Question 1. Solve $u_{tt}=u_{xx}$, $-\infty<x<\infty$ with the initial conditions $u(x,0)=0$, $u_t(x,0)=x$ using d'Alembert formula.

Solution. $u(x,t)=xt$.

$$u(x,t)=\frac{1}{2}\left[f(x+t)+f(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t}g(s)\,ds$$
$$=\frac{1}{2}\left[u(x+t,0)+u(x-t,0)\right]+\frac{1}{2}\int_{x-t}^{x+t}u_t(s,0)\,ds$$
$$=\frac{1}{2}\int_{x-t}^{x+t}s\,ds=\frac{1}{2}\left[\frac{1}{2}s^2\right]_{x-t}^{x+t}$$
$$=\frac{1}{4}\left[x^2+2xt+t^2-x^2+2xt-t^2\right]$$
$$=xt$$

Question 2. Solve the following PDEs using the method of characteristics.

1) $u_t+u_x=0$, $-\infty<x<\infty$, $t>0$ with the initial condition $u(x,0)=\cos x$, $-\infty<x<\infty$.

2) Solve $(t+1)u_t+xu_x=0$, $-\infty<x<\infty$, $t>1$ with the initial condition $u(x,0)=x^2$, $-\infty<x<\infty$.

Solution. 1) $u(x,t)=\cos(x-t)$. 2) $u(x,t)=\left(\frac{x}{t+1}\right)^2$.

(1) $u_t+u_x=0$

$a(x,t)=b(x,t)=1$, $c(x,t)=0$

characterstic curve: $\frac{d\tilde{x}}{ds}=1\Rightarrow\tilde{x}=s+A$, $\tilde{x}(0)=x_0\Rightarrow A=x_0$

$\frac{d\tilde{t}}{ds}=1\Rightarrow\tilde{t}=s+B$, $\tilde{t}(0)=0\Rightarrow B=0$

$x=\tilde{x}(s_1)$, $t=\tilde{t}(s_1)\Rightarrow x=s_1+x_0$, $t=s_1\Rightarrow x_0=x-t$

Along the curve, u remains constant

$\Rightarrow u(x,t)=u(x_0,0)=\cos(x_0)=\cos(x-t)$

2) $xu_x+(t+1)u_t=0$

$a=x$, $b=t+1$, $c=0$

Characteristic curve: $\frac{d\tilde{x}}{ds}=x\Rightarrow\tilde{x}=Ae^s$, $\tilde{x}(0)=x_0\Rightarrow A=x_0$

$\frac{d\tilde{t}}{dt}=t+1\Rightarrow\tilde{t}=-1+Be^t$, $\tilde{t}(0)=0\Rightarrow B=1$

for (x,t) on the curve, $x=\tilde{x}(s_1)$, $t=\tilde{t}(s_1)$

$=x_0e^{s_1}$ $\quad=-1+e^{s_1}$

$x_0=\frac{x}{e^{s_1}}=\frac{x}{t+1}$

So $u(x,t)=u(x_0,0)=x_0^2=\frac{x^2}{(t+1)^2}$

Question 3. Solve $u_t-cu_x=x^2$, $-\infty<x<\infty$, $t>0$ with the initial condition $u(x,0)=x$, $-\infty<x<\infty$. using the method of characteristics.

Solution. $u(x,t)=\frac{(x+ct)^3-x^3}{3c}+x+ct$.

