$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\frac{z}{\sqrt{x^2+y^2}}\right) \\ \varphi = \arctan\left(\frac{y}{x}\right) \end{cases}$$

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = \boxed{\frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin\theta}(\sin\theta\, u_\theta)_\theta + \frac{1}{r^2\sin^2\theta} u_{\varphi\varphi}}$$

Def Legendre polynomial

sol $\Theta_k(\theta) = P_k(\cos\theta)$ to Legendre eq.

$$(\sin\alpha\, \Theta')' + (k(k+1)\sin\alpha)\Theta = 0$$

satisfying $\Theta(-\pi) = \Theta(\pi)$, $\Theta'(-\pi) = \Theta'(\pi)$ 称为 Legendre polynomial

Legendre eq is equiv to:

(2) $\Theta'' + \cot\alpha\, \Theta' + k(k+1)\Theta = 0$

(3) $((1-s^2)y')' + k(k+1)y = 1$

(4) $(1-s^2)y'' - 2sy' + k(k+1)y = 0$

(set $s = \cos\alpha$)

(1) $\Leftrightarrow$ (2):

Pf $(\sin(\theta)\, \varphi'(\theta))' + k(k+1)\sin\theta\, \varphi(\theta) = 0$

$\Leftrightarrow \cos\theta\, \varphi'(\theta) + \sin\theta\, \varphi''(\theta) + k(k+1)\sin\theta\, \varphi(\theta) = 0$

$\Leftrightarrow \frac{\cos\theta}{\sin\theta}\varphi'(\theta) + \varphi''(\theta) + k(k+1)\varphi(\theta) = 0$

$\Leftrightarrow \varphi''(\theta) + \cot\theta\, \varphi'(\theta) + k(k+1)\varphi(\theta) = 0$

(3) $\Leftrightarrow$ (4):

Pf $((1-s^2)y')' + k(k+1)y = 0$

Since $((1-s^2)y')' = \frac{d}{ds}[(1-s^2)y'] = -2sy' + (1-s^2)y''$

$\Leftrightarrow (1-s^2)y'' - 2sy' + k(k+1)y = 0$

(2) $\Leftrightarrow$ (3):

Pf Let $s = \cos\theta \Rightarrow \theta = \arccos s$, $y(s) = \varphi(\theta)$

$$\frac{dy}{d\theta} = y'(s)(-\sin\theta), \quad \frac{d^2y}{d\theta^2} = -\cos\theta\, y'(s) - \sin\theta \underbrace{\frac{dy'(s)}{d\theta}}_{= y''(s)(-\sin\theta)}$$

$$= -\cos\theta\, y'(s) + \sin^2(\theta) y''(s)$$

So eq(2) $\Leftrightarrow$

$y''(s)\sin^2\theta - y'(s)\cos\theta + \cot\theta(-\sin\theta\, y'(s)) + k(k+1)y(s) = 0$

$\Leftrightarrow y''(s)\sin^2(\theta) - 2y'(s)\cos\theta + k(k+1)y(s) = 0$

$\Leftrightarrow (1-s^2)y''(s) - 2sy'(s) + k(k+1)y(s) = 0$

note $((1-s^2)y')' = \frac{d}{ds}[(1-s^2)y'] = -2sy' + (1-s^2)y''$

$\Leftrightarrow (1-s^2)y'' - 2sy' + k(k+1)y = 0$

Prop Let $s = \cos\alpha \Rightarrow$

$P(s)$ is a $\boxed{\text{polynomial in } s \text{ of degree } k}$

Pf Suppose the Taylor series of $P(s)$ is $P(s) = \sum_{n=0}^{\infty} a_n s^n$

$\Rightarrow y' = \sum_{n=0}^{\infty} n a_n s^{n-1}$

$\Rightarrow (1-s^2)y' = \sum_{n=0}^{\infty} n a_n s^{n-1} - \sum_{n=0}^{\infty} n a_n s^{n+1} = \sum_{n=-2}^{-1} (n+2)a_{n+2} s^{n+1} + \sum_{n=0}^{\infty}((n+2)a_{n+2} - na_n)s^{n+1}$

$\Rightarrow ((1-s^2)y')' + k(k+1)y = \sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + (k(k+1) - n(n+1))a_n]s^n = 0$

