Recall

$\forall P_1, P_2,\ L(f, P_1) \le U(f, P_2)$

in particular, $\{L(f,P)\}$ is bdd above

$\{U(f,P)\}$ is bdd below

Def Let B be a box, $f: B \to \mathbb{R}$ bdd

(a) $\underline{\int_B} f(x)\,dx = \sup_{P \in \mathcal{P}(B)} L(f,P)$

$\overline{\int_B} f(x)\,dx = \inf_{P \in \mathcal{P}(B)} U(f,P)$

is called the lower/upper integral of f over B

(P is from all partitions)

exercise always have $\underline{\int_B} f \le \overline{\int_B} f$

(b) f is called Riemann integrable if

$$\underline{\int_B} f(x)\,dx = \overline{\int_B} f(x)\,dx$$

Then call $\int_B f(x)\,dx = \underline{\int_B} f(x)\,dx = \overline{\int_B} f(x)\,dx$

the Riemann integral of f over B

Rmk Strictly it is the def of Darboux integrability

True def of Rm integrability is:

if $\exists A \in \mathbb{R}$ s.t. ($\leftarrow$ integral)

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. Riemann sum

$$\left|\sum_{B_\alpha} f(x_\alpha)\, v(B_\alpha) - A\right| < \varepsilon$$

whenever $\|P\| < \delta$

($\{B_\alpha\}$ is subboxes of the arbitrary partition P with $\|P\| < \delta$

x_α is arbitrary pt. in B_α)

But Rm intbility $\Leftrightarrow$ Db intbility (hw)

We will use Db's def

ex Let $f: [0,1]^2 \to \mathbb{R}$ be

$$f(x,y) = \begin{cases} 0, & \text{if } x, y \text{ rationally dependent} \\ 1, & \text{else} \end{cases}$$

i.e. $\exists k_1, k_2 \in \mathbb{Z}$ not both 0 s.t. $k_1 x + k_2 y = 0$

For any partition P of $B = [0,1]^2$

any subbox S has $m_S(f) = 0$

(since S a box, $S \cap \mathbb{Q} \ne \emptyset$)

Also $M_S(f) = 1$ (since $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ also dense)

So $L(f,P) = 0$, $U(f,P) = 1$

not Rm intble.

Thm The Riemann condition

Let B be a box, $f: B \to \mathbb{R}$ bdd

Then f is Rm intble iff

$\forall \varepsilon > 0, \exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Pf $\Leftarrow$ dir exercise

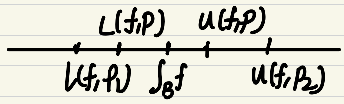
$\Rightarrow$ dir if f int. $\Rightarrow \exists P_1, P_2$ s.t.

$$\left|L(f,P_1) - \underline{\int_B} f\right| < \frac{\varepsilon}{2}, \quad \left|U(f,P_2) - \overline{\int_B} f\right| < \frac{\varepsilon}{2}$$

Let P be a common refinement of P_1, P_2

$\Rightarrow L(f,P_1) \le L(f,P) < U(f,P) \le U(f,P_2)$

$\Rightarrow |U(f,P) - L(f,P)| < \varepsilon$ by tri. eq.



$\square$

Def Let B be a box

$R(B)$ be the set of all Riem intble function $f: B \to \mathbb{R}$

Lemma $f, g \in R(B) \Rightarrow f + g \in R(B)$

Pf $\forall$ set $S \subseteq B$

$\inf_S f + \inf_S g \le \inf_S (f+g)$ ①

$\sup_S (f+g) \le \sup_S f + \sup_S g$ ②

Given $\varepsilon > 0$, pick P_f, P_g s.t. $U(f,P_f) - L(f,P_f) < \frac{\varepsilon}{2}$

$U(g,P_g) - L(g,P_g) < \frac{\varepsilon}{2}$

Let P be the common refinement

$L(f,P_f) + L(g,P_g) \le L(f,P) + L(g,P) \le L(f+g, P)$ (by ①)

$\le U(f+g, P) \le U(f,P) + U(g,P)$ (by ②)

$\le U(f,P_f) + U(g,P_g)$

So $U(f+g,P) - L(f+g,P) \le U(f,P_f) - L(f,P_f)$

$+ U(g,P_g) - L(g,P_g) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ $\square$

exercise $\forall f \in R(B)$, $c \in \mathbb{R} \Rightarrow cf \in R(B)$

($\Rightarrow$ $R(B)$ is a vector space)

Rmk const functions $\in R(B)$

Def $A \subseteq \mathbb{R}^n$ has (Lebesgue) measure 0 ($m(A)=0$)

if $\forall \varepsilon > 0$, $\exists$ a ctbl cover of boxes $\{B_1, B_2, \dots\}$

s.t. $\sum_{i=1}^{\infty} V(B_i) < \varepsilon$

Recall:

1. does not matter if boxes are open/closed
2. ctbl union of measure 0 sets has measure 0. (TBL) (thus ctbl sets must have measure 0)

Thm Characterization of Riem integrability

$B \subseteq \mathbb{R}^n$ be a box

$f: B \to \mathbb{R}$ bdd.

Let $D = \{x \mid f \text{ is not ctn. at } x\}$

$\Rightarrow$ $\boxed{f \text{ is Riem intble} \iff D \text{ has measure } 0}$

ex $f: [0,1] \to \mathbb{R}$

$x \mapsto \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

$\Rightarrow D_f = [0,1]$, $m^*(D_f) = 1 \Rightarrow$ not Riem intble

ex2 $f: [0,1] \to \mathbb{R}$

$x \mapsto \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ for some } m,n \text{ in lowest term} \\ 0 \end{cases}$

$D_f = \mathbb{Q} \cap [0,1]$ ctbl $\Rightarrow$ measure 0 $\Rightarrow$ Riem intble

Pf of Characterization of Riem intbility

$\Rightarrow$ dir: D measure 0 $\Rightarrow f$ int.

Pf Choose $M > 0$ s.t. $|f(x)| \le M$ on B

Let $\varepsilon > 0$. WTS: exhibit P s.t. $U(f,P) - L(f,P) < \varepsilon$

Find a cover of D_f by open boxes $\{B_1, B_2, \dots\}$

with $\sum_{i=1}^{\infty} V(B_i) < \frac{\varepsilon}{4M}$

For each $x \in B \setminus D_f$, find an open box $Q_x \ni x$

note: f ctn at x s.t. $\sup_{Q_x} f - \inf_{Q_x} f < \frac{\varepsilon}{2V(B)}$

Since we can always assume B closed $\{B_i\}_{i=0}^{\infty} \cup \{Q_x\}_{x \in B \setminus D}$ is an open cover of B

Since Rm sum of boundary can always be arbitrarily close to 0.

$\Rightarrow \exists$ finite subcover, call it $\{B_{i_1}, \dots, B_{i_p}\} \cup \{Q_{x_1}, \dots, Q_{x_m}\}$

Let P be a partition of B s.t. each subbox is contained in an $\overline{B_i}$ or $\overline{Q_j}$

Let S_1 be the set of subboxes contained in B_i

S_2 be the rest of subboxes $\Rightarrow$ contained in Q_j

$$U(f,P) - L(f,P) = \sum_{S \in S_1} [M_S(f) - m_S(f)] V(S) + \sum_{S \in S_2} [M_S(f) - m_S(f)] V(S)$$

$$\le 2M \sum_{S \in S_1} V(S) + \frac{\varepsilon}{2V(B)} \sum_{S \in S_2} V(S)$$

$$\le 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$