Fubini's Thm

Let $Q = A\times B$, $A\subseteq \mathbb{R}^k$, $B\subseteq \mathbb{R}^l$ are boxes

Let $f: Q\to\mathbb{R}$ bdd

If f is intble over Q, then

$$x \mapsto \underline{\int}_B f(x,y)\,dy \quad \text{and}$$

$$x \mapsto \overline{\int}_B f(x,y)\,dy$$

are intble over A and $\int_Q f = \int_A \underline{\int}_B f(x,y)\,dy\,dx$

$$= \int_A \overline{\int}_B f(x,y)\,dy\,dx$$

Pf Let $\overline{I}(x) = \overline{\int}_B f(x,y)\,dy$ and $\underline{I}(x) = \underline{\int}_B f(x,y)\,dy$

Let P be a partition of Q

Write $P = (P_A, P_B)$ where P_A & P_B are partitions of A & B respectively

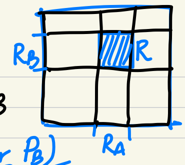
Let R be a subbox of P

So $\exists$ subbox R_A of A, R_B of B s.t. $R_A\times R_B = R$

Fix $x_0\in R_A$ we have $\overset{\inf_{x,y\in R} f(x,y)}{m_R(f)} \le \inf_{y\in R_B} f(x_0,y) = m_{R_B} f(x_0,\cdot)$

Multiplying by $v(R_B)$

& summing over all possible R_B



(have R_A fixed, R_B ranges over P_B)

$$\sum_{R_B\in P_B} m_{R_A\times R_B}(f)\,v(R_B) \le \sum_{R_B\in P_B} m_{R_B}(f(x_0,\cdot))\,v(R_B)$$

$$= L(f(x_0,\cdot), P_B) \le \underline{I}(x)$$

Taking the inf over $x\in R_A$,

$$\sum_{R_B} m_{R_A\times R_B}(f)\,v(R_B) \le m_{R_A}(\underline{I})$$

Multiplying by $v(R_A)$ & summing over all R_A

$$\sum_{R_A,R_B} m_{R_A\times R_B}(f)\,v(R_A)\,v(R_B) \le \sum_{R_A} m_{R_A}(\underline{I})\,v(R_A)$$

$$\|$$

$$L(f,P) \le L(\underline{I}, P_A) \quad ①$$

An exactly similar argument gives

$$U(f,P) \ge U(\overline{I}, P_A) \quad ②$$

Now:

$$L(f,P) \overset{①}{\le} L(\underline{I},P_A) \overset{L\le U}{\le} U(\underline{I},P_A) \overset{\underline{I}\le\overline{I}}{\le} U(\overline{I},P_A) \overset{②}{\le} U(f,P)$$

Similarly we have

$$L(f,P) \le L(\underline{I},P_A) \le L(\overline{I},P_A) \le U(\overline{I},P_A) \le U(f,P)$$

This proves that $\forall P$, the upper & lower P_A Riem sums for $\underline{I}$ & $\overline{I}$ are in $[L(f,P), U(f,P)]$

So $\forall \varepsilon > 0$,

since f intble, $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Then $U(\underline{I},P_A) - L(\underline{I},P_A) < \varepsilon$

$U(\overline{I},P_A) - L(\overline{I},P_A) < \varepsilon$

So $\left|\int_Q f - U(\overline{I},P_A)\right| < \varepsilon$

$(U(\underline{I},P_A),\ L(\overline{I},P_A),\ L(\underline{I},P_A))$

Therefore $\int_A \underline{I} = \int_A \overline{I} = \int_Q f$

□

Integral over bounded sets.

Def Let $S\subseteq\mathbb{R}^n$ be bdd. and $f: S\to\mathbb{R}$ be bdd

Define $f_S(x) = \begin{cases} f(x), & \text{if } x\in S \\ 0, & \text{else}\end{cases}$

Choose a box $Q\supseteq S$

Define: $\int_S f(x)\,dx = \int_Q f_S(x)\,dx$ provided that last integral exists.

