Question 1

Find the asymptotic function for Question 4 in the Homework Problem Set 5 and justify your answer.

Solution.

$$u(t,z)\to U(z) \quad \text{where} \quad U(z) \text{ is the steady solution.}$$

$$\lim_{t\to\infty} u(z,t) = T_1+\phi_2 z + 0 = U(z)$$

Question 2

Find the solution of the nonhomogeneous heat equation

$$u_t = Ku_{zz} + ve^{-at}\sin\frac{\pi z}{L}, \quad 0<z<L, \quad t>0,$$

with $u(0,t)=u(L,t)=u(z,0)=0$. Here a, v, K are positive constants.

Solution. If $a\neq\frac{\pi^2K}{L^2}$, then

$$u(z,t) = -v\sin\frac{\pi z}{L}\,\frac{e^{-at}-e^{-\frac{\pi^2Kt}{L^2}}}{a-\frac{\pi^2K}{L^2}}.$$

If $a=\frac{\pi^2K}{L^2}$, then

$$u(z,t) = v\sin\frac{\pi z}{L}\,te^{-\frac{\pi^2Kt}{L^2}}.$$

Step 1 Solve $\begin{cases} U_{zz}(z,t)=0 \Rightarrow U(z,t)=A(t)+B(t)z \\ U(0,t)=0 \\ U(L,t)=0 \end{cases}$

$\Rightarrow A(t)=0,\ B(t)\cdot L=0 \Rightarrow B(t)=0 \Rightarrow U(z,t)=0$

$\Rightarrow U_t(z,t)=0,\ U(z,0)=0$

$F(z)=f(z)$

Step 2

Define $v(z,t)=r(z,t)-U(z,t)$

$R(z,t)=r(z,t)-U_t(z,t)=r(z,t) = ve^{-at}\sin\frac{\pi z}{L}$

$F(z)=f(z)-U(z,0)=f(z)=0$

We solve $\begin{cases} v_t = Kv_{zz}+ve^{-at}\sin\frac{\pi z}{L} \\ v(0,t)=0 \\ v(L,t)=0 \\ v(z,0)=0 \end{cases}$ for $v(z,t)$

Step 3 We compute the eigenvalue & eigenfunctions of

$A: \varphi \mapsto -\partial_{zz}\varphi$, $\text{Dom}\,A=\{\varphi(z) \mid \varphi(0)=\varphi(L)=0\}$

Here $p(x)=1$, $s(x)=1$,

SL: $A\varphi=\lambda\varphi$

$\Rightarrow -\varphi''=\lambda\varphi$, $\varphi''+\lambda\varphi=0$

$\begin{cases}\varphi(0)=0\\ \varphi(L)=0\end{cases}$

$\Rightarrow \varphi_n(z)=\sin\frac{n\pi z}{L}$, $\varphi_n''(z)=(\frac{n\pi}{L})^2\varphi$

$\Rightarrow \lambda_n=(\frac{n\pi}{L})^2$

Step 4 expand $R=ve^{-at}\sin\frac{\pi z}{L}$ and $F=0$ by eigenfunctions

We have $v(z,t)=\sum_{n=1}^{\infty} v_n(t)\varphi_n(z)$

$R(z,t)=\sum_{n=1}^{\infty} R_n(t)\varphi_n(z)$

$F(z)=\sum_{n=1}^{\infty} F_n(t)\varphi_n(z)$

where $R_n(t)=\dfrac{\int_0^L (ve^{-at}\sin\frac{\pi z}{L})\sin\frac{n\pi z}{L}\,dz}{\int_0^L (\sin\frac{n\pi z}{L})^2\,dz}$, $F_n=0$

$= \begin{cases} ve^{-at}, & n=1 \\ 0, & n\neq 1\end{cases}$ by orthogonality

$\Rightarrow v_n(t) = F_n e^{-\lambda_n Kt} + \int_0^t R_n(s)e^{-\lambda_n K(t-s)}\,ds$

$= \int_0^t R_n(s)e^{-\lambda_n K(t-s)}\,ds$

$= \begin{cases} 0, & n\neq 1 \\ \int_0^t ve^{-at-K(\frac{\pi}{L})^2(t-s)}\,ds, & n=1\end{cases}$

$\Rightarrow u(z,t) = v(z,t)+0 = \sum_{n=1}^{\infty} v_n(t)\varphi_n(z)$

$= v_1(t)\sin\frac{\pi z}{L}$

$a=\frac{\pi^2K}{L^2} \Rightarrow v_1(t)=vte^{-\frac{\pi^2Kt}{L^2}}$

$a\neq\frac{\pi^2K}{L^2} \Rightarrow v_1(t)=\dfrac{v}{\frac{\pi^2K}{L^2}-a}\left(e^{-at}-e^{-\frac{\pi^2Kt}{L^2}}\right)$

