Homework 8

Question 1. Let $\{x_n\}$ be the nonnegative solutions to $J_m(x_n)\cos\beta + x_n J_m'(x_n)\sin\beta = 0$, where $m \geq 0$ and $0 \leq \beta \leq \pi/2$. Prove

$$\int_0^1 J_m(x x_{n_1}) J_m(x x_{n_2})\, x dx = 0 \quad n_1 \neq n_2.$$

Solution. If we define $y_i(x) = J_m(x x_{n_i})$, then the Bessel equation becomes $(x y_i')' + \left(x x_{n_i}^2 - \frac{m^2}{x}\right) y_i = 0$. We then multiply the equation for y_1 by y_2 and integrate both sides over x. Interchanging the roles of y_1 and y_2 and subtracting the resulting equations leaves

$$\left.(y_1' y_2 - y_1 y_2')\right|_{x=1} + \left(x_{n_1}^2 - x_{n_2}^2\right)\int_0^1 x y_1(x) y_2(x) dx = 0.$$

Sol Define $y_i(x) = J_m(x x_{n_i})$

Then Bessel's eq for y_1:

$$(x y_1')' + \left(x x_{n_1}^2 - \frac{m^2}{x}\right) y_1 = 0$$

$$\Rightarrow y_2 (x y_1')' + \left(x x_{n_1}^2 - \frac{m^2}{x}\right) y_1 y_2 = 0$$

Similarly we get

$$y_1 (x y_2')' + \left(x x_{n_2}^2 - \frac{m^2}{x}\right) y_1 y_2 = 0$$

$$\Rightarrow \underbrace{\left[(x y_1')' y_2 - (x y_2')' y_1\right]}_{= \frac{d}{dx}(x y_1' y_2 - x y_2' y_1)} + x(x_{n_1}^2 - x_{n_2}^2) y_1 y_2 = 0$$

$$\xrightarrow[\text{both sides}]{\text{integrate}} \underbrace{\left[x y_1' y_2 - x y_2' y_1\right]_0^1}_{①} + (x_{n_1}^2 - x_{n_2}^2)\int_0^1 x y_1 y_2 dx = 0$$

$$① = y_1'(1) y_2(1) - y_2'(1) y_1(1)$$

$$= J_m'(x_1) J_m(x_2) - J_m'(x_2) J_m(x_1)$$

$$= \cancel{\tfrac{1}{2}\left(J_{m-1}(x_1) - J_{m+1}(x_1)\right) J_m(x_2)} - \cancel{\tfrac{1}{2}\left(J_{m-1}(x_2) - J_{m+1}(x_2)\right) J_m(x_1)}$$

$$= 0$$

Therefore $\underbrace{(x_{n_1}^2 - x_{n_2}^2)}_{\neq 0 \text{ since } n_1 \neq n_2} \int_0^1 x y_1 y_2 dx = 0$

$$\Rightarrow \int_0^1 J_m(x x_{n_1}) J_m(x x_{n_2})\, x\, dx = 0$$

Question 2. Find the solution of the vibrating membrane problem (i.e., the edges are fixed) in the case where $u(\rho, \varphi, 0) = 0$ and $u_t(\rho, \varphi, 0) = 1, 0 < \rho < a$.

Solution. $u(\rho, \varphi, t) = \frac{2a}{c}\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/a)}{x_n^2 J_1(x_n)} \sin\frac{ctx_n}{a}, J_0(x_n) = 0.$

Sol Suppose $u(\rho, \varphi, t) = R(\rho)\phi(\varphi)T(t)$

By apply same procedure in lecture we obtain:

$$\begin{cases} \phi'' + \mu\phi = 0, \quad \phi(-\pi) = \phi(\pi), \ \phi'(-\pi) = \phi'(\pi) & ① \\ R'' + \frac{1}{\rho}R' + \left(\lambda - \frac{\mu}{\rho^2}\right) R = 0, \quad R(a) = 0 \\ T'' + \lambda c^2 T = 0 & ③ \end{cases}$$