This is well-defined thanks to:

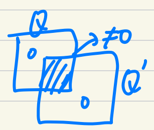
f supported on A means $f(x)\neq 0$ iff $x\in A$

Lemma Let Q, Q' be two (closed) boxes in $\mathbb{R}^n$

Let $f:\mathbb{R}^n\to\mathbb{R}$ be supported on $Q\cap Q'$

Then f intble over $Q \iff$ intble over Q' and

$$\int_Q f = \int_{Q'} f$$



Pf It suffices to assume $Q\subseteq Q'$

(if not, find $Q''\supseteq Q, Q'$. ~~P~~ we proved it for set containing another set, we can note that $\int_Q f = \int_{Q''} f$) $= \int_{Q'} f = \int_{Q''} f$

Fst assume $Q \subseteq Q'$

Suppose f is intble over Q

Write $Q = I_1 \times I_2 \times \dots \times I_n$

$Q' = I_1' \times I_2' \times \dots \times I_n' \Rightarrow I_k \subseteq I_k'$

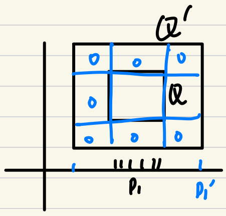
Given $\varepsilon > 0$, pick a partition P of Q with

$U(f,P) - L(f,P) < \varepsilon$

If $P = (P_1, \dots, P_n)$

Define $P' = (P_1', \dots, P_n')$, a partition of Q'

by letting P_i' be P_i with the endpts of I_i' added in



Note $U(f,P) = U(f,P')$

& $L(f,P) = L(f,P')$

Since $f=0$ on new subboxes $\Rightarrow f$ intble on Q'

& $\int_{Q'} f = \int_Q f$

Conversely

Say f is intble over Q'

Let P' be a partition of Q'

Let $\widetilde{P'}$ be obtained by setting $\widetilde{P_i'}$ to be P_i' plus the end pts of I

Let P be the associated partition of Q

Note $L(f,P') \le L(f,\widetilde{P'}) = L(f,P) \le U(f,P)$

$= U(f,\widetilde{P'}) \le U(f,P')$

$\square$

Thm Properties of integrals

Let $S \subseteq \mathbb{R}^n$ be bdd.

$f, g : S \to \mathbb{R}$ intble

$\Rightarrow$ (a) Linearity $\forall c \in \mathbb{R}$, $f + cg$ is intble

and $\int_S (f + cg) = \int_S f + c \int_S g$

(b) $M(x) = \max(f(x), g(x))$ &

$m(x) = \min(f(x), g(x))$ are intble

(c) if $f(x) \le g(x)$ $\forall x \in S$

$\Rightarrow \int_S f \le \int_S g$

(d) $|f|$ is intble & $|\int_S f| \le \int_S |f|$

(e) Monotonicity Let $T \subseteq S$

If nonnegtive f is intble on T & S

$\Rightarrow \int_T f \le \int_S f$

(f) Additivity If f is intble on S_1 & S_2

$\Rightarrow$ it is intble on $S_1 \cup S_2$ and $S_1 \cap S_2$

and $\int_{S_1 \cup S_2} f = \int_{S_1} f + \int_{S_2} f - \int_{S_1 \cap S_2} f$

(g) Let $S_1, \dots, S_n$ be bdd. subsets of $\mathbb{R}^n$

Assume $m(S_i \cap S_j) = 0$, $\forall i \ne j$

$\Rightarrow f$ is intble over $\bigcup_n S_n$ and

$$\int_{\bigcup_n S_n} f = \sum_n \int_{S_n} f$$

Pf Some hw some exercise

Rmk. f is intble on S $\Leftrightarrow m(D_{f_S}) = 0$