Problem A: Let S be a bounded set in $\mathbb{R}^n$; and let $f : S \to \mathbb{R}$ be a bounded function; let $A = \mathrm{Int}(S)$ be the interior of S. Suppose that f is integrable on S.

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1. (1) Show that f is integrable on A and that $\int_A f = \int_S f$.
2. (2) Deduce that if S is Jordan measurable, then A is Jordan measurable as well and $m(A) = m(S)$.

Warning: A is always Lebesgue measurable, since A is open and open sets are Lebesgue measurable. However, for example when $S = [0,1] \setminus \mathbb{Q}$, it's possible to have S Lebesgue measurable but for A and S to have different Lebesgue measure. Such examples must have S not Jordan measurable.

(1) Define $D_f := \{x \in dom(f) \mid f \text{ discontinuous on } x\}$

By Thm on Lec18,

f Riem intble $\iff m(D_f) = 0$,

$m(\{x_0 \in \partial S \mid \lim_{x\to x_0} f(x) \neq 0\}) = 0$

Since $A \subseteq S \Rightarrow D_{f|_A} \subseteq D_f \Rightarrow m(D_{f|_A}) = 0$ ① by monotonicity of Lebesgue measure

Let $x \in \partial A \Rightarrow x \in \bar{A} \setminus A$ since A open

$\bar{A} \subseteq \bar{S}$ since $A \subseteq S \Rightarrow x \in \bar{S} \setminus A \Rightarrow x \in \partial S$

Therefore $\partial A \subseteq \partial S$, so

$\{x_0 \in \partial A \mid \lim_{x\to x_0} f|_A(x) \neq 0\} \subseteq \{x_0 \in \partial S \mid \lim_{x \to x_0} f(x) \neq 0\}$

Thus we have $m(\{x_0 \in \partial A \mid \lim_{x\to x_0} f|_A(x) \neq 0\}) = 0$ ②

By ①②, f is (ordinarily) integrable on A.

Since f Riemble integrable on S and $A \Rightarrow$ it is Riemann integrable on $S \setminus A \Rightarrow$ so does $|f|$.

$\int_S f = \int_A f + \int_{S\setminus A} f$ by additivity of Riemman integral

WTS: $\int_{S\setminus A} f = 0$

it suffices to show $\int_{S\setminus A} |f| = 0$ since if so then

$-\int_{S\setminus A} |f| = \int_{S\setminus A} -|f| = 0$ and then $\int_{S\setminus A} f = 0$ as $\forall x \in S\setminus A$ $-|f(x)| \le f(x) \le |f(x)|$

Since f is ctn, for all non-isolated point x_0 in $S\setminus A$, $\lim_{x\to x_0} f(x) = f(x_0)$

We divide $S \setminus A$ into

$S_1 = \{x \in S\setminus A \mid x_0 \text{ is isolated pt.}\}$

$S_2 = \{x \in S\setminus A \mid x_0 \text{ is not isolated pt and } \lim_{x\to x_0} f(x) = f(x_0) \neq 0\}$

$S_3 = \{x \in S\setminus A \mid x_0 \text{ is not isolated pt and } \lim_{x\to x_0} f(x) = f(x_0) = 0\}$

Then $S\setminus A = S_1 \sqcup S_2 \sqcup S_3$ $\Rightarrow \int_{S_3} |f| = 0$

Since $S \setminus A \subseteq \partial S \Rightarrow S_2 \subseteq \{x_0 \in \partial S \mid \lim_{x\to x_0} f(x) \neq 0\}$

$\Rightarrow m(S_2) = 0$

Since isolated points of any set in $\mathbb{R}^n$ is ctbl $\Rightarrow m(S_1) = 0$

Since $|f|$ is bdd. Suppose $|f| \le M \Rightarrow 0 \le \int_{S_1} |f| \le \int_{S_1} M$

$\int_{S_1} M = M \int_{S_1} 1 = 0$ since S_1 has measure 0, so is $\int_{S_2} M$

$\Rightarrow \int_{S_1} |f| = \int_{S_2} |f| = 0$

$\Rightarrow \int_{S\setminus A} |f| = \int_{S_1} |f| + \int_{S_2} |f| + \int_{S_3} |f| = 0 + 0 + 0 = 0$

