Thm open $A, B \subseteq \mathbb{R}^n$, $n \geq 2$

$g: A \to B$ diffeo

$\Rightarrow \forall a \in A, \exists$ nbh $U_0 \ni a$ and seq. of primitive diffeos

$$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} \cdots \xrightarrow{h_k} U_k \subseteq B$$

s.t. $h_k \circ \cdots \circ h_1 = g|_{U_0}$

(can locally factor a diffeo into primitive diffeos)

Pf Claim 1 Any non-singular linear map can be factored into primitives

let $T: \mathbb{R}^n \to \mathbb{R}^n$ linear

$T(x) = Cx$ for nonsingular $C \in M_{n\times n}(\mathbb{R})$

C factors as a product of elementary matrices

(1) swap 2 coords

(2) replace some i^{th} coord with i^{th} coord $+ kj^{th}$ coord for some k, j

(3) multiply some coord with some $k \in \mathbb{R}$

note (2) and (3) are primitive (actually stronger than primitive, with multiple coords)

(1) is primitive when $n \geq 3$

So need to prove (1) is product of primitives when $n=2$

Pf algorithm

coord 1	coord 2
a	b
$a-b$	b
$a-b$	$b+(a-b)=a$
$b-a$	a
$b-a+a$ $=b$	a

$\square$

Claim 2 translation can be factorized into primitives

$t: \mathbb{R}^n \to \mathbb{R}^n$

$x \mapsto x + c$

$\Rightarrow t = t_2 \circ t_1$

$t_1(x) = x + (c_1, \dots, c_{n-1}, 0)$

$t_2(x) = x + (0, \dots, 0, c_n)$

Claim 3 when $a=0$, $g(a)=0$, $Dg(a)=Id$, g can be factorized into primitives near a

Define $h: A \to \mathbb{R}^n$ by

$h(x) = (g_1(x)\ g_2(x) \dots g_{n-1}(x), x_n)$

$$\Rightarrow Dh(x) = \begin{bmatrix} \frac{\partial(g_1, \dots, g_{n-1})}{\partial x} \\ \hline 0\ 0\ \cdots\ 0\ 1 \end{bmatrix}$$

So $Dh(0) = Id$

By IFT, $\exists$ some $U_0 \ni 0$ s.t. $h|_{V_0}: U_0 \to V_1$ is a diffeo

$(g(0) \in V_1 \subseteq B$ open in $\mathbb{R}^n)$

Define $k: V_1 \to \mathbb{R}$

$y \mapsto (y_1, \dots, y_{n-1}, g_n(h^{-1}(y)))$

So $k(0) = 0$

and $Dk(y) = \begin{bmatrix} I_{n-1} & 0 \\ \hline \multicolumn{2}{c}{D(g_n \circ h^{-1})} \end{bmatrix}$

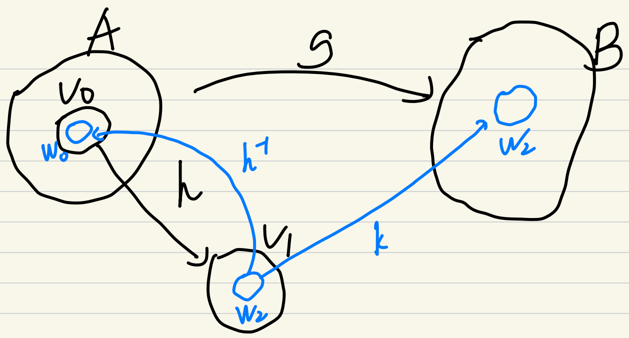
The chain rule gives:

$D(g_n \circ h^{-1})(0) = Dg_n(0)\, Dh^{-1}(0)$

$= Dg_n(0)(Dh(0))^{-1}$

$Dk(0) = I_n$ $\quad = [0 \dots 0\ 1]\, Id_n = [0 \dots 0\ 1]$

$\Rightarrow$ By IFT, k is a diffeo from some $W_1 \ni g(0) = 0$ s.t. $W_1 \subseteq V_1$



So let $W_0 = h^{-1}(W_1)$

$\Rightarrow W_0 \xrightarrow{h|_{W_0}} W_1 \xrightarrow{k|_{W_1}} W_2$ are primitive, diffeos

And notice: $k|_{W_1} \circ h|_{W_0} = g|_{W_0}$

Since $\forall x \in W_0 \Rightarrow h(x) = (g_1(x) \dots g_{n-1}(x)\ x_n)$

$\Rightarrow k \circ h(x) = (g_1(x) \dots g_{n-1}(x), g_n(h^{-1}(x_n)))$

$= g(x)$ $\square$

Step 4 The general case.

