Recall: multinomial Thm

$$(x_1+\dots+x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha$$

Lemma Higher order product rule

$\forall \alpha \in \mathbb{Z}^n_{\geq 0}$ and any $f,g:\mathbb{R}^n \to \mathbb{R}$

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

whenever f,g are $C^{|\alpha|}$

ex

(1) $|\alpha|=0$, i.e. $\alpha=(0,\dots,0)$

$$\partial^\alpha(fg) = fg = \frac{0!}{0!0!}(\partial^0 f)(\partial^0 g) = fg$$

(2) $|\alpha|=1$, i.e. $\alpha=(0,\dots,1,\dots,0)=e_i$, $i\in\{0,\dots,n\}$

Consider $n=2$, $\alpha=(1,0)$

$$\partial^\alpha(fg) = \frac{\partial}{\partial x_1}(fg) = \left(\frac{\partial f}{\partial x_1}\right)g + \left(\frac{\partial g}{\partial x_1}\right)f = \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$$

$$= \partial^{(1,0)} f\, \partial^{(0,0)} g + \partial^{(0,0)} f\, \partial^{(1,0)} g$$

$$= g\frac{\partial}{\partial x_1} f + f\frac{\partial}{\partial x_1} g$$

(3) $n=2$, $\alpha=(2,0)$

$$\partial^\alpha(fg) = \frac{\partial^2}{\partial x_1^2}(fg)$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1} g + f\frac{\partial g}{\partial x_1}\right)$$

$$= \frac{\partial^2 f}{\partial x_1^2} g + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_1} + f\frac{\partial^2 g}{\partial x_1^2}$$

$$= (\partial^{(2,0)} f)(\partial^{(0,0)} g) + 2(\partial^{(1,0)} f)(\partial^{(1,0)} g) + (\partial^{(0,0)} f)(\partial^{(2,0)} g)$$

Pf of product rule:

Base case $n=1$: On HW5 G

Inductive step: Assume true for $N-1$

这时可以方便写一个更多信息的结论, for $f_1,\dots,f_m$ $\mathbb{R}\to\mathbb{R}$

$$\frac{d^k(fg)}{dx^k} = \sum_{\beta+\gamma=k}\frac{k!}{\beta!\gamma!}\left(\frac{d^\beta f}{dx^\beta}\right)\left(\frac{d^\gamma g}{dx^\gamma}\right)$$

更 generally:

$$\frac{d^k(f_1\cdot\ldots\cdot f_m)}{dx^k} = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\left(\frac{d^{\alpha_1} f_1}{dx^{\alpha_1}}\right)\cdots\left(\frac{d^{\alpha_m} f_m}{dx^{\alpha_m}}\right)$$

Then for N, $f,g:\mathbb{R}^N\to\mathbb{R}$

WTS: $\forall \alpha\in\mathbb{Z}^N_{\geq 0}$, $\partial^\alpha(fg) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma\in\mathbb{Z}^n_{\geq 0}}}\frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g)$

Write $\alpha=(a,\theta)$ where $a\in\mathbb{N}_0$, $\theta\in\mathbb{N}_0^{n-1}$

当 f,g are $C^{|\alpha|}$, we have:

note: product rule 是局部适用的 即使 $f:\mathbb{R}^{100}\to\mathbb{R}$, 如果 $\alpha=(0,\dots,3,\dots)$ 我们也可使用 $\alpha\in\mathbb{Z}^1_{\geq 0}$ 时的 product rule, 即只和求 partial 的变量有关.

$$\partial^\alpha(fg) = \partial^a\partial^\theta(fg) = \partial^a\left(\sum_{u+v=\theta}\frac{\theta!}{u!v!}(\partial^u f)(\partial^v g)\right)$$

$$= \sum_{u+v=\theta}\frac{\theta!}{u!v!}\partial^a[(\partial^u f)(\partial^v g)]$$

by case $n=1$, $= \sum_{u+v=\theta}\frac{\theta!}{u!v!}\sum_{m+k=a}\frac{a!}{m!k!}[\partial^m(\partial^u f)\,\partial^k(\partial^v g)]$

$$= \sum_{u+v=\theta}\sum_{m+k=a}\frac{\overbrace{\theta!a!}^{=\alpha!}}{u!m!v!k!}(\partial^{m+u} f)(\partial^{k+v} g)$$

