Contents

HW 1 on metric spaces

1.1 rooted metric gives $(0, 1)$ infinite length

Suppose (X, d) is a metric space. For $0 < \epsilon < 1$, show that d^ϵ is also a metric on X.

If $X = [0, 1]$ is the unit interval and d is the usual metric, show that X has "infinite length" using the metric d^ϵ, in that

$$\sup_{0=t_0<t_1<\cdots<t_n=1} \sum_{i=1}^{n} d^\epsilon(t_i, t_{i-1}) = \infty.$$

Here the supremum is taken over all n and all $n + 1$ tuples of points t_i as in the subscript.

Optional context: If you'd like to know where the metrics d^ϵ appear, try looking up the Assouad Embedding Theorem. If you'd like to know more about the notion of length used here, try looking up rectifiable curves.

1.2 Matrix chain multiplication

If X is an $a \times b$ matrix, and Y is a $b \times c$ matrix, it takes abc multiplications to compute XY according to the usual formula for matrix multiplication. (There are ac entries in XY, and each is a sum of b products.) Thus, let's estimate the time it takes to multiply these two matrices as abc.

Say A_1 is a 5×1 matrix, A_2 is a 1×5 matrix, A_3 is a 5×2 matrix, A_4 is a 2×5 matrix, A_5 is a 5×1 matrix, and A_6 is a 1×10 matrix.

If you want to compute $A_1A_2A_3A_4A_5A_6$, how should you bracket this product so that the sum of the time estimates for the multiplications is as small as possible? For example, should you do

$$(A_1(A_2A_3))((A_4A_5)A_6)?$$

Or

$$(A_1(A_2(A_3(A_4(A_5A_6)))))?$$

Or something else? HintL: Wikipedia entry titled "Matrix chain multiplication."

Sol. Dynamic Programming. I have no idea why it appears here, but it is dynamic programming.

HW 1

DUE FRIDAY AUGUST 30

5/5

Problem A: Suppose (X, d) is a metric space. For $0 < \epsilon < 1$, show that d^ϵ is also a metric on X.

If $X = [0,1]$ is the unit interval and d is the usual metric, show that X has "infinite length" using the metric d^ϵ, in that

$$\sup_{0=t_0<t_1<\cdots<t_n=1} \sum_{i=1}^{n} d^\epsilon(t_i, t_{i-1}) = \infty.$$

Here the sup is taken over all n and all $n+1$ tuples of points t_i as in the subscript.

Optional context: If you'd like to know where the metrics d^ϵ appear, try looking up Assouad Embedding Theorem. If you'd like to know more about the notion of length used here, try looking up rectifiable curves.

① Pf Assume the hypothesis.
Take $x, y, z \in X$.

(a) $d^\varepsilon(x,y) = (d(x,y))^\varepsilon \geq 0$ since $d(x,y) \geq 0$ and $0 < \varepsilon < 1$

(b) $d^\varepsilon(x,y) = (d(x,y))^\varepsilon = (d(y,x))^\varepsilon = d^\varepsilon(y,x)$ by symmetry of d

let $f(x) = x^\varepsilon$, $x \geq 0$, so $f''(x) = \underbrace{\varepsilon(\varepsilon-1)}_{<0} x^{\varepsilon-2} \leq 0$ for $x \geq 0$

So f is concave for $x \geq 0$
and $d(x,y), d(y,z), d(x,z) \geq 0$

Hence $f(d(x,y)) + f(d(y,z)) \geq f(d(x,y) + d(y,z))$

Since f is increasing on domain and $d(x,y) + d(y,z) \geq d(x,z)$,
we have $f(d(x,y)) + f(d(y,z)) \geq f(d(x,y) + d(y,z)) \geq f(d(x,z))$

Hence (c) $d^\varepsilon(x,y) + d^\varepsilon(y,z) \geq d^\varepsilon(x,z)$ □

(a),(b),(c) shows that d^ε is a metric on X.

② Pf It suffices to show that for any $M \in \mathbb{N}$, $\exists$ some choices of partition $\{t_i\}_I$ s.t. $\sup_{0=t_0<t_1<\dots<t_n=1} \sum_{i=1}^{n} d^\varepsilon(t_i, t_{i+1}) > M$

let $M \in \mathbb{N}$. Consider equally partition the interval into n subintervals

Then $\sup_{0=t_0<t_1<\dots<t_n=1} \sum_{i=1}^{n} d^\varepsilon(t_i, t_{i+1}) = n \left|\frac{1}{n}\right|^\varepsilon = n^{1-\varepsilon}$. Since $1-\varepsilon > 0$, it is unbounded above, so $\exists$ some $n \in \mathbb{N}$ s.t. $n^{1-\varepsilon} > M$ □

Bonus problem: If X is a a by b matrix, and Y is a b by c matrix, it takes abc multiplications to compute XY according to the usual formula for matrix multiplication. (There are ac entries in XY, and each is a sum of b products.) Thus, let's estimate the time it takes to multiply these two matrices as abc.

0.75/1

Say A_1 is a 5 by 1 matrix, A_2 is a 1 by 5 matrix, A_3 is a 5 by 2 matrix, A_4 is a 2 by 5 matrix, A_5 is a 5 by 1 matrix, and A_6 is a 1 by 10 matrix.

If you want to compute $A_1A_2A_3A_4A_5A_6$, how should you bracket this product so that the sum of the time estimates for the multiplications is as small as possible? For example, should you do

$$(A_1(A_2A_3))((A_4A_5)A_6)?$$

Or

$$(A_1(A_2(A_3(A_4(A_5A_6)))))?$$

Or something else?

In general you will have to show your work, but for this first bonus problem only **you only need to submit the final answer**.

Hint 1: See hint posted on my office door (EH 5848).

Hint 2: Only use this Hint 2 if you can't figure it out on your own after spending at least 10 minutes looking at Hint 1. In general, you can't use wikipedia or internet resources unless explicitely allowed, but for this first bonus problem you can. See the wikipedia entry titled "Matrix chain multiplication".

$(A_1((A_2A_3)(A_4A_5))A_6$

DP table is in the next page.

Denote the minimal cost of multiplying A_i through A_j by $m(i,j)$

Recursively, $m(i,j) = \min_{i \leq k \leq j} \{ m(i,k) + m(k+1,j) + \text{row}(A_i) \cdot \text{col}(A_k) \cdot \text{col}(A_j) \}$

DP Table

	1 (5,1)	2 (1,5)	3 (5,2)	4 (2,5)	5 (5,1)	6 (1,10)
1	0	(5,5) 25	(5,2) 20 1(23)	(5,5) 45 1(24)	(5,1) 27 1(25)	(5,10) 77 (15)6
2		0	(1,2) 10	(1,5) 20 (23)4	(1,1) 22 (23)(45)	(1,10) 32 (25)6
3			0	(5,5) 50	(5,1) 20 3(45)	(5,10) 70 3(56)
4				0	(2,1) 10	(2,10) 30 (45)6
5					0	(5,10) 50
6						0

$m(1,3) = \min(25 + 5\times5\times2,\ 10 + 5\times1\times2) = 20$

$m(2,4) = \min(25+50,\ 10+10) = 20$

$m(3,5) = \min(50+25,\ 10+10) = 20$

$m(4,6) = \min(10+20,\ 50+100) = 30$

$m(1,4) = \min(\underset{(13)4}{20+50},\ \underset{(12)(34)}{25+50+125},\ \underset{1(24)}{20+25}) = 45$

$m(2,5) = \min(\underset{(24)5}{25},\ \underset{(23)(45)}{10+10+2},\ \underset{2(35)}{25}) = 22$

$m(3,6) = \min(\underset{(35)6}{20+50},\ \underset{(34)(56)}{50+50+250},\ \underset{3(46)}{30+100}) = 70$

$m(1,5) = \min(\underset{1(25)}{22+5},\ \underset{(12)(35)}{\dots},\ \underset{(13)(45)}{\dots},\ \underset{(14)5}{\dots}) = 27$

$m(2,6) = \min(\underset{2(36)}{70+50},\ \underset{(23)(46)}{10+30+20},\ \underset{(24)(56)}{20+50+\dots},\ \underset{(25)6}{32}) = 32$

$m(1,6) = \min(\underset{1(26)}{32+50}\ \ \underset{(12)(36)}{25+70+\dots}\ \ \underset{(13)(46)}{20+30+100}\ \ \underset{(14)(56)}{45+50+\dots}\ \ \underset{(15)6}{27+50}) = 77$

HW 2 on ttl bdd, sup norm and cptness

2.1 metric induced by vector norm

Def 2.1

normed vector space:

vector space 上可以定义 norm 来表示每个 vector 的“大小”，norm 的定义是满足

(1) positive definiteness; (2) homogeneity (3) triangle ineq 的 $||.|| : V \to \mathbb{R}_{\geq 0}$ ♣

If $\|\cdot\|$ is a norm on a vector space V, show that $d(x, y) = \|x - y\|$ defines a metric on V.

2.2 operator norm: linear map is ctn iff bdd (因而有限维 vector spaces 间的 linear operator 一定 ctn 且 bdd)

Def 2.2 (operator norm)

Let $T : V_1 \to V_2$ be a linear map between normed vector spaces. The norm on V_i will be denoted $\|\cdot\|_i$. Define

$$\|T\| = \sup_{v \in V_1, v \neq 0} \frac{\|Tv\|_2}{\|v\|_1} = \sup_{v \in V_1, \|v\|_1 = 1} \|Tv\|_2.$$

This is either a non-negative real number or infinity. The linear map is called bounded if it is not infinity. Show that T is continuous if and only if it is bounded. ♣

Remark 给定两个 normed vector spaces V_1, V_2，我们实际上也 induce 出了 $Hom(V_1, V_2)$ 上的一个 norm，即此处的 operator norm.

$Hom(V_1, V_2)$ 上还可以定义 Frobenious norm (treat like $\mathbb{R}^{n \times n}$) 以及 sup norm (sup norm in $\mathbb{R}^{n \times n}$).

Remark 我们可以用 $||T(v) - T(w)||_2 \to 0$ as $||v - w||_1 \to 0$（或者 sequentially）来定义 **continuous by norm**.

Theorem 2.1 (Finite dim VS 上 linear map bdd 且 ctn)

(1) **linear map bounded** 当且仅当它 **continuous.** (2) 有限维度的 **vector space** 之间的任意 **linear map** 一定 **bounded**（因为一定可以选择 base 然后把线性变换用 matrix 来表示），所以一定 continuous. ♡

2.3 unbounded linear map in infinite dim vector space

1. Give an example of an unbounded linear map.

   Sol. 一个 infinite dimension 的 vector space 的 unbounded linear map 的例子:

$$T = \frac{d}{dx}|_{x=0} \in Hom(C[0, 1], \mathbb{R})$$

2. Give an example of a sequence (T_i) of diagonalizable 2×2 real matrices whose eigenvalues stay bounded but for which $\|T_i\| \to \infty$. (Here the matrices define linear maps from $\mathbb{R}^2$ to itself, and we use the Euclidean norm on $\mathbb{R}^2$.)

2.4 ttl bdd metric space 一定 separable (有 ctbl dense subset)

Show that if a subset of a metric space is totally bounded, then it is also separable (i.e., there exists a countable dense subset).

Remark 如果一个 metric space X 是 separable 的，那么随意 enumerate 一个 dense sequence (p_n)，这个 sequence 的所有 subsequential limit 就是整个 X. 即对于 X 中任意一个元素，都可以找到 (p_n) 的一个 subsequence 使得它的 limit 是整个元素

2.5 quotient topology

Let X be defined as infinitely many copies of $[0, 1]$ with all their left endpoints glued together, with the natural metric d.

Formally, we can first define $\hat{X} = \mathbb{N} \times [0, 1]$, and define an equivalence relation on $\hat{X}$ by $(i, x) \sim (j, y)$ if and only if $(i, x) = (j, y)$ or $x = y = 0$. Let X be the set of equivalence classes, and define a metric d by setting

$$d([(i, x)], [(j, y)]) = |x| + |y| \quad \text{if } i \neq j \quad \text{and} \quad d([(i, x)], [(j, y)]) = |x - y| \quad \text{if } i = j.$$

You should convince yourself that this makes sense but don't have to write this up.

Prove that (X, d) is bounded but not totally bounded.

2.6 space of converging sequence 中 ttl bdd 的判断标准

Let c_0 be the subspace of $\ell^\infty(\mathbb{N})$ of sequences that converge to zero, with the sup metric. Show that a subset Q of c_0 is totally bounded if and only if it is bounded and for all $\epsilon > 0$, there exists $N > 0$ such that for all $(x_n) \in Q$ and all $n \geq N$, we have $|x_n| < \epsilon$.

Remark $l^\infty(\mathbb{N})$ 中的所有收敛到 0 的序列构成的集合，称之为 $l_0^\infty(\mathbb{N})$，这个集合 $l_0^\infty(\mathbb{N})$ **从 bdd 提升到 ttl bdd 所需要的条件是其中所有序列是 uniformly 收敛的.**

2.7 Bonus: isometry embedding

A map $f : X \to Y$ between metric spaces is called an isometric embedding if

$$d(f(x_1), f(x_2)) = d(x_1, x_2)$$

for all $x_1, x_2 \in X$. If such a map exists, we say X embeds isometrically in Y.

Show that every separable metric space embeds isometrically into $\ell^\infty(\mathbb{N})$.

Def 2.3

isometric embedding:

如果两个 metric space 之间的一个 map 保留 **distance** (即 $\forall x, y \in X, d(f(x), f(y)) = d(x, y)$) 那么就称这个 map 为一个 isometric embedding.

♣

Theorem 2.2

任意 **separable metric space** 都可以 **embedds isometrically into** $l^\infty(\mathbb{N})$.

♡

Proof 构造：首先 enumeate X 的一个 dense sequence (p_n)，并取它的首项 p_1，然后对于每个 $x \in X$，induce 出 $(d_X(x, p_n) - d_X(p_0, p_n))_{n\in\mathbb{N}}$ 这个序列. 可以发现从 $f : x \mapsto (d_X(x, p_n) - d_X(p_0, p_n))_{n\in\mathbb{N}}$ 是一个 isometry embedding.

HW 2

DUE FRIDAY SEPTEMBER 6

def of norm:
1. $\|x\| \geq 0$, $=0$ iff $x=0$
2. $\forall \alpha \in \mathbb{C}$, $\|\alpha x\| = |\alpha|\|x\|$
3. $\|x+y\| \leq \|x\| + \|y\|$ $\forall x, y$

For hints see office door. But try without the hints first.

Problem A: If $\|\cdot\|$ is a norm on a vector space V, show that $d(x,y) = \|x-y\|$ defines a metric on V.

Proof Let V be a normed vector space with norm $\|\cdot\|$
Let $x, y, z \in V$
By positivity of norm: $\|x-y\| \geq 0$ and $\|x-y\| = 0$ iff $x=y$
so $d(x,y) \geq 0$ and $=0$ iff $x=y$ ①
By homogeneity of norm: $\|-(x-y)\| = |-1| \cdot \|x-y\|$
$\Rightarrow \|y-x\| = \|x-y\|$, so $d(x,y) = d(y,x)$ ②
By triangular inequality of norm:
$\|x-y\| = \|(x-z)+(z-y)\| \geq \|x-z\| + \|z-y\|$
So $d(x,y) \geq d(x,z) + d(z,y)$ ③
Hence $d(x,y) = \|x-y\|$ defines a metric on V. $\square$
Conclusion: norm induces metric on a vector space

Problem B: Let $T : V_1 \to V_2$ be a linear map between normed vector spaces. The norm on V_i will be denoted $\|\cdot\|_i$. Define

$$\|T\| = \sup_{v \in V_1, v \neq 0} \frac{\|Tv\|_2}{\|v\|_1} = \sup_{v \in V_1, \|v\|_1 = 1} \|Tv\|_2.$$

$\|T(v_n) - T(v)\|_2 \to 0$ as $\|v_n - v\| \to 0$

This is either a non-negative real number or infinity. The linear map is called bounded if it is not infinity. Show that T is continuous if and only if it is bounded.

Pf ① Suppose T is bounded
Let $\varepsilon > 0$
Since T is bounded we have $\|T\| = \sup_{v \in V_1, v \neq 0} \frac{\|Tv\|_2}{\|v\|_1} = c$ for some $c > 0 \in \mathbb{R}$
Thus for all $v \in V_1$, $0 \leq \|Tv\|_2 \leq c\|v\|_1$
Let $\delta = \frac{\varepsilon}{c}$.
Take $w \in V_1$ s.t. $\|w - v\| < \delta \Rightarrow c\|w - v\| < \varepsilon$
Then $\|Tw - Tv\|_2 = \|T(w-v)\|_2 \leq c\|w - v\|_1 < \varepsilon$
Hence T is continuous by the δ-ε formulation of continuity in metric space

② Suppose T is continuous
By continuity at 0, $\exists \delta > 0$ s.t. $\|Tv\|_2 < 37$ whenever $\|v\|_1 < \delta$
Take $w \in V_1$ s.t. $\|w\|_1 = 1$
Then $\|\frac{\delta}{2} w\|_1 = \frac{\delta}{2} < \delta \Rightarrow \|T(\frac{\delta}{2}w)\|_2 < 37$, $\frac{\delta}{2}\|Tw\| < 37$ $\Rightarrow \|Tw\| < \frac{74}{\delta}$
So $\sup_{v \in V_1, \|v\|_1 = 1} \|Tv\|_2 \leq \frac{74}{\delta} < \infty$
Hence T is bounded. $\square$
Conclusion: A linear map between two normed vector spaces is continuous iff it is bounded.

Problem C: Give an example of an unbounded linear map.

Consider $T = \frac{d}{dx}\Big|_{x=0} \in Hom(C^1[0,1], \mathbb{R})$ with
$\|f\|_1 = \|f\|_\infty = \sup_{[0,1]} |f|$ and $\|x\|_2 = |x|$, $\forall f \in C^1[0,1]$ and $x \in \mathbb{R}$
(I think we have already shown that T is a linear map and $\|\cdot\|_1$, $\|\cdot\|_2$ are valid norms)
Consider a sequence of functions $(f_n(x) = \sin\frac{nx}{n})_{n \in \mathbb{N}}$ in $C^1[0,1]$

$$\sup_{n \in \mathbb{N}} \frac{\|Tf_n(x)\|_2}{\|f_n(x)\|_1} = \sup_{n \in \mathbb{N}} \frac{\lim_{h \to 0} \frac{\frac{\sin(nh)}{n} - 0}{h}}{\left|\sup_{x \in [0,1]} \frac{\sin(nx)}{n}\right|} = \sup_{n \in \mathbb{N}} \frac{1}{\frac{1}{n}} = \sup_{n \in \mathbb{N}} n \to \infty$$

So $\sup_{f \in V_1, \|f\| \neq 0} \frac{\|Tf\|_2}{\|f\|_1} \geq \sup_{n \in \mathbb{N}} \frac{\|Tf_n(x)\|_2}{\|f_n(x)\|_1} \to \infty$
Thus T is an unbounded linear map.

Problem D: Given an example of a sequence (T_i) of diagonalizable 2 by 2 real matrices whose eigenvalues stay bounded but for which $\|T_i\| \to \infty$. (Here the matrices define linear maps from $\mathbb{R}^2$ to itself, and we use the Euclidean norm on $\mathbb{R}^2$.)

Consider $(T_i)_{i \in \mathbb{N}}$ while $T_i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$ for each $i \in \mathbb{N}$
Notice that $\forall i \in \mathbb{N}$, eigenvalue of T_i is $\lambda_1 = \lambda_2 = 1$
Consider the vector $v_i = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$
Then for each $i \in \mathbb{N}$, $\frac{\|T_i v_i\|_2}{\|v_i\|_2} = \sqrt{(1+i)^2 + 1} = \sqrt{i^2 + 2i + 2} > i$
So $\|T_i\| = \sup_{v \in \mathbb{R}^2, v \neq 0} \frac{\|T_i v\|_2}{\|v\|_2} \geq \frac{\|T_i v_i\|_2}{\|v_i\|_2} > i$
Hence $\|T_i\| \to \infty$

Problem E: Show that if a subset of a metric space is totally bounded, then it is also separable (i.e. there exists a countable dense subset).

Proof Let (X,d) be a metric space with $S \subseteq X$ is totally bounded
For each $n \in \mathbb{N}$, we apply a finite cover $U_n = \{B_{\frac{1}{n}}(x_i^{(n)}) \mid i = 1, \dots, k_n\}$ to cover S, guaranteed by totally boundedness.
We denote the centers of balls in U_n as $x_i^{(n)}$ $i = 1, \dots, k_n$
Consider the set $\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}$
This set is countable since it is a countable union of finitely many points.
Claim: $\overline{\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}} = X$
We show that by showing that $\forall x \in X$, either $x \in \bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}$ or x is a limit point of it

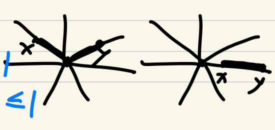
Let $x \in X$
if $x \in \bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}$, it is done.
if $x \notin \bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}$, $x \in B_1(x_{i_0}^{(1)})$ for some $x_{i_0}^{(1)} \in \{x_i^{(1)} \mid i = 1, \dots, k_1\}$
So $d(x, x_{i_0}^{(1)}) < 1$
Given $x_{i_1}^{(1)}, x_{i_2}^{(2)}, \dots, x_{i_n}^{(n)}$, $x \in B_{\frac{1}{n+1}}(x_{i_{n+1}}^{(n+1)})$ for some $x_{i_{n+1}}^{(n+1)} \in \{x_i^{(n+1)} \mid i = 1, \dots, k_{n+1}\}$
Then $d(x, x_{i_{n+1}}^{(n+1)}) < \frac{1}{n+1}$
Hence the sequence $(x_{i_n}^{(n)})_{n \in \mathbb{N}} \to x$ since for all $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x, x_{i_n}^{(n)}) < \frac{1}{N} < \varepsilon$ for all $n \geq N$
So x is a limit point of $\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}$
This finishes the proof that $\overline{\bigcup_{n=1}^{\infty} \{x_i^{(n)} \mid i = 1, \dots, k_n\}} = X$
Hence this countable subset is dense in X, showing that X is separable. $\square$

Problem F: Let X be define as infinitely many copies of $[0,1]$ with all their left endpoints glued together, with the natural metic d.

Formally, we can first define $\hat{X} = \mathbb{N} \times [0,1]$, and define an equivalence relation on $\hat{X}$ by $(i,x) \sim (j,y)$ if and only if $(i,x) = (j,y)$ or $x = y = 0$. Let X be the set of equivalence classes, and define a metric d by setting $d([(i,x)],[(j,y)])$ to be $|x| + |y|$ if $i \neq j$ and $|x - y|$ if $i = j$. You should convince yourself that this makes sense but don't have to write this up.

Prove that (X,d) is bounded but not totally bounded.

Pf Take arbitrary $[(i,x)] \in X$
if $i = 0$ then $d([(0,x)],[(0,0)]) = |x - 0| \leq 1$
if $i \neq 0$ then $d([(i,x)],[(0,0)]) = |x| + 0 \leq 1$



So $X \subseteq B_1([(0,0)]) \Rightarrow (X,d)$ is bounded

To show that (X,d) is not totally bounded, we take $\varepsilon = \frac{1}{2}$

Claim: any open ball of radius $\frac{1}{2}$ can cover at most one point of form $[(i,1)]$ where $i \in \mathbb{N}$

Suppose for contradiction that the claim does not hold, then $\exists [(i_0,x_0)], [(i,1)], [(j,1)] \in X$ s.t. $\{[(i,1)],[(j,1)]\} \subseteq B_{\frac{1}{2}}([(i_0,x_0)])$

which would imply that $d([(i_0,x_0)],[(i,1)]),\ d([(i_0,x_0)],[(j,1)]) < \frac{1}{2}$

So $d([(i,1)],[(j,1)]) < d([(i_0,x_0)],[(i,1)]) + d([(i_0,x_0)],[(j,1)]) < 1$

which contradicts with $d([(i,1)],[(j,1)]) = 2$

Thus the claim is true

Hence in order to cover all points of the form $[(i,1)]$, $i \in \mathbb{N}$, we need infinitely many open balls of radius ε.

This finishes the proof that (X,d) is not ttl. bdd.

-1

Problem G: Let c_0 be the subspace of $\ell^\infty(\mathbb{N})$ of sequences that converge to zero, with the sup metric. Show that a subset Q of c_0 is totally bounded if and only if it is bounded and for all $\epsilon > 0$ there exists $N > 0$ such that for all $(x_n) \in Q$ and all $n \geq N$ we have $|x_n| < \epsilon$.

Pf ① Suppose Q is totally bounded

Take $\delta = 1$. By ttl bddness, we can use finitely many, say k_δ, δ-balls to cover Q.

Then $\operatorname{diam} \bigcup_{n=1}^{k_\delta} B_n \leq 2\delta k_\delta$ ✗ actually not bounded by the max of finitely many bounds.

So by taking any point $q \in Q$, $Q \subseteq B_{2\delta k_\delta}(q)$

Thus Q is bounded

Let $\varepsilon > 0$

Suppose such N does not exist, i.e. $(\forall N > 0, \exists (x_n) \in Q \text{ s.t. } \exists n \geq N, |x_n| \geq \varepsilon)$ for contradiction

Thus for each $N > 0$, we can pick such sequence to make a sequence $((x_t)^{(N)})_{N\in\mathbb{N}}$ of sequences in Q

For each term $(x_t)^{(N)}$, since it converges to 0, $\exists T \in \mathbb{N}$ st $x_t < \frac{\varepsilon}{2}$ whenever $t \geq T$

So $\forall M > T$, $d((x_t)^{(M)}, (x_t)^{(N)}) > \frac{\varepsilon}{2}$

Thus we can make a subsequence of $((x_t)^{(N)})_{N\in\mathbb{N}}$ st. $\forall N \in \mathbb{N}$ and $M \in \mathbb{N}$, $d((x_t)^{(N)}, (x_t)^{(M)}) > \frac{\varepsilon}{2}$

Thus Q can not be covered by finitely many $\frac{\varepsilon}{2}$-balls, contradicting ttl bddness.

Thus by contradiction we have proved the existence of such $N \in \mathbb{N}$.

This finishes the proof that ttl bddness implies bddness and the other condition

② Next we show that the two conditions can imply ttl bddness.

Assume the hypothesis

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ s.t. $(|x_n| < \frac{\varepsilon}{2}\ \forall (x_n) \in Q$ and $n \geq N)$

By boundedness we have $\sup_{n\in\mathbb{N}} |x_n| \leq M$ for all $(x_n) \in Q$

For the first N terms of sequences in Q, the possible range of any term of any sequence is $[-M,M]$.

So we can mesh N of $[-M,M]$ into $\lceil \frac{4M}{\varepsilon} \rceil^N$ intervals with each one of $\frac{\varepsilon}{2}$ length. $[a_0 \lceil \frac{4M}{\varepsilon} \rceil^N, (a_0+1)\lceil \frac{4M}{\varepsilon} \rceil^N]$

for each small interval, if $\exists$ some term of some sequence in Q whose position is N_0 and value lies in the interval, $(x_n)^{(t)}$ pick one such sequence and add $B_{\frac{\varepsilon}{2}}((x_n)^{(t)})$ to covering

if no such sequence that has such term in that interval, then continue.

Since there are only finitely many small intervals, the covering is finite.

For any $(y_n) \in Q$, the first N terms lies in the range of some $B_{\frac{\varepsilon}{2}}((x_n)^{(t)})$ in covering.

Take that $(x_n)^{(t)}$, $d((y_n), (x_n)^{(t)}) = \sup |y_n - x_n^{(t)}| < \varepsilon$

Since if $\sup |y_n - x_n^{(t)}| = \max_{1\leq n\leq N} |y_n - x_n^{(t)}| \Rightarrow \sup |y_n - x_n^{(t)}| \leq \frac{\varepsilon}{2}$

if not, then $\sup |y_n - x_n^{(t)}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$,

as $|y_n|, |x_n|$ when $n \geq N$ are bounded by $\frac{\varepsilon}{2}$. □

4/4

DUE FRIDAY SEPTEMBER 6

Bonus problem: A map $f : X \to Y$ between metric spaces is called an isometric embedding if

$$d(f(x_1), f(x_2)) = d(x_1, x_2)$$

for all $x_1, x_2 \in X$. If such a map exists we say X embedds isometrically in Y.

Show that every seperable metric space embedds isometrically into $\ell^\infty(\mathbb{N})$.

Let X be a separable metric space,

Take a countable dense subset $E \subseteq X$ and enumerate it as $(p_n)_{n\in\mathbb{N}}$

This induces a seq. in $\ell^\infty(X)$: $(d(x,p_n))_{n\in\mathbb{N}}$ for each $x \in X$

Then we construct $f : X \to \ell^\infty(\mathbb{N})$ by $x \mapsto (d_X(x,p_n))_{n\in\mathbb{N}}$

Let $x,y \in X \Rightarrow d_\infty(f(x), f(y)) = \sup_{n\in\mathbb{N}} |d_X(x,p_n) - d_X(y,p_n)|$

By triangular inequality, $|d_X(x,p_n) - d_X(y,p_n)| \leq d_X(x,y)$ for all $n \in \mathbb{N}$

And since E is dense in X, x is a subsequential limit of (p_n)

So $\sup_{n\in\mathbb{N}} |d_X(x,p_n) - d_X(y,p_n)| = |d_X(x,y) - 0| = d_X(x,y)$

Therefore f is an isometric embedding between X and $\ell^\infty(\mathbb{N})$

9/7 Fix. the sequence $(d_X(x,p_n))_{n\in\mathbb{N}}$ can be unbounded in X, causing it not in $\ell^\infty(\mathbb{N})$

but we can pick an arbitrary term in (p_n), name it as p_0 then fix it

And we induce another sequence $(d_X(x,p_n) - d_X(x_0,p_n))_{n\in\mathbb{N}}$

which is bounded ensured by triangular inequality:

$\forall n$, $|d_X(x,p_n) - d_X(x_0,p_n)| \leq d(x,x_0)$

So $\forall x \in X$, $(d_X(x,p_n) - d_X(x_0,p_n))_{n\in\mathbb{N}} \in \ell^\infty$

Then we construct $f_{\text{modified}} : X \to \ell^\infty(\mathbb{N})$ mapping $x \mapsto (d_X(x,p_n) - d_X(x_0,p_n))_{n\in\mathbb{N}}$

So $\forall x,y \in X$, $d_\infty(f_{\text{modified}}(x), f_{\text{modified}}(y))$

$= \sup_{n\in\mathbb{N}} |d_X(x,p_n) - d_X(y,p_n)|$

$= d_X(x,y)$ as shown above.

This completes the proof.

□

HW 3 on Lipschitz map and differentiation

3.1 Lipschitz condition: stronger than uniformly ctn

Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We say that a function $f : X \to Y$ is Lipschitz with constant C if for any $x, y \in X$, we have

$$d_2(f(x), f(y)) \leq C d_1(x, y).$$

1. Show that Lipschitz maps are uniformly continuous, i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d_1(x_1, x_2) < \delta$, then $d_2(f(x_1), f(x_2)) < \epsilon$.
2. Let $f_n : X_1 \to X_2$ be Lipschitz maps with common Lipschitz constant C. Suppose that f_n converges uniformly to f, i.e., for all $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$ and all $x \in X_1$,

$$d_2(f_n(x), f(x)) < \epsilon.$$

   Is f Lipschitz? What if we only assume that the f_n are Lipschitz (without giving a common Lipschitz constant)?

3.2 ctn map between topological spaces preserves connectness

We say that a metric space X is connected if it cannot be written as $X = A \cup B$ where A and B are nonempty disjoint open subsets of X.

1. Show that if $f : X \to Y$ is a continuous function between metric spaces X and Y, then $f(X)$ is connected if X is connected.
2. Conclude that if $f : X \to \mathbb{R}$ and X is a connected metric space, then f admits all intermediate values $m \in (\inf f, \sup f)$. That is, for any such m, there exists $x_0 \in X$ such that $f(x_0) = m$.

3.3 ctn bijective map from a cpt topological space to a Hausdorff space is a homeomorphism

Let $f : X \to Y$ be a continuous bijective (one-to-one and onto) mapping between metric spaces X and Y.

1. Suppose that X is compact. Show that the inverse function $f^{-1} : Y \to X$ is also continuous.
2. Give an example to show that the requirement that X is compact is necessary.

Remark 实际上这个条件可更宽松. 任意一个 **continuous bijective map from a compact topological space to a Hausdorff space** 都是一个 **homeomorphism.**

这是因为 continuous function maps compact set to compact set, 而 **compact topological space** 中的 **closed set** 一定 **compact**; 且 **Hausdorff space** 中 **compact set** 一定 **closed.**

所以任取 closed set in X, 由于 X compact, 这个集合也 compact; 且它的 preimage by f^{-1} is closed since f ctn. 因而 f^{-1} 是 ctn 的.

3D: an example of ctn but non-diffble point

Let f be a real-valued function defined on $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$). Recall that the directional derivative $D_v f(p)$ of f at p in the direction v is the vector

$$D_v f(p) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$$

if this limit exists.

1. If $c \in \mathbb{R}$ and $D_v f(p)$ exists, prove that $D_{cv} f(p)$ exists and $D_{cv} f(p) = c \cdot D_v f(p)$.
2. For $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sqrt{|xy|}$$

   and $v = (1, 0)$, $v' = (0, 1)$, show that $D_v f(0, 0)$ and $D_{v'} f(0, 0)$ exist but $D_{v+v'} f(0, 0)$ does not exist.
3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \frac{xy^2}{x^2 + y^2}$$

   for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Prove that $D_v f(0, 0)$ exists for every $v = (a, b) \in \mathbb{R}^2$, vanishing if $v = 0$ and equal to

$$\frac{ab^2}{a^2 + b^2}$$

   otherwise.

Using polar coordinates, it is easy to see that f is continuous at $(0, 0)$.

Remark 如果 f 在 x 处 differentiable, 那么它的 $D_v(x)$ 是关于 v linear 的; 如果在 x 处不 differentiable 则不然 (即便各个方向的偏导数都存在也不能保证.)

3.4 Bonus: uniformly disconnectness and ultrametric

A metric space (X, d) is said to be uniformly disconnected if there is $\epsilon_0 > 0$ so that no pair of distinct points $x, y \in X$ can be connected by an ϵ_0-chain, where an ϵ_0-chain connecting x and y is a sequence of points

$$x = x_0, x_1, \ldots, x_m = y$$

satisfying

$$d(x_i, x_{i+1}) \leq \epsilon_0 d(x, y).$$

1. Show that the Cantor set is uniformly disconnected.
2. Show that a metric space (X, d) is uniformly disconnected if and only if there is an ultrametric d' on X for which there is some $C > 1$ such that

$$\frac{d'(x, y)}{C} \leq d(x, y) \leq C d'(x, y).$$

An ultrametric is a metric which satisfies the following improvement of the triangle inequality:

$$d(x, z) \leq \max(d(x, y), d(y, z))$$

for all x, y, z. The discrete metric, where the distance between any pair of distinct points is 1, is an example of an ultrametric. Many other more interesting and important examples exist.

Problem A: Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We say that a function $f : X \to Y$ is Lipschitz with constant C if for any $x, y \in X$, we have

$$d_2(f(x), f(y)) \leq C d_1(x, y).$$

(1) Show that Lipschitz maps are uniformly continuous, i.e. for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d_1(x_1, x_2) < \delta$ then $d_2(f(x_1), f(x_2)) < \epsilon$.

(2) Let $f_n : X_1 \to X_2$ be Lipschitz maps with common Lipschitz constant C. Suppose that the f_n converge uniformly to f, i.e. for all $\epsilon > 0$ there exists $N > 0$ such that for all $n > N$, and all $x \in X_1$,

$$d_2(f_n(x), f(x)) < \epsilon.$$

Is f Lipschitz? What if we only assume that the f_n are Lipschitz (without giving a common Lipshitz constant)?

