hw 5

Problem A: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$F(x, y, z) = (\exp(x^2 + 2y^2), \sin(z^2 - y^2) \cdot (x^2 + 2z^2), (x^2 + y^2 + z^2)^9).$$

Explain why F is differentiable, and then prove why $DF(x, y, z)$ always has zero determinant. You may not actually compute any derivatives your solution.

① $F_1(x,y,z) = \exp(x^2+2y^2)$

let $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ and $f : \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto x^2+2y^2$

g is exponential, in $C^\infty(\mathbb{R})$; f is polynomial, in $C^\infty(\mathbb{R}^3)$

So $F_1 = g \circ f \in C^\infty(\mathbb{R}^3)$

(note: f is C^r and g is C^r $\Rightarrow$ $g \circ f$ is C^r, by applying chain rule and product rule recursively.)

② $F_2(x,y,z) = \left(\sin(z^2-y^2)\right)(x^2+2z^2)$

let $g : \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$, $x \mapsto \sin x$ (trig), $f : \mathbb{R}^3 \to \mathbb{R} \in C^\infty(\mathbb{R}^3)$, $(x,y,z) \mapsto z^2-y^2$ (poly), $h : \mathbb{R}^3 \to \mathbb{R} \in C^\infty(\mathbb{R}^3)$, $(x,y,z) \mapsto x^2+2z^2$ (poly)

So $F_2 = (g \circ f) \cdot h$ is C^∞

③ $F_3(x,y,z) = (x^2+y^2+z^2)^9$

let $g : \mathbb{R} \to \mathbb{R} \in C^\infty(\mathbb{R})$, $x \mapsto x^9$ (positive power), $f : \mathbb{R}^3 \to \mathbb{R} \in C^\infty(\mathbb{R}^3)$, $(x,y,z) \mapsto x^2+y^2+z^2$ (poly)

So $F_3 = g \circ f$ is C^∞

Thus all entries of the Jacobian matrix of F are in C^∞, thus F is differentiable.

Let $F_1 : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x^2+2y^2 \\ x^2+2z^2 \\ z^2-y^2 \end{pmatrix}$$

$F_2 : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} \exp(a) \\ b\sin(c) \\ \left(\frac{1}{2}a + \frac{1}{2}b\right)^9 \end{pmatrix}$$

note: $F = F_2 \circ F_1$

Thus by the chain rule: $\forall a \in \mathbb{R}^3$, $DF(a) = DF_2(F_1(a)) \cdot DF_1(a)$

So $\det(DF(a)) = \det(DF_2(F_1(a))) \det(DF_1(a))$

note that $\forall a = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, $DF_1(a) = \begin{pmatrix} 2x & 4y & 0 \\ 2x & 0 & 4z \\ 0 & -2y & 2z \end{pmatrix}$

where $row_3 = \frac{1}{2}(row_2 - row_1) \Rightarrow$ linearly dependent $\Rightarrow$ row rank < 3

$\Rightarrow \det(DF_1(a)) = 0$

This finishes the proof the $\forall a \in \mathbb{R}^3$, $\det(DF(a)) = 0$

$\square$

Problem B: Suppose

$$F : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$$

and

$$G : B \subset \mathbb{R}^m \to A \subset \mathbb{R}^n$$

are both differentiable and are inverses of each other (A, B open). Show that $n = m$ and that, for all pairs $a \in A, b \in B$ with $F(a) = b$

$$DG(b) = DF(a)^{-1}.$$

Pf WLOG suppose $n<m$

Then take arbitrary $x \in A$

Since F, G are differentiable and inverse of each other,

we have $GF(x) = x \Rightarrow D(GF)(x) = I_n$

$\Rightarrow DG(F(x))DF(x) = I_n$ by chain rule ①

Similary we have:

$\forall y \in B, FG(y) = y \Rightarrow D(FG)(y) = I_m$

$\Rightarrow DF(G(y))DG(y) = I_m$ by chain rule

By taking $y = F(x) \Rightarrow DF(x)DG(F(x)) = I_m$ ②

Claim: if matrix $AB = I_m$ and $BA = I_n$ then we must have $m = n$ and $A = B^{-1}$.

Pf of claim $AB = I_m \Rightarrow m = \text{rank}(AB) \le \text{rank}(A) \le \min\{n, m\}$

$BA = I_n \Rightarrow n = \text{rank}(BA) \le \text{rank}(B) \le \min\{m, n\}$

if $m \ge n \Rightarrow$ must have $m \le n$; if $m \le n \Rightarrow$ must have $m \ge n$

Therefore $m = n$

By claim, we have $m = n$ and $DF(x)^{-1} = DG(F(x))$

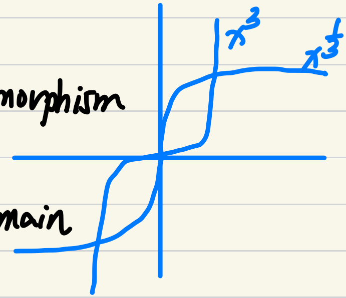
Since x is taken arbitrary, this proves that $\forall a \in A, b \in B$ s.t. $F(a) = b$

we have $DF(a)^{-1} = DG(b)$ $\square$

Problem C: Give an example of a differentiable homeomorphism from $\mathbb{R}$ to itself whose inverse is not differentiable at every point. (For your example, you need to find only a single point where the inverse isn't differentiable.)

ex $f: \mathbb{R} \to \mathbb{R}$

sending $x \mapsto x^3$ is a differentiable homeomorphism since it is invertible and differentiable on the whole domain



its inverse: $f^{-1}: x \mapsto \sqrt[3]{x}$ is not differentiable at $x=0$

Since $\frac{d}{dx} f^{-1}(x) = \frac{1}{3\sqrt[3]{x^2}}$ does not exist at $x=0$

Problem D: Suppose $F: \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin. Show

$$\lim_{h\to 0}\lim_{k\to 0} F(h,k) = \lim_{k\to 0}\lim_{h\to 0} F(h,k),$$

assuming all these limits exit. Give an example where F is not continuous, both double limits exist, but the two double limits are not equal.

Pf By continuity at the origin we have $\lim_{\sqrt{h^2+k^2}\to 0} F(h,k) = F(0,0)$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall \sqrt{h^2+k^2} < \delta, |F(h,k) - F(0,0)| < \varepsilon$

Let $\varphi_1(h) = \lim_{k\to 0} F(h,k)$, $h \in \mathbb{R}$

Claim 1 $\varphi_1(0) = \lim_{k\to 0} F(0,k) = F(0,0)$

Let $\varepsilon > 0$. By continuity of F at origin,

$\exists \delta > 0$ s.t. $\forall \sqrt{h^2+k^2} < \delta, |F(h,k) - F(0,0)| < \varepsilon$

take the same $\delta \Rightarrow |F(0,k) - F(0,0)| < \varepsilon$ for all $k \in B_0(\delta)$

Thus $\varphi_1(0) = F(0,0)$

Claim 2 $\lim_{h\to 0} \varphi_1(h) = \varphi_1(0) = F(0,0)$

(i.e. φ_1 is continuous at $h=0$)

Let $\varepsilon > 0$. By continuity of F at origin,

take $\delta > 0$ s.t. $\forall \sqrt{h^2+k^2} < \delta$, $|F(h,k) - F(0,0)| < \varepsilon$ ①

WTS: $\exists \delta > 0$ s.t. $|\varphi_1(h) - \varphi_1(0)| < \varepsilon$ for all $|h| < \delta$

Consider $\delta_2 = \frac{\delta}{\sqrt{2}}$. Let $|h| < \delta_2$

$\Rightarrow |\varphi_1(h) - \varphi(0)| = |\lim_{k\to 0} F(h,k) - F(0,0)| < \varepsilon$

since $\forall |k| < \delta_2$, we always have $\sqrt{h^2+k^2} < \delta$

thus $F(h,k) \in B_\varepsilon(F(0,0)) \Rightarrow \lim_{k\to 0} F(h,k) \in B_\varepsilon(F(0,0))$

(since the limit exists, it is bounded by all values of $F(h,k)$ near $(0,0)$)

