Problem A: Show that there is no injective smooth function

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

with $n > m$.

We first prove a corollary of IFT that we will need for the proof.

Lemma constant rank theorem

if f: open $U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^p and $\forall x \in U$, $\text{rank}(\text{Det } Df(x)) = r$ is const

$\Rightarrow \forall x_0 \in U$, $\exists$ some nbh $V \ni x_0$ and $W \ni f(x_0)$ and C^p diffeo $\varphi : V \to V' \subseteq \mathbb{R}^n$, $\psi: W \to W' \subseteq \mathbb{R}^m$

st. $\psi \circ f \circ \varphi^{-1} : V' \to W'$

$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ where the $r+1^{th} \sim n^{th}$ coordinate of image is 0

Pf of Lemma

By hw10: $\forall x \in U$, $Df(x)$ has a nonsingular $r \times r$ minor and all larger minors are singular

WLOG suppose the left upper $r \times r$ minor is non-singular

Let $\varphi(x) = (f_1(x), \dots, f_r(x), x^{r+1}, \dots, x^n)$

$\Rightarrow \forall x \in U$, $\det D\varphi(x) \neq 0$

$\Rightarrow$ by IFT, φ is a C^p diffeo around every point in U

and $f \circ \varphi^{-1}(v) = (v_1, \dots, v_r, g^{r+1}(v), \dots, g^m(v))$

Since φ^{-1} invertible, the $\text{rank}(Dg) = \text{ran}(Df \circ \varphi^{-1}) = r$

$$Dg = \begin{pmatrix} I_r & 0_{r \times (n-r)} \\ \left(\frac{\partial g_i}{\partial v_j}\right)_{\substack{i=r+1,\dots,m \\ j=1,\dots,r}} & \left(\frac{\partial g_i}{\partial v_j}\right)_{\substack{i=r+1,\dots,m \\ j=r+1,\dots,n}} \end{pmatrix}$$

So $\left(\frac{\partial g_i}{\partial v_j}\right)_{\substack{i=r+1,\dots,m \\ j=r+1,\dots,n}}$ is 0 matrix, $g(v)$

let $\psi(y) = (y_1, \dots, y_r, y_{r+1} - g_{r+1}(y), \dots, y_m - g_m(y))$

$\Rightarrow \det D\psi = 1$, so it is locally C^p diffeo around $f(x_0)$

and $\psi \circ f \circ \varphi^{-1}(v) = (v_1, \dots, v_r, 0, \dots, 0)$ locally $\square$

Then suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is injective and smooth with $n > m$

Since $\text{rank}(Df)$ is lower semi-ctn and take values from finite set $\{0, \dots, m\}$,

it must be locally const on some $x_0 \in \mathbb{R}^n$

Thus $\exists$ C^p diffeo $\varphi : V \to V' \ni x_0$ and $\psi : W \to W' \ni f(x_0)$

s.t. $\psi \circ f \circ \varphi^{-1} : \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ \vdots \\ v_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ not injective

Since ψ, φ^{-1} are diffeo thus bijective and f injective, $\psi \circ f \circ \varphi^{-1}$ should be injective, contradicts. $\square$

Problem B: Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. Suppose that f and g are 0 outside a compact set. Define the convolution $f * g$ by:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$

Show that

$$\int_{\mathbb{R}^n} f * g = \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right).$$

Also show that convolution is commutative and associative.

Pf integrating both sides on $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} (f * g) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) g(y) \, dy \right) dx$$

Since f, g are supported only on a cpt set.

Let $B = B_1 \times B_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be box st. $\text{supp}(f), \text{supp}(g) \subseteq B$

$$\Rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y) g(y) = \int_B f(x-y) g(y)$$

$$= \int_{B_1} \int_{B_2} f(x-y) g(y) \text{ by Fubini's Thm}$$

$$\text{So } \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) g(y) \, dy \, dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y) g(y)$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y) g(y) \, dx \right) dy$$

since we can interchange the order by Fubini

$$= \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} f(x-y) \, dx \right) dy = \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} f(z) \, dz \right) dy$$

$$= \left(\int_{\mathbb{R}^n} g(y) \, dy \right) \left(\int_{\mathbb{R}^n} f(z) \, dz \right) = \int_{\mathbb{R}^n} f \int_{\mathbb{R}^n} g$$

Pf of commutativity

Let $x \in \mathbb{R}^n$

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y) \, dy$$

$$= \int_{\mathbb{R}^n} f(z) g(x-z) \, dz = \int_{\mathbb{R}^n} g(x-z) f(z) \, dz$$

$$= g * f(x)$$

Pf of associativity

let $x \in \mathbb{R}^n$

$$((f * g) * h)(x) = \int_{\mathbb{R}^n} f * g(x-y) \, h(y) \, dy$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y-z) g(z) \, dz \right) h(y) \, dy$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x-y-z) \, g(z) \, h(y) \, dz \, dy$$

by Fubini as previous

$$= \int_{\mathbb{R}^n} f(x-y-z) \left(\int_{\mathbb{R}^n} g(z) h(y) dy \right) dz$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-u) \, g(u-v) \, h(v) \, du \, dv$$

$$(f * (g * h))(x) = \int_{\mathbb{R}^n} f(x-y) (g * h(y)) \, dy$$

$$= \int_{\mathbb{R}^n} f(x-y) \left(\int_{\mathbb{R}^n} g(y-z) h(z) \, dz \right) dy$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-u) g(u-v) h(v) \, du \, dv = ((f * g) * h)(x) \quad \square$$