(1) Pf. Suppose $f: X \to Y$ is Lipschitz.

Let $\varepsilon > 0$

By Lipschitz $\exists C$ s.t. $d_2(f(x), f(y)) \leq C d_1(x,y)$

Take $\delta = \frac{\varepsilon}{C} \Rightarrow \forall x, y \in X_1$ st. $d_1(x,y) < \delta$,

we have $d_2(f(x), f(y)) \leq \frac{\varepsilon}{\delta} d(x,y) < \varepsilon$ $\square$

(2) Claim: If the assumptions hold true then f is Lipschitz.

Pf Let $x, y \in X_1$. Take $\varepsilon = d_1(x,y)$

So $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(f_n(x), f(x)) < \varepsilon$

and by Lipschitz, $d(f_n(x), f_n(y)) < C\, d(x,y)$

Then by triangular inequality of metric space,

$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y))$

This proves that f is Lipschitz with $< C d_1(x,y) + 2 d_1(x,y)$

constant $C+2$ (can get arbitrary close to C by diff ε)

Claim: Without common Lipschitz condition, the proposition is false.

Consider $\left(f_n(x) = \sqrt{x + \frac{1}{n}}\right)_{n \in \mathbb{N}} \to f(x) = \sqrt{x}$, $x \in (0, +\infty)$

Obviously the convergence is uniform.

Claim $f: A \to \mathbb{R}$ is Lipschitz if and only if it is differentiable and f' is bounded.
(A open)

Pf of claim: Assume f is Lipschitz with constant C

Then $\forall x \in \mathbb{R}$, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \leq \lim_{h \to 0} \frac{Ch}{h} = C$, exists and is bounded

Assume f is diff'ble and f' is bounded by $M \geq 0$.

Then $\forall x, y \in A$, by FTC, $\int_x^y f' dx = f(y) - f(x) \leq M(y-x)$ $\square$

$A = \bigcup_{i=1}^{n} (x_i, y_i)$ for some $x_1, \dots, x_n, y_1, \dots, y_n \in A$

$= \sum \int_{x_i}^{y_i} f' dx = \sum f(y_i) - f(x_i)$

Note: $\forall n \in \mathbb{N}$, $f_n \in C^1$

For any $n \in \mathbb{N}$, $f_n'(x) = \frac{1}{\sqrt{x + \frac{1}{n}}}$ is bounded by $\sqrt{n}$, so Lipschitz

But $f'(x) = \frac{1}{\sqrt{x}}$ is not bounded ($f' \to \infty$ when $x \to 0$), so not Lipschitz

2.5/5

Problem B: We say that a metric space X is connected if it cannot be written as $X = A \cup B$ where A and B are nonempty disjoint open subsets of X.

(1) Show that if $f : X \to Y$ is a continuous function between metric spaces X and Y, then $f(X)$ is connected if X is connected.

(2) Conclude that if $f : X \to \mathbb{R}$ and X is a connected metric space, then f admits all *intermediate* values $m \in (\inf f, \sup f)$. That is, for any such m, there exists $x_0 \in X$ such that $f(x_0) = m$.

(1) Pf Suppose X is connected

Assume for contradiction that $f(X)$ is not connected

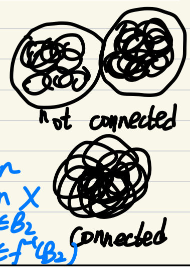
Then $\exists B_1, B_2$ open in Y s.t. $Y = B_1 \sqcup B_2$

nonemptiness -2

By continuity of f, $f^{-1}(B_1)$, $f^{-1}(B_2)$ are open in X

Let $x \in X \Rightarrow f(x) \in Y \Rightarrow f(x) \in B_1$ or $f(x) \in B_2$

$\Rightarrow x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$



Then $X = f^{-1}(B_1) \sqcup f^{-1}(B_2)$ (disjoint by well-definedness of the function)

So contradicts.

This finishes the proof that $f(X)$ is connected if X is connected $\square$

(2) Conclusion: if $f: X \to \mathbb{R}$ with X being a connected metric space, then $\text{im}(f)$ is connected, i.e. f admits all intermediate value $m \in (\inf f, \sup f)$

more justification of $f(X) = (\inf f, \sup f)$ -0.5

$(\forall m \in (\inf f, \sup f), m \in \inf)$

Problem C: Let $f : X \to Y$ be a continuous bijective (one-to-one and onto) mapping between metric spaces X and Y.

(1) Suppose that X is compact. Show that the inverse function $f^{-1} : Y \to X$ is also continuous.

(2) Give an example to show that the requirement that X is compact is necessary.

(1) Pf Suppose X is compact

let $B \subseteq X$ be closed $\Rightarrow$ then B is compact since it is closed subset of a compact MS.

$(f^{-1})^{-1}(B) = f(B)$

Note: $f(B)$ is compact since f ctn, and B cpt. (Lec 4) thus closed

So $\forall$ closed $B \in X$, $(f^{-1})^{-1}(B)$ is closed in Y $\square$

Claim C.1: For $f: X \to Y$ between topological spaces, f is ctn. iff $\forall$ close $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(Pf) Suppose $f: X \to Y$ be ctn. $C \subseteq Y$ be closed

So $Y \setminus C$ is open in $Y \Rightarrow f^{-1}(Y \setminus C)$ is open in X

$\Rightarrow f^{-1}(C) = X \setminus f^{-1}(Y \setminus C)$ is closed in X.

Suppose $\forall$ closed $C \subseteq Y$, $f^{-1}(C)$ is closed

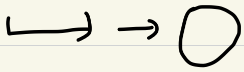
Let $B \subseteq Y$ be open $\Rightarrow Y \setminus B$ is closed in $Y \Rightarrow f^{-1}(Y \setminus B)$ is open

$\Rightarrow f^{-1}(B) = X \setminus f^{-1}(Y \setminus B)$ is open

By claim C.1, f^{-1} is continuous. This finishes the proof. $\square$

(2) Consider $f: [0, 2\pi) \to S^1$

mapping $t \mapsto e^{it}$



Claim f is continuous and bijective

For bijective: $e^{it} = \cos t + i \sin t$

So for distinct $t_1, t_2 \in [0, 2\pi)$, $e^{it_1} \neq e^{it_2}$

For continuous: Let $\varepsilon > 0$, $t_1 \in [0, 2\pi)$

Take $\delta > 0$ s.t. $\forall t_2 \in B_\delta(t_1)$, $|\cos t_1 - \cos t_2| < \frac{\varepsilon}{\sqrt{2}}$

and $|\sin t_1 - \sin t_2| < \frac{\varepsilon}{\sqrt{2}}$ (can be done since sin, cos are ctn.)

Then $|e^{it_1} - e^{it_2}| = \sqrt{(\cos t_1 - \cos t_2)^2 + (\sin t_1 - \sin t_2)^2} < \varepsilon$

But clearly f^{-1} is not ctn at $1 = e^{0i}$.

Rmk: Seems that the proposition can be extended to general topological spaces since we did not use special properties of metric space during proof?

Problem D: Let f be a real valued function defined on $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$). Recall that the directional derivative $D_v f(p)$ of f at p in the direction v is vector

$$D_v f(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t}$$

if this limit exists.

(1) If $c \in \mathbb{R}$ and $D_v f(p)$ exists, prove that $D_{cv} f(p)$ exists and $D_{cv} f(p) = c \cdot D_v f(p)$.

(2) For $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \sqrt{|xy|}$$

and $v = (1,0)$, $v' = (0,1)$, show that $D_v f(0,0)$ and $D_{v'} f(0,0)$ exist but $D_{v+v'} f(0,0)$ does not exist.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{xy^2}{x^2+y^2}$$

for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Prove that $D_v f(0,0)$ exists for every $v = (a,b) \in \mathbb{R}^2$, vanishing if $v = 0$ and equal to

$$\frac{ab^2}{a^2+b^2}$$

otherwise.

Remark 1. This formula for $D_v f(0,0)$ is not linear in v.

Remark 2. Using polar coordinates, it is easy to see that f is continuous at $(0,0)$.

(1) Pf Suppose $c\in\mathbb{R}$ and $\exists D_v f(p) = \lim_{t\to 0}\frac{f(p+tv)-f(p)}{t}$

then if $c=0$, $\lim_{t\to 0}\frac{f(p+ctv)-f(p)}{t} = \lim_{t\to 0} 0 = 0 = cD_v f(p)$

if $c\neq 0$, $\frac{f(p+t(cv))-f(p)}{t} = \frac{f(p+ctv)-f(p)}{ct}\cdot c$

So $\lim_{t\to 0}\frac{f(p+t(cv))-f(p)}{t} = \lim_{h\to 0}\frac{f(p+hv)-f(p)}{h}\cdot c = cD_v f(p)$

Therefore for all $c\in\mathbb{R}$, $D_{cv} f(p)$ exists and is equal to $cD_v f(p)$

(2) $f: \binom{x}{y} \mapsto \sqrt{|xy|}$, $v=(1,0)$, $v'=(0,1)$

$D_v f(0,0) = \lim_{t\to 0}\frac{f(\binom{0}{0}+t\binom{1}{0}) - f\binom{0}{0}}{t} = \lim_{t\to 0}\frac{\sqrt{0}-\sqrt{0}}{t} = 0$

$D_{v'} f(0,0) = \lim_{t\to 0}\frac{f(\binom{0}{0}+t\binom{0}{1}) - f\binom{0}{0}}{t} = \lim_{t\to 0}\frac{\sqrt{0}-\sqrt{0}}{t} = 0$

$D_{v+v'} f(0,0) = \lim_{t\to 0}\frac{f(\binom{0}{0}+t\binom{1}{1}) - f\binom{0}{0}}{t} = \lim_{t\to 0}\frac{\sqrt{t^2}}{t} = \lim_{t\to 0}\frac{|t|}{t}$

does not exist since $\lim_{t\to 0^+}\frac{|t|}{t} = 1$, $\lim_{t\to 0^-}\frac{|t|}{t} = -1$

(3) Pf

For $v=0$, $D_v f(0,0) = \lim_{t\to 0}\frac{f(0,0)-f(0,0)}{0} = 0$

For $v\neq 0$, $D_v f(0,0) = \lim_{t\to 0}\frac{f(ta,tb)-f(0,0)}{t} = \lim_{t\to 0}\frac{\frac{(ta)(tb)^2}{(ta)^2+(tb)^2}-0}{t}$

$= \lim_{t\to 0}\frac{t^3ab^2}{t^3(a^2+b^2)} = \frac{ab^2}{a^2+b^2}$

So $D_v f(0,0)$ exists for every $v\in\mathbb{R}^2$

Problem E: Give the statement of the Baire Category Theorem (from Worksheet 1). (Test yourself by seeing if you can write it down from memory!)

For complete metric space (X,d), any sequence of open dense sets in $(U_n)_{n\in\mathbb{N}}$ in X has $\bigcap_{n=1}^{\infty} U_n$ also dense in X.

Problem F: Submit a writeup of Problem B from Worksheet 2.

WS2 Problem B

Contradict that $m: P(\mathbb{R}^d) \to [0,\infty)$ satisfying:

(a) $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$ for countable $(E_n)_{n\in\mathbb{N}}$ where $E_n \subset \mathbb{R}^d$

(b) $m(E) = m(F)$ if E congruent to F.

(c) $m([0,1)^d) = 1$

does not exist.

By the construction: say $x\sim y$ if $x-y\in\mathbb{Q}$

Take $N\subseteq[0,1)$ s.t. N contains exactly one element of each congruent class.

$R = [0,1)\cap\mathbb{Q}$

For each $r\in R$, define $N_r = \{x+r : x\in N\cap[0,1-r)\} \cup \{x+r-1 : x\in N\cap[1-r,1)\}$

(1) $[0,1) = \bigsqcup_r N_r$

Pf Claim 1.1: $[0,1) = \bigcup_r N_r$

Denote the set of all congruent class as $con(\mathbb{R})$

let $x\in[0,1)$

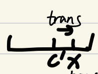
if $x\in N \Rightarrow x\in N_0 \Rightarrow$ done

if $x\notin N$, at least $x\in[c]$ for some $[c]\in con(\mathbb{R})$

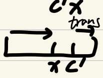
And by def of N, $\exists$ some $c'\in N$ s.t. $c'\in[c] \Rightarrow x-c'\in\mathbb{Q}$

Notice that $c'\in[0,1)$.

if $x<c' \Rightarrow x\in N_{c'-x}$



if $x\leq c' \Rightarrow x\in N_{1-c'+x}$



Claim 1.2 $\forall r_1,r_2\in R$, $N_{r_1}=N_{r_2}$ or $N_{r_1}\cap N_{r_2}=\emptyset$ (i.e. N_{r_1}, N_{r_2} are disjoint)

Suppose $N_{r_1}\cap N_{r_2}\neq\emptyset$

let $x\in N_{r_1}\cap N_{r_2}$

Note: $x\in[c]$ for some congruent class $[c]$

Clearly, each N_r must and can only has one element from each congruent class, otherwise if $x_1,x_2\in[c]$, $x_1,x_2\in N_r \Rightarrow$ by translation there will be two elements from $[c]$ in N.

Therefore $x\in N_{r_1}$ and $x\in N_{r_2}$ are translated from the same element from N.

$\Rightarrow x+r \equiv x+s \mod 1$

Since $r_1,r_2\in[0,1)$, we must have $r=s$

By claim 1.1 & 1.2, $[0,1) = \bigsqcup_{r\in R} N_r$

(2) If $m: P(\mathbb{R}^d)\to[0,\infty)$ satisfying (a)(b)(c),

then $m(N) = m(N_r)$ for every $r\in R$

let $r\in R$. Denote $A=[0,1-r)$, $B=[1-r,1)$

$N = (N\cap A)\sqcup(N\cap B)$

Note that $N_r = (N\cap A)+r \sqcup (N\cap B)+(1-r)$

So by property (b) of the measure function we assume, we must have $m(N\cap A) = m(N\cap A+r)$

$m(N\cap B) = m(N\cap B+1-r)$

Then by property (a), $m(N) = m(N\cap A)+m(N\cap B)$

$= m(N\cap A+r)+m(N\cap B+1-r)$

$= m(N_r)$

(c) Arrive at a contradiction

Pf If $m(N)=0$, then since $\mathbb{R}$ is infinite,

$$m([0,1)) = m(\bigsqcup_r N_r) = \sum_{r\in\mathbb{R}} N_r = \sum_{r\in\mathbb{R}} N = +\infty$$

If $m(N)\neq 0$, then $m([0,1]) = \sum_{r\in\mathbb{R}} N = 0$

Thus in whatever way we define $m(N)$, the propery (c) will fail to be true. (Basically if m defines N to be measurable, then property (a)(b) contradicts (c).)

Bonus problem: A metric space (X, d) is said to be uniformly disconnected if there is $\epsilon_0 > 0$ so that no pair of distinct points $x, y \in X$ can be connected by an ϵ_0-chain, where an ϵ_0-chain connecting x and y is a sequence of points

$$x = x_0, x_1, \ldots, x_m = y$$

satisfying

$$d(x_i, x_{i+1}) \leq \epsilon_0 d(x, y).$$

(1) Show that the Cantor set is uniformly disconnected.

(2) Show that a metric space (X, d) is uniformly disconnected if and only if there is an ultrametric d' on X for which there is some $C > 1$ such that

$$d'(x, y)/C \leq d(x, y) \leq Cd'(x, y).$$

An ultrametric is a metric which satisfies the following improvement of the triangule inequality:

$$d(x, z) \leq \max(d(x, y), d(y, z))$$

for all x, y, z. The discrete metric, where the distance between any pair of distinct points is 1, is an example of an ultrametric. Many other more interesting and important examples exist.

For a hint on the bonus, see office door. But try without first.

(1) Pf Consider $\varepsilon = \frac{9}{30}$

Denote the Cantor set as Cat

Let $x,y \in$ Cat with WLOG $y > x$

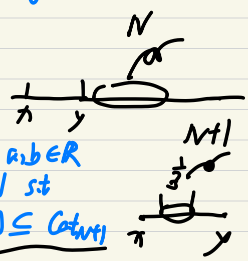
Let $n\in\mathbb{N}$

For $n=1$, we take away $(\frac{1}{3},\frac{2}{3})$ from $[0,1]$ and get Cat_1

Suppose that the union of the middle parts of all disjoint inverals in $[0,1]$ we take away is I_n and we get

$$Cat_n = Cat_{n-1} \setminus I_n$$

Then $Cat = [0,1] \setminus \bigsqcup_{n\in\mathbb{N}} I_n$



Write $|I| = b-a$ if $I=[a,b]$ for some $a,b\in\mathbb{R}$

Since $x,y \in$ Cantor, $\exists$ some $N\in\mathbb{N}$ s.t. $[x,y] \subseteq Cat_N$ but $[x,y] \subseteq Cat_{N+1}$

Then by definition of Cantor set,

$$|I_{N+1} \cap [x,y]| \geq \frac{1}{3}(y-x)$$

And $I_{N+1}\cap[x,y] = I'_{N+1} = [b-a]$ for some $x\leq a\leq b\leq y$

Let $x=x_1,\ldots,x_m=y$ be arbitrary ε-chain in Cat, with $\varepsilon=\frac{9}{30}$, between y and x

Then $\exists$ some x_{m_1}, x_{m_2} in the chain s.t. $x_{m_1}\leq a\leq b\leq x_{m_2}$

So $d(x_{m_2}, x_{m_1}) \geq \frac{1}{3}d(x,y) > \varepsilon\, d(x,y)$

This finishes the proof that Cat is uniformly disconnected. □

(2) Hint: $d'(x,y) = \inf(\gamma : \exists$ a $\frac{\gamma}{d(x,y)}$-chain from x to $y)$

For $\Rightarrow$: (Uniformly disconnected implies $\exists$ such ultrametric d')

Pf Assume (X,d) is uniformly disconnected.

Construct $d': X\to\mathbb{R}$ sending

$(x,y)\mapsto \inf(\gamma : \exists$ a $\frac{\gamma}{d(x,y)}$-chain from x to $y)$

d' is nonnegative and $d'(x,y)=0$ iff $x=y$ since X is uniformly disconnected and $d'(x,y)=d'(y,x)$ for all $x,y\in X$ since an ε-chain is an ε-chain from x to y iff it is an ε-chain from y to x.

So it suffices to show that d' satisfies the improved triangular ineq. in order to show that d' is an ultrametric

Let $x,y,z\in X$.

Suppose for contradiction that

$d'(x,z) > d'(x,y)$ and $d'(x,z) > d'(y,z)$

WLOG let $\gamma = d'(x,y) \geq d'(y,z)$

Then $\exists$ $\frac{\gamma}{d(x,y)}$ chain $x=x_1,\ldots,x_m=y$, and

$\frac{\gamma}{d(x,z)}$ chain $y=y_1,\ldots,y_m=z$

Thus $x=x_1,\ldots,x_m, y_1,\ldots,y_m=z$ is a $\frac{\gamma}{d(x,z)}$ chain from x to z, so $d'(x,z)\leq\gamma=d'(x,y)$, reaching contradiction

① This proves that d' is an ultrametric on X induced by d

Now we want to show that $\exists C>1$ s.t.

$\frac{d'(x,y)}{C} \leq d(x,y) \leq C d'(x,y)$ for all $x,y\in X$.

Since (X,d) is uniformly disconnected, we can take ε_0 s.t. $\forall x,y\in X$, there is no ε_0-chain between x,y

Take $C=\frac{1}{\varepsilon_0}$

Let $x,y\in X$

WTS: $\frac{d'(x,y)}{C} \leq d(x,y) \leq C d'(x,y)$ and we must have $C>1$

This part I have not finished yet.

For $(\Leftarrow)$: Such ultrametric exists imply that X is uniformly disconnected.

Pf Assume $\exists$ ultrametric d' on X and $C>1$ s.t.

$$\frac{d'(x,y)}{C} \leq d(x,y) \leq C d'(x,y)$$

Then let $\varepsilon = \frac{1}{2C}$

Let $x,y\in X$

Suppose for contradiction that $\exists$ an ε_0-chain

$x=x_0, x_1,\ldots,x_m=y$ s.t. $d(x_i,x_{i+1})\leq\varepsilon\, d(x,y)$ for each $i=0,\ldots,m$

Then $d'(x_i,x_{i+1}) \leq C d(x_i,x_{i+1}) \leq C\varepsilon\, d(x,y) = \frac{1}{2}d(x,y)$ for each i

Then by the ultrametric inequality,

$$d'(x,y) \leq \max_i d'(x_i,x_{i+1}) \leq \frac{1}{2}d'(x,y)$$

$\Longrightarrow d'(x,y)=0$, reaching a contradiction

Therefore for $\varepsilon=\frac{1}{2C}$, no pair of distinct points in X can be connected by a ε-chain

$\Longrightarrow X$ is uniformly disconnected □

This finishes the proof of the iff statement.

HW 4 on partial derivatives

4.1 在原点可导且保留 scalar 的函数一定是 linear map

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ satisfy

$$F(tx) = tF(x)$$

for all positive real numbers t and all $x \in \mathbb{R}^n$. Assume F is differentiable at the origin. Show that F is linear.

4.2 all partials exist 且 bdd on $A \implies$ ctn on A

Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ (for $1 \le i \le m$ and $1 \le j \le n$) exist and are bounded on A. Show that f is continuous on A.

4.3 example of all partials exist and bounded 但不可导

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation:

$$f(r, \theta) = (r\cos\theta, r\sin\theta).$$

1. Calculate Df and $\det Df$.
2. Let $S = [1, 2] \times [0, \pi/2]$. Find $f(S)$ and sketch it.
3. Show that f is a homeomorphism from S onto $f(S)$ and compute the inverse function f^{-1}.
4. Compute Df^{-1} and $\det Df^{-1}$.
5. What relation can you find between Df and Df^{-1}?

4D: Give an example of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that, at the origin, all directional derivatives exist and are zero, but F is not differentiable at the origin.

4E: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting $f(0) = 0$ and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}.$$

1. Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0, 0)$.
2. Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x, y) \neq 0$.
3. Show that $f \in C^1(\mathbb{R}^2)$.
4. Show that $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ and $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ exist everywhere on $\mathbb{R}^2$, but they are not equal at $(0, 0)$.

4.4 Bonus: ultrametric on graph

Recall that an ultrametric space is a metric space where one has the following stronger than usual form of the triangle inequality:

$$d(x, z) \le \max(d(x, y), d(y, z)).$$

1. Show that, in an ultrametric space, open balls are closed.
2. Show that, in an ultrametric space, if two balls intersect, one of the two must be contained in the other.
3. Show that, in an ultrametric space, every point of a ball is the center of the ball. That is, if $y \in B_r(x)$, then $B_r(x) = B_r(y)$.
4. Let G be a connected weighted undirected graph. (The weighting is the assignment of a positive number to each edge.) Let $V(G)$ be the set of vertices. Given a path in the graph (a sequence of adjacent edges), define the length of the path to be the largest weight of an edge crossed by the path. Given $v, w \in V(G)$, define $d(v, w)$ to be the smallest length of a path from v to w. Show that d is an ultrametric on $V(G)$.
5. Show that any finite ultrametric arises as in the previous part.

Just for fun (don' t hand in): Imagine you have an electric car, and you live in a country that provides free charging stations, and you' re not in a hurry. Why might you end up thinking about an ultrametric?

Problem A: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be satisfy
$$F(tx) = tF(x)$$
for all all positive real numbers t and all $x \in \mathbb{R}^n$. Assume F is differentiable at the origon. Show F is linear.

Pf Construct $r_0(h) = F(0+h) - F(0) - Df(0)h = F(h) - Df(0)h$

Then that for $t \in \mathbb{R}_{>0}$

$r_0(th) = F(th) - Df(0)(th) = tF(h) - tDf(0)h = t r_0(h)$

Claim: $\forall h \in \mathbb{R}^n$, $r_0(h) = 0$

(pf) Suppose for contradiction that for some $h_0 \in \mathbb{R}^n$, $r_0(h_0) \neq 0$

Then $\frac{\|r_0(h_0)\|}{\|h_0\|} = c$ for some $c > 0$

By homogeneity, for any $t > 0$, $\frac{\|r_0(th_0)\|}{\|th_0\|} = \frac{t\|r_0 h_0\|}{t\|h_0\|} = \frac{\|r_0(h_0)\|}{\|h_0\|} = c$

then for $t \to 0$, $\|th_0\| \to 0$ while $\frac{\|r_0(h_0)\|}{\|h_0\|} = c$

contradicting that $\lim_{\|h\| \to 0} \left\| \frac{r_0(h)}{h} \right\| = 0$

This proves the claim.

So $\forall h \in \mathbb{R}^n$, $F(h) = Df(0)h$ where $Df(0) \in Hom(\mathbb{R}^m, \mathbb{R}^n)$

Thus F is a linear transformation. $\square$

Problem B: Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}^m$. Suppose that the partial derivatives $\frac{\partial f_i}{\partial x_j}$ $(1 \le i \le m, 1 \le j \le n)$ exist are are bounded on A. Show that f is continuous on A.

Pf Claim $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m = \begin{pmatrix} f_1 : \mathbb{R}^n \to \mathbb{R} \\ \vdots \\ f_m : \mathbb{R}^n \to \mathbb{R} \end{pmatrix}$ is continuous

at $x_0 \in A$ iff $\forall i \in \{1, \dots, n\}$, f_i is continuous at x_0.

This directly follows from $\|f(x) - f(x_0)\|_2 = \sqrt{\sum_i (f_i(x) - f_i(x_0))^2}$

(if $\forall x \in B_\delta(x_0)$ we have $f(x) \in B_\varepsilon(f(x_0))$, then $\forall f_i$, $f_i(x) \in B_\varepsilon(f_i(x_0))$

if for all i, $\forall x \in B_\delta(x_0)$ we have $f_i(x) \in B_{\frac{\varepsilon}{\sqrt{n}}}(f_i(x_0)) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$)

There WLOG we can set $m = 1$

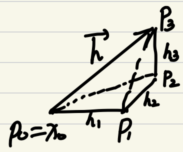
Assume $\frac{\partial f}{\partial x_j}$ $(1 \le j \le n)$ exist (for all $x \in A$) and bounded

WTS: f is continuous on A.

Let $\varepsilon > 0$.

Let $x = x_0 + h$ where $h \in \mathbb{R}^n$

Then $h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ for some $h_1, \dots, h_n \in \mathbb{R}$



Let $p_0 = x_0$

$p_1 = p_0 + h_1 e_1$

$\vdots$

$p_n = p_{n-1} + h_n e_n = x_0 + h$

For each $i = 1, \dots, n$, let $\varphi_i : [0, h_i] \to \mathbb{R}$

map $s \longmapsto f(p_{i-1} + s e_i)$

Then $\forall s_0 \in (0, h_i)$, $\frac{d}{ds}\Big|_{s=s_0} \varphi_i(s) = \frac{d}{ds}\Big|_{s=s_0} f(p_{i-1} + s e_i) = \frac{\partial}{\partial x_i} f(p_{i-1} + s_0 e_i)$

Since all partials exist on A and bounded,

all φ_i are differentiable on $(0, h_i)$,

So by MVT, $\forall i$, $\varphi_i(h_i) - \varphi_i(0) = \left(\frac{\partial}{\partial x_i} f(p_{i-1} + s_i e_i)\right) \cdot h_i$

for some $s_i \in (0, h_i)$, we write $p_{i-1} + s_i e_i$ as q_i

Then $|f(x+h) - f(x)| = \left| \sum_{i=1}^n (f(p_i) - f(p_{i-1})) \right| = \left| \sum_{i=1}^n (\varphi_i(h_i) - \varphi_i(0)) \right|$

$= \left| \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} f(q_i) h_i\right) \right|$

Since all partials are bounded by some $M \in \mathbb{R}$ and

$|h_i| \le \|h\| = \|x - x_0\|$

Thus we have:

$|f(x) - f(x_0)| \le nM \|x - x_0\|$

This implies that f is Lipschitz on A, thus (uniformly) continuous (by hw 3). $\square$

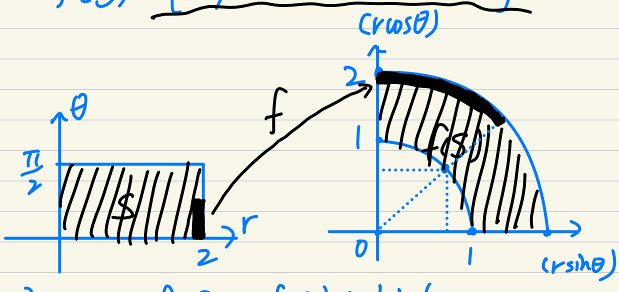
Problem C: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation:
$$f(r, \theta) = (r\cos\theta, r\sin\theta).$$

(1) Calculate Df and $\det Df$.
(2) Let $S = [1, 2] \times [0, \pi/2]$. Find $f(S)$ and sketch it.
(3) Show that f is a homeomorphism from S on $f(S)$ and compute the inverse function f^{-1}.
(4) Compute Df^{-1} and $\det Df^{-1}$.
(5) What relation can you find between Df and Df^{-1}?

(1) $Df = \begin{pmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$\det Df = r\cos^2\theta + r\sin^2\theta = r$

(2) $f(S) = \{(x, y) : 1 \le x^2 + y^2 \le 4 \text{ and } x, y \ge 0\}$



(3) Claim ①: $f : S \to f(S)$ is bijective

Pf it is surjective since we take $f(S)$ to be in place of codomain

Now prove injectivity: suppose $f(r_1, \theta_1) = f(r_2, \theta_2)$

then $r_1\cos\theta_1 = r_2\cos\theta_2$

$r_1\sin\theta_1 = r_2\sin\theta_2$

$\Rightarrow r_1^2\cos^2\theta_1 + r_1^2\sin^2\theta = r_2^2\cos^2\theta_2 + r_2^2\sin^2\theta_2 \Rightarrow r_1^2 = r_2^2 \Rightarrow r_1 = r_2$ since $r_1, r_2 > 0$

$\Rightarrow \cos\theta_1 = \cos\theta_2 \Rightarrow \theta_1 = \theta_2$ since $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$

Claim ②. f is continuous.

Since $f(r,\theta) = \begin{pmatrix} f_1(r,\theta) = r\cos\theta \\ f_2(r,\theta) = r\sin\theta \end{pmatrix}$ where f_1, f_2 are all continuous functions, f is always continuous.

Claim ③ f^{-1} is continuous

Let $f(r,\theta) = \binom{x}{y} \Rightarrow x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2+y^2 = r^2$

$\Rightarrow r = \sqrt{x^2+y^2}$ since $r>0$;

and $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}(\frac{y}{x})$ since $x, y > 0$, $x \in [0, \frac{\pi}{2}]$

So $f^{-1}: f(S) \to S$

sending $\binom{x}{y} \mapsto \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}(\frac{y}{x}) \end{pmatrix}$, which is continuous since f_1^{-1}, f_2^{-1} are continuous.

Claim ①②③ proves that f is a homeomorphism.

(4) $\frac{\partial}{\partial x} r = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial}{\partial y} r = \frac{y}{\sqrt{x^2+y^2}}$, $\frac{\partial}{\partial x}\theta = \frac{\partial}{\partial x}\arctan(\frac{y}{x}) = \frac{-y}{x^2+y^2}$, $\frac{\partial}{\partial y}\theta = \frac{x}{x^2+y^2}$

$$\Rightarrow Df^{-1}(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}, \quad \det Df^{-1} = \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$$

(5) $$Df \cdot Df^{-1} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{\sin\theta}{-r} & \frac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (Df)(Df^{-1}) = I_2$$

Problem D: Give an example of a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that, at the origin, all directions derivatives exist and are zero, but F is not differentiable at the origin.

Consider $F\binom{x}{y} = \begin{pmatrix} \frac{x^2y}{x^2+y^2} \\ \frac{x^2y}{x^2+y^2} \end{pmatrix}$ for $\binom{x}{y} \neq \binom{0}{0}$ and $F\binom{0}{0} = \binom{0}{0}$

$$D_{e_1}F(0,0) = \lim_{t\to 0} \frac{F\binom{0+t}{0} - F\binom{0}{0}}{t} = \lim_{t\to 0}(0,0) = (0,0)$$

Similarly $D_{e_2}F(0,0) = \lim_{t\to 0}(0,0) = (0,0)$

Since $\forall u \in \mathbb{R}^2$, u is a linear comb of e_1, e_2 and $D_uF(\binom{0}{0})$ is linear in u, all directional derivatives exist and are 0 at origin:

$$D_uf\binom{0}{0} = D_{u_1e_1+u_2e_2}f\binom{0}{0} = u_1D_{e_1}f\binom{0}{0} + u_2D_{e_2}f\binom{0}{0} = \binom{0}{0}$$

The Jacobian matrix $J_f(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\lim_{\|(x,y)\|\to 0} \frac{f(x,y) - f(0,0) - J_f(0)\binom{x}{y}}{\sqrt{x^2+y^2}} = \lim_{\|(x,y)\|\to 0} \begin{pmatrix} \frac{xy}{(x^2+y^2)^{\frac{3}{2}}} \\ \frac{x^2y}{(x^2+y^2)^{\frac{3}{2}}} \end{pmatrix}$$

Consider the sequence $((x_n, y_n) = (\frac{1}{n}, \frac{1}{n}))_{n\in\mathbb{N}}$

this sequence converge to 0 by norm.

But $\lim_{n\to\infty} \frac{x_n^2 y_n}{(x_n^2+y_n^2)^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{1}{n^3} \Big/ \frac{2\sqrt{2}}{n^3} = \frac{\sqrt{2}}{4} \neq 0$

Hence the Jacobian matrix is not the derivative of f at 0 which suffices to indicate that f is not differentiable at 0 □

Problem E: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by setting $f(0) = 0$ and

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}.$$

(1) Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at 0.
(2) Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $(x,y) \neq 0$.
(3) Show that $f \in C^1(\mathbb{R}^2)$.
(4) Show that

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$$

exist everywhere on $\mathbb{R}^2$, but they are not equal at $(x,y) = 0$.

(1) $\frac{\partial f}{\partial x}(0) = \lim_{t\to 0} \frac{f\binom{t}{0} - f\binom{0}{0}}{t} = \lim_{t\to 0} \frac{0}{t} = 0$

$\frac{\partial f}{\partial y}(0) = \lim_{t\to 0} \frac{f\binom{0}{t} - f\binom{0}{0}}{t} = \lim_{t\to 0} \frac{0}{t} = 0$

(2) Claim product and quotient rule of differentiation holds for partial derivatives.

Pf $\frac{\partial}{\partial x}(u(x,y)v(x,y)) = \lim_{t\to 0} \frac{u(x+t,y)v(x+t,y) - u(x,y)v(x,y)}{t}$

$= \lim_{t\to 0} \frac{u(x+t,y)(v(x+t,y) - v(x,y))}{t} + \frac{v(x,y)(u(x+t,y) - u(x,y))}{h}$

$= u(x,y)\frac{\partial v(x,y)}{\partial x} + v(x,y)\frac{\partial u(x,y)}{\partial x}$

Quotient rule follows from the product rule.

Now we use the rules for calculation:

for $\binom{x}{y} \neq 0$, $\frac{\partial}{\partial x}f(x,y) = \frac{x^2-y^2}{x^2+y^2}\frac{\partial}{\partial x}(xy) + xy\frac{\partial}{\partial x}(\frac{x^2-y^2}{x^2+y^2})$

where $\frac{\partial}{\partial x}(\frac{x^2-y^2}{x^2+y^2}) = \frac{(x^2+y^2)2x - (x^2-y^2)2x}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$

$\Rightarrow \frac{\partial}{\partial x}f(x,y) = y\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2}$

$\frac{\partial}{\partial y}f(x,y) = \frac{x^2-y^2}{x^2+y^2}\frac{\partial(xy)}{\partial y} + xy\frac{\partial}{\partial y}(\frac{x^2-y^2}{x^2+y^2})$

$= x\frac{x^2-y^2}{x^2+y^2} + x\frac{-4x^2y^2}{(x^2+y^2)^2}$

(3) Since we have shown that for all $(x,y) \in \mathbb{R}^2$, all partials at (x,y) exist, it suffices to show that $\forall (x,y) \in \mathbb{R}^2$, all partials are continuous, in order to show that $f \in C^1(\mathbb{R}^2)$

And since any directional derivative $D_uf(x_0)$ is linear in u, it suffices to show that $\forall (x,y) \in \mathbb{R}^2$, $\frac{\partial}{\partial x}f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous.

