Def support set of a function

$\varphi: A \to \mathbb{R}$, A is a topological space

$\mathrm{supp}(\varphi) = \overline{\{x \in A \mid \varphi(x) \neq 0\}}$ note: taking closure

Def Partition of unity (general)

Let A be a topological space

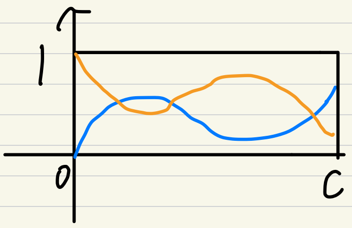
a partition of unity on A is a collection of

(1) ctn nonnegative function $\{\varphi_\alpha : A \to [0,1]\}_\alpha$ s.t.

(2) support contained: $\forall \alpha$, $\mathrm{supp}(\varphi_\alpha) \subseteq A$

(3) local finiteness: $\forall a \in A$, $\exists$ nbh of a that intersects only finitely many $\mathrm{supp}(\varphi_\alpha)$

(4) $\boxed{\sum_\alpha \varphi_\alpha = 1}$ (const function 1 from A to $\mathbb{R}$)



a simple POU on $[0,C]$ $\{\varphi_1, \varphi_2\}$

Thm Partition of unity Thm on $\mathbb{R}^n$

Let $\mathcal{A}$ be a collection of open sets in $\mathbb{R}^n$

Let $A = \bigcup_{U \in \mathcal{A}} U$ be its union

$\Rightarrow \exists$ a ctbl partition of unity on A s.t.

(5) each φ_i is C^∞

(6) each $\mathrm{supp}(\varphi_i)$ is cpt (say: the POU has cpt supp)
(and J-meas, actually closed cubes)

(7) each $\mathrm{supp}(\varphi_i)$ is contained in some element of $\mathcal{A}$
(say: the POU is dominated by $\mathcal{A}$)

这个 Thm 即：$\mathbb{R}^n$ 中任意的 collection of open sets $\mathcal{A}$ 上都存在一个 C^∞ partion of unity with cpt supp, dominated by $\mathcal{A}$.

Pf easy.

$A = \bigcup_{n=1}^{\infty} C_n$, C_n cpt with $C_n \subseteq C_{n+1}^\circ$

Partition A with a seq of locally finite closed cubes $\{S_i\}_{i=1}^\infty$ with $A = \bigcup_i S_i^\circ$ and each $S_i \subseteq U_\alpha$ for some $U_\alpha \in \mathcal{A}$

This is done by cube partition Lemma,

And for each $i \in \mathbb{N}$, let $\psi_i : \mathbb{R}^n \to \mathbb{R}$ be C^∞ with $\psi_i(x) > 0$ iff $x \in S_i^\circ$ ($\mathrm{supp}(\psi_i) = S_i$)

This is done by chad bump function Lemma

Here we already done everything except (4)

And by local finiteness of the partition,

$\forall x \in A$, only finite ψ_i are positive at x

Define $\lambda = \sum_{i=1}^{\infty} \psi_i \Rightarrow \lambda$ is bdded ensured by

And λ is C^∞

So define

$$\varphi_i(x) = \begin{cases} \dfrac{\psi_i(x)}{\lambda(x)}, & x \in A \\ 0, & x \in A \end{cases}$$

for each $i \in \mathbb{N}$

$\Rightarrow \sum_{i=1}^{\infty} \varphi_i = 1$

This finishes the construction of the POU

Integration by POU

POU 的有用之处在哪里

Lemma Let f : open $A \subseteq \mathbb{R}^n \to \mathbb{R}$ be ctn

if f vanishes outside cpt $C \subseteq A$

$\Rightarrow \int_A f$ exists

并且 $\int_A f = \int_C f$ ord.

rmk 我们说的积分都指代 extended 积分
如果 ord intble 那么一定一样

Pf C bdd $\Rightarrow \int_C f$ exists

Let $A = \bigcup_{n=1}^{\infty} C_n$, $C_n \subseteq C_{n+1}^\circ$

Let $C_M \supseteq C$. Since f vanishes outside C

$\Rightarrow \int_C f = \int_{C_M} f$

且 $\forall n \geq M$, $\int_{C_n} f = \int_{C_M} f$ (trivial)

So $\int_A f = \lim_{n \to \infty} \int_{C_n} f = \int_{C_M} f = \int_C f$ $\square$

Lemma 给我们的帮助:

对于 A上的 POU $\{\varphi_i\}$ with cpt supp (cube)

其中任意 φ_i, $\int_A \varphi_i = \int_{\mathrm{supp}(\varphi_i)} \varphi_i$ → a cube

把它变为了最简单的 box 上的积分

并且下面Thm:

Thm integration via POU

Let f: open $A \subseteq \mathbb{R}^n \to \mathbb{R}$ ctn

Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a POU on A with cpt supp

$\Rightarrow \int_A f$ exists iff $\boxed{\sum_{i=1}^{\infty} \int_A \varphi_i f = \sum_{i=1}^{\infty} \int_{\mathrm{supp}(\varphi_i)} \varphi_i f}$ converges

In this case, $\boxed{\int_A f = \sum_{i=1}^{\infty} \int_{\mathrm{supp}(\varphi_i)} \varphi_i f}$

Pf Case 1 $f \geq 0$

(1) Suppose $\sum \int_A \varphi_i f$ converges

Let $D \in \mathcal{J}_c$ and $D \subseteq A$

By local finiteness $\exists M$ s.t. $\forall i > M$, φ_i vanishes on D

So $f|_D = \sum_{i=1}^{M} \varphi_i|_D f|_D$

$\Rightarrow \int_D f = \sum_{i=1}^{M} \int_D \varphi_i f$

$\leq \sum_{i=1}^{M} \int_{D \cup \mathrm{supp}(\varphi_i)} \varphi_i f$ by monotonicity

$= \sum_{i=1}^{M} \int_A \varphi_i f \leq \sum_{i=1}^{\infty} \int_A \varphi_i f$

So $\sum_{i=1}^{\infty} \int_A \varphi_i f$ conv $\Rightarrow \int_D f \ \exists$ for any $D \in \mathcal{J}_c$ in A

$\Rightarrow \int_A f \ \exists$

& $\int_A f \leq \sum_{i=1}^{\infty} \int_A \varphi_i f$

(2) Suppose $\int_A f \ \exists$

$\forall N$, $\sum_{i=1}^{N} \int_A \varphi_i f = \int_A \sum_{i=1}^{N} \varphi_i f$ by linearity

$\leq \int_A f$ since $\sum_{i=1}^{N} \varphi_i f \leq f$

$\Rightarrow \sum_{i=1}^{\infty} \int_A \varphi_i f \leq \int_A f$ so it conv.

By (1)(2) $\Rightarrow \sum_{i=1}^{\infty} \int_A \varphi_i f$ conv iff $\int_A f \ \exists$

and they equals.

Case 2 General f

$f = f_+ - f_-$

$\int_A f = \int_A f_+ - \int_A f_-$

$= \sum_{i=1}^{\infty} \int_A \varphi_i f_+ - \sum_{i=1}^{\infty} \int_A \varphi_i f_-$

$= \sum_{i=1}^{\infty} \int_A \varphi_i f$

□