$\in\mathbb{N}_0^{N-1}$, 没有 θ 所有 entry

(Let $u+m=\beta$, $v+k=\gamma$)

$$= \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}(\partial^\beta f)(\partial^\gamma g) \qquad \square$$

Lemma Single variable Taylor Thm

(即不用 C^{k+1}, C^k 且 $k+1$ order derivative 存在就可以)

If $f: I\subseteq\mathbb{R}\to\mathbb{R} \in C^k(I)$ (interval) 且 f $(k+1)$-times diffble

$\Rightarrow$ 任取一点 $a\in I$

the k^{th} Taylor polynomial of f centered at a

$$T_{k,a}(x) = \sum_{j=0}^k \frac{f^{(j)}(a)}{j!}(x-a)^j + R_{a,k}(x)$$

是对 f 的 best k-order polynomial approximation

即 $\forall x\in I$, $|T_{k,a}(x)-f(x)| = o(|x-a|^k)$

with remainder $\boxed{R_{a,k}(x) = \frac{f^{(k+1)}(c_x)}{(k+1)!}(x-a)^{k+1}}$ for some $c_x\in[a,x]$ for each $x\in I$

note: 对于每个 x, 这个 c_x 都不同,是由 Cauchy Mean value Thm 得到 我们并不关心它是什么.

(但这并不代表 $k\to\infty$ 时 $T(x)\to f(x)$

即便 $f\in C^\infty(I)$, 仍需满足 $R_{a,k}(x)\to 0$ when $k\to\infty$

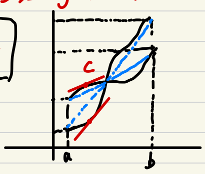
具体条件为: $\forall$ compact $K\subseteq I$, $\exists m\in\mathbb{R}$ s.t $\forall x\in K$, $\forall j$ $\left|\frac{d^j f}{dx^j}(x)\right| \leq m^{k+1}\cdot j!$)

(满足则称 f real analytic)

Pf (Recall Cauchy MVT, the corollary of Lagrange's MVT:

if $f:[a,b]\to\mathbb{R}$ ctn. 且 diffble on (a,b), 且 $g'(x)\neq 0$ for all $x\in(a,b)$

$\Rightarrow \exists c\in(a,b)$ s.t. $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$



Let $f\in C^k(I)$

Let $x\in I$ $\underbrace{\;a\quad x\;}_{I}$

define $g(t) = f(t) - T_{k,a}(t)$

We have $g(a)=0$, $g'(a)=0$, ...

g 的任意阶导数在 a 处都为 0

Let $h(t) = (t-a)^{k+1}$

Fix x, consider $\frac{g(t)}{h(t)}$ on $[a,x]$

By Cauchy MVT: $\exists c\in[a,x]$ s.t $\frac{g'(c)}{h'(c)} = \frac{g(x)-\cancel{g(a)}}{h(x)-\cancel{h(a)}}$

且 $\exists c_1, c_2, \dots c_k$ s.t. $\frac{g'(c)}{h'(c)} = \frac{g''(c_1)}{h''(c_2)} = \dots = \frac{g^{(k)}(c_k)}{h^{(k)}(c_k)} = \frac{g(x)}{h(x)}$

$$\Rightarrow \frac{f(x) - \sum_{j=0}^k \frac{f^{(j)}(a)}{j!}(x-a)^j}{(x-a)^{k+1}} = \frac{f^{(k+1)}(c)}{(k+1)!}$$

$$\Rightarrow f(x) = \sum_{j=0}^k \frac{f^{(j)}(a)}{j!}(x-a)^j + \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$$

Lemma Let $f: A\subseteq\mathbb{R}^n\to\mathbb{R}$ be C^k

let $a\in A$, $u\in\mathbb{R}^n$

$\Rightarrow \varphi: (-\varepsilon,\varepsilon)\to\mathbb{R}$, $t\mapsto f(a+tu)$ is C^k

$\varphi: \mathbb{R}\to\mathbb{R}^n\to\mathbb{R}$, $t\mapsto a+tu\mapsto f(a+tu)$

φ 即

且 $\varphi^{(k)}(0) = \sum_{|\alpha|=k}\frac{k!}{\alpha!}u^\alpha\partial^\alpha f(a)$

