Metric space

Boundedness

$A \subseteq X$ is bdd if $\sup_{x,y \in A} d(x,y) < \infty$

$A \subseteq X$ is totally bdd. if $\forall \varepsilon > 0, \exists$ finite $S \subseteq A$ s.t. $\bigcup_{s \in S} B_\varepsilon(s) \supseteq A$

Completeness:

A is cplt if $\forall$ Cauchy seq in A conv. to some pt. in A

compactness:

A is cpt. if $\forall$ open cover of A has a finite subcover

A is seq. cpt. if $\forall$ seq in A has a subseq. that conv. to some pt. in A

In M.S. seq. cpt $\Leftrightarrow$ cpt $\Leftrightarrow$ cplt + ttl. bdd.

In $\mathbb{R}^n$, cplt $\Leftrightarrow$ closed

bdd $\Leftrightarrow$ ttl. bdd.

Continuous

$f: X \to Y$ is ctn. if $\forall$ open $U \subseteq Y$, $f^{-1}(U)$ is open
($\Leftrightarrow$ if $\forall$ closed $V \subseteq Y$, $f^{-1}(V)$ is closed)

Thm

closed subset of cpt set is cpt.

Thm

if f ctn.

f(cpt set) is cpt.

Differentiation

$f: E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diffble at x_0 if

$\exists$ linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\lim_{|h| \to 0} \frac{|f(x_0+h) - f(x_0) - Ah|}{|h|} = 0$$

Say $Df(x_0) = A$ unique.

Should know how to prove

$p_k = (x_1^{(k)}, \ldots, x_n^{(k)}) \in \mathbb{R}^n$

then $(p_k) \to p \Leftrightarrow \forall i, (p_k^{(i)}) \to p^{(i)}$
$\Leftrightarrow (|p - p_k|) \to 0$

$Df(x_0)$ is the best linear approximation to $h \mapsto f(x_0+h) - f(x_0)$

Directional derivative

1. (def) $D_u f(x_0) = \lim_{t \to 0} \dfrac{f(x_0+tu) - f(x_0)}{t}$
$= \dfrac{d}{dt}\Big|_{t=0} f(x_0 + tu)$

2. If $Df(x_0)$ exists, $D_u f(x_0) = Df(x_0)\, u$

3. (def) $\dfrac{\partial f}{\partial x_i} = D_{e_i} f$

4. If $Df(x_0)$ exists, it is the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_m \end{pmatrix}$$

5. **Thm** if all partials of f exists and ctn. near x_0 $\Rightarrow$ f diffble at x_0
(i.e. C^1 near x_0)

mixed partial Thm

if f is C^k, then any k-order partials of f can change any partial order

ex f is $C^3 \Rightarrow \dfrac{\partial^3 f}{\partial x \partial y \partial z} = \dfrac{\partial^3 f}{\partial z \partial x \partial y}$

Multi index notation:

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$

$|\alpha| = \sum \alpha_i$

$\alpha! = \alpha_1! \cdots \alpha_n!$

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

$\partial^\alpha f = \left(\dfrac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial x_n}\right)^{\alpha_n} f$

Chain rule

if $Df(x_0)$, $D_g(f(x_0))$ exists

$\Rightarrow D(g \circ f)(x_0) = D_g(f(x_0)) Df(x_0)$

typical ex. $f: \mathbb{R}^2 \to \mathbb{R}^2$, $g: \mathbb{R}^2 \to \mathbb{R}$

Let $G(x) = \begin{pmatrix} x \\ g(x,y) \end{pmatrix} \Rightarrow f(x, g(x)) = f(G(x,y))$

$Df(x, g(x)) = D(f \circ G)(x,y) = Df(G(x,y))\, DG(x,y)$

Product Rule

$\alpha = e_i + \alpha'$

$|\alpha'| = |\alpha| + 1$

$\partial^\alpha = \partial^{e_i}\partial^{\alpha'} = \partial^{\alpha'}\partial^{e_i}$

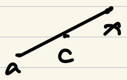
Taylor's Thm

if $f \in C^{r+1}(A)$ and A convex

$\forall a \in A$

$\forall x \in A,\ f(x) = \underbrace{\sum_{|\alpha| \le k} \frac{\partial^\alpha f}{\alpha!}(x-a)^\alpha}_{T_{a,k}(x)} + R_{a,k}(x)$

$R_{a,k}(x) = \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(c)}{\alpha!}(x-a)^\alpha$



$T_{a,k}$ is the uniq. poly. of deg $\le k$ centered at a that best approximates f.

