Question 1. Compute the Fourier transform of the following functions.

1) Find the Fourier transform of $f(x)$, where $f(x) = 1$ for $-2 < x < 2$ and $f(x) = 0$ otherwise.
2) Find the Fourier transform of $f(x)$, where $f(x) = e^{-x^2/2}$.
3) Find the Fourier transform of $f(x)$, where $f(x) = e^{-(x-2)^2/2}$.
4) Find the Fourier transform of $f(x) = \frac{1}{1+(x-3)^2}$.

Solution. 1) $\tilde{f}(\mu) = \frac{\sin 2\mu}{\pi\mu}$. 2) $\tilde{f}(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2}$. 3) $\tilde{f}(\mu) = \frac{1}{\sqrt{2\pi}} e^{-2i\mu} e^{-\mu^2/2}$. 4) $\tilde{f}(\mu) = \frac{1}{2} e^{-3i\mu} e^{-|\mu|}$.

Sol (1) $\tilde{f}(\mu) = \int_{-\infty}^{\infty} f(x) e^{-i\mu x} dx = \int_{-2}^{2} e^{-i\mu x} dx$

$$= \left[\frac{e^{-i\mu x}}{-i\mu}\right]_{-2}^{2} = \frac{e^{-2i\mu} - e^{2i\mu}}{-i\mu} = \frac{-2i\sin(2\mu)}{-i\mu} = \frac{2\sin(2\mu)}{\mu}$$

(2) This is Gaussian function centered at 0, $\sigma = 1$, multiplied by $\sqrt{2\pi}$

$$\Rightarrow \tilde{f}(\mu) = \frac{\sqrt{2\pi}}{2\pi} e^{-\frac{\mu^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2}}$$

(3) This function is a Gaussian centered at 2, $\sigma = 1$, multiplied by $\sqrt{2\pi}$

So $\tilde{f}(\mu) = \frac{\sqrt{2\pi}}{2\pi} e^{-2i\mu} e^{-\mu^2/2} = \frac{1}{\sqrt{2\pi}} e^{-2i\mu} e^{-\frac{\mu^2}{2}}$

(4) $\tilde{f}(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+(x-3)^2} e^{-i\mu x} dx = \frac{1}{2\pi} \cdot \pi e^{-|\mu|} e^{-3i\mu}$

$$= \frac{1}{2} e^{-|\mu|} e^{-3i\mu}$$

by Lorentzian function centered at 3.

Question 2. Consider the following initial-value problem for a diffusion equation with the absorption coefficient a (>0):

$$\begin{cases} u_t = K u_{xx} - au & t > 0, -\infty < x < \infty, \\ u = f(x) = e^{-x^2} & t = 0, -\infty < x < \infty. \end{cases}$$

Find $u(x,t)$ using the Fourier transform. (Hint: You can directly use the Fourier transform. Also you can use the transformation $u(x,t) = e^{-at} w(x,t)$ then use the Fourier transform. Either method is fine.)

Solution. $u(x,t) = \frac{1}{\sqrt{4Kt+1}} \exp\left[-\frac{x^2}{4Kt+1}\right] e^{-at}$.

Sol $u_t = K u_{xx} - au$

$$\Rightarrow \frac{\partial}{\partial t} \tilde{u}(\mu,t) = K(-\mu^2)\tilde{u}(\mu,t) - a\tilde{u}(\mu,t)$$
$$= (-K\mu^2 + a)\tilde{u}(\mu,t)$$

$$\Rightarrow \tilde{u}(\mu,t) = \tilde{u}(\mu,0) e^{-(K\mu^2+a)t}$$

And $\tilde{u}(\mu,0) = \int_{-\infty}^{\infty} e^{-x^2} e^{-i\mu x} dx = \sqrt{\pi} e^{-\mu^2 (\frac{1}{\sqrt{2}})^2/2} = \sqrt{\pi} e^{-\frac{\mu^2}{4}}$

So $\tilde{u}(\mu,t) = \sqrt{\pi} e^{-\mu^2(\frac{1}{4}+Kt)} e^{-at} = \sqrt{\pi} e^{-\frac{\mu^2}{4}(4Kt+1)} e^{-at}$

$$\Rightarrow u(x,t) = \frac{1}{2\sqrt{\pi}} e^{-at} \int_{-\infty}^{\infty} e^{-\frac{\mu^2}{4}(4Kt+1)} e^{i\mu x} d\mu$$
$$= \frac{1}{2\sqrt{\pi}} e^{-at} \sqrt{\frac{4\pi}{4Kt+1}} \exp\left(-\frac{x^2}{4Kt+1}\right)$$
$$= \frac{e^{-at}}{\sqrt{4Kt+1}} \exp\left(-\frac{x^2}{4Kt+1}\right)$$

