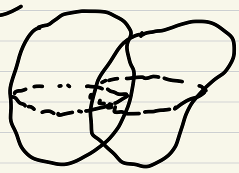
ex of implicit function Thm

Let $f, g: \mathbb{R}^3 \to \mathbb{R}$ be C^1

Typically we expect $f=0$ & $g=0$ to each define a surface and their intersection define a curve.



Say $(x_0, y_0, z_0) \in \{f=0\} \cap \{g=0\}$

Set $F(x,y,z) = (f(x,y,z), g(x,y,z))$

$F: \mathbb{R}^3 \to \mathbb{R}^2$

Note $\{f=0\} \cap \{g=0\} = F^{-1}(0)$

If $DF(x): \mathbb{R}^3 \to \mathbb{R}^2$ has rank 2

So WLOG $\frac{\partial F}{\partial(y,z)}$ is non-singular

Then the Implicit Function Thm says:

we can solve y, z in terms of x.

. ex $y = \varphi(x)$, $z = \psi(x)$

note $\{(x, \varphi(x), \psi(x))\}$ is a curve

Rmk (Most 2×3 matrices have rank 2.)

Riemann Integration

Def Box

A box B in $\mathbb{R}^n$ is the product of n intervals.

$B = I_1 \times \dots \times I_n$

(any interval can be open/close, not much difference, we will use closed intervals)

$B_n = [a_1, b_1] \times \dots \times [a_n, b_n]$

define $v(B) = \prod (b_i - a_i)$

Def partition (on $\mathbb{R}$)

Given an interval $I = [a,b]$, a partition of I is a finite collections of pts

$a = x_0 < x_1 < \dots < x_k = b$

So each $[x_{i-1}, x_i]$ has length $\Delta x_i = x_i - x_{i-1}$

Define the mesh (or say norm)

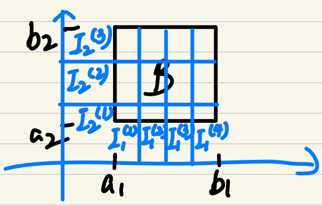
$\|P\| = \max_i \Delta x_i$

Def Partition ($\mathbb{R}^n$)

A partition of box $B \subseteq \mathbb{R}^n$ is an n-tuple

$P = (P_1, \dots, P_n)$

where P_j is a partition of I_j



each P_j divides I_j into subintervals.

$I_j^{(1)}, \dots, I_j^{(k_j)}$ with disjoint interiors

This gives a decomposition of B into subboxes of the form $J_1 \times \dots \times J_n$

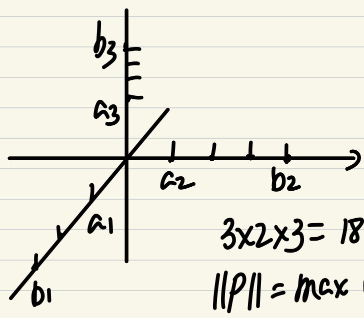
where each $J_i \in \{I_i^{(1)}, \dots, I_i^{(k_i)}\}$

The mesh (norm) of $P = (P_1, \dots, P_n)$ is

$$\boxed{\|P\| = \max_{1 \le j \le n} \|P_j\|}$$

(= the max width in any dimension of a subbox)

ex



$3\times2\times3 = 18$ subintervals

$\|P\| = \max$ of $3+2+3=8$ #number

Def Let B be a box

P be a partition of B

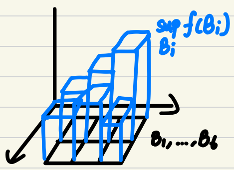
$f: B \to \mathbb{R}$ be bdd.

Let the subboxes by P be

$B_1, \dots, B_N$ (s.t. $\bigcup_{i}^{N} B_i = B$)

Let $m_{B_j}(f) = \inf_{B_j} f$

$M_{B_j}(f) = \sup_{B_j} f$



We define lower & upper sums of f by P as

$$L(f, P) = \sum_{i=1}^{N} m_{B_i} v(B_i)$$

$$U(f, P) = \sum_{i=1}^{N} M_{B_i} v(B_i)$$

Def refinement

Let B be a box & $P = (P_1, \dots, P_n)$, $Q = (Q_1, \dots, Q_n)$ be two partitions of B

We say that Q is a refinement of P if

$$P_j \subseteq Q_j \quad \forall j$$

Given partitions P and P', their common refinement is $Q = (P_1 \cup P_1', \dots, P_n \cup P_n')$

Lemma Let B be a box

$f: B \to \mathbb{R}$ bdd

Let P be a partition of B

& Q be a refinement of P

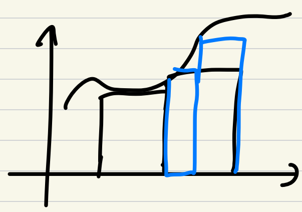
$\Rightarrow$ Then we have:

$$L(f, P) \le L(f, Q)$$

and

$$U(f, P) \geq U(f, Q)$$

ex



Pf (exercise $U(f, P) = -L(-f, P)$)

By exercise, enough to prove the result for lower sums.

Since Q can be obtained from P by successively adding a pt. to one of the partitions.

we can assume Q is obtained by adding a pt. to P_k.

WLOG $k = 1$

Suppose $B = [a_1, b_1] \times \dots \times [a_n, b_n]$

and P_1 : $a_1 = x_0 < x_1 < \dots < x_k = b_1$

and Q is obtained by adding a pt. to (x_{p-1}, x_p)

The subboxes of P are of the form

$$[x_{i-1}, x_i] \times \underset{\underset{[a_2, b_2] \times \dots \times [a_n, b_n]}{\uparrow}}{S}$$

determined by $(P_2, \dots, P_n)$

So the subboxes for Q are either of form

$$[x_{i-1}, x_i] \times S$$

or of the form:

$$[x_{p-1}, q] \times S$$

or $[q, x_p] \times S$

$$(*) \; L(Q, f) - L(P, f) = \sum_{S \in S} \Big[m_{[x_{p-1}, q] \times S} v([x_{p-1}, q] \times S) + m_{[q, x_p] \times S} v([q, x_p] \times S) - m_{[x_{p-1}, x_p] \times S} v([x_{p-1}, x_p] \times S) \Big]$$

Since $[x_{p-1}, x_q] \times S \subseteq [x_{p-1}, x_p] \times S$

$$m_{[x_{q-1}, x_q] \times S} \le m_{[x_{p-1}, x_p] \times S}$$

$\square$

Corollary Let B be a box and

$f: B \to \mathbb{R}$ bdd

$\forall$ partition P, P'

we also have $L(f, P) \le U(f, P')$

Pf Let $Q = P \cup P'$

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P')$$

(lower sum 总比 upper sum 小, 即使不同 P)