Therefore $\int_{S\setminus A} f = 0$ which proves that $\int_S f = \int_A f$ $\square$

(2) Suppose S is Jordan measurable $\Rightarrow m(\partial S) = 0$

In (1) we have shown $\partial A \subseteq \partial S$

$\Rightarrow m(\partial A) = 0$ by monotonicity

$\Rightarrow A$ is Jordan measurable

Since $m(S) = m(A) + m(S\setminus A)$

and $S \setminus A \subseteq \partial S \Rightarrow m(S\setminus A) \le m(\partial S) = 0$

we have $m(S) = m(A) + 0 = m(A)$ $\square$

Problem B: Let $B^n_a(x)$ denote the ball in $\mathbb{R}^n$ centered at x and radius a.

1. (1) Show that
$$\mathrm{vol}\,(B^n_a(x)) = \Gamma_n a^n$$
where $\Gamma_n = \mathrm{vol}\,(\cancel{B^n_0(1)})$. $B_1^n(0)$
2. (2) What is Γ_1 and Γ_2?
3. (3) Compute Γ_n in terms of Γ_{n-2}, and deduce a formula for Γ_n for any n.

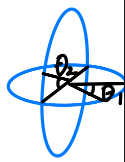
Just for fun, don't hand it: As the dimension gets big, what happens to the volume of the unit ball divided by the volume of $[-1,1]^n$ (the smallest box containing the unit ball)? If you pick a point at random in the box, is it likely to be in the ball?

$B^n_a(x) = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n (y_i - x_i)^2 < a^2\}$

By volume of $B^n_a(x)$ we mean $\int_{B^n_a(x)} 1$, by IBLSA we know $B^n_a(x)$ is Jordan measurable thus 1 is Riem integrable on $B^n_a(x)$

let $g : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} y_1 = r \\ y_2 = \theta_1 \\ y_3 = \theta_2 \\ \vdots \\ y_n = \theta_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 = r\cos\theta_1 \\ x_2 = r\sin\theta_1 \cos\theta_2 \\ x_3 = r\sin\theta_1 \sin\theta_2 \cos\theta_3 \\ \vdots \\ x_{n-1} = r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2}\cos\theta_{n-1} \\ x_n = r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2}\sin\theta_{n-1} \end{pmatrix}$$ x_n



By induction we can get that

$\det(Dg) = r^{n-1} \prod_{k=1}^{n-2} \sin^k(\theta_k)$ for $n \ge 2$

Let $A_a^n(0) = \{(r,\theta) \mid 0<r<a,\ 0\le\theta_i\le 2\pi \text{ for } 1\le i\le n\}$

$\widetilde{A}_a^n(0) = \{(r,\theta)\mid 0<r<a,\ 0<\theta_i<2\pi \text{ for } 1\le i\le n\}$

We know that $g|_{\widetilde{A}_a^n(0)}: \widetilde{A}_a^n(0)\to \widetilde{B}_a^n(0)$ is a diffeomorphism where $\widetilde{B}_a^n(0)$ is $B_a^n(0)$ minus an $n-1$ dim hyperplane which has measure 0 in $\mathbb{R}^n$ so does not affect the integral

By change of variable thm,

$$\int_{B_a^n(0)} 1 = \int_{\widetilde{B}_a^n(0)} 1 = \int_{g(\widetilde{A})} 1 = \int_{\widetilde{A}} 1\,|\det Dg|$$

$$= \int_A |\det Dg| = \int_A r^{n-1}\prod_{k=1}^{n-2}\sin^k(\theta_k) \quad (n\ge 2)$$

$$= \int_0^a\left(\int_0^\pi\cdots\int_0^{2\pi} r^{n-1}\prod_{k=1}^{n-2}\sin^k(\theta_k)\,d\theta_{n-1}\cdots d\theta_1\right)dr \quad (n\ge2)$$

by Fubini's thm since constant function is cts.

$$= \int_0^a r^{n-1}dr\underbrace{\left(\int_0^\pi\cdots\int_0^{2\pi}\prod_{k=1}^{n-2}\sin^k(\theta_k)\,d\theta_{n-1}\cdots d\theta_1\right)}_{\text{some constant } C,\text{ same for all } n\text{-dim ball}} \quad (n\ge2)$$

$= \frac{a^n}{n}$

$= C\frac{a^n}{n}$, and $\Gamma_n = \int_{B_1^n(0)} 1 = \frac{C}{n}$

This proves that $Vol(B_a^n(0)) = \int_{B_a^n(0)} 1 = a^n\Gamma_n \quad (n\ge2)$

And for $n=1$ the ball is just box, $vol(B_a^1(0)) = a\Gamma_1$ for sure.