By SL, for ① we get

$$\phi_m(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi, \quad \mu = m^2, \ m = 0, 1, 2, \ldots$$

By Bessel's eq, for each m, $R_m(\rho) = J_m(\rho\sqrt{\lambda})$

For $R(a) = 0$ we obtain $\sqrt{\lambda} = \frac{x_n^{(m)}}{a}$

$$\Rightarrow u_{mn}(\rho, \varphi, t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\left(\hat{A}_{mn}\cos\frac{ctx_n^{(m)}}{a} + \tilde{B}_{mn}\sin\frac{ctx_n^{(m)}}{a}\right)$$

$$u_t(\rho, \varphi, 0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\left[A_{mn}\cos m\varphi + B_{mn}\sin m\varphi\right]\left(\frac{c x_n^{(m)}}{a}\right)\tilde{B}_{mn} = 1$$

$u_t(\rho, \varphi, 0) = 1 \Rightarrow A_{mn}, B_{mn} = 0$ for all $m \geq 1$

$$\Rightarrow u(\rho, \varphi, t) = \sum_{n=1}^{\infty} J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\left[\tilde{A}_n \cos\left(\frac{c x_n^{(0)}}{a}t\right) + \tilde{B}_n \sin\left(\frac{c x_n^{(0)}}{a}t\right)\right]$$

By $u(\rho, \varphi, 0) = 0$, $\sum_{n=1}^{\infty} J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\tilde{A}_n = 0 \ \forall n \Rightarrow \tilde{A}_n = 0$

By $u_t(\rho, \varphi, 0) = 1 \Rightarrow \sum_{n=1}^{\infty} J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\left(\frac{c x_n}{a}\right)\tilde{B}_n = 1$

expand 1 by Fourier-Bessel series:

$$1 = \sum_{n=0}^{\infty} c_n J_0\left(\frac{\rho x_n^{(0)}}{a}\right)$$

$$\Rightarrow \int_0^a \rho J_0\left(\frac{\rho x_k^{(0)}}{a}\right) d\rho = c_k \frac{a^2}{2}\left[J_1(x_k^{(0)})\right]^2$$

$$\frac{a^2 J_1(x_k^{(0)})}{x_k^{(0)}} = c_k \frac{a^2}{2}\left[J_1(x_k^{(0)})\right]^2$$

$$\Rightarrow c_k = \frac{2}{x_k J_1(x_k)}$$

$$\text{So } 1 = \sum_{n=1}^{\infty} \frac{2}{x_n J_1(x_n)} J_0\left(\frac{\rho x_n^{(0)}}{a}\right) = \sum_{n=1}^{\infty} J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\left(\frac{c x_n^{(0)}}{a}\right)\tilde{B}_n$$

$$\Rightarrow \tilde{B}_n = \frac{2a}{c x_n^2 J_1(x_n)}$$

$$\Rightarrow u(\rho, \varphi, t) = \frac{2a}{c}\sum_{n=1}^{\infty} \frac{J_0\left(\frac{\rho x_n^{(0)}}{a}\right)}{x_n^2 J_1(x_n^{(0)})} \sin\left(\frac{c x_n^{(0)}}{a}t\right)$$

Question 3. Find the solution of the vibrating membrane problem in the case where $u(\rho, \varphi, 0) = 0$ and $u_t(\rho, \varphi, 0) = a^2 - \rho^2, 0 < \rho < a$.

