Problem A: Write the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as a product of 2 by 2 matrices that are primitive diffeomorphisms. You need only give the final answer.

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$(a,b) \to (b,a)$ | $(-b,a) \to (b,a)$ | $(a-b,a) \to (-b,a)$ | $(a-b,b) \to (a-b,a)$ | $(a,b) \to (a-b,b)$

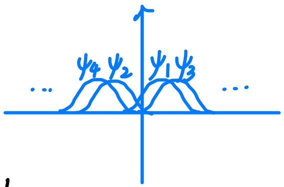
Problem B: Explicitely give a partition of unity of $\mathbb{R}$ dominated by the cover by all open intervals of length 7.

define $\psi : \mathbb{R} \to [0,1]$

$$x \mapsto \begin{cases} \exp\left(-\frac{1}{1-\left(\frac{x}{3.5}\right)^2}\right), & |x|<3.5 \\ 0, & \text{elsewhere} \end{cases}$$

Then $\mathrm{supp}(\psi) = [-3.4, 3.4]$

and ψ is non-negative (1)



for each $n \in \mathbb{N}$ define

$$\psi_n(x) = \begin{cases} \psi(x-n), & n \text{ odd} \\ \psi(x+n), & n \text{ even} \end{cases} \quad (2)$$

So $\mathrm{supp}(\psi_n) = \begin{cases} [n-3.4, n+3.4], & n \text{ odd} \\ [-n-3.4, -n+3.4], & n \text{ even} \end{cases}$

Then $\forall x \in \mathbb{R}$, at most four ψ_n are supported at x (3)

(ex: $x=0$, $\mathrm{supp}(\psi_1)=[-2.4, 4.4]$, $\mathrm{supp}(\psi_2) = [-5.4, 1.4]$, $\mathrm{supp}(\psi_3) = [-0.4, 5.4]$, other ψ_n not supported at x)

So define $\lambda = \sum_{n \in \mathbb{N}} \psi_n$ is a C^∞ function, positive on the whole $\mathbb{R}$

define for each $n \in \mathbb{N}$ $\varphi_n = \frac{\psi_n}{\lambda}$

So $\sum_{n \in \mathbb{N}} \varphi_n(x) = 1 \ \forall x \in \mathbb{R}$ (4)

Then $\{\varphi_n\}_{n=1}^\infty$ is a partition of unity on $\mathbb{R}$ by (1)(2)(3)(4)

and it is dominated by {open intervals of length 7}

since each $\mathrm{supp}(\varphi_i) = [m-3.4, m+3.4]$ for some $m \in \mathbb{Z}$ $\subseteq (m-3.5, m+3.5)$

Problem C: Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = e^{-1/x}$$

when $x > 0$ and $f(x) = 0$ otherwise is C^∞. (For hints, see page 143 of the text.)

f is of C^∞ class both on $x>0$ since it is composed of fundamental smooth functions and on $x<0$ since $f=0$

So it suffices to prove that f is infinite-times differentiable at 0

Pf for $x>0$, $f(x) = e^{-\frac{1}{x}}$, we know that

$f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-\frac{1}{x}}$ where P_n is a polynomial of degree $2n$

We will prove by induction that $f^{(n)}(0) = 0$

Base case $f'(0)$ exists and is 0

Inductive step: suppose $f^{(n-1)}(0) = 0$ for $n \geq 2$

WTS: $f^{(n)}(0)$ exists and is 0

$$\lim_{x \to 0^-} \left|\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right| = \lim_{x\to 0^-} \frac{0-0}{x} = 0$$

$$\lim_{x \to 0^+} \left|\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right| = \left|P_{n-1}\left(\tfrac{1}{x}\right)e^{-\frac{1}{x}} \cdot \tfrac{1}{x}\right|$$

Let $\frac{1}{x} := t \Rightarrow f^{(n)}(0) = P_{n-1}(t)e^{-t} \cdot t$

$= Q_n(t)e^{-t}$ for some polynomial Q_n of degree $2n$

Since $Q_n(t)$ is degree $2n$, $\exists$ some const C_n s.t. $|Q_n(t)| \leq C_n t^{2n}$

$$\text{So } \lim_{x \to 0^+} \left|\frac{f^{(n-1)}(x) - f^{(n-1)}(0)}{x-0}\right| \leq \lim_{x \to 0^+} C_n \frac{\left(\frac{1}{x}\right)^{2n}}{e^{\frac{1}{x}}} = \lim_{t \to \infty} C_n \frac{t^{2n}}{e^t} = 0$$

easily obtained by L'Hopital's Rule

Therefore the upper, lower limit agrees as 0

$\Rightarrow f^{(n)}(0)$ exists and is 0

This finishes the proof that f is infinitely-times diff'ble at 0, thus is C^∞ on $\mathbb{R}$. □

Problem D: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $n < m$, show that the image cannot contain a non-empty open set.