And also, $\forall x\in\mathbb{R}^n$, we fix x and define translation by x

$$t_x: B_a^n(0)\to B_a^n(x)$$
$$v\mapsto v+x$$

The translation function t_x is a diffeomorphism, and its $\det(Dt) = Id_n$

By Change of variable Thm we we can get

$$\int_{B_a^n(x)} 1 = \int_{t_x(B_a^n(0))} 1 = \int_{B_a^n(0)} 1 = a^n\Gamma_n$$

This finishes the proof that $vol(B_a^n(x)) = a^n\Gamma_n$ for all $x\in\mathbb{R}^n$

(2) $\Gamma_1 = V((-1,1)) = 2$

$\Gamma_2 = \int_0^1 r\,dr\int_0^{2\pi}1 = \frac12(2\pi)=\pi$

(3) for $n\ge3$

We decompose $x\in\mathbb{R}^n$ into $(z\in\mathbb{R}^2, w\in\mathbb{R}^{n-2})$

So $\Gamma_n = \int_{[-1,1]^n}\chi_{B_1^n(0)} = \int_{[-1,1]^2}\int_{[-1,1]^{n-2}}\chi_{B_1^n(0)}\,dw\,dz$ by Fubini's Thm

For each fixed z, the set of w s.t. $\|w\|<\sqrt{1-|z|^2}$ form a $(n-2)$-dim ball

$$\Rightarrow \Gamma_n = \Gamma_{n-2}\int_{B_1^2(0)}(1-|z|^2)^{\frac{n-2}{2}}dz$$

By polar coordinate,

$$\int_{B_1^2(0)}(1-|z|^2)^{\frac{n-2}{2}}dz = \int_0^{2\pi}\int_0^1(1-r^2)^{\frac{n-2}{2}} r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^1\frac12(1-u)^{\frac{n-2}{2}}du$$

$$= 2\pi\,\frac{1}{\frac{n-2}{2}+1} = 2\pi\cdot\frac12\cdot\frac{2}{n} = \frac{2\pi}{n}$$

Therefore $\Gamma_n = \frac{2\pi}{n}\Gamma_{n-2}$. So $\Gamma_n = \begin{cases}\frac{(2\pi)^{k-1}}{(2k)!!}\pi, & n=2k \text{ for some } k\in\mathbb{N}\\ \frac{2^{k+1}\pi^k}{(2k+1)!!}, & n=2k+1 \text{ for some } k\in\mathbb{N}\cup\{0\}\end{cases}$

Problem C: Let A be an open Jordan measurable set in $\mathbb{R}^{n-1}$. Given a point $p\in\mathbb{R}^n$ with $p_n>0$, let S be the subset of $\mathbb{R}^n$ defined by the equation

$$S=\{(1-t)a+tp : a\in A\times 0, 0<t<1\}.$$

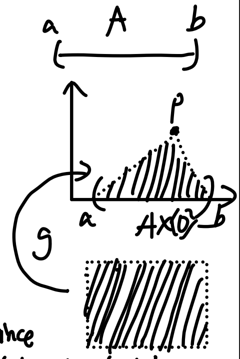
(S is the union of all open line segments in $\mathbb{R}^n$ joining p to points of $A\times 0$. You might think of it as a cone over A.)

(1) Define a diffeomorphism g of $A\times(0,1)$ with S.

(2) Find the volume of S interms of the area of A.