Let $(x,y) \in \mathbb{R}^2$

Since $\frac{\partial}{\partial x}f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are rational functions (thus ctn.) except at $x=0$, we only need to show that $\frac{\partial}{\partial x}f(x,y)$, $\frac{\partial}{\partial y}f(x,y)$ are ctn. at $x=0$.

$$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial x}f(x,y) = \lim_{(x,y)\to(0,0)} y\overbrace{\frac{x^2-y^2}{x^2+y^2}}^{\in[-1,1]} + y\frac{4x^2y^2}{x^4+2x^2y^2+y^4} \in (0,2]$$

the expression is bounded by $|3y|$, so its limit when $(x,y)\to 0$ is 0.

Similarly, $$\lim_{(x,y)\to(0,0)} \frac{\partial}{\partial y}f(x,y) = \lim_{(x,y)\to(0,0)} x\underbrace{\frac{x^2-y^2}{x^2+y^2}}_{\in[-1,1]} + x\frac{-4x^2y^2}{x^4+2x^2y^2+y^4} \in [-2,0]$$

the expression is bounded by $|3x|$, so its limit when $(x,y)\to 0$ is 0.

Notice that $\mathbb{R}^2$ has no isolated pt., so f is ctn. at origin and thus ctn. since it is rational elsewhere

(4) Let $(x,y) \in \mathbb{R}^2$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial x}\left(x\frac{x^2-y^2}{x^2+y^2} + x\frac{-4x^2y^2}{(x^2+y^2)^2}\right) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial y}\left(y\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2}\right) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

So $\frac{\partial^2}{\partial x \partial y}$ and $\frac{\partial^2}{\partial y \partial x}$ exists everywhere and equal except on the origin

On the origin: $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(0,0)\right) = \lim_{t\to 0}\frac{\frac{\partial}{\partial y}f(t,0) - \frac{\partial}{\partial y}f(0,0)}{t} = \lim_{t\to 0}\frac{t-0}{t} = 1$

but $\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(0,0)\right) = \lim_{t\to 0}\frac{\frac{\partial}{\partial x}(0,t) - \frac{\partial}{\partial x}(0,0)}{t} = \lim_{t\to 0}\frac{-t-0}{t} = -1$

Bonus: Recall that an ultametric space is a metric space where one has the following stronger than usual form of the triangle inquality:

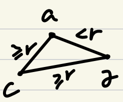
$$d(x,z) \le \max(d(x,y), d(y,z)).$$

(1) Show that, in an ultrametric space, open balls are closed.
(2) Show that, in an ultrametric space, if two balls intersect, one of the two must be contained in the other.
(3) Show that, in an ultrametric space, every point of a ball is the center of the ball. That is, if $y \in B_r(x)$, then $B_r(x) = B_r(y)$.
(4) Let G be a connected weighted undirected graph. (The weighting is the assignment of a positive number to each edge). Let $V(G)$ be the set of vertices.
Given a path in the graph (a sequence of adjacent edges), define the length of the path to be the largest weight of an edge crossed by the path.
Given $v, w \in V(G)$, define $d(v,w)$ to be the smallest length of a path from v to w.
Show that d is an ultrametric on $V(G)$.
(5) Show that any finite ultrametric arises as in the previous part.

Just for fun (don't hand in): Imagine you have an electric car, and you live in a country that provides free charging stations, and you're not in a hurry. Why might you end up thinking about an ultrametric?

(1) Let (X,d) be an ultra-metric space.
Suppose $B=\{x : d(x,c)<r\}$ is an open ball in X centered at $c \in X$
Let $z \in X \setminus B \Rightarrow d(z,c) \ge r$

Consider $B_r(z)$: let $a \in B_r(z)$, then $d(a,z)<r$
By ultrametric, $d(z,c) \le \max\{d(a,c), d(a,z)\}$

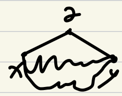


And since $d(z,c) \ge r \Rightarrow \max\{d(c,a), d(a,z)\} \ge r$
We already know that $d(a,z)<r$
Therefore $d(c,a) \ge r \Rightarrow a \in X \setminus B_r(c)$
$\Rightarrow B_r(z) \subseteq X \setminus B_r(c)$
Since z is arbitrary, this proves that $X \setminus B_r(c)$ is open
$\Rightarrow B_r(c)$ is closed
Then we can conclude that every open ball is also closed in X.

(2) Let (X,d) be an ultrametric space
Let $B_r(x), B_s(y) \subseteq X$ be two open balls with $B_r(x) \cap B_s(y) \neq \emptyset$
We only need to consider the case when $x \neq y$ since if $x=y$ then one ball must contain the other one.
WLOG suppose $r \le s$.
Let $a \in B_r(x)$, $z \in B_r(s) \cap B_r(y)$
$\Rightarrow d(z,x)<s$, $d(z,y)<r$
$\Rightarrow d(a,y) \le \max\{d(a,z), d(z,y)\} < r \Rightarrow a \in B_s(y)$
$\Rightarrow B_r(x) \subseteq B_s(y)$ □

(3) This directly follows from (2):
Let $y \in B_r(x) \Rightarrow B_r(y) \cap B_r(x) \neq \emptyset$
$\Rightarrow B_r(y) \subseteq B_r(x)$ and $B_r(x) \subseteq B_r(y)$
$\Rightarrow B_r(y) = B_r(x)$

(4) Positivity follows from the definition of the graph and $\forall x,y \in V(G)$, $d(x,y)=d(y,x)$ since the graph is undirected (every path commutes)



So it suffices to show d is an ultrametric by showing the ultra-triangular property

Let $x,y,z \in V(G)$ s.t. there is at least one path from x to z and z to y and x to y
We write the weight of an edge e as $w(e)$ and the smallest weight of an edge cross a path p as $L(p)$

Case 1: the smallest-length path between x,y, say P_{xy}, goes through z.
Then $P_{xy} = P_{xz} \cup P_{zy}$ where P_{xz} is a path between x,z and P_{zy} is a path between z,y
Then $d(x,y) = L(P_{xy}) = \max\{L(P_{xz}), L(P_{zy})\}$
and $d(x,z) = \min\{L(P) : \text{path through } x,z\}$
$d(z,y) = \min\{L(P) : \text{path through } z,y\}$
so $L(P_{xz}) \le d(x,z)$, $L(P_{zy}) \le d(z,y)$
Thus $d(x,y) = L(P_{xy}) \le \max\{d(x,z), d(z,y)\}$

Case 2: the smallest-length path between x,y, say P_{xy}, does not go through z.
Take path P_{xz}, P_{zy} s.t. $L(P_{xz}) = d(x,z)$, $L(P_{zy}) = d(z,y)$
Then let $P'_{xy} = P_{xz} \cup P_{zy}$, we have $L(P'_{xy}) = \max\{L(P_{xz}), L(P_{zy})\}$
Since $d(x,y) = L(P_{xy}) \Rightarrow L(P_{xy}) \ge L(P'_{xy})$
$\Rightarrow d(x,y) = L(P_{xy}) > \max\{L(P_{xz}), L(P_{zy})\}$
$= \max\{d(x,z), d(z,y)\}$
In both case the ultra-triangular ineq. holds true
This finishes the proof that d is an ultrametric on $V(G)$

(5) Let (X, d_u) be a finite ultrametric field with $\#|X| = C$
WTS: we can construct a graph $G=(X, E(G))$ endowed with metric d_g in (4), s.t. (X, d_u) is isometrically embedded into (G, d_g)

Construction: for each $v,w \in X$, add an edge $e(v,w)$ to $E(G)$ with $w(e(v,w)) = d_u(v,w)$
Then the graph will be a complete C-graph
By d_u, $\forall x \in X$, $d(v,w) \le \max\{d(v,x), d(x,w)\}$
So every path P through v,w has $L(P) \ge w(e(v,w))$
So $d_g(v,w) = w(e(v,w)) = d_u(v,w)$

HW 5 on chain rule, prod rule and Taylor's Thm

5.1 applying chain rule to compute derivative

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x, y, z) = (\exp(x^2 + 2y^2), \sin(z^2 - y^2) \cdot (x^2 + 2z^2), (x^2 + y^2 + z^2)^9).$$

Explain why F is differentiable, and then prove why $DF(x, y, z)$ always has zero determinant. You may not actually compute any derivatives in your solution.

5.2 diffeo 维度必须相同且 derivatives 互为 inverse matrix

Suppose

$$F : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$$

and

$$G : B \subset \mathbb{R}^m \to A \subset \mathbb{R}^n$$

are both differentiable and are inverses of each other (with A, B open). Show that $n = m$ and that, for all pairs $a \in A$, $b \in B$ with $F(a) = b$,

$$DG(b) = DF(a)^{-1}.$$

5.3 differentiable homeomorphism 未必是 diffeomorphism

Give an example of a differentiable homeomorphism from $\mathbb{R}$ to itself whose inverse is not differentiable at every point. (For your example, you need to find only a single point where the inverse isn' t differentiable.)

5.4 f 在 x_0 处 ctn $\implies$ 取多元极限可换序

Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin. Show

$$\lim_{h\to 0} \lim_{k\to 0} F(h, k) = \lim_{k\to 0} \lim_{h\to 0} F(h, k),$$

assuming all these limits exist. Give an example where F is not continuous, both double limits exist, but the two double limits are not equal.

Just for fun (don' t hand in): Give an example where $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin but $\lim_{h\to 0} \lim_{k\to 0} F(h, k)$ does not exist.

Just for fun (don' t hand in): Also note that for $a_{n,m} = 2^{n-m}$ it is not true that

$$\lim_{n\to\infty} \lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty} \lim_{n\to\infty} a_{n,m}.$$

5.5 number of terms in Taylor series

If F is a function of 4 variables, how many terms (in general) are in the degree 10 Taylor series of F? (Degree 10 means you use multi-indices α of degree at most 10.) You do not need to show your work; just give the final answer.

5.6 open connected set 上 determinant 为 0 $\implies$ constant function

Suppose that $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with A open and connected and $Df(a) = 0$ for all $a \in A$. Show that F is constant.

5.7 multifunction product rule in 1d

Let $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ be C^k. Show that

$$\frac{\partial^k}{\partial x^k}(f_1 \cdot f_2 \cdot \ldots \cdot f_m) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^{\alpha_1} f_1 \cdot \ldots \cdot \partial^{\alpha_m} f_m.$$

$$\sum_{i=1}^{m} \frac{k!}{\beta_1! \cdots (\beta_i - 1)! \cdots \beta_m!}$$

is it equal to

$$\frac{(k+1)!}{\beta_1! \cdots \beta_m!}$$

5.8 proof: Taylor series is the best polynomial approximation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be in C^{k+1}. Show that the Taylor polynomial of degree k centered at $x_0 \in \mathbb{R}^n$ is the best polynomial approximation of $f(x)$ near x_0 in the following sense: Suppose that $P(x)$ is a polynomial of degree k. Then

$$P(x) - f(x) = o(|x - x_0|^k)$$

if and only if P is the Taylor polynomial of degree k centered at x_0. (Recall that a quantity Q is $o(|x - x_0|^k)$ if $\lim_{x \to x_0} \frac{Q}{|x - x_0|^k} = 0$.)

5I: computation problem 1 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x^2 + y^2)$, so F is differentiable.

1. Prove that $x\frac{\partial F}{\partial y} = y\frac{\partial F}{\partial x}$.
2. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are differentiable. Define $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ via

   $$\phi(x, y, z) = (f(h(x), g(x, y), z), g(y, z)).$$

   Find a formula for the matrix of the derivative $D\phi(p) : \mathbb{R}^3 \to \mathbb{R}^2$ at an arbitrary point $p \in \mathbb{R}^3$ in terms of the partial derivatives of f, g, and h at p.
3. In (ii), compute $D\phi(1, 1, 1)$ when $f(x, y, z) = x^2 + yz$, $g(x, y) = y^3 + xy$, and $h(x) = e^x$. Do this in two ways: using your general formula in (ii) and also by explicitly computing ϕ in this case and directly

computing the Jacobian matrix from this.

5J: computation problem 2 Problem 2(a) on page 63 of the text.

5K: computation problem 3 Find the 3rd order Taylor series of $F(x, y) = e^{x+y^2}$ about the origin.

5.9 Bonus: positive-definite matrix

A real symmetric $n \times n$ matrix A is called positive definite if $x^T Ax > 0$ for all $x \in \mathbb{R}^n$.

1. Show that a real symmetric matrix is positive definite if and only if it is invertible and, for all non-zero x, the angle between Ax and x is less than 90 degrees.
2. Show that a real symmetric matrix is positive definite if and only if all its eigenvalues are positive.
3. Let A_d be the top left $d \times d$ minor of A. Show that if A is positive definite, so is each A_d, $1 \leq d \leq n$.
4. Prove that A is positive definite if and only if $\det(A_d) > 0$ for all $1 \leq d \leq n$.

You may use without proof that a real symmetric matrix has an orthonormal basis of eigenvectors. A hint for the last part is available on the office door, but as always, try it without first!

hw 5 40.75/46

Problem A: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$F(x, y, z) = (\exp(x^2 + 2y^2), \sin(z^2 - y^2) \cdot (x^2 + 2z^2), (x^2 + y^2 + z^2)^9)$.
Explain why F is differentiable, and then prove why $DF(x, y, z)$ always has zero determinant. You may not actually compute any derivatives your solution.

① $F_1(x,y,z) = \exp(x^2+2y^2)$

let $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ and $f: \mathbb{R}^3 \mapsto \mathbb{R}$, $(x,y,z) \mapsto x^2+2y^2$

g is exponential, in $C^\infty(\mathbb{R})$; f is polynomial, in $C^\infty(\mathbb{R}^3)$

So $F_1 = g \circ f \in C^\infty(\mathbb{R}^3)$

(note: f is C^r and g is $C^r \Rightarrow g\circ f$ is C^r, by applying chain rule and product rule recursively.)

② $F_2(x,y,z) = (\sin(z^2-y^2))(x^2+2z^2)$

let $g: \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$, $x \mapsto \sin x$ (trig), $f: \mathbb{R}^3 \to \mathbb{R} \in C^\infty(\mathbb{R}^3)$, $(x,y,z) \mapsto z^2-y^2$ (poly), $h: \mathbb{R}^3 \to \mathbb{R} \in C^\infty(\mathbb{R}^3)$, $(x,y,z) \mapsto x^2+2z^2$ (poly)

So $F_2 = (g\circ f)\cdot h$ is C^∞

③ $F_3(x,y,z) = (x^2+y^2+z^2)^9$

let $g: \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$, $x \mapsto x^9$ (positive power), $f: \mathbb{R}^3 \to \mathbb{R} \in C^\infty(\mathbb{R}^3)$, $(x,y,z) \mapsto x^2+y^2+z^2$ (poly)

So $F_3 = g \circ f$ is C^∞

Thus all entries of the Jacobian matrix of F are in C^∞, thus F is differentiable.

Let $F_1: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} x\\y\\z \end{pmatrix} \mapsto \begin{pmatrix} x^2+2y^2 \\ x^2+2z^2 \\ z^2-y^2 \end{pmatrix}$ $F_2: \mathbb{R}^3 \to \mathbb{R}^3$, $\begin{pmatrix} a\\b\\c \end{pmatrix} \mapsto \begin{pmatrix} \exp(a) \\ b\sin(c) \\ (\frac{1}{2}a+\frac{1}{2}b)^9 \end{pmatrix}$

note: $F = F_2 \circ F_1$

Thus by the chain rule: $\forall a \in \mathbb{R}^3$, $DF(a) = DF_2(F_1(a)) \cdot DF_1(a)$

So $\det(DF(a)) = \det(DF_2(F_1(a)))\det(DF_1(a))$

note that $\forall a = \begin{pmatrix} x\\y\\z \end{pmatrix} \in \mathbb{R}^3$, $DF_1(a) = \begin{pmatrix} 2x & 4y & 0 \\ 2x & 0 & 4z \\ 0 & -2y & 2z \end{pmatrix}$

where $row_3 = \frac{1}{2}(row_2 - row_1) \Rightarrow$ linearly dependent $\Rightarrow$ row rank <3

$\Rightarrow \det(DF_1(a)) = 0$

This finishes the proof the $\forall a \in \mathbb{R}^3$, $\det(DF(a)) = 0$ $\square$

Problem B: Suppose
$$F : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$$
and
$$G : B \subset \mathbb{R}^m \to A \subset \mathbb{R}^n$$
are both differentiable and are inverses of each other (A, B open). Show that $n = m$ and that, for all pairs $a \in A, b \in B$ with $F(a) = b$
$$DG(b) = DF(a)^{-1}.$$

Pf WLOG suppose $n<m$

Then take arbitrary $x \in A$

Since F, G are differentiable and inverse of each other,

we have $GF(x) = x \Rightarrow D(GF)(x) = I_n$

$\Rightarrow DG(F(x))DF(x) = I_n$ by chain rule ①

Similary we have:

$\forall y \in B$, $FG(y) = y \Rightarrow D(FG)(y) = I_m$

$\Rightarrow DF(G(y))DG(y) = I_m$ by chain rule

By taking $y = F(x) \Rightarrow DF(x)DG(F(x)) = I_m$ ②

Claim: if matrix $AB = I_m$ and $BA = I_n$ then we must have $m=n$ and $A = B^{-1}$.

Pf of claim $AB = I_m \Rightarrow m = \text{rank}(AB) \le \text{rank}(A) \le \min\{n,m\}$

$BA = I_n \Rightarrow n = \text{rank}(BA) \le \text{rank}(B) \le \min\{m,n\}$

if $m \ge n \Rightarrow$ must have $m \le n$; if $m \le n \Rightarrow$ must have $m \ge n$

Therefore $m = n$

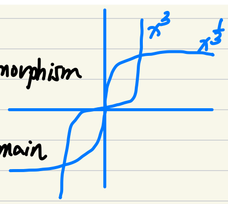
By claim, we have $m=n$ and $DF(x)^{-1} = DG(F(x))$

Since x is taken arbitrary, this proves that $\forall a \in A, b \in B$ s.t. $F(a) = b$ we have $DF(a)^{-1} = DG(b)$ $\square$

Problem C: Give an example of a differentiable homeomorphism from $\mathbb{R}$ to itself whose inverse is not differentiable at every point. (For your example, you need to find only a single point where the inverse isn't differentiable.)

ex $f: \mathbb{R} \to \mathbb{R}$

sending $x \mapsto x^3$ is a differentiable homeomorphism since it is invertible and differentiable on the whole domain



its inverse: $f^{-1}: x \mapsto \sqrt[3]{x}$ is not differentiable at $x=0$

Since $\frac{d}{dx}f^{-1}(x) = \frac{1}{3\sqrt[3]{x^2}}$ does not exist at $x=0$

Problem D: Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin. Show
$$\lim_{h\to0}\lim_{k\to0} F(h, k) = \lim_{k\to0}\lim_{h\to0} F(h, k),$$
assuming all these limits exit. Give an example where F is not continuous, both double limits exist, but the two double limits are not equal.

Pf By continuity at the origin we have $\lim_{\sqrt{h^2+k^2}\to 0} F(h,k) = F(0,0)$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall \sqrt{h^2+k^2} < \delta$, $|F(h,k) - F(0,0)| < \varepsilon$

Let $\varphi_1(h) = \lim_{k\to 0} F(h,k)$, $h \in \mathbb{R}$

Claim 1 $\varphi_1(0) = \lim_{k\to0} F(0,k) = F(0,0)$

let $\varepsilon > 0$. By continuity of F at origin,

$\exists \delta > 0$ s.t. $\forall \sqrt{h^2+k^2} < \delta$, $|F(h,k) - F(0,0)| < \varepsilon$

take the same $\delta \Rightarrow |F(0,k) - F(0,0)| < \varepsilon$ for all $k \in B_0(\delta)$

Thus $\varphi_1(0) = F(0,0)$

Claim 2 $\lim_{h\to 0}\varphi_1(h) = \varphi_1(0) = F(0,0)$

(i.e. φ_1 is continuous at $h=0$)

Let $\varepsilon>0$. By continuity of F at origin,

take $\delta>0$ s.t. $\forall \sqrt{h^2+k^2}<\delta$, $|F(h,k)-F(0,0)|<\varepsilon$ ①

WTS: $\exists \delta>0$ s.t. $|\varphi_1(h)-\varphi_1(0)|<\varepsilon$ for all $|h|<\delta$

Consider $\delta_2=\frac{\delta}{\sqrt{2}}$. Let $|h|<\delta_2$

$\Rightarrow |\varphi_1(h)-\varphi(0)| = \left|\lim_{k\to 0}F(h,k)-F(0,0)\right|<\varepsilon$

since $\forall |k|<\delta_2$, we always have $\sqrt{h^2+k^2}<\delta$

thus $F(h,k)\in B_\varepsilon(F(0,0)) \Rightarrow \lim_{k\to 0}F(h,k)\in B_\varepsilon(F(0,0))$

(since the limit exists, it is bounded by all values of $F(h,k)$ near $(0,0)$)

$\Rightarrow \left|\lim_{k\to 0}F(h,k)-F(0,0)\right|<\varepsilon$

that is, $|\varphi_1(h)-\varphi_1(0)|<\varepsilon$

This proves that $\lim_{h\to 0}\varphi_1(h)=\varphi_1(0)$

i.e. $\lim_{h\to 0}\lim_{k\to 0}F(h,k)=F(0,0)$

By taking $\varphi_2(k)=\lim_{h\to 0}F(h,k)$, $k\in\mathbb{R}$

we can dually prove that $\varphi_2(0)=F(0,0)$

and $\lim_{k\to 0}\varphi_2(k)=\varphi_2(0)$

thus $\lim_{k\to 0}\lim_{h\to 0}F(h,k)=F(0,0)$

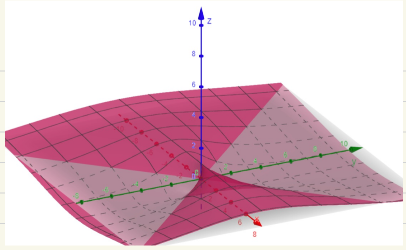
This finishes the proof that $\lim_{k\to 0}\lim_{h\to 0}F(h,k)=\lim_{h\to 0}\lim_{k\to 0}F(h,k)$ □

counterexample when f is not continuous:

$$F(h,k)=\begin{cases}\frac{h^2-k^2}{h^2+k^2}, & (h,k)\neq(0,0)\\ 0, & (h,k)=(0,0)\end{cases}$$

$\lim_{h\to 0}\lim_{k\to 0}F(h,k)=\frac{1}{1}=1$

$\lim_{k\to 0}\lim_{h\to 0}F(h,k)=\frac{-1}{1}=-1$



Just for fun (don't hand in): Give an example where $F:\mathbb{R}^2\to\mathbb{R}$ is continuous at the origin but $\lim_{h\to 0}\lim_{k\to 0}F(h,k)$ does not exist.

Just for fun (don't hand in): Also note that for $a_{n,m}=2^{n-m}$ it is not true that

$$\lim_{n\to\infty}\lim_{m\to\infty}a_{n,m}=\lim_{m\to\infty}\lim_{n\to\infty}a_{n,m}.$$

Problem E: If F is a function of 4 variables, how many terms (in general) are in the degree 10 Taylor series of F? (Degree 10 means you use multi-indices α of degree at most 10.) You do not need to show your work; just give the final answer.

0/3

actually: 每个$|\alpha|=k$, 都有$\#\alpha=\binom{k-1}{4-1}$, 即把$k$个位置分给4个vars, 等于在$k+1$个间隙里放3个隔板(一个里可放多个)

$\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$

$|\alpha|\le 10$ $\binom{10+4}{4}$

$\{\alpha \mid \text{feasible}\}=\binom{10+4-1}{4-1}=286$

而$\#\{\alpha \mid |\alpha|\le 10\}=\sum_{k=0}^{10}\binom{k+1+3-1}{3}=\sum_{k=0}^{10}\binom{k+3}{3}=\binom{10+3+1}{3+1}$

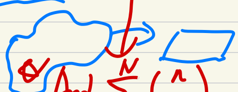
0|00|0|0|0000

$\#\alpha=\binom{|\alpha|+3-1}{3}$

Problem F: Suppose that $F: A\subseteq\mathbb{R}^n\to\mathbb{R}^m$ is differentiable with A open and connected and $Df(a)=0$ for all $a\in A$. Show that F is constant.

Claim 1 for all $a\in A$, F is locally constant on A

($\exists\,\varepsilon>0$ s.t. $f[B_\varepsilon(a)]$ is const.)

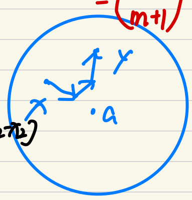


And $\sum_{n=0}^{N}\binom{n}{m}=\binom{N+1}{m+1}$

Pf of claim 1

Let $a\in A$. Take $\varepsilon>0$ s.t. $B_\varepsilon(a)\subseteq A$ (by openness)

Let $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}, y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}\in B_\varepsilon(a)$.



Let $S_0=x$, $\varphi_0: t\mapsto F(S_0+te_1)$, $t\in[0,y_1-x_1]$

$S_1=x+(y_1-x_1)$, $\varphi_1: t\mapsto F(S_1+te_2)$, $t\in[0,y_2-x_2]$

$\vdots$

$S_n=x+(y_1-x_1)+(y_2-x_2)+\dots+(y_n-x_n)=y$

Since for all points $c\in A$, all entries of $Df(c)$ are 0,

every partial is constant 0 (thus continuous) on A

Note: For each φ_i, $0\le i\le n-1$, we have $\varphi_i'(t)=\frac{\partial F}{\partial x_i}(S_i+te_{i+1})=0$ for all $t\in[0,y_{i+1}-x_{i+1}]$

$\Rightarrow \forall i=1,\dots,n-1$, $F(S_i)-F(S_{i-1})=(y_i-x_i)\varphi_{i-1}'(t)=0$ by MVT

Thus $F(y)-F(x)=\sum_{i=1}^{n}\left(F(S_i)-F(S_{i-1})\right)=0$

$\Rightarrow F(y)=F(x)$

Since x,y are arbitrary, we have proved that $\forall x,y\in B_\varepsilon(a)$, $F(x)=F(y)$

Therefore F is locally constant around a.

Claim 2 $S=\{x: F(x)=F(a)\}$ is both closed and open in A

Pf of Claim 2

S is open since ($\forall a\in S$, $\exists\,\varepsilon>0$ s.t. $B_\varepsilon(a)\subseteq S$) proved in claim 1

Let (a_n) be a sequence in S s.t. $(a_n)\to x$ for some $x\in A$

Since F is diff'ble thus ctn, $F(x)=\lim_{n\to\infty}F(a_n)=F(a)\Rightarrow x\in S$

Thus S is closed

The fact that A is connected implies that the only set both open and closed in A is A itself.

$\Rightarrow S=A$.

$\Rightarrow \forall x\in A$, $F(x)=F(a)$, which shows that F is constant. □

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Problem G: Let $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ be C^k. Show that

$$\partial^k(f_1 \cdots f_m) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^{\alpha_1} f_1 \cdots \partial^{\alpha_m} f_m.$$

Pf We prove by induction on $k \in \mathbb{N}$

Base case: $k=1$, $\partial(f_1 \ldots f_m) = \partial(f_1 \cdot (f_2 \cdots f_m))$

$= (f_2 \cdots f_m)\partial f_1 + f_1 \partial(f_2 \cdots f_m)$ by product rule

$= (f_2 \cdots f_m)\partial f_1 + (f_1 f_3 \cdots f_m)\partial f_2 + f_1 f_2 \partial(f_3 \ldots f_m)$

$= \left(\sum_{i=1}^{k} \partial f_i \prod_{j \neq i} f_j\right) + \left(\prod_{i=1}^{k} f_i\right)\left(\partial \prod_{i=k+1}^{m}\right)$

$= \sum_{i=1}^{m} \left(\partial f_i \prod_{j\neq i} f_j\right)$

$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} \partial^{\alpha_1} f_1 \ldots \partial^{\alpha_m} f_m$, statement holds true

Inductive step: Suppose the equality holds for $1, 2, \ldots, k$

Then $\partial^{k+1}(f_1 f_2 \ldots f_m) = \partial(\partial^k(f_1 f_2 \ldots f_m))$

$= \partial\left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^{\alpha_1} f_1 \ldots \partial^{\alpha_m} f_m\right)$

$= \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial(\partial^{\alpha_1} f_1 \ldots \partial^{\alpha_m} f_m)$ $= \sum_{|\beta|=1} \partial^{\beta_1}(\partial^{\alpha_1} f_1) \ldots \partial^{\beta_m}(\partial^{\alpha_m} f_m)$

$= \sum_{|\alpha|=k} \sum_{|\beta|=1} \frac{k!}{\alpha!} (\partial^{\beta_1+\alpha_1} f_1) \ldots (\partial^{\beta_m+\alpha_m} f_m)$ $= \sum_{|\beta|=1} \partial^{\beta_1+\alpha_1} f_1 \cdots \partial^{\beta_m+\alpha_m} f_m$ by base case

$= \sum_{|\alpha|=k} \frac{k!}{\alpha!}\left(\sum_{i=1}^{m} \partial^{\alpha_1} f_1 \cdots \partial^{\alpha_i+1} f_i \ldots \partial^{\alpha_m} f_m\right)$

every term correspond to a multi-index β s.t.

for some $j \in \{1, \ldots, m\}$, $\beta_j = \alpha_j + 1$ while for $i \neq j$, $\beta_i = \alpha_i$

So each β has $|\beta| = k+1$

Thus $\partial^{k+1}(f_1 \ldots f_m) = \sum_{|\beta|=k+1} (\text{coeff})\, \partial^{\beta_1} f_1 \ldots \partial^{\beta_m} f_m$

The (coeff) $= \sum_i$ coeff of $(\partial^{\beta_1} f_1 \ldots \partial^{\beta_i - 1} f_i \ldots \partial^{\beta_m} f_m)$

$= \sum_{i=1}^{m} \frac{k!}{\beta_1! \ldots (\beta_i - 1)! \ldots \beta_m!}$

$= \sum_{i=1}^{m} \frac{k!\beta_i}{\beta!} = \frac{k!(k+1)}{\beta!} = \frac{(k+1)!}{\beta!}$

Therefore the expression simplifies to:

$$\partial^{k+1}\left(\prod_{i=1}^{m} f_i\right) = \sum_{|\beta|=k+1} \left(\frac{(k+1)!}{\beta!} \prod_{i=1}^{m} \partial^{\beta_i} f_i\right)$$

This finishes the proof by induction. □

3/5

Problem H: Let $f : \mathbb{R}^n \to \mathbb{R}$ be in C^{k+1}. Show that the Taylor polynomial of degree k centered at $x_0 \in \mathbb{R}^n$ is the best polynomial approximation of $f(x)$ near x_0 in the following sense: Suppose that $P(x)$ is a polynomial of degree k. Then

$$P(x) - f(x) = o(|x - x_0|^k)$$

if and only if P is the Taylor polynomial of degree k centered at x_0. (Recall that a quantity Q is $o(|x - x_0|^k)$ if $\lim_{x \to x_0} \frac{Q}{|x-x_0|^k} = 0$.)

Pf Backward direction

Let $f: \mathbb{R}^n \to \mathbb{R}$ be in C^{k+1}

Let $T_k : \mathbb{R}^n \to \mathbb{R}$ be the Taylor polynomial of degree k centered at x_0

i.e. $T_k(x) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha$, $x \in \mathbb{R}^n$

Then by Taylor's Theorem we have

$T_k(x) - f(x) = R_{x_0,k}(x) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f}{\alpha!}(c)(x - x_0)^\alpha$

for some c on the line segment connecting c, x.

note that $\{\alpha \mid |\alpha| = k+1\}$ is finite, $\frac{\partial^\alpha f}{\alpha!}(c)$ is constant for all α

It suffices to show that $T_k(x) - f(x)$ is $o(|x - x_0|^k)$ by

showing that for any α s.t. $|\alpha| = k+1$,

$\lim_{x \to x_0} \frac{(x - x_0)^\alpha}{\|x - x_0\|^k} = 0$, i.e. $\lim_{x \to 0} \frac{x^\alpha}{\|x\|^k} = 0$

note: $\forall x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} = |x_1|^{\alpha_1} \ldots |x_n|^{\alpha_n} \leq \|x\|^{\alpha_1 + \ldots + \alpha_n} = \|x\|^{k+1}$

So $\lim_{x \to 0} \frac{x^\alpha}{\|x\|^k} \leq \lim_{x \to 0} \|x\| = 0$ □

Forward Direction

let $P(x)$ be a polynomial of degree k that is not T_k

$f(x) - P(x) = C_1 x^{\alpha^{(1)}} + \ldots + C_m x^{\alpha^{(m)}}$ for some

constants $C_1, \ldots, C_m$ and multi-index $\alpha^{(1)}, \ldots, \alpha^{(m)}$ s.t. $|\alpha^{(i)}| \leq k$ for each i

WTS: $\lim_{x \to 0} \frac{C_1 x^{\alpha^{(1)}} + \ldots + C_m x^{\alpha^{(m)}}}{|x|^k} \neq 0$

Case 1 $\sum_{i=1}^{m} C_m \neq 0$

Consider the sequence $(t_n = \frac{1}{n})_{n \in \mathbb{N}}$ in $\mathbb{R}$

for each t_n, let $x_n = (t_n, t_n, \ldots, t_n)$

Then $(x_n) \to 0$ in $\mathbb{R}^d$

Here $\sum_{i=1}^{m} C_i x^{\alpha^{(i)}} = \sum_{i=1}^{m} C_i t^n = t^n \sum_{i=1}^{m} C_i$

$\Rightarrow \frac{C_1 x^{\alpha^{(1)}} + \ldots + C_m x^{\alpha^{(m)}}}{|x|^k} = \frac{\sum_{i=1}^{m} C_i t^d}{\sqrt{d}\, t^d} = \frac{\sum_{i=1}^{m} C_i}{\sqrt{d}}$ is constant while $x \to 0$

This suffices to show that $\lim_{x \to 0} \frac{C_1 x^{\alpha^{(1)}} + \ldots + C_m x^{\alpha^{(m)}}}{|x|^k} \neq 0$

Case 2 $\sum_{i=1}^{m} C_m = 0$

idk :(

Try another way to prove the forward direction:

Claim a polynomial homogeneous of degree k in $\mathbb{R}^d$ is not $o(|x|^k)$

(by homogeneous of degree k we mean: $\forall t \in \mathbb{R}, x \in \mathbb{R}^n$, $P(tx) = t^k P(x)$)

Let $P(x)$ be a polynomial homogeneous of degree k in $\mathbb{R}^d$

let $x_0 \in \mathbb{R}$, $(t_n = \frac{1}{n})_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$

So $((t_n x_0, \ldots, t_n x_0))_{n \in \mathbb{N}} \to 0$ in $\mathbb{R}^d$

Denote each term of this seq as x_n

$\Rightarrow \frac{P(x_n)}{|x_n|^k} = \frac{t^k P(x_0)}{t^k |x_0|^k} = \frac{P(x_0)}{|x_0|^k}$ is const.

This implies that $\lim_{|x| \to 0} \frac{P(x)}{|x|^k} \neq 0$

Note that a polynomial of degree k is a homogeneous of degree k.

Problem I: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x^2 + y^2)$, so F is differentiable.