Pf Base case $k=0$, $\varphi(t) = f(a+tu)$

Thm Taylor's Thm

Let $G \subseteq \mathbb{R}^n$ be open and convex

$f: G \to \mathbb{R}$ be C^{k+1}

$\Rightarrow \forall a, x \in G,\ \exists c$ on the line segment from a to x

s.t. $f(x) = \sum_{|\alpha| \le k} \frac{(x-a)^\alpha}{\alpha!} \partial^\alpha f(a) + R_{a,k}(x)$

where $R_{a,k}(x) = \sum_{|\alpha| = k+1} \frac{(x-a)^\alpha}{\alpha!} \partial^\alpha f(c)$

Let $a, x \in A$

For any c on the line segment between,

都有 $\boxed{c = a + (1-t)x}$ for some $t \in [0,1]$

Pf Let $\varphi: [0,1] \to \mathbb{C}$

map $t \overset{\varphi}{\mapsto} f(a + t(x-a))$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "t \mapsto a+t(x-a)"] \arrow[rr, bend right, "\varphi"'] & \mathbb{R}^n \arrow[r, "x \mapsto f(x)"] & \mathbb{R} \end{tikzcd}$$

Note that $\varphi(0) = f(a),\ \varphi(1) = f(x)$

And $\varphi \in C^{k+1}([0,1])$ since $f \in C^{k+1}([0,1])$

于是我们把 multi var 转化为 single var，可以使用 single var Taylor

$$\Rightarrow f(x) = \varphi(1) = \sum_{p=0}^{k} \frac{\varphi^{(p)}(0) \cdot (1-0)^p}{p!} + R_{0,k}(1) = \sum_{p=0}^{k} \frac{\varphi^{(p)}(0)}{p!} + \frac{\varphi^{(k+1)}(c)}{(k+1)!}$$

$$\left(R_{0,k} = \frac{\varphi^{(k+1)}(c)}{(k+1)!} \cdot 1^{k+1} \text{ for some } c \in [0,1] \right)$$

$$\varphi'(t) = \frac{d}{dt} f(a + t(x_0 - a))$$

$$= \left(Df(a + t(x_0-a))\right)\left(x_0 - a\right) \quad \text{by chain rule}$$

$$= D_{x_0 - a} f(a + t(x_0 - a))$$

Set $u = x_0 - a$

we have $\varphi'(t) = \left(u_1 \frac{\partial}{\partial x_1} + \dots + u_n \frac{\partial}{\partial x_n}\right) f(a+tu)$

$\left(D_u f(x) = \sum_i u_i \frac{\partial}{\partial x_i} f(x)\right)$ $= \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i} f(a+tu)$

$$\varphi''(t) = \frac{d}{dt}\left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j} f(a+tu)\right)$$

即 $\sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}$ 这一 operator 对 f 的作用在 $a+tu$ 上

$$= \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right) D_u f(a+tu) = \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right)^2 f(a+tu)$$

By induction 可得到 $\varphi^{(p)}(t) = \left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right)^p f(a+tu)$

并且我们发现 by multinomial Thm:

$$\boxed{\left(\sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}\right)^p = \sum_{|\alpha| = p} \frac{p!}{\alpha!} u^\alpha \partial^\alpha}$$

因而 $\varphi^{(p)}(0) = \sum_{|\alpha| = p} \frac{p!}{\alpha!} u^\alpha \partial^\alpha f(a)$

$$\varphi^{(k+1)}(\theta) = \sum_{|\alpha| = k+1} \frac{(k+1)!}{\alpha!} u^\alpha \partial^\alpha f(a + \theta u)$$

因而 $f(x) = \varphi(1) = \sum_{p=0}^{k} \frac{\varphi^{(p)}(0)}{p!} + \frac{\varphi^{(k+1)}(c)}{(k+1)!}$ for some $c \in [0,1]$

$$= \sum_{p=0}^{k} \frac{1}{p!}\left(p! \sum_{|\alpha| = p} \frac{1}{\alpha!} u^\alpha \partial^\alpha f(a)\right)$$

$$+ \cancel{\frac{1}{(k+1)!}} \sum_{|\alpha| = k+1} \frac{\cancel{(k+1)!}}{\alpha!} u^\alpha \partial^\alpha (c)$$

$$= \sum_{|\alpha| \le k} \frac{\partial^\alpha f(a)}{\alpha!} u^\alpha + \sum_{|\alpha| = k+1} \frac{\partial^\alpha (c)}{\alpha!} u^\alpha$$

□