Question 3. Consider the initial-value problem

$$\begin{cases} u_t = K u_{xx} & t > 0, x > 0, \\ u_x(0,t) = 0 & t > 0, \\ u(x,0) = f(x) = \begin{cases} 1 & 0 \le x \le L_1, \\ 0 & x > L_1. \end{cases} \end{cases}$$

Find $u(x,t)$ using the method of images. (Hint: Define $f_E(x) = f(x)(x \ge 0)$, $f(-x)(x \le 0)$. Then deri $u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi Kt}} e^{-(x-y)^2/4Kt} f_E(y) dy$.)

Solution. $u(x,t) = \frac{1}{\sqrt{4\pi Kt}} \int_0^{L_1} \left\{\exp\left[-\frac{(x-y)^2}{4Kt}\right] + \exp\left[-\frac{(x+y)^2}{4Kt}\right]\right\} dy$.

Sol We extend $f(x)$ as $f_E(x) = \begin{cases} f(x), & x \ge 0 \\ f(-x), & x \le 0 \end{cases}$

Then we have $\begin{cases} u_t = K u_{xx}, & t > 0, -\infty < x < \infty \\ u(x,0) = f_E(x), & -\infty < x < \infty \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{4\pi Kt}} \int_{-\infty}^{\infty} f_E(y) e^{-(x-y)^2/4Kt} dy$$
$$= \left(\int_{-L_1}^{0} + \int_0^{L_1}\right) \frac{e^{-(x-y)^2/4Kt}}{\sqrt{4\pi Kt}} dy$$
$$= \int_0^{L_1} \frac{\exp\left(\frac{-(x-y)^2}{4Kt}\right) + \exp\left(\frac{-(x+y)^2}{4Kt}\right)}{4\sqrt{\pi} Kt} dy$$

Question 4. Consider the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} & t > 0, -\infty < x < \infty, \\ u = f_1(x) & t = 0, -\infty < x < \infty, \\ u_t = f_2(x) & t = 0, -\infty < x < \infty. \end{cases}$$

Derive d'Alembert's formula.

Solution. $u(x,t) = \frac{1}{2}\left[f_1(x+ct) + f_1(x-ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} f_2(y) dy$.

Sol We use the Fourier transformation:

$$\begin{cases} u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\mu,t) e^{i\mu x} d\mu, & \tilde{u}(\mu,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t) e^{-i\mu x} dx \\ f_1(x) = \int_{-\infty}^{\infty} \tilde{f}_1(\mu) e^{i\mu x} d\mu, & \tilde{f}_1(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x) e^{-i\mu x} dx \\ f_2(x) = \int_{-\infty}^{\infty} \tilde{f}_2(\mu) e^{i\mu x} d\mu, & \tilde{f}_2(\mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(x) e^{-i\mu x} dx \end{cases}$$

Then the equation reduces to

$$\begin{cases} \tilde{u}_{tt} + c^2 \mu^2 \tilde{u} = 0 \\ \tilde{u}(\mu,0) = \tilde{f}_1(\mu), & -\infty < \mu < \infty \\ \tilde{u}_t(\mu,0) = \tilde{f}_2(\mu), & -\infty < \mu < \infty \end{cases}$$

$$\Rightarrow \tilde{u}(\mu,t) = A(\mu)\cos(\mu ct) + B(\mu)\sin(\mu ct)$$

By initial conditions, $A(\mu) = \tilde{f}_1(\mu)$, $B(\mu) = \frac{\tilde{f}_2(\mu)}{\mu c}$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} \left[\tilde{f}_1(\mu)\cos(\mu ct) + \tilde{f}_2(\mu)\frac{\sin(\mu ct)}{\mu c}\right] e^{i\mu x} d\mu$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \tilde{f}_1(\mu)\left[e^{i\mu(x+ct)} + e^{i\mu(x-ct)}\right] d\mu + \frac{1}{2c}\int_{-\infty}^{\infty} \tilde{f}_2(\mu) \int_{x-ct}^{x+ct} e^{i\mu y} dy\, d\mu$$
$$= \frac{1}{2}\left[f_1(x+ct) + f_1(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(y) dy$$

