# Question 1

Find the solution of the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 1, t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = x(1-x), \quad 0 < x < 1,$$
$$u_t(x,0) = 0, \quad 0 < x < 1.$$

**Solution.**

$$u(x,t) = \sum_{k=1, k \text{ odd}}^{\infty} \frac{8}{k^3\pi^3} \sin(k\pi x)\cos(ck\pi t).$$

Sol Suppose $u = \varphi(x)T(t)$

$\Rightarrow \varphi T'' = c^2 T \varphi''$

$\frac{\varphi''}{\varphi} = c^2 \frac{T''}{T} = -\lambda$ for some $\lambda$

$① \begin{cases} \varphi'' + \lambda\varphi = 0 \\ \varphi(0) = \varphi(L) = 0 \end{cases}, \quad ② \begin{cases} T'' + \lambda c^2 T = 0 \\ T'(0) = 0 \end{cases}$

By ① we have $\varphi_n(x) = \sin(\sqrt{\lambda_n}x)$, $\lambda_n = (\frac{n\pi}{L})^2 = (n\pi)^2$

Then By ② we have $T_n(t) = C_1 \sin(\sqrt{\lambda_n}ct) + C_2\cos(\sqrt{\lambda_n}ct)$

By $T'(0) = 0 \Rightarrow C_1 = 0 \Rightarrow T_n(t) = \cos(\sqrt{\lambda_n}ct)$

(omitting coeff)

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct)\sin(n\pi x)$

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = x(1-x) \Rightarrow B_n = \frac{2}{L}\int_0^L x(1-x)\sin(n\pi x)\,dx$

$B_n = 2\left(\int_0^1 x\sin(n\pi x)dx - \int_0^1 x^2 \sin(n\pi x)dx\right)$

$= 2\left(\frac{(-1)^{n+1}}{\pi n} - \left(\frac{-1}{n\pi} + \frac{2}{(n\pi)^3}\right)(-1)^n + \frac{2}{(n\pi)^3}\right) = \frac{4}{(n\pi)^3}\left(-(-1)^n + 1\right)$

Therefore $u(x,t) = \sum_{n=1, n \text{ odd}}^{\infty} \frac{8}{(n\pi)^3}\cos(n\pi ct)\sin(n\pi x)$

## Question 2

Compute the multidimensional sine Fourier series of $f(x,y) = xy$, $x, y \in [-\pi,\pi] \times [-\pi,\pi]$.

**Solution.**

$$f(x,y) = \sum_{n,m=1}^{\infty} \frac{4}{nm}(-1)^{n+m} \sin(nx)\sin(my).$$

We only need to consider $x \in [0,\pi]$ and $y \in [0,\pi]$

since $\forall x \in [-\pi, 0]$, $f(x,y) = -f(-x,y)$

$\forall y \in [-\pi, 0]$, $f(x,y) = -f(x,-y)$

So $\forall x, y \in [0,\pi]$, $f(x,y) = \sum_{n=1}^{\infty} B_{mn} \sin(mx)\sin(ny) = xy$

where $B_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} xy \sin(mx)\sin(ny)\, dx\, dy$

$= \frac{4}{\pi^2}\int_0^{\pi} x \sin mx \, dx \int_0^{\pi} y \sin ny \, dy$

$= \frac{4}{\pi^2}\frac{1}{nm}(1-(-1)^m)\pi(1-(-1)^n)\pi$

$= \frac{4}{nm}(1-(-1)^m)(1-(-1)^n)$

Therefore $f(x,y) = \sum_{n,m=1}^{\infty} \frac{4}{nm}(-1)^{n+m} \sin(mx)\sin(ny)$

## Question 3

Solve the initial-value problem for the heat equation $u_t = K\Delta u$ in the column $0 < x < L_1, 0 < y < L_2$ with the boundary conditions $u(0,y,t) = 0$, $u_x(L_1,y,t) = 0$, $u(x,0,t) = 0$, $u_y(x,L_2,t) = 0$, and the initial condition $u(x,y,0) = 1$.

**Solution.**

$$u(x,y,t) = \frac{4}{\pi^2} \sum_{m,n=1}^{\infty} u_{mn}(x,y,t),$$

where

$$u_{mn}(x,y,t) = \sin\left(\left(m - \frac{1}{2}\right)\frac{\pi x}{L_1}\right)\frac{1}{m-\frac{1}{2}} \sin\left(\left(n - \frac{1}{2}\right)\frac{\pi y}{L_2}\right)\frac{1}{n-\frac{1}{2}} e^{-\lambda_{mn}Kt}.$$

Here,

$$\lambda_{mn} = \left(m - \frac{1}{2}\right)^2\left(\frac{\pi}{L_1}\right)^2 + \left(n - \frac{1}{2}\right)^2\left(\frac{\pi}{L_2}\right)^2.$$

Sol suppose $u = X(x)Y(y)T(t)$

$u_t = K\Delta u \Rightarrow XYT' = K(X''Y + XY'')T$

$\Rightarrow \frac{T'}{KT} = \frac{X''}{X} + \frac{Y''}{Y}$

$\Rightarrow \frac{T'}{KT} = -\lambda, \frac{X''}{X} = -\lambda_x, \frac{Y''}{Y} = -\lambda_y$ for some const $\lambda = \lambda_x + \lambda_y$

$\Rightarrow ① T = e^{-\lambda Kt}$, $② X'' + \lambda_x X = 0$, $③ Y'' + \lambda_y Y = 0$

$②: X_n = A_n \sin(\sqrt{\lambda_{x_n}}x) + B_n \cos(\sqrt{\lambda_{x_n}}x)$

By $u(0,y,t) = 0 \Rightarrow X_n(0) = 0 \Rightarrow B_n = 0 \Rightarrow X_n = A_n \sin(\sqrt{\lambda_{x_n}}x)$

And by $u_x(L_1,y,t) = 0 \Rightarrow \sqrt{\lambda_{x_n}}L_1 = (n - \frac{1}{2})\pi$

$\lambda_{x_n} = (n-\frac{1}{2})^2\left(\frac{\pi}{L_1}\right)^2$

$③: Y_m = C_m \sin(\sqrt{\lambda_{y_m}}x) + D_m \cos(\sqrt{\lambda_{y_m}}x)$

By $u(y,0,t) = 0 \Rightarrow Y_m(0) = 0 \Rightarrow D_m = 0 \Rightarrow Y_m = C_m \sin(\sqrt{\lambda_{y_m}}x)$

By $u_y(y,L_2,t) = 0 \Rightarrow \lambda_{y_m} = (m-\frac{1}{2})^2\left(\frac{\pi}{L_2}\right)^2$

$\Rightarrow u(x,y,t) = \sum_{m,n=1}^{\infty} a_{mn} \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right) \sin\left((m-\frac{1}{2})\frac{\pi y}{L_2}\right) e^{-\lambda_{mn}Kt}$

Apply the initial condition:

$$1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{mn}\sin\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\tfrac{1}{2})\frac{\pi y}{L_2}\right)$$

$$\Rightarrow d_{mn}=\frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} 1\cdot\sin\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\tfrac{1}{2})\frac{\pi y}{L_2}\right)\,dx\,dy$$

$$\Rightarrow d_{mn}=\left(\int_0^{L_1}\sin\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)dx\right)\left(\int_0^{L_2}\sin\left((m-\tfrac{1}{2})\frac{\pi x}{L_2}\right)dy\right)$$

$$=4\frac{1}{(m-\frac{1}{2})\pi}\;\frac{1}{(n-\frac{1}{2})\pi}=\frac{4}{\pi^2}\frac{1}{(m-\frac{1}{2})(n-\frac{1}{2})}$$

$$\left(\text{since } \int_0^{L_1}\sin\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)dx=\int_0^{(n-\frac{1}{2})\pi}\sin\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)d\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)=\left[-\cos t\right]_0^{(n-\frac{1}{2})\pi}=1\right)$$

$$\Rightarrow u(x,y,t)=\frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{1}{(n-\frac{1}{2})(m-\frac{1}{2})}\sin\left((n-\tfrac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\tfrac{1}{2})\frac{\pi x}{L_2}\right)e^{-\lambda_{mn}kt}$$

where $\lambda_{mn}=\lambda_{X_n}+\lambda_{Y_m}=(n-\frac{1}{2})\left(\frac{\pi}{L_1}\right)^2+(m-\frac{1}{2})\left(\frac{\pi}{L_2}\right)^2$