(1) Prove

4.75/5 (minor mistake)

$$x\frac{\partial F}{\partial y} = y\frac{\partial F}{\partial x}.$$

(2) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are differentiable. Define $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ via

$$\phi(x, y, z) = (f(h(x), g(x, y), z), g(y, z)).$$

Find a formula for the matrix of the derivative $D\phi(p) : \mathbb{R}^3 \to \mathbb{R}^2$ at an arbitrary point $p \in \mathbb{R}^3$ in terms of the partial derivatives of f, g, and h at p.

(3) In (ii), compute $D\phi(1, 1, 1)$ when $f(x, y, z) = x^2 + yz$, $g(x, y) = y^3 + xy$, and $h(x) = e^x$. Do this in two ways: using your general formula in (ii) and also by explicitly computing ϕ in this case and directly computing the Jacobian matrix from this.

(1) Pf Let $(x,y)\in\mathbb{R}^2$

$$\frac{\partial F}{\partial x} = 2x\,Df(x^2+y^2) \text{ by chain rule}$$

$$\frac{\partial F}{\partial y} = 2y\,Df(x^2+y^2) \text{ by chain rule}$$

$$\Rightarrow y\frac{\partial F}{\partial x} = x\frac{\partial F}{\partial y} = 2xyDf(x^2+y^2)$$

(2) $\varphi: \mathbb{R}^3 \to \mathbb{R}^2$

Let $\varphi_m: \mathbb{R}^3 \to \mathbb{R}^4$ map $\begin{pmatrix} x\\ y\\ z\end{pmatrix} \mapsto \begin{pmatrix} h(x)\\ g(x,y)\\ z\\ g(y,z)\end{pmatrix}$

$\varphi_n: \mathbb{R}^4 \to \mathbb{R}^2$ map $\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} \mapsto \begin{pmatrix} f(a,b,c)\\ d\end{pmatrix}$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "\varphi"] \arrow[dr, "\varphi_m"'] & \mathbb{R} \\ & \mathbb{R}^4 \arrow[u, "\varphi_n"'] \end{tikzcd}$$

Then $\varphi = \varphi_n \circ \varphi_m$

$$\varphi: p = \begin{pmatrix} x\\ y\\ z\end{pmatrix} \xmapsto{\varphi_m} \begin{pmatrix} a=h(x)\\ b=g(x,y)\\ c=z\\ d=g(y,z)\end{pmatrix} \xmapsto{\varphi_n} \begin{pmatrix} f(a,b,c)\\ d\end{pmatrix}$$

$\Rightarrow D\varphi(p) = D\varphi_n(\varphi_m(p))\,D\varphi_m(p)$ by chain rule.

$$= \begin{pmatrix} \frac{\partial f}{\partial x_1}(h(p_1), g(p_1,p_2), p_3) & \frac{\partial f}{\partial x_2}(h(p_1), g(p_1,p_2), p_3) & \frac{\partial f}{\partial x_3}(h(p_1), g(p_1,p_2), p_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h'(p_1) & 0 & 0 \\ \frac{\partial g}{\partial x_1}(p_1,p_2) & \frac{\partial g}{\partial x_2}(p_1,p_2) & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\partial g}{\partial x_1}(p_2,p_3) & \frac{\partial g}{\partial x_2}(p_2,p_3) \end{pmatrix}$$

(3) $p = \begin{pmatrix} 1\\ 1\\ 1\end{pmatrix}$, $f(x,y,z) = x^2+yz$, $g(x,y) = y^3+xy$, $h(x) = e^x$

$h(p_1) = e$, $g(p_1,p_2) = 2$, $\frac{\partial f}{\partial x_1}(x,y,z) = 2x$, $\frac{\partial f}{\partial x_2}(x,y,z) = z$, $\frac{\partial f}{\partial x_3}(x,y,z) = y$

$h'(x) = e^x$, $\frac{\partial g}{\partial x_1}(x,y) = y$, $\frac{\partial g}{\partial x_2}(x,y) = 3y^2+y$

By the formula in (2),

$$D\varphi(1,1,1) = \begin{pmatrix} 2e & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2e^2+\overset{1}{\cancel{2}} & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

Compute directly:

$$\varphi(x,y,z) = \begin{pmatrix} e^{2x} + z(y^3+xy) \\ z^3+yz \end{pmatrix}$$

$$D\varphi(1,1,1) = \begin{pmatrix} 2\times 1\, e^{2\cdot 1} + 1\cdot 2 & 3\cdot 1 + 1\cdot 1 & 1^3 + 1\cdot 1 \\ 0 & 1 & 3\cdot 1 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^2+2 & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

same result. :)

Problem J: Problem 2(a) on page 63 of the text.

Problem $f: \mathbb{R}^2 \to \mathbb{R}^3$ $\begin{pmatrix} x_1\\ x_2\end{pmatrix} \mapsto \begin{pmatrix} e^{2x_1+x_2} \\ 3x_2 - \cos x_1 \\ x_1^2+x_2+2 \end{pmatrix}$ $g: \mathbb{R}^3 \to \mathbb{R}^2$ $\begin{pmatrix} y_1\\ y_2\\ y_3\end{pmatrix} \mapsto \begin{pmatrix} 3y_1+2y_2+y_3^2 \\ y_1^2 - y_3 + 1 \end{pmatrix}$

$F(x) = g\circ f(x)$. Find $DF(0)$

Sol $DF(0) = Dg(f(0))\,Df(0)$ by chain rule

$$Df = \begin{pmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ \sin x_1 & 3 \\ 2x_1 & 1 \end{pmatrix} \qquad Dg = \begin{pmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{pmatrix}$$

$$f(0) = \begin{pmatrix} e^0 = 1 \\ 0 - \cos 0 = -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow DF(0) = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 13 \\ 4 & 1 \end{pmatrix}$$

Problem K: Find the 3rd order Taylor series of $F(x, y) = e^{x+y^2}$ about the origin.

($F(x,y) = e^{x+y^2}$ is in C^∞ so we can do this)

$$T_3(x,y) = \sum_{|\alpha|\le 3} \frac{\partial^\alpha f(0)}{\alpha!}(x,y)^\alpha \qquad \frac{\partial}{\partial y}f \quad 2ye^{x+y}$$

possible α: $|\alpha| = 0 : (0,0) \Rightarrow f(0,0) = e^0 = 1$

$|\alpha| = 1 : (0,1), (1,0) \Rightarrow \partial^{(0,1)}f(0,0) = 2ye^{x+y^2}\big|_{(0,0)} = 0$

$\partial^{(1,0)}f(0,0) = e^{x+y^2}\big|_{(0,0)} = 1$

$|\alpha| = 2 : (0,2), (2,0), (1,1)$

$\Rightarrow \partial^{(0,2)}f(0,0) = 2e^{x+y^2} + 4y^2e^{x+y^2}\big|_{(0,0)} = 2$

$\partial^{(2,0)}f(0,0) = e^{x+y^2}\big|_{(0,0)} = 1$

$\partial^{(1,1)}f(0,0) = 2ye^{x+y^2}\big|_{(0,0)} = 0$

$|\alpha| = 3 : (1,2), (2,1), (3,0), (0,3)$

$\Rightarrow \partial^{(1,2)}f(0,0) = 2e^{x+y^2} + 4y^2e^{x+y^2}\big|_{(0,0)} = 2$

$\partial^{(2,1)}f(0,0) = 2ye^{x+y^2}\big|_{(0,0)} = 0$

$\partial^{(3,0)}f(0,0) = e^{x+y^2}\big|_{(0,0)} = 1$

$\partial^{(0,3)}f(0,0) = 4ye^{x+y^2} + 8ye^{x+y^2} + 8y^3e^{x+y^2}\big|_{(0,0)} = 0$

Thus $T_3(x,y) = 1 + x + \frac{1}{2}x^2 + y^2 + xy^2 + \frac{x^3}{6}$

6.5/8

Bonus: A real symmetric n by n matrix A is called positive definite if $x^T A x > 0$ for all $x \in \mathbb{R}^n$.

(1) Show that a real symmetric matrix is positive definite if and only if it is invertible and, for all non-zero x, the angle between Ax and x is less than 90 degrees.

(2) Show that a real symmetric matrix is positive definite if and only if all its eigenvalues are positive.

(3) Let A_d be the top left d by d minor of A. Show that if A is positive definite, so is each $A_d, 1 \le d \le n$.

(4) Prove that A is positive definite if and only if $\det(A_d) > 0$ for all $1 \le d \le n$.

You may use without proof that a real symmetric matrix has an orthonormal basis of eigenvectors. A hint for the last part is available on the office door, but as always try it without first!

(1) Let $A \in \mathbb{R}^{n\times n}$ be positive define

Suppose A is not invertible $\Rightarrow \exists x \in \mathbb{R}^n$ s.t. $Ax = 0$

Thus A is invertible. $\qquad \Rightarrow x^T A x = 0$, contradicts

let $x \in \mathbb{R}^n$, θ be the angle between Ax and x

$$\Rightarrow \cos\theta = \frac{x \cdot Ax}{\|x\|\,\|Ax\|} = \frac{x^T A x}{\|x\|\,\|Ax\|} > 0 \Rightarrow \theta \in (0, \tfrac{\pi}{2})$$

Then we prove the backward direction.

let A be invertible with $\forall x \in \mathbb{R}^n$,

angle between Ax and x $\theta \in (0, \frac{\pi}{2})$

let $x \in \mathbb{R}^n$, we have $\frac{x \cdot Ax}{\|x\|\,\|Ax\|} = \frac{x^T A x}{\|x\|\,\|Ax\|} > 0 \Rightarrow x^T A x > 0$

This proves the iff statement

(2) Suppose $A \in \mathbb{R}^{n\times n}$ is positive define

let λ be an eigenvalue of it.

$\Rightarrow$ for some $x \in \mathbb{R}^n$, $Ax = \lambda x \Rightarrow x^T A x = \lambda x^T x = \lambda \|x\|^2 > 0$

$\Rightarrow \lambda > 0$

This proves the forward direction.

For the backward direction:

Suppose all eigenvalues of A are positive.

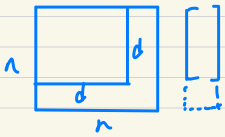
Since A is real symmetric it has an orthonormal eigenbasis $\{b_1, \dots, b_n\}$

So for any $x \in \mathbb{R}^n$, $x = \sum_{i=1}^{n} c_i b_i$ for some $c_1, \dots, c_n \in \mathbb{R}$

$$\Rightarrow x^T A x = \sum_{i=1}^{n} \lambda_i b_i^2 > 0 \text{ since each } \lambda_i > 0$$

This proves the proof. $\square$

(3) Suppose $A \in \mathbb{R}^{n\times n}$ is positive define.



let $1 \le d \le n$, $x \in \mathbb{R}^d$.

Take $\tilde{x} = (x, \underbrace{0, 0, \dots, 0}_{n-d}) \in \mathbb{R}^n$

By A being positive define, we have $\tilde{x}^T A \tilde{x} > 0$

Note that
$$\tilde{x}^T A \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_d \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \mathrm{row}(A)_1 \cdot \tilde{x} \\ \vdots \\ \mathrm{row}(A)_d \cdot \tilde{x} \\ \vdots \end{bmatrix} = \sum_{i=1}^{d} \tilde{x}_i (\mathrm{row}(A)_i \cdot \tilde{x})$$
$$= \sum_{i=1}^{d} x_i (\mathrm{row}(A_d)_i \cdot x) = x^T A_d x > 0$$

Therefore A_d is positive define, $1 \le d \le n$ $\square$

(4) Pf

Forward direction:

Suppose A is positive define $\Rightarrow \forall\, 1 \le d \le n$, A_d is positive define,

By (2), each A_d has its all eigenvalues positive

So $\det(A_d) = \prod \lambda_{d_i} > 0$.

Backward direction:

.... (dirty work at all)

$a_{ij} = \mathrm{row}_i \cdot \mathrm{row}_j^T$

$\sum_{r=1}^{n} l_{ir} l_{jr}$

$a_{ij} = \sum_{r=1}^{n} l_{ri} l_{rj}$

$\sqrt{a_{11}}$

$r_1 \cdot r_1 \quad r_1 \cdot r_2 \quad r_1 \cdot r_j$

$r_2 \cdot r_2 \quad r_2 r_j$

r_j

$a_{11} = l_{11}^2$

$a_{21} = l_{21} l_{11} + l_{22} l_{21}$

$l_{21}(l_{11} + l_{22})$

$a_{31} = l_{31} l_{11} + l$

$\begin{pmatrix} l_{11} & 0 \\ l_{21} & \end{pmatrix}$

HW 6 on IVT

6.1 MVT does not hold for $f : \mathbb{R} \to \mathbb{R}^n$

Let $f : [a, b] \to \mathbb{R}^n$ be continuous on the closed interval $[a, b] \subset \mathbb{R}$ and differentiable on (a, b).

1. Show that there is a $c \in (a, b)$ such that

$$|f(a) - f(b)| \leq |f'(c)| \cdot |a - b|.$$

2. Give an example when $n = 2$ to show that it is possible the inequality is strict for all $c \in (a, b)$. (In particular, the Mean Value Theorem does not hold for f.)

6B Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the equation

$$f(x, y) = (e^x \cos y, e^x \sin y).$$

1. Show that f is one-to-one on the set $A = \{(x, y) : x \in \mathbb{R}, 0 < y < 2\pi\}$.
2. What is $B = f(A)$?
3. If g is the inverse function of f restricted to A, find $Dg(0, 1)$.
4. What is $f(\mathbb{R}^2)$?
5. Show that the Jacobian matrix of f is nonsingular for any $(x, y) \in \mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood on which f is one-to-one. Nonetheless, show that f is not one-to-one on $\mathbb{R}^2$.
6. Find an explicit formula for the inverse function g of f in the neighborhood of $(0, 1)$. Use this formula to check your answer in part (3).

6.2 仅限 single var: 处处 locally invertible 则 globally invertible

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and locally invertible. Show that the image of f is open, and that a global inverse for f exists defined on the image.

Remark: The analogue of Problem C is false in higher dimensions. You should pause to note what your solution uses that wouldn’t be available in higher dimensions.

Just for fun (don’t hand in): Give an example of a continuous $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is locally invertible but not injective.

6.3 real-valued function Taylor series 收敛于自身的条件

Say that $f : \mathbb{R}^m \to \mathbb{R}$ is C^∞, and there is a number $W > 0$ such that for all $x \in B_r(0)$ and all α, we have

$$|\partial^\alpha f(x)| \leq W|\alpha|.$$

Show that

$$\lim_{k\to\infty} \sum_{|\alpha|\leq k} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha = f(x)$$

for $x \in B_r(0)$. Show also that the infinite series

$$\sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha$$

converges absolutely, so that without any ambiguity we can write

$$f(x) = \sum_\alpha \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha.$$

Just for fun (don' t hand in): $|\partial^\alpha f(x)| \leq W|\alpha|$ is not optimal. Can you phrase a natural assumption that is closer to optimal?

6.4 local min $\implies$ singular

If $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ is C^1 and has a local minimum at x, prove $Df(x) = 0$. (Points x with $Df(x) = 0$ are called critical points.)

6.5 正定矩阵与 Hessian

Let $f : A \to \mathbb{R}$ be a C^2 function, $A \subset \mathbb{R}^n$ open, and let $x \in A$. The Hessian $Hf(x)$ of f at x is the n-by-n symmetric matrix with entry (j, k) equal to $\partial e_i + e_k f(x)$.

A symmetric matrix S is called positive definite if $x^T Sx > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$, in which case we write $S > 0$. If the same condition holds with a non-strict inequality $x^T Sx \geq 0$, we say S is positive semi-definite and write $S \geq 0$. The negative of a positive (semi-)definite matrix is called negative (semi-)definite, and we similarly write $S < 0$ or $S \leq 0$.

Assume that A is convex, and $Df(x_0) = 0$. Prove that:

1. if $Hf(x) \geq 0$ for all $x \in A$, then $f(x) \geq f(x_0)$ for all $x \in A$.
2. if $Hf(x) > 0$ for all $x \in A$, then $f(x) > f(x_0)$ for all $x \in A$.
3. if $Hf(x) \leq 0$ for all $x \in A$, then $f(x) \leq f(x_0)$ for all $x \in A$.
4. if $Hf(x) < 0$ for all $x \in A$, then $f(x) < f(x_0)$ for all $x \in A$.
5. if $Hf(x_0) \ngeq 0$, then f does not have a local minimum at x_0.
6. if $Hf(x_0) \nleq 0$, then f does not have a local maximum at x_0.

You only need to submit your proofs for parts (1) and (5); you don' t have to write up the rest.

6.6 Bonus: Matrix exponential

Let $M_n(\mathbb{R})$ denote the set of n-by-n real matrices. It has a topology and metric by identifying with $\mathbb{R}^{n^2}$ using the entries of the metric.

1. For any $A \in M_n(\mathbb{R})$, show that

$$\lim_{K\to\infty} \sum_{k=0}^{K} \frac{A^k}{k!}$$

exists. This limit is denoted

$$\exp(X) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

2. Compute $\exp$ of the following matrices:

$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}, \quad \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}.$$

3. Show that $\exp(A+B) = \exp(A)\exp(B)$ when A and B commute.
4. Prove that $\exp$ is differentiable at the origin and compute its derivative there.
5. Is $\exp$ surjective?
6. Is $\exp$ injective?

HW 7 on IFT and Implicit Function Thm

For hints, see office door. But try without the hints first. There are no IBL problems this week.

7.1 directional derivative bounded by the norm of gradient

Suppose $A \subset \mathbb{R}^n$ is open and $f : A \to \mathbb{R}$ is differentiable at $x \in A$. Show that if u is a unit vector in $\mathbb{R}^n$, then

$$D_u f(x) \leq |Df(x)|,$$

with equality if and only if $u = \frac{Df(x)}{|Df(x)|}$. Show that $D_u f(x) = 0$ if and only if u is orthogonal to $Df(x)$.

Remark: Keeping in mind that for functions to $\mathbb{R}$, the Jacobian matrix is also called the gradient, this shows that the gradient is the direction of fastest change of the function. Similarly, the set of directions where the function does not change (to first order) is the perpendicular space to the gradient.

7.2 Lagrange Multiplier

Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ and $c : U \to \mathbb{R}$ are C^1. Set

$$M = c^{-1}(0).$$

Assume that f restricted to M has a local minimum at p, and that Dc is surjective at p. Then prove that there exists a real number λ such that

$$Df(p) = \lambda Dc(p).$$

(This means the gradients of f and c are parallel at p. The number λ is called a *Lagrange multiplier*.)

Hints: how that M can be parameterized, by implicit function theorem: M = (x, g(x)) for a $g : \mathbb{R}^{n-1} \to \mathbb{R}$, show that kerDc = ImDg

7C: explaining Lagrange Multiplier

In at most a few sentences, give a non-rigorous, intuitive explanation for Problem B.

7D: applying Lagrange Multiplier

Using Problem B, find the minimum of the function $f(x, y) = 3x + y$ on the unit circle centered at the origin in $\mathbb{R}^2$.

7E: generalizing Lagrange Multiplier

Formulate and prove a generalization of Problem B when c maps to $\mathbb{R}^k$ rather than $\mathbb{R}$. (Whereas Problem B allows you to do certain optimization problems subject to one constraint, this lets you do some optimization problems subject to k constraints. Your generalization will feature numbers $\lambda_1, \ldots, \lambda_k$.)

Remark: You must do Problem B first and then Problem E. You may not reference E in your solution to B.

7.3 所有正定矩阵在对称矩阵中 open

Prove that the set of positive definite matrices is open in the set of $n \times n$ symmetric matrices. (You may not use the bonus from HW5.)

7.4 critical pt + 正定 Hessian $\implies$ strict local min

Suppose $f : A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open. Suppose that x_0 is a critical point of f and the Hessian of f is positive definite at x_0. Prove that x_0 is a strict local minimum for f.

7.5 Cramer's rule

Let A be an invertible $n \times n$ matrix. Let C be its cofactor matrix, so $C_{ij} = (-1)^{i+j} \det(A_{ij})$, where A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A. Prove the following version of Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} C^T,$$

where C^T denotes the transpose of C. (You may use the cofactor expansion of the determinant.)

7.6 Differentiating with dot product

Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable. Show that

$$(f \cdot g)'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t),$$

where $\cdot$ denotes dot product and $f'(t)$ denotes $Df(t)$ (which in this case is a vector).

7.7 Bonus: graph 上方 convex $\Leftrightarrow$ Hessian 正定

Suppose $f : A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open and convex. Show that the region above f, i.e.

$$\{(x, y) \in A \times \mathbb{R} : y \geq f(x)\},$$

is convex if and only if the Hessian of f is positive semi-definite at each point of A.

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Problem A: Suppose $A \subset \mathbb{R}^n$ is open and $f : A \to \mathbb{R}$ is differentiable at $x \in A$. Show that if u is a unit vector in $\mathbb{R}^n$, then

$$D_u f(x) \le |Df(x)|,$$

with equality if and only if $u = Df(x)/|Df(x)|$. Show that $D_u f(x) = 0$ if and only if u is orthogonal to $Df(x)$.

Remark: Keeping in mind that for functions to $\mathbb{R}$ the Jacobian matrix is also called the gradient, this shows that the gradient is the direction of fastest change of the function. Similarly the set of directions where the function does not change (to first order) is the perp space of the gradient.

Pf (1) $D_u f(x) = \underset{\in\mathbb{R}^n}{Df(x)} \cdot \underset{\in\mathbb{R}^n}{u} = \nabla f(x)\cdot u$ since f is diff'ble at x

By Cauchy Schwarz we have $|\nabla f(x)\cdot u| \le |\nabla f(x)|\cdot|u|$

Since $|u|=1$, it shows that $D_u f(x) \le |Df(x)|$

The equality holds true iff $\nabla f(x)$ is parallel to u

i.e. $\exists\, \lambda > 0$ s.t. $u = \lambda \nabla f(x)$

Suppose $u = \lambda\nabla f(x) \Rightarrow \lambda|\nabla f(x)| = 1$, $\lambda = \frac{1}{|\nabla f(x)|}$

Therefore $u = \frac{\nabla f(x)}{|\nabla f(x)|} = \frac{Df(x)}{|Df(x)|}$

(2) WTS: $D_u f(x) = 0$ iff u is orthogonal to $Df(x)$

Suppose $D_u f(x) = 0 \Rightarrow Df(x)\cdot u = 0 \Rightarrow Df(x)$ is orthogonal to u

Suppose u is orthogonal to $Df(x) \Rightarrow D_u f(x) = u \cdot Df(x) = 0$ $\square$

Problem B: Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ and $c : U \to \mathbb{R}$ are C^1. Set

$$M = c^{-1}(0).$$

Assume that f restricted to M has a local minimum at p, and that Dc is surjective at p. Then prove that there exist a real number λ such that

$$Df(p) = \lambda Dc(p).$$

(This means the gradients of f and ~~λ~~ c are parallel at p. The number λ is called a Lagrange multiplier.)

Pf Assume the hypotheses

Since Dc is surjective at p, $Dc(p) \ne 0$ (otherwise $\forall v \in \mathbb{R}^n$, $Dc(p)v = 0$, im $Dc(p) = 0$, can not be surjective)

So at least one entry of

$Dc(p) = \left(\frac{\partial c}{\partial x_1}(p), \dots, \frac{\partial c}{\partial x_n}(p)\right)$ is not 0

WLOG assume $\frac{\partial c}{\partial x_n}(p) \ne 0$. (can always reorder coordinates)

Write c in the form $c(x,y)$ with $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$

let $p = (a,b)$ with $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$

$\Rightarrow 0 \ne \frac{\partial f}{\partial y}(a,b) \in \mathbb{R}$, so by Implicit function theorem

$\exists$ nbh $B_\varepsilon(a) \subseteq \mathbb{R}^{n-1}$ and unique ctn. $g: B_\varepsilon(a) \to \mathbb{R}^n$

s.t. $g(a) = b$ and $c(x, g(x)) = 0$ $\forall x \in B_\varepsilon(a)$

Therefore M can be locally parametrized by g

i.e. $\exists$ some open nbh $U_p \subseteq M$ s.t. $U_p = \{(x, g(x)) \mid x \in B_\varepsilon(p)\}$

Let $h(x) = f(x, g(x))$ be a function from $U_p \to \mathbb{R}$

Since h reaches a local minimum at $x = a$ (where $(x, g(x)) = p$),

we have $\nabla h(a) = 0$

So $\frac{\partial f}{\partial x_i}(p) + \frac{\partial f}{\partial x_n}(p)\cdot\frac{\partial g}{\partial x_i}(a) = 0$ ①

for each $i = 1, \dots, n-1$

differentiating $c(x, g(x)) = 0$, we get

$\frac{\partial c}{\partial x_i}(p) + \frac{\partial c}{\partial x_n}(p)\cdot\frac{\partial g}{\partial x_i}(a) = 0$ for each $1 \le i \le n-1$

So $\frac{\partial g}{\partial x_i}(a) = \frac{-\frac{\partial c}{\partial x_i}(p)}{\frac{\partial c}{\partial x_n}(p)}$ ②

Combining ①② we have $\frac{\partial f}{\partial x_i}(p) - \frac{\partial f}{\partial x_n}(p)\frac{\frac{\partial c}{\partial x_i}(p)}{\frac{\partial c}{\partial x_n}(p)} = 0$ for each $i = 1, \dots, n-1$

So $\frac{\partial f}{\partial x_i}(p) - \frac{\frac{\partial f}{\partial x_n}(p)}{\frac{\partial c}{\partial x_n}(p)}\frac{\partial c}{\partial x_i}(p) = 0$ for each $i = 1, \dots, n-1$

Also when $i = n$, $\frac{\partial f}{\partial x_i}(p) - \frac{\frac{\partial f}{\partial x_n}(p)}{\frac{\partial c}{\partial x_n}(p)}\frac{\partial c}{\partial x_i}(p) = \frac{\partial f}{\partial x_i}(p) - \frac{\partial f}{\partial x_i}(p) = 0$

Note that $\frac{\frac{\partial f}{\partial x_n}(p)}{\frac{\partial c}{\partial x_n}(p)}$ is a constant, we name it λ

Therefore $\forall i = 1, \dots, n$, $\frac{\partial f}{\partial x_i}(p) = \lambda\frac{\partial c}{\partial x_i}(p)$

This finishes the proof that $Df(p) = \lambda Dc(p)$ for some const λ. $\square$

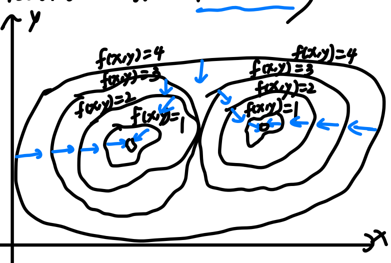
Problem C: In at most a few sentences, give a non-rigorous, intuitive explantion for Problem B.

we view f as a function to minimize

c as a function of contraint to this optimization

(constraint: $c(p) = 0$, we restrict dom(f) to $M = c^{-1}(0)$)

For the minimum pt. p, the gradient of f at p must be normal to all possible directions you can move it. (locally on M)



The gradient of c at p: the direction of the greatest rate of change of c (normal to M) at p

Therefore the gradient of f must be parallel to the gradient of c at p

Problem D: Using Problem B, find the mimum of the function $f(x, y) = 3x + y$ on the unit circle centered at the origin in $\mathbb{R}^2$.

Constraint: $c(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (3, 1)$

$\nabla c = \left(\frac{\partial c}{\partial x}, \frac{\partial c}{\partial y}\right) = (2x, 2y)$

By B. $\nabla f(p) = \lambda\nabla c(p)$ at minimum pt. p for some $\lambda \in \mathbb{R}$

$\Rightarrow 3 = 2\lambda x$, $1 = 2\lambda y \Rightarrow y = \frac{x}{3}$ at p

Take $(x, \frac{x}{3})$ into constraint we get: $x^2 + (\frac{x}{3})^2 = 1 \Rightarrow \frac{10}{9}x^2 = 1$,

$x = \pm\frac{3}{\sqrt{10}}$

Therefore critical points are $\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$ and $\left(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f\left(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right) = -\frac{10}{\sqrt{10}}$

$f\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = \frac{10}{\sqrt{10}}$

Thus the minimum of f on the unit circle is taken at $\left(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$, with value $-\sqrt{10}$.

Problem E: Formulate and prove a generalization of Problem B when c maps to $\mathbb{R}^k$ rather than $\mathbb{R}$ (Whereas Problem B allows you to do certain optimization problems subject to one constraint, this lets you do some optimization problems subject to k constraints. Your generalization will feature numbers $\lambda_1, \ldots, \lambda_k$.)

Remark: You must do B first and then E. You may not reference E in your solution to B.

Generalization: Let $U \subseteq \mathbb{R}^n$ be open

$f: U \to \mathbb{R}$, $c: U \to \mathbb{R}^k$ be in C^1

Restrict f to $M = c^{-1}(0)$.

Then if $f|_M$ has a local minimum at $p \in M$ and $Dc(p)$ has rank k, we must have

$Df(p) = \sum_{i=1}^{k} \lambda_i Dc_i(p)$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$

where $Dc(p) = \begin{pmatrix} Dc_1(p) \\ \vdots \\ Dc_k(p) \end{pmatrix}$

Pf Write $f(x_1, \ldots, x_n)$ as $f(x, y)$ with $x \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^k$

Write $p = (a, b)$ for $a \in \mathbb{R}^{n-k}$, $b \in \mathbb{R}^k$

Since $\operatorname{rank}(Dc(p)) = k$, we can WLOG suppose the $k \times k$ submatrix $\frac{\partial c}{\partial y}$ is invertible (non-singular) (can also get that by reordering variables)

Thus by the Implicit Function Thm,

$\exists$ nbh $B_\varepsilon(a) \subseteq \mathbb{R}^{n-k}$ s.t.

$g: B_\varepsilon(a) \to \mathbb{R}^k$ is C^1

with $\forall x \in B_\varepsilon(a)$, $c(x, g(x)) = 0$

Therefore locally around p, $\exists$ some nbh $U_p = \{(x, g(x)) \mid x \in V\} \subseteq M$

Define $h: B_\varepsilon(a) \to \mathbb{R}$

mapping $x \mapsto f(x, g(x))$

Since f reaches local minimum at $p \Rightarrow h$ reaches local minimum at a.

So $\nabla h(a) = 0$

Thus $\forall i = 1, 2, \ldots, n-k$, we have

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial x_i}(p) + \sum_{j=1}^{k} \frac{\partial f}{\partial x_{n-k+j}}(p) \cdot \frac{\partial g_j}{\partial x_i}(a) = 0 \quad ①$$

Similarly by differentiating $c(x, y) = 0$

$$\frac{\partial c}{\partial x_i}(p) + \frac{\partial c}{\partial y}(p)^T \frac{\partial g}{\partial x_i}(a) = 0 \quad ②$$

$$\Rightarrow \frac{\partial g}{\partial x_i}(a) = -\left(\frac{\partial c}{\partial y}(p)\right)^{T^{-1}} \frac{\partial c}{\partial x_i}(p)$$

Combining ①② we have

$$\frac{\partial f}{\partial x_i}(p) = \sum_{j=1}^{k} \frac{\partial f}{\partial x_{n-k+j}}(p) \cdot \left(\left(\frac{\partial c}{\partial y}(p)\right)^{T^{-1}} \frac{\partial c}{\partial x_i}(p)\right)$$

$$= \underbrace{\left(\frac{\partial f}{\partial y}(p)\right)^T \left(\frac{\partial c}{\partial y}(p)\right)^{-1}}_{\text{take as } \lambda \in \mathbb{R}^k} \cdot \frac{\partial c}{\partial x_i}(p)$$

Problem F: Prove that the set of positive definite matrices is open in the set of n by n symmetric matrices. (You may not use the bonus from HW5.)

Pf Let PD_n denote all positive definite $n \times n$ matrices

Sym_n denote all $n \times n$ symmetric matrices.

WTS: PD_n is open in Sym_n

Let $A \in PD_n$

Consider $Q: S^{n-1} \to \mathbb{R}$

$x \mapsto x^T A x$

Q is ctn. since $Q(x) = x \cdot (Ax)$ where $x \mapsto Ax$ is ctn. and dot product is ctn. function so the composition is ctn

Since S^{n-1} is compact and Q is ctn

$\Rightarrow \{x^T A x \mid x \in S^{n-1}\}$ is compact in $\mathbb{R}$

Therefore we can take $m = \min_{x \in S^{n-1}} x^T A x > 0$ by positive definiteness

Let $\varepsilon = m$

Let $B \in Sym_n$ with $\|A - B\| < \varepsilon$

$x^T B x = x^T A x + x^T (B - A) x$

Since we have $|(B-A)x| \leq \|B - A\| |x|$

And by Cauchy Swartz,

$|x^T (B-A) x| = |x \cdot (B-A)x| \leq |x| \cdot |(B-A)x| \leq |x| \|B-A\| |x| = \|B - A\| < \varepsilon$

Therefore $x^T B x = x^T A x + x^T (B-A) x \geq m - |\varepsilon| > 0$

So $x^T B x$ is positive definite

This proves that PD_n is open in Sym_n. $\square$

Problem G: Suppose $f: A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open. Suppose that x_0 is critical point of f and the Hessian of f is positive definite at x_0. Prove that x_0 is a strict local minimum for f.

Pf x_0 being critical point $\Rightarrow \nabla f(x_0) = 0$

$$H_f(x_0) = \begin{pmatrix} \frac{\partial^2 f(x_0)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x_0)}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f(x_0)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x_0)}{\partial x_n^2} \end{pmatrix}$$

Since $H_f(x_0)$ is positive definite, we have $v^T H_f(x_0) v > 0$ for all $v \in \mathbb{R}^n$

Since A is open, take $\varepsilon > 0$ s.t. $B_\varepsilon(x_0) \subseteq A$

Note the open ball $B_\varepsilon(x_0)$ is an open and convex set

Apply the 1st-order Taylor expansion around x_0 for $x \in B_\varepsilon(x_0)$

we have $f|_{B_\varepsilon(x_0)}(x) = T_{x_0, 1}(x) + R_{x_0, 1}(x)$

$$= f(x_0) + \sum_{|\alpha|=1} \partial^\alpha f(x_0) (x - x_0)^\alpha + \sum_{|\alpha|=2} \frac{\partial^\alpha f}{\alpha!}(c) (x - x_0)^\alpha$$

$$= f(x_0) + \underbrace{\nabla f(x_0) \cdot (x - x_0)}_{=0} + \frac{1}{2} (x - x_0)^T H_f(c) (x - x_0)$$

for some $c \in B_\varepsilon(x_0)$, on the line segment between x, x_0.

So for $x \in B_\varepsilon(x_0)$, $f(x) - f(x_0) = \frac{1}{2} (x - x_0)^T H_f(c) (x - x_0)$

Note that $f \in C^2$, so every entry of H_f is ctn., so H_f is ctn.

Therefore we can choose $\delta > 0$ s.t. $H_f(x)$ is positive definite for all $x \in B_\delta(x_0)$

So we update $\varepsilon' = \min(\delta, \varepsilon)$ and find that $\forall x \in B_{\varepsilon'}(x_0)$, $f(x) - f(x_0) = \frac{1}{2}(x - x_0)^T H_f(c')(x - x_0)$ for some $c' \in B_{\varepsilon'}(x_0)$

Since $H_f(c')$ is positive definite, this proves that $f(x) > f(x_0)$ for all $x \in B_{\varepsilon'}(x_0)$

Hence x_0 is a strict local minimum of f. $\square$

Problem H: Let A be an invertible n by n matrices. Let C be its cofactor matrix, so $C_{ij} = (-1)^{i+j}\det(A_{ij})$, where A_{ij} is the $n-1$ by $n-1$ matrix obtained by deleting row i and column j from A. Prove the following version of Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} C^T,$$

where C^T denotes the transpose of C. (You may use the cofactor expansion of the determinant.)

Pf $(C)_{ij} = (-1)^{i+j}\det(A_{ij})$

Cofactor expansion: $\det(A) = \sum_{j=1}^{n} (A)_{ij}(C)_{ij}$, $\forall$ row i

Consider the product AC^T (note: $(A)_{ij}$ denote the ij^{th} element of A, while A_{ij} denote the $(n-1)\times(n-1)$ submatrix)

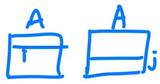
$(AC^T)_{ij} = \sum_{k=1}^{n} (A)_{ik}(C^T)_{kj} = \sum_{k=1}^{n} (A)_{ik}(C)_{jk}$

Thus

Case 1 for all $1 \le i = j \le n$, $(AC^T)_{ij} = \det(A)$ by cofactor expansion

Case 2 for all $1 \le i \ne j \le n$, $(AC^T)_{ij} = \sum_{k=1}^{n} (-1)^{j+k}(A)_{ik}\det(A_{jk})$

Consider matrix B with the j^{th} row identical to the i^{th} row of A, while the other rows of B identical to the corresponding of A



i.e. B is a copy of A with the j^{th} row replaced by the i^{th} row

Then $\det(B) = 0$ as $\operatorname{rank}(B) = n-1$

Cofactor expand of B at row j: $\det(B) = \sum_{k=1}^{n} (-1)^{j+k}(B)_{jk}\det(B_{jk})$

$= \sum_{k=1}^{n} (-1)^{j+k}(A)_{ik}\det(A_{jk})$ since all rows but j^{th} of B are same as A's

$= (AC^T)_{ij}$

Thus $(AC^T)_{ij} = 0$ for all $1 \le i \ne j \le n$

Therefore $AC^T = \begin{bmatrix} \det A & & & \\ & \det A & & \\ & & \ddots & \\ & & & \det A \end{bmatrix}$

$= (\det A) I_n$

$\Rightarrow A^{-1} = \frac{1}{\det A} C^T$ $\square$

Problem I: Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable. Show that

$$(f \cdot g)'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t),$$

where $\cdot$ denotes dot product and $f'(t)$ denotes $Df(t)$ (which in this case is a vector).

Pf $(f\cdot g)(t) = f(t)\cdot g(t) = \sum_{i=1}^{n} f_i(t) g_i(t)$

$\frac{d}{dt}(f\cdot g)(t) = \frac{d}{dt}\sum_{i=1}^{n} f_i(t) g_i(t)$

$= \sum_{i=1}^{n}\left(f_i'(t) g_i(t) + f_i(t) g_i'(t)\right)$

$= \sum_{i=1}^{n} f_i'(t) g_i(t) + \sum_{i=1}^{n} f_i(t) g_i'(t)$

$= f'(t)\cdot g(t) + f(t)\cdot g'(t)$ $\square$

Bonus: Suppose $f : A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open and convex. Show that the region above f, i.e.

$$\{(x, y) \in A \times \mathbb{R} : y \ge f(x)\},$$

is convex if and only if the Hessian of f is positive semi-definite at each point of A.

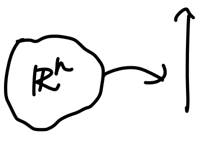
Pf We first show that:

Claim 1 f is convex function iff $H_f(x)$ is positive semidefinite for each $x \in A$

i.e. $\forall x, y \in A, \forall \lambda \in [0,1], f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

($\Leftarrow$) Assume $\forall x \in A$, $H_f(x)$ is positive semi-definite

Let $x, y \in A$



Consider function $g : [0,1] \to \mathbb{R}^n$

$g(t) = x + t(y - x)$

and $\varphi : [0,1] \to \mathbb{R}$

$\varphi(t) = f(g(t))$

So

$\Rightarrow \varphi'(t) = \nabla f(g(t))^T g'(t) = \nabla f(g(t))^T (y-x)$

$\Rightarrow \varphi''(t) = (y-x)^T H_f(g(t))(y-x) \ge 0$ by positive semi-definite of $H_f(g(t))$ and f being C^2

In note that φ is a function from $[0,1]$ to $\mathbb{R}$

By result of one-variable analysis we know that φ is convex function

So $\forall t_1, t_2 \in [0,1]$ and $\lambda \in [0,1]$,

$\varphi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\varphi(t_1) + (1-\lambda)\varphi(t_2)$

Set $t_1 = 0, t_2 = 1$, we have $\varphi(\lambda) \le \lambda\varphi(0) + (1-\lambda)\varphi(1)$

i.e. $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

Since x, y is arbitrary, we proved that f is convex on A

($\Rightarrow$) Assume f is convex on A

Let $x \in A$, $v \in \mathbb{R}^n$

Define $\varphi : \mathbb{R} \to \mathbb{R}$

$h \mapsto f(x_0 + hv)$

Since $f \in C^2(A)$, $h \mapsto x_0 + hv$ is $C^2 \Rightarrow \varphi$ is also C^2

Claim 1.1 φ is convex

Pf of Claim 1.1 Let $a, b \in \mathbb{R}$, $\lambda \in [0,1]$

Let $g : \mathbb{R} \to \mathbb{R}^n$ $\square \mapsto \square$

$g(h) = hv + x_0 = vh + x_0$ (g is an affine function)

So $g(\lambda a + (1-\lambda)b) = v(\lambda a + (1-\lambda)b) + x_0 = \lambda g(a) - \lambda x_0 + (1-\lambda)g(b) - (1-\lambda)x_0 + \lambda x_0$

$= \lambda g(a) + (1-\lambda) g(b)$

Since f is convex,

$\varphi(\lambda a + (1-\lambda)b) = f(g(\lambda a + (1-\lambda)b))$

$= f(\lambda g(a) + (1-\lambda)g(b)) \le \lambda f(g(a)) + (1-\lambda) f(g(b))$

$= \lambda f\circ g(a) + (1-\lambda) f\circ g(b)$

$= \lambda\varphi(a) + (1-\lambda)\varphi(b)$

Claim 1.1 $\square$

By $\varphi''(h)$ being C^2, $\varphi'(h) = \frac{d}{dh}(f(x_0 + hv)) = \nabla f(x_0 + hv)^T v$

$\varphi''(h) = v^T H_f(x_0 + hv) v$

Since φ is convex $\Rightarrow \varphi''(0) = v^T H_f(x_0) v \ge 0$

Forward direction $\square$

Claim 1 $\square$

Now we use claim 1 to prove the statement.

We use upgraph(f) to refer to the set.

Assume $\forall x \in A$ we have $H_f(x)$ being positive semidefinite

Let $(x_1, y_1), (x_2, y_2) \in$ upgraph(f), $\lambda \in [0,1]$

$$y_i \geq f(x_i) \Rightarrow \lambda y_1 + (1-\lambda) y_2 \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geq f(\lambda x_1 + (1-\lambda)(x_2)$$

HW 8 on Implicit Function Thm

8A: applying Implicit FT

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be of class C^1 and such that $f(1, 2, 3) = 0$ and

$$Df(1,2,3) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

1. Does the equation $f(x, y, z) = 0$ define implicitly a function of some of the variables in terms of the rest? If so, what variables can be expressed in terms of what others? Discuss all the possibilities.
2. Suppose there is a function $g : B \to \mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}$ such that $f(x, g(x)) = 0$ for $x \in B$ and $g(1) = (2, 3)$. Compute $Dg(1)$.

8B: applying Implicit FT

Let $f : \mathbb{R}^{k+n} \to \mathbb{R}^n$ be of class C^1 and suppose that $f(a) = 0$ and $Df(a)$ has rank n. Show that if $c \in \mathbb{R}^n$ is sufficiently close to 0, then the equation $f(x) = c$ has a solution.

8.1 closed box 上的连续函数 intble

Let B be a closed box, and $f : B \to \mathbb{R}$ be a continuous function. Show that f is integrable.

Bonus

A map T from a metric space (X, d) to itself is called a contraction mapping if there is a $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq c \cdot d(x, y)$$

for all $x, y \in X$.

1. Show that every contraction mapping of a complete metric space has a unique fixed point.
2. Suppose that $f : \mathbb{R}^d \to \mathbb{R}^d$ is C^1, and $Df(0)$ is invertible. Show that for $\epsilon > 0$ sufficiently small, if B is the closed ball of radius ϵ around 0, then there is a $\delta > 0$ such that if $|y| < \delta$

   $$T(x) = Df(0)(x) + y - f(x)$$

   defines a contraction mapping from B to itself. (As part of this, you' ll have to show $T(B) \subset B$.)
3. Explain why this immediately implies that the image of f contains a neighborhood of $f(0)$.

Remark: The point of this problem is to give the idea for a different proof of the Inverse Function Theorem. (This proof can be found in many textbooks, but don' t look!) Studying the proof from class will help you solve this question, and you can use the lemmas from class if you want to.

Problem A: Let $f:\mathbb{R}^3\to\mathbb{R}^2$ be of class C^1 and such that $f(1,2,3)=0$ and

$$Df(1,2,3)=\begin{bmatrix}1&2&1\\1&-1&1\end{bmatrix}.$$

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(1) Does the equation $f(x,y,z)=0$ define and implicitly a function of some of the variables in terms of the rest? If so, what variables can be expressed in terms what others? Discuss all the possibilities.

(2) Suppose there is a function $g:B\to\mathbb{R}^2$ of class C^1 defined on an open set B of $\mathbb{R}$ such that $f(x,g(x))=0$ for $x\in B$ and $g(1)=(2,3)$. Compute $Dg(1)$.

(1) $f:\mathbb{R}^{1+2}\to\mathbb{R}^2$ is C^1

Implicit function theorem tells us that we can solve two variable in terms of the other one under certain conditions around $(1,2,3)$

$\det\left(\frac{\partial f}{\partial(x,y)}(1,2,3)\right)=\begin{vmatrix}1&2\\1&-1\end{vmatrix}=-3\neq0,$

so (x,y) can be expressed in terms of z around $(1,2,3)$

$\det\left(\frac{\partial f}{\partial(y,z)}(1,2,3)\right)=\begin{vmatrix}2&1\\-1&1\end{vmatrix}=3\neq0$

So (y,z) can be expressed in terms of x around $(1,2,3)$

$\det\left(\frac{\partial f}{\partial(x,z)}(1,2,3)\right)=\begin{vmatrix}1&1\\1&1\end{vmatrix}=0$

(x,z) cannot be expressed in terms of y around $(1,2,3)$

② Let $h:B\to\mathbb{R}^3$

$x\mapsto\binom{x}{g(x)}$

Then $f(x,g(x))=0\Rightarrow f\circ h(x)=0$

$\Rightarrow Df(h(x))\,Dh(x)=0$ by chain rule

at $x=1$, we have $\left[\frac{\partial f}{\partial x}\ \vdots\ \frac{\partial f}{\partial(y,z)}\right]_{h(1)}\begin{bmatrix}I_1\\Dg\end{bmatrix}_{x=1}=0$

$\frac{\partial f}{\partial x}(1,2,3)+\frac{\partial f}{\partial(y,z)}(1,2,3)\,Dg(1)=0$

Since $\frac{\partial f}{\partial(y,z)}$ is non-singular,

$Dg(1)=-\left[\frac{\partial f}{\partial(y,z)}(1,2,3)\right]^{-1}\frac{\partial f}{\partial x}(1,2,3)$

$=-\begin{bmatrix}2&1\\-1&1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}$

By Cramer's rule, $\begin{bmatrix}2&1\\-1&1\end{bmatrix}^{-1}=-\frac{1}{3}\begin{bmatrix}1&-1\\1&2\end{bmatrix}$

So $Dg(1)=-\frac{1}{3}\begin{bmatrix}0\\3\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$

Problem B: Let $f:\mathbb{R}^{k+n}\to\mathbb{R}^n$ be of class C^1 and suppose that $f(a)=0$ and $Df(a)$ has rank n. Show that if $c\in\mathbb{R}^n$ is sufficiently close to 0, then the equation $f(x)=c$ has a solution.

Pf Since $Df(a)$ has rank n,

We can divide a into $b\in\mathbb{R}^k$, $c\in\mathbb{R}^n$

s.t. $\frac{\partial f}{\partial y}(a_1,a_2)\neq0$

where we divide variable in $\mathbb{R}^{k+n}$ into $x\in\mathbb{R}^k$ and $y\in\mathbb{R}^n$ with reordering s.t. $\frac{\partial f}{\partial y}(a)\neq0$

Define $g:\mathbb{R}^n\to\mathbb{R}^n$

$y\mapsto f(a_1,y)$

So $g(a_2)=f(a)=0$. $Dg(a_2)=Df(a)$

Let $h:\mathbb{R}^n\to\mathbb{R}^{k+n}$

$y\mapsto\binom{a_1}{y}$

By chain rule: $Dg(y)=Df(a_1,y)\,Dh(y)$

$\Rightarrow Dg(a_2)=Df(a)$

$=\left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right)\Big|_a\begin{pmatrix}0\\I_n\end{pmatrix}=I_n\frac{\partial f}{\partial y}(a)$

$=\frac{\partial f}{\partial y}(a)\neq0$

Therefore $Dg(a_2)$ is nonsingular

and since f is $C^1\Rightarrow g$ is C^1,

by IVT: $\exists$ some nbh $B_\varepsilon(a_2)\subseteq\mathbb{R}^n$ s.t.

$g|_{B_\varepsilon(a_2)}$ is invertible

So $\exists$ some nbh $V\ni f(a)=0\in\mathbb{R}^n$

and some function $\varphi:V\subseteq\mathbb{R}^n\to B_\varepsilon(a_2)\subseteq\mathbb{R}^n$

s.t. $\varphi=g|_{B_\varepsilon(a_2)}^{-1}$

$\Rightarrow\forall c\in V,\ \exists$ some $y\in\mathbb{R}^n$ s.t. $f(a_1,y)=c$

This proves that if $c\in\mathbb{R}^n$ is sufficiently close to 0, $f(x)=c$ has solution in $\mathbb{R}^{k+n}$ □

Problem C: Let B be a closed box, and $f:B\to\mathbb{R}$ is a continuous function. Show that f is integrable.

Pf B is a closed box (thus also bounded in $\mathbb{R}^n$) $\Rightarrow B$ compact.

$\Rightarrow f$ is uniformly ctn.

Let $\varepsilon>0$

By uniform ctnity, $\exists\delta>0$ s.t. $\forall x,y\in B$,

$|f(x)-f(y)|<\frac{\varepsilon}{V(B)}$ whenever $\|x-y\|<\delta$ ①

Set partitions $P_1,\dots,P_n$ on $B_1,\dots,B_n$ s.t. $\max_{1\le i\le n}\|P_i\|<\delta$

By ①, $\forall$ subbox B_i we have $\left|\sup_{x\in B_i}f(x)-\inf_{x\in B_i}f(x)\right|<\frac{\varepsilon}{V(B)}$

Let $P=(P_1,\dots,P_n)$,

$|U(f,P)-L(f,P)|=\sum_{\text{subbox index } j}\left|\sup_{x\in B_j}f(x)-\inf_{x\in B_j}f(x)\right|V(B_j)$

$\Rightarrow\overline{\int_B}f=\int_B f\quad<\sum_j\frac{\varepsilon}{V(B)}V(B_j)=\varepsilon$

This finishes the proof that f is Riemann integrable □

HW 9 on Riemann Integrability

9.1 metric space 之间函数的连读点集合一定是 ctbl open sets 的交

Let $f : X \to Y$ be a function between metric spaces. Show that the set of points at which f is continuous is a countable intersection of open sets.

9.2 single var: 递增且 bdd 一定 intble

Suppose that $f(x) : [a,b] \to \mathbb{R}$ is non-decreasing and bounded. Show that f is integrable.

9.3 同一 intble function 作为两个 variables 相乘: 乘积也 intble

Suppose that $f, g : [0,1] \to \mathbb{R}$ are two integrable functions. Show that the function $F(x,y) : [0,1]^2 \to \mathbb{R}$ given by $F(x,y) = f(x)g(y)$ is also integrable.

9.4 Thomas function is Riem intble

Let $f : [0,1] \to \mathbb{R}$ be defined by setting $f(x) = 1/q$ if $x = p/q$ where p and q are positive integers with no common factor, and $f(x) = 0$ otherwise.

1. Without using the characterization of Riemann integrable functions, show that f is integrable.
2. Show that f is continuous except at a set of measure 0.

9.5 supp 零测: Riem intble

Let Q be a box in $\mathbb{R}^n$ and $f : Q \to \mathbb{R}$ be a bounded function. Show that if f vanishes except on a closed set B of measure zero, then f is integrable.

Bonus

For this question, you can use without proof that if (a,b) is an open interval in $\mathbb{R}$, there is a smooth function which is positive on (a,b) and zero elsewhere. (You can see explicit examples of such functions on the Wikipedia page on bump functions.)

1. If B is an open box in $\mathbb{R}^d$, show that there is a smooth function which is positive on B and zero elsewhere.
2. If U is an open set in $\mathbb{R}^d$, show that there is a smooth function which is positive on U and zero elsewhere.
3. So in particular, nasty sets like the Cantor set can be level sets of smooth functions. Why doesn' t this contradict the Implicit Function Theorem? What more would you have to assume about f to get that $f^{-1}(0)$ can' t be a nasty set like the Cantor set?

4. Show that there exist two smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that the intersection of their graphs is the Cantor set in the x-axis.

hw 5

Problem A: Let $f : X \to Y$ be a function between metric spaces. Show that the set of points at which f is continuous is a countable intersection of open sets.

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Pf Define

$O_n = \{x\in X \mid \exists \delta_x>0 \text{ s.t. } \forall x_1,x_2\in B_{\delta_x}(x), \text{ we have } d_Y(f(x_1),f(x_2))<\frac{1}{n}\}$

$C_f = \{x \mid f \text{ is ctn. at } x\}$

Claim 1 $C_f=\bigcap_{i=1}^{\infty} O_i$

Pf Suppose $x_0\in C_f$. Let $n\in\mathbb{N} \Rightarrow \frac{1}{n}>0$

By continuity, $\exists\delta>0$ s.t. $f(x)\in B_{\frac{1}{2n}}(f(x_0))$ whenever $x\in B_\delta(x_0)$

So $\forall x_1,x_2\in B_\delta(x_0)$, $d_Y(f(x_1),f(x_2))\le d_Y(f(x_1),f(x_0))+d_Y(f(x_0),f(x_2)) < \frac{1}{2n}+\frac{1}{2n} < \frac{1}{n}$

Thus $x_0\in O_n$

Since n is arbitrary $\Rightarrow \forall i,\ x_0\in O_i \Rightarrow x_0\in\bigcap_{i=0}^{\infty} O_i$

$\Rightarrow$ Thus $C_f\subseteq \bigcap_{i=1}^{\infty} O_i$

Suppose $x_0\in\bigcap_{i=0}^{\infty} O_i \Rightarrow \forall\varepsilon>0$, can take

Let $\varepsilon>0$. Take $n\in\mathbb{N}$ s.t. $\frac{1}{n}<\varepsilon \Rightarrow x_0\in O_n$

So $\exists\delta>0$ s.t. $d(f(x_1),f(x_2))<\frac{1}{n}$ for all $x_1,x_2\in B_\delta(x_0)$

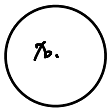
$\Rightarrow d(f(x'),f(x_0))<\frac{1}{n}<\varepsilon$ for all $x'\in B_\delta(x_0)$

$\Rightarrow f$ ctn. at $x_0 \Rightarrow x_0\in C_f$

Thus $\bigcap_{i=1}^{\infty} O_i\subseteq C_f$

Therefore $C_f=\bigcap_{i=1}^{\infty} O_i$. Claim proved.

Claim 2 $\forall i\in\mathbb{N}$, O_i is open



Pf Let $n\in\mathbb{N}$. Let $x_0\in O_n$

$\Rightarrow \exists\delta>0$ s.t. $\forall x_1,x_2\in B_\delta(x_0)$, $d(f(x_1),f(x_2))<\frac{1}{n}$

Consider $B_{\frac{\delta}{2}}(x_0)$: let $x_1\in B_{\frac{\delta}{2}}(x_0)$,

we have $B_{\frac{\delta}{2}}(x_1)\subset B_\delta(x_0)$ by triangular ineq.

$\Rightarrow \forall y_1,y_2\in B_{\frac{\delta}{2}}(x_1)$, $d(f(y_1),f(y_2))<\frac{1}{n} \Rightarrow x_1\in O_n$

Thus $B_{\frac{\delta}{2}}(x_0)\subseteq O_n$

Since x_0 is arbitrary in O_n, this finishes the proof of Claim 2.

Claim 2 + Claim 1 proves exactly what the statement is.

Problem B: Suppose that $f(x) : [a,b]\to\mathbb{R}$ is non-decreasing and bounded. Show that f is integrable.

Pf Let $m=\inf_{[a,b]} f(x)$, $M=\sup_{[a,b]} f(x)$

For all $q\in\mathbb{Q}\cap[m,M]$, define

$D_q=\{x\in[a,b] \mid \lim_{t\to x^-} f(t)\le q\le \lim_{t\to x^+} f(t)\}$

Write $D_f=\{x\in[a,b]\mid f \text{ is not ctn. at } x\}$

Claim 1 $D_f=\bigcup_{q\in\mathbb{Q}\cap[m,M]} D_q$

Pf Let $x\in D_q$ for some $q\in\mathbb{Q}\cap[m,M] \Rightarrow x\in D_f$

Let $x\in D_f \Rightarrow$ Since $[a,b]$ has not isolated pt., we must have $\lim_{t\to x^-}f(t)\ne\lim_{t\to x^+}f(t)$

Then by density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ some $q\in\mathbb{Q}\cap[\lim_{t\to x^-}f(t),\lim_{t\to x^+}f(t)]$

Thus $x\in D_q$ for some $q\in\mathbb{Q}\cap[m,M]$ (since f is non-decreasing)

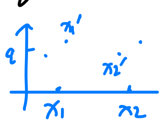
This proves Claim 1

Claim 2 $\forall q\in\mathbb{Q}\cap[m,M]$, D_q has at most one element

Pf Let $q\in\mathbb{Q}\cap[m,M]$

Suppose for contradiction that $x_1,x_2\in D_q$ with $x_1\ne x_2$

WLOG suppose $x_1<x_2$



$\Rightarrow \lim_{t\to x_1^-}f(t)<q$, $\lim_{t\to x_2^-}f(t)<q$

$\lim_{t\to x_1^+}f(t)>q$, $\lim_{t\to x_2^+}f(t)>q$

So $\exists x_1'>x_1$ s.t. $f(x_1')>q$ and $x_2'<x_2$ st. $f(x_2')<q$

Contradicts with f being non-decreasing.

This proves claim 2

By Claim 1,2, D_f is countable.

Therefore D_f is Riem integrable.

□

Problem C: Suppose that $f,g : [0,1]\to\mathbb{R}$ are two integrable functions. Show that the function $F(x,y) : [0,1]^2\to\mathbb{R}$ given by $F(x,y) = f(x)g(y)$ is also integrable.

Pf Since f,g are bounded $\exists M_f,M_g\in\mathbb{R}$ s.t.

$|f(x)|\le M_f$, $|g(y)|\le M_g$ for all $x,y\in[0,1]$

$\Rightarrow |F(x,y)|\le M_fM_g$ for all $(x,y)\in[0,1]^2$. it is bounded

Let D_f,D_g be the set of points in $[0,1]$ where f,g is discontinuous at.

Since $F=f\cdot g$ is ctn. at (x_0,y_0) whenever f is ctn. at x_0 and g is ctn. at y_0

$\Rightarrow$ we have $D_F\subseteq (D_f\times[0,1])\cup([0,1]\times D_g)$

Claim D_F has measure 0

it suffices to prove that $D_f\times[0,1]$ has measure 0

since if so, then dually $[0,1]\times D_g$ has measure 0 and thus $(D_f\times[0,1])\cup([0,1]\times D_g)$ has measure 0, so by monotonicity D_F has measure 0.

Pf Let $\varepsilon>0$

Since D_f is Riem integrable $\Rightarrow D_f$ has measure 0.

$\Rightarrow \forall\varepsilon>0$, $\exists$ covering $\{Q_k\mid k\in\mathbb{N}\}$ s.t. $D_f\subseteq\bigcup_{k=1}^{\infty}Q_k$ and $\sum_{k=1}^{\infty}v(Q_k)<\varepsilon$

So take $\{I_k=Q_k\times[0,1]\mid k\in\mathbb{N}\} \Rightarrow D_f\times[0,1]\subseteq\bigcup_{k=1}^{\infty}I_k$

and $\sum_{k=1}^{\infty}v(I_k)=\sum_{k=1}^{\infty}(v(Q_k)\times 1)<\varepsilon$

This finishes the proof of $D_f\times[0,1]$ having measure 0, thus proving D_F has measure 0.

$\Rightarrow F$ is Riem integrable. □

Problem D: Let $f : [0,1] \to \mathbb{R}$ be defined by setting $f(x) = 1/q$ if $x = p/q$ where p and q are positive integers with no common factor, and $f(x) = 0$ otherwise.

(1) Without using the characterization of Riemann integrable functions, show that f is integrable.

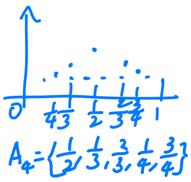
(2) Show that f is continuous except at a set of measure 0.

(1) Pf Let $\varepsilon > 0$. WTS: $\exists$ partition P s.t. $U(P,f) - L(P,f) < \varepsilon$

Take $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$

Define

$A_N = \{x \in [0,1] \mid x = \frac{p}{q}$ in lowest term for some $q \le N\}$



$A_4 = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}\}$

$\Rightarrow \forall x \in A_N, f(x) \ge \frac{1}{N}$

and $\# A_N \le \frac{N(N-1)}{2} < \frac{N^2}{2}$ since $\forall q \in \mathbb{N}$, there can at most be $(q-1)$ terms with q as denominator in A_N.

Let P be a partition on $[0,1]$ s.t. $A_N \subseteq P$ and $\|P\| < \frac{\varepsilon}{N^2}$

For any subbox S created by P, we have

① if $S \cap A_N = \emptyset$, then $\sup_{x\in S} f < \frac{1}{N}$ since $\forall x = \frac{p}{q}$ in lowest term in S, $q > N$

② if $S \cap A_N \neq \emptyset$, then $\sup_{x \in S} f < 1$

Therefore $U(f,P) < \frac{N^2}{2}\|P\| + \frac{1}{N} < \frac{N^2}{2}\cdot\frac{\varepsilon}{N^2} + \frac{\varepsilon}{2} = \varepsilon$

Note that we always have $L(f,P) = 0$ since $(\mathbb{R}\setminus\mathbb{Q}) \cap [0,1]$ is dense in $[0,1]$, and $\forall x \in (\mathbb{R}\setminus\mathbb{Q})\cap[0,1]$ we have $f(x) = 0$.

So $U(f,P) - L(f,P) < \varepsilon$.

This proves that the function is Riem integrable. □

(2) Claim: $\forall x \in (\mathbb{R}\setminus\mathbb{Q}) \cap [0,1]$, f is ctn. at x.

Pf Let $x_0 \in (\mathbb{R}\setminus\mathbb{Q}) \cap [0,1]$

Let $(x_n)_{n\in\mathbb{N}}$ be a seq of irrationals converging to x_0

The convergence is ensured by density of $(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]$ in $[0,1]$

Then $f(x_n) \to 0$ since $\forall n \in \mathbb{N}$, $f(x_n) = 0$

So $\lim_{x\to x_0} f(x) = 0 = f(x_0)$

This proves that $\forall x \in (\mathbb{R}\setminus\mathbb{Q})\cap[0,1]$, f is ctn. at x.

So $D_f \subseteq \mathbb{Q} \cap [0,1]$ is at most countable, thus has measure 0. □

Problem E: Let Q be a box in $\mathbb{R}^n$ and $f : Q \to \mathbb{R}$ be a bounded function. Show that if f vanishes except on a closed set B of measure zero, then f is integrable.

Pf Suppose $|f(x)| \le M$ for some $M > 0$, $\forall x \in Q$

and $f(x) = 0$ $\forall x \in Q \setminus B$, where B is closed and has measure 0.

Let $\varepsilon > 0$. WTS: $\exists$ partition P s.t. $|U(f,P) - L(f,P)| < \varepsilon$

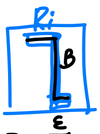
Since B has measure 0, we cover B with $\{R_i \mid i = 1, \dots, k\}$ of closed boxes s.t. $B \subseteq \bigcup_{i=1}^{k} R_i$ and $\sum_{i=1}^{k} v(R_i) < \frac{\varepsilon}{2M}$

Let P be a partition on Q s.t. P has all R_i, $i = 1, \dots, k$ as subboxes obtained by P.

For any subbox S created by P,

if S intersects $B \Rightarrow |f(x)| \le M$ for all $x \in S$

if S does not intersect $B \Rightarrow f(x) = 0$ for all $x \in S$



Therefore we have $L(P,f) \ge -M\sum_{i=1}^{k} v(R_i) > -M\cdot\frac{\varepsilon}{2M} = -\frac{\varepsilon}{2}$

$U(P,f) \le M\sum_{i=1}^{k} v(R_i) < M\cdot\frac{\varepsilon}{2M} = \frac{\varepsilon}{2}$

$\Rightarrow U(P,f) - L(P,f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since ε is arbitrary, this proves that f is Riem integrable. □

Problem F: Show that if $Q_1, Q_2, \dots$ is a countable collection of closed boxes covering some box Q, then $v(Q) \le \sum v(Q_i)$.

Pf It suffices to assume that $\forall i$, Q_i is open since if they are closed then we can replace the i^{th} box with a slightly bigger open box of volume at most $v(Q_i) + \frac{\varepsilon}{2^i}$

And can WLOG assume Q is closed since $v(Q)$ is the same no matter it is open or closed.

So Q is compact $\Rightarrow \exists$ finite subcover $\{Q_{k_1}, \dots, Q_{k_n}\} \subseteq \{Q_j \mid j \in \mathbb{N}\}$ s.t. $Q \subseteq \bigcup_{i=1}^{n} Q_{k_i}$

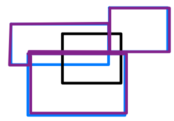
We have $\sum_{i=1}^{n} v(Q_{k_i}) \le \sum_{j=1}^{\infty} v(Q_j)$ since $\{Q_{k_i} \mid i \in \mathbb{N}\} \subseteq \{Q_j \mid j \in \mathbb{N}\}$ and each $v(Q_j) \ge 0$ ①

Let $Q'_{k_n} = Q_{k_n} \setminus (\bigcup_{i=1}^{n-1} Q_{k_i})$

$Q'_{k_{n-1}} = Q_{k_{n-1}} \setminus (\bigcup_{i=1}^{n-2} Q_{k_i})$

$\vdots$

$Q'_{k_1} = Q_{k_1}$



Then $\{Q'_{k_1}, \dots, Q'_{k_n}\}$ is a disjoint cover of Q, since $\bigcup_{i=1}^{n} Q'_{k_i} = Q_{k_1} \cup Q_{k_2} \cup \dots \cup Q_{k_n} = \bigcup_{i=1}^{n} Q_{k_i}$

By $\{Q'_{k_i}\}$ being disjoint, $Q = \bigcup_{i=1}^{n} (Q'_{k_i} \cap Q)$

$\Rightarrow v(Q) = \sum_{i=1}^{n} v(Q'_{k_i} \cap Q) \le \sum_{i=1}^{n} v(Q'_{k_i})$ ②

Since $\forall i \in \{1, \dots, n\}$, $Q'_{k_i} \subseteq Q_{k_i} \Rightarrow \forall i \in \{1, \dots, n\}, v(Q'_{k_i}) \le v(Q_{k_i})$

$\Rightarrow \sum_{i=1}^{n} v(Q'_{k_i}) \le \sum_{i=1}^{n} v(Q_{k_i})$ ③

By ①②③, $v(Q) \le \sum_{i=1}^{n} v(Q'_{k_i}) \le \sum_{i=1}^{n} v(Q_{k_i}) \le \sum_{j=1}^{\infty} v(Q_j)$, exactly what we want. □

Problem G: Write out a proof of the following special case of the Implicit Function Theorem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^1, and suppose that $(x_0, y_0) \in \mathbb{R}^2$ is such that $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Prove that there is an interval I containing x_0 and a C^1 function $g : I \to \mathbb{R}$ such that

$$f(x, g(x)) = 0$$

for all $x \in I$. (You may not write a proof of the general case of the Implicit Function Theorem; the point is to specialize the general proof to this specific case where the notation is a bit simpler to better understand the proof.)

Pf Define an auxiliary function

$$F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$$

$$\Rightarrow DF(x,y) = \begin{pmatrix} DF_1 \\ DF_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\Rightarrow \det DF(x,y) = \det \frac{\partial f}{\partial y}$$

So $\det DF(x_0, y_0) \neq 0$

$\Rightarrow$ By IFT, $\exists U \times V \ni (a,b)$ s.t. U, V open in $\mathbb{R}$ and $F|_{U\times V}: U \times V \to W$ is a local C^1 diffeo for some $W \subseteq \mathbb{R}^2$

Let $G: W \to U \times V$ be the inverse of $F|_{U\times V}$

$\Rightarrow \forall (x,y) \in U \times V$, $(x,y) = G(x, f(x,y))$

So $\forall (x,z) \in W$, $(x,z) = F \circ G(x,z)$

This shows that $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ where G_1 is identity function

Since G is C^1 (by IFT), G_2 must be C^1.

Now we construct the implicit function g:

Let $I \ni x_0$ s.t. $I \times \{0\} \subseteq W$

$\Rightarrow$ Then $\forall (x,y) \in I \times V$,

$f(x,y) = 0$ iff $F(x,y) = (x, 0)$

iff $(x,y) = G(x,0) = (x, G_2(x,0))$

Define $g: I \to \mathbb{R}$

map $x \mapsto G_2(x,0)$

Then we have $f(x,y) = 0$ iff $y = g(x)$, $\forall x \in B$

Note that g is C^1 since G_2 is C^1 $\square$

Bonus: For this question, you can use without proof that if (a, b) is an open interval in $\mathbb{R}$, there is a smooth function with is positive on (a, b) and zero elsewhere. (You can see explicit examples of such functions on the Wikipedia page on bump functions.)

(1) If B is an open box in $\mathbb{R}^d$, show that there is a smooth function which is positive on B and zero elsewhere.
(2) If U is an open set in $\mathbb{R}^d$, show that there is a smooth function which is positive on U and zero elsewhere.
(3) So in particular, nasty sets like the Cantor set can be level sets of smooth functions. Why doesn't this contradict the Implicit Function Theorem? What more would you have to assume about f to get that $f^{-1}(0)$ can't be a nasty set like the Cantor set?
(4) Show that there exist two smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that the intersection of their graphs is the Cantor set in the x-axis.

(1) $B = (a_1, b_1) \times \dots \times (a_d, b_d)$ for some $a_1, \dots, a_d, b_1, \dots b_d \in \mathbb{R}$

For each pair a_i, b_i we define $\varphi_i: \mathbb{R} \to \mathbb{R}$ be a smooth function that is positive on (a_i, b_i) and zero elsewhere

Then define $f: \mathbb{R}^d \to \mathbb{R}$

$$x \mapsto \prod_{i=1}^{d} \varphi_i(x_i)$$

Then f is positive on B and 0 elsewhere

Note that f is smooth since each φ_i are smooth.

(2) Since $\mathbb{R}^d$ is second countable, we can find a countable cover $\{B_n \mid n \in \mathbb{N}\}$ of U s.t. each B_n is an open ball with $B_n \subseteq U$

For each B_n, suppose it is centered at x_n with radius r_n

Define $\psi_n(t): [0,\infty) \to [0,\infty)$

$$t \mapsto \begin{cases} e^{\frac{-1}{r_n^2 - t^2}}, & t < r_n \\ 0, & t \geq r_n \end{cases}$$

Then define $\varphi_n: \mathbb{R}^d \to \mathbb{R}$

$$x \mapsto \psi_n(\|x - x_n\|)$$

Note that ψ_n is smooth, so φ_n is also smooth

Now define $f: \mathbb{R}^d \to \mathbb{R}$

$$x \mapsto \sum_{n=1}^{\infty} \varphi_n(x)$$

Since $\forall \varphi_n$, φ_n is positive for $x \in B_n$ and 0 for $x \notin B_n$

and $\forall B_n$, $B_n \subseteq U$

$\Rightarrow f(x)$ is positive iff $x \in U$

And smoothness of f is ensured by smoothness of each φ_n.

