38/38

Problem A: Suppose $A \subset \mathbb{R}^n$ is open and $f : A \to \mathbb{R}$ is differentiable at $x \in A$. Show that if u is a unit vector in $\mathbb{R}^n$, then

$$D_u f(x) \leq |Df(x)|,$$

with equality if and only if $u = Df(x)/|Df(x)|$. Show that $D_u f(x) = 0$ if and only if u is orthogonal to $Df(x)$.

Remark: Keeping in mind that for functions to $\mathbb{R}$ the Jacobian matrix is also called the gradient, this shows that the gradient is the direction of fastest change of the function. Similarly the set of directions where the function does not change (to first order) is the perp space of the gradient.

Pf (1) $D_u f(x) = \underset{\in \mathbb{R}^n}{Df(x)} \cdot \underset{\in \mathbb{R}^n}{u} = \nabla f(x) \cdot u$ since f is diff'ble at x

By Cauchy Schwarz we have $|\nabla f(x) \cdot u| \leq |\nabla f(x)| \cdot |u|$

Since $|u| = 1$, it shows that $D_u f(x) \leq |Df(x)|$

The equality holds true iff $\nabla f(x)$ is parallel to u

i.e. $\exists \lambda > 0$ s.t. $u = \lambda \nabla f(x)$

Suppose $u = \lambda \nabla f(x) \Rightarrow \lambda |\nabla f(x)| = 1$, $\lambda = \frac{1}{|\nabla f(x)|}$

Therefore $u = \frac{\nabla f(x)}{|\nabla f(x)|} = \frac{Df(x)}{|Df(x)|}$

(2) WTS: $D_u f(x) = 0$ iff u is orthogonal to $Df(x)$

Suppose $D_u f(x) = 0 \Rightarrow Df(x) \cdot u = 0 \Rightarrow Df(x)$ is orthogonal to u

Suppose u is orthogonal to $Df(x) \Rightarrow D_u f(x) = u \cdot Df(x) = 0$ □

Problem B: Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$ and $c : U \to \mathbb{R}$ are C^1. Set

$$M = c^{-1}(0).$$

Assume that f restricted to M has a local minimum at p, and that Dc is surjective at p. Then prove that there exist a real number λ such that

$$Df(p) = \lambda Dc(p).$$

(This means the gradients of f and ~~λ~~ c are parallel at p. The number λ is called a Lagrange multiplier.)

Pf Assume the hypotheses

Since Dc is surjective at p, $Dc(p) \neq 0$ (otherwise $\forall v \in \mathbb{R}^n$, $Dc(p)v = 0$, im $Dc(p) = 0$, can not be surjective)

So at least one entry of

$Dc(p) = \left(\frac{\partial c}{\partial x_1}(p), \dots, \frac{\partial c}{\partial x_n}(p)\right)$ is not 0

WLOG assume $\frac{\partial c}{\partial x_n}(p) \neq 0$. (can always reorder coordinates)

Write c in the form $c(x,y)$ with $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$

let $p = (a,b)$ with $a \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$

$\Rightarrow 0 \neq \frac{\partial f}{\partial y}(a,b) \in \mathbb{R}$, so by Implicit function theorem

$\exists$ nbh $B_\varepsilon(a) \subseteq \mathbb{R}^{n-1}$ and unique ctn. $g: B_\varepsilon(a) \to \mathbb{R}^n$

s.t. $g(a) = b$ and $c(x, g(x)) = 0$ $\forall x \in B_\varepsilon(a)$

Therefore M can be locally parametrized by g

i.e. $\exists$ some open nbh $U_p \subseteq M$ s.t. $U_p = \{(x, g(x)) \mid x \in B_\varepsilon(p)\}$