Given $a \in A$.

Define $t_1, t_2, T: \mathbb{R}^n \to \mathbb{R}$

$t_1: x \mapsto x + a$

$t_2: x \mapsto x - g(a)$

$T: x \mapsto (Dg(a))^{-1} x$

Let $\tilde{g} = T \circ t_2 \circ g \circ t_1$

Note: $\tilde{g}(0) = T \circ t_2 \circ g(a)$

$= T(0) = 0$

& $D\tilde{g}(0) = M^{-1} I M I I = I$

By step 3, $\tilde{g}$ can be factored into two primitive diffeos

Also, $T^{-1}, t_1^{-1}, t_2^{-1}$ can be factored into primitives diffeos

So $g = t_2^{-1} \circ T^{-1} \circ \tilde{g} \circ t_1^{-1}$ can be factored into primitive diffeos

$\square$

Partitions of unity

Lemma Let Q be a box in $\mathbb{R}^n$

$\Rightarrow \exists C^\infty$ $\varphi: \mathbb{R}^n \to \mathbb{R}$

s.t. $\varphi(x) > 0$ for all $x \in Q^\circ$

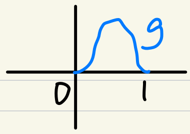
$\varphi(x) = 0$ for all $x \in \mathbb{R}^n \setminus Q^\circ$

Pf note $f: \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & \text{else} \end{cases}$$

it is C^∞

Define $g(x) = f(x) \cdot f(1-x)$



If $Q = [a_1, b_1] \times \dots \times [a_n, b_n]$

Define $\varphi(x) = g\left(\frac{x_1 - a_1}{b_1 - a_1}\right) \times \dots \times g\left(\frac{x_n - a_n}{b_n - a_n}\right)$

$\square$

Lemma Let $\mathcal{A}$ be a collection of open sets in $\mathbb{R}^n$

Let A be be union of all elts of $\mathcal{A}$

$\Rightarrow \exists$ a seq. of closed boxes (cubes) $Q_1, Q_2, \dots$ s.t.

(1) $\bigcup_{i \in \mathbb{N}} Q_i \supseteq A$

(2) $\forall Q_i$, $Q_i \subseteq A_k$ for some $A_k \in \mathcal{A}$

(3) (Local finiteness)

$\forall a \in A$, $\exists$ nbh $U \ni a$ s.t.

U only intersects finitely many Q_i

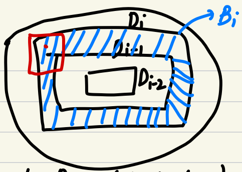
Pf First take $\{D_i\}$ be the seq of cpt sets

s.t. $\bigcup_i D_i = A$ and $D_i \subseteq D_{i+1}^\circ$ $\forall i$

Let $D_0 = \emptyset$, $B_1 = D_1$

$B_i = D_i - D_{i-1}^\circ$ $\forall i \geq 2$

B_i is closed & bdd thus cpt.



Note that $B_i \cap D_{i-2} = \emptyset$!

And can find $\delta > 0$ s.t. the δ-nbh of B_i does not intersect D_{i-2} (last lec)

Cover B_i by cubes:

$\forall x \in B_i$ can choosed closed cube C_x centered at x s.t.

① $C_x \subseteq A_k$ for some $A_k \in \mathcal{A}$

② C_x disjoint from D_{i-2}

Then $\bigcup_{x \in B_i} C_x^\circ \supseteq B_i \Rightarrow \exists$ finite subcover

Let C_i be the collection of cubes finitely covering B_i

Claim $A = \bigcup_{n=1}^{\infty} C_n$

Can easily show that $A \subseteq \bigcup_{n=1}^{\infty} C_n$ since

$\forall D_i$ is covered by $\bigcup_{n=1}^{i} C_n$

and $\bigcup_{n=1}^{\infty} C_n \subseteq A$ since $\forall C_n, \forall Q_x \in C_n, Q_x \subseteq A_k$ for some $A_k \in \mathcal{A}$

It remains to show local finiteness

Let $x \in A \Rightarrow x \in D_i^\circ$ for some D_i

And D_i° can only intersect cubes in $C_1, \dots, C_{i+1}$

thus finite $\square$