hw 5

19/19

Problem A: Let $f : X \to Y$ be a function between metric spaces. Show that the set of points at which f is continuous is a countable intersection of open sets.

Pf Define

$O_n = \{x \in X \mid \exists \delta_x > 0 \text{ s.t } \forall x_1, x_2 \in B_{\delta_x}(x), \text{ we have } d_Y(f(x_1), f(x_2)) < \frac{1}{n}\}$

$C_f = \{x \mid f \text{ is ctn. at } x\}$

Claim 1 $C_f = \bigcap_{i=1}^{\infty} O_i$

Pf Suppose $x_0 \in C_f$. Let $n \in \mathbb{N} \Rightarrow \frac{1}{n} > 0$

By continuity, $\exists \delta > 0$ s.t. $f(x) \in B_{\frac{1}{2n}}(f(x_0))$ whenever $x \in B_\delta(x_0)$

So $\forall x_1, x_2 \in B_\delta(x_0)$, $d_Y(f(x_1), f(x_2)) \le d_Y(f(x_1), f(x_0)) + d_Y(f(x_0), f(x_2))$

Thus $x_0 \in O_n$ $\qquad < \frac{1}{2n} + \frac{1}{2n} < \frac{1}{n}$

Since n is arbitrary $\Rightarrow \forall i,\ x_0 \in O_i \Rightarrow x_0 \in \bigcap_{i=0}^{\infty} O_i$

$\Rightarrow$ Thus $C_f \subseteq \bigcap_{i=1}^{\infty} O_i$

Suppose $x_0 \in \bigcap_{i=0}^{\infty} O_i \Rightarrow \forall \varepsilon > 0$, can take

Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ s.t $\frac{1}{n} < \varepsilon \Rightarrow x_0 \in O_n$

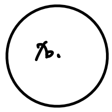
So $\exists \delta > 0$ s.t. $d(f(x_1), f(x_2)) < \frac{1}{n}$ for all $x_1, x_2 \in B_\delta(x_0)$

$\Rightarrow d(f(x'), f(x_0)) < \frac{1}{n} < \varepsilon$ for all $x' \in B_\delta(x_0)$

Thus $\bigcap_{i=1}^{\infty} O_i \subseteq C_f$ $\qquad \Rightarrow f$ ctn. at $x_0 \Rightarrow x_0 \in C_f$

Therefore $C_f = \bigcap_{i=1}^{\infty} O_i$. Claim proved.

Claim 2 $\forall i \in \mathbb{N}, O_i$ is open



Pf Let $n \in \mathbb{N}$. Let $x_0 \in O_n$

$\Rightarrow \exists \delta > 0$ s.t $\forall x_1, x_2 \in B_\delta(x_0)$, $d(f(x_1), f(x_2)) < \frac{1}{n}$

Consider $B_{\frac{\delta}{2}}(x_0)$: let $x_1 \in B_{\frac{\delta}{2}}(x_0)$,

we have $B_{\frac{\delta}{2}}(x_1) \subset B_\delta(x_0)$ by triangular ineq.

$\Rightarrow \forall y_1, y_2 \in B_{\frac{\delta}{2}}(x_1)$, $d(f(y_1), f(y_2)) < \frac{1}{n} \Rightarrow x_1 \in O_n$

Thus $B_{\frac{\delta}{2}}(x_0) \subseteq O_n$

Since x_0 is arbitrary in O_n, this finishes the proof of Claim 2.

Claim 2 + Claim 1 proves exactly what the statement is.

Problem B: Suppose that $f(x) : [a, b] \to \mathbb{R}$ is non-decreasing and bounded. Show that f is integrable.