Let $h(x) = f(x, g(x))$ be a function from $U_p \to \mathbb{R}$

Since h reaches a local minimum at $x = a$ (where $(x, g(x)) = p$),

we have $\nabla h(a) = 0$

So $\frac{\partial f}{\partial x_i}(p) + \frac{\partial f}{\partial x_n}(p) \cdot \frac{\partial g}{\partial x_i}(a) = 0$ ①

for each $i = 1, \dots, n-1$

differentiating $c(x, g(x)) = 0$, we get

$\frac{\partial c}{\partial x_i}(p) + \frac{\partial c}{\partial x_n}(p) \cdot \frac{\partial g}{\partial x_i}(a) = 0$ for each $1 \leq i \leq n-1$

So $\frac{\partial g}{\partial x_i}(a) = \frac{-\frac{\partial c}{\partial x_i}(p)}{\frac{\partial c}{\partial x_n}(p)}$ ②

Combining ①② we have $\frac{\partial f}{\partial x_i}(p) - \frac{\partial f}{\partial x_n}(p) \frac{\frac{\partial c}{\partial x_i}(p)}{\frac{\partial c}{\partial x_n}(p)} = 0$ for each $i = 1, \dots, n-1$

So $\frac{\partial f}{\partial x_i}(p) - \frac{\frac{\partial f}{\partial x_n}(p)}{\frac{\partial c}{\partial x_n}(p)} \frac{\partial c}{\partial x_i}(p) = 0$ for each $i = 1, \dots, n-1$

Also when $i = n$, $\frac{\partial f}{\partial x_i}(p) - \frac{\frac{\partial f}{\partial x_n}(p)}{\frac{\partial c}{\partial x_n}(p)} \frac{\partial c}{\partial x_i}(p) = \frac{\partial f}{\partial x_i}(p) - \frac{\partial f}{\partial x_i}(p) = 0$

Note that $\frac{\frac{\partial f}{\partial x_n}(p)}{\frac{\partial c}{\partial x_n}(p)}$ is a constant we name it λ

Therefore $\forall i = 1, \dots, n$, $\frac{\partial f}{\partial x_i}(p) = \lambda \frac{\partial c}{\partial x_i}(p)$

This finishes the proof that $Df(p) = \lambda Dc(p)$ for some const λ.

□

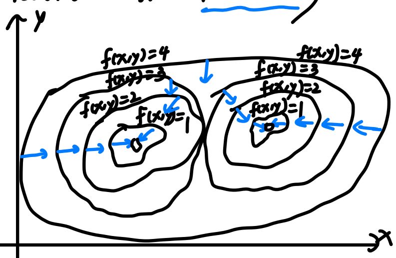
Problem C: In at most a few sentences, give a non-rigorous, intuitive explantion for Problem B.

we view f as a function to minimize

c as a function of contraint to this optimization

(constraint: $c(p) = 0$, we restrict dom(f) to $M = c^{-1}(0)$)

For the minimum pt. p, the gradient of f at p must be normal to all possible directions you can move it. (locally on M)



The gradient of c at p: the direction of the greatest rate of change of c (normal to M) at p

Therefore the gradient of f must be parallel to the gradient of c at p

Problem D: Using Problem B, find the mimum of the function $f(x,y) = 3x + y$ on the unit circle centered at the origin in $\mathbb{R}^2$.

Constraint: $c(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (3, 1)$

$\nabla c = \left(\frac{\partial c}{\partial x}, \frac{\partial c}{\partial y}\right) = (2x, 2y)$

By B, $\nabla f(p) = \lambda \nabla c(p)$ at minimum pt. p for some $\lambda \in \mathbb{R}$

$\Rightarrow 3 = 2\lambda x$, $1 = 2\lambda y \Rightarrow y = \frac{x}{3}$ at p

Take $(x, \frac{x}{3})$ into constraint we get: $x^2 + (\frac{x}{3})^2 = 1 \Rightarrow \frac{10}{9}x^2 = 1$,

$x = \pm \frac{3}{\sqrt{10}}$

Therefore critical points are $\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$ and $\left(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f\left(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right) = -\frac{10}{\sqrt{10}}$

$f\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = \frac{10}{\sqrt{10}}$

Thus the minimum of f on the unit circle is taken at $\left(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$, with value $-\sqrt{10}$.