(3) Denote Cantor set by Cat.

Cat is closed. Consider Cat^2 which is also closed,

so the complement $C_{\mathbb{R}^2}(Cat^2)$ is open.

Then by (2), there exists a smooth function f that is positive on $C_{\mathbb{R}^2}(Cat^2)$ and zero on Cat^2

Therefore $f^{-1}(0) = Cat^2$

This seems to contradict with the Implicit function theorem since within appropriate conditions, there can exist some pt. $(x_0, y_0) \in Cat^2$ s.t. $\exists$ some open nbh $B \ni x_0$ and

C^∞ function $g: B \to \mathbb{R}$ s.t. $g(x_0) = y_0$ and

$\forall x \in B \quad f(x, g(x)) = 0$

But that cannot happen since if so then $B \subseteq Cat$

but Cat can contain no open set

However this does not necessarily contradict the Implicit Function Theorem since it requires $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

So we would assume f to be singular on boundary of U.

(4) By (2) we can let $h: \mathbb{R} \to \mathbb{R}$ be a function s.t.

$h(x) = 0$ for all $x \in Cot$

$h(x) > 0$ for all $x \notin Cot$

Define $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto y^2$, $g: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto h(x)$

$f(x,y) = g(x,y) \Leftrightarrow y^2 = h(x)$

HW 10 on Riemann Integration

10.1 max function given two intble functions is intble

Let B be a box in $\mathbb{R}^n$. Show that if $f, g : B \to \mathbb{R}$ are both integrable, then so is the function M defined by

$$M(x) = \max(f(x), g(x)).$$

10.2 intble $\implies$ absolute intble

Let B be a box in $\mathbb{R}^n$. Show that if $f : B \to \mathbb{R}$ is integrable, then so is $|f|$ and moreover

$$\int_B f(x)\,dx \leq \int_B |f(x)|\,dx.$$

10.3 与任意 line 都至多交一点的 dense subset of bR^2

1. Prove that there exists a dense subset $S \subset [0,1]^2$ such that the intersection of S with any vertical line is at most one point, and the intersection of S with any horizontal line is at most one point.
2. Let $f(x,y) : [0,1]^2 \to \mathbb{R}$ be the characteristic function of S (i.e., $f(x,y) = 1$ for $(x,y) \in S$ and 0 otherwise). Show that $f(x,\cdot)$ is an integrable function of y and $f(\cdot,y)$ is an integrable function of x, but f is not integrable on $[0,1]^2$ as a function of two variables.

10.4 C^2 mixed partials: 积分可换序

Let A be an open subset of $\mathbb{R}^2$; and let $f : A \to \mathbb{R}$ be C^2. Let Q be a box contained in A.

1. Use Fubini' s Theorem and the Fundamental Theorem of Calculus to show that
$$\int_Q \frac{\partial^2}{\partial x \partial y} f(x,y)\,dx\,dy = \int_Q \frac{\partial^2}{\partial y \partial x} f(x,y)\,dy\,dx.$$
2. Provide a new proof (other than the one given in class) of the equality of mixed partials $\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y)$.

10.5 证明 Darboux 和 Riemann 条件等价

Show that $f : B \to \mathbb{R}$ is Darboux integrable if and only if f is Riemann integrable.

10.6 Gradient and Hessian Transformations under Linear Map

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2 and an $n \times n$ matrix A, define a new function $g(x) = f(Ax)$. Calculate $Dg(0)$ in terms of $Df(0)$ and A. Also, calculate the Hessian of g at 0 in terms of the Hessian of f at 0.

Proof 注意到:

$$(H_f)_{\text{row i}} = D\frac{\partial f}{\partial x_i}$$

这里我们用到 chain rule 的一个平凡 corollary:

$$\frac{\partial u}{\partial x_i} = \sum_i \frac{\partial u}{\partial y_j}\frac{\partial y_j}{\partial x_i}$$

10.7 deg 2 Taylor polynomial by Hessian

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2, give a formula for the degree two Taylor polynomial of f at 0 in terms of the derivative and Hessian of f at 0.

Bonus: nonexistence of function disctn on irrationals and ctn on rationals

Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous on the irrationals and continuous on the rationals.

Problem A: Let B be a box in $\mathbb{R}^n$. Show that if $f, g : B \to \mathbb{R}$ are both integrable then so is the function m defined by

$$M(x) = \max(f(x), g(x)).$$

Pf Assume the hypothesis.

Denote the set of discontinuity of f, g, M by D_f, D_g, D_m respectively

Let $x \in B \setminus (D_f \cup D_g)$

Let $\varepsilon > 0$

Since f, g ctn at x, $\exists \delta_1, \delta_2$ st.

$$|f(y) - f(x)| < \varepsilon, \ \forall y \in B_{\delta_1}(x)$$
$$|g(y) - g(x)| < \varepsilon, \ \forall y \in B_{\delta_2}(x)$$

So $|M(y) - M(x)| < \varepsilon, \ \forall y \in B_{\min(\delta_1, \delta_2)}(x)$

Therefore M is ctn at x

This proves that $B \setminus D_m \supseteq (B \setminus D_f) \cap (B \setminus D_g) = B \setminus (D_f \cup D_g)$

$\Rightarrow D_m \subseteq D_f \cup D_g$

Since $m(D_f) = m(D_g) = 0$

$\Rightarrow m(D_f \cup D_g) \le m(D_f) + m(D_g) = 0$

$\Rightarrow m^*(D_m) \le m(D_f \cup D_g) = 0$

$\Rightarrow m(D_m) = 0$

This proves that M is Riem integrable.

Problem B: Let B be a box in $\mathbb{R}^n$. Show that if $f : B \to \mathbb{R}$ is integrable then so is $|f|$ and moreover

$$\int_B f(x)dx \le \int_B |f(x)|dx.$$

Pf Denote the set of discontinuities of $f, |f|$ by $D_f, D_{|f|}$ respectively

Claim $D_{|f|} \subseteq D_f$

Let $x_0 \in D_{|f|} \Rightarrow \exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists y \in B_\delta(x_0)$ s.t. $\big||f(x_0)| - |f(y)|\big| \ge \varepsilon$

Fix ε_0

Let $\delta > 0$

Take $y \in B_\delta(x_0)$ st. $\big||f(x_0)| - |f(y)|\big| \ge \varepsilon$

By triangular ineq., we have $|f(x_0) - f(y)| \ge \big||f(x_0)| - |f(y)|\big| \ge \varepsilon$

Since δ is arbitrary, this proves the discontinuity of f at x_0

Thus $D_{|f|} \subseteq D_f$

By linearity of Lebesgue outer measure, $m^*(D_{|f|}) \le m^*(D_f) = 0$

So $m(D_{|f|}) = 0 \Rightarrow |f|$ Riem intble. $\square$

Let P be a partition on B

$$\Rightarrow U(f,P) = \sum_{\text{subbox } S_i} \sup_{S_i} f \cdot v(S_i)$$
$$U(|f|,P) = \sum_{\text{subbox } S_i} \sup_{S_i} |f| \cdot v(S_i)$$

By triangular ineq., $|U(f,P)| \le U(|f|,P)$

So $\left|\int_B f\right| = \left|\overline{\int_B} f\right| = \left|\inf_P U(f,P)\right| \le \inf_P |U(|f|,P)| = \overline{\int_B} |f| = \int_B |f|$ $\square$

Problem C:

(1) Prove that there exists a dense subset $S \subset [0,1]^2$ such that the intersection of S with any vertical line is at most one point, and the intersection of S with any horizontal line is at most one point.

(2) Let $f(x,y) : [0,1]^2 \to \mathbb{R}$ be the characteristic function (aka indicator function) of S (i.e. $f(x,y) = 1$ for $(x,y) \in S$ and 0 otherwise). Show that $f(x,\cdot)$ is an integrable function of y and $f(\cdot,y)$ is an integrable function of x, but f is not integrable on $[0,1]^2$ as a function of two variables.

Just for fun bonus, not to hand it: In part (1) above, show that you can pick S so its intersection with any horizontal or vertical line is *exactly* one point.

Pf let $R = \begin{pmatrix} \cos\sqrt{2}\pi & \sin\sqrt{2}\pi \\ -\sin\sqrt{2}\pi & \cos\sqrt{2}\pi \end{pmatrix}$

Note that $\mathbb{Q}^2 \cap [0,1]^2$ is dense in $[0,1]^2$

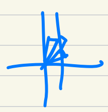
Now define $S = \{R(a,b) \mid (a,b) \in \mathbb{Q}^2 \cap [0,1]^2\}$

i.e. $S = \operatorname{im}(R[\mathbb{Q}^2 \cap [0,1]^2])$

R is a rotation so S is still dense in $[0,1]^2$

Claim $\forall x_0 \in [0,1]$, the vertical line $x = x_0$ intersects S at most once

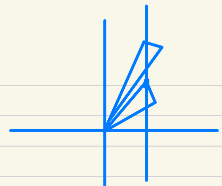
Pf Suppose $(x_0, y_0), (x_0, y_0') \in S$



$\Rightarrow \exists (r_1, s_1), (r_2, s_2) \in \mathbb{Q}^2 \cap [0,1]^2$ st

$R(r_1, s_1) = (x_0, y_0)$, $R(r_2, s_2) = (x_0, y_0')$

$\Rightarrow \cos\sqrt{2}\pi(r_1 - r_2) - \sin\sqrt{2}\pi(s_1 - s_2) = 0$



Note that $r_1 - r_2, s_1 - s_2 \in \mathbb{Q}$

while $\cos\sqrt{2}\pi, \sin\sqrt{2}\pi \in \mathbb{R} \setminus \mathbb{Q}$

$\Rightarrow r_1 = r_2, s_1 = s_2$

$\Rightarrow y_0 = y_0'$

This proves that S intersects any vertical line for at most one point

For similar reasoning, S intersects any horizontal line for at most one point.

(2) Since S intersects any horizontal, vertical line for at most one point,

$f(x,y) = 1$ for at most one y in $[0,1]$ for fixed x,

So $f(x,\cdot)$ is 0 a.e on $[0,1] \Rightarrow$ Riem intble

Same for $f(\cdot, y)$

For f on $[0,1]^2$, by density of S, we always have

$$U(f,P) = \sum_S v(S) \cdot 1 = 1$$
$$L(f,P) = \sum_S v(S) \cdot 0 = 0$$

so $M(f) \ne m(f)$.

So f not Riem intble on $[0,1]^2$.

Problem D: Let A be an open subset of $\mathbb{R}^2$; and let $f : A \to \mathbb{R}$ be C^2. Let Q be a box contained in A.

(1) Use Fubini's Theorem and the Fundamental Theorem of Calculus to show that
$$\int_Q \frac{\partial^2}{\partial x \partial y} f(x,y)dxdy = \int_Q \frac{\partial^2}{\partial y \partial x} f(x,y)dydx.$$

(2) Give a new proof (other than the one we gave earlier in class) of the equality of mixed partials $\frac{\partial^2}{\partial x\partial y}f(x,y) = \frac{\partial^2}{\partial y\partial x}f(x,y)$.

(1) Pf WLOG suppose Q is closed, then

$Q=[a_1,b_1]\times[a_2,b_2]$ for some endpoints $a_1,a_2,b_1,b_2\in\mathbb{R}$

Since $f\in C^2(A) \Rightarrow \frac{\partial^2}{\partial x\partial y}f(x,y)$, $\frac{\partial^2}{\partial y\partial x}f(x,y)$ are ctn

thus Riem intble (their set of discontinuity has measure 0)

$\Longrightarrow$ By Fubini's Thm,
$$\int_Q \frac{\partial^2}{\partial x\partial y} f(x,y)\,dxdy = \int_{a_2}^{b_2}\int_{a_1}^{b_1}\frac{\partial^2}{\partial x\partial y}f(x,y)\,dx\,dy$$
$$\int_{a_1}^{b_1}\frac{\partial^2}{\partial x\partial y}f(x,y)\,dx = \left[\frac{\partial f}{\partial y}(x,y)\right]_{a_1}^{b_1} = \frac{\partial f}{\partial y}(b_1,y)-\frac{\partial f}{\partial y}(a_1,y)$$
by FTC

Then $\int_Q \frac{\partial^2}{\partial x\partial y}f(x,y)\,dxdy = \int_{a_2}^{b_2}\frac{\partial f}{\partial y}(b_1,y)-\frac{\partial f}{\partial y}(a_1,y)\,dy$

$= f(b_1,b_2)-f(b_1,a_2)-f(a_1,b_2)+f(a_1,a_2)$ ①

by FTC

Similarly we have:
$$\int_Q \frac{\partial^2}{\partial y\partial x}f(x,y)\,dydx = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\partial^2}{\partial y\partial x}f(x,y)\,dy\,dx$$
$$= \int_{a_1}^{b_1}\left[\frac{\partial}{\partial x}f(x,y)\right]_{a_2}^{b_2}dx = \int_{a_1}^{b_1}\left(\frac{\partial}{\partial x}f(x,b_2)-\frac{\partial}{\partial x}f(x,a_2)\right)dx$$
$$= f(b_1,b_2)-f(a_1,b_2)-f(b_1,a_2)+f(a_1,a_2) \text{ ②}$$

Note that ① = ② which is exactly what we want to show □

(2) Let $(a,b)\in A$.

Let $\varepsilon>0$, $Q_\varepsilon=[a,a+\varepsilon]\times[b,b+\varepsilon]$

$\Rightarrow \int_b^{b+\varepsilon}\int_a^{a+\varepsilon}\left(\frac{\partial^2 f}{\partial x\partial y}(x,y)-\frac{\partial^2 f}{\partial y\partial x}(x,y)\right)dx\,dy=0$

Since $\left(\frac{\partial^2 f}{\partial x\partial y}(x,y)-\frac{\partial^2 f}{\partial y\partial x}(x,y)\right)$ is ctn both regard to x and y,

we can apply the integral MVT to get

$\int_a^{a+\varepsilon}\left(\frac{\partial^2 f}{\partial x\partial y}(x,y)-\frac{\partial^2 f}{\partial y\partial x}(x,y)\right)dx = \varepsilon\left(\frac{\partial^2 f}{\partial x\partial y}(x_0,y)-\frac{\partial^2 f}{\partial y\partial x}(x_0,y)\right)$ for some $x_0\in[a,a+\varepsilon]$

Then get $\int_b^{b+\varepsilon}\left(\frac{\partial^2 f}{\partial x\partial y}(x_0,y)-\frac{\partial^2 f}{\partial y\partial x}(x_0,y)\right)dy = \varepsilon\left(\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)-\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\right)$ for some $y_0\in[b,b+\varepsilon]$

Therefore $\varepsilon^2\left(\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)-\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)\right)=0$

$\Rightarrow \frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)=\frac{\partial^2 f}{\partial y\partial x}(x_0,y_0)$

Problem E: Show that $f : B \to \mathbb{R}$ is Darboux integrable if and only if f is Riemann integrable.

Pf Backward direction

Suppose f is Riemann integrable

let $A\in\mathbb{R}$ be the integral

let $\varepsilon>0$

Take $\delta>0$ st. $\forall\|P\|<\delta$, $\left|\sum_{B_\alpha \text{ by } P} f(x_\alpha)v(B_\alpha)-A\right|<\frac{\varepsilon}{2}$

for any $x_\alpha\in B_\alpha$ for each α

Fix one arbitrary P st. $\|P\|<\delta$, let $\{B_1,\dots,B_k\}$ be subboxes of B obtained by P

So $\forall i=1,\dots,k$, take arbitrary $x_i\in B_i$

we have $\left|\sum_{i=1}^n f(x_i)v(B_i)-A\right|<\frac{\varepsilon}{2}$

Since each x_i is take arbitrarily,
$$|L(f,P)-A| = \left|\sum_{i=1}^n m(B_i)v(B_i)-A\right|\le\frac{\varepsilon}{2}$$
$$|U(f,P)-A| = \left|\sum_{i=1}^n M(B_i)v(B_i)-A\right|\le\frac{\varepsilon}{2}$$

So $|U(f,P)-L(f,P)|\le|L(f,P)-A|+|U(f,P)-A|\le\varepsilon$

Since ε is arbitrary and we always have $U(f,P)\ge L(f,P)$

this proves that $\inf_P U(f,P)=\sup_P L(f,P)$ where P indexes over all partitions of B

Thus f is Darboux integrable

Forward direction

Lemma Let $B'\subseteq B$ be a box

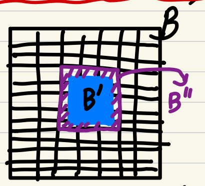
$\forall\varepsilon>0,\exists\delta>0$ s.t. any partition P on B s.t. $\|P\|<\delta$,

we have $\left(\sum_{S\cap B'\neq\emptyset} v(S)\right)-v(B')\le\varepsilon$,

where S indexes on subboxes of B by P

Pf of Lemma

Fix $\varepsilon>0$. Consider $\delta=\frac{\sqrt[n]{\varepsilon}}{2}$



Let P be arbitrary partition st $\|P\|<\delta$

$B'=[a_1,b_1]\times\dots\times[a_n,b_n]$

Let $G=\{S_1,S_2,S_3,\dots,S_k\}$ be set of subboxes of B by P that have non empty intersection with B'

write each $S_i:=[\alpha_1^{(i)},\beta_1^{(i)}]\times\dots\times[\alpha_n^{(i)},\beta_n^{(i)}]$

Let $\alpha_j=\min\{\alpha_j^{(i)}\mid 1\le i\le k\}$

$\beta_j=\max\{\beta_j^{(i)}\mid 1\le i\le k\}$ for each $j=1,\dots,n$

Since $\|P\|<\delta \Rightarrow \forall j=1,\dots,n$, $a_j-\alpha_j\le\delta$ and $b_j-\beta_j\le\delta$

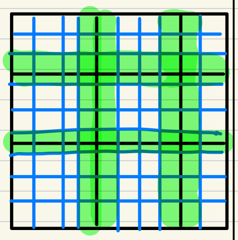
($\alpha_j\le a_j$, $\beta_j\ge b_j$ $\forall j$)

Consider $B''=[\alpha_1,\beta_1]\times\dots\times[\alpha_n,\beta_n]\supseteq B'$

$\Rightarrow B'\subseteq\bigcup_G S\subseteq B''$ so $v(B')\le\sum_G v(S)\le v(B'')$

since subboxes in G are almost disjoint and $v(B'') - v(B') \le (2\delta)^n = (\sqrt[n]{\varepsilon})^n \le \varepsilon$

So $\sum_{S\in G} v(S) - v(B') < \varepsilon$ □ Lemma



Now we have come to

Pf of Darboux integrable $\Rightarrow$ Riem integrable

Suppose f is Darboux integrable

So $\inf_P U(f,P) = \sup_P L(f,P) = A \in \mathbb{R}$

Let $\varepsilon > 0$

Let P_ε be a partition with subboxes collection $G = \{S_1, \dots, S_k\}$

s.t. $\left|\sum_{B_i\in G} m(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$ and $\left|\sum_{B_i\in G} M(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$

For each subbox B_i by P_ε, by Lemma, we can pick $\delta_i > 0$ that for any partition P on B s.t. $\|P\| < \delta_i$ we have $\left(\sum_{S\cap B_i \ne \emptyset} v(S)\right) - v(B_i) < \frac{\varepsilon}{2k(\sup_B f - \inf_B f)}$

where S indexes over subboxes of B by P

Take $\delta = \min\{\delta_1, \dots, \delta_k\}$ and fix it

Let P_{new} be arbitrary partition of B s.t. $\|P_{new}\| < \delta$

Let G_{new} be subboxes of B by P_{new}

Let $G_a \subseteq G_{new}$ be the collection of subboxes by P_{new} not completely contained in some subboxes by P_ε

For each $B_\alpha \in G_{new}$, take arbitrary $x_\alpha \in B_\alpha$

We have $\sum_{G_{new}} f(x_\alpha) v(B_\alpha)$

$$= \sum_{G_a} f(x_\alpha) v(B_\alpha) + \sum_{G_{new}\setminus G_a} f(x_\alpha) v(B_\alpha)$$

This is between $\sum_{G_a} \inf_B f \, v(B_\alpha) + \sum_{G_{new}\setminus G_a} m(B_\alpha) v(B_\alpha)$

and $\sum_{G_a} \sup_B f \, v(B_\alpha) + \sum_{G_{new}\setminus G_a} M(B_\alpha) v(B_\alpha)$

which is within $\left[\sum_{B_i\in G} m(B_i)v(B_i) - \frac{\varepsilon}{2}, \sum_{B_i\in G} M(B_i)v(B_i) + \frac{\varepsilon}{2}\right]$

by our assumption of P_{new} through lemma

Since we also have

$\left|\sum_{B_i\in G} m(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$ and $\left|\sum_{B_i\in G} M(B_i)v(B_i) - A\right| < \frac{\varepsilon}{2}$,

we obtain that

$\left|\sum_{G_{new}} f(x_\alpha) v(B_\alpha) - A\right| < \varepsilon$ by triangular ineq

Since P_{new} s.t. $\|P_{new}\| < \delta$ is taken arbitrarily, this shows the Riemann's condition

Then we finishes the proof of Darboux intble $\Rightarrow$ Riem intble since ε is arbitrary. □

Problem F: Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2 and given a n by n matrix A, define a new function by $g(x) = f(Ax)$. Calculate $Dg(0)$ in terms of $Df(0)$ and A. Calculate the Hessian of g at 0 in terms of the Hessian of f at 0.

By chain rule, $Dg(x) = D(f\circ A)(x)$

$$= Df(Ax)\underbrace{DA(x)}_{=A}$$

So $Dg(0) = Df(0)A$

$$H_g(x) = \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2}(x) & \frac{\partial^2 g}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 g}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 g}{\partial x_2 \partial x_1}(x) & \ddots & & \vdots \\ \vdots & & & \\ \frac{\partial^2 g}{\partial x_n \partial x_1}(x) & \cdots & & \frac{\partial^2 g}{\partial x_n^2}(x) \end{pmatrix} = \begin{pmatrix} D\frac{\partial g}{\partial x_1}(x) \\ D\frac{\partial g}{\partial x_2}(x) \\ \vdots \\ D\frac{\partial g}{\partial x_n}(x) \end{pmatrix}$$

Let $y(x) = Ax \ \forall x \in \mathbb{R}^n$

Denote the Jacobian matrix of f by $J_f = \left(\frac{\partial f}{\partial y_1} \cdots \frac{\partial f}{\partial y_n}\right)$

the Jacobian matrix of y by $J_y = \left(\frac{\partial y}{\partial x_1} \cdots \frac{\partial y}{\partial x_n}\right)$

Note that $\frac{\partial g}{\partial x_i} = (J_f)(A)_{col\, i} = \sum_{j=1}^n A_{ji} \frac{\partial f}{\partial y_j}$

So $D\frac{\partial g}{\partial x_i}(0) = \left(\frac{\partial}{\partial x_1}\left(\frac{\partial g}{\partial x_i}(0)\right) \cdots \frac{\partial}{\partial x_n}\left(\frac{\partial g}{\partial x_i}(0)\right)\right)$

$$= \left(\sum_{j=1}^n A_{ji} \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial y_j}(0)\right) \cdots \sum_{j=1}^n A_{ji} \frac{\partial}{\partial x_n}\left(\frac{\partial f}{\partial y_j}(0)\right)\right)$$

$$= \left(\sum_{j=1}^n \sum_{k=1}^n A_{ji} \frac{\partial^2 f}{\partial y_k \partial y_j}(0) \overset{=A_{1k}}{\frac{\partial y_1}{\partial x_k}(0)} \cdots \sum_{j=1}^n \sum_{k=1}^n A_{ji} \frac{\partial^2 f}{\partial y_k \partial y_j}(0) \overset{=A_{nk}}{\frac{\partial y_n}{\partial x_k}(0)}\right)$$

since $\frac{\partial f}{\partial x_n} = \sum_{k=1}^n \frac{\partial f}{\partial y_k}\frac{\partial y_k}{\partial x_n}$ implied by chain rule.

Therefore $(H_g(0))_{im} = \sum_{j=1}^n \sum_{k=1}^n A_{ji} (H_f(0))_{jk} A_{km}$

$$= (A)_{row\, i} (H_f(0)) (A)_{col\, m}$$

This shows that $H_g(0) = A^T H_f(0) A$

Problem G: Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2, give a formula for the degree two Taylor polynomial of f at 0 in terms of derivative and Hessian of f at 0.

$$f(x) = \sum_{|\alpha| \le n} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha + R_{a,k}(x)$$

$$T_{a,2}(x) = \sum_{|\alpha|\le 2} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha$$

$$= f(0) + \sum_{|\alpha|=1} \partial^\alpha f(0)\, x^\alpha + \sum_{|\alpha|=2} \frac{\partial^\alpha f(0)}{2} x^\alpha$$

$$= f(0) + \underbrace{\sum_{i=1}^n \frac{\partial f}{\partial x_i}(0) x_i}_{= Df(0)x} + \frac{1}{2} \sum_{j=1}^n \frac{\partial f}{\partial x_j}\left(\sum_{k=1}^n \frac{\partial f}{\partial x_k}(0) x_k \right) x_j$$

$$= f(0) + Df(0)\, x + \frac{1}{2} \underbrace{\sum_{j=1}^n x_j \left(\sum_{k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k}(0) x_k \right)}_{= x^T H_f(0) x \text{ by } C^2} \quad (\text{by } C^2)$$

$$= f(0) + Df(0)\, x + \frac{1}{2} x^T H_f(0) x$$

Bonus: Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous on the irrationals and continuous on the rationals. (Hint: There is a short solution. You're welcome to use IBL results if you want to.)

Recall in topological sense: a G_δ set means a countable intersection of open sets.

Claim 1 $\forall f : X \to Y$ between metric spaces, $C_f = \{x \mid f \text{ ctn at } x\}$ is G_δ.

Pf of claim 1: exactly hw 9 problem A $\square$

Claim 2 Corollary of Baire Category Thm: a complete metric space can not be a countable union of nowhere denses sets in it.

Pf of claim 2 Let K be a complete metric space. Assume the contrary:

Let (N_n) be a seq of nowhere dense sets and $X = \bigcup_n N_n$

$\Rightarrow \{O_n = X \setminus \overline{N_n}\}_{n\in\mathbb{N}}$ is dense and open in X

But $\bigcap_n O_n = \emptyset$, contradicts BCT. $\square$

Claim 3 $\mathbb{Q}$ is not G_δ.

Pf of Claim 3 Suppose for contradiction that $\mathbb{Q}$ is G_δ

Then $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$ for some countable collection of open sets $\{U_n\}$

$\Rightarrow \mathbb{R}\setminus\mathbb{Q} = \mathbb{R} \setminus \bigcap_{n=1}^\infty U_n = \bigcup_{n=1}^\infty (\mathbb{R}\setminus U_n)$ is a countable union of closed sets.

Note that each $\mathbb{R}\setminus U_n$ is nowhere set as its interior is empty (otherwise it contains an open set, which in $\mathbb{R}$ means that it contains open interval)

And $\mathbb{Q}$ as a countable set, is a ctbl union of singleton sets which are also nowhere dense

$\Rightarrow \mathbb{R} = \mathbb{Q} \cup (\mathbb{R}\setminus\mathbb{Q})$ is a ctbl union of nowhere dense sets which contradicts with the Baire Category Thm $\square$

By claim 1, 3

we proved that $\nexists f : \mathbb{R}\to\mathbb{R}$ s.t. $D_f = \{\mathbb{R}\setminus\mathbb{Q}\}$ as that implies $\mathbb{Q}$ being G_δ. $\square$

HW 11 on J-meas and Fubini's Thm

11.1 Taylor polynomial after composing linear map

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ that is C^2 and satisfies $Df(0) = 0$, show that there exists an invertible $n \times n$ matrix such that the function $g(x) = f(Ax)$ has a degree two Taylor polynomial at 0 of the form

$$C + \sum a_i x_i^2$$

where each a_i is a real number and C is a constant.

11.2 sum of doubly infinite seq

Let $a : \mathbb{N}^2 \to \mathbb{R}^+$ be a non-negative doubly infinite sequence with $a_{n,m}$ as the (n, m) term. Show that

$$\sum_{(n,m)\in\mathbb{N}^2} a_{n,m} = \sum_{n\in\mathbb{N}} \sum_{m\in\mathbb{N}} a_{n,m}.$$

11.3 extended half $\mathbb{R}$ is semiring

Check that $\mathbb{R} = [0, \infty) \cup \{\infty\}$ is a semiring, with the expected definitions of addition and multiplication, and the definition $0 \cdot \infty = 0$.

11.4 oscillation is upper semi-ctn

Let X be a metric space. A function $f : X \to \mathbb{R}$ is upper semi-continuous at x_0 if

$$\limsup_{x\to x_0} f(x) \leq f(x_0).$$

Show that if $g : \mathbb{R} \to \mathbb{R}$ is a bounded function, then $x \mapsto \operatorname{osc} g(x)$ is upper semi-continuous.

11.5 Fubini's theorem for simple regions

Consider a region $S \subset \mathbb{R}^{n+1}$ that is simple if there exists a compact Jordan measurable set C and two bounded continuous functions $\varphi, \psi : C \to \mathbb{R}$ such that

$$S = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in C; \varphi(x) \leq t \leq \psi(x)\}.$$

Here we assume that $\phi \leq \psi$ on C.

1. Explain why S is compact.
2. Show that $\partial S = E_1 \cup E_2 \cup E_3$ where

$$E_1 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in \partial C; \phi(x) \leq t \leq \psi(x)\},$$

$$E_2 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in C; t = \phi(x)\},$$

$$E_3 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : x \in C; t = \psi(x)\}.$$

3. Show that E_1, E_2, E_3 all have Lebesgue measure 0.
4. Conclude that S is Jordan measurable.
5. Prove Fubini's theorem for simple regions: Let $f : S \to \mathbb{R}$ be a continuous function. Then f is integrable over S, and
$$\int_S f = \int_C \left(\int_{t=\phi(x)}^{t=\psi(x)} f(x,t)\, dt \right) dx.$$

11.6 extended intbility over open sets

Let A be an open subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be continuous. Prove that $\int_A f$ exists in the extended sense if and only if $\int_{U_N} |f|$ is bounded, where $U_1 \subset U_2 \subset \ldots$ is a sequence of open sets whose union is A.

11.7 Bonus: nowhere dense subset with positive measure

Construct a closed, nowhere dense subset of $[0,1]$ that does not have measure 0.

HW 12 applications of COV

12.1 integration of ctn function on bounded sets

Let S be a bounded set in $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a bounded continuous function; let $A = \text{Int}(S)$ be the interior of S. Suppose that f is integrable on S.

1. Show that f is integrable on A and that

$$\int_A f = \int_S f.$$

2. Deduce that if S is Jordan measurable, then A is Jordan measurable as well and $m(A) = m(S)$.

Warning: A is always Lebesgue measurable, since A is open and open sets are Lebesgue measurable. However, for example, when $S = [0, 1] \setminus \mathbb{Q}$, it' s possible to have S Lebesgue measurable but for A and S to have different Lebesgue measures. Such examples must have S not Jordan measurable.

Proof f Riemann integrable on S, so $m(D_f) = 0$ and $m(\{x_0 \in \partial S | \lim_{x\to x_0} f(x) \neq 0\}) = 0$.

My question: Since f is ctn on S, does it mean that for all $x_0 \in S - \partial S$, $\lim_{x\to x_0} f(x) = f(x_0)$?

12.2 volume of Balls in $\mathbb{R}^n$

Let $B^n_a(x)$ denote the ball in $\mathbb{R}^n$ centered at x and radius a.

1. Show that

$$\text{vol}(B^n_a(x)) = \Gamma_n a^n$$

where $\Gamma_n = \text{vol}(B^n_1(0))$.

2. What is Γ_1 and Γ_2?

3. Compute Γ_n in terms of Γ_{n-2}, and deduce a formula for Γ_n for any n.

Just for fun, don' t hand it in: As the dimension gets large, what happens to the volume of the unit ball divided by the volume of $[-1, 1]^n$ (the smallest box containing the unit ball)? If you pick a point at random in the box, is it likely to be in the ball?

12.3 volume of a cone over an open set

Let A be an open Jordan measurable set in $\mathbb{R}^{n-1}$. Given a point $p \in \mathbb{R}^n$ with $p_n > 0$, let S be the subset of $\mathbb{R}^n$ defined by

$$S = \{(1 - t)a + tp : a \in A \times \{0\}, 0 < t < 1\}.$$

(S is the union of all open line segments in $\mathbb{R}^n$ joining p to points of $A \times \{0\}$. You might think of it as a cone over A.)

1. Define a diffeomorphism g of $A \times (0, 1)$ with S.

2. Find the volume of S in terms of the area of A.

12.4 volume of Ellipsoid

Compute the volume inside the ellipsoid

$$\frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}+\frac{(z-w)^2}{c^2}=1$$

where $a, b, c > 0$ and $u, v, w \in \mathbb{R}$.

12.5 volume of a solid bounded by a surface and plane

Compute the volume of the solid in $\mathbb{R}^3$ bounded below by the surface $z = x^2 + 2y^2$ and above by the plane $z = 2x + 6y + 1$.

12.6 Gaussian Integral

Evaluate

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

12.7 integrability of $|x|^e$ over the unit ball and its complement

For which exponents $e \in \mathbb{R}$ is $f(x) = |x|^e$ integrable over the unit ball in $\mathbb{R}^n$? For which exponents is it integrable over the complement of the closed unit ball?

12.8 Bonus: a case of Sard’ s Theorem

The purpose of this bonus is to prove the easiest case of Sard’ s Theorem, a central tool in the study of manifolds.

1. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and I is a compact interval, then $f(I)$ is a compact interval. (You don’ t need to prove this; it follows easily from the fact that a continuous image of a compact set is compact and a continuous image of a connected set is connected.) Show that if f is differentiable with $|f'| \le \delta$ on I, then

$$|f(I)| \le \delta |I|,$$

where $|I|$ denotes the length of an interval I.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1. Show that

$$f(\{x \in \mathbb{R} : f'(x) = 0\})$$

has measure zero. You should do this by first showing that for each n,

$$f(\{x \in [-n, n] : f'(x) = 0\})$$

has measure zero, using that f' is continuous and hence uniformly continuous on $[-n, n]$.

Problem A: Let S be a bounded set in $\mathbb{R}^n$; and let $f : S \to \mathbb{R}$ be a bounded function; let $A = \mathrm{Int}(S)$ be the interior of S. Suppose that f is integrable on S.

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(1) Show that f is integrable on A and that $\int_A f = \int_S f$.
(2) Deduce that if S is Jordan measurable, then A is Jordan measurable as well and $m(A) = m(S)$.

Warning: A is always Lebesgue measurable, since A is open and open sets are Lebesgue measurable. However, for example when $S = [0,1]\setminus\mathbb{Q}$, it's possible to have S Lebesgue measurable but for A and S to have different Lebesgue measure. Such examples must have S not Jordan measurable.