$\Rightarrow |\lim_{k\to 0} F(h,k) - F(0,0)| < \varepsilon$

that is, $|\varphi_1(h) - \varphi_1(0)| < \varepsilon$

This proves that $\lim_{h\to 0} \varphi_1(h) = \varphi_1(0)$

i.e. $\lim_{h\to 0} \lim_{k\to 0} F(h,k) = F(0,0)$

By taking $\varphi_2(k) = \lim_{h\to 0} F(h,k)$, $k \in \mathbb{R}$

we can dually prove that $\varphi_2(0) = F(0,0)$

and $\lim_{k\to 0} \varphi_2(k) = \varphi_2(0)$

thus $\lim_{k\to 0} \lim_{h\to 0} F(h,k) = F(0,0)$

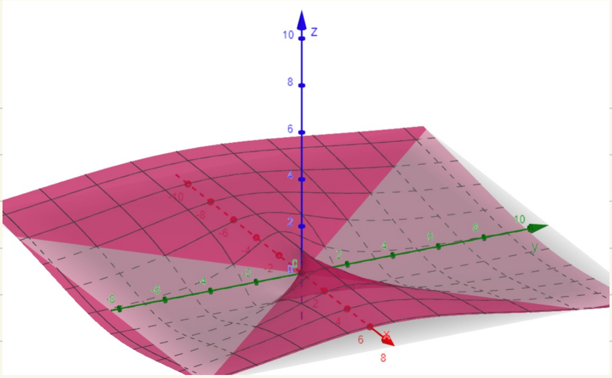
This finishes the proof that $\lim_{k\to 0} \lim_{h\to 0} F(h,k) = \lim_{h\to 0} \lim_{k\to 0} F(h,k)$ □

counterexample when f is not continuous:

$$F(h,k) = \begin{cases} \frac{h^2-k^2}{h^2+k^2}, & (h,k) \neq (0,0) \\ 0, & (h,k) = (0,0) \end{cases}$$

$$\lim_{h\to 0}\lim_{k\to 0} F(h,k) = \frac{1}{1} = 1$$

$$\lim_{k\to 0}\lim_{h\to 0} F(h,k) = \frac{-1}{1} = -1$$



Just for fun (don't hand in): Give an example where $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous at the origin but $\lim_{h\to 0}\lim_{k\to 0} F(h,k)$ does not exist.

Just for fun (don't hand in): Also note that for $a_{n,m} = 2^{n-m}$ it is not true that

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} a_{n,m}.$$

Problem E: If F is a function of 4 variables, how many terms (in general) are in the degree 10 Taylor series of F? (Degree 10 means you use multi-indices α of degree at most 10.) You do not need to show your work; just give the final answer.

$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

$|\alpha| \leq 10$

$\{\alpha \mid \text{feasible}\} = \binom{10+4-1}{4-1} = 286$

$10+4-1$

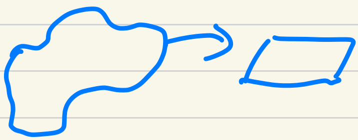
$0|00\,??|?|0\,??0$

$\frac{11 \times 12 \times 13}{3!}$

Problem F: Suppose that $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with A open and connected and $Df(a) = 0$ for all $a \in A$. Show that F is constant.

Claim 1 for all $a \in A$, f is locally constant on A

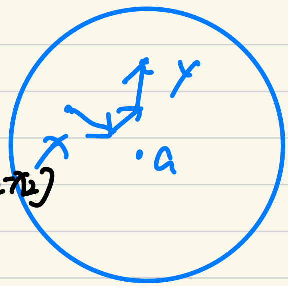
($\exists\, \varepsilon > 0$ s.t. $f[B_\varepsilon(a)]$ is const.)



Pf of claim 1

Let $a \in A$. Take $\varepsilon > 0$ s.t. $B_\varepsilon(a) \subseteq A$ (by openness)

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in B_\varepsilon(a)$.



Let $S_0 = x$, $\varphi_0 : t \mapsto F(S_0 + te_1)$, $t \in [0, y_1 - x_1]$

$S_1 = x + (y_1 - x_1)$, $\varphi_1 : t \mapsto F(S_1 + te_2)$, $t \in [0, y_2 - x_2]$

$\vdots$

$S_n = x + (y_1 - x_1) + (y_2 - x_2) + \dots + (y_n - x_n) = y$

Since for all points $c \in A$, all entries of $Df(c)$ are 0,
every partial is constant 0 (thus continuous) on A

Note: For each φ_i, $0 \leq i \leq n-1$, we have $\varphi_i'(t) = \frac{\partial F}{\partial x_i}(S_i + te_{i+1}) = 0$
for all $t \in [0, y_{i+1} - x_{i+1}]$

$\Rightarrow \forall i=1,\dots,n-1,\ F(s_i)-F(s_{i-1}) = (y_i - x_i)\varphi_{i-1}'(t) = 0$ by MVT

Thus $F(y)-F(x) = \sum_{i=1}^{n}\left(F(s_i)-F(s_{i-1})\right) = 0$

$\Rightarrow F(y)=F(x)$

Since x,y are arbitrary, we have proved that $\forall x,y \in B_\varepsilon(a)$, $F(x)=F(y)$

Therefore F is locally constant around a.