(1) Define $g: A\times(0,1)\to S$

$(a',t)\mapsto(1-t)(a',0)+tp$



Write $p=(p',p_n)$ where $p'\in\mathbb{R}^{n-1}$, $p_n\in\mathbb{R}$

$\Rightarrow g(a',t) = ((1-t)a'+tp', tp_n)$

g is surjective since $\forall s\in S$, $s=(1-t)a+tp$ for some $a\in A\times\{0\}$ and $t\in(0,1)$

and injective since suppose $g(a_1',t_1)=g(a_2',t_2)$

$\Rightarrow t_1p_n=t_2p_n \Rightarrow t_1=t_2\Rightarrow a_1'=a_2'$ since $(1-t)a_1'+tp'=(1-t)a_2'+tp'$

and g should be C^1 since it consists of only linear operations on a', t.

(2) For $i,j=1,\dots,n-1$: $\frac{\partial g_i}{\partial a_j'}=(1-t)\delta_{ij}$ and $\frac{\partial g_i}{\partial t} = -a_i'+p_i'$

For $i=n$, $\frac{\partial g_n}{\partial a_j'}=0\ \forall j$, $\frac{\partial g_n}{\partial t}=p_n \Rightarrow$ upper triangular, $\det Dg=(1-t)^{n-1}p_n$

here we can apply Fubini as we can decompose A into ctbl almost disjoint boxes

$$vol(S) = \int_S 1 = \int_{g(A\times(0,1))}1 = \int_{A\times(0,1)}|\det Dg(x)|\,dx$$

$$=\int_0^1\int_A(1-t)^{n-1}p_n\,da'\,dt = p_n\cdot area(A)\int_0^1(1-t)^{n-1}dt = \frac1n p_n\cdot area(A)$$

Problem D: Compute the volume inside the ellipsoid

$$\frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} + \frac{(z-w)^2}{c^2} = 1$$

where $a, b, c > 0$ and $u, v, w \in \mathbb{R}$.

Let volume inside the ellipsoid is

$$E = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{(x-u)^2}{a^2} + \frac{(y-v)^2}{b^2} + \frac{(z-w)^2}{c^2} < 1 \right\}$$

Define $g: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{x-u}{a} \\ \frac{y-z}{b} \\ \frac{z-w}{c} \end{pmatrix} \Rightarrow g^{-1}(X,Y,Z) = (aX+u, bY+z, cZ+w)$$

$$Dg^{-1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \Rightarrow \det(Dg^{-1}) = abc$$

Observe that g is bijective and C^1, and g^{-1} also C^1
(as it only concerns scaling and translation)

So $\text{vol}(E) = \int_E 1 = \int_{g^{-1}(E)} 1 \, |\det Dg^{-1}| = abc \int_{g^{-1}(E)} 1$

where $g^{-1}(E) = \{ (X,Y,Z) \in \mathbb{R}^3 \mid X^2+Y^2+Z^2 \le 1 \}$

Note that $g^{-1}(E)$ is a 3-dim unit ball centered at 0.

So by problem b, $\int_{g^{-1}(E)} 1 = \Gamma_3 = \frac{4}{3}\pi$

$\Rightarrow v(E) = \frac{4}{3} abc\pi$

Problem E: Compute the volume of the the solid in $\mathbb{R}^3$ bounded below by the surface $z = x^2 + 2y^2$ and above by the plane $z = 2x + 6y + 1$.

The solid region is

$$S = \{ (x,y,t) \in \mathbb{R}^3 \mid x^2+y^2 < z < 2x+6y+1 \}$$

Let $\varphi(x,y) = x^2+2y^2$, $\psi(x,y) = 2x+6y+1$

So we get S restricted to $\mathbb{R}^2$ is

$$C = \{ (x,y) \in \mathbb{R}^2 \mid \varphi(x,y) < \psi(x,y) \}$$

By Fubini's Thm on simple region, we get

$$\text{volume}(S) = \int_S 1 = \int_C \left(\int_{t=\varphi(x,y)}^{t=\psi(x,y)} 1 \right)$$

$$= \int_C 2x+6y+1 - x^2 - 2y^2$$

Since $(x,y) \in C \Leftrightarrow x^2+2y^2 < 2x+6y+1$
$\Leftrightarrow (x-1)^2 + 2(y-1.5)^2 = 6.5$

