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Math 454: Boundary Value Problems for Partial Differential Equations
No deadline

Review 2

Question 1. Redo all the problems in HW5 - HW7.

Question 2. Prove the following facts

1) The following operator is symmetric
   (a) $A = -\frac{d^2}{dx^2}$, $\mathrm{Dom}(A) = \{f(x) : f(0) = f(1) = 0\}$.
   (b) $A\varphi = -\frac{1}{\rho(x)}\Big([s(x)\varphi'(x)]' + [\lambda\rho(x) - q(x)]\varphi(x)\Big)$,
   $\mathrm{Dom}(A) = \{\varphi(x) :\ \varphi(a)\cos\alpha - L\varphi'(a)\sin\alpha = 0, \quad \varphi(b)\cos\beta + L\varphi'(b)\sin\beta = 0\}$.
2) If A is symmetric, then its eigenvalues are real and its eigenvectors are orthogonal.
3) If $f(x) = g(y)$, then they must be a constant independent of x and y.
4) On $x = 0$ part of the domain $\{(x, y) : x \geq 0, 0 \leq y \leq 1\}$, the boundary condition $a(x, y)\mathbf{n} \cdot \nabla u + b(x, y)u = c(x, y)$ can be reduced to $-a(0, y)u_x + b(0, y)u = c(0, y)$.

Solution. 1b) This is Sturm-Liouville operator, you need to prove it using integration by parts as in the class. 2), 3) are theorems in the class. 4) The normal vector on $x = 0$ is $\mathbf{n} = (-1, 0)$.

Question 3. Find the solution of the following heat equations.

1) Find the solution and the asymptotics when $t \to \infty$

$$\begin{cases} u_t = Ku_{zz}, \quad 0 < z < L, t > 0, \\ u(0, t) = T, \quad u_z(L, t) = \Phi, \\ u(z, 0) = \Phi_0 z. \end{cases}$$

2) Find the solution of

$$\begin{cases} u_t = u_{xx} + \dfrac{1}{2}e^{-t}, \quad 0 < x < 1, t > 0, \\ u(0, t) = \dfrac{1}{2}e^{-t}, \quad u(1, t) = \dfrac{1}{2}e^{-t}, \\ u(x, 0) = \dfrac{3}{2}. \end{cases}$$

Solution. 1) $u(z, t) = T + \Phi z + \sum_{n=1}^{\infty} A_n \sin\frac{(n-1/2)\pi z}{L} \exp\left\{-\left[\frac{(n-1/2)\pi}{L}\right]^2 Kt\right\}$, where $A_n = -\frac{2T}{(n-1/2)\pi} + \frac{2L(\Phi_0-\Phi)(-1)^{n-1}}{(n-1/2)^2\pi^2}$. Asymptotics is $u_\infty(z, t) = T + \Phi z$. 2) $u(x, t) = \frac{1}{2}e^{-t} + \sum_{n=0}^{\infty} \frac{2}{n\pi}(1 - (-1)^n)e^{-n^2\pi^2 t} \sin n\pi x + \sum_{n=0}^{\infty} \frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)}(e^{-t} - e^{-n^2\pi^2 t}) \sin n\pi x$.

Question 4. For the solution of the following wave equation.

1) Solve the following equation using separation of variables.

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < 1 \ \textit{and}\ t > 0 \\ u(0, t) = u(1, t) = 0 & t > 0 \\ u(x, 0) = x & 0 < x < 1 \\ u_t(x, 0) = 0 & 0 < x < 1 \end{cases}$$

2) Solve the following equation using eigenfunction expansion method.

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < 1 \ \textit{and}\ t > 0 \\ u(0, t) = 0, \quad u_z(1, t) = 0 & t > 0 \\ u(x, 0) = x & 0 < x < 1 \\ u_t(x, 0) = 0 & 0 < x < 1 \end{cases}$$

Solution. 1) $u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)\cos(cn\pi t)$. 2) $\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi^2\left(n-\frac{1}{2}\right)^2} \cos((n-\frac{1}{2})\pi ct)\sin((n-\frac{1}{2})\pi z)$.

Question 5. Find the solution of the 2D homogeneous wave equation using separation of variables

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} & t > 0, \quad (x,y) \in [0,\pi]\times[0,\pi] \\ u(0,y,t) = 0, \quad u(\pi,y,t) = 0, & t > 0, \quad y \in [0,\pi] \\ u(x,0,t) = 0, \quad u(x,\pi,t) = 0 & t > 0, \quad x \in [0,\pi] \\ u(x,y,0) = 0, \quad u_t(x,y,0) = xy & x,y \in [0,\pi]\times[0,\pi] \end{cases}$$

Solution. $u(x,y,t) = \sum_{n,m=1}^{\infty} \frac{4(-1)^{n+m}}{nm\sqrt{n^2+m^2}} \sin\left(\sqrt{n^2+m^2}\,t\right)\sin(nx)\sin(my)$.