## Question 4

Find the solution of the homogeneous wave equation

$$u_{tt} = u_{xx} + u_{yy}, \quad t > 0,\ (x, y) \in [0, \pi] \times [0, \pi],$$
$$u(0, y, t) = 0, \quad u(\pi, y, t) = 0, \quad t > 0,\ y \in [0, \pi],$$
$$u(x, 0, t) = 0, \quad u(x, \pi, t) = 0, \quad t > 0,\ x \in [0, \pi],$$
$$u(x, y, 0) = xy, \quad u_t(x, y, 0) = 0, \quad x, y \in [0, \pi] \times [0, \pi].$$

**Solution.**

$$u(x, y, t) = \sum_{n,m=1}^{\infty} \frac{4}{nm}(-1)^{n+m} \cos\left(\sqrt{n^2 + m^2}t\right) \sin(nx) \sin(my).$$

Sol Suppose $U = XYT$

$\Rightarrow XYT'' = X''YT + XY''T$

$\Rightarrow \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$ for some const $\lambda$

$\Rightarrow \begin{cases} X'' + \lambda_x X = 0, \\ X(0) = X(\pi) = 0 \end{cases} \begin{cases} Y'' + \lambda_y Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases}$

$\Rightarrow X_n = \sin(nx), \quad Y_m = \sin(my)$

$\lambda_n = n^2 \quad \lambda_m = m^2$

So $T_{nm}(t) = A_{nm}\cos(\sqrt{n^2+m^2}\,t) + B_{nm}\sin(\sqrt{n^2+m^2}\,t)$

$U_t(x,y,t) = XYT' = \sum_{m,n=1}^{\infty}\left(-A_{nm}\sqrt{n^2+m^2}\sin(\sqrt{n^2+m^2}\,t) + B_{nm}\sqrt{n^2+m^2}\cos(\sqrt{n^2+m^2}\,t)\right)\sin(nx)\sin(my)$

$U_t(x,y,0) = 0 \Rightarrow \underline{B_{nm} = 0}$

$U(x,y,0) = xy \Rightarrow \sum_{m,n=1}^{\infty} A_{nm}\cos(\sqrt{n^2+m^2}\,t)\sin(nx)\sin(my) = xy$

$A_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\sin(nx)\sin(my)\,dx\,dy$

$= \frac{4}{\pi^2}\int_0^\pi x\sin(nx)\,dx\int_0^\pi y\sin(my)\,dy = \frac{4}{\pi^2}$

$= \frac{4}{\pi^2}\left(\frac{\pi^2}{n}(-1)^{n+1}\right)\left(\frac{\pi^2}{m}(-1)^{m+1}\right) = \underline{\frac{4}{nm}(-1)^{n+m}}$

$\Rightarrow \underline{U(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{4}{nm}(-1)^{n+m}\cos(\sqrt{n^2+m^2}\,t)\sin(nx)\sin(my)}$

## Question 5

Find the solution of the homogeneous wave equation

$$u_{tt} = u_{xx} + u_{yy}, \quad t > 0,\ (x, y) \in [0, \pi] \times [0, \pi],$$
$$u_x(0, y, t) = 0, \quad u_x(\pi, y, t) = 0, \quad t > 0,\ y \in [0, \pi],$$
$$u(x, 0, t) = 0, \quad u(x, \pi, t) = 0, \quad t > 0,\ x \in [0, \pi],$$
$$u(x, y, 0) = x(\pi - x)y, \quad u_t(x, y, 0) = 0, \quad x, y \in [0, \pi] \times [0, \pi].$$

**Solution.**

$$u(x, y, t) = -\sum_{m=1}^{\infty} \frac{\pi^2(-1)^m}{3m} \cos(mt) \sin(my) + \sum_{n,m=1}^{\infty} \frac{4(1 + (-1)^n)(-1)^m}{n^2 m} \cos\left(\sqrt{n^2 + m^2}\, t\right) \cos(nx) \sin(my).$$

