Problem A: Construct a sequence of smooth functions $\psi_i : \mathbb{R} \to \mathbb{R}$ with disjoint supports such that the function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by $\lambda(x) = \sum_{i=1}^{\infty} \psi_i(x)$ is not continuous.

Hint: Try letting ψ_i be a bump function supported on $[1/2^{2i+2}, 1/2^{2i+1}]$.

Remark: The point of this problem is to illustrate why, in the definition of a partition of unity, we require that each point has a *neighbourhood* that intersects only finitely many supports, rather than just saying that for each x only finitely many of the functions are non-zero at x.

Pf For each $i \in \mathbb{N}$, define

$$a_i = \frac{1}{2^{2i+2}},\ b_i = \frac{1}{2^{2i+1}}$$

$$I_i = [a_i, b_i],\ L_i = |I_i| = \frac{1}{4^{i+1}},\ M_i = 4^{i+1}$$

Define $\varphi : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} \exp\left(-\frac{1}{1-t^2}\right), & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

, we verified that φ is smooth

and define $\varphi_i(x) = M_i \varphi\left(\frac{2(x - \frac{a_i+b_i}{2})}{L_i}\right)$ for each $i \in \mathbb{N}$

So φ_i is supported on I_i for each $i \in \mathbb{N}$

and smooth since it is just translating and streching a smooth function

This $(\varphi_i)_{i\in\mathbb{N}}$ is a construction of smooth functions with disjoint support.

Define $\lambda(x) = \sum_{i=1}^{\infty} \varphi_i(x)$

Since $\varphi_i\left(\frac{a_i+b_i}{2}\right) = M_i \varphi(0) = M_i e^{-1} \to \infty$ as $i \to \infty$

$\left(\frac{a_i+b_i}{2}\right)_{i\in\mathbb{N}} \to 0$ as $i \to \infty$

$\Rightarrow \lim_{x\to 0^+} \lambda(x) = \infty$, while $\lambda(0) = 0$

This creates a discontinuity at $x = 0$

Problem B: Prove the Change of Variable Theorem for linear diffeomorphisms and continuous functions with compact support. You can use results proved in class up to and including the result that every linear diffeomorphism is a composition of primitive linear diffeomorphisms. You cannot use results proved after that.

Remark: This will not take very long. Because the functions have compact support, the integrals can be interpreted as ordinary integrals, and one can procede directly to the punchline with Fubini. The purpose of this problem is to help you study the "core" of the proof of Change of Variables without getting lost in the technical details.

Pf We prove by induction

Base case $n = 1$

Let $g : A \subseteq \mathbb{R} \to B \subseteq \mathbb{R}$ is linear diffeo

$f : B \to \mathbb{R}$ ctn with cpt supp.

Let $x \in A$, interval $I \ni x \Rightarrow J := g(I)$ is an interval

By partition of unity, it suffices to prove the statement for $g|_{I^\circ} : I^\circ \to J^\circ$, $f|_{J^\circ} : J^\circ \to \mathbb{R}$

And this is true by the change of variable theorem in single variable analysis.

Inductive step suppose the statement holds in dim $n-1$

WTS: it holds for dim n

Claim for dim $n > 1$, it suffices to prove the theorem for primitive $h : U \to V$ and $f|_V$

This is because for each $x \in A$,

$\exists$ nbh $U_0 \ni x$ and a finite seq of primitive diffeos

$$U_0 \xrightarrow{h_1} U_1 \to \dots \xrightarrow{h_k} U_k$$

s.t. $g|_{U_0} = h_k \circ \dots \circ h_1$

WLOG suppose h preserves the last coord

Let $p \in U$, $Q \subseteq V$ be a box st $h(p) \in Q^\circ$, $S := h^{-1}(Q)$

Since p is arbitrary, it suffices to prove the statement for $h : S^\circ \to Q^\circ$ and $f|_{Q^\circ}$

Since $(f \circ h)|\det Dh|$ is ctn and has cpt support in S°, it is intble on S°

WTS: $\int_{Q^\circ} f = \int_{S^\circ}$ and by cpt support we can treat the two integrals as ordinary integrals.