Pf In class we have shown: for smooth $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with $m>n$ and A open, we have $m(f(A)) = 0$

So $m(f(\mathbb{R}^n)) = 0$ in this context

Assume for contradiction that $f(\mathbb{R}^n)$ contains an non-empty open set U

Take arbitary $a \in U$, $\exists \varepsilon > 0$ st. $B_\varepsilon(a) \subseteq U \subseteq f(\mathbb{R}^n)$

An open ball of radius always has volume >0 (last hw)

i.e $\int_{B_\varepsilon(a)} 1 > 0$

An open ball is Jordan-measurable, thus its volume agree with Jordan measure and agrees with Lebesgue measure

So $\int_{B_\varepsilon(a)} 1 = m_J(B_\varepsilon(a)) = m(B_\varepsilon(a)) > 0$

By monotonicity of Lebesgue measure, $m(f(\mathbb{R}^n)) \geq m(B_\varepsilon(a)) > 0$, contradicts

Thus the image cannot contain non-empty open set. □

Problem E: Define a diffeomorphism to be super-primitive if it preserves all but one coordinate. Show that the theorem proved in class on locally factoring diffeos remains true if one replaces primitive with super-primitive.

Pf <u>Claim 1</u> a diffeomorphism $g: \overset{\text{open}}{U} \subseteq \mathbb{R}^n \to \overset{\text{open}}{V} \subseteq \mathbb{R}^n$ s.t. $g(0)=0$, $Dg(0) = Id_n$ can be decomposed near $x=0$ into $k \circ h$ where

$$h(x) = (g_1(x), \dots, g_{i-1}(x), x_i, g_{i+1}(x), \dots, g_n(x))$$

$$k(y) = (y_1, \dots, y_{i-1}, g_i(h^{-1}(y)), y_{i+1}, \dots, y_n)$$

for any $i = 1, \dots, n$

Pf Assume the hypothesis.

$$Dh(x) = \begin{bmatrix} \partial(g_1, \dots, g_{i-1})/\partial x \\ 0 \cdots \underset{ith}{1} \cdots 0 \\ \partial(g_{i+1}, \dots, g_n)/\partial x \end{bmatrix}$$

Since $Dg(0) = I_n \Rightarrow Dh(0) = I_n$

By IFT, $\exists$ some open $V_0 \ni 0$ in U s.t. $h|_{V_0}: V_0 \to V_1$ is a diffeomorphism

Define $k: V_1 \to \mathbb{R}^n$ as above

$$\Rightarrow Dk(y) = \begin{bmatrix} I_{i-1} & 0 \\ \multicolumn{2}{c}{D(g_i \circ h^{-1})(y)} \\ 0 & I_{i+1} \end{bmatrix}$$

$$D(g_i \circ h^{-1})(0) = Dg_i(0) \cdot Dh^{-1}(0) = [0 \dots 1 \dots 0] I_n = [0 \dots \overset{ith}{1} \dots 0]$$

So for some open nbh $W_1 \ni 0$, $k|_{W_1}: W_1 \to W_2$ for some open $W_2 \subseteq \mathbb{R}^n$ is a diffeomorphism

Let $W_0 = h^{-1}(W_1)$, then

$$g|_{W_0} = k|_{W_1} \circ h|_{W_0}$$

Here $k|_{W_1}$ is super-primitive diffeo □

<u>Claim 2</u> a diffeomorphism $g: \overset{\text{open}}{U} \subseteq \mathbb{R}^n \to \overset{\text{open}}{V} \subseteq \mathbb{R}^n$ s.t. $g(0)=0$, $Dg(0) = Id_n$ can be decomposed into finite super-primitive diffeos