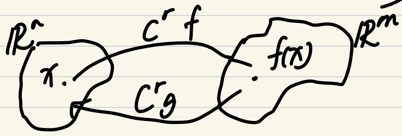
$f(x) = Ax$ then f inv. $\Leftrightarrow$ A inv. (non singular)

Inverse function Thm

If $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^r $(r \geq 1)$ and $Df(x_0) \neq 0$

$\Rightarrow f$ is a local diffeo at x_0

(for $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, it is true without C^r)



(note) so that we can solve $\begin{cases} y_1 = f_1(x_1, \dots x_n) \\ \vdots \\ y_n = f_n(x_1, \dots x_n) \end{cases}$

for x_i in terms of y_i in some nbhs

so can use $(y_1, \dots, y_n)$ as local coordinates.

Implicit Function Thm

If $f: \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is linear

i.e. $f(x,y) = Ax + By$

and B is invertible

$\Rightarrow y = B^{-1}(f(x,y) - Ax)$

If $f: A \subseteq \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is C^r

& $f(a,b)$ & $\frac{\partial f}{\partial y}(a,b) \neq 0$

$\Rightarrow \exists\, g: B_\varepsilon(a,b) \subseteq \mathbb{R}^k \to \mathbb{R}^n$ C^r

s.t. $\{ f|_{B_\varepsilon(a,b)}(x,y) = 0 \} = \{ (x, g(x)) \mid x \in B_\varepsilon(a,b) \}$

AND $\exists$ a formula for Dg

Integration

$f: B \to \mathbb{R}$

$P = (P_1, \dots, P_n)$ be a partition on $B \subseteq \mathbb{R}^n$

$L(f,P) = \sum_{\text{subbox } S} m_S\, v(S)$

$G(x) = \begin{pmatrix} x \\ g(x) \end{pmatrix}$

$DG = \begin{pmatrix} I_k \\ \hline Dg \end{pmatrix}$

*Pf ① if $f:\mathbb{R}^2\to\mathbb{R}$ C^1, then diff'ble

Pf f $C^1 \Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ ctn. on $\mathbb{R}^2$

Let $x_0\in\mathbb{R}^2$

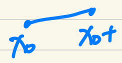
Let $r>0$

Let $h\in\mathbb{R}^2$ s.t. $|h|<r$

Set $p_0=x_0$

$p_1=x_0+h_1e_1$

$p_2=p_1+h_2e_2$



with function $\varphi_1: B_{h_1}(0)\subseteq\mathbb{R}\to\mathbb{R}$

$s\mapsto f(p_0+se_1)$

$\varphi_2: B_{h_2}(0)\subseteq\mathbb{R}\to\mathbb{R}$

$s\mapsto f(p_1+se_2)$

note: $\varphi_1'(s)=\frac{\partial}{\partial x}f(p_0+se_1)$

$\varphi_2'(s)=\frac{\partial}{\partial y}f(p_1+se_2)$

By MVT (since φ_1,φ_2 ctn)

$\varphi_1(h_1)-\varphi_1(0)=\frac{\partial f}{\partial x}(p_0+c_1e_1)\,h_1$ for some $c_1<h_1$