Solution. To consider the initial conditions, we need to compute the Fourier-Bessel series of $a^2 - \rho^2$. To this end, we begin with writing $a^2 - \rho^2 = \sum_{n=1}^{\infty} A_n J_0(\rho x_n/a)$ (The expansion $a^2 - \rho^2 = \sum_{n=1}^{\infty} B_n J_0(\rho x_n)$ is possible but $J_0(\rho x_n/a)$ is desired because this Bessel function shows up in the general solution). By defining $x = \rho/a$, we have $1 - x^2 = \sum_{n'=1}^{\infty}(A_{n'}/a^2) J_0(x x_{n'})$. Thus we obtain $\int_0^1 (1 - x^2) J_0(x x_n) x dx = \sum_{n'=1}^{\infty}(A_{n'}/a^2)\int_0^1 J_0(x x_{n'}) J_0(x x_n) x dx$. For the left-hand side we introduce $t = x x_n$, and we have $(1/x_n^4)\int_0^{x_n}(x_n^2 - t^2) J_0(t) t dt$. We note that $t J_0(t) = \frac{d}{dt}[t J_1(t)]$ and $J_0'(t) = -J_1(t)$. By integration by parts we obtain $\int_0^{x_n}(x_n^2 - t^2) J_0(t) t dt = 4 x_n J_1(x_n)$. In the end, we obtain $u(\rho, \varphi, t) = \frac{8a^3}{c}\sum_{n=1}^{\infty} \frac{J_0(\rho x_n/a)}{x_n^4 J_1(x_n)} \sin\frac{ctx_n}{a}, J_0(x_n) = 0.$

Sol Suppose $u(\rho, \varphi, t) = R(\rho)\phi(\varphi)T(t)$

Same as problem 2 we get

$$\begin{cases} \phi'' + \mu\phi = 0, \quad \phi(-\pi) = \phi(\pi), \ \phi'(-\pi) = \phi'(\pi) & ① \\ R'' + \frac{1}{\rho}R' + \left(\lambda - \frac{\mu}{\rho^2}\right) R = 0, \quad R(a) = 0 \\ T'' + \lambda c^2 T = 0 & ③ \end{cases}$$

$$\Rightarrow u_{mn}(\rho, \varphi, t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\left(\hat{A}\cos\frac{ctx_n^{(m)}}{a} + \tilde{B}\sin\frac{ctx_n^{(m)}}{a}\right)$$

$$u_t(\rho, \varphi, 0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\left[A_{mn}\cos m\varphi + B_{mn}\sin m\varphi\right]\left(\frac{c x_n^{(m)}}{a}\right)\tilde{B}_{mn} = 1$$

$R(a) = 0 \Rightarrow A_{mn} = B_{mn} = 0$ for $m \geq 1$, A_{0n} is const

$u(\rho, \varphi, 0) = 0 \Rightarrow \tilde{A}_n = 0 \ \forall n$

$$\Rightarrow u(\rho, \varphi, t) = \sum_{n=1}^{\infty} \tilde{B}_n J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\sin\left(\frac{c x_n^{(0)}}{a}t\right)$$

Applying $u_t(\rho, \varphi, 0) = a^2 - \rho^2 \Rightarrow \sum_{n=1}^{\infty} \tilde{B}_n \frac{c x_n^{(0)}}{a} J_0\left(\frac{\rho x_n^{(0)}}{a}\right) = a^2 - \rho^2$

Expanding $a^2 - \rho^2 = \sum_{n=1}^{\infty} C_n J_0\left(\frac{\rho x_n^{(0)}}{a}\right)$

$$C_n = \frac{\int_0^a \rho(a^2 - \rho^2) J_0\left(\frac{\rho x_n^{(0)}}{a}\right) d\rho}{\int_0^a \rho\left[J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\right]^2 d\rho} = \frac{\int_0^a \rho(a^2 - \rho^2) J_0\left(\frac{\rho x_n}{a}\right) d\rho}{\frac{a^2}{2} J_1^2(x_n)}$$

Setting $x=\frac{\rho}{a} \Rightarrow 1-x^2 = \sum_{n'=1}^{\infty} \frac{C_{n'}}{a^2} J_0(x x_{n'}^{(0)})$

$$\Rightarrow \int_0^1 (1-x^2) J_0(x x_n)\, x\, dx = \sum_{n'=1}^{\infty} \frac{C_{n'}}{a^2}\int_0^1 J_0(x x_{n'}) J_0(x x_n)\, x\, dx = \frac{C_n}{2a^2} J_1(x_n)^2$$

