(WTS) IFT

$f: A \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be C^r, $r \geq 1$

If $\underbrace{\det Df(x_0) \neq 0}_{\text{(non-singular)}} \Rightarrow f$ is a local C^r diffeo at x_0

(invertible, and $f|_U^{-1}$ also C^r)

Last time we proved:

If f C^1 且 $Df(x_0) \neq 0 \Rightarrow f$ is locally inj.

and for some $\alpha > 0$, $\underset{\text{(open)}}{U} \ni x_0$, $|f(x) - f(y)| \geq \alpha |x - y|$ $\forall x, y \in U$

Lemma: if $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is C^1 (U open)

且 (1) f inj.

(2) $\forall x \in U$, $\det Df(x) \neq 0$

$\Rightarrow$ $f(U)$ is open

(Rmk easy if $n=1$)

这-Lemma 用处:

$Df(x_0) \neq 0 \Rightarrow \exists U \ni x_0$ st.

locally inj 且 x_0 周围 $Df(x) \neq 0$ $(\forall x \in U)$

(⇒bij, 有 f^{-1})

$\Rightarrow$ $\forall U_0 \subseteq U$ open, $f(U_0)$ 也 open

$\Downarrow$

$f^{-1}|_{f(U)}$ is ctn.

Pf Set $V = f(U)$ (inj + derivative never 0)

Consider $y \in V$

So $y = f(x)$ for some $x \in U$

WTS: $\exists \varepsilon > 0$ s.t. $B_\varepsilon(y) \subset V$

Pick $\delta > 0$ st $\overline{B_\delta(x)} \subseteq U$ (can pick $B_{2\delta}(x) \subseteq U \Rightarrow \overline{B_\delta(x)} \subseteq U$)

$\Rightarrow \partial \overline{B_\delta(x)} = \{z \in \mathbb{R}^n : |z - x| = \delta\}$ is cpt.

So $\Gamma = f(\partial \overline{B_\delta(x)})$ is cpt. by ctn.

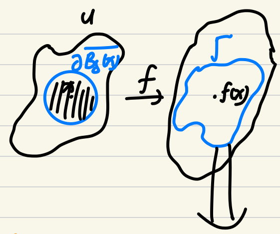
Since f is inj., $y \notin \Gamma$

Since $\mathbb{R}^n \setminus \Gamma$ is open,

(as Γ is closed by cpt.)

$\exists \varepsilon > 0$ s.t.

$B_{2\varepsilon}(y) \subseteq \mathbb{R}^n \setminus \Gamma$



Claim 1 $B_\varepsilon(y) \subseteq V$

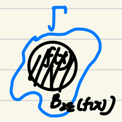
Fix $c \in B_\varepsilon(y)$ and define

$\varphi: \overline{B_\delta(x)} \to \mathbb{R}$

$\varphi(z) = |f(z) - c|^2$ (distance)

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{R} \\ & \mathbb{R}^n \arrow[u, "|a-c|^2"'] \end{tikzcd}$$

($a \mapsto |a-c|^2$)



$\overline{B_\delta(x)}$ is cpt. so φ achieve a min at some pt. $z_* \in \overline{B_\delta(x)}$

Claim 1.1 $z_* \notin \partial \overline{B_\delta(x)}$ (so $z_* \in B_\delta(x)$)

Pf If $z_* \in \partial \overline{B_\delta(x)}$ then $f(z_*) \in \Gamma$

Now $|f(z_*) - c|^2 = |f(z_*) - y + y - c|^2 \geq \Big(\underbrace{|f(z_*) - y|}_{\geq 2\varepsilon} - \underbrace{|y - c|}_{< \varepsilon}\Big)^2$

$\Rightarrow \sqrt{\varphi(z_*)} > \varepsilon$, contradicts (should $< \varepsilon$)

This proves the claim 1.1

Note: The chain rule gives that $D\varphi(z) = 2(f(z) - c)\, Df(z)$

$a, c \in \mathbb{R}^n$, $g: a \mapsto (a - c)^2$

$Dg(a) = \big(2(a_1 - c_1) \ldots 2(a_n - c_n)\big) = 2(a - c)$ ✿

Since $z_* \in B_\delta(z)$, HW gives $D\varphi(z_*) = 0$

Lemma if $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ diff'ble
若 f 在 x_0 处 extreme $\Rightarrow Df(x_0) = 0$