So $C = \{ (x,y) \mid (x-1)^2 + 2(y-1.5)^2 < 6.5 \}$

Let $u = x-1$, $v = y-1.5 \Rightarrow C = \{ (u,v) \mid u^2 + 2v^2 < 6.5 \}$

and $f(u,v) = 6.5 - u^2 - 2v^2$

Apply another change of variable $g: \begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{6.5}\, r\cos\theta \\ \sqrt{\frac{6.5}{2}}\, r\sin\theta \end{pmatrix}$

So $g|_{(0,1)\times(0,2\pi)}$ is a diffeo mapping to $C \setminus \{x\text{-axis}\}$

$$Dg = \begin{pmatrix} \sqrt{6.5}\cos\theta & -\sqrt{6.5}\, r\sin\theta \\ \sqrt{\frac{6.5}{2}}\sin\theta & \sqrt{\frac{6.5}{2}}\, r\cos\theta \end{pmatrix}$$

Then $\int_C f = \int_{C \setminus \{x\text{-axis}\}} f = \int_{(0,1)\times(0,2\pi)} f(g(r,\theta)) \det(Dg) \, r \, dr \, d\theta$

$$= \frac{(6.5)^2}{\sqrt{2}} \int_0^{2\pi} \int_0^1 (1-r^2) r \, dr \, d\theta \quad \text{by Fubini}$$

$$= \frac{(6.5)^2}{\sqrt{2}} \cdot \frac{1}{4} \cdot 2\pi = \frac{\frac{169}{4}\pi}{2\sqrt{2}} = \frac{169\sqrt{2}\pi}{16}$$

Problem F: Evaluate $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Consider $\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx$

Since $e^{-(x^2+y^2)}$ is continuous and bounded by 1,
it is extendedly integrable

Let B_n be the ball centered at 0 with radius n, $n \ge 1$

i.e. $B_n = \{ (x,y) \in \mathbb{R}^2 \mid 0 < x^2+y^2 < n^2 \}$

Then $\overline{B_n}$ is cpt & $\overline{B_n} \subseteq \mathring{B}_{n+1}$ $\forall n \in \mathbb{N}$ & $\bigcup_n \overline{B_n} = \mathbb{R}^2$

We then have $\int_{\mathbb{R}^2} e^{-(x^2+y^2)} = \lim_{n\to\infty} \int_{\overline{B_n}} e^{-(x^2+y^2)} = \lim_{n\to\infty} \int_{B_n} e^{-(x^2+y^2)}$

And let $g: \mathbb{R}^2 \to \mathbb{R}^2$
$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$

Let $A_n = \{ (r,\theta) \mid 0 < r < n, 0 \le \theta \le 2\pi \}$
$\tilde{A}_n = \{ (r,\theta) \mid 0 < r < n, 0 < \theta < 2\pi \}$

Note that $g|_{\tilde{A}}$ is a diffeomorphism between open sets

$$\int_{B_n} e^{-(x^2+y^2)} = \int_{B_n \setminus \{\text{post } x\text{-axis}\}} e^{-(x^2+y^2)} = \int_{g(\tilde{A})} e^{-(x^2+y^2)}$$

$$= \int_{\tilde{A}} e^{-r^2} r \quad \text{by change of variable Thm}$$

$$= \int_0^{2\pi} \int_0^n e^{-r^2} r \, dr \, d\theta \quad \text{by Fubini's Thm for}$$

So $\int_{\mathbb{R}^2} e^{-x^2+y^2} = \lim_{n\to\infty}\int_0^{2\pi}\int_0^n e^{-r^2} r\,dr\,d\theta$ by cov

$$= \int_0^{2\pi}\left(0-\left(-\tfrac{1}{2}\right)\right)d\theta = \pi$$

Also, consider $C_n = [-n,n]\times[-n,n]$

And get $\int_{\mathbb{R}^2} e^{-x^2+y^2} = \lim_{n\to\infty}\int_{C_n} e^{-x^2+y^2}$

$$= \lim_{n\to\infty}\int_{-n}^{n}\int_{-n}^{n} e^{-x^2+y^2}\,dy\,dx$$

$$= \lim_{n\to\infty}\int_{-n}^{n} e^{-x^2}\left(\int_{-n}^{n} e^{-y^2}dy\right)dx$$

$$= \lim_{n\to\infty}\left(\int_{-n}^{n} e^{-y^2}dy\right)\left(\int_{-n}^{n} e^{-x^2}dx\right)$$

$$= \left(\int_{\mathbb{R}} e^{-x^2}dx\right)^2$$

So $\int_{\mathbb{R}} e^{-x^2}dx = \sqrt{\int\int_{\mathbb{R}^2} e^{-x^2+y^2}} = \sqrt{\pi}$