$\varphi_2(h_2)-\varphi_2(0)=\frac{\partial f}{\partial y}(p_1+c_2e_2)\,h_2$ for some c_2

$$f(x_0+h)-f(x_0)=f(p_1+h_2)-f(x_0)$$
$$=\big(f(p_1+h_2)-f(p_0+h_1)\big)+\big(f(p_0+h_1)-f(p_0)\big)$$
$$=\big(\varphi_2(h_2)-\varphi_2(0)\big)+\big(\varphi_1(h_1)-\varphi_1(0)\big)$$

$$\Rightarrow f(x_0+h)-f(x_0)=\sum_{i=1,2}\left(\frac{\partial f}{\partial x_i}(q_i)-\frac{\partial f}{\partial x_i}(p_i)\right)h_i$$

So $|f(x_0+h)-f(x_0)-\underset{\substack{\parallel\\ \left(\frac{\partial f}{\partial x_1}(p_0)\ \frac{\partial f}{\partial x_2}(p_0)\right)\binom{h_1}{h_2}}}{Ah}|\le|h|\sum_{i=1,2}\left(\frac{\partial f}{\partial x_i}(q_i)-\frac{\partial f}{\partial x_i}(p_0)\right)$

$$\frac{|f(x_0+h)-f(x_0)-Ah|}{|h|}\le\sum_{i=1,2}(\cdot\ -)\to0 \text{ by ctnty.}$$

② mixed partial thm for $f:\mathbb{R}^2\to\mathbb{R}$

i.e. if $f\in C^r$, then all r-order partials can change order

Pf it suffices to show the C^2 case (之后[illegible]即可)

Take $x\in A$

Let $G(h,k)=f(x_1+h,x_2+k)-f(x_1+h,x_2)-f(x_1,x_2+k)+f(x_1,x_2)$

Define $\varphi(t)=f(x_1+h,t)-f(x_1,t)$

So $G(h,k)=\varphi(x_2+k)-\varphi(x_2)$

用两次MVT, $G(h,k)=kh\frac{\partial^2}{\partial x_1\partial x_2}f(s_0,t_0)$

换序再来一次, $G(h,k)=hk\frac{\partial^2}{\partial x_1\partial x_2}f(s_0',t_0')$

By C_2, $h,k\to0$ 时 $(s_0,t_0),(s_0',t_0')\to(x_1,x_2)$ □

③ The chain rule $D(g\circ f)(x_0)=Dg(f(x_0))\,Df(x_0)$

Pf Let $x_0\in A$

We prove it by proving the remainder function at x_0

$R_{g\circ f}(h)\to0$ as $R_f(h)\to0$

Define: $R_f(h)=\dfrac{f(x_0+h)-f(x_0)-Df(x_0)h}{|h|}$

f diff'ble $\Rightarrow|R_f(h)|\to0$ as $|h|\to0$

Let $y_0=f(x_0)$

Define $R_g(k)=\dfrac{g(y_0+k)-g(y_0)-Dg(y_0)k}{|k|}$

Again, $|R_g(k)|\to0$ as $|k|\to0$

Set $A=Dg(y_0)\,Df(x_0)$

Define $R_{g\circ f}(h)=\dfrac{g\circ f(x_0+h)-g\circ f(x_0)-Ah}{|h|}$

$=\dfrac{g(f(x_0+h))-g(y_0)-Ah}{|h|}$

note:

$g(f(x_0+h))=g\big(|h|R_f(h)+f(x_0)+Df(x_0)h\big)$

So $R_{g\circ f}(h)=\dfrac{g(y_0+|h|R_f(h)+Df(x_0)h)-g(y_0)-Ah}{|h|}$

自然想到 set $k=|h|R_f(h)+Df(x_0)h$

So for h small enough,

$|k|\le\|Df(x_0)\|\,|h|+|h|$ ①

(note: $|Cx|\le\|C\|\,|x|$)

同样手法得到

$R_{g\circ f}(h)=\dfrac{|h|\,Dg(y_0)R_f(h)+|k|R_g(h)}{|h|}$

$\le\|Dg(y_0)\|\cdot|R_f(h)|+\frac{|k|}{|h|}|R_g(k)|$

$\to0$ as $h\to0$ by ①

④ higher-order product rule

$\partial^\alpha(fg)=\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}(\partial^\beta f)(\partial^\gamma g)$, p.t. f,g are $C^{|\alpha|}$