Problem E: Formulate and prove a generalization of Problem B when c maps to $\mathbb{R}^k$ rather than $\mathbb{R}$. (Whereas Problem B allows you to do certain optimization problems subject to one constraint, this lets you do some optimization problems subject to k constraints. Your generalization will feature numbers $\lambda_1, \ldots, \lambda_k$.)

Remark: You must do B first and then E. You may not reference E in your solution to B.

Generalization: Let $U \subseteq \mathbb{R}^n$ be open

$f: U \to \mathbb{R}$, $c: U \to \mathbb{R}^k$ be in C^1

Restrict f to $M = c^{-1}(0)$.

Then if $f|_M$ has a local minimum at $p \in M$ and $Dc(p)$ has rank k, we must have

$Df(p) = \sum_{i=1}^{k} \lambda_i Dc_i(p)$ for some $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$

where $Dc(p) = \begin{pmatrix} Dc_1(p) \\ \vdots \\ Dc_k(p) \end{pmatrix}$

Pf Write $f(x_1, \ldots, x_n)$ as $f(x, y)$ with $x \in \mathbb{R}^{n-k}$, $y \in \mathbb{R}^k$

Write $p = (a, b)$ for $a \in \mathbb{R}^{n-k}$, $b \in \mathbb{R}^k$

Since $\operatorname{rank}(Dc(p)) = k$, we can WLOG suppose the $k \times k$ submatrix $\frac{\partial c}{\partial y}$ is invertible (non-singular) (can also get that by reordering variables)

Thus by the Implicit Function Thm,

$\exists$ nbh $B_\varepsilon(a) \subseteq \mathbb{R}^{n-k}$ s.t.

$g: B_\varepsilon(a) \to \mathbb{R}^k$ is C^1

with $\forall x \in B_\varepsilon(a)$, $c(x, g(x)) = 0$

Therefore locally around p, $\exists$ some nbh $U_p = \{(x, g(x)) \mid x \in V\} \subseteq M$

Define $h: B_\varepsilon(a) \to \mathbb{R}$

mapping $x \mapsto f(x, g(x))$

Since f reaches local minimum at $p \Rightarrow h$ reaches local minimum at a.

So $\nabla h(a) = 0$

Thus $\forall i = 1, 2, \ldots, n-k$, we have

$$\frac{\partial h}{\partial x_i} = \frac{\partial f}{\partial x_i}(p) + \sum_{j=1}^{k} \frac{\partial f}{\partial x_{n-k+j}}(p) \cdot \frac{\partial g_j}{\partial x_i}(a) = 0 \quad (1)$$

Similarly by differentiating $c(x, y) = 0$

$$\frac{\partial c}{\partial x_i}(p) + \frac{\partial c}{\partial y}(p)^T \frac{\partial g}{\partial x_i}(a) = 0 \quad (2)$$

$$\Rightarrow \frac{\partial g}{\partial x_i}(a) = -\left(\frac{\partial c}{\partial y}(p)\right)^{T^{-1}} \frac{\partial c}{\partial x_i}(p)$$

Combining (1)(2) we have

$$\frac{\partial f}{\partial x_i}(p) = \sum_{j=1}^{k} \frac{\partial f}{\partial x_{n-k+j}}(p) \cdot \left(\frac{\partial c}{\partial y}(p)\right)^{T^{-1}} \frac{\partial c}{\partial x_i}(p)$$

$$= \left(\frac{\partial f}{\partial y}(p)\right)^T \left(\frac{\partial c}{\partial y}(p)\right)^{-1} \cdot \frac{\partial c}{\partial x_i}(p)$$

take as $\lambda \in \mathbb{R}^k$

Problem F: Prove that the set of positive definite matrices is open in the set of n by n symmetric matrices. (You may not use the bonus from HW5.)