Problem G. For which exponents $e \in \mathbb{R}$ is $f(x) = |x|^e$ integrable over the unit ball in $\mathbb{R}^n$? For which exponents is it integrable over the complement of the closed unit ball?

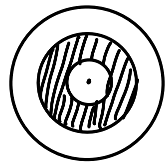
For $e \geq 0$, f is bounded on $B_1(0)$. Since it is also continuous, it is always (ordinarily) integrable as $m(D_f)=0$ and $m(\partial B_1(0))=0$

For $e<0$:

f is unbounded at 0. So the $f|_{B_1(0)}$ is at most extendedly integrable

Claim 1 f is ext. integrable on $B_1(0)$ for $e>-n$

Define $B_n = B_{1-\frac{1}{\sqrt{n+1}}}(0) - B_{\frac{1}{\sqrt{n+1}}}(0)$



We have $\overline{B_n} \subseteq B_{n+1}^{\circ}$ for each n and

$$\bigcup_{n=1}^{\infty} \overline{B_n} = B_1(0)\setminus\{0\}$$

f ext. intble on $B_1(0)$ is equivalent to ext. intble on $B_1(0)\setminus\{0\}$ since changing one point does not affect integrability.

WTS: $\left\{\int_{B_i} f\right\}_{i=1}^{\infty}$ is bdd.

Define $C_n = \overline{B_{i+1}}\setminus B_i \ \forall i$

So $\{C_i\}_{i=1}^{\infty}$ are disjoint and closed, $\bigcup_{i=1}^{N} C_i = \overline{B_N}$

Then $\int_{B_N} f = \sum_{i=1}^{N}\int_{C_i} f \leq \sum_{i=1}^{N}\int_{C_i}\max_{C_i} f$

$$= \Gamma_n \sum_{n=1}^{N}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}+1-\frac{1}{\sqrt{n+2}}-\left(1-\frac{1}{\sqrt{n+1}}\right)\right)\max_{C_n} f$$

$$= \Gamma_n \sum_{i=1}^{N}\frac{2}{\sqrt{n+1}\sqrt{n+2}}\left(\frac{1}{n+2}\right)^{n+e}$$

$$\leq \Gamma_n \sum_{i=1}^{N}\frac{2}{n+1}\left(\frac{1}{n+2}\right)^{n+e}$$

So $\left\{\int_{B_N} f\right\}$ is bdd by $\Gamma_n \sum_{i=1}^{\infty}\frac{2}{n+2}\left(\frac{1}{n+2}\right)^{n+e}$

When $e>-n$, $n+e>0 \Rightarrow$ This series converges

$\Rightarrow \left\{\int_{B_N} f\right\}$ bdded, thus ext. intble

Claim 2 f is not ext. intble on $B_1(0)$ for $e\leq -n$

$$\int_{B_N} f = \sum_{i=1}^{N}\int_{C_i} f \geq \sum_{i=1}^{N}\int_{C_i}\max_{C_i} f$$

$$= \Gamma_n \sum_{n=1}^{N}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n+2}}+1-\frac{1}{\sqrt{n+2}}-\left(1-\frac{1}{\sqrt{n+1}}\right)\right)\min_{C_n} f$$

$$= \Gamma_n \sum_{i=1}^{N}\frac{2}{\sqrt{n+1}\sqrt{n+2}}\underbrace{\left(1-\frac{1}{n+2}\right)^{n+e}}_{\geq 1}$$

$$\geq \Gamma_n \sum_{i=1}^{N}\frac{2}{n+2}\left(\frac{1}{n+2}\right)^{n+e}$$ ≥ 1 when $e\leq -n$

diverges when $N\to\infty$

Conclusion f is integrable on $B_1(0)$ iff $e>-n$

Symmetrically can get f is integrable on $\mathbb{R}^n\setminus\overline{B_1(0)}$ iff $e<-n$

Bonus: The purpose of this bonus is to prove the easiest case of Sard's Theorem, which is something we might prove in general in Math 396 and is a central tool in the study of manifolds.