Claim 2 $S = \{x : F(x) = F(a)\}$ is both closed and open in A

Pf of Claim 2

S is open since $(\forall a \in S, \exists \varepsilon > 0$ s.t. $B_\varepsilon(a) \subseteq S)$ proved in claim 1

Let (a_n) be a sequence in S s.t. $(a_n) \to x$ for some $x \in A$

Since F is diff'ble thus ctn, $F(x) = \lim_{n\to\infty} F(a_n) = F(a) \Rightarrow x \in S$

Thus S is closed

The fact that A is connected implies that the only set both open and closed in A is A itself.

$\Rightarrow S = A.$

$\Rightarrow \forall x \in A, F(x) = F(a)$, which shows that F is constant. $\square$

Problem G: Let $f_1, f_2, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ be C^k. Show that

$$\partial^k(f_1 \cdots f_m) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^{\alpha_1} f_1 \cdots \partial^{\alpha_m} f_m.$$

Pf We prove by induction on $k \in \mathbb{N}$

Base case: $k=1$, $\partial(f_1 \ldots f_m) = \partial(f_1 \cdot (f_2 \cdots f_m))$

$= (f_2 \cdots f_m)\partial f_1 + f_1 \partial(f_2 \cdots f_m)$ by product rule

$= (f_2 \cdots f_m)\partial f_1 + (f_1 f_3 \cdots f_m)\partial f_2 + f_1 f_2 \partial(f_3 \ldots f_m)$

$= \left(\sum_{i=1}^{k} \partial f_i \prod_{j \neq i} f_j\right) + \left(\prod_{i=1}^{k} f_i\right)\left(\partial \prod_{i=k+1}^{m}\right)$

$= \underbrace{\sum_{i=1}^{m}\left(\partial f_i \prod_{j \neq i} f_j\right)}$

$= \sum_{|\alpha|=1} \frac{1!}{\alpha!} \partial^{\alpha_1} f_1 \ldots \partial^{\alpha_m} f_m$, statement holds true

Inductive step: Suppose the equality holds for $1, 2, \ldots, k$

Then $\partial^{k+1}(f_1 f_2 \ldots f_m) = \partial(\partial^k(f_1 f_2 \ldots f_m))$

$= \partial\left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial^{\alpha_1} f_1 \ldots \partial^{\alpha_m} f_m\right)$

$= \sum_{|\alpha|=k} \frac{k!}{\alpha!} \partial\underbrace{(\partial^{\alpha_1} f_1 \ldots \partial^{\alpha_m} f_m)}$ $= \sum_{|\beta|=1} \partial^{\beta_1}(\partial^{\alpha_1} f_1) \ldots \partial^{\beta_m}(\partial^{\alpha_m} f_m)$

$= \sum_{|\alpha|=k} \sum_{|\beta|=1} \frac{k!}{\alpha!} (\partial^{\beta_1+\alpha_1} f_1) \ldots (\partial^{\beta_m+\alpha_m} f_m)$ $= \sum_{|\beta|=1} \partial^{\beta_1+\alpha_1} f_1 \ldots \partial^{\beta_m+\alpha_m} f_m$

by base case

$= \sum_{|\alpha|=k} \frac{k!}{\alpha!}\left(\sum_{i=1}^{m} \partial^{\alpha_1} f_1 \ldots \partial^{\alpha_i+1} f_i \ldots \partial^{\alpha_m} f_m\right)$

every term corespond to a multi-index β s.t.

for some $j \in \{1, \ldots, m\}$, $\beta_i = \alpha_i + 1$ while for $i \neq j$, $\beta_i = \alpha_i$

So each β has $|\beta| = k+1$

Thus $\partial^{k+1}(f_1 \dots f_m) = \sum_{|\beta|=k+1} (\text{coeff})\, \partial^{\beta_1} f_1 \dots \partial^{\beta_m} f_m$

The $(\text{coeff}) = \underbrace{\sum_i \text{coeff of} \left(\partial^{\beta_1} f_1 \dots \partial^{\beta_i - 1} f_i \dots \partial^{\beta_m} f_m\right)}$

$$= \sum_{i=1}^{m} \frac{k!}{\beta_1! \dots (\beta_i - 1)! \dots \beta_m!}$$

$$= \sum_{i=1}^{m} \frac{k!\,\beta_i}{\beta!} = \frac{k!(k+1)}{\beta!} = \frac{(k+1)!}{\beta!}$$