Pf Let $m = \inf_{[a,b]} f(x)$, $M = \sup_{[a,b]} f(x)$

For all $q \in \mathbb{Q} \cap [m, M]$, define

$D_q = \{x \in [a,b] \mid \lim_{t \to x^-} f(t) \le q \le \lim_{t \to x^+} f(t)\}$

Write $D_f = \{x \in [a,b] \mid f \text{ is not ctn. at } x\}$

Claim 1 $D_f = \bigcup_{q \in \mathbb{Q} \cap [m,M]} D_q$

Pf Let $x \in D_q$ for some $q \in \mathbb{Q} \cap [m,M] \Rightarrow x \in D_f$

Let $x \in D_f \Rightarrow$ Since $[a,b]$ has not isolated pt., we must have $\lim_{t \to x^-} f(t) \ne \lim_{t \to x^+} f(t)$

Then by density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ some $q \in \mathbb{Q} \cap [\lim_{t \to x^-} f(t), \lim_{t \to x^+} f(t)]$

Thus $x \in D_q$ for some $q \in \mathbb{Q} \cap [m, M]$ (since f is non-decreasing)

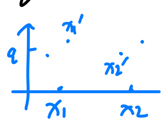
This proves Claim 1

Claim 2 $\forall q \in \mathbb{Q} \cap [m, M]$, D_q has at most one element

Pf Let $q \in \mathbb{Q} \cap [m, M]$

Suppose for contradiction that $x_1, x_2 \in D_q$ with $x_1 \ne x_2$

WLOG suppose $x_1 < x_2$



$\Rightarrow \lim_{t \to x_1^-} f(t) < q$, $\lim_{t \to x_2^-} f(t) < q$

$\lim_{t \to x_1^+} f(t) > q$, $\lim_{t \to x_2^+} f(t) > q$

So $\exists x_1' > x_1$ s.t $f(x_1') > q$ and $x_2' < x_2$ st. $f(x_2') < q$

Contradicts with f being non-decreasing.

This proves claim 2

By Claim 1, 2, D_f is countable.

Therefore D_f is Riem integrable.

$\square$

Problem C: Suppose that $f, g : [0, 1] \to \mathbb{R}$ are two integrable functions. Show that the function $F(x, y) : [0, 1]^2 \to \mathbb{R}$ given by $F(x, y) = f(x)g(y)$ is also integrable.

Pf Since f, g are bounded $\exists M_f, M_g \in \mathbb{R}$ s.t.

$|f(x)| \le M_f$, $|g(y)| \le M_g$ for all $x, y \in [0,1]$

$\Rightarrow |F(x,y)| \le M_f M_g$ for all $(x,y) \in [0,1]^2$. it is bounded

Let D_f, D_g be the set of points in $[0,1]$ where f, g is discontinuous at.

Since $F = f \cdot g$ is ctn. at (x_0, y_0) whenever f is ctn. at x_0 and g is ctn. at y_0

$\Rightarrow$ we have $D_F \subseteq (D_f \times [0,1]) \cup ([0,1] \times D_g)$

Claim D_F has measure 0

it suffices to prove that $D_f \times [0,1]$ has measure 0

since if so, then dually $[0,1] \times D_g$ has measure 0 and thus $(D_f \times [0,1]) \cup ([0,1] \times D_g)$ has measure 0, so by monotonicity D_F has measure 0.

Pf Let $\varepsilon > 0$

Since D_f is Riem integrable $\Rightarrow D_f$ has measure 0.

$\Rightarrow \forall \varepsilon > 0, \exists$ covering $\{Q_k \mid k \in \mathbb{N}\}$ s.t. $D_f \subseteq \bigcup_{k=1}^{\infty} Q_k$ and $\sum_{k=1}^{\infty} v(Q_k) < \varepsilon$

So take $\{I_k = Q_k \times [0,1] \mid k \in \mathbb{N}\} \Rightarrow D_f \times [0,1] \subseteq \bigcup_{k=1}^{\infty} I_k$

and $\sum_{k=1}^{\infty} v(I_k) = \sum_{k=1}^{\infty} (v(Q_k) \times 1) < \varepsilon$

This finishes the proof of $D_f \times [0,1]$ having measure 0, thus proving D_F has measure 0.

$\Rightarrow F$ is Riem integrable.

$\square$

Problem D: Let $f : [0,1] \to \mathbb{R}$ be defined by setting $f(x) = 1/q$ if $x = p/q$ where p and q are positive integers with no common factor, and $f(x) = 0$ otherwise.

(1) Without using the characterization of Riemann integrable functions, show that f is integrable.

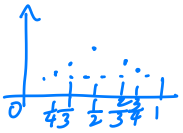
(2) Show that f is continuous except at a set of measure 0.

(1) Pf Let $\varepsilon > 0$. WTS: $\exists$ partition P s.t. $U(P,f) - L(P,f) < \varepsilon$

Take $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{2}$

Define

$A_N = \{x \in [0,1] \mid x = \frac{p}{q}$ in lowest term for some $q \le N\}$



$A_4 = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}\}$

$\Rightarrow \forall x \in A_N,\ f(x) \ge \frac{1}{N}$

and $\# A_N \le \frac{N(N-1)}{2} < \frac{N^2}{2}$ since $\forall q \in \mathbb{N}$, there can at most be $(q-1)$ terms with q as denominator in A_N.

Let P be a partition on $[0,1]$ s.t. $A_N \subseteq P$ and $\|P\| < \frac{\varepsilon}{N^2}$

For any subbox S created by P, we have

① if $S \cap A_N = \emptyset$, then $\sup_{x \in S} f < \frac{1}{N}$ since $\forall x = \frac{p}{q}$ in lowest term in S, $q > N$

② if $S \cap A_N \ne \emptyset$, then $\sup_{x \in S} f < 1$

Therefore $U(f,P) < \frac{N^2}{2}\|P\| + \frac{1}{N} < \frac{N^2}{2} \cdot \frac{\varepsilon}{N^2} + \frac{\varepsilon}{2} = \varepsilon$

Note that we always have $L(f,P) = 0$ since $(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ is dense in $[0,1]$, and $\forall x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ we have $f(x) = 0$.

So $U(f,P) - L(f,P) < \varepsilon$.

This proves that the function is Riem integrable. □

(2) Claim: $\forall x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$, f is ctn. at x.

Pf Let $x_0 \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$

Let $(x_n)_{n \in \mathbb{N}}$ be a seq of irrationals converging to x_0

The convergence is ensured by density of $(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$ in $[0,1]$

Then $f(x_n) \to 0$ since $\forall n \in \mathbb{N}$, $f(x_n) = 0$

So $\lim_{x \to x_0} f(x) = 0 = f(x_0)$

This proves that $\forall x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]$, f is ctn. at x.

So $D_f \subseteq \mathbb{Q} \cap [0,1]$ is at most countable, thus has measure 0. □

Problem E: Let Q be a box in $\mathbb{R}^n$ and $f : Q \to \mathbb{R}$ be a bounded function. Show that if f vanishes except on a closed set B of measure zero, then f is integrable.

Pf Suppose $|f(x)| \le M$ for some $M > 0$, $\forall x \in Q$

and $f(x) = 0$ $\forall x \in Q \setminus B$, where B is closed and has measure 0.

Let $\varepsilon > 0$. WTS: $\exists$ partition P s.t. $|U(f,P) - L(f,P)| < \varepsilon$

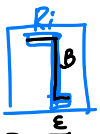
Since B has measure 0, we cover B with $\{R_i \mid i = 1, \dots, k\}$ of closed boxes s.t. $B \subseteq \bigcup_{i=1}^{k} R_i$ and $\sum_{i=1}^{k} v(R_i) < \frac{\varepsilon}{2M}$

Let P be a partition on Q s.t. P has all R_i, $i = 1, \dots, k$ as subboxes obtained by P.

For any subbox S created by P,

if S intersects $B \Rightarrow |f(x)| \le M$ for all $x \in S$

if S does not intersect $B \Rightarrow f(x) = 0$ for all $x \in S$



Therefore we have $L(P,f) \ge -M \sum_{i=1}^{k} v(R_i) > -M \cdot \frac{\varepsilon}{2M} = -\frac{\varepsilon}{2}$

$U(P,f) \le M \sum_{i=1}^{k} v(R_i) < M \cdot \frac{\varepsilon}{2M} = \frac{\varepsilon}{2}$

$\Rightarrow U(P,f) - L(P,f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Since ε is arbitrary, this proves that f is Riem integrable. □

Problem F: Show that if $Q_1, Q_2, \dots$ is a countable collection of closed boxes covering some box Q, then $v(Q) \le \sum v(Q_i)$.

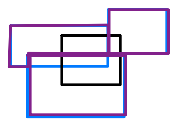
Pf It suffices to assume that $\forall i$, Q_i is open since if they are closed then we can replace the i^{th} box with a slightly bigger open box of volume at most $v(Q_i) + \frac{\varepsilon}{2^i}$

And can WLOG assume Q is closed since $v(Q)$ is the same no matter it is open or closed.