Pf Define $h^{(1)}(x) = (x_1, g_2(x), \dots, g_n(x))$

$$k^{(1)}(x) = (g_1(h^{(1)^{-1}}(y)), y_2, \dots, y_n)$$

So $\exists$ open $W_0^{(1)}, W_1^{(1)}$ s.t.

$$g|_{W_0^{(1)}} = k^{(1)}|_{W_1^{(1)}} \circ h|_{W_0^{(1)}}$$

For each $i = 2, \dots, n-1$, define

$$h^{(i)}(x) = (h_1^{(i-1)}(x), \dots, h_{i-1}^{(i-1)}(x), x_i, h_{i+1}^{(i-1)}(x), \dots, h_n^{(i-1)}(x))$$

$$k^{(i)}(y) = (y_1, \dots, y_{i-1}, g_i(h^{(i-1)^{-1}}(y)), y_{i+1}, \dots, y_n)$$

For each i, $h^{(i)}$ is a primitive diffeo on some open $W_0^{(i)} \subseteq W_0^{(i-1)}$

$k^{(i)}$ is a super primitive diffeo on some open $W_1^{(i)} \subseteq W_1^{(i-1)}$

By induction we can get: for all $i = 1, \dots, n-1$

$$h^{(i)}(x) = (x_1, \dots, x_i, h_{i+1}^{(i-1)}(x), \dots, h_n^{(i-1)}(x))$$

Since each $h^{(i-1)}(x)$ preserves the $i-1$ th coord and this preservation preceeds in $h^{(i)}(x)$

Thus $h^{(n-1)}(x) = (x_1, x_2, \dots, x_{n-1}, h_n^{(n-1)}(x))$ is a super-primitive diffeo on $W_0^{(n-1)}$

So $g = k^{(1)} \circ \dots \circ k^{(n-1)} \circ h^{(n-1)}$ on $W_0^{(n-1)}$ is composed into super-primitive diffeos

In class we proved that for general diffeomorphism $f: A \to B$ and given $a \in A$, f around a can be locally decomposed into $T \circ t_2 \circ g \circ t_1$ where g defined in claim 2, $t_1: x \mapsto x + a$, $t_2: x \mapsto x - g(a)$, $T: x \mapsto (Dg(a))^{-1} x$

t_1, t_2 are translations so can be decomposed respectively into n super-primitive translations, each translate one coordinate

And T is a invertible linear transformation, so can be decomposed into linear transformations represented by elementary matrices. Elementary matrices are all super-primitive diffeos except swapping two rows, but swapping two rows can also be decomposed into four matrices that are super-primitives (procedure shown in class)

Finally, we can decompose

$$f = E_1 \circ \dots \circ E_n \circ t_2^{(n)} \circ \dots \circ t_2^{(1)} \circ k^{(1)} \circ \dots \circ k^{(n-1)} \circ h^{(n-1)} \circ t_1^{(n)} \circ \dots \circ t_1^{(1)}$$

around a locally, each function is super-primitive diffeomorphism □

Problem F: Show that there is no injective smooth function $f : \mathbb{R}^2 \to \mathbb{R}$.

Pf Suppose for contradiction that $f: \mathbb{R}^2 \to \mathbb{R}$ is smooth and injective

If $\forall v \in \mathbb{R}^2$ $\frac{\partial f}{\partial x}(v) = \frac{\partial f}{\partial y}(v) = 0$

$\Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are constant 0 so inductively any order partial is constant 0

$\Rightarrow f$ is constant function by Taylor's Thm, not injective

Thus $\exists\, v_0 \in \mathbb{R}^2$ s.t. at least one of $\frac{\partial f}{\partial x}(v_0), \frac{\partial f}{\partial y}(v_0) \neq 0$
$= (a_0, b_0)$

WLOG suppose $\frac{\partial f}{\partial x}(v_0) \neq 0$

Since f is smooth and $\det \frac{\partial f}{\partial x}(a_0, b_0) = \frac{\partial f}{\partial x}(a_0, b_0) \neq 0$

$\Rightarrow$ by IFT, $\exists$ nbh $B \ni a_0$ and $g: B \to \mathbb{R}$ s.t.