Problem C: Find the maximum and minimum values of

$$f(x,y) = 4x^2 + 10y^2$$

on the disk $x^2 + y^2 \leq 4$.

f is ctn and defined on cpt area in $\mathbb{R}^2$, thus must have both max, min points.

On the interior: any extreme value point (x_0,y_0) must have

$$\nabla f(x_0,y_0) = (8x_0, 20y_0) = (0,0) \Rightarrow (x_0,y_0) = (0,0)$$

$$f(0,0) = 0$$

On the boundary: $x^2+y^2=4$

$$f(x,y) = 4(x^2+y^2)+6y^2 = 16+6y^2,$$

so f takes maximum when $|y|$ reaches maximum

$\Rightarrow f_{max} = 16+6\times 2^2 = 40$

on ∂D with $y_0 = \pm 2$

So $x_0 = 0$

Thus $f_{min} = 0$ at $(0,0)$

$f_{max} = 40$ at $(0,2)$ and $(0,-2)$

Problem D: Determine which of the following are tensors on $\mathbb{R}^4$, and express those that are in terms of the elementary tensors:

$$\begin{aligned} f(x,y) &= 3x_1y_2 + 5x_2x_3, \\ g(x,y) &= x_1y_2 + x_2y_4 + 1, \\ h(x,y) &= x_1y_1 - 7x_2y_3. \end{aligned}$$

f is not a tensor since

$$f(\lambda x, y) = 3\lambda x_1 y_2 + 5\lambda^2 x_2 x_3$$

so $f(2x,y) = 6x_1y_2 + 20x_2x_3$

$\neq 2f(x,y)$

g is not a tensor since

$$g(\lambda x, y) = \lambda x_1 y_2 + \lambda x_2 y_4 + 1$$

so $g(2x,y) = 2g(x,y) - 1 \neq 2g(x,y)$

h is a tensor since it is linear in both x, y

$$\begin{aligned} h(\lambda x + z, y) &= \lambda x_1 y_1 - 7\lambda x_2 y_3 + z_1 y_1 - 7 z_2 y_3 \\ &= (\lambda x_1 + z_1) y_1 - 7(\lambda x_2 + z_2) y_3 \\ &= \lambda h(x,y) + h(z,y) \end{aligned}$$

Similarly it is linear in y.

Define $e^i : \mathbb{R}^4 \to \mathbb{R}$

$x \mapsto x_i$ for each $i = 1, \dots, 4$,

Then $h(x,y) = x_1y_1 - 7x_2y_3 = \underbrace{e^1 \otimes e^1 - 7e^2 \otimes e^3}_{\text{elementary tensors}}$

Problem E: Pick one fact related to tensors whose proof was omitted in class. State it and prove it carefully.

Pick: Let V be a vector space, then

$\forall k$, $L^k(V)$ is also a vector space

by defining $\begin{cases} (f+g)(v_1,\dots,v_k) = f(v_1,\dots,v_k) + g(v_1,\dots,v_k) \\ (cf)(v_1,\dots,v_k) = c(f(v_1,\dots,v_k)) \end{cases}$

Pf ① Let $f,g,h \in L^k(V)$, $c \in \mathbb{F}$ (scalar field of V)

Let $v_1,\dots,v_k \in V$

$(cf+g)(v_1,\dots,v_k) = cf(v_1,\dots,v_k) + g(v_1,\dots,v_k)$

Let $\alpha \in \mathbb{F}$, $w \in V$, $i \in \{1,\dots,k\}$

$\Rightarrow (cf+g)(v_1,\dots,\alpha v_i + w,\dots,v_k)$

$= c\alpha f(v_1,\dots,v_k) + f(v_1,\dots,w,\dots,v_k) + c\alpha g(v_1,\dots,v_k) + g(v_1,\dots,w,\dots,v_k)$

$= \alpha(cf+g)(v_1,\dots,v_k) + (cf+g)(v_1,\dots,w,\dots,v_k)$ is linear in i^{th} coord

$\Rightarrow cf+g \in L^k(V)$

$\Rightarrow L^k(V)$ is closed under addition and scalar multiplication

② commutativity, associativity of tensor addition follows from def

③ additive identity: $f_0 : V \times \dots \times V \to \mathbb{F}$ sending every element to $0_{\mathbb{F}}$, $f_0 \in L^k(V)$