So Q is compact $\Rightarrow \exists$ finite subcover $\{Q_{k_1}, \dots, Q_{k_n}\} \subseteq \{Q_j \mid j \in \mathbb{N}\}$ s.t. $Q \subseteq \bigcup_{i=1}^{n} Q_{k_i}$

We have $\sum_{i=1}^{n} v(Q_{k_i}) \le \sum_{j=1}^{\infty} v(Q_j)$ since $\{Q_{k_i} \mid i \in \mathbb{N}\} \subseteq \{Q_j \mid j \in \mathbb{N}\}$ and each $v(Q_j) \ge 0$ ①

Let
$Q'_{k_n} = Q_{k_n} \setminus (\bigcup_{i=1}^{n-1} Q_{k_i})$
$Q'_{k_{n-1}} = Q_{k_{n-1}} \setminus (\bigcup_{i=1}^{n-2} Q_{k_i})$
$\vdots$
$Q'_{k_1} = Q_{k_1}$



Then $\{Q'_{k_1}, \dots, Q'_{k_n}\}$ is a disjoint cover of Q, since $\bigcup_{i=1}^{n} Q'_{k_i} = Q_{k_1} \cup Q_{k_2} \cup \dots \cup Q_{k_n} = \bigcup_{i=1}^{n} Q_{k_i}$

By $\{Q'_{k_i}\}$ being disjoint, $Q = \bigcup_{i=1}^{n} (Q'_{k_i} \cap Q)$

$\Rightarrow v(Q) = \sum_{i=1}^{n} v(Q'_{k_i} \cap Q) \le \sum_{i=1}^{n} v(Q'_{k_i})$ ②

Since $\forall i \in \{1, \dots, n\}$, $Q'_{k_i} \subseteq Q_{k_i} \Rightarrow \forall i \in \{1, \dots, n\}, v(Q'_{k_i}) \le v(Q_{k_i})$

$\Rightarrow \sum_{i=1}^{n} v(Q'_{k_i}) \le \sum_{i=1}^{n} v(Q_{k_i})$ ③

By ①②③, $v(Q) \le \sum_{i=1}^{n} v(Q'_{k_i}) \le \sum_{i=1}^{n} v(Q_{k_i}) \le \sum_{j=1}^{\infty} v(Q_j)$, exactly what we want. □

Problem G: Write out a proof of the following special case of the Implicit Function Theorem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^1, and suppose that $(x_0, y_0) \in \mathbb{R}^2$ is such that $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$. Prove that there is an interval I containing x_0 and a C^1 function $g : I \to \mathbb{R}$ such that

$$f(x, g(x)) = 0$$

for all $x \in I$. (You may not write a proof of the general case of the Implicit Function Theorem; the point is to specialize the general proof to this specific case where the notation is a bit simpler to better understand the proof.)

Pf Define an auxiliary function

$$F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$$

$$\Rightarrow DF(x,y) = \begin{pmatrix} DF_1 \\ DF_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$$

$$\Rightarrow \det DF(x,y) = \det \frac{\partial f}{\partial y}$$

So $\det DF(x_0, y_0) \neq 0$

$\Rightarrow$ By IFT, $\exists U \times V \ni (a,b)$ s.t. U, V open in $\mathbb{R}$ and $F|_{U\times V} : U \times V \to W$ is a local C^1 diffeo for some $W \subseteq \mathbb{R}^2$

Let $G : W \to U \times V$ be the inverse of $F|_{U\times V}$

$\Rightarrow \forall (x,y) \in U \times V$, $(x,y) = G(x, f(x,y))$

So $\forall (x,z) \in W$, $(x,z) = F \circ G(x,z)$

This shows that $G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ where G_1 is identity function

Since G is C^1 (by IFT), G_2 must be C^1.

Now we construct the implicit function g:

Let $I \ni x_0$ s.t. $I \times \{0\} \subseteq W$

$\Rightarrow$ Then $\forall (x,y) \in I \times V$,

$f(x,y) = 0$ iff $F(x,y) = (x, 0)$

iff $(x,y) = G(x,0) = (x, G_2(x,0))$

Define $g : I \to \mathbb{R}$

map $x \mapsto G_2(x,0)$

Then we have $f(x,y) = 0$ iff $y = g(x)$, $\forall x \in B$

Note that g is C^1 since G_2 is C^1 $\square$

Bonus: For this question, you can use without proof that if (a, b) is an open interval in $\mathbb{R}$, there is a smooth function with is positive on (a, b) and zero elsewhere. (You can see explicit examples of such functions on the Wikipedia page on bump functions.)