Pf Base case on $h=s$ $(f,g:\mathbb{R}\to\mathbb{R})$

Inductive step: Assume true for any $f,g:\mathbb{R}^{n-1}\to\mathbb{R}$

Write $\alpha=(a,\theta)$ with $a\in\mathbb{N}_0$, $\theta\in\mathbb{N}_0^{n-1}$

$$\partial^\alpha(fg)=\partial^a\partial^\theta(fg)=\partial^a\Big(\sum_{u+v=\theta}\frac{\theta!}{u!\,v!}(\partial^u f)(\partial^v g)\Big)$$
$$=\sum_{u+v=\theta}\frac{\theta!}{u!\,v!}\partial^a\big((\partial^u f)(\partial^v g)\big)$$
$$=\sum_{u+v=\theta}\sum_{m+k=a}\frac{\theta!\,a!}{u!\,m!\,v!\,k!}(\partial^{m+u}f)(\partial^{k+v}g)$$

set $u+m=\beta$, $v+k=\gamma$ $\Rightarrow=\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}(\partial^\beta f)(\partial^\gamma g)$

(Here I prove base case) $\partial^k(fg)=\sum_{|\alpha|=k}\frac{k!}{\alpha!}\partial^{\alpha_1}f\,\partial^{\alpha_2}g$

We also need induction

Base case $k=1$: $(fg)'=f'g+g'f$

Inductive step: Assume $(fg)^{(k)}=\sum_{a+b=k}\frac{k!}{a!b!}f^{(a)}g^{(b)}$

$$(fg)^{(k+1)}=(fg^{(k)})'$$
$$=\sum_{a+b=k}\frac{k!}{a!b!}\left(f^{(a)}g^{(b+1)}+f^{(a+1)}g^{(b)}\right)$$
$$=\sum_{a+b=k+1}\frac{k!}{a!b!}f^{(a)}g^{(b)} \qquad \square$$

(5) Let f be C^1, $Df(x_0)\neq 0$

$\Rightarrow \exists\alpha>0$, open $U\ni x_0$ s.t.

$\forall x,y\in U$, $|f(x)-f(y)|\geq\alpha|x-y|$ (A strong version of local inj.)

Pf Let $H(x)=f(x)-Df(x_0)$

So $DH(x)=Df(x)-Df(x_0)$

$DH(x_0)=0$, $\|DH(x_0)\|=0$

Since H is C^1 $\Rightarrow$ $x\mapsto DH(x)$, $x\mapsto\|DH(x)\|$ is ctn.

So can take $\varepsilon>0$ s.t. $\forall x\in B_\varepsilon(x_0)$, $\|DH(x)\|\leq\frac{1}{2\|Df^{-1}(x_0)\|}$

Let $x,y\in B_\varepsilon(x_0)$,

By MVT (Taylor's Thm of order 0),

$H(x)-H(y)=DH(c)\cdot(x-y)$ for some c on the line seg of x,y

So $|H(x)-H(y)|=|DH(c)\cdot(x-y)|$
$$\leq\|DH(c)\|\cdot|x-y|$$
$$\leq\frac{|x-y|}{2\|Df^{-1}(x_0)\|}$$

Note $|H(x)-H(y)|=|f(x)-f(y)-Df(x-y)|$
$$\geq|Df(x-y)|-|f(x)-f(y)| \text{ by tri eq.}$$
$$\Rightarrow |f(x)-f(y)|\geq|Df(x-y)|-\frac{|x-y|}{2\|Df^{-1}(x_0)\|}$$

Claim: if $E\in\mathbb{R}^{n\times n}$ invertible, then $\forall v\neq 0$, $\boxed{|Ev|\geq\frac{|v|}{\|E^{-1}\|}}$

(Pf) $|v|=|E^{-1}Ev|\leq\|E^{-1}\|\,|Ev|$

Therefore $|f(x)-f(y)|\geq\frac{|x-y|}{\|Df^{-1}(x_0)\|}-\frac{|x-y|}{2\|Df^{-1}(x_0)\|}$
$$=\frac{|x-y|}{2\|Df^{-1}(x_0)\|} \qquad \square$$

