Recall (*) $\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$

$u = g(x)$, $du = g'(x)\, dx$

Note that this uses signed integral

$$\int_a^b f = \begin{cases} \int_{[a,b]} f, & \text{if } a \le b \\ -\int_{[b,a]} f, & \text{if } a > b \end{cases}$$

(*) follows from the chain rule

Pf of (*) If $F(x) = \int_c^x f(t)\, dt$

by chain rule $(F\circ g)'(x) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$ by FTC + chain rule

$$\int_a^b f(g(x))g'(x)\,dx = \int_a^b (F\circ g)'(x)\,dx$$

$$= F\circ g(b) - F\circ g(a) = \int_c^{g(b)} f - \int_c^{g(a)} f$$

$$= \int_{g(a)}^{g(b)} f$$

Sketch of a different proof

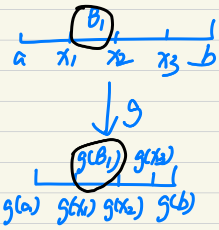
for monotonic ↗ f:

$$\int_a^b f(g(x))\, g'(x)\, dx \approx \sum_B f(g(x_B^*))\, g'(x_B^*)\, v(B)$$

Apply g to a partition of $[a,b]$ to get a partition of $[g(a), g(b)]$

Then there is a corresponse of boxes

$\boxed{B \leftrightarrow g(B)}$



Key fact: $v(g(B)) \approx g'(x_B^*)\, v(B)$

Since for $\|P\|$ small enough

$g'|_B$ is about const $g'(x_B^*)$

Taylor: $g(x) \approx g(x_B^*) + (x - x_B)\, g'(x_B^*)$

If $\varphi(x) = mx + b$, $m > 0$

$\Rightarrow \varphi((c,d)) = (mc+b, md+b)$

$\text{vol}((c,d)) = d - c$

$\text{vol}(\varphi(c,d)) = (md+b) - (mc+b) = m(d-c) = m(\text{vol}(c,d))$

Using the key fact

$$\int_a^b f(g(x))\, g'(x)\, dx \approx \sum_B f(g(x_B^*))\, g'(x_B^*)\, v(B)$$

Now: change of variable thm in higher dim

Note: no longer have nice notation as signed integrals in higher dim.

So we will need a version without signed integral

We will get this for monotone g 1D:

if $g' \ge 0$, $\int_{[a,b]} f(g(x))\, g'(x)\, dx = \int_{g([a,b])} f(u)\, du$

if $g' \le 0$, $\int_{[a,b]} f(g(x))\, g'(x)\, dx = -\int_{g([a,b])} f(u)\, du$

So either way:

$$\int_{[a,b]} f(g(x))\, |g'(x)|\, dx = \int_{g([a,b])} f$$

1D	higher D
$[a,b]$	set A
$g([a,b])$	$g(A)$
g monotonic and C^1	g C^1 diffeo $A \to g(A)$
$u = g(x)$, $du = \lvert g'(x)\rvert dx$	$u = g(x)$, $du = \lvert\det Dg(x)\rvert dx$

(some symbolic nonsense)

Thm Change of variable

Let $g: A \to B$ be C^1 diffeo of open sets in $\mathbb{R}^n$

Let $f: B \to \mathbb{R}$ be ctn

$\Rightarrow$ f intble over B iff $f(g(x))\,|\det Dg(x)|$ intble over A

And $f\circ g: \mathbb{R}^n \to \mathbb{R}$

$$\int_A f(g(x))\, |\det(Dg(x))|\, dx = \int_B f(u)\, du$$

$Dg: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ $g(A)$

$\det \circ Dg: \mathbb{R}^n \to \mathbb{R}$

(Recall the geometric interpretation of det in $\mathbb{R}^{3\times 3}$

$\det(T)$ = the factor of volumn change of applying T

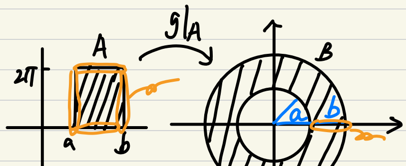
$= \dfrac{v(T(\text{unit cube}))}{v(\text{unit cube})}$

ex polar coord

$B = \{(x,y) \in \mathbb{R}^2 \mid a^2 < x^2 + y^2 < b^2\}$

$g: \mathbb{R}^2 \to \mathbb{R}^2$

$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$



Consider $A = \{(r,\theta) \mid a < r < b,\ 0 \le \theta \le 2\pi\}$

A bit annoyance : A not open

But consider $\tilde{A} = \{(r,\theta) \mid a<r<b,\ 0<\theta<2\pi\}$ open

$g(\tilde{A}) = B \setminus \{\text{pos } x\text{-axis}\}$

Note that the x-axis has meas 0

$$\Rightarrow \int_B f = \int_{\tilde{B}} f = \int_{g(\tilde{A})} f$$

$$= \int_{\tilde{A}} f \circ g(r,\theta) \underbrace{|\det Dg(r,\theta)|}_{\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}} \, ds \quad \det = r$$

$$= \int_{\tilde{A}} f(g(r,\theta))\, r \, ds$$

$$= \int_0^{2\pi}\int_a^b f(r\cos\theta, r\sin\theta)\, dr\, d\theta \quad \text{by Fubini}$$

ex <u>spherical coords</u>

Let $B = \left\{(x,y,z) \,\middle|\, \begin{array}{l} x>0,\ y>0,\ z>0 \\ x^2+y^2+z^2<a^2 \end{array}\right\}$

Consider

$$g: \begin{pmatrix} \rho \\ \varphi \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} \rho\sin\varphi\cos\theta \\ \rho\sin\varphi\sin\theta \\ \rho\cos\theta \end{pmatrix}$$

$\det Dg = \rho^2 \sin\theta$, >0 if $\rho>0$ & $0<\theta<\pi$

Let $A = \left\{(\rho,\varphi,\theta) \,\middle|\, \begin{array}{l} 0<\rho<a \\ 0<\varphi<\frac{\pi}{2} \\ 0<\theta<\frac{\pi}{2} \end{array}\right\}$

Note that $\det Dg > 0$ on A and $g|_A$ is diffeo

$$\int_B f = \int_{g(A)} f = \int_A f(\rho\sin\varphi\cos\theta, \rho\sin\rho\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi$$

$$= \int_{\rho=0}^a \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}}$$