Let $t = x x_n \Rightarrow \text{LHS} = \frac{1}{x_n^4}\int_0^{x_n}(x_n^2-t^2)J_0(t)\,t\,dt$

$$= \frac{1}{x_n^4} 4 x_n J_1(x_n) = \frac{4}{x_n^3}J_1(x_n)$$

$$\Rightarrow C_n = \frac{8a^2}{J_1(x_n)x_n^3} \Rightarrow \tilde{B}_n = \frac{C_n}{\frac{c x_n}{a}} = \frac{8a^3}{c x_n^4}$$

Therefore

$$u(\rho,\varphi,t) = \sum_{n=1}^{\infty} \tilde{B}_n J_0\left(\frac{\rho x_n^{(0)}}{a}\right) \sin\left(\frac{c x_n^{(0)}}{a} t\right)$$

$$= \frac{8a^3}{c}\sum_{n=1}^{\infty} \frac{J_0\left(\frac{\rho x_n}{a}\right)}{x_n^4 J_1(x_n)} \sin\frac{c t x_n}{a}$$

Question 4. Find the solution of the heat equation $u_t = K\nabla^2 u$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max},\varphi,t) = 0$ and the initial condition $u(\rho,\varphi,0) = \rho_{\max}^2 - \rho^2$.

Solution. $u(\rho,\varphi,t) = 8\rho_{\max}^2 \sum_{n=1}^{\infty} \frac{J_0(\rho x_n/\rho_{\max})}{x_n^3 J_1(x_n)} e^{-x_n^2 Kt/\rho_{\max}^2}$, where $J_0(x_n) = 0$.

$$\begin{cases} u_t = K\nabla^2 u, \ t>0,\ 0\le\rho<\rho_{max}, \\ u(\rho_{max},\varphi,t)=0, \ t>0 \\ u(\rho,\varphi,0) = \rho_{max}^2-\rho^2, \ 0\le\rho<\rho_{max}, \end{cases}$$

Let $u(\rho,\varphi,t) = R(\rho)\Phi(\varphi)T(t)$, we get

$$u(\rho,\varphi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{\rho_{max}}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\exp\left(-\frac{(x_n^{(m)})^2}{\rho_{max}^2}Kt\right)$$

Since $u(\rho,\varphi,0) = \rho_{max}^2-\rho^2$ ind. of φ

$\Rightarrow$ only $m=0$ term contributes to the sol.

So $u(\rho,\varphi,t) = \sum_{n=1}^{\infty} \tilde{A}_n J_0\left(\frac{x_n^{(0)}\rho}{\rho_{max}}\right)\exp\left(-\left(\frac{x_n^{(0)}}{\rho_{max}}\right)^2 Kt\right)$

Apply B.C. $\Rightarrow$

$$\rho_{max}^2-\rho^2 = \sum_{n=1}^{\infty} \tilde{A}_n J_0\left(\frac{x_n^{(0)}\rho}{\rho_{max}}\right)$$

$$\tilde{A}_n = \frac{\int_0^{a}\rho(\rho_{max}^2-\rho^2)J_0\left(\frac{\rho x_n^{(0)}}{\rho_{max}}\right)d\rho}{\int_0^{a}\rho\left[J_0\left(\frac{\rho x_n^{(0)}}{\rho_{max}}\right)\right]^2 d\rho}$$

$$= \frac{8\rho_{max}^2}{J_1(x_n)x_n^3}, \text{ (same as last problem)}$$

$$\Rightarrow u(\rho,\varphi,t) = 8\rho_{max}^2\sum_{n=1}^{\infty} \frac{J_0\left(\frac{x_n^{(0)}\rho}{\rho_{max}}\right)}{J_1(x_n)x_n^3}\exp\left(-\left(\frac{x_n^{(0)}}{\rho_{max}}\right)^2 Kt\right)$$

Question 5. Find the solution of the heat equation $u_t = K\nabla^2 u + \sigma$ in the infinite cylinder $0 \le \rho < \rho_{\max}$ satisfying the boundary condition $u(\rho_{\max},\varphi,t) = T_1$ and the initial condition $u(\rho,\varphi,0) = T_2\left(1-\rho^2/\rho_{\max}^2\right)$. Here K,σ,T_1,T_2 are positive constants.