Therefore the expression simplifies to:

$$\underline{\partial^{k+1}\left(\prod_{i=1}^{m} f_i\right) = \sum_{|\beta|=k+1} \left(\frac{(k+1)!}{\beta!} \prod_{i=1}^{m} \partial^{\beta_i} f_i\right)}$$

This finishes the proof by induction. $\square$

Problem H: Let $f : \mathbb{R}^n \to \mathbb{R}$ be in C^{k+1}. Show that the Taylor polynomial of degree k centered at $x_0 \in \mathbb{R}^n$ is the best polynomial approximation of $f(x)$ near x_0 in the following sense: Suppose that $P(x)$ is a polynomial of degree k. Then

$$P(x) - f(x) = o(|x - x_0|^k)$$

if and only if P is the Taylor polynomial of degree k centered at x_0. (Recall that a quantity Q is $o(|x - x_0|^k)$ if $\lim_{x \to x_0} \frac{Q}{|x - x_0|^k} = 0$.)

Pf <u>Backward direction</u>

Let $f : \mathbb{R}^n \to \mathbb{R}$ be in C^{k+1}

Let $T_k : \mathbb{R}^n \to \mathbb{R}$ be the Taylor polynomial of degree k centered at x_0

i.e. $T_k(x) = \sum_{|\alpha| \le k} \frac{\partial^\alpha f(x_0)}{\alpha!} (x - x_0)^\alpha$, $x \in \mathbb{R}^n$

Then by Taylor's Theorem we have

$$T_k(x) - f(x) = R_{x_0,k}(x) = \sum_{|\alpha|=k+1} \frac{\partial^\alpha f}{\alpha!}(c)\,(x-x_0)^\alpha$$

for some c on the line segment connecting a, x.

note that $\{\alpha \mid |\alpha| = k+1\}$ is finite, $\frac{\partial^\alpha f}{\alpha!}(c)$ is constant for all α

It suffices to show that $T_k(x) - f(x)$ is $o(|x-x_0|^k)$ by showing that for any α s.t. $|\alpha| = k+1$,

$$\lim_{x\to x_0} \frac{(x-x_0)^\alpha}{\|x-x_0\|^k} = 0 \quad , \text{ i.e. } \lim_{x\to 0} \frac{x^\alpha}{\|x\|^k} = 0$$

note: $\forall x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = |x_1|^{\alpha_1} \cdots |x_n|^{\alpha_n} \leq \|x\|^{\alpha_1+\dots+\alpha_n} = \|x\|^{k+1}$

So $\lim_{x\to 0} \frac{x^\alpha}{\|x\|^k} \leq \lim_{x\to 0} \|x\| = 0$ $\square$

Forward Direction

let $P(x)$ be a polynomial of degree k that is not T_k

$f(x) - P(x) = C_1 x^{\alpha^{(1)}} + \dots + C_m x^{\alpha^{(m)}}$ for some constants $C_1, \dots, C_m$ and multi-index $\alpha^{(1)}, \dots, \alpha^{(m)}$ s.t. $|\alpha^{(i)}| \leq k$ for each i

WTS: $\lim_{x\to 0} \frac{C_1 x^{\alpha^{(1)}} + \dots + C_m x^{\alpha^{(m)}}}{|x|^k} \neq 0$

Case 1 $\sum_{i=1}^m C_m \neq 0$

Consider the sequence $(t_n = \frac{1}{n})_{n\in\omega}$ in $\mathbb{R}$

for each t_n, let $x_n = (t_n, t_n, \dots, t_n)$

Then $(x_n) \to 0$ in $\mathbb{R}^d$

Here $\sum_{i=1}^m C_i x^{\alpha^{(i)}} = \sum_{i=1}^m C_i t^n = t^n \sum_{i=1}^m C_i$

$$\Rightarrow \frac{C_1 x^{\alpha^{(1)}} + \dots + C_m x^{\alpha^{(m)}}}{|x|^k} = \frac{\sum_{i=1}^m C_i t^d}{\sqrt{d}\, t^d} = \frac{\sum_{i=1}^m C_i}{\sqrt{d}} \text{ is constant while } x \to 0$$

This suffices to show that $\lim_{x\to 0} \frac{C_1 x^{\alpha^{(1)}} + \dots + C_m x^{\alpha^{(m)}}}{|x|^k} \neq 0$

Case 2 $\sum_{i=1}^m C_m = 0$

idk ¨⌒

Try another way to prove the forward direction:

