HW 1

DUE FRIDAY AUGUST 30

5/5

Problem A: Suppose (X, d) is a metric space. For $0 < \epsilon < 1$, show that d^ϵ is also a metric on X.

If $X = [0, 1]$ is the unit interval and d is the usual metric, show that X has "infinite length" using the metric d^ϵ, in that

$$\sup_{0=t_0<t_1<\cdots<t_n=1} \sum_{i=1}^{n} d^\epsilon(t_i, t_{i-1}) = \infty.$$

Here the sup is taken over all n and all $n+1$ tuples of points t_i as in the subscript.

Optional context: If you'd like to know where the metrics d^ϵ appear, try looking up Assouad Embedding Theorem. If you'd like to know more about the notion of length used here, try looking up rectifiable curves.

① Pf Assume the hypothesis.
Take $x, y, z \in X$.

(a) $d^\varepsilon(x,y) = (d(x,y))^\varepsilon \geq 0$ since $d(x,y) \geq 0$ and $0 < \varepsilon < 1$

(b) $d^\varepsilon(x,y) = (d(x,y))^\varepsilon = (d(y,x))^\varepsilon = d^\varepsilon(y,x)$ by symmetry of d

let $f(x) = x^\varepsilon$, $x \geq 0$, so $f''(x) = \underbrace{\varepsilon(\varepsilon-1)}_{<0} x^{\varepsilon-2} \leq 0$ for $x \geq 0$

So f is concave for $x \geq 0$
and $d(x,y), d(y,z), d(x,z) \geq 0$

Hence $f(d(x,y)) + f(d(y,z)) \geq f(d(x,y) + d(y,z))$

Since f is increasing on domain and $d(x,y) + d(y,z) \geq d(x,z)$,
we have $f(d(x,y)) + f(d(y,z)) \geq f(d(x,y) + d(y,z)) \geq f(d(x,z))$

Hence (c) $d^\varepsilon(x,y) + d^\varepsilon(y,z) \geq d^\varepsilon(x,z)$

(a), (b), (c) shows that d^ε is a metric on X. $\square$

② Pf It suffices to show that for any $M \in \mathbb{N}$, $\exists$ some choices of partition $\{t_i\}_I$ s.t. $\sup_{0=t_0<t_1<\dots<t_n=1} \sum_{i=1}^{n} d^\varepsilon(t_i, t_{i+1}) > M$

let $M \in \mathbb{N}$. Consider equally partition the interval into n subintervals

Then $\sup_{0=t_0<t_1<\dots<t_n=1} \sum_{i=1}^{n} d^\varepsilon(t_i, t_{i+1}) = n\left|\frac{1}{n}\right|^\varepsilon = n^{1-\varepsilon}$. Since $1-\varepsilon > 0$, it is unbounded above, so $\exists$ some $n \in \mathbb{N}$ s.t. $n^{1-\varepsilon} > M$ $\square$

Bonus problem: If X is a a by b matrix, and Y is a b by c matrix, it takes abc multiplications to compute XY according to the usual formula for matrix multiplication. (There are ac entries in XY, and each is a sum of b products.) Thus, let's estimate the time it takes to multiply these two matrices as abc.

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Say A_1 is a 5 by 1 matrix, A_2 is a 1 by 5 matrix, A_3 is a 5 by 2 matrix, A_4 is a 2 by 5 matrix, A_5 is a 5 by 1 matrix, and A_6 is a 1 by 10 matrix.

If you want to compute $A_1A_2A_3A_4A_5A_6$, how should you bracket this product so that the sum of the time estimates for the multiplications is as small as possible? For example, should you do

$$(A_1(A_2A_3))((A_4A_5)A_6)?$$

Or

$$(A_1(A_2(A_3(A_4(A_5A_6)))))?$$

Or something else?

In general you will have to show your work, but for this first bonus problem only **you only need to submit the final answer**.

Hint 1: See hint posted on my office door (EH 5848).

Hint 2: Only use this Hint 2 if you can't figure it out on your own after spending at least 10 minutes looking at Hint 1. In general, you can't use wikipedia or internet resources unless explicitely allowed, but for this first bonus problem you can. See the wikipedia entry titled "Matrix chain multiplication".

$(A_1((A_2A_3)(A_4A_5))A_6$

DP table is in the next page.

Denote the minimal cost of multiplying A_i through A_j by $m(i,j)$

Recursively, $m(i,j) = \min_{i \leq k \leq j} \{m(i,k) + m(k+1,j) + \text{row}(A_i) \cdot \text{col}(A_k) \cdot \text{col}(A_j)\}$

DP Table

	(5,1) 1	(1,5) 2	(5,2) 3	(2,5) 4	(5,1) 5	(1,10) 6
1	0	(5,5) 25	(5,2) 20 1(23)	(5,5) 45 1(24)	(5,1) 27 1(25)	(5,10) 77 (15)6
2		0	(1,2) 10	(1,5) 20 (23)4	(1,1) 22 (23)(45)	(1,10) 32 (25)6
3			0	(5,5) 50	(5,1) 20 3(45)	(5,10) 70 3(56)
4				0	(2,1) 10	(2,10) 30 (45)6
5					0	(5,10) 50
6						0

$m(1,3) = \min(25 + 5\times5\times2,\ 10 + 5\times1\times2) = 20$

$m(2,4) = \min(25+50,\ 10+10) = 20$

$m(3,5) = \min(50+25,\ 10+10) = 20$

$m(4,6) = \min(10+20,\ 50+100) = 30$

$m(1,4) = \min(\underset{(13)4}{20+50},\ \underset{(12)(34)}{25+50+125},\ \underset{1(24)}{20+25}) = 45$

$m(2,5) = \min(\underset{(24)5}{25},\ \underset{(23)(45)}{10+10+2},\ \underset{2(35)}{25}) = 22$

$m(3,6) = \min(\underset{(35)6}{20+50},\ \underset{(34)(56)}{50+50+250},\ \underset{3(46)}{30+100}) = 70$

$m(1,5) = \min(\underset{1(25)}{22+5},\ \underset{(12)(35)}{\dots},\ \underset{(13)(45)}{\dots},\ \underset{(14)5}{\dots}) = 27$

$m(2,6) = \min(\underset{2(36)}{70+50},\ \underset{(23)(46)}{10+30+20},\ \underset{(24)(56)}{20+50+\dots},\ \underset{(25)6}{32}) = 32$

$m(1,6) = \min(\underset{1(26)}{32+50}\ \ \underset{(12)(36)}{25+70+\dots}\ \ \underset{(13)(46)}{20+30+100}\ \ \underset{(14)(56)}{45+50+\dots}\ \ \underset{(15)6}{27+50}) = 77$