Solving 1st order PDE by MoC

linear homo

linear (no u) ex: $a = x+1$, $b = -(y+1)$ homo

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = 0 \\ u(x,0) = f(x) \end{cases}$$ ex: $f(x) = x+1$

Sol step1 Characteristic curve

$$\begin{cases} \frac{d\tilde{x}}{ds} = a(\tilde{x},\tilde{y}) \;\; (\tilde{x}+1) \\ \frac{d\tilde{y}}{ds} = b(\tilde{x},\tilde{y}) \;\; -(\tilde{y}+1) \end{cases} \Rightarrow \begin{cases} \tilde{x}(s) = \dots \; (-1+Ae^{s}), & \tilde{x}(0) = x_0 \\ \tilde{y}(s) = \dots \; (-1+Be^{-s}), & \tilde{y}(0) = 0 \end{cases}$$

$\int \frac{1}{y+1} = \int 1$, $-\ln|y+1| = -s$, $y+1 = Be^{-s}$

$A = x_0 + 1$, $B = 1$

step2 for (x,y) on the curve

$$\begin{cases} x = x(s_1) = -1+(x_0+1)e^{s_1} \\ y = y(s_1) = -1+e^{-s_1} \end{cases} \Rightarrow x_0 = \dots(x,y)$$

$(x+1)(y+1) = x_0+1 \Rightarrow x_0 = (x+1)(y+1)-1$

$$u(x,y) = u(x_0,0) = f(x_0) = f(\dots(x,y))$$

$= x_0 + 1 = (x+1)(y+1)$

Thm u is const on one characteristic curve

Linear

$$\begin{cases} a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \\ u(x,0) = f(x) \end{cases}$$ ($a=\gamma$, $b=1$, $c=\alpha$ for ex; $f = \sin x$ for ex)

Step1 Characteristic curve:

$$\begin{cases} \frac{d\tilde{x}}{ds} = a \;(\gamma) \\ \frac{d\tilde{t}}{ds} = b \;(1) \end{cases} \Rightarrow \begin{cases} \tilde{x} = \dots(s) \;(\gamma s + A), & \text{and } \tilde{x}(0) = x_0 \\ \tilde{t} = \dots(s) \;(s+B), & \text{and } \tilde{t}(0) = 0 \end{cases}$$

$\Rightarrow A = x_0$, $\tilde{x} = \gamma s + x_0$; $\Rightarrow B = 0$, $\tilde{t} = s$

On curve $x = \tilde{x}(s_1)$, $t = \tilde{t}(s_1)$ $\Rightarrow$ express x_0, s

$= \gamma s_1 + x_0$, $= s_1$; $x_0 = x - \gamma s_1 = x - \gamma t$, $s = t$

we have $\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = b u_t + a u_x = -cu$ $\;= -\alpha u$

$$\Rightarrow \begin{cases} \frac{du}{ds} + cu = 0, & s > 0 \\ u = f(x_0), & s = 0 \end{cases} \Rightarrow \text{solve } u(s)$$

$u(s) = \sin(x_0)e^{-\alpha s}$

$\Rightarrow u(x,t) = \sin(x-\gamma t)e^{-\alpha t}$

(if $c = 0$, then $u(x,t) = f(x_0) = \sin(x_0) = \sin(x-\gamma t)$

quasi-linear

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \\ u(x, v(x)) = f(x) \end{cases}$$ ex $a = u^2$, $b = 1$, $c = u$; ex $u(x,0) = x$

Step1
$$\begin{cases} \frac{d\tilde{x}}{ds} = a \;(\tilde{u}^2) \\ \frac{d\tilde{y}}{ds} = b \;(1) \\ \frac{d\tilde{u}}{ds} = c \;(\tilde{u}) \end{cases} \quad \begin{cases} \tilde{x}(0) = x_0 \\ \tilde{y}(0) = y_0 \;(0) \\ \tilde{u}(0) = f(x_0) \;(x_0) \end{cases} \Rightarrow \begin{cases} \tilde{x}(s) = x_0 + \frac{1}{2}x_0^2(e^{2s}-1) \\ \tilde{y}(s) = s \\ \tilde{u}(s) = x_0 e^{s} \end{cases}$$

on curve
$$\begin{cases} x = \tilde{x}(s_1) \\ y = \tilde{y}(s_1) \\ u = \tilde{u}(s_1) \end{cases} \Rightarrow \begin{cases} x = x_0 + \frac{1}{2}x_0^2(e^{2s_1}-1) \\ y = s_1, \quad u = x_0 e^{s_1} \end{cