctn

suppose $f(x) = \varepsilon \Rightarrow \exists$ open box $B \ni x$ s.t. $f(x) \geq \frac{\varepsilon}{2}$ on B

Let P be a partition that includes B as a subbox

$\Rightarrow L(f,P) \geq \frac{\varepsilon}{2} v(B) > 0$

Note that $0 < L(f,P) \leq \underline{\int_B} f = 0$, contradicts

So $f = 0$ almost everywhere $\square$

Thm Fubini's Thm

Let $Q = A \times B$, $A \subseteq \mathbb{R}^k$, $B \subseteq \mathbb{R}^l$

Write $(x,y) \in Q$ if $x \in A$, $y \in B$

Let $f: A \times B \to Q$ be bdd & intble.

Then the functions $A \to \mathbb{R}$

$$\boxed{\begin{aligned} x &\mapsto \underline{\int_B} f(x,y)\, dy \\ x &\mapsto \overline{\int_B} f(x,y)\, dy \end{aligned}}$$ are intble

and $\int_Q f = \int_A \left[\underline{\int_B} f(x,y)\, dy \right] dx = \int_A \left[\overline{\int_B} f(x,y)\, dy \right] dx$

Pf next time

Corollary With assumptions in Fubini's Thm,

(a) $\int_B f(x,y)\, dy$ exists a.e (almost everywhere) in A

i.e. $\{x \mid y \mapsto f(x,y)$ is not intble over $B\}$ has measure 0

(b) If we assume $\int_B f(x,y)\, dy$ exists $\forall x$

$\Rightarrow \int_Q f = \int_A \int_B f(x,y)\, dy\, dx$

(c) Let $Q = \prod [a_i, b_i]$

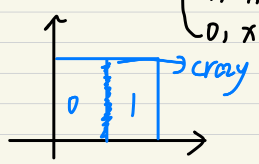
If $f: Q \to \mathbb{R}$ ctn

$\Rightarrow \int_Q f = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f\, dx_n \cdots dx_1$

Pf next time

ex Let $f: [0,1]^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} 0, & x < \frac{1}{2} \\ 1, & x > \frac{1}{2} \\ 1, & x = \frac{1}{2} \text{ \& } y \in \mathbb{Q} \\ 0, & x = \frac{1}{2} \text{ \& } y \notin \mathbb{Q} \end{cases}$$



f is intble since $D_f = \{\frac{1}{2}\} \times [0,1]$. $m(D_f) = 0$

We'd like to say $\int_{[0,1]^2} f = \int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx$

Though $\int_0^1 f(x,y)\, dy$ seems to make no sense as it is not defined for $\frac{1}{2}$, that should not matter since this set has measure 0.

Solution: Use upper or lower integrals

i.e. $\int_0^1 \underline{\int_0^1} f(x,y)\, dy\, dx$ or $\int_0^1 \overline{\int_0^1} f(x,y)\, dy\, dx$

(allowed by Fubini's.)