Question 1. Solve $u_{tt} = u_{xx}$, $-\infty < x < \infty$ with the initial conditions $u(x,0) = 0$, $u_t(x,0) = x$ using d'Alembert formula.
Solution. $u(x,t) = xt$.

$$u(x,t) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds$$

$$= \frac{1}{2}\left[u(x+t,0) + u(x-t,0)\right] + \frac{1}{2}\int_{x-t}^{x+t} u_t(s,0)\, ds$$

$$= \frac{1}{2}\int_{x-t}^{x+t} s\, ds \quad = \frac{1}{2}\left[\frac{1}{2}s^2\right]_{x-t}^{x+t}$$

$$= \frac{1}{4}\left[x^2 + 2xt + t^2 - x^2 + 2xt - t^2\right]$$

$$= xt$$

Question 2. Solve the following PDEs using the method of characteristics.

1) $u_t + u_x = 0$, $-\infty < x < \infty$, $t > 0$ with the initial condition $u(x,0) = \cos x$, $-\infty < x < \infty$.
2) Solve $(t+1)u_t + xu_x = 0$, $-\infty < x < \infty$, $t > 1$ with the initial condition $u(x,0) = x^2$, $-\infty < x < \infty$.

Solution. 1) $u(x,t) = \cos(x-t)$. 2) $u(x,t) = \left(\frac{x}{t+1}\right)^2$.

(1) $u_t + u_x = 0$

$a(x,t) = b(x,t) = 1,\ c(x,t) = 0$

characterstic curve: $\frac{d\tilde{x}}{ds} = 1 \Rightarrow \tilde{x} = s + A,\ \tilde{x}(0) = x_0 \Rightarrow A = x_0$

$\frac{d\tilde{t}}{ds} = 1 \Rightarrow \tilde{t} = s + B,\ \tilde{t}(0) = 0 \Rightarrow B = 0$

$x = \tilde{x}(s_1),\ t = \tilde{t}(s_1) \Rightarrow \begin{array}{l} x = s_1 + x_0 \\ t = s_1 \end{array} \Rightarrow x_0 = x - t$

Along the curve, u remains constant

$\Rightarrow u(x,t) = u(x_0, 0) = \cos(x_0) = \cos(x-t)$

2) $xu_x + (t+1)u_t = 0$

$a = x,\ b = t+1,\ c = 0$

Characteristic curve: $\begin{cases} \frac{d\tilde{x}}{ds} = x \Rightarrow \tilde{x} = Ae^s,\ \tilde{x}(0) = x_0 \Rightarrow A = x_0 \\ \frac{d\tilde{t}}{dt} = t+1 \Rightarrow \tilde{t} = -1 + Be^t,\ \tilde{t}(0) = 0 \Rightarrow B = 1 \end{cases}$

for (x,t) on the curve, $x = \tilde{x}(s_1) = x_0 e^{s_1}$, $t = \tilde{t}(s_1) = -1 + e^{s_1}$

$x_0 = \frac{x}{e^{s_1}} = \frac{x}{t+1}$

So $u(x,t) = u(x_0, 0) = x_0^2 = \frac{x^2}{(t+1)^2}$

Question 3. Solve $u_t - cu_x = x^2$, $-\infty < x < \infty$, $t > 0$ with the initial condition $u(x,0) = x$, $-\infty < x < \infty$. using the method of characteristics.

Solution. $u(x,t) = \frac{(x+ct)^3 - x^3}{3c} + x + ct$.

Sol $u_t - cu_x = x^2$ $\Rightarrow \begin{cases} \tilde{x} = -cs + x_0 \\ \tilde{t} = s \end{cases}$