Pf Let PD_n denote all positive definite $n \times n$ matrices

Sym_n denote all $n \times n$ symmetric matrices.

WTS: PD_n is open in Sym_n

Let $A \in PD_n$

Consider $Q: S^{n-1} \to \mathbb{R}$

$x \mapsto x^T A x$

Q is ctn. since $Q(x) = x \cdot (Ax)$ where $x \mapsto Ax$ is ctn. and dot product is ctn. function so the composition is ctn

Since S^{n-1} is compact and Q is ctn

$\Rightarrow \{x^T A x \mid x \in S^{n-1}\}$ is compact in $\mathbb{R}$

Therefore we can take $m = \min_{x \in S^{n-1}} x^T A x > 0$ by positive definiteness

Let $\varepsilon = m$

Let $B \in Sym_n$ with $\|A - B\| < \varepsilon$

$x^T B x = x^T A x + x^T (B - A) x$

Since we have $|(B-A)x| \leq \|B - A\| |x|$

And by Cauchy Swartz,

$|x^T (B-A) x| = |x \cdot (B-A)x| \leq |x| \cdot |(B-A)x| \leq |x| \|B - A\| |x| = \|B - A\| < \varepsilon$

Therefore $x^T B x = x^T A x + x^T (B-A) x \geq m - |\varepsilon| > 0$

So $x^T B x$ is positive definite

This proves that PD_n is open in Sym_n. $\square$

Problem G: Suppose $f: A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open. Suppose that x_0 is critical point of f and the Hessian of f is positive definite at x_0. Prove that x_0 is a strict local minimum for f.

Pf x_0 being critical point $\Rightarrow \nabla f(x_0) = 0$

$$H_f(x_0) = \begin{pmatrix} \frac{\partial^2 f(x_0)}{\partial x_1^2} & \cdots & \frac{\partial^2 f(x_0)}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f(x_0)}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(x_0)}{\partial x_n^2} \end{pmatrix}$$

Since $H_f(x_0)$ is positive definite, we have $v^T H_f(x_0) v > 0$ for all $v \in \mathbb{R}^n$

Since A is open, take $\varepsilon > 0$ s.t. $B_\varepsilon(x_0) \subseteq A$

Note the open ball $B_\varepsilon(x_0)$ is an open and convex set

Apply the 1st-order Taylor expansion around x_0 for $x \in B_\varepsilon(x_0)$

we have $f|_{B_\varepsilon(x_0)}(x) = T_{x_0,1}(x) + R_{x_0,1}(x)$

$$= f(x_0) + \sum_{|\alpha|=1} \partial^\alpha f(x_0)(x - x_0)^\alpha + \sum_{|\alpha|=2} \frac{\partial^\alpha f}{\alpha!}(c)(x - x_0)^\alpha$$

$$= f(x_0) + \underbrace{\nabla f(x_0) \cdot (x - x_0)}_{=0} + \frac{1}{2}(x - x_0)^T H_f(c)(x - x_0)$$

for some $c \in B_\varepsilon(x_0)$, on the line segment between x, x_0.

So for $x \in B_\varepsilon(x_0)$, $f(x) - f(x_0) = \frac{1}{2}(x - x_0)^T H_f(c)(x - x_0)$

Note that $f \in C^2$, so every entry of H_f is ctn., so H_f is ctn.