(1) Define $D_f := \{x \in \mathrm{dom}(f) \mid f \text{ discontinuous on } x\}$

By Thm on Lec 18,

f Riem intble $\iff m(D_f) = 0$,

$m(\{x_0 \in \partial S \mid \lim_{x\to x_0} f(x) \neq 0\}) = 0$

Since $A \subseteq S \Rightarrow D_{f|_A} \subseteq D_f \Rightarrow m(D_{f|_A}) = 0$ ① by monotonicity of Lebesgue measure

Let $x \in \partial A \Rightarrow x \in \bar{A}\setminus A$ since A open

$\bar{A} \subseteq \bar{S}$ since $A \subseteq S \Rightarrow x \in \bar{S}\setminus A \Rightarrow x \in \partial S$

Therefore $\partial A \subseteq \partial S$, so

$\{x_0 \in \partial A \mid \lim_{x\to x_0} f|_A(x) \neq 0\} \subseteq \{x_0 \in \partial S \mid \lim_{x\to x_0} f(x) \neq 0\}$

Thus we have $m(\{x_0 \in \partial A \mid \lim_{x\to x_0} f|_A(x) \neq 0\}) = 0$ ②

By ①②, f is (ordinarily) integrable on A.

Since f Riemble integrable on S and $A \Rightarrow$ it is Riemann integrable on $S\setminus A \Rightarrow$ so does $|f|$.

$\int_S f = \int_A f + \int_{S\setminus A} f$ by additivity of Rienman integral

WTS: $\int_{S\setminus A} f = 0$

it suffices to show $\int_{S\setminus A} |f| = 0$ since if so then $-\int_{S\setminus A}|f| = \int_{S\setminus A} -|f| = 0$ and then $\int_{S\setminus A} f = 0$ as $\forall x \in S\setminus A$ $-|f(x)| \le f(x) \le |f(x)|$

Since f is ctn, for all non-isolated point x_0 in $S\setminus A$, $\lim_{x\to x_0} f(x) = f(x_0)$

We divide $S\setminus A$ into

$S_1 = \{x \in S\setminus A \mid x_0 \text{ is isolated pt.}\}$

$S_2 = \{x \in S\setminus A \mid x_0 \text{ is not isolated pt and } \lim_{x\to x_0} f(x) = f(x_0) \neq 0\}$

$S_3 = \{x \in S\setminus A \mid x_0 \text{ is not isolated pt and } \lim_{x\to x_0} f(x) = f(x_0) = 0\}$

Then $S\setminus A = S_1 \sqcup S_2 \sqcup S_3$ $\Rightarrow \int_{S_3} |f| = 0$

Since $S\setminus A \subseteq \partial S \Rightarrow S_2 \subseteq \{x_0 \in \partial S \mid \lim_{x\to x_0} f(x) \neq 0\}$

$\Rightarrow m(S_2) = 0$

Since isolated points of any set in $\mathbb{R}^n$ is ctbl $\Rightarrow m(S_1) = 0$

Since $|f|$ is bdd. Suppose $|f| \le M \Rightarrow 0 \le \int_{S_1} |f| \le \int_{S_1} M$

$\int_{S_1} M = M\int_{S_1} 1 = 0$ since S_1 has measure 0, so is $\int_{S_2} M$

$\Rightarrow \int_{S_1} |f| = \int_{S_2} |f| = 0$

$\Rightarrow \int_{S\setminus A} |f| = \int_{S_1}|f| + \int_{S_2}|f| + \int_{S_3}|f| = 0+0+0 = 0$

Therefore $\int_{S\setminus A} f = 0$ which proves that $\int_S f = \int_A f$ □

(2) Suppose S is Jordan measurable $\Rightarrow m(\partial S) = 0$

In (1) we have shown $\partial A \subseteq \partial S$

$\Rightarrow m(\partial A) = 0$ by monotonicity

$\Rightarrow A$ is Jordan measurable

Since $m(S) = m(A) + m(S\setminus A)$

and $S\setminus A \subseteq \partial S \Rightarrow m(S\setminus A) \le m(\partial S) = 0$

we have $m(S) = m(A) + 0 = m(A)$ □

Problem B: Let $B^n_a(x)$ denote the ball in $\mathbb{R}^n$ centered at x and radius a.

(1) Show that

$$\mathrm{vol}\,(B^n_a(x)) = \Gamma_n a^n$$

where $\Gamma_n = \mathrm{vol}\,(\cancel{B^n_0(1)})$. $B_1^n(0)$

(2) What is Γ_1 and Γ_2?
(3) Compute Γ_n in terms of Γ_{n-2}, and deduce a formula for Γ_n for any n.

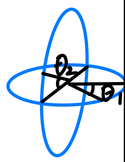
Just for fun, don't hand it: As the dimension gets big, what happens to the volume of the unit ball divided by the volume of $[-1,1]^n$ (the smallest box containing the unit ball)? If you pick a point at random in the box, is it likely to be in the ball?

$B_a^n(x) = \{(y_1,\dots,y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n (y_i - x_i)^2 < a^2\}$

By volume of $B_a^n(x)$ we mean $\int_{B_a^n(x)} 1$, by IBLSA we know $B_a^n(x)$ is Jordan measurable thus 1 is Riem integrable on $B_a^n(x)$

let $g: \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} y_1 = r \\ y_2 = \theta_1 \\ y_3 = \theta_2 \\ \vdots \\ y_n = \theta_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1\cos\theta_2 \\ x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \\ \vdots \\ x_{n-1} = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1} \\ x_n = r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1} \end{pmatrix}$$ x_n



By induction we can get that

$\det(Dg) = r^{n-1}\prod_{k=1}^{n-2}\sin^k(\theta_k)$ for $n \ge 2$

Let $A_a^n(0) = \{(r,\theta) \mid 0<r<a,\ 0\le\theta_i\le 2\pi \text{ for } 1\le i\le n\}$

$\widetilde{A_a^n}(0) = \{(r,\theta)\mid 0<r<a,\ 0<\theta_i<2\pi \text{ for } 1\le i \le n\}$

We know that $g|_{\widetilde{A_a^n}(0)}: \widetilde{A_a^n}(0)\to \widetilde{B_a^n}(0)$ is a diffeomorphism where $\widetilde{B_a^n}(0)$ is $B_a^n(0)$ minus an n-1 dim hyperplane which has measure 0 in $\mathbb{R}^n$ so does not affect the integral

By change of variable thm,

$$\int_{B_a^n(0)} 1 = \int_{\widetilde{B_a^n}(0)} 1 = \int_{g(\tilde A)} 1 = \int_{\tilde A} 1\, |\det Dg|$$

$$= \int_A |\det Dg| = \int_A r^{n-1}\prod_{k=1}^{n-2}\sin^k(\theta_k) \quad (n\ge2)$$

$$= \int_0^a\left(\int_0^\pi\cdots\int_0^{2\pi} r^{n-1}\prod_{k=1}^{n-2}\sin^k(\theta_k)\,d\theta_{n-1}\cdots d\theta_1\right)dr \quad (n\ge 2)$$

by Fubini's thm since constant function is cts.

$$= \int_0^a r^{n-1}dr \underbrace{\left(\int_0^\pi\cdots\int_0^{2\pi}\prod_{k=1}^{n-2} \sin^k(\theta_k)\, d\theta_{n-1}\cdots d\theta_1\right)}_{\text{some constant } C\text{, same for all n-dim ball}}\quad(n\ge2)$$

$= \frac{a^n}{n}$

$= C\frac{a^n}{n}$, and $\Gamma_n = \int_{B_1^n(0)} 1 = \frac{C}{n}$

This proves that $Vol(B_a^n(0)) = \int_{B_a^n(0)} 1 = a^n\Gamma_n \quad (n\ge2)$

And for $n=1$ the ball is just box, $vol(B_a^1(0)) = a\Gamma_1$ for sure.

And also, $\forall x\in\mathbb{R}^n$, we fix x and define translation by x

$$t_x: B_a^n(0)\to B_a^n(x)$$
$$v\mapsto v+x$$

The translation function t_x is a diffeomorphism, and its $\det(Dt) = Id_n$

By Change of variable Thm we we can get

$$\int_{B_a^n(x)} 1 = \int_{t_x(B_a^n(0))} 1 = \int_{B_a^n(0)} 1 = a^n\Gamma_n$$

This finishes the proof that $vol(B_a^n(x)) = a^n\Gamma_n$ for all $x\in\mathbb{R}^n$

(2) $\Gamma_1 = V((-1,1)) = 2$

$\Gamma_2 = \int_0^1 r\,dr\int_0^{2\pi} 1 = \frac12(2\pi)=\pi$

(3) for $n\ge3$

We decompose $x\in\mathbb{R}^n$ into $(z\in\mathbb{R}^2, w\in\mathbb{R}^{n-2})$

So $\Gamma_n = \int_{[-1,1]^n}\chi_{B_1^n(0)} = \int_{[-1,1]^2}\int_{[-1,1]^{n-2}}\chi_{B_1^n(0)}\,dw\,dz$ by Fubini's Thm

For each fixed z, the set of w s.t. $\|w\|<\sqrt{1-|z|^2}$ form a $(n-2)$-dim ball

$$\Rightarrow \Gamma_n = \Gamma_{n-2}\int_{B_1^2(0)}(1-|z|^2)^{\frac{n-2}{2}}dz$$

By polar coordinate,

$$\int_{B_1^2(0)}(1-|z|^2)^{\frac{n-2}{2}}dz = \int_0^{2\pi}\int_0^1(1-r^2)^{\frac{n-2}{2}} r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^1\frac12(1-u)^{\frac{n-2}{2}}du$$

$$= 2\pi\frac{1}{\frac{n-2}{2}+1} = 2\pi\cdot\frac12\cdot\frac2n = \frac{2\pi}{n}$$

Therefore $\Gamma_n = \frac{2\pi}{n}\Gamma_{n-2}$. So $\Gamma_n = \begin{cases}\frac{(2\pi)^{k-1}}{(2k)!!}\pi, & n=2k \text{ for some } k\in\mathbb{N}\\ \frac{2^{k+1}\pi^k}{(2k+1)!!}, & n=2k+1 \text{ for some } k\in\mathbb{N}\cup\{0\}\end{cases}$

Problem C: Let A be an open Jordan measurable set in $\mathbb{R}^{n-1}$. Given a point $p\in\mathbb{R}^n$ with $p_n>0$, let S be the subset of $\mathbb{R}^n$ defined by the equation

$$S=\{(1-t)a+tp : a\in A\times 0, 0<t<1\}.$$

(S is the union of all open line segments in $\mathbb{R}^n$ joining p to points of $A\times 0$. You might think of it as a cone over A.)

(1) Define a diffeomorphism g of $A\times(0,1)$ with S.

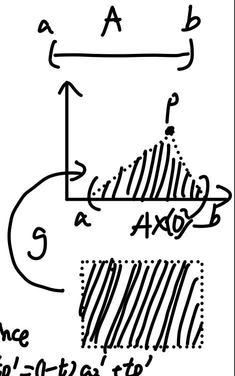
(2) Find the volume of S interms of the area of A.

(1) Define $g: A\times(0,1)\to S$

$(a',t)\mapsto(1-t)(a',0)+tp$

Write $p=(p',p_n)$ where $p'\in\mathbb{R}^{n-1}, p_n\in\mathbb{R}$

$\Rightarrow g(a',t) = ((1-t)a'+tp', tp_n)$



g is surjective since $\forall s\in S$, $s=(1-t)a+tp$ for some $a\in A\times\{0\}$ and $t\in(0,1)$

and injective since suppose $g(a_1',t_1)=g(a_2',t_2)$

$\Rightarrow t_1p_n=t_2p_n \Rightarrow t_1=t_2\Rightarrow a_1'=a_2'$ since $(1-t)a_1'+tp'=(1-t)a_2'+tp'$

and g should be C^1 since it consists of only linear operations on a',t.

(2) For $i,j=1,\dots,n-1$: $\frac{\partial g_i}{\partial a_j'} = (1-t)\delta_{ij}$ and $\frac{\partial g_i}{\partial t} = -a_i'+p_i'$

For $i=n$, $\frac{\partial g_n}{\partial a_j'}=0\ \forall j$, $\frac{\partial g_n}{\partial t}=p_n \Rightarrow$ upper triangular, $\det Dg = (1-t)^{n-1}p_n$

here we can apply Fubini as we can decompose A into ctbl almost disjoint boxes

$$vol(S) = \int_S 1 = \int_{g(A\times(0,1))} 1 = \int_{A\times(0,1)}|\det Dg(x)|\,dx$$

$$= \int_0^1\int_A(1-t)^{n-1}p_n\,da'dt = p_n\cdot area(A)\int_0^1(1-t)^{n-1}dt = \frac1n p_n\cdot area(A)$$

Problem D: Compute the volume inside the ellipsoid

$$\frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}+\frac{(z-w)^2}{c^2}=1$$

where $a,b,c>0$ and $u,v,w\in\mathbb{R}$.

Let volume inside the ellipsoid is

$$E=\left\{(x,y,z)\in\mathbb{R}^3 \;\middle|\; \frac{(x-u)^2}{a^2}+\frac{(y-v)^2}{b^2}+\frac{(z-w)^2}{c^2}<1\right\}$$

Define $g:\mathbb{R}^3\to\mathbb{R}^3$ $\Rightarrow g^{-1}(X,Y,Z)$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}\frac{x-u}{a}\\ \frac{y-z}{b}\\ \frac{z-w}{c}\end{pmatrix} \quad =(aX+u,\ bY+z,\ cZ+w)$$

$$D(g^{-1})=\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix}\Rightarrow \det(D(g^{-1}))=abc$$

Observe that g is bijective and C^1, and g^{-1} also C^1 (as it only concerns scaling and translation)

So $\mathrm{vol}(E)=\int_E 1=\int_{g^{-1}(E)}1|\det Dg^{-1}|=abc\int_{g^{-1}(E)}1$

where $g^{-1}(E)=\{(X,Y,Z)\in\mathbb{R}^3 \mid X^2+Y^2+Z^2\le 1\}$

Note that $g^{-1}(E)$ is a 3-dim unit ball centered at 0.

So by problem b, $\int_{g^{-1}(E)}1=\Gamma_3=\frac{4}{3}\pi$

$\Rightarrow v(E)=\frac{4}{3}abc\pi$

Problem E: Compute the volume of the the solid in $\mathbb{R}^3$ bounded below by the surface $z=x^2+2y^2$ and above by the plane $z=2x+6y+1$.

The solid region is

$$S=\{(x,y,t)\in\mathbb{R}^3 \mid x^2+y^2<z<2x+6y+1\}$$

Let $\varphi(x,y)=x^2+2y^2$, $\psi(x,y)=2x+6y+1$

So we get S restricted to $\mathbb{R}^2$ is

$$C=\{(x,y)\in\mathbb{R}^2 \mid \varphi(x,y)<\psi(x,y)\}$$

By Fubini's Thm on simple region, we get

$$\mathrm{volume}(S)=\int_S 1=\int_C\left(\int_{t=\varphi(x,y)}^{t=\psi(x,y)}1\right)$$

$$=\int_C 2x+6y+1-x^2-2y^2$$

Since $(x,y)\in C \Leftrightarrow x^2+2y^2<2x+6y+1$

$\Leftrightarrow (x-1)^2+2(y-1.5)^2=6.5$

So $C=\{(x,y)\mid (x-1)^2+2(y-1.5)^2<6.5\}$

Let $u=x-1$, $v=y-1.5 \Rightarrow C=\{(u,v)\mid u^2+2v^2<6.5\}$

and $f(u,v)=6.5-u^2-2v^2$

Apply another change of variable $g:\begin{pmatrix}r\\\theta\end{pmatrix}\mapsto\begin{pmatrix}\sqrt{6.5}\,r\cos\theta\\ \sqrt{\frac{6.5}{2}}\,r\sin\theta\end{pmatrix}$

So $g|_{(0,1)\times(0,2\pi)}$ is a diffeo mapping to $C\setminus\{x\text{-axis}\}$

$$Dg=\begin{pmatrix}\sqrt{6.5}\cos\theta & -\sqrt{6.5}\,r\sin\theta\\ \sqrt{\frac{6.5}{2}}\sin\theta & \sqrt{\frac{6.5}{2}}\,r\cos\theta\end{pmatrix}$$

Then $\int_C f=\int_{C\setminus\{x\text{-axis}\}}f=\int_{(0,1)\times(0,2\pi)}f(g(r,\theta))\det(Dg)\,r\,dr\,d\theta$

$$=\frac{(6.5)^2}{\sqrt2}\int_0^{2\pi}\int_0^1(1-r^2)r\,dr\,d\theta \quad \text{by Fubini}$$

$$=\frac{(6.5)^2}{\sqrt2}\cdot\frac14\cdot2\pi=\frac{\frac{169}{4}\pi}{2\sqrt2}=\frac{169\sqrt2\pi}{16}$$

Problem F: Evaluate $\int_{-\infty}^{\infty}e^{-x^2}dx$.

Consider $\int_{\mathbb{R}^2}e^{-(x^2+y^2)}dx$

Since $e^{-(x^2+y^2)}$ is continuous and bounded by 1, it is extendedly integrable

Let B_n be the ball centered at 0 with radius n, $n\ge1$

i.e. $B_n=\{(x,y)\in\mathbb{R}^2 \mid 0<x^2+y^2<n^2\}$

Then $\overline{B_n}$ is cpt & $\overline{B_n}\subseteq \mathring{B}_{n+1}$ $\forall n\in\mathbb{N}$ & $\bigcup_n \overline{B_n}=\mathbb{R}^2$

We then have $\int_{\mathbb{R}^2}e^{-(x^2+y^2)}=\lim_{n\to\infty}\int_{\overline{B_n}}e^{-(x^2+y^2)}=\lim_{n\to\infty}\int_{B_n}e^{-(x^2+y^2)}$

And let $g:\mathbb{R}^2\to\mathbb{R}^2$

$(r,\theta)\mapsto(r\cos\theta, r\sin\theta)$

Let $A_n=\{(r,\theta)\mid 0<r<n, 0\le\theta\le2\pi\}$

$\tilde{A}_n=\{(r,\theta)\mid 0<r<n, 0<\theta<2\pi\}$

Note that $g|_{\tilde{A}}$ is a diffeomorphism between open sets

$$\int_{B_n}e^{-(x^2+y^2)}=\int_{B_n\setminus\{\text{post }x\text{-axis}\}}e^{-(x^2+y^2)}=\int_{g(\tilde{A})}e^{-(x^2+y^2)}$$

$$=\int_{\tilde{A}}e^{-r^2}r \quad \text{by change of variable Thm}$$

$$=\int_0^{2\pi}\int_0^n e^{-r^2}r\,dr\,d\theta \quad \text{by Fubini's Thm for}$$

So $\int_{\mathbb{R}^2} e^{-x^2+y^2} = \lim_{n\to\infty}\int_0^{2\pi}\int_0^n e^{-r^2} r\,dr\,d\theta$ by cov

$$= \int_0^{2\pi}\left(0-\left(-\tfrac{1}{2}\right)\right)d\theta = \pi$$

Also, consider $C_n = [-n,n]\times[-n,n]$

And get $\int_{\mathbb{R}^2} e^{-x^2+y^2} = \lim_{n\to\infty}\int_{C_n} e^{-x^2+y^2}$

$$= \lim_{n\to\infty}\int_{-n}^{n}\int_{-n}^{n} e^{-x^2+y^2}\,dy\,dx$$

$$= \lim_{n\to\infty}\int_{-n}^{n} e^{-x^2}\left(\int_{-n}^{n} e^{-y^2}dy\right)dx$$

$$= \lim_{n\to\infty}\left(\int_{-n}^{n} e^{-y^2}dy\right)\left(\int_{-n}^{n} e^{-x^2}dx\right)$$

$$= \left(\int_{\mathbb{R}} e^{-x^2}dx\right)^2$$

So $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\iint_{\mathbb{R}^2} e^{-x^2+y^2}} = \sqrt{\pi}$

Problem G. For which exponents $e \in \mathbb{R}$ is $f(x) = |x|^e$ integrable over the unit ball in $\mathbb{R}^n$? For which exponents is it integrable over the complement of the closed unit ball?

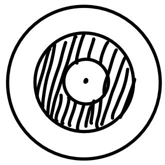
For $e \ge 0$, f is bounded on $B_1(0)$. Since it is also continuous, it is always (ordinarily) integrable as $m(D_f)=0$ and $m(\partial B_1(0))=0$

For $e<0$:

f is unbounded at 0. So the $f|_{B_1(0)}$ is at most extendedly integrable

Claim 1 f is ext. integrable on $B_1(0)$ for $e>-n$

Define $B_n = B_{1-\frac{1}{\sqrt{n+1}}}(0) - B_{\frac{1}{\sqrt{n+1}}}(0)$



We have $\overline{B_n} \subseteq B_{n+1}^{\circ}$ for each n and

$$\bigcup_{n=1}^{\infty} \overline{B_n} = B_1(0)\setminus\{0\}$$

f ext. intble on $B_1(0)$ is equivalent to ext. intble on $B_1(0)\setminus\{0\}$ since changing one point does not affect integrability.

WTS: $\left\{\int_{\overline{B_i}} f\right\}_{i=1}^{\infty}$ is bdd.

Define $C_n = \overline{B_{i+1}}\setminus B_i \ \forall i$

So $\{C_i\}_{i=1}^{\infty}$ are disjoint and closed, $\bigcup_{i=1}^{N} C_i = \overline{B_N}$

Then $\int_{B_N} f = \sum_{i=1}^{N}\int_{C_i} f \le \sum_{i=1}^{N}\int_{C_i}\max_{C_i} f$

$$= \Gamma_n \sum_{n=1}^{N}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}+1-\frac{1}{\sqrt{n+2}}-\left(1-\frac{1}{\sqrt{n+1}}\right)\right)\max_{C_n} f$$

$$= \Gamma_n \sum_{i=1}^{N}\frac{2}{\sqrt{n+1}\sqrt{n+2}}\left(\frac{1}{n+2}\right)^{n+e}$$

$$\le \Gamma_n \sum_{i=1}^{N}\frac{2}{n+1}\left(\frac{1}{n+2}\right)^{n+e}$$

So $\left\{\int_{B_N} f\right\}$ is bdd by $\Gamma_n \sum_{i=1}^{\infty}\frac{2}{n+2}\left(\frac{1}{n+2}\right)^{n+e}$

When $e>-n$, $n+e>0 \Rightarrow$ This series converges

$\Rightarrow \left\{\int_{B_N} f\right\}$ bdded, thus ext. intble

Claim 2 f is not ext. intble on $B_1(0)$ for $e \le -n$

$$\int_{B_N} f = \sum_{i=1}^{N}\int_{C_i} f \ge \sum_{i=1}^{N}\int_{C_i}\max_{C_i} f$$

$$= \Gamma_n \sum_{n=1}^{N}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}+1-\frac{1}{\sqrt{n+2}}-\left(1-\frac{1}{\sqrt{n+1}}\right)\right)\min_{C_n} f$$

$$= \Gamma_n \sum_{i=1}^{N}\frac{2}{\sqrt{n+1}\sqrt{n+2}}\underbrace{\left(1-\frac{1}{n+2}\right)^{n+e}}_{\ge 1}$$

$$\ge \Gamma_n \sum_{i=1}^{N}\frac{2}{n+2}\underbrace{\left(\frac{1}{n+2}\right)^{n+e}}_{\ge 1 \text{ when } e\le -n}$$

diverges when $N\to\infty$

Conclusion f is integrable on $B_1(0)$ iff $e>-n$

Symmetrically can get f is integrable on $\mathbb{R}^n\setminus\overline{B_1(0)}$ iff $e<-n$

Bonus: The purpose of this bonus is to prove the easiest case of Sard's Theorem, which is something we might prove in general in Math 396 and is a central tool in the study of manifolds.

(1) If $f:\mathbb{R}\to\mathbb{R}$ is continuous and I is a compact interval, then $f(I)$ is a compact interval. (You don't need to prove this; it follows easily from the fact that a continuous image of a compact set is compact and a continuous image of a connected set is connected.) Show that if f is differentiable with $|f'|\le\delta$ on I, then

$$|f(I)| \le \delta|I|,$$

where $|I|$ denotes the length of an interval I.

(2) Suppose $f:\mathbb{R}\to\mathbb{R}$ is C^1. Show that

$$f(\{x\in\mathbb{R} : f'(x)=0\})$$

has measure 0. You should do this by first showing that for each n,

$$f(\{x\in[-n,n] : f'(x)=0\})$$

has measure zero, using that f' is continuous and hence uniformly continuous on $[-n,n]$.

(1) Since $f(I)$ is cpt, it contains extreme points, Say $\min(f(I))=\alpha$ and $\max(f(I))=\beta$

So $|f(I)|=\beta-\alpha$

There exists some pt. $a\in I$ and $b\in I$ s.t. $f(a)=\alpha$, $f(b)=\beta$

WLOG suppose $a\le b$

Since $f|_I$ is differentiable

by MVT, $\exists\, c\in[a,b]$ s.t. $f'(c)(b-a)=f(b)-f(a)=\beta-\alpha$

So $|f(I)|=\beta-\alpha=f'(c)(b-a) \le \delta(b-a)\le\delta|I|$

Write $A = \{x \in \mathbb{R} \mid f'(x) = 0\}$

(2) Let $A_n = \{x \in [-n,n] \mid f'(x) = 0\} \Rightarrow A = \bigcup_{n=1}^{\infty} A_n \Rightarrow f(A) = \bigcup_{n=1}^{\infty} f(A_n)$

Since $f \in C^1(\mathbb{R})$, f' is ctn thus uniformly ctn. on $[-n,n]$

Let $\varepsilon > 0 \Rightarrow \exists \delta > 0$ s.t. $f'(x) - f'(y) < \varepsilon$ for all $x, y \in [-n,n]$ s.t. $|x-y| < \delta$

For each $x \in A_n$, take open interval $I_x \in (x - \frac{\delta}{2}, x + \frac{\delta}{2}) \cap (-n,n)$

$\Rightarrow \forall y \in I_x, |f'(y) - f'(x)| < \varepsilon$

Then $\{I_x \mid x \in A_n\}$ form an open cover of A_n

Notice that A_n is cpt. since $f'^{-1}(\{0\})$ is closed and $A_n = f'^{-1}\{0\} \cap [-n,n]$ is bounded.

So $\exists$ finite subcover $\{I_1, I_2, \ldots, I_k\}$ with $\bigcup_{i=1}^{k} I_i \subseteq [-n,n]$

so $A_n \subseteq \bigcup_{i=1}^{k} I_i$, thus $f(A_n) \subseteq f(\bigcup_{i=1}^{k} I_i)$

Since $\bigcup_{i=1}^{k} I_i$ is finite union of intervals thus still interval,

$$\left|f\left(\bigcup_{i=1}^{k} I_i\right)\right| = \left|f\left(\overline{\bigcup_{i=1}^{k} I_i}\right)\right|$$

$$\leq \varepsilon \left|\overline{\bigcup_{i=1}^{k} I_i}\right| \leq \varepsilon |[-n,n]| \leq 2n\varepsilon \quad \text{by (1)}$$

$\Rightarrow m(f(A_n)) \leq 2n\varepsilon$

Since ε is arbitrary $\Rightarrow m(f(A_n)) = 0$

So $m(f(A)) = m\left(\bigcup_{n=1}^{\infty} f(A_n)\right) \leq \sum_{n=1}^{\infty} m(f(A_n)) = 0$

$\square$

HW 13 on smooth functions

13.1 Matrix decomposition

Write the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as a product of 2×2 matrices that are primitive diffeomorphisms. You need only give the final answer.

Sol. This is just the matrix of swapping row 1 and row 2.

Apply procedure at page 157.

13.2 POU sDominated by open cover

Explicitly give a partition of unity of $\mathbb{R}$ dominated by the cover by all open intervals of length 7.

Sol. just the bump function supported on $[-3,4,3.4]$, for each n translated by n (onto left if odd, onto right if even).

13.3 smoothness of $e^{-1/x}$

Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

is C^∞. (For hints, see page 143 of the text.)

Proof It is smooth on $x < 0$, $X > 0$ since it is composed of fundamental smooth functions.

So it remains to show that the function is infinite-times differentiable at $x = 0$.

13D

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $n < m$, show that the image cannot contain an open set.

13.4 stronger factorization: 任意 diffeo 都可拆成 super-primitive diffeos

Define a diffeomorphism to be super-primitive if it preserves all but one coordinate. Show that the theorem proved in class on locally factoring diffeomorphisms remains true if one replaces primitive with super-primitive.

Proof

13F

Show that there is no injective smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

13.5 smooth point on arbitrary set

(Mun Page 144) For an arbitrary subset $S \subset \mathbb{R}$ and a function $f : S \to \mathbb{R}$, we say that f is smooth at $x \in S$ if there is an open set U_x containing x and a smooth function $f_x : U_x \to \mathbb{R}$ such that f and f_x agree on $U_x \cap S$. Show that if f is smooth at every point of S, then there is an open set V containing S and a smooth function $g : V \to \mathbb{R}$ that agrees with f on S.

13.6 rank = the max size of non-singular minor

Let A be a matrix. Show that the rank of A is the maximum value of k such that a $k \times k$ minor of A has a non-zero determinant. (A $k \times k$ minor is a matrix obtained by deleting all but k rows and all but k columns.)

13.7 Bonus: Composition of Primitive Diffeomorphisms

Prove or disprove: Every diffeomorphism between connected open subsets of $\mathbb{R}^n$ (where $n \geq 2$) can be (globally) expressed as a composition of primitive diffeomorphisms between open subsets of $\mathbb{R}^n$.

Problem A: Write the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as a product of 2 by 2 matrices that are primitive diffeomorphisms. You need only give the final answer.

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$(a,b) \to (b,a)$ $\quad$ $(-b,a) \to (b,a)$ $\quad$ $(a-b,a) \to (-b,a)$ $\quad$ $(a-b,b) \to (a-b,a)$ $\quad$ $(a,b) \to (a-b,b)$

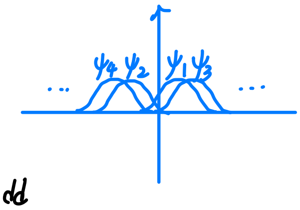
Problem B: Explicitely give a partition of unity of $\mathbb{R}$ dominated by the cover by all open intervals of length 7.

define $\psi : \mathbb{R} \to [0,1]$

$$x \mapsto \begin{cases} \exp\left(-\frac{1}{1-\left(\frac{x}{3.5}\right)^2}\right), & |x|<3.5 \\ 0, & \text{elsewhere} \end{cases}$$

Then $\operatorname{supp}(\psi) = [-3.4, 3.4]$

and ψ is non-negative (1)



for each $n \in \mathbb{N}$ define

$$\psi_n(x) = \begin{cases} \psi(x-n), & n \text{ odd} \\ \psi(x+n), & n \text{ even} \end{cases}$$

(2)

So $\operatorname{supp}(\psi_n) = \begin{cases} [n-3.4, n+3.4], & n \text{ odd} \\ [-n-3.4, -n+3.4], & n \text{ even} \end{cases}$

(3)

Then $\forall x \in \mathbb{R}$, at most four ψ_n are supported at x

(ex: $x=0$, $\operatorname{supp}(\psi_1) = [-2.4, 4.4]$, $\operatorname{supp}(\psi_2) = [-5.4, 1.4]$, $\operatorname{supp}(\psi_3) = [-0.4, 5.4]$, other ψ_n not supported at x)

So define $\lambda = \sum_{n \in \mathbb{N}} \psi_n$ is a C^∞ function, positive on the whole $\mathbb{R}$

define for each $n \in \mathbb{N}$ $\varphi_n = \frac{\psi_n}{\lambda}$

So $\sum_{n \in \mathbb{N}} \varphi_n(x) = 1 \quad \forall x \in \mathbb{R}$ (4)

Then $\{\varphi_n\}_{n=1}^{\infty}$ is a partition of unity on $\mathbb{R}$ by (1)(2)(3)(4)

and it is dominated by {open intervals of length 7}

since each $\operatorname{supp}(\varphi_i) = [m-3.4, m+3.4]$ for some $m \in \mathbb{Z}$ $\subseteq (m-3.5, m+3.5)$

Problem C: Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-1/x}$$

when $x > 0$ and $f(x) = 0$ otherwise is C^∞. (For hints, see page 143 of the text.)

f is of C^∞ class both on $x > 0$ since it is composed of fundamental smooth functions and on $x<0$ since $f = 0$

So it suffices to prove that f is infinite-times differentiable at 0

Pf for $x>0$, $f(x) = e^{-\frac{1}{x}}$, we know that

$f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$ where P_n is a polynomial of degree $2n$

We will prove by induction that $f^{(n)}(0) = 0$

Base case $f'(0)$ exists and is 0

Inductive step: suppose $f^{(n-1)}(0) = 0$ for $n \geq 2$

WTS: $f^{(n)}(0)$ exists and is 0

$$\left|\lim_{x \to 0^-} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right| = \lim_{x \to 0^-} \frac{0-0}{x} = 0$$

$$\left|\lim_{x \to 0^+} \frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right| = \left|P_{n-1}\left(\tfrac{1}{x}\right)e^{-\frac{1}{x}} \cdot \tfrac{1}{x}\right|$$

Let $\frac{1}{x} := t \Rightarrow f^{(n)}(0) = P_{n-1}(t)e^{-t} \cdot t$

$= Q_n(t)e^{-t}$ for some polynomial Q_n of degree $2n$

Since $Q_n(t)$ is degree $2n$, $\exists$ some const C_n s.t. $|P_n(t)| \leq C_n t^{2n}$

$$\text{So } \lim_{x \to 0^+} \left|\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right| \leq \lim_{x \to 0^+} C_n \frac{\left(\frac{1}{x}\right)^{2n}}{e^{\frac{1}{x}}} = \lim_{t \to \infty} C_n \frac{t^{2n}}{e^t} = 0$$

easily obtained by L'Hopital's Rule

Therefore the upper, lower limit agrees as 0

$\Rightarrow f^{(n)}(0)$ exists and is 0

This finishes the proof that f is infinitely-times diff'ble at 0, thus is C^∞ on $\mathbb{R}$. □

Problem D: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $n < m$, show that the image cannot contain a non-empty open set.

Pf In class we have shown: for smooth $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with $m > n$ and A open, we have $m(f(A)) = 0$

So $m(f(\mathbb{R}^n)) = 0$ in this context

Assume for contradiction that $f(\mathbb{R}^n)$ contains an non-empty open set U

Take arbitrary $a \in U$, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(a) \subseteq U \subseteq f(\mathbb{R}^n)$

An open ball of radius always has volume >0 (last hw)

i.e $\int_{B_\varepsilon(a)} 1 > 0$

An open ball is Jordan-measurable, thus its volume agree with Jordan measure and agrees with Lebesgue measure

So $\int_{B_\varepsilon(a)} 1 = m_J(B_\varepsilon(a)) = m(B_\varepsilon(a)) > 0$

By monotonicity of Lebesgue measure, $m(f(\mathbb{R}^n)) \geq m(B_\varepsilon(a)) > 0$, contradicts

Thus the image cannot contain non-empty open set. □

Problem E: Define a diffeomorphism to be super-primitive if it preserves all but one coordinate. Show that the theorem proved in class on locally factoring diffeos remains true if one replaces primitive with super-primitive.

Pf **Claim 1** a diffeomorphism $g: \overset{\text{open}}{U} \subseteq \mathbb{R}^n \to \overset{\text{open}}{V} \subseteq \mathbb{R}^n$ s.t. $g(0)=0$, $Dg(0) = Id_n$ can be decomposed near $x=0$ into $k \circ h$ where

$$h(x) = (g_1(x), \dots g_{i-1}(x), x_i, g_{i+1}(x), \dots, g_n(x))$$

$$k(y) = (y_1, \dots, y_{i-1}, g_i(h^{-1}(y)), y_{i+1}, \dots, y_n)$$

for any $i = 1, \dots, n$

Pf Assume the hypothesis.

$$Dh(x) = \begin{bmatrix} \partial(g_1, \dots, g_{i-1})/\partial x \\ 0 \cdots \underset{i\text{th}}{1} \cdots 0 \\ \partial(g_{i+1}, \dots, g_n)/\partial x \end{bmatrix}$$

Since $Dg(0) = I_n \Rightarrow Dh(0) = I_n$

By IFT, $\exists$ some open $V_0 \ni 0$ in U s.t.