So $2(f(z_*) - c)\, Df(z_*) = 0$

Since $Df(z_*)$ is invertible $\Rightarrow f(z_*) - c = 0 \Rightarrow c \in V$

Since c is arbitrary $\Rightarrow B_\varepsilon(y) \subseteq V$

Since y is arbitrary $\Rightarrow V$ open

□

Rmk M square matrix

$\det M = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod M_{i\sigma(i)}$ ✿

is a continuous function of the entries, i.e. $\det: \mathbb{R}^{n \times n} \to \mathbb{R}$

So $\det Df(x_0) \neq 0$ implies that the det of $Df(x)$ near x_0 ($x \in B_\varepsilon(x_0)$) are all $\neq 0$ if $f \in C^1$

Pf of IFT

Now we come to pf. of IFT:

Pf Pick open $U \subseteq A$ small enough s.t. $x_0 \in U$ and $\forall x \in U$, $Df(x) \neq 0$; 且 $f|_U$ is inj (by Lemma 1)

By today's Lemma, we have $V = f(U)$ is open

and the map $f: U \to V$ sends open sets to open sets

Thus the inverse map (it exists from (1))

$g: V \to U$ is ctn. $\Rightarrow f|_U$ is homeo

现在已证明: $g = f|_U^{-1}$ 且 f, g 为 homeomorphism

还差: g 是 C^r 的 (即 $f|_U$ 为一个 C^r-diffeomorphism)

Claim 1 g is diff'ble

let $y = f(x) \in V$, $E = Df(x)$ → for k small enough, $y + k \in V$ since V open.

WTS: $\lim_{k \to 0} \dfrac{g(y+k) - g(y) - E^{-1}(k)}{|k|} = 0$

(which shows $\boxed{Dg(y) = E^{-1} = (Df(x))^{-1}}$)

Pf of claim 1

set $h = g(y+k) - x = g(y+k) - g(y) \Rightarrow h \to 0$ as $k \to 0$ by g ctn.

Note $f(x+h) = f(g(y+k)) = y + k$ (✿)

Since f is diff'ble at x,

$f(x+h) - f(x) - Eh = r(h)$ s.t. $\dfrac{r(h)}{h} \to 0$ as $h \to 0$

So $k - Eh = r(h)$ by (✿)

So $E^{-1}k - h = E^{-1}r(h)$

i.e. $E^{-1}k - (g(y+k) - g(y)) = E^{-1}r(h)$

$\Rightarrow \dfrac{g(y+k) - g(y) - E^{-1}k}{|k|} = \dfrac{-E^{-1}r(h)}{|k|}$

$= \dfrac{-E^{-1}r(h)}{|h|} \cdot \dfrac{|h|}{|k|}$

By Lemma 1, $|k| = |f(x+h) - f(x)| \geq \alpha |h|$, so $\dfrac{|h|}{|k|}$ bounded by $\dfrac{1}{\alpha}$

So $\lim_{k\to 0} \dfrac{g(y+k)-g(y)-E^{-1}k}{|k|} = 0$

This proves the claim 1.

Claim 2 g is C^r (最后一步)

already shown: $\forall y \in V$, $Dg(y) = [Df(x)]^{-1}$, where $y = f(x)$, $x = g(y)$

$\Rightarrow$ 即 $Dg(y) = Df^{-1}(g(y))$

因而 operator $Dg = Df^{-1} \circ g$

By Cramer's Rule, $[Df]^{-1}$ 为一个 rational function of $\frac{\partial f_i}{\partial x_j}$

$M^{-1} = \frac{1}{\det M}[\mathrm{Adj}\, M]$, $[\mathrm{Adj}\, M]_{ij} = \det(M_{ij})$

因而 $f \in C^r \Rightarrow Df \in C^{r-1} \Rightarrow [Df^{-1}] \in C^{r-1}$

And Since $g \in C^0$ $\Rightarrow Dg \in C^0$

$\Downarrow$

$g \in C^1$

$\downarrow$

recursively together with $Df \in C^{r-1}$, we get $g \in C^r$

$\square$

Amazing.

(FYI) Mean Value Thm 推广

if $H: \mathbb{R}^n \to \mathbb{R}$ diff'ble

$\forall x, y \in \mathbb{R}^n$, $\exists c$ on the line segment between x, y

s.t. $H(y) - H(x) = DH(c)\cdot(y-x)$

Pf Take $\varphi: [0,1] \to \mathbb{R}$

$t \mapsto H(x + t(y-x))$

By MVT, $\exists c_0 \in [0,1]$ s.t.

$\varphi(1) - \varphi(0) = \varphi'(t)\cdot(1-0)$

i.e. $H(y) - H(x) = \varphi'(t)$

By chain rule, $\varphi'(t) = DH(c)\cdot(y-x)$

(Actually just Taylor's Thm for degree $k=0$)