Sol Suppose $U = XYT$

$$\Rightarrow XYT'' = X''YT + XY''T$$

$$\Rightarrow \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \text{ for some const } \lambda$$

$$\Rightarrow \begin{cases} X'' + \lambda_x X = 0, \\ X'(0) = X'(\pi) = 0 \end{cases} \quad \begin{cases} Y'' + \lambda_y Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases}$$

$$\Rightarrow X_n = \cos(nx), \quad Y_m = \sin(my)$$

$$\lambda_n = n^2, \quad \lambda_m = m^2,\ m \in \mathbb{N}$$

$$n \in \mathbb{N} \cup \{0\}$$

$$\Rightarrow T_{nm}(t) = A_{nm} \cos(\sqrt{n^2+m^2}\, t) + B_{nm} \sin(\sqrt{n^2+m^2}\, t)$$

note as $W_{nm}$

So $u(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(A_{nm} \cos(W_{nm} t) + B_{nm} \sin(W_{nm} t)\right) \cos nx \sin my$

$u_t(x,y,t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(-A_{nm} W_{nm} \sin(W_{nm} t) + B_{nm} W_{nm} \cos(W_{nm} t)\right) \cos nx \sin my$

$u_t(x,y,0) = 0 \Rightarrow B_{nm} = 0 \quad \forall n, m$

Now we determine $A_{nm}$:

$$x(\pi - x)y = \sum_{n,m=1}^{\infty} A_{nm} \cos nx \sin my$$

$$\text{where } A_{nm} = \frac{\int_0^\pi \int_0^\pi x(\pi-x)y \cos nx \sin my \, dx\, dy}{\int_0^\pi \cos^2 nx\, dx \int_0^\pi \sin^2 my\, dy}$$

$$\int_0^{\pi} \cos^2 nx \, dx = \begin{cases} \pi, & n=0 \\ \frac{\pi}{2}, & n \geq 1 \end{cases}, \quad \int_0^{\pi} \sin^2 my \, dy = \frac{\pi}{2}$$

for $n=0$, $\int_0^{\pi}\int_0^{\pi} x(\pi-x)\, y \cos nx \sin my \, dx dy$

$$= \left(\int_0^{\pi} x(\pi-x)\,dx\right)\left(\int_0^{\pi} y \sin my \, dy\right)$$

$$= \frac{\pi^3}{6}\left(\frac{-\pi(-1)^m}{m}\right)$$

$$\Rightarrow A_{0m} = \frac{\frac{\pi^3}{6}\left(-\frac{\pi(-1)^m}{m}\right)}{\frac{1}{2}\pi^2} = \frac{-\pi^2(-1)^m}{3m}$$

for $n \geq 1$, $\int_0^{\pi}\int_0^{\pi} x(\pi-x)\, y \cos nx \sin my \, dx dy$

$$= \left(\int_0^{\pi} x(\pi-x)\cos nx \, dx\right)\left(\int_0^{\pi} y \sin my \, dy\right)$$

$$= \left(-\pi\left(\left(\frac{\pi}{n\pi}\right)^2 - (-1)^n\left(\frac{\pi}{n\pi}\right)^2 - \frac{2\pi^3}{(n\pi)^2}(-1)^n\right)\right)\left(\frac{\pi^2(-1)^{m+1}}{m\pi}\right)$$

$$= \left(-\pi\frac{1}{n^2}(1-(-1)^n) - \frac{2\pi}{n^2}(-1)^n\right)\frac{\pi}{m}(-1)^{m+1}$$

$$= \left(\frac{-\pi}{n^2} - \frac{\pi}{n^2}(-1)^n\right)\frac{\pi}{m}(-1)^{m+1} = \frac{-\pi^2(1+(-1)^n)(-1)^{m+1}}{n^2 m}$$

$$\Rightarrow A_{nm} = \frac{4}{\pi^2}\,\frac{-\pi^2(1+(-1)^n)(-1)^{m+1}}{n^2 m} = \frac{4(1+(-1)^n)(-1)^m}{n^2 m}$$

Therefore

$$u(x,y,t) = -\sum_{m=1}^{\infty}\frac{\pi^2(-1)^m}{3m}\cos mt \sin my + \sum_{n,m=1}^{\infty}\frac{4(-1)^n(-1)^m}{n^2 m}\cos\left(\sqrt{n^2+m^2}\,t\right)\cos nx \sin my$$