Define $F : \mathbb{R}^n \to \mathbb{R}$

$$x \mapsto \begin{cases} f(h(x))|\det Dh|, & x \in S^\circ \\ 0, & \text{elsewhere} \end{cases}$$

so WTS: $\int_Q f = \int_S F$

Pf Let $Q = D \times I$, where D is a box in $\mathbb{R}^{n-1}$, I is interval in $\mathbb{R}$

Since S cpt and h preserves the last coord,

$S \subseteq E \times I$ for some box $E \subseteq \mathbb{R}^{n-1}$

By Fubini's Thm, the statement to be proved is

$$\int_I \int_D f(y,t)\, dy\, dt = \int_I \int_E F(x,t)\, dx\, dt$$

So it suffices to prove that $\int_D f(y,t)dy = \int_E F(x,t)\,dx$ for each $t \in I$

Fix $t \in I$, $h(x,t) = (k(x,t), t)$ for some C^1 function $k: U \to \mathbb{R}^{n-1}$

We define $h_t : x \in \mathbb{R}^{n-1} \mapsto k(x,t) \in \mathbb{R}^{n-1}$

Since $Dh = \left[\begin{array}{c|c} \partial k/\partial x & \partial k/\partial t \\ \hline 0 & 1 \end{array}\right] = \partial k/\partial k = Dh_t$,

we have $\int_{V_t} f(y,t)\,dy = \int_{U_t} f(h_t(x))\,|\det Dh_t|\,dx$

$= \int_{U_t} F(x,t)\,dx$

by inductive hypothesis. □

Problem C: Let $M_{n,m}$ be the space of n by m matrices. Show that the rank function is lower semi-continuous on $M_{n,m}$. Give an example to show it need not be continuous.

Hint: To prove lower semi-contintity, you need to show every matrix A has a neighbourhood whose rank is at least as big as the rank of A. You can do that quickly with the problem from the last homework on rank. For the example, you may want to consider small multiples of the identity.

Pf Let $A \in M_{n,m}$

WTS: $\forall \varepsilon > 0$, $\exists$ nbh $U \ni A$ s.t.

$\forall B \in U$, $\text{rank}(B) > \text{rank}(A) + \varepsilon$

It suffices to show that $\exists$ nbh $U \ni A$ s.t.

$\forall B \in U$, $\text{rank}(B) \geq \text{rank}(A)$ WTS

This statement is trivially true when $\text{rank}(A) = 0$ since rank is nonnegative

So suppose $\text{rank}(A) = r \neq 0$

Then $\exists$ a $r \times r$ minor B of A s.t. $\det(B) \neq 0$

WLOG suppose $\det(B) > 0$

Since the det function is ctn, $\exists \varepsilon > 0$ s.t

$\forall B' \in M_{r,r}$ st $\|B' - B\|_F \leq \varepsilon$, $\det(B') > 0$

So consider $B_\varepsilon(A) \subseteq M_{n,m}$ with respect to the Frebenius norm

Let $A' \in B_\varepsilon(A) \Rightarrow \|A' - A\|_F \leq \varepsilon$

then the $r \times r$ minor of A' on the same position as B, call it B', has $\|B' - B\|_F \leq \varepsilon \Rightarrow \det(B') > 0 \Rightarrow \text{rank}(A') \geq r$

This finishes the proof that the rank function in $M_{n,m}$ is lower semi-ctn. □

Counterex to show that rank function is not ctn.

Consider $\left(A_k = \frac{1}{k} I_n\right)_{k \in \mathbb{N}}$

$(A_k) \to 0 \in M_{n,m}$ as $k \to \infty$

But $\forall k \in \mathbb{N}$, $\text{rank}(A_k) = n$

so $\lim_{k \to \infty} \text{rank}(A_k) = n \neq 0 = \text{rank}(0)$

So rank function is not ctn at 0.