Therefore $u(z,t)=\begin{cases} vte^{-\frac{\pi^2Kt}{L^2}}\sin(\frac{\pi z}{L}), & \text{if } a\neq\frac{\pi^2K}{L^2} \\ \dfrac{v}{\frac{\pi^2K}{L^2}-a}\left(e^{-at}-e^{-\frac{\pi^2Kt}{L^2}}\right)\sin(\frac{\pi z}{L}), & \text{if } a\neq\frac{\pi^2K}{L^2}\end{cases}$

Question 3

Find the solution of the nonhomogeneous heat equation

$$u_t = u_{xx}+\frac{1}{2}e^{-t}, \quad 0<x<1, \quad t>0,$$

$$u(0,t)=\frac{1}{2}e^{-t}, \quad u(1,t)=\frac{1}{2}e^{-t},$$

$$u(x,0)=x+\frac{1}{2}.$$

Solution.

$$u(x,t)=\frac{1}{2}e^{-t}+\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}e^{-n^2\pi^2t}\sin n\pi x+\sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)}\left(e^{-t}-e^{-n^2\pi^2t}\right)\sin n\pi x.$$

Step 1 Solve $\begin{cases} U_{xx}(z,t)=0,\ 0<x<1 \Rightarrow U(x,t)=A(t)x+B(t) \\ U(0,t)=\frac{1}{2}e^{-t} \Rightarrow B(t)=\frac{1}{2}e^{-t} \\ U(1,t)=\frac{1}{2}e^{-t} \Rightarrow A(t)+B(t)=\frac{1}{2}e^{-t} \Rightarrow A(t)=0\end{cases}$

$\Rightarrow U(x,t)=\frac{1}{2}e^{-t}$

Step 2 Define: $v(x,t)=u(x,t)-U(x,t)=u(x,t)-\frac{1}{2}e^{-t}$

$R(x,t)=r(x,t)-U_t(x,t)=e^{-t}$

$F(x)=u(x,0)-U(x,0)=x+\frac{1}{2}-\frac{1}{2}=x$

We solve $\begin{cases} v_t=v_{xx}+e^{-t},\ 0<x<1,\ t>0 \\ v(0,t)=v(1,t)=0,\ t>0 \\ v(x,0)=x,\ 0<x<1\end{cases}$

Step 3 Define $A:\varphi\mapsto -\varphi''$

Consider SL problem $\begin{cases}\varphi''+\lambda\varphi=0,\ 0<x<1 \\ \varphi(0)=\varphi(1)=0\end{cases}$

We get $\varphi_n(x)=\sin(n\pi x) \Rightarrow \lambda_n=(n\pi)^2$

Step 4 expand $v(x,t)$, $R(x,t)$, $F(x)$ by eigenfunction

$$V(x,t)=\sum_{n=1}^{\infty} v_n(t)\varphi_n(x),\quad R(x,t)=\sum_{n=1}^{\infty} R_n(t)\varphi_n(x),$$

$$F(x)=\sum_{n=1}^{\infty} F_n\varphi_n(x)$$

where $F_n = \dfrac{\int_0^1 F(x)\varphi_n(x)\,dx}{\int_0^1 \varphi_n^2(x)\,dx} = 2\int_0^1 x\sin(n\pi x)\,dx$

$$\int_0^1 x\sin(n\pi x)\,dx = \frac{-x\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{\cos(n\pi)}{-n\pi} + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1$$

$$= -\frac{(-1)^n}{n\pi}$$

$$\Rightarrow F_n = \frac{2(-1)^{n+1}}{n\pi}$$

Similarly, $R_n(t) = 2e^{-t}\int_0^1 \sin(n\pi x)\,dx = \dfrac{2(1-(-1)^n)e^{-t}}{n\pi}$

So $V_n(t) = F_n e^{-\lambda_n kt} + \int_0^t R_n(s)e^{-\lambda_n k(t-s)}\,ds$

$$= \frac{2(-1)^{n+1}}{n\pi}e^{-n^2\pi^2 t} + \int_0^t e^{-(n\pi)^2(t-s)}\,ds$$

$$= \frac{2(-1)^{n+1}e^{-n^2\pi^2 t}}{n\pi} + \frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)}\left(e^{-t}-e^{-n^2\pi^2 t}\right)$$

Therefore

$$u(x,t) = U(x,t)+V(x,t) = U(x,t)+\sum_{n=1}^{\infty} v_n(t)\varphi_n(x)$$

$$= \frac12 e^{-t} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}e^{-n^2\pi^2 t}}{n\pi} + \frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)}\left(e^{-t}-e^{-n^2\pi^2 t}\right)\right)$$

Question 4

The energy of a vibrating string of tension T_0 and density $\rho = \frac{m}{L}$ is defined by

$$E = \frac12\int_0^L \left(\rho y_t^2 + T_0 y_s^2\right) ds.$$

Let

$$y(s,t) = \sum_{n=1}^{\infty}\left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)\sin\frac{n\pi s}{L}$$

be a solution of the wave equation with $\omega_n = \frac{n\pi c}{L}$, where $c^2 = \frac{T_0}{\rho}$. Show that E is independent of t (conservation of energy) by using Parseval's theorem to write E as an infinite series involving A_n, B_n.

Solution.

$$E = \frac{L}{4}\sum_{n=1}^{\infty}\left(\rho\omega_n^2 B_n^2 + T_0\left(\frac{n\pi}{L}\right)^2 A_n^2\right).$$

Pf $y_t = \sum_{n=1}^{\infty}\left(-A_n\omega_n\sin(\omega_n t) + B_n\omega_n\cos(\omega_n t)\right)\sin\left(\frac{n\pi s}{L}\right)$

By Parseval's Thm for Fourier sine series.

① $\dfrac1L\int_0^L y_t^2\,ds = \dfrac12\sum_{n=1}^{\infty}\left(-A_n\omega_n\sin(\omega_n t)+B_n\omega_n\cos(\omega_n t)\right)^2$

$$= \frac12\sum_{n=1}^{\infty}\Big(\omega_n^2A_n^2\sin^2(\omega_n t)+\omega_n^2B_n^2\cos^2(\omega_n t)\underbrace{-2A_nB_n\sin\omega_n t\cos\omega_n t}_{=0 \text{ by orthogonality}}\Big)$$

$$= \frac12\sum_{n=1}^{\infty}\omega_n^2\left(A_n^2\sin^2(\omega_n t)+B_n^2\cos^2(\omega_n t)\right)$$

$$y_s = \sum_{n=1}^{\infty}\left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)\frac{n\pi}{L}\cos\frac{n\pi s}{L}$$

By Parseval's Thm for Fourier cosine series.

② $\dfrac1L\int_0^L y_s^2\,ds = \dfrac12\sum_{n=1}^{\infty}\left(\dfrac{n\pi}{L}\right)^2\left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)^2$

$$= \frac12\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(A_n^2\cos^2\omega_n t + B_n^2\sin^2\omega_n t\right)$$

$$\Rightarrow E = \frac{L}{2}\left(\rho① + T_0②\right) = \frac{L}{4}\sum_{n=1}^{\infty}\left(T_0\left(\frac{n\pi}{L}\right)^2A_n^2 + \rho\omega_n^2B_n^2\right)$$

Question 5

Consider the following initial-value problem for the wave equation $y_{tt} = c^2 y_{ss}$ for $t>0$, $0<s<L$ with $y(0,t) = y(L,t) = 0$ for $t>0$ and $y(s,0)=0$, $y_t(s,0)=1$ for $0<s<L$. Find the Fourier representation of the solution.

Solution.

$$y(s,t) = \frac{2L}{\pi^2 c}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin\frac{n\pi s}{L}\sin\frac{n\pi ct}{L}.$$

Sol Suppose $y(s,t) = X(s)T(t)$

$$\Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = -\lambda \text{ for some } \lambda$$

① $X''(s)+\lambda X(s)=0$, ② $T''(t)+c^2\lambda T(t)=0$

For ①, by the boundary conditions $X(0)=X(L)=0$

$$\Rightarrow X_n(s) = \sin\left(\frac{n\pi s}{L}\right),\quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Then for ② we have $T_n(t) = A_n\cos(\sqrt{\lambda_n}ct) + B_n\sin(\sqrt{\lambda_n}ct)$

$$= A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)$$

By $T(0)=0 \Rightarrow A_n=0$

$$\Rightarrow y(s,t) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi s}{L}\right)$$

$$y_t(s,t) = \sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi s}{L}\right)$$

$$y_t(s,0) = 1 \Rightarrow \sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\sin\frac{n\pi s}{L} = 1$$

Let $m\in\mathbb{N}$. Multiply both sides by $\sin\frac{m\pi s}{L}$ and integrate $\Rightarrow$

$$\int_0^L y_t(s,0)\sin\frac{m\pi s}{L}\,ds = \sum_{n=1}^{\infty} B_n\frac{n\pi c}{L}\int_0^L \sin\frac{n\pi s}{L}\sin\frac{m\pi s}{L}$$

By orthogonality $\Rightarrow \dfrac{L}{2}B_m\dfrac{m\pi c}{L} = \int_0^L \sin\left(\dfrac{m\pi s}{L}\right)ds = \dfrac{L}{m\pi}\left(1-(-1)^m\right)$

$$\Rightarrow B_m = \frac{2L}{(m\pi)^2c}\left(1-(-1)^m\right)$$

Thus $y(s,t) = \sum_{n=1}^{\infty}\dfrac{2L}{(n\pi)^2c}\left(1-(-1)^n\right)\sin\dfrac{n\pi s}{L}\sin\dfrac{n\pi ct}{L}$.

$$= \frac{2L}{\pi^2c}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin\frac{n\pi s}{L}\sin\frac{n\pi ct}{L}$$