(6) if $f:U\subseteq\mathbb{R}^n\to\mathbb{R}^n$ is C^1, f inj.; $\forall x\in U$ $Df(x)\neq 0$

$\Rightarrow f(U)$ is open

Pf Write $f(U)=V$

Let $y\in V$, then $f(x)=y$ for some $x\in U$

WTS: $\exists\varepsilon>0$ s.t. $B_\varepsilon(y)\subseteq V$ (then V is open)

Take $\delta>0$ s.t. $\overline{B_\delta(x)}\subseteq U$

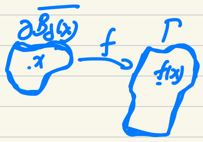
Since $\overline{B_\delta(x)}$ closed $\Rightarrow$ $\partial B_\delta(x)$ is closed

and since $\partial B_\delta(x)=\{z\in\mathbb{R}^n \mid |z-x|=\delta\}$ is bdd.

$\Rightarrow \partial B_\delta(x)$ is cpt.

Since f is C^1 $\Rightarrow$ f ctn $\Rightarrow$ $f(\partial B_\delta(x))$ is cpt.

Write $f(\partial B_\delta(x))=\Gamma$



Since f inj $\Rightarrow y\notin\Gamma$

Since Γ is closed $\Rightarrow$ $\mathbb{R}^n\setminus\Gamma$ is open

$\Rightarrow \exists\varepsilon>0$ s.t. $\boxed{B_{2\varepsilon}(y)\subseteq\mathbb{R}^n\setminus\Gamma}$

Now lets show $B_\varepsilon(y)\subseteq V$

Let $c\in B_\varepsilon(y)$ WTS: $c\in V$ 思路: 证明 $\varphi_{min}=0$ 即 $c=f(z_*)$ for some $z_*\in B_\delta(x)$ 从而 $\Rightarrow c\in V$

Let $\varphi:\overline{B_\delta(x)}\to\mathbb{R}$

$z\mapsto|z-c|^2$ So φ is bounded by ε^2

Since $\overline{B_\delta(x)}$ cpt $\Rightarrow$ φ reaches min val at some pt $z_*\in\overline{B_\delta(x)}$

Claim $z_*\notin\partial B_\delta(x)$ (i.e. $z_*\in B_\delta(x)$)

Assume that for contradiction that $z_*\in\partial B_\delta(x)$

$\Rightarrow f(z_*)\in\Gamma \Rightarrow |f(z_*)-c|^2=|f(z_*)-y+y-c|^2$
$$\geq(|f(z_*)-y|-|y-c|)^2>\varepsilon^2\text{, contradicts}$$

Then we can say $D\varphi(z_*)=0$ (since $z_*\in B_\delta(x)$)

By chain rule, $D\varphi(z)=2(f(z)-c)\,Df(z)$

Since $D\varphi(z_*)=0 \Rightarrow 2(f(z_*)-c)\underbrace{Df(z_*)}_{\neq 0}=0$

$\Rightarrow B_\varepsilon(y)\subseteq V$ $\qquad$ $\Rightarrow f(z_*)=c\Rightarrow c\in V$

$\Rightarrow V$ is open $\qquad\square$

exercise: pf of $\forall C\in\mathbb{R}^{m\times n}$,
$$\|C\|\leq mn\max_{i,j}|C_{ij}|$$

By def $\|C\|=\sup_{x\neq 0}\frac{|Cx|}{|x|}$

$\Rightarrow \forall x$, $|Cx|\leq\|C\|\cdot x$

Also, $\|C\|=\sup_{|x|=1}|Cx|$
$$=\sup_{|x|=1}\sum_{i=1}^{m}\left(\sum_{j=1}^{n}C_{ij}x_i\right)\leq\sum_{i=1}^{m}n\left(\max_{1\leq j\leq n}|C_{ij}|\right)$$
$$\leq mn\max_{\substack{1\leq i\leq m\\1\leq j\leq n}}|C_{ij}|$$