Claim a polynomial homogeneous of degree k in $\mathbb{R}^d$ is not $o(|x|^k)$

(by homogeneous of degree k we mean: $\forall t \in \mathbb{R}, x \in \mathbb{R}^n, \; P(tx) = t^k P(x)$)

Let $P(x)$ be a polynomial homogeneous of degree k in $\mathbb{R}^d$

let $x_0 \in \mathbb{R}$, $(t_n = \frac{1}{n})_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$

So $((t_n x_0, \dots, t_n x_0))_{n \in \mathbb{N}} \to 0$ in $\mathbb{R}^d$

Denote each term of this seq as x_n

$$\Rightarrow \frac{P(x_n)}{|x_n|^k} = \frac{t^k P(x_0)}{t^k |x_0|^k} = \frac{P(x_0)}{|x_0|^k} \text{ is const.}$$

This implies that $\lim_{|x| \to 0} \frac{P(x)}{|x|^k} \neq 0$

Note that a polynomial of degree k is a homogeneous of degree k.

Problem I: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x^2 + y^2)$, so F is differentiable.

(1) Prove
$$x\frac{\partial F}{\partial y} = y\frac{\partial F}{\partial x}.$$

(2) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$, $g : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are differentiable. Define $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ via
$$\phi(x, y, z) = (f(h(x), g(x, y), z), g(y, z)).$$
Find a formula for the matrix of the derivative $D\phi(p) : \mathbb{R}^3 \to \mathbb{R}^2$ at an arbitrary point $p \in \mathbb{R}^3$ in terms of the partial derivatives of f, g, and h at p.

(3) In (*ii*), compute $D\phi(1, 1, 1)$ when $f(x, y, z) = x^2 + yz$, $g(x, y) = y^3 + xy$, and $h(x) = e^x$. Do this in two ways: using your general formula in (*ii*) and also by explicitly computing ϕ in this case and directly computing the Jacobian matrix from this.

(1) Pf Let $(x,y) \in \mathbb{R}^2$

$$\frac{\partial F}{\partial x} = 2x\, Df(x^2+y^2) \text{ by chain rule}$$

$$\frac{\partial F}{\partial y} = 2y\, Df(x^2+y^2) \text{ by chain rule}$$

$$\Rightarrow y\frac{\partial F}{\partial x} = x\frac{\partial F}{\partial y} = 2xyDf(x^2+y^2)$$

(2) $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$

let $\varphi_m : \mathbb{R}^3 \to \mathbb{R}^4$ map $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} h(x) \\ g(x,y) \\ z \\ g(y,z) \end{pmatrix}$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[rr, "\varphi"] \arrow[dr, "\varphi_m"'] & & \mathbb{R} \\ & \mathbb{R}^4 \arrow[ur, "\varphi_n"'] & \end{tikzcd}$$

$\varphi_n : \mathbb{R}^4 \to \mathbb{R}^2$ map $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} f(a,b,c) \\ d \end{pmatrix}$

Then $\varphi = \varphi_n \circ \varphi_m$

$$\varphi : p = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\varphi_m} \begin{pmatrix} a = h(x) \\ b = g(x,y) \\ c = z \\ d = g(y,z) \end{pmatrix} \xmapsto{\varphi_n} \begin{pmatrix} f(a,b,c) \\ d \end{pmatrix}$$

$\Rightarrow D\varphi(p) = D\varphi_n(\varphi_m(p))\, D\varphi_m(p)$ by chain rule.

$$= \begin{pmatrix} \frac{\partial f}{\partial x_1}(h(p_1), g(p_1,p_2), p_3) & \frac{\partial f}{\partial x_2}(h(p_1), g(p_1,p_2), p_3) & \frac{\partial f}{\partial x_3}(h(p_1), g(p_1,p_2), p_3) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h'(p_1) & 0 & 0 \\ \frac{\partial g}{\partial x_1}(p_1,p_2) & \frac{\partial g}{\partial x_2}(p_1,p_2) & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\partial g}{\partial x_1}(p_2,p_3) & \frac{\partial g}{\partial x_2}(p_2,p_3) \end{pmatrix}$$

(3) $p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $f(x,y,z) = x^2 + yz$, $g(x,y) = y^3 + xy$, $h(x) = e^x$

$h(p_1) = e$, $g(p_1,p_2) = 2$ $\quad \frac{\partial f}{\partial x_1}(x,y,z) = 2x$, $\frac{\partial f}{\partial x_2}(x,y,z) = z$, $\frac{\partial f}{\partial x_3}(x,y,z) = y$