Sol $u_t-cu_x=x^2$ $\quad\Rightarrow\begin{cases}\tilde{x}=-cs+x_0\\ \tilde{t}=s\end{cases}$

Characteristic curve:

$\frac{d\tilde{x}}{ds}=-c$, $\frac{d\tilde{t}}{ds}=1$ $\Rightarrow$ $\tilde{x}=-cs+A$, $\tilde{x}(0)=x_0\Rightarrow A=x_0$; $\tilde{t}=s+B$, $\tilde{t}(0)=0\Rightarrow B=0$

$\frac{d\hat{u}}{ds}=\tilde{x}^2$ $\qquad\hat{u}(0)=x_0\Rightarrow C=x_0$

$\frac{d\hat{u}}{ds}=(-cs+x_0)^2\Rightarrow\int_0^s d\hat{u}=\int_0^s(-ct+x_0)^2\,dt+x_0$

$\tilde{u}(s)=\int_0^s c^2t^2-2cx_0t+x_0^2\,dt+x_0$

$=\left[\frac{1}{3}c^2t^3-cx_0t^2+x_0^2t\right]_0^s+x_0$

$=\frac{1}{3}c^2s^3-cx_0s^2+x_0^2s+x_0$

on curve: $x=\tilde{x}(s_1)$, $t=\tilde{t}(s_1)$, $u=\tilde{u}(s_1)$

$=-cs_1+x_0$ $\quad=s_1$ $\quad=\frac{1}{3}c^2s_1^3-cx_0s_1^2+x_0^2s_1+x_0$

$=-ct+x_0$ $\quad=\frac{1}{3}c^2t^3-c(x+ct)t^2+(x+ct)^2t+x+ct$

$=\frac{1}{3}c^2t^3-\cancel{cxt^2}-\cancel{c^2t^3}+x^2t+\cancel{2}cxt^2+\cancel{c^2t^3}+x+ct$

$=\frac{c^2}{3}t^3+x^2t+cxt^2+x+ct$

Thus $u(x,t)$ $=\frac{c^2}{3}t^3+x^2t+cxt^2+x+ct$

Question 4. Solve $u_t+uu_x=u$, $-\infty<x<\infty$, $t>0$ with the initial condition $u(x,0)=x^2$, $-\infty<x<\infty$. using the method of characteristics.

Solution. $u(x,t)=\frac{(\sqrt{1+4(e^t-1)x}-1)^2}{4(e^t-1)^2}e^t$. When you solve the equation $x_0^2+\frac{1}{e^t-1}x_0-\frac{1}{e^t-1}x=0$, you should choose the "−" sign in $x_0=\frac{1\pm\sqrt{1+4(e^t-1)x}}{2(e^t-1)}$. Because if we take $t=0$, the choice of "+" sign will give $x_0=\infty$, which is impossible.

Sol $u_t+uu_x=u$

Characteristic curves

$\frac{d\tilde{x}}{ds}=\tilde{u}$, $\tilde{x}(0)=x_0$

$\frac{d\tilde{t}}{ds}=1$, $\tilde{t}(0)=0$

$\frac{d\hat{u}}{ds}=\tilde{u}$, $\hat{u}(0)=x_0^2$

$\Rightarrow\begin{cases}\tilde{t}(s)=s+B,\ B=0\\ \tilde{u}(s)=Ce^s,\ C=x_0^2\end{cases}$

$\Rightarrow\begin{cases}\tilde{t}(s)=s\\ \tilde{u}(s)=x_0^2e^s\end{cases}$

So $\frac{d\tilde{x}}{ds}=x_0^2e^s\Rightarrow\tilde{x}(s)=\int_0^s x_0^2e^y\,dy+A$

$=x_0^2e^s+A$

$\tilde{x}(0)=x_0\Rightarrow A=x_0-x_0^2$

So $\tilde{x}(s)=x_0^2(e^s-1)+x_0$

On curve: $x=\tilde{x}(s_1)$, $t=\tilde{t}(s_1)$

$=x_0^2(e^{s_1}-1)+x_0$ $\quad=s_1$

$=x_0^2(e^t-1)+x_0\Rightarrow(e^t-1)x_0^2+x_0-x=0$

$x_0=\frac{-1-\sqrt{1+4(e^t-1)}}{2(e^t-1)}$ $\quad$ since $x_0\to\infty$ if $t\to0$ if +

So $u(x,t)=\hat{u}(s_1)=x_0^2e^s=\left(\frac{-1-\sqrt{1+4(e^t-1)}}{2(e^t-1)}\right)^2e^t$