Characteristic curve:

$$\begin{cases} \dfrac{d\tilde{x}}{ds} = -c \\ \dfrac{d\tilde{t}}{ds} = 1 \end{cases} \Rightarrow \begin{cases} \tilde{x} = -cs + A, \quad \tilde{x}(0) = x_0 \Rightarrow A = x_0 \\ \tilde{t} = s + B, \quad \tilde{t}(0) = 0 \Rightarrow B = 0 \end{cases}$$

$$\frac{d\hat{u}}{ds} = \tilde{x}^2 \qquad \hat{u}(0) = x_0 \Rightarrow C = x_0$$

$$\frac{d\hat{u}}{ds} = (-cs + x_0)^2 \Rightarrow \int_0^s d\hat{u} = \int_0^s (-ct + x_0)^2\, dt + x_0$$

$$\tilde{u}(s) = \int_0^s c^2t^2 - 2cx_0t + x_0^2\, dt + x_0$$

$$= \left[\frac{1}{3}c^2t^3 - cx_0t^2 + x_0^2 t\right]_0^s + x_0$$

$$= \frac{1}{3}c^2s^3 - cx_0s^2 + x_0^2 s + x_0$$

on curve: $x = \tilde{x}(s_1)$, $t = \tilde{t}(s_1)$, $u = \tilde{u}(s_1)$

$x = -cs_1 + x_0 = -ct + x_0$, $t = s_1$

$$u = \frac{1}{3}c^2s_1^3 - cx_0s_1^2 + x_0^2 s_1 + x_0$$

$$= \frac{1}{3}c^2t^3 - c(x+ct)t^2 + (x+ct)^2 t + x + ct$$

$$= \frac{1}{3}c^2t^3 - \cancel{cxt^2} - \cancel{c^2t^3} + x^2t + \cancel{2}cxt^2 + \cancel{c^2t^3} + x + ct$$

$$= \frac{c^2}{3}t^3 + x^2t + cxt^2 + x + ct$$

Thus $u(x,t) = \frac{c^2}{3}t^3 + x^2t + cxt^2 + x + ct$

Question 4. Solve $u_t + uu_x = u$, $-\infty < x < \infty$, $t > 0$ with the initial condition $u(x, 0) = x^2$, $-\infty < x < \infty$. using the method of characteristics.

Solution. $u(x,t) = \frac{(\sqrt{1+4(e^t-1)x}-1)^2}{4(e^t-1)^2}e^t$. When you solve the equation $x_0^2 + \frac{1}{e^t-1}x_0 - \frac{1}{e^t-1}x = 0$, you should choose the "$-$" sign in $x_0 = \frac{1 \pm \sqrt{1+4(e^t-1)x}}{2(e^t-1)}$. Because if we take $t = 0$, the choice of "+" sign will give $x_0 = \infty$, which is impossible.

Sol $u_t + uu_x = u$

Characteristic curves

$$\begin{cases} \frac{d\tilde{x}}{ds} = \tilde{u}, & \tilde{x}(0) = x_0 \\ \frac{d\tilde{t}}{ds} = 1, & \tilde{t}(0) = 0 \\ \frac{d\tilde{u}}{ds} = \tilde{u}, & \tilde{u}(0) = x_0^2 \end{cases} \Rightarrow \begin{cases} \tilde{t}(s) = s + B, \ B = 0 \\ \tilde{u}(s) = Ce^s, \ C = x_0^2 \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{t}(s) = s \\ \tilde{u}(s) = x_0^2 e^s \end{cases}$$

So $\frac{d\tilde{x}}{ds} = x_0^2 e^s \Rightarrow \tilde{x}(s) = \int_0^s x_0^2 e^y dy + A$

$= x_0^2 e^s + A$

$\tilde{x}(0) = x_0 \Rightarrow A = x_0 - x_0^2$

So $\tilde{x}(s) = x_0^2(e^s - 1) + x_0$

On curve: $x = \tilde{x}(s_1)$, $t = \tilde{t}(s_1)$

$= x_0^2(e^{s_1} - 1) + x_0 = s_1$

$= x_0^2(e^t - 1) + x_0 \Rightarrow (e^t - 1)x_0^2 + x_0 - x = 0$

$$x_0 = \frac{-1 - \sqrt{1 + 4(e^t - 1)}}{2(e^t - 1)}$$

since $x_0 \to \infty$ if $t \to \infty$ if t

So $u(x,t) = \tilde{u}(s_1) = x_0^2 e^s = \left(\frac{-1 - \sqrt{1+4(e^t-1)}}{2(e^t-1)}\right)^2 e^t$