$h'(x) = e^x$, $\frac{\partial g}{\partial x_1}(x,y) = y$, $\frac{\partial g}{\partial x_2}(x,y) = 3y^2 + y$

By the formula in (2),

$$D\varphi(1,1,1) = \begin{pmatrix} 2e & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2e^2+2 & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

Compute directly:

$$\varphi(x,y,z) = \begin{pmatrix} e^{2x} + z(y^3 + xy) \\ z^3 + yz \end{pmatrix}$$

$$D\varphi(1,1,1) = \begin{pmatrix} 2\times 1\, e^{2\cdot 1} + 1\cdot 2 & 3\cdot 1 + 1\cdot 1 & 1^3 + 1\cdot 1 \\ 0 & 1 & 3\cdot 1 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^2+2 & 4 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

same result. :)

Problem J: Problem 2(a) on page 63 of the text.

Problem $f: \mathbb{R}^2 \to \mathbb{R}^3$ $g: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{2x_1+x_2} \\ 3x_2 - \cos x_1 \\ x_1^2 + x_2 + 2 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} 3y_1 + 2y_2 + y_3^2 \\ y_1^2 - y_3 + 1 \end{pmatrix}$$

$F(x) = g \circ f(x)$. Find $DF(0)$

Sol $DF(0) = Dg(f(0))\, Df(0)$ by chain rule

$$Df = \begin{pmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ \sin x_1 & 3 \\ 2x_1 & 1 \end{pmatrix} \qquad Dg = \begin{pmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{pmatrix}$$

$$f(0) = \begin{pmatrix} e^0 = 1 \\ 0 - \cos 0 = -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow DF(0) = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 13 \\ 4 & 1 \end{pmatrix}$$

Problem K: Find the 3rd order Taylor series of $F(x,y) = e^{x+y^2}$ abou the origin.

$\left(F(x,y) = e^{x+y^2} \text{ is in } C^\infty \text{ so we can do this} \right)$

$$T_3(x,y) = \sum_{|\alpha|\le 3} \frac{\partial^\alpha f(0)}{\alpha!}(x,y)^\alpha \qquad \frac{\partial}{\partial y} f \quad 2ye^{x+y}$$

possible α: $|\alpha| = 0 : (0,0) \Rightarrow f(0,0) = e^0 = 1$

$|\alpha| = 1 : (0,1), (1,0) \Rightarrow \partial^{(0,1)} f(0,0) = 2ye^{x+y^2}\big|_{(0,0)} = 0$

$\partial^{(1,0)} f(0,0) = e^{x+y^2}\big|_{(0,0)} = 1$

$|\alpha| = 2 : (0,2), (2,0), (1,1)$

$\Rightarrow \partial^{(0,2)} f(0,0) = 2e^{x+y^2} + 4y^2 e^{x+y^2}\big|_{(0,0)} = 2$

$\partial^{(2,0)} f(0,0) = e^{x+y^2}\big|_{(0,0)} = 1$

$\partial^{(1,1)} f(0,0) = 2ye^{x+y^2}\big|_{(0,0)} = 0$

$|\alpha| = 3 : (1,2), (2,1), (3,0), (0,3)$

$\Rightarrow \partial^{(1,2)} f(0,0) = 2e^{x+y^2} + 4y^2 e^{x+y^2}\big|_{(0,0)} = 2$

$\partial^{(2,1)} f(0,0) = 2ye^{x+y^2}\big|_{(0,0)} = 0$

$\partial^{(3,0)} f(0,0) = e^{x+y^2}\big|_{(0,0)} = 1$

$\partial^{(0,3)} f(0,0) = 4ye^{x+y^2} + 8ye^{x+y^2} + 8y^3 e^{x+y^2}\big|_{(0,0)}$

$= 0$

Thus $T_3(x,y) = 1 + x + \frac{1}{2}x^2 + y^2 + xy^2 + \frac{x^3}{6}$

Bonus: A real symmetric n by n matrix A is called positive definite if $x^TAx > 0$ for all $x \in \mathbb{R}^n$.

(1) Show that a real symmetric matrix is positive definite if and only if it is invertible and, for all non-zero x, the angle between Ax and x is less than 90 degrees.

(2) Show that a real symmetric matrix is positive definite if and only if all its eigenvalues are positive.

(3) Let A_d be the top left d by d minor of A. Show that if A is positive definite, so is each $A_d, 1 \leq d \leq n$.