$g(a_0) = b_0$ and $f(x, g(x)) = 0$ for all $x \in B$

Thus f is not injective, contradicts

Therefore such function does not exist

$\square$

Problem G: For an arbitrary subset S of $\mathbb{R}$ and a function $f : S \to \mathbb{R}$, we say that f is smooth at $x \in S$ if there is a open set U_x containing x and a smooth function $f_x : U_x \to \mathbb{R}$ such that f and f_x agree on $U_x \cap S$. Show that if f is smooth at every point of S then there is an open set V containing S and a smooth function $g : V \to \mathbb{R}$ that agrees with f on S.

Pf Suppose f is smooth at every point of S

Let $\{U_x\}_{x\in S}$ be the collection of open nbh of each $x \in S$ s.t. $f_x: U_x \to \mathbb{R}$ is smooth and agrees with f

Let $A = \bigcup_{x\in S} U_x \Rightarrow A$ open

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a partition of unity on A of C^∞ class, dominated by $\{U_x\}_{x\in S}$ with each $\operatorname{supp}(\varphi_n) \subseteq U_{x_n}$ for some $U_{x_n} \in \{U_x\}_{x\in S}$

For each φ_n, define

$$h_n(x) := \begin{cases} \varphi_n f_{x_n}, & x \in U_x \\ 0, & \text{elsewhere} \end{cases}$$

So $\operatorname{supp}(h_n) \subseteq U_{x_n}$ and $\{h_n\}_{n\in\mathbb{N}}$ is locally finite for all $x \in A$ by the locally finiteness of $\{\varphi_n\}_{n\in\mathbb{N}}$

So $\sum_{n=1}^{\infty} h_n(x)$ converges

and each $h_n(x)$ is smooth since φ_n, f_{x_n} is smooth: $\varphi_n f_{x_n}$ is smooth on U_{x_n} and φ_n reaches 0 on the boundary of U_x, which agrees with the value of h_n outside U_{x_n}

So $\lambda(x) = \sum_{n=1}^{\infty} h_n(x)$ is smooth function on A

Let $x_0 \in S$

$\lambda(x_0) = \varphi_{n_1}(x_0) f_{x_{n_1}}(x_0) + \dots + \varphi_{n_k}(x_0) f_{x_{n_k}}(x_0)$ for some $n_1, \dots, n_k$ by local finiteness

By smoothness: $f_{x_{n_i}}(x_0) = f(x_0)$ for $i = 1, \dots, k$

By partition of unity: $\sum_{i=1}^{k} \varphi_{x_i}(x_0) = 1$

So $\lambda(x_0) = f(x_0)$

This proves λ is a smooth function that agrees with f on S

$\square$

Problem H: Let A be a matrix. Show that the rank of A is the maximum value of k so that a k by k minor of A has non-zero determinant. (A k by k minor is a matrix obtained by deleting all but k rows and all but columns.)

Pf Suppose A is $n \times m$ matrix, $\operatorname{rank}(A) = k$

Claim1 There exists an $k \times k$ minor with non-zero determinant

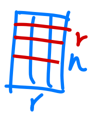
Since there are k linearly independent column,

We take the k columns out, forming a $n \times k$ matrix M

Since the k columns are linearly independent, $\operatorname{rank}(M) = k$

So there are k linearly independent rows in M

We take the k rows out to get a $k \times k$ minor of M, call it M_2



Since the rows of M_2 are lin. ind $\Rightarrow \operatorname{rank}(M_2) = k$
$\Rightarrow \det(M_2) \neq 0$

Claim2 any minor of A has rank less than or equal to k

Let M be a minor of A

Let $S = \{V_1, \dots, V_r\}$ be the columns of M extended to the whole column in A

Since $\operatorname{rank}(A) = k \Rightarrow$ there are at most k linearly independent vectors in S

Let $N = [V_1 \cdots V_r] \Rightarrow \operatorname{rank}(N) \leq k$

Since M is obtains by deleting some rows in N, the row rank of M is less than or equal to that of N

So $\text{rank}(M) \leq k$

Claim 2 implies that for any $r \times r$ minor M of A s.t
$r > k \Rightarrow \det(M) = 0$

Therefore, the rank of A is the max value of k s.t. $\exists$ a $k \times k$ minor of A that has non-zero determinant.