(1) If $f:\mathbb{R}\to\mathbb{R}$ is continuous and I is a compact interval, then $f(I)$ is a compact interval. (You don't need to prove this; it follows easily from the fact that a continuous image of a compact set is compact and a continuous image of a connected set is connected.) Show that if f is differentiable with $|f'|\leq\delta$ on I, then

$$|f(I)|\leq\delta|I|,$$

where $|I|$ denotes the length of an interval I.

(2) Suppose $f:\mathbb{R}\to\mathbb{R}$ is C^1. Show that

$$f(\{x\in\mathbb{R} : f'(x)=0\})$$

has measure 0. You should do this by first showing that for each n,

$$f(\{x\in[-n,n] : f'(x)=0\})$$

has measure zero, using that f' is continuous and hence uniformly continuous on $[-n,n]$.

(1) Since $f(I)$ is cpt, it contains extreme points,

Say $\min(f(I))=\alpha$ and $\max(f(I))=\beta$

So $|f(I)|=\beta-\alpha$

There exists some pt. $a\in I$ and $b\in I$ s.t. $f(a)=\alpha$, $f(b)=\beta$

WLOG suppose $a\leq b$

Since $f|_I$ is differentiable

by MVT, $\exists\, c\in[a,b]$ s.t. $f'(c)(b-a)=f(b)-f(a)=\beta-\alpha$

So $|f(I)|=\beta-\alpha=f'(c)(b-a)$

$\leq \delta(b-a)\leq\delta|I|$

Write $A = \{x \in \mathbb{R} \mid f'(x) = 0\}$

(2) Let $A_n = \{x \in [-n,n] \mid f'(x) = 0\} \Rightarrow A = \bigcup_{n=1}^{\infty} A_n \Rightarrow f(A) = \bigcup_{n=1}^{\infty} f(A_n)$

Since $f \in C^1(\mathbb{R})$, f' is ctn thus uniformly ctn. on $[-n,n]$

Let $\varepsilon > 0 \Rightarrow \exists \delta > 0$ s.t. $f'(x) - f'(y) < \varepsilon$ for all $x, y \in [-n,n]$ s.t. $|x-y| < \delta$

For each $x \in A_n$, take open interval $I_x \in (x - \frac{\delta}{2}, x + \frac{\delta}{2}) \cap (-n,n)$

$\Rightarrow \forall y \in I_x, |f'(y) - f'(x)| < \varepsilon$

Then $\{I_x \mid x \in A_n\}$ form an open cover of A_n

Notice that A_n is cpt. since $f'^{-1}(\{0\})$ is closed and $A_n = f'^{-1}\{0\} \cap [-n,n]$ is bounded.

So $\exists$ finite subcover $\{I_1, I_2, \dots, I_k\}$ with $\bigcup_{i=1}^{k} I_i \subseteq [-n,n]$

so $A_n \subseteq \bigcup_{i=1}^{k} I_i$, thus $f(A_n) \subseteq f(\bigcup_{i=1}^{k} I_i)$

Since $\bigcup_{i=1}^{k} I_i$ is finite union of intervals thus still interval,

$$\left|f\left(\bigcup_{i=1}^{k} I_i\right)\right| = \left|f\left(\overline{\bigcup_{i=1}^{k} I_i}\right)\right| \leq \varepsilon \left|\overline{\bigcup_{i=1}^{k} I_i}\right| \leq \varepsilon |[-n,n]| \leq 2n\varepsilon \quad \text{by (1)}$$

$\Rightarrow m(f(A_n)) \leq 2n\varepsilon$

Since ε is arbitrary $\Rightarrow m(f(A_n)) = 0$

So $m(f(A)) = m\left(\bigcup_{n=1}^{\infty} f(A_n)\right) \leq \sum_{n=1}^{\infty} m(f(A_n)) = 0$

$\square$