(4) Prove that A is positive definite if and only if $\det(A_d) > 0$ for all $1 \leq d \leq n$.

You may use without proof that a real symmetric matrix has an orthonormal basis of eigenvectors. A hint for the last part is available on the office door, but as always try it without first!

(1) Let $A \in \mathbb{R}^{n\times n}$ be positive define

Suppose A is not invertible $\Rightarrow \exists x \in \mathbb{R}^n$ s.t. $Ax = 0$

$\Rightarrow x^TAx = 0$, contradicts

Thus A is invertible.

let $x \in \mathbb{R}^n$, θ be the angle between Ax and x

$$\Rightarrow \cos\theta = \frac{x \cdot Ax}{\|x\| \|Ax\|} = \frac{x^TAx}{\|x\| \|Ax\|} > 0 \Rightarrow \theta \in [0, \tfrac{\pi}{2})$$

Then we prove the backward direction.

let A be invertible with $\forall x \in \mathbb{R}^n$,

angle between Ax and x $\theta \in [0, \frac{\pi}{2})$

let $x \in \mathbb{R}^n$. we have $\dfrac{x \cdot Ax}{\|x\| \|Ax\|} = \dfrac{x^TAx}{\|x\| \|Ax\|} > 0 \Rightarrow x^TAx > 0$

This proves the iff statement

(2) Suppose $A \in \mathbb{R}^{n\times n}$ is positive define

let λ be an eigenvalue of it.

$\Rightarrow$ for some $x \in \mathbb{R}^n$, $Ax = \lambda x \Rightarrow x^T A x = \lambda x^T x = \lambda \|x\|^2 > 0$

$\Rightarrow \lambda > 0$

This proves the forward direction.

For the backward direction:

Suppose all eigenvalues of A are positive.

Since A is real symmetric it has an orthonormal eigenbasis $\{b_1, \dots, b_n\}$

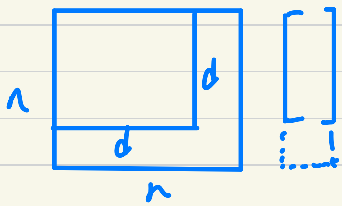
So for any $x \in \mathbb{R}^n$, $x = \sum_{i=1}^{n} c_i b_i$ for some $c_1, \dots, c_n \in \mathbb{R}$

$\Rightarrow x^T A x = \sum_{i=1}^{n} \lambda_i b_i^2 > 0$ since each $\lambda_i > 0$

This proves the proof. $\square$

(3) Suppose $A \in \mathbb{R}^{n\times n}$ is positive define.

let $1 \le d \le n$, $x \in \mathbb{R}^d$.



Take $\tilde{x} = (x, \underbrace{0, 0, \dots, 0}_{n-d}) \in \mathbb{R}^n$

By A being positive define, we have $\tilde{x}^T A \tilde{x} > 0$

Note that $\tilde{x}^T A \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_d \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \text{row}(A)_1 \cdot \tilde{x} \\ \vdots \\ \text{row}(A)_d \cdot \tilde{x} \\ \vdots \end{bmatrix} = \sum_{i=1}^{d} \tilde{x}_i (\text{row}(A)_i \cdot \tilde{x})$

$= \sum_{i=1}^{d} x_i (\text{row}(A_d)_i \cdot x) = x^T A_d x > 0$

Therefore A_d is positive define, $1 \le d \le n$ $\square$

(4) Pf

Forward direction:

Suppose A is positive define $\Rightarrow \forall 1 \le d \le n$, A_d is positive define.

By (2), each A_d has its all eigenvalues positive

So $\det(A_d) = \prod_i \lambda_{d_i} > 0$.

Backward direction:

$$a_{ij} = \text{row}_i \cdot \text{row}_j^T$$

$$\sum_{y=1}^{n} l_{iy} l_{jy}$$

a

$$a_{ij} = \sum_{x=1}^{n} l_{xi} l_{xj}$$

$\sqrt{a_{11}}$

$\underset{\sim}{r_1 \cdot r_1}$ $\quad r_1 \cdot r_2 \quad r_1 \cdot r_3$

$\underset{\sim}{r_2 \cdot r_2} \quad r_2 r_3$

$\underset{\sim}{r_3}$

$$a_{11} = l_{11}^2$$

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} \end{pmatrix}$$

$$a_{21} = l_{21} l_{11} + l_{22} l_{21}$$

$$l_{21}(l_{11} + l_{22})$$

$$a_{31} = l_{31} l_{11} + l$$