Today's topic:

Preparations for the proof of the change of variable thm

Lemma Let $A \subseteq \mathbb{R}^n$ open

$g: A \to \mathbb{R}^n \quad \in C^1(A)$

if $E \subseteq A$ has $m(E) = 0$

$\Rightarrow m(g(E)) = 0$

Pf Step 1

Claim $\forall \varepsilon, \delta > 0$ and $\forall S \subseteq \mathbb{R}^n$ s.t. $m(S) = 0$

$\Rightarrow \exists \{B_n\}_{n=1}^{\infty}$ of cubes s.t. each width $\leq \delta$ and $\sum_n |B_n| \leq \varepsilon$

equal width box s.t. $S \subseteq \bigcup_n B_n$

(Pf of step 1: idea is that each box can be covered with finitely many tiny cubes of width $\leq \delta$, with little loss)

Here we use notation

(sup norm) $|x = (x_1, \dots, x_n)| = \max_i |x_i|$

$\|x = (x_1, \dots, x_n)\| = \sqrt{\sum_i x_i^2}$

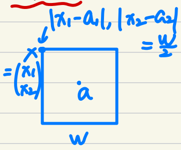
(sup norm) $|A| = \max_{i,j} |A_{ij}|$

$\|A\| =$ operator norm of $A = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| = 1}} |Ax|$

Step 2 Let C be a closed cube in A and suppose $\|Dg(x)\| \leq M \ \forall x \in C$

Claim if C has width w, then $g(C)$ is contained in a cube of width nMw

Pf of Step 2: Let a be the center of C



$\Rightarrow C = \{x : |x - a| \leq \frac{w}{2}\}$
sup norm

MVT $\Rightarrow \forall x \in C \ \forall j \ \exists c_j \in C$ s.t.

$g_j(x) - g_j(a) = Dg_j(c_j)(x - a)$

So $|g_j(x) - g_j(a)| \leq nM\frac{w}{2}$

So $g_j(x) \in [g_j(a) - \frac{nMw}{2}, g_j(a) + \frac{nMw}{2}]$

$\Rightarrow g(C) \subseteq \prod_{i=1}^{n} [g_i(a) - \frac{nMw}{2}, g_i(a) - \frac{nMw}{2}]$

Step 3 Remainder of the proof

Let $(C_i)_{i=1}^{\infty}$ be a seq. of cpt sets with

$\bigcup_{i=1}^{\infty} C_i = A$ and $C_i \subseteq C_{i+1}^{\circ} \ \forall i$

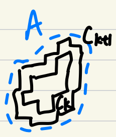
Let $E_k = C_k \cap E \Rightarrow \bigcup_k E_k = E$ note: union of ctbly measure 0 sets has measure 0

It suffices to show $m(g(E_k)) = 0$

(Since union of ctbl measure 0 sets has measure 0)

exercise Show that for fixed k,

$\exists \delta > 0$ st. the δ-nbh of C_k in the sup metric is in C_{k+1}°



i.e. $\{x \mid \exists x' \in C_k \text{ st } d_{sup}(x, x') < \delta\} \subseteq C_{k+1}^{\circ}$

Pf (exercise)

Then chose M s.t. $|Dg(x)| \leq M$ on C_{k+1}

Using step 1, cover E_k by ctbly many cubes $D_1, D_2, \dots$

each of width $< \delta$ with total volume $< \frac{\varepsilon}{(nM)^n}$

WLOG each D_i intersects $E_k \subseteq C_k$, so contained in C_{k+1}

By step 2, $g(D_i)$ is contained in some cube

D_i' with width$(D_i') = nM \cdot$ width(D_i)

so $v(D_i') = (nM)^n v(D_i)$

So $m(g(E_k)) \leq \sum v(D_i')$

$= (nM)^n \sum v(D_i)$

$< (nM)^n \frac{\varepsilon}{(nM)^n} = \varepsilon$

So $m(g(E_k)) = 0$

$\square$

Corollary If $A \subseteq \mathbb{R}^n$ open

$f: A \to \mathbb{R}^m \in C^1(A)$ with $m > n$

$\Rightarrow m(f(A)) = 0$

Pf Define $g: \underbrace{A \times \mathbb{R}^{m-n}}_{\subseteq \mathbb{R}^m} \to \mathbb{R}^m$

$g(a, b) = f(a)$ (ignoring b)

$\Rightarrow f(A) = g(A \times \{0\})$

$\Rightarrow m(f(A)) = m(g(\underbrace{A \times \{0\}}_{\text{measure } 0})) = 0$

Rmk 1. not true if f is C^0 or not even C^0

counterex space-filling curve

2. not true for $m \leq n$

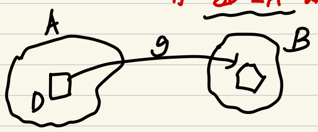
ex $f: \mathbb{R}^2 \to \mathbb{R}$

$(x, y) \to x$

the x-axis has measure 0 in $\mathbb{R}^2$ but is mapped to whole $\mathbb{R}$

Thm Let $A, B \subseteq \mathbb{R}^n$ open
$g: A \to B$ be a diffeomorphism
Let $D \subseteq A$ be cpt.
$\Rightarrow$ (a) $g(D^\circ) = (g(D))^\circ$
$g(\partial D) = \partial(g(D))$
(b) if D is J-meas $\Rightarrow$ $g(D)$ is J-meas

These result also hold when D is not cpt.
if $\partial D \subseteq A$ and $\partial(g(D)) \subseteq B$



Pf of (a) By diffeo, $g^{-1}: B \to A$ is ctn. (actually C^1)
so $g(D^\circ) = (g^{-1})^{-1}(D^\circ)$
is open & contained in $g(D)$
$\Rightarrow$ $g(D^\circ) \subseteq (g(D))^\circ$
Similarly, $g((A-D)^\circ) \subseteq (B - g(D))^\circ$
it follows that $\partial(g(D)) \subseteq g(\partial D)$
By symmetry (g^{-1} instead of g), we get the opposite containments. $\square$

Pf of (b) D is J-meas $\Rightarrow$ $m(\partial D) = 0$
By the previous thm, $g(\partial D) = \partial(g(D))$ has measure 0. $\square$

Rmk boundary of J-meas set has L-measure 0
But boundary of L-meas set does not.

ex. $\mathbb{Q}$ is L-meas but not J-meas
$m(\mathbb{Q}) = 0$
but $\partial \mathbb{Q} = \mathbb{R}$
$m(\partial \mathbb{Q}) \neq 0$

Def we say $h: A \to B$ preserves the i^{th} coord
if $h_i(x) = x_i$
and then call h primitive if it preserves any coord

Thm Let $A, B \subseteq \mathbb{R}^n$ open, $n \geq 2$
$g: A \to B$ be a diffeo
$\Rightarrow \forall a \in A$, $\exists$ open $U_0 \ni a$ s.t. $U_0 \subseteq A$
and $\exists$ a finite seq. of primitive diffeos $h_1, \dots, h_k$
s.t. $U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} \dots \xrightarrow{h_k} U_k \subseteq B$
and $g|_{U_0} = h_k \circ \dots \circ h_1$ ☆
(note: not true for $n=1$. the only primitive map is identity)

Pf next time