Functions on bdd sets

Thm1 Let $S \subseteq \mathbb{R}^n$ be bdd.

$f: S \to \mathbb{R}$ be bdd.

Let $D_f = \{x \in S \mid f \text{ not ctn at } x\}$

Let $E = \{x_0 \in \partial S \mid \lim_{x \to x_0} f(x) \neq 0\}$

$\int_S f$ exists $\iff m(D) = m(E) = 0$

Pf by def

Thm2 Let $S \subseteq \mathbb{R}^n$ be bdd. TFAE.

(1) S is Jordan measurable

(2) The const function 1 is Riem intble on S

(3) $m(\partial S) = 0$

(4) $\overline{m_J}(\partial S) = 0$

Rmk Moreover, S is Jordan measurable $\Rightarrow$ $\boxed{m_J(E) = \int_S 1\, dx}$ ☆

Pf $1 \iff 4$, $3 \Rightarrow 4$ by IBL

$4 \Rightarrow 3$: $m_J(\partial S) \le \overline{m_J}(\partial S) = 0$

$2 \iff 3$: Follows from Thm 1

for $f(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$ which is disctn on ∂S

Rmk: $\chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases} \Rightarrow D_{\chi_S} = \partial S$ ☆

Corollary S is Jordan measurable

$\iff$ all ctn functions from $S \to \mathbb{R}$ are Riem intble on S

Pf of corollary

all ctn functions $S \to \mathbb{R}$ Riem intble on S

$\Rightarrow f = 1$ Riem intble on S $\Rightarrow$ S Jordan measurable

S Jordan measurable $\Rightarrow m(\partial S) = 0$

$\Rightarrow m(E) = 0 \Rightarrow$ all const functions $(D_f = \phi)$ Riem intble

□

note

Jordan measurable

又称为

rectifiable

New topic:

Improper integrals

Goal: Define $\int_S f$ sometimes when S or f is unbdd

Def $\mathcal{J} := \{\text{all Jordan measurable sets on } \mathbb{R}^n\}$

$\mathcal{J}_c := \{\text{all cpt Jordan measurable sets on } \mathbb{R}^n\}$

For a function $f: S \to \mathbb{R}$, its positive & negative parts are

$f_+(x) = \max(f(x), 0) \quad (S \to \mathbb{R})$

$f_-(x) = \max(-f(x), 0) \quad (S \to \mathbb{R})$

Note $f_+(x), f_-(x) \geq 0$, $f = f_+ - f_-$ ☆, $|f| = f_+ + f_-$

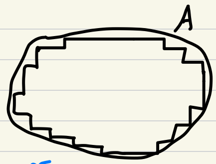
Rmk if f is ctn $\Rightarrow$ so is f_+, f_-

Def Let A be an open subset of $\mathbb{R}^N$

Let $f: A \to \mathbb{R}$ be ctn.

If f is non-neg on A

Define the extended integral of f over A as



$$\boxed{\int_A f = \sup_{\substack{D \subseteq A \\ D \in \mathcal{J}_c}} \int_D f}$$

(if the sup exists)

If $f: A \to \mathbb{R}$ is ctn and f_+, f_- both integrable in the extended sense

We say f is integrable in the extended sense

In this case we define

$$\boxed{\int_A f = \int_A f_+ - \int_A f_-}$$

Rmk1 Now we have two defs of "$\int_A f$" for open bdd A and ctn bdd f

We will see later that they agree when both define

Rmk2 The extended integral can be defined without the ordinary integral existing.

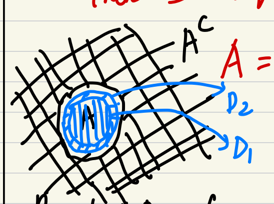
ex A bdd, open, not J mble

$f = 1$ on A not Riem intble by Thm 2

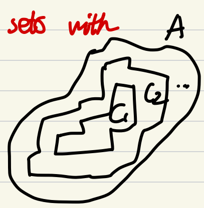
but have extended integral

Lemma Let $A \subseteq \mathbb{R}^n$ be open

Then $\exists$ a seq. $C_1, C_2, \ldots$ of cpt elem sets with



$A = \bigcup_{i=1}^{\infty} C_i$, $C_i \subseteq C_{i+1}^{\circ}$



Pf Let $D_N = \{x \in \mathbb{R}^n \mid \operatorname{dist}(A^c, x) \geq \frac{1}{N} \text{ and } |x| \le N\}$

Recall: $\operatorname{dist}(x, B) = \inf_{y \in B} |x - y|$ is a ctn. function of x ☆

Note that D_N is bdd by def

and closed since $x \mapsto |x|$

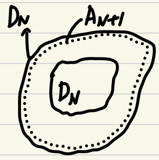
$x \mapsto \operatorname{dist}(x, A^c)$ are ctn.

So D_N is cpt

易得 $\bigcup_N D_N = A$

let $A_{N+1} = \{x \in \mathbb{R}^n \mid \text{dist}(x, A^c) > \frac{1}{N+1} \text{ \& } |x| < N+1\}$

Then A_{N+1} is open



Note $D_N \subseteq A_{N+1} \subseteq D_{N+1}$

since $\geq \frac{1}{N} \Rightarrow > \frac{1}{N+1}$, $\leq N \Rightarrow < N+1$; since $< \Rightarrow \leq$

D_N may not be J-meable

To fix this, $\forall x \in D_N$, pick an closed cube contained in A_{N+1} centered at x

The interiors of these cubes are an open cover for D_N

Define C_N to be the union of the closed cubes in a finite cover

So C_N is cpt and $D_N \subseteq C_N \subseteq A_{N+1}$ $(\forall x \in D_N \Rightarrow x \in C_N)$

Note $\bigcup_N C_N = A$ since $\bigcup_N D_N = A$ & $C_N \supseteq D_N$

$\Rightarrow C_N \subseteq A_{N+1} \subseteq D_{N+1} \subseteq C_{N+1}$

So since A_{N+1} is open, $C_N \subseteq C_{N+1}^{\circ}$

Properties of extended integrals

$A \subseteq \mathbb{R}^n$ open, $f, g : A \to \mathbb{R}$ ctn & ext intble

$\Rightarrow$ (a) $\forall c \in \mathbb{R}$, $f + cg$ is also ext intble

且 $\int_A f + cg = \int_A f + c\int_A g$

(ext $\int$ 和 ord $\int$ 一样 也有 linearity)

(b) if $f \leq g \Rightarrow \int_A f \leq \int_A g$

(In particular, $\int_A f \leq |\int_A f| \leq \int_A |f|$)

(c) if B open 且 $A \subseteq B$

且 f intble & non-neg over B

$\Rightarrow \int_A f \leq \int_B f$

(ext $\int$ 和 ord $\int$ 一样都有 monotonicity)

(d) if A, B open & f ctn on $A \cup B$

& f intble on A & B

$\Rightarrow f$ intble on $A \cup B$ 及 $A \cap B$

(union/intersection 下保持可积性, 前提是 ctn)

Thm Let $A \subseteq \mathbb{R}^n$ be open

$f : A \to \mathbb{R}$ ctn.

Choose seq $(C_N) \in J_c$ s.t. $A = \bigcup_{N=1}^{\infty} C_N$ and $C_N \subseteq C_{N+1}^{\circ}$

Then f is Riem intble on $A \iff \left\{\int_{C_N} |f|\right\}_{N=1}^{\infty}$ is bdd

in this case $\int_A f = \lim_{N \to \infty} \int_{C_N} f$

In particular, f is Riem intble over A iff $|f|$ is

Pf next time