Therefore we can choose $\delta > 0$ s.t. $H_f(x)$ is positive definite for all $x \in B_\delta(x_0)$

So we update $\varepsilon' = \min(\delta, \varepsilon)$ and find that $\forall x \in B_{\varepsilon'}(x_0)$, $f(x) - f(x_0) = \frac{1}{2}(x - x_0)^T H_f(c')(x - x_0)$ for some $c' \in B_{\varepsilon'}(x_0)$

Since $H_f(c')$ is positive definite, this proves that $f(x) > f(x_0)$ for all $x \in B_{\varepsilon'}(x_0)$

Hence x_0 is a strict local minimum of f. $\square$

Problem H: Let A be an invertible n by n matrices. Let C be its cofactor matrix, so $C_{ij} = (-1)^{i+j}\det(A_{ij})$, where A_{ij} is the $n-1$ by $n-1$ matrix obtained by deleting row i and column j from A. Prove the following version of Cramer's rule:

$$A^{-1} = \frac{1}{\det(A)} C^T,$$

where C^T denotes the transpose of C. (You may use the cofactor expansion of the determinant.)

Pf $(C)_{ij} = (-1)^{i+j}\det(A_{ij})$

cofactor expansion: $\det(A) = \sum_{j=1}^{n} (A)_{ij}(C)_{ij}$, $\forall$ row i

Consider the product AC^T

(note: $(A)_{ij}$ denote the ij^{th} element of A, while A_{ij} denote the $(n-1)\times(n-1)$ submatrix)

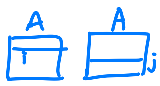
$$(AC^T)_{ij} = \sum_{k=1}^{n} (A)_{ik}(C^T)_{kj} = \sum_{k=1}^{n} (A)_{ik}(C)_{jk}$$

Thus

Case 1 for all $1 \le i = j \le n$, $(AC^T)_{ij} = \det(A)$ by cofactor expansion

Case 2 for all $1 \le i \ne j \le n$, $(AC^T)_{ij} = \sum_{k=1}^{n} (-1)^{j+k}(A)_{ik}\det(A_{jk})$

Consider matrix B with the j^{th} row identical to the i^{th} row of A, while the other rows of B identical to the corresponding of A



i.e. B is a copy of A with the j^{th} row replaced by the i^{th} row

Then $\det(B) = 0$ as $\operatorname{rank}(B) = n-1$

Cofactor expand of B at row j: $\det(B) = \sum_{k=1}^{n} (-1)^{j+k}(B)_{jk}\det(B_{jk})$

$= \sum_{k=1}^{n} (-1)^{j+k}(A)_{ik}\det(A_{jk})$ since all rows but j^{th} of B are same as A &

$= (AC^T)_{ij}$

Thus $(AC^T)_{ij} = 0$ for all $1 \le i \ne j \le n$

Therefore $AC^T = \begin{bmatrix} \det A & & & \\ & \det A & & \\ & & \ddots & \\ & & & \det A \end{bmatrix}$

$= (\det A) I_n$

$\Rightarrow A^{-1} = \frac{1}{\det A} C^T$ $\square$

Problem I: Let $f, g : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ be differentiable. Show that

$$(f \cdot g)'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t),$$

where $\cdot$ denotes dot product and $f'(t)$ denotes $Df(t)$ (which in this case is a vector).

Pf $(f\cdot g)(t) = f(t)\cdot g(t) = \sum_{i=1}^{n} f_i(t) g_i(t)$

$\frac{d}{dt}(f\cdot g)(t) = \frac{d}{dt}\sum_{i=1}^{n} f_i(t) g_i(t)$

$= \sum_{i=1}^{n} \left(f_i'(t) g_i(t) + f_i(t) g_i'(t)\right)$

$= \sum_{i=1}^{n} f_i'(t) g_i(t) + \sum_{i=1}^{n} f_i(t) g_i'(t)$

$= f'(t)\cdot g(t) + f(t)\cdot g'(t)$ $\square$

Bonus: Suppose $f : A \to \mathbb{R}$ is C^2, with $A \subset \mathbb{R}^n$ open and convex. Show that the region above f, i.e.

$$\{(x, y) \in A \times \mathbb{R} : y \ge f(x)\},$$

is convex if and only if the Hessian of f is positive semi-definite at each point of A.