$h|_{V_0}: V_0 \to V_1$ is a diffeomorphism

Define $k: V_1 \to \mathbb{R}^n$ as above

$$\Rightarrow Dk(y) = \begin{bmatrix} I_{i-1} & 0 \\ \multicolumn{2}{c}{D(g_i \circ h^{-1})(y)} \\ 0 & I_{i+1} \end{bmatrix}$$

$$D(g_i \circ h^{-1})(0) = Dg_i(0) \cdot Dh^{-1}(0)$$

$$= [0 \dots 1 \dots 0] I_n = [0 \dots \overset{i\text{th}}{1} \dots 0]$$

So for some open nbh $W_1 \ni 0$, $k|_{W_1}: W_1 \to W_2$ for some open $W_2 \subseteq \mathbb{R}^n$ is a diffeomorphism

Let $W_0 = h^{-1}(W_1)$, then

$$g|_{W_0} = k|_{W_1} \circ h|_{W_0}$$

Here $k|_{W_1}$ is super-primitive diffeo □

Claim 2 a diffeomorphism $g: \overset{\text{open}}{U} \subseteq \mathbb{R}^n \to \overset{\text{open}}{V} \subseteq \mathbb{R}^n$ s.t. $g(0)=0$, $Dg(0) = Id_n$ can be decomposed into finite super-primitive diffeos

Pf Define $h^{(1)}(x) = (x_1, \dots g_2(x), \dots, g_n(x))$

$$k^{(1)}(x) = (g_1(h^{(1)-1}(y)), y_2, \dots, y_n)$$

So $\exists$ open $W_0^{(1)}, W_1^{(1)}$ s.t.

$$g|_{W_0^{(1)}} = k^{(1)}|_{W_1^{(1)}} \circ h|_{W_0^{(1)}}$$

For each $i = 2, \dots, n-1$, define

$$h^{(i)}(x) = (h_1^{(i-1)}(x), \dots h_{i-1}^{(i-1)}(x), x_i, h_{i+1}^{(i-1)}(x), \dots, h_n^{(i-1)}(x))$$

$$k^{(i)}(y) = (y_1, \dots, y_{i-1}, g_i(h^{(i-1)-1}(y)), y_{i+1}, \dots, y_n)$$

For each i, $h^{(i)}$ is a primitive diffeo on some open $W_0^{(i)} \subseteq W_0^{(i-1)}$

$k^{(i)}$ is a super primitive diffeo on some open $W_1^{(i)} \subseteq W_1^{(i-1)}$

By induction we can get: for all $i = 1, \dots, n-1$

$$h^{(i)}(x) = (x_1, \dots, x_i, h_{i+1}^{(i-1)}(x), \dots, h_n^{(i-1)}(x))$$

Since each $h^{(i-1)}(x)$ preserves the $i-1$ th coord and this preservation preceeds in $h^{(i)}(x)$

Thus $h^{(n-1)}(x) = (x_1, x_2, \dots, x_{n-1}, h_n^{(n-1)}(x))$ is a super-primitive diffeo on $W_0^{(n-1)}$

So $g = k^{(1)} \circ \dots \circ k^{(n-1)} \circ h^{(n-1)}$ on $W_0^{(n-1)}$ is composed into super-primitive diffeos

In class we proved that for general diffeomorphism $f: A \to B$ and given $a \in A$, f around a can be locally decomposed into $T \circ t_2 \circ g \circ t_1$ where g defined in claim 2, $t_1: x \mapsto x + a$, $t_2: x \mapsto x - g(a)$, $T: x \mapsto (Dg(a))^{-1} x$

t_1, t_2 are translations so can be decomposed respectively into n super-primitive translations, each translate one coordinate

And T is a invertible linear transformation, so can be decomposed into linear transformations represented by elementary matrices. Elementary matrices are all super-primitive diffeos except swapping two rows, but swapping two rows can also be decomposed into four matrices that are super-primitives (procedure shown in class)

Finally, we can decompose

$$f = E_1 \circ \dots \circ E_n \circ t_2^{(n)} \circ \dots \circ t_2^{(1)} \circ k^{(1)} \circ \dots \circ k^{(n-1)} \circ h^{(n-1)} \circ t_1^{(n)} \circ \dots \circ t_1^{(1)}$$

around a locally, each function is super-primitive diffeomorphism □

Problem F: Show that there is no injective smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

Pf Suppose for contradiction that $f: \mathbb{R}^2 \to \mathbb{R}$ is smooth and injective

If $\forall v \in \mathbb{R}^2$ $\frac{\partial f}{\partial x}(v) = \frac{\partial f}{\partial y}(v) = 0$

$\Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are constant 0 so inductively any order partial is constant 0

$\Rightarrow f$ is constant function by Taylor's Thm, not injective

Thus $\exists v_0 \in \mathbb{R}^2$ s.t. at least one of $\frac{\partial f}{\partial x}(v_0), \frac{\partial f}{\partial y}(v_0) \neq 0$
$= (a_0, b_0)$

WLOG suppose $\frac{\partial f}{\partial x}(v_0) \neq 0$

Since f is smooth and $\det \frac{\partial f}{\partial x}(a_0, b_0) = \frac{\partial f}{\partial x}(a_0, b_0) \neq 0$

$\Rightarrow$ by IFT, $\exists$ nbh $B \ni a_0$ and $g: B \to \mathbb{R}$ s.t.

$g(a_0) = b_0$ and $f(x, g(x)) = 0$ for all $x \in B$

Thus f is not injective, contradicts

Therefore such function does not exist

$\square$

Problem G: For an arbitrary subset S of $\mathbb{R}$ and a function $f : S \to \mathbb{R}$, we say that f is smooth at $x \in S$ if there is a open set U_x containing x and a smooth function $f_x : U_x \to \mathbb{R}$ such that f and f_x agree on $U_x \cap S$. Show that if f is smooth at every point of S then there is an open set V containing S and a smooth function $g : V \to \mathbb{R}$ that agrees with f on S.

Pf Suppose f is smooth at every point of S

Let $\{U_x\}_{x \in S}$ be the collection of open nbh of each $x \in S$ s.t. $f_x : U_x \to \mathbb{R}$ is smooth and agrees with f

Let $A = \bigcup_{x \in S} U_x \Rightarrow A$ open

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a partition of unity on A of C^∞ class, dominated by $\{U_x\}_{x \in S}$ with each $\operatorname{supp}(\varphi_n) \subseteq U_{x_n}$ for some $U_{x_n} \in \{U_x\}_{x \in S}$

For each φ_n, define

$$h_n(x) := \begin{cases} \varphi_n f_{x_n}, & x \in U_x \\ 0, & \text{elsewhere} \end{cases}$$

So $\operatorname{supp}(h_n) \subseteq U_{x_n}$ and $\{h_n\}_{n \in \mathbb{N}}$ is locally finite for all $x \in A$ by the locally finiteness of $\{\varphi_n\}_{n \in \mathbb{N}}$

So $\sum_{n=1}^{\infty} h_n(x)$ converges

and each $h_n(x)$ is smooth since φ_n, f_{x_n} is smooth : $\varphi_n f_{x_n}$ is smooth on U_{x_n} and φ_n reaches 0 on the boundary of U_x, which agrees with the value of h_n outside U_{x_n}

So $\lambda(x) = \sum_{n=1}^{\infty} h_n(x)$ is smooth function on A

Let $x_0 \in S$

$\lambda(x_0) = \varphi_{n_1}(x_0) f_{x_{n_1}}(x_0) + \dots + \varphi_{n_k}(x_0) f_{x_{n_k}}(x_0)$ for some $n_1, \dots, n_k$ by local finiteness

By smoothness: $f_{x_{n_i}}(x_0) = f(x_0)$ for $i = 1, \dots, k$

By partition of unity: $\sum_{i=1}^{k} \varphi_{x_i}(x_0) = 1$

So $\lambda(x_0) = f(x_0)$

This proves λ is a smooth function that agrees with f on S

$\square$

Problem H: Let A be a matrix. Show that the rank of A is the maximum value of k so that a k by k minor of A has non-zero determinant. (A k by k minor is a matrix obtained by deleting all but k rows and all but columns.)

Pf Suppose A is $n \times m$ matrix, $\operatorname{rank}(A) = k$

Claim1 There exists an $k \times k$ minor with non-zero determinant

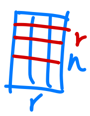
Since there are k linearly independent column,

We take the k columns out, forming a $n \times k$ matrix M

Since the k columns are linearly independent, $\operatorname{rank}(M) = k$

So there are k linearly independent rows in M

We take the k rows out to get a $k \times k$ minor of M, call it M_2



Since the rows of M_2 are lin. ind $\Rightarrow \operatorname{rank}(M_2) = k$

$\Rightarrow \det(M_2) \neq 0$

Claim2 any minor of A has rank less than or equal to k

Let M be a minor of A

Let $S = \{v_1, \dots, v_r\}$ be the columns of M extended to the whole column in A

Since $\operatorname{rank}(A) = k \Rightarrow$ there are at most k linearly independent vectors in S

Let $N = [v_1 \cdots v_r] \Rightarrow \operatorname{rank}(N) \leq k$

Since M is obtains by deleting some rows in N, the row rank of M is less than or equal to that of N

So $\text{rank}(M) \leq k$

Claim 2 implies that for any $r \times r$ minor M of A s.t

$r > k \Rightarrow \det(M) = 0$

Therefore, the rank of A is the max value of k s.t. $\exists$ a $k \times k$ minor of A that has non-zero determinant.

HW 14 on POU and COV

14.1 seq of smooth functions that sum to be disctn

Construct a sequence of smooth functions $\psi_i : \mathbb{R} \to \mathbb{R}$ with disjoint supports such that the function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$\lambda(x) = \sum_{i=1}^{\infty} \psi_i(x)$$

is not continuous.

Hint: Try letting ψ_i be a bump function supported on $[1/2^{2i+2}, 1/2^{2i+1}]$.

Remark: This problem illustrates why, in the definition of a partition of unity, we require that each point has a neighborhood that intersects only finitely many supports, rather than just saying that for each x only finitely many of the functions are non-zero at x.

14.2 Proving COV for linear diffeos and ctns functions with cpt supp

Prove the Change of Variable Theorem for linear diffeomorphisms and continuous functions with compact support. You can use results proved in class up to and including the result that every linear diffeomorphism is a composition of primitive linear diffeomorphisms. You cannot use results proved after that.

Remark: This will not take very long. Because the functions have compact support, the integrals can be interpreted as ordinary integrals, and one can proceed directly to the punchline with Fubini. The purpose of this problem is to help you study the "core" of the proof of Change of Variables without getting lost in the technical details.

14.3 rank function is semi-continuous on $M_{n,m}$

Let $M_{n,m}$ be the space of $n \times m$ matrices. Show that the rank function is lower semi-continuous on $M_{n,m}$. Provide an example to show it need not be continuous.

Hint: To prove lower semi-continuity, you need to show every matrix A has a neighborhood whose rank is at least as big as the rank of A. You can do this quickly using the problem from the last homework on rank. For the example, consider small multiples of the identity.

Problem A: Construct a sequence of smooth functions $\psi_i : \mathbb{R} \to \mathbb{R}$ with disjoint supports such that the function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by $\lambda(x) = \sum_{i=1}^{\infty} \psi_i(x)$ is not continuous.

Hint: Try letting ψ_i be a bump function supported on $[1/2^{2i+2}, 1/2^{2i+1}]$.

Remark: The point of this problem is to illustrate why, in the definition of a partition of unity, we require that each point has a *neighbourhood* that intersects only finitely many supports, rather than just saying that for each x only finitely many of the functions are non-zero at x.

Pf For each $i \in \mathbb{N}$, define

$a_i = \frac{1}{2^{2i+2}}$, $b_i = \frac{1}{2^{2i+1}}$

$I_i = [a_i, b_i]$, $L_i = |I_i| = \frac{1}{4^{i+1}}$, $M_i = 4^{i+1}$

Define $\varphi : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} \exp\left(-\frac{1}{1-t^2}\right), & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$, we verified that φ is smooth

and define $\varphi_i(x) = M_i \varphi\left(\frac{2(x - \frac{a_i+b_i}{2})}{L_i}\right)$ for each $i \in \mathbb{N}$

So φ_i is supported on I_i for each $i \in \mathbb{N}$

and smooth since it is just translating and streching a smooth function

This $(\varphi_i)_{i\in\mathbb{N}}$ is a construction of smooth functions with disjoint support.

Define $\lambda(x) = \sum_{i=1}^{\infty} \varphi_i(x)$

Since $\varphi_i\left(\frac{a_i+b_i}{2}\right) = M_i \varphi(0) = M_i e^{-1} \to \infty$ as $i \to \infty$

$\left(\frac{a_i+b_i}{2}\right)_{i\in\mathbb{N}} \to 0$ as $i \to \infty$

$\Rightarrow \lim_{x\to 0^+} \lambda(x) = \infty$, while $\lambda(0) = 0$

This creates a discontinuity at $x = 0$

Problem B: Prove the Change of Variable Theorem for linear diffeomorphisms and continuous functions with compact support. You can use results proved in class up to and including the result that every linear diffeomorphism is a composition of primitive linear diffeomorphisms. You cannot use results proved after that.

Remark: This will not take very long. Because the functions have compact support, the integrals can be interpreted as ordinary integrals, and one can procede directly to the punchline with Fubini. The purpose of this problem is to help you study the "core" of the proof of Change of Variables without getting lost in the technical details.

Pf We prove by induction

Base case $n = 1$

Let $g : A \subseteq \mathbb{R} \to B \subseteq \mathbb{R}$ is linear diffeo

$f : B \to \mathbb{R}$ ctn with cpt supp.

let $x \in A$, interval $I \ni x \Rightarrow J := g(I)$ is an interval

By partition of unity, it suffices to prove the statement for $g|_{I^\circ} : I^\circ \to J^\circ$; $f|_{J^\circ} : J^\circ \to \mathbb{R}$

And this is true by the change of variable theorem in single variable analysis.

Inductive step suppose the statement holds in dim $n-1$

WTS: it holds for dim n

Claim for dim $n > 1$, it suffices to prove the theorem for primitive $h : U \to V$ and $f|_V$

This is because for each $x \in A$,

$\exists$ nbh $U_0 \ni x$ and a finite seq of primitive diffeos

$U_0 \xrightarrow{h_1} U_1 \to \cdots \xrightarrow{h_k} U_k$

s.t. $g|_{U_0} = h_k \circ \cdots \circ h_1$

WLOG suppose h preserves the last coord

Let $p \in U$, $Q \subseteq V$ be a box st $h(p) \in Q^\circ$, $S := h^{-1}(Q)$

Since p is arbitrary, it suffices to prove the statement for $h : S^\circ \to Q^\circ$ and $f|_{Q^\circ}$

Since $(f \circ h)|\det Dh|$ is ctn and has cpt support in S°, it is intble on S°

WTS: $\int_{Q^\circ} f = \int_{S^\circ}$ and by cpt support we can treat the two integrals as ordinary integrals.

Define $F : \mathbb{R}^n \to \mathbb{R}$

$$x \mapsto \begin{cases} f(h(x))|\det Dh|, & x \in S^\circ \\ 0, & \text{elsewhere} \end{cases}$$

so WTS: $\int_Q f = \int_S F$

Pf Let $Q = D \times I$, where D is a box in $\mathbb{R}^{n-1}$, I is interval in $\mathbb{R}$

Since S cpt and h preserves the last coord,

$S \subseteq E \times I$ for some box $E \subseteq \mathbb{R}^{n-1}$

By Fubini's Thm, the statement to be proved is

$$\int_I \int_D f(y,t)\, dy\, dt = \int_I \int_E F(x,t)\, dx\, dt$$

So it suffices to prove that $\int_D f(y,t) dy = \int_E F(x,t) dx$ for each $t \in I$

Fix $t \in I$, $h(x,t) = (k(x,t),t)$ for some C' function $k: U \to \mathbb{R}^{n-1}$

We define $h_t : x \in \mathbb{R}^{n-1} \mapsto k(x,t) \in \mathbb{R}^{n-1}$

Since $Dh = \left[\begin{array}{c|c} \partial k/\partial x & \partial k/\partial t \\ 0 & 1 \end{array}\right] = \partial k/\partial k = Dh_t$,

we have $\int_{V_t} f(y,t) dy = \int_{U_t} f(h_t(x)) |\det Dh_t| dx$

$= \int_{U_t} F(x,t) dx$

by inductive hypothesis. $\square$

Problem C: Let $M_{n,m}$ be the space of n by m matrices. Show that the rank function is lower semi-continuous on $M_{n,m}$. Give an example to show it need not be continuous.

Hint: To prove lower semi-contininity, you need to show every matrix A has a neighbourhood whose rank is at least as big as the rank of A. You can do that quickly with the problem from the last homework on rank. For the example, you may want to consider small multiples of the identity.

Pf Let $A \in M_{n,m}$

WTS: $\forall \varepsilon > 0$, $\exists$ nbh $U \ni A$ s.t.
$\forall B \in U$, $\text{rank}(B) > \text{rank}(A) + \varepsilon$

It suffices to show that $\exists$ nbh $U \ni A$ s.t.
$\forall B \in U$, $\text{rank}(B) \geq \text{rank}(A)$ WTS

This statement is trivially true when $\text{rank}(A) = 0$ since rank is nonnegative

So suppose $\text{rank}(A) = r \neq 0$

Then $\exists$ a $r \times r$ minor B of A s.t. $\det(B) \neq 0$
WLOG suppose $\det(B) > 0$

Since the det function is ctn, $\exists \varepsilon > 0$ s.t
$\forall B' \in M_{r,r}$ st $\|B' - B\|_F \leq \varepsilon$, $\det(B') > 0$

So consider $B_\varepsilon(A) \subseteq M_{n,m}$ with respect to the Frebenius norm

Let $A' \in B_\varepsilon(A) \Rightarrow \|A' - A\|_F \leq \varepsilon$

then the $r \times r$ minor of A' on the same position as B, call it B', has $\|B' - B\|_F \leq \varepsilon \Rightarrow \det(B') > 0 \Rightarrow \text{rank}(A') \geq r$

This finishes the proof that the rank function in $M_{n,m}$ is lower semi-ctn. $\square$

Counterex to show that rank function is not ctn.

Consider $(A_k = \frac{1}{k} I_n)_{k \in \mathbb{N}}$

$(A_k) \to 0 \in M_{n,m}$ as $k \to \infty$

But $\forall k \in \mathbb{N}$, $\text{rank}(A_k) = n$

so $\lim_{k \to \infty} \text{rank}(A_k) = n \neq 0 = \text{rank}(0)$

So rank function is not ctn at 0.

HW 15 on tensors

15.1 Nonexistence of injective smooth function from higher dim to lower dim

Prove that there does not exist an injective smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$ when $n > m$.

15.2 Convolution of continuous functions

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions that are zero outside a compact set. Define their convolution as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy.$$

Show the following: 1. $\int_{\mathbb{R}^n} f * g = \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right)$, 2. Convolution is commutative and associative.

15.3 Optimization of a quadratic function on a disk

Find the maximum and minimum values of $f(x, y) = 4x^2 + 10y^2$ on the disk $x^2 + y^2 \leq 4$.

15.4 expressing tensors

Determine which of the following functions on $\mathbb{R}^4$ are tensors, and express those that are as elementary tensors:

$$f(x, y) = 3x_1y_2 + 5x_2x_3, \quad g(x, y) = x_1y_2 + x_2y_4 + 1, \quad h(x, y) = x_1y_1 - 7x_2y_3.$$

15.5 $\mathcal{L}^k(V)$ is a vector space

Let V be a vector space, and $\mathcal{L}^k(V)$ be the space of all k-tensors on V. Prove: $\mathcal{L}^k(V)$ is a vector space.

15F: compute the sign of a permutation

Let $\sigma \in S_k$ be the permutation described by

$$1 \to 2 \to 3 \to \cdots \to k \to 1.$$

Compute the sign of σ.

15.6 dual of alternating k-tensor is also alternating k-tensor

Let $T : V \to W$ be a linear transformation. If $f \in A^k(W)$, show that $T^*(f) \in A^k(V)$.

15.7 Reproducing Theorem 27.7

Prove Theorem 27.7: Let ψ_I be an elementary alternating tensor on $\mathbb{R}^n$ correspoding to the usal basis for $\mathbb{R}^n$, where $I = (i_1, \cdots, i_k)$.

Given vectors $x_1, \cdots, x_k$ of $\mathbb{R}^n$, let X be the matrix $X = [x_1, \cdots, x_k]$. Then we have:

$$\psi_I(x_1, \cdots, x_k) = \det X_I$$

15.8 Bonus: Analytic functions and series convergence

Consider the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(p) = \sum_{n=1}^{\infty} a_n p^{k_n} (1-p)^{n-k}.$$

where $a_n \in \{0,1\}$ and $0 \leq k_n \leq n$. Show the following: 1. f is real analytic. 2. For a power series $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, define the radius of convergence as

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}},$$

and show that the series converges uniformly and absolutely for $x \in (x_0 - R, x_0 + R)$ and diverges for $x \notin [x_0 - R, x_0 + R]$. 3. If the series converges on $(x_0 - r, x_0 + r)$ for some $r > 0$, then for any $0 < \rho < r$, there exists a constant C such that $|c_n| \leq C\rho^n$. 4. Conclude that $f'(x_0)$ exists and is equal to c_1.

Remark: The final part demonstrates that real analytic functions are smooth. The proof involves showing that f is C^∞ by defining $g(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}$, and leveraging results about uniform convergence of derivatives.

Problem A: Show that there is no injective smooth function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

with $n > m$.

We first prove a corollary of IFT that we will need for the proof.

Lemma constant rank theorem

if f: open $U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^p and $\forall x \in U$, $\text{rank}(\text{Det } Df(x)) = r$ is const

$\Rightarrow \forall x_0 \in U$, $\exists$ some nbh $V \ni x_0$ and $W \ni f(x_0)$ and C^p diffeo $\varphi : V \to V' \subseteq \mathbb{R}^n$

$\psi : W \to W' \subseteq \mathbb{R}^m$

s.t. $\psi \circ f \circ \varphi^{-1} : V' \to W'$

$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ where the $r+1^{th} \sim n^{th}$ coordinate of image is 0

Pf of Lemma

By hw 10: $\forall x \in U$, $Df(x)$ has a nonsingular $r \times r$ minor and all larger minors are singular

WLOG suppose the left upper $r \times r$ minor is non-singular

Let $\varphi(x) = (f_1(x), \ldots, f_r(x), x^{r+1}, \ldots, x^n)$

$\Rightarrow \forall x \in U$, $\det D\varphi(x) \neq 0$

$\Rightarrow$ by IFT, φ is a C^p diffeo around every point in U

and $f \circ \varphi^{-1}(v) = (v_1, \ldots, v_r, g^{r+1}(v), \ldots, g^m(v))$

Since φ^{-1} invertible, the $\text{rank}(Dg) = \text{ran}(Df \circ \varphi^{-1}) = r$

$$Dg = \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ \left(\frac{\partial g_i}{\partial v_j}\right)_{\substack{i=r+1,\ldots,m \\ j=1,\ldots,r}} & \left(\frac{\partial g_i}{\partial v_j}\right)_{\substack{i=r+1,\ldots,m \\ j=r+1,\ldots,n}} \end{pmatrix}$$

So $\left(\frac{\partial g_i}{\partial v_j}\right)_{\substack{i=r+1,\ldots,m \\ j=r+1,\ldots,n}}$ is 0 matrix, $g(v)$

let $\psi(y) = (y_1, \ldots, y_r, y_{r+1} - g_{r+1}(y), \ldots, y_m - g_m(y))$

$\Rightarrow \det D\psi = 1$, so it is locally C^p diffeo around $f(x_0)$

and $\psi \circ f \circ \varphi^{-1}(v) = (v^1, \ldots, v^r, 0, \ldots, 0)$ locally $\square$

Then suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is injective and smooth with $n > m$

Since $\text{rank}(Df)$ is lower semi-ctn and take values from finite set $\{0, \ldots, m\}$,

it must be locally const on some $x_0 \in \mathbb{R}^n$

Thus $\exists$ C^p diffeo $\varphi : V \to V' \ni x_0$ and $\psi : W \to W' \ni f(x_0)$

s.t. $\psi \circ f \circ \varphi^{-1} : \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ not injective

Since ψ, φ^{-1} are diffeo thus bijective and f injective, $\psi \circ f \circ \varphi^{-1}$ should be injective, contradicts. $\square$

Problem B: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. Suppose that f and g are 0 outside a compact set. Define the convolution $f * g$ by:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

Show that

$$\int_{\mathbb{R}^n} f * g = \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right).$$

Also show that convolution is commutative and associative.

Pf integrating both sides on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} (f*g) = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\right)dx$$

Since f, g are supported only on a cpt set.

Let $B = B_1 \times B_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be box s.t. $\text{supp}(f), \text{supp}(g) \subseteq B$

$\Rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y)g(y) = \int_B f(x-y)g(y)$

$= \int_{B_1}\int_{B_2} f(x-y)g(y)$ by Fubini's Thm

So $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)\,dy\,dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y)g(y)$

$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x-y)g(y)\,dx\right)dy$

since we can interchange the order by Fubini

$= \int_{\mathbb{R}^n} g(y)\left(\int_{\mathbb{R}^n} f(x-y)\,dx\right)dy = \int_{\mathbb{R}^n} g(y)\left(\int_{\mathbb{R}^n} f(z)\,dz\right)dy$

$= \left(\int_{\mathbb{R}^n} g(y)\,dy\right)\left(\int_{\mathbb{R}^n} f(z)\,dz\right) = \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g$

Pf of commutativity

Let $x \in \mathbb{R}^n$

$f*g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$

$= \int_{\mathbb{R}^n} f(z)g(x-z)\,dz = \int_{\mathbb{R}^n} g(x-z)f(z)\,dz$

$= g*f(x)$

Pf of associativity

let $x \in \mathbb{R}^n$

$((f*g)*h)(x) = \int_{\mathbb{R}^n} f*g(x-y)\,h(y)\,dy$

$= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x-y-z)g(z)\,dz\right)h(y)\,dy$

$= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y-z)\,g(z)\,h(y)\,dz\,dy$

by Fubini as previous

$= \int_{\mathbb{R}^n} f(x-y-z)\left(\int_{\mathbb{R}^n} g(z)h(y)\,dy\right)dz$

$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-u)\,g(u-v)\,h(v)\,du\,dv$

$(f*(g*h))(x) = \int_{\mathbb{R}^n} f(x-y)(g*h(y))\,dy$

$= \int_{\mathbb{R}^n} f(x-y)\left(\int_{\mathbb{R}^n} g(y-z)h(z)\,dz\right)dy$

$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-u)g(u-v)h(v)\,du\,dv = ((f*g)*h)(x)$ $\square$

Problem C: Find the maximum and minimum values of

$$f(x,y) = 4x^2 + 10y^2$$

on the disk $x^2 + y^2 \le 4$.

f is cts and defined on cpt area in $\mathbb{R}^2$, thus must have both max, min points.

On the interior: any extreme value point (x_0,y_0) must have

$$\nabla f(x_0,y_0) = (8x_0, 20y_0) = (0,0) \Rightarrow (x_0,y_0) = (0,0)$$

$$f(0,0) = 0$$

On the boundary: $x^2+y^2=4$

$$f(x,y) = 4(x^2+y^2) + 6y^2 = 16 + 6y^2,$$

so f takes maximum when $|y|$ reaches maximum

$$\Rightarrow f_{max} = 16 + 6\times 2^2 = 40$$

on ∂D with $y_0 = \pm 2$

So $x_0 = 0$

Thus $f_{min} = 0$ at $(0,0)$

$f_{max} = 40$ at $(0,2)$ and $(0,-2)$

Problem D: Determine which of the following are tensors on $\mathbb{R}^4$, and express those that are in terms of the elementary tensors:

$$\begin{aligned} f(x,y) &= 3x_1y_2 + 5x_2x_3, \\ g(x,y) &= x_1y_2 + x_2y_4 + 1, \\ h(x,y) &= x_1y_1 - 7x_2y_3. \end{aligned}$$

f is not a tensor since

$$f(\lambda x, y) = 3\lambda x_1 y_2 + 5\lambda^2 x_2 x_3$$

so $f(2x,y) = 6x_1y_2 + 20x_2x_3$

$\neq 2f(x,y)$

g is not a tensor since

$$g(\lambda x, y) = \lambda x_1 y_2 + \lambda x_2 y_4 + 1$$

so $g(2x,y) = 2g(x,y) - 1 \neq 2g(x,y)$

h is a tensor since it is linear in both x, y

$$\begin{aligned} h(\lambda x + z, y) &= \lambda x_1 y_1 - 7\lambda x_2 y_3 + z_1 y_1 - 7 z_2 y_3 \\ &= (\lambda x_1 + z_1) y_1 - 7(\lambda x_2 + z_2) y_3 \\ &= \lambda h(x,y) + h(z,y) \end{aligned}$$

Similarly it is linear in y.

Define $e^i : \mathbb{R}^4 \to \mathbb{R}$

$x \mapsto x_i$ for each $i = 1, \dots, 4$,

Then $h(x,y) = x_1y_1 - 7x_2y_3 = \underbrace{e^1 \otimes e^1 - 7e^2 \otimes e^3}_{\text{elementary tensors}}$

Problem E: Pick one fact related to tensors whose proof was omitted in class. State it and prove it carefully.

Pick: Let V be a vector space, then $\forall k$, $L^k(V)$ is also a vector space

by defining $\begin{cases} (f+g)(v_1, \dots, v_k) = f(v_1, \dots, v_k) + g(v_1, \dots, v_k) \\ (cf)(v_1, \dots, v_k) = cf(v_1, \dots, v_k) \end{cases}$

Pf ① Let $f, g, h \in L^k(V)$, $c \in \mathbb{F}$ (scalar field of V)

Let $v_1, \dots, v_k \in V$

$$(cf+g)(v_1, \dots, v_k) = cf(v_1, \dots, v_k) + g(v_1, \dots, v_k)$$

Let $\alpha \in \mathbb{F}$, $w \in V$, $i \in \{1, \dots, k\}$

$$\begin{aligned} \Rightarrow (cf+g)(v_1, \dots, \alpha v_i + w, \dots, v_k) & \\ = c\alpha f(v_1, \dots, v_k) + f(v_1, \dots, w, \dots, v_k) + c\alpha g(v_1, \dots, v_k) & + g(v_1, \dots, w, \dots, v_k) \\ = \alpha(cf+g)(v_1, \dots, v_k) + (cf+g)(v_1, \dots, w, \dots, v_k) & \text{ is linear in } i^{th} \text{ coord} \end{aligned}$$

$\Rightarrow cf+g \in L^k(V)$

$\Rightarrow L^k(V)$ is closed under addition and scalar multiplication

② commutativity, associativity of tensor addition follows from def

③ additive identity: $f_0 : V \times \dots \times V \to \mathbb{F}$ sending every element to $0_{\mathbb{F}}$ $\in L^k(V)$

④ additive inverse: let $f \in L^k(V) \Rightarrow -f : (v_1, \dots, v_k) \mapsto -f(v_1, \dots, v_k)$ satisfies $(f+(-f))(v_1, \dots, v_k) = 0$

⑤ distributivity: follows from def and ①

$\square$

Problem F: Let $\sigma \in S_k$ be the permutation described by

$$1 \mapsto 2 \mapsto 3 \mapsto \cdots \mapsto k \mapsto 1.$$

Compute the sign of σ.

$t_1 \quad t_2 \quad t_3 \quad \dots \quad t_{k-1}$

$1 \quad 2 \quad 3 \quad 4 \quad \dots \quad k$

$2 \quad 3 \quad \dots \quad 1$

number of transpositions: $k-1$

So $\operatorname{sgn}\sigma = (-1)^{k-1}$

Problem G: Let $T : V \to W$ be a linear transformation. If $f \in \mathcal{A}^k(W)$, show $T^*(f) \in \mathcal{A}^k(V)$.

Pf let $f \in A^k(W)$,

$T^*(f)(v_1, \dots, v_k) = f(Tv_1, Tv_2, \dots, Tv_k)$ $\forall (v_1, \dots, v_k) \in V^k$

So $T^*(f) \in \mathcal{L}^k(V)$ since it it linear in each coordinate, by linearity of T

it remains to show the alternating property of $T^*(f)$

let $(v_1, \dots, v_k) \in V^k$, $\sigma \in S_k$

$$\Rightarrow T^*(f)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(Tv_{\sigma(1)}, \dots, Tv_{\sigma(k)})$$
$$= \operatorname{sgn}(\sigma) f(Tv_1, \dots, Tv_k)$$
$$= \operatorname{sgn}(\sigma) T^*(f)(v_1, \dots, v_k)$$

$\square$

Problem H: Read Theorem 27.7 and its proof in the text. Then, without looking at it, write out the statement and its proof.

Thm Let φ_I be an elementary alternating tensor on $\mathbb{R}^n$ with respect to the usual basis of $\mathbb{R}^n$, $I = (i_1, \dots, i_k)$

Given $x_1, \dots, x_k \in \mathbb{R}^n$, let $X = [\vec{x}_1 \dots \vec{x}_k]$

$\Rightarrow \varphi_I(x_1, \dots, x_k) = \det X_I$ where X_I denotes the matrix whose rows are row $i_1, i_2, \dots, i_k$ of X

Pf $\varphi_I(x_1, \dots, x_k) = \sum_\sigma (\operatorname{sgn}\sigma) \varphi_I(x_{\sigma(1)}, \dots, x_{\sigma(k)})$

$= \sum_\sigma (\operatorname{sgn}\sigma) x_{i_1,\sigma(1)} \cdot x_{i_2\sigma(2)} \cdot \ldots \cdot x_{i_k\sigma(k)}$

which is the expansion formula of $\det X_I$

Bonus: Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a function. We say that f is (real) analytic if for all $x_0 \in I$ there are real numbers $c_n, n \geq 0$ such that the series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

converges and is equal to $f(x)$ in a neighborhood of x_0.

(1) For each $n \geq 1$ let $a_n \in \{0, 1\}$ and let $0 \leq k_n \leq n$. Consider the function $f : (0,1) \to \mathbb{R}$ defined by

$$f(p) = \sum_{n=1}^{\infty} a_n p^{k_n} (1-p)^{n-k_b}.$$

Note that this sum converges by comparison to a geometric series. Show that f is (real) analytic.

(2) Given a power sets of the form $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, define the radius of convergence R by the equation

$$R = 1/ \limsup_{n \to \infty} |c_n|^{\frac{1}{n}},$$

where in this specific instance we use the convention $1/0 = \infty$ and $1/\infty = 0$. Show that the series converges uniformly and absolutely for $x \in (x_0 - R, x_0 + R)$ and diverges for $x \notin [x_0 - R, x_0 + R]$. (No statement is made about $x = x_0 + \pm R$.)

(3) Conclude that if the series converges on $(x_0 - r, x_0 + r)$ for some $r > 0$, then for any $0 < \rho < r$ there is a constant C such that $|c_n| \leq \frac{C}{\rho^n}$.

(4) Conclude that if the series converges on $(x_0 - r, x_0 + r)$ for some $r > 0$ then $f'(x_0)$ exists and is equal to c_1.

Remark The final part of the bonus should give you some intuition for the fact that real analytic functions are smooth. It wouldn't take all that much more work to prove this now, but the bonus is already long enough as it is! The idea is to define $g(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}$. You can show that g has the same radius of convergence as f. Using that the uniform limit of continuous functions is continuous, you can show it defines a continuous function on $(x_0 - R, x_0 + R)$.

The main claim now is that $f'(x)$ and is equal to $g(x)$. With this claim in hand we see that f is C^1, and repeating we get that f is C^∞. The main claim is proved via the following result, which could be a regular HW question for us: If ϕ_n are differentiable functions, and ϕ_n converge uniformly to ϕ, and ϕ_n' converge uniformly to ψ, then ϕ' exists and is equal to ψ. (All of this is on an interval.)