Solution. $u(\rho,\varphi,t) = T_1 + \frac{\sigma\left(\rho_{\max}^2-\rho^2\right)}{4K} + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n}{\rho_{\max}}\right) e^{-x_n^2 Kt/\rho_{\max}^2}$, where $J_0(x_n) = 0$, $A_n = \frac{8\left[T_2-\sigma\rho_{\max}^2/4K\right]}{x_n^3 J_1(x_n)}$ $\frac{2T_1}{x_n J_1(x_n)}$.

Sol By B.C the sol should be independent of φ

$\Rightarrow u_t = K(u_{\rho\rho}+\frac{1}{\rho}u_\rho)+\sigma$

Decompose it into steady-state sol $U(\rho)$ and transient sol $V(\rho,t)$

$0 = K(U_{\rho\rho}+\frac{1}{\rho}U_\rho)+\sigma$

$\Rightarrow U_{\rho\rho}+\frac{1}{\rho}U_\rho = -\frac{\sigma}{K}$, let $W=U_\rho$

$\Rightarrow W_\rho + \frac{1}{\rho}W = -\frac{\sigma}{K} \Rightarrow W = -\frac{\sigma}{2K}\rho + \frac{C_1}{\rho}$

$\Rightarrow U = -\frac{\sigma}{4K}\rho^2 + C_1\ln\rho + C_2$

To bound $\ln\rho \Rightarrow C_1=0$, $U(\rho_{max},t)=T_1 \Rightarrow C_2 = T_1+\frac{\sigma}{4K}\rho_{max}^2$

$\Rightarrow U(\rho) = T_1+\frac{\sigma}{4K}(\rho_{max}^2-\rho^2)$

Then we solve for $\begin{cases} V_t = K(V_{\rho\rho}+\frac{1}{\rho}V_\rho) \\ V(\rho_{max},t)=0 \\ V(\rho,0) = \left(T_2-\frac{\sigma\rho_{max}^2}{4K}\right)\left(1-\frac{\rho^2}{\rho_{max}^2}\right) \end{cases}$

The separated sol should be

$$V(\rho,t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{x_n\rho}{\rho_{max}}\right) e^{-\left(\frac{x_n}{\rho_{max}}\right)^2 Kt}$$

$$V(\rho,0) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{x_n\rho}{\rho_{max}}\right) = \left(T_2-\frac{\sigma\rho_{max}^2}{4K}\right)\left(1-\frac{\rho^2}{\rho_{max}^2}\right)$$

$$\Rightarrow A_n = \left(T_2-\frac{\sigma\rho_{max}^2}{4K}\right)\frac{\int_0^{a}\rho\left(1-\frac{\rho^2}{\rho_{max}^2}\right)J_0\left(\frac{\rho x_n}{\rho_{max}}\right)d\rho}{\int_0^{a}\rho\left[J_0\left(\frac{\rho x_n}{\rho_{max}}\right)\right]^2 d\rho}$$

$$= \frac{8\left(T_2-\frac{\sigma\rho_{max}^2}{4K}\right)}{x_n^3 J_1(x_n)}$$

Therefore $V(\rho,t) = \sum_{n=1}^{\infty}\frac{8\left(T_2-\frac{\sigma\rho_{max}^2}{4K}\right)}{x_n^3 J_1(x_n)} J_0\left(\frac{\rho x_n}{\rho_{max}}\right)\exp\left(\frac{-x_n^2 Kt}{\rho_{max}^2}\right)$

$$u(\rho,\varphi,t) = T_1 + \frac{\sigma(\rho_{max}^2-\rho^2)}{4K} + \sum_{n=1}^{\infty}\frac{8\left(T_2-\frac{\sigma\rho_{max}^2}{4K}\right)}{x_n^3 J_1(x_n)} J_0\left(\frac{\rho x_n}{\rho_{max}}\right)\exp\left(\frac{-x_n^2 Kt}{\rho_{max}^2}\right)$$