(1) If B is an open box in $\mathbb{R}^d$, show that there is a smooth function which is positive on B and zero elsewhere.

(2) If U is an open set in $\mathbb{R}^d$, show that there is a smooth function which is positive on U and zero elsewhere.

(3) So in particular, nasty sets like the Cantor set can be level sets of smooth functions. Why doesn't this contradict the Implicit Function Theorem? What more would you have to assume about f to get that $f^{-1}(0)$ can't be a nasty set like the Cantor set?

(4) Show that there exist two smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ such that the intersection of their graphs is the Cantor set in the x-axis.

(1) $B = (a_1, b_1) \times \dots \times (a_d, b_d)$ for some $a_1, \dots, a_d, b_1, \dots b_d \in \mathbb{R}$

For each pair a_i, b_i we define $\varphi_i : \mathbb{R} \to \mathbb{R}$ be a smooth function that is positive on (a_i, b_i) and zero elsewhere

Then define $f : \mathbb{R}^d \to \mathbb{R}$

$$x \mapsto \prod_{i=1}^{d} \varphi_i(x_i)$$

Then f is positive on B and 0 elsewhere

Note that f is smooth since each φ_i are smooth.

(2) Since $\mathbb{R}^d$ is second countable, we can find a countable cover $\{B_n \mid n \in \mathbb{N}\}$ of U s.t. each B_n is an open ball with $B_n \subseteq U$

For each B_n, suppose it is centered at x_n with radius r_n

Define $\psi_n(t) : [0, \infty) \to [0, \infty)$

$$t \mapsto \begin{cases} e^{\frac{-1}{r_n^2 - t^2}}, & t < r_n \\ 0, & t \geq r_n \end{cases}$$

Then define $\varphi_n : \mathbb{R}^d \to \mathbb{R}$

$x \mapsto \psi_n(\|x - x_n\|)$

Note that ψ_n is smooth, so φ_n is also smooth

Now define $f : \mathbb{R}^d \to \mathbb{R}$

$$x \mapsto \sum_{n=1}^{\infty} \varphi_n(x)$$

Since $\forall \varphi_n$, φ_n is positive for $x \in B_n$ and 0 for $x \notin B_n$ and $\forall B_n$, $B_n \subseteq U$

$\Rightarrow f(x)$ is positive iff $x \in U$

And smoothness of f is ensured by smoothness of each φ_n.

(3) Denote Cantor set by Cot.

Cat is closed. Consider Cat^2 which is also closed,

so the complement $C_{\mathbb{R}^2}(Cot^2)$ is open

Then by (2), there exists a smooth function f that is positive on $C_{\mathbb{R}^2}(Cat^2)$ and zero on Cat^2

Therefore $\underbrace{f^{-1}(0) = Cat^2}$

This seems to contradict with the Implicit function theorem since within appropriate conditions, there can exist some pt. $(x_0, y_0) \in Cat^2$ s.t. $\exists$ some open nbh $B \ni x_0$ and C^∞ function $g: B \to \mathbb{R}$ s.t. $g(x_0) = y_0$ and $\forall x \in B$ $f(x, g(x)) = 0$

But that cannot happen since if so then $B \subseteq Cat$ but Cat can contain no open set

However this does not necessarily contradict the Implicit Function Theorem since it requires $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

So we would assume f to be singular on boundary of U.

(4) By (2) we can let $h: \mathbb{R} \to \mathbb{R}$ be a function s.t.

$h(x) = 0$ for all $x \in Cot$

$h(x) > 0$ for all $x \notin Cot$

Define $f: \mathbb{R}^2 \to \mathbb{R}$, $g: \mathbb{R}^2 \to \mathbb{R}$

$\underbrace{(x,y) \mapsto y^2}$ $\underbrace{(x,y) \mapsto h(x)}$

$f(x,y) = g(x,y) \Leftrightarrow y^2 = h(x)$