④ additive inverse: let $f \in L^k(V) \Rightarrow -f : (v_1,\dots,v_k) \mapsto -f(v_1,\dots,v_k)$ satisfies $(f+(-f))(v_1,\dots,v_k) = 0$

⑤ distributivity: follows from def and ①

□

Problem F: Let $\sigma \in S_k$ be the permutation described by

$$1 \mapsto 2 \mapsto 3 \mapsto \dots \mapsto k \mapsto 1.$$

Compute the sign of σ.

t_1 t_2 t_3 t_{k-1}

1 2 3 4 ... k

2 3 1

number of transpositions : $k-1$

So $\operatorname{sgn}\sigma = (-1)^{k-1}$

Problem G: Let $T : V \to W$ be a linear transformation. If $f \in \mathcal{A}^k(W)$, show $T^*(f) \in \mathcal{A}^k(V)$.

Pf let $f \in A^k(W)$,

$$T^*(f)(v_1, \dots, v_k) = f(Tv_1, Tv_2, \dots, Tv_k) \quad \forall (v_1, \dots, v_k) \in V^k$$

So $T^*(f) \in \mathcal{L}^k(V)$ since it it linear in each coordinate, by linearity of T

it remains to show the alternating property of $T^*(f)$

let $(v_1, \dots, v_k) \in V^k$, $\sigma \in S_k$

$$\Rightarrow T^*(f)(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(Tv_{\sigma(1)}, \dots, Tv_{\sigma(k)})$$
$$= \operatorname{sgn}(\sigma) f(Tv_1, \dots, Tv_k)$$
$$= \operatorname{sgn}(\sigma) T^*(f)(v_1, \dots, v_k)$$

$\square$

Problem H: Read Theorem 27.7 and its proof in the text. Then, without looking at it, write out the statement and its proof.

Thm Let φ_I be an elementary alternating tensor on $\mathbb{R}^n$ with respect to the usual basis of $\mathbb{R}^n$, $I = (i_1, \dots, i_k)$

Given $x_1, \dots, x_k \in \mathbb{R}^n$, let $X = [\vec{x}_1 \dots \vec{x}_k]$

$\Rightarrow \varphi_I(x_1, \dots, x_k) = \det X_I$ where X_I denotes the matrix whose rows are row $i_1, i_2, \dots, i_k$ of X

Pf $$\varphi_I(x_1, \dots, x_k) = \sum_\sigma (\operatorname{sgn}\sigma) \varphi_I(x_{\sigma(1)}, \dots, x_{\sigma(k)})$$
$$= \sum_\sigma (\operatorname{sgn}\sigma)\, x_{i_1 \sigma(1)} \cdot x_{i_2 \sigma(2)} \cdot \ldots \cdot x_{i_k \sigma(k)}$$

which is the expansion formula of $\det X_I$

Bonus: Let I be an open interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$ be a function. We say that f is (real) analytic if for all $x_0 \in I$ there are real numbers $c_n, n \geq 0$ such that the series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

converges and is equal to $f(x)$ in a neighborhood of x_0.

(1) For each $n \geq 1$ let $a_n \in \{0, 1\}$ and let $0 \leq k_n \leq n$. Consider the function $f : (0, 1) \to \mathbb{R}$ defined by

$$f(p) = \sum_{n=1}^{\infty} a_n p^{k_n} (1 - p)^{n - k_b}.$$

Note that this sum converges by comparison to a geometric series. Show that f is (real) analytic.

(2) Given a power sets of the form $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, define the radius of convergence R by the equation

$$R = 1 / \limsup_{n \to \infty} |c_n|^{\frac{1}{n}},$$

where in this specific instance we use the convention $1/0 = \infty$ and $1/\infty = 0$. Show that the series converges uniformly and absolutely for $x \in (x_0 - R, x_0 + R)$ and diverges for $x \notin [x_0 - R, x_0 + R]$. (No statement is made about $x = x_0 + \pm R$.)

(3) Conclude that if the series converges on $(x_0 - r, x_0 + r)$ for some $r > 0$, then for any $0 < \rho < r$ there is a constant C such that $|c_n| \leq \frac{C}{\rho^n}$.

(4) Conclude that if the series converges on $(x_0 - r, x_0 + r)$ for some $r > 0$ then $f'(x_0)$ exists and is equal to c_1.

Remark The final part of the bonus should give you some intuition for the fact that real analytic functions are smooth. It wouldn't take all that much more work to prove this now, but the bonus is already long enough as it is! The idea is to define $g(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{n-1}$. You can show that g has the same radius of convergence as f. Using that the uniform limit of continuous functions is continuous, you can show it defines a continuous function on $(x_0 - R, x_0 + R)$.

The main claim now is that $f'(x)$ and is equal to $g(x)$. With this claim in hand we see that f is C^1, and repeating we get that f is C^∞. The main claim is proved via the following result, which could be a regular HW question for us: If ϕ_n are differentiable functions, and ϕ_n converge uniformly to ϕ, and ϕ_n' converge uniformly to ψ, then ϕ' exists and is equal to ψ. (All of this is on an interval.)