$\Rightarrow a_{n+2} = \frac{(n-k)(n+k+1)}{(n+1)(n+2)} a_n$ ∎

4.2 Rodrigue formula

$$P_k(s) = \frac{1}{2^k k!}\left(\frac{d}{ds}\right)^k (s^2-1)^k$$

Lemma if $j < k \Rightarrow \left(\frac{d}{ds}\right)^j (s^2-1)^k \Big|_{s=\pm 1} = 0$

Pf it is proportional to the j^{th} Taylor coeff of (s^2-1) at $s=1$

and $(s^2-1)^k = 2^k(s-1)^k(1+\frac{s-1}{2})^k \Rightarrow j^{th}$ Taylor coeff is 0

Lemma (important in computing Legendre)

$$\left(\frac{d}{ds}\right)^k \Big|_{s=0} s^l = l!\,\delta_{kl}$$

Prop Recursion formula for P_k

$$nP_n(s) = (2n-1)sP_{n-1}(s) - (n-1)P_{n-2}(s),$$
$$P_0(s) = 1, \quad P_1(s) = s$$

Orthogonality

SL problem: $\begin{cases} (\sin(\alpha)\Theta')' + k(k+1)(\sin\alpha)\Theta = 0 \\ \Theta(0), \Theta(\pi) < \infty \end{cases}$

Equivly $\begin{cases} ((1-s^2)y')' + k(k+1)y = 0 \\ y(-1), y(1) < \infty \end{cases}$ Sol: $y(s) = P_k(s)$

Applying SL Thm $\Rightarrow \int_{-1}^{1} P_k(s)P_q(s)\,ds = \int_0^\pi P_k(\cos\theta)P_q(\cos\theta)\sin\theta\,d\theta = \frac{2}{2k+1}\delta_{kq}$

Pf 0 part is by SL Thm

It suffices to show: $\int_{-1}^{1}(P_k(s))^2 ds = \frac{2}{2k+1}$ ∎

$$\int_{-1}^{1}(P_k(s))^2 = \frac{1}{2^{2k}(k!)^2}\int_{-1}^{1}\left(\left(\frac{d}{ds}\right)^k(s^2-1)^k\right)^2 ds$$
$$= \frac{(2k)!}{2^{2k}(k!)^2}\int_{-1}^{1}(s^2-1)^k ds \quad \text{(by int by part } k \text{ times)}$$

note: $\boxed{\left(\frac{d}{ds}\right)^{2k}(s^2-1)^k = (2k)!}$ ∎

and $\int_{-1}^{1}(s-1)^k(s+1)^k = \frac{(k!)^2}{(2k)!}\frac{2^{2k+1}}{2k+1}$ by int by part k times

$$\Rightarrow \int_{-1}^{1}(P_k(s))^2 ds = \frac{2k}{2^{2k}(k!)^2}\frac{(k!)^2}{(2k)!}\frac{2^{2k+1}}{2k+1} = \frac{2}{2k+1}$$

Note P_k is even function for even k

odd ~ for odd k ∎

4.3 Fourier-Legendre series

pws $\Rightarrow f(\theta) = \sum_{n=0}^{\infty} A_k P_k(\cos\theta)$, $0<\theta<\pi$; $\quad f(s) = \sum_{n=0}^{\infty} A_k P_k(s)$, $-1<s<1$

$$A_k = \frac{2k+1}{2}\int_0^{\pi} f(\theta) P_k(\cos\theta)\sin\theta\, d\theta \qquad A_k = \frac{2k+1}{2}\int_0^{\pi} f(s) P_k(s)\, ds$$

ex compute F-L series of $f = \begin{cases} 1, & s\in(0,1) \\ -1, & s\in(-1,0) \end{cases}$

$$f(s) = \sum_{k=0}^{\infty} a_k P_k(s), \quad a_k = \frac{2k+1}{2}\int_{-1}^{1} f(s) P_k(s)\, ds$$

Since f odd $\Rightarrow a_{2m} = 0 \Rightarrow f(s) = \sum_{m=0}^{\infty} a_{2m+1} P_{2m+1}(s)$

$$a_{2m+1} = \frac{4m+3}{2}\int_{-1}^{1} P_{2m+1}(s)\, ds = 4m+3\int_0^1 P_{2m+1}$$

$$\boxed{\int_0^1 P_{2m+1}(s)\,ds} = \frac{1}{2^{2m+1}(2m+1)!}\int_0^1 \left(\frac{d}{ds}\right)^{2m+1}(s^2-1)^{2m+1}\, ds$$

$$= \sim \left[\left(\frac{d}{ds}\right)^{2m}(s^2-1)^{2m+1}\right]_0^1 = \sim \left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}(s^2-1)^{2m+1}$$

note: $(a+b)^k = \sum_{l=0}^{k}\binom{k}{l}a^l b^{k-l}$, $\left(\frac{d}{ds}\right)^k\Big|_{s=0} s^l = l!\,\delta_{kl}$

$$\Rightarrow = \sim\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\left(\sum_{l=0}^{2m+1}\binom{2m+1}{l}(-1)^{2m+1-l}s^{2l}\right)$$

$$= \sim\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\binom{2m+1}{m}(-1)^{m+1}s^{2m}$$ only this not 0.