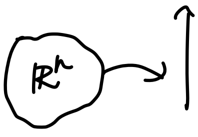
Pf We first show that:

Claim 1 f is convex function iff $H_f(x)$ is positive semidefinite for each $x \in A$

i.e. $\forall x, y \in A, \forall \lambda \in [0,1]$, $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

($\Leftarrow$) Assume $\forall x \in A$, $H_f(x)$ is positive semidefinite

Let $x, y \in A$



Consider function $g : [0,1] \to \mathbb{R}^n$

$g(t) = x + t(y-x)$

and $\varphi : [0,1] \to \mathbb{R}$

$\varphi(t) = f(g(t))$

So

$\Rightarrow \varphi'(t) = \nabla f(g(t))^T g'(t) = \nabla f(g(t))^T (y-x)$

$\Rightarrow \varphi''(t) = (y-x)^T H_f(g(t))(y-x) \ge 0$ by positive semi-definite of $H_f(g(t))$ and f being C^2

In note that φ is a function from $[0,1]$ to $\mathbb{R}$

By result of one-variable analysis we know that φ is convex function

So $\forall t_1, t_2 \in [0,1]$ and $\lambda \in [0,1]$,

$\varphi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\varphi(t_1) + (1-\lambda)\varphi(t_2)$

Set $t_1 = 0, t_2 = 1$, we have $\varphi(\lambda) \le \lambda\varphi(0) + (1-\lambda)\varphi(1)$

i.e. $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$

Since x, y is arbitrary, we proved that f is convex on A

($\Rightarrow$) Assume f is convex on A

Let $x \in A, v \in \mathbb{R}^n$

Define $\varphi : \mathbb{R} \to \mathbb{R}$

$h \mapsto f(x_0 + hv)$

Since $f \in C^2(A)$, $h \mapsto x_0 + hv$ is $C^2 \Rightarrow \varphi$ is also C^2

Claim 1.1 φ is convex

Pf of Claim 1.1 Let $a, b \in \mathbb{R}, \lambda \in [0,1]$

Let $g : \mathbb{R} \to \mathbb{R}^n$

$g(h) = hv + x_0 = vh + x_0$ (g is an affine function)

So $g(\lambda a + (1-\lambda)b) = v(\lambda a + (1-\lambda)b) + x_0 = \lambda g(a) - \lambda x_0 + (1-\lambda)g(b) - (1-\lambda)x_0 + x_0$

$= \lambda g(a) + (1-\lambda) g(b)$

Since f is convex,

$\varphi(\lambda a + (1-\lambda)b) = f(g(\lambda a + (1-\lambda)b))$

$= f(\lambda g(a) + (1-\lambda)g(b)) \le \lambda f(g(a)) + (1-\lambda) f(g(b))$

$= \lambda f\circ g(a) + (1-\lambda) f\circ g(b)$

$= \lambda\varphi(a) + (1-\lambda)\varphi(b)$

Claim 1.1 $\square$

By $\varphi''(h)$ being C^2, $\varphi'(h) = \frac{d}{dh}(f(x+hv)) = \nabla f(x_0+hv)^T v$

$\varphi''(h) = v^T H_f(x_0+hv) v$

Since φ is convex $\Rightarrow \varphi''(0) = v^T H_f(x_0) v \ge 0$

Forward direction $\square$

Claim 1 $\square$

Now we use claim 1 to prove the statement.

We use upgraph(f) to refer to the set.

Assume $\forall x \in A$ we have $H_f(x)$ being positive semidefinite

Let $(x_1, y_1), (x_2, y_2) \in$ upgraph(f), $\lambda \in [0,1]$

$$y_i \geq f(x_i) \Rightarrow \lambda y_1 + (1-\lambda) y_2 \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$
$$\geq f(\lambda x_1 + (1-\lambda) x_2)$$