Question 5. Use d'Alembert's formula to solve the wave equation $u_{tt} = c^2u_{xx}$ with initial conditions $u(x,0) = 0$ and $u_t(x,0) = 4\cos 5x$.
Solution. $u(x,t) = (4/5c)\cos 5x \sin 5ct$.

$$u(x,t) = \frac{1}{2}\left[f_1(x+ct)+f_1(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(y)dy$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} 4\cos 5y \, dy$$

$$= \frac{2}{5c}\int_{5x-5ct}^{5x+5ct} \cos z \, dz$$

$$= \frac{2}{5c} \sin(5x+5ct) - \sin(5x-5ct)$$

$$= \frac{2}{5c}\left(2\cos\left(\frac{10x}{2}\right)\sin\left(\frac{10ct}{2}\right)\right) = \frac{4}{5c}\cos 5x \sin 5ct$$

Question 6. Find the solution of the wave equation $u_{tt} = c^2u_{xx}$ for $t > 0$ and $x > 0$ satisfying the boundary conditions $u(0,t) = 0$ and the initial conditions $u(x,0) = 0$ and $u_t(x,0) = g(x)$.
Solution. Since $u(x,t) = 0$, we extend the function $g(x)$ as

$$g_O(x) = \begin{cases} g(x) & x > 0, \\ 0 & x = 0, \\ -g(-x) & x < 0. \end{cases}$$

Then we use d'Alemberts' formula for

$$\begin{cases} u_{tt} = c^2u_{xx} & t > 0, -\infty < x < \infty, \\ u(x,0) = 0 & -\infty < x < \infty, \\ u_t(x,0) = g_O(x) & -\infty < x < \infty. \end{cases}$$

We obtain $u(x,t) = \frac{1}{2c}\int_{ct-x}^{ct+x} g(y)dy$ for $0 < x < ct$ and $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$ for $x > ct$.

Sol odd extend $g(x)$ as $g_0(x) = \begin{cases} g(x), & x>0 \\ 0, & x=0 \\ -g(-x), & x<0 \end{cases}$

$\Rightarrow$ The equations $\begin{cases} u_{tt} = c^2u_{xx}, & t>0, -\infty<x<\infty \\ u(x,0)=0, & -\infty<x<\infty \\ u_t(x,0)=g_0(x), & -\infty<x<\infty \end{cases}$

By Alembert's formula,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(y)\,dy$$

if $x<ct$: $= \frac{1}{2c}\left(\int_{x-ct}^{0} -g(-y)\,dy + \int_{0}^{x+ct} g(y)dy\right)$

$$= \frac{-1}{2c}\int_0^{ct-x} g(z)\,dz + \int_0^{x+ct} g(y)dy = \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)\,dy$$

if $x\geq ct$: $= \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$

Question 7. Solve the following initial-value problem for a diffusion equation with the absorption coefficient a (> 0) using Green function method,

$$\begin{cases} u_t = Ku_{xx} - au & t > 0, -\infty < x < \infty, \\ u = f(x) & t = 0, -\infty < x < \infty. \end{cases}$$

Solution. $G(x,x',t) = \frac{1}{\sqrt{4\pi Kt}}e^{-\frac{(x-x')^2}{4Kt}}e^{-at}$ and $u(x,t) = \frac{e^{-at}}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{4Kt}} f(x')dx'$.

Sol The Green's function for the equation without absorption:

$$G_0(x,x',t) = \frac{1}{\sqrt{4\pi Kt}}\exp\left(-\frac{(x-x')^2}{4Kt}\right)$$

The absorption term cause the sol. to decay exponentially over time

$$G_t = KG_{xx} - aG$$

$$\Rightarrow G(x,x',t) = e^{-at}G_0(x,x',t)$$

So $G(x,x',t) = e^{-at}\frac{1}{\sqrt{4\pi Kt}}\exp\left(-\frac{(x-x')^2}{4Kt}\right)$

So $u(x,t) = \int_{-\infty}^{\infty} G(x,x',t) f(x')\,dx'$

$$= e^{-at}\frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-x')^2}{4Kt}\right) f(x')dx'$$