$\left(\frac{d}{ds}\right)^{2m} s^{2m} = (2m)!$ $\Rightarrow$ it $= \frac{1}{2^{2m+1}(2m+1)!}\binom{2m+1}{m}(-1)^{m+1}(2m)!$

4.4 Laplace eq in 3D $(0\le r<a,\ 0\le\alpha\le\pi)$

$$\begin{cases} \Delta u = 0 \Rightarrow \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin\theta}(\sin\theta\, u_\theta)_\theta = 0 \\ u(a,\alpha) = G(\theta): \begin{cases} 0, & 0<\alpha<\frac{\pi}{2} \\ 1, & \frac{\pi}{2}<\alpha<\pi \end{cases} \end{cases}$$

$$u = R\alpha,\ \frac{(r^2R')'}{R} = -\frac{(\sin(\alpha)\Theta')'}{\sin(\alpha)\Theta} = \mu$$

$$\Rightarrow \begin{cases} (\sin\alpha\,\Theta')' + \mu\sin\alpha\,\Theta = 0, \quad \Theta(0), \Theta(\pi) < \infty & ① \\ (r^2R')' - \mu R = 0 & ② \end{cases}$$

change of var: let $r = e^s \Rightarrow$ ② becomes $R''(s) + R'(s) - \mu R(s) = 0$

The characteristic eq: $x^2 + x - \mu = 0 \Rightarrow x = \frac{-1\pm\sqrt{1+4\mu}}{2}$

So sol: $R = Ae^{x_+ s} + Be^{x_- s} = Ar^{x_+} + Br^{x_-}$

$x_\pm$ are required to be integers to make it smooth

$\Rightarrow \mu = k(k+1)$, $x_+ = k$, $x_- = -k-1$ Finiteness gives $B=0$ otherwise $r^{-(k+1)} \to \infty$ as $r\to 0$

And $\Theta(\alpha) = P_k(\cos\alpha)$

Therefore $u(r,\alpha) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\alpha)$

$u(a,\alpha) = G(\alpha) \Rightarrow G(\alpha) = \sum_{k=0}^{\infty} A_k a^k P_k(\cos\alpha)$

$$\Rightarrow A_k a^k = \frac{2k+1}{2a^k}\int_0^{\pi} G(\alpha)P_k(\cos\alpha)\sin\theta\, d\theta$$

$$= \frac{2k+1}{2a^k}\int_{-1}^{1} G(s)P_k(s)\, ds = \frac{2k+1}{2a^k}\int_0^1 P_k(s)\, ds$$

by Rodrigue, $\int_0^1 P_k(s)\,ds = \frac{1}{2^k k!}\int_0^1 \left(\frac{d}{ds}\right)^k (s^2-1)^k\, ds$

$$= \frac{1}{2^k k!}\left(\frac{d}{ds}\right)^{k-1}\Big|_{s=0}(s^2-1)^k$$

$$= \frac{1}{2^k k!}\left(\frac{d}{ds}\right)^{k-1}\Big|_{s=0}\left(\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}s^{2l}\right)$$

Since $\left(\frac{d}{ds}\right)^k\Big|_{s=0} s^l = l!\,\delta_{kl}$

$\Rightarrow \left(\frac{d}{ds}\right)^{k-1}\Big|_{s=0} s^{2l} = l!\,\delta_{(k-1),2l}$ $\Rightarrow$ if $k=2m$, all vanish; if $k=2m+1$, only m-term supports

$$\Rightarrow \int_0^1 P_{2m+1}(s)\,ds = \frac{1}{2^{2m+1}(2m+1)}\binom{2m+1}{m}(-1)^{m+1}$$

$$\Rightarrow u(r,\theta) = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}(4m+3)}{2^{2m+2}(2m+1)}\binom{2m+1}{m}\left(\frac{r}{a}\right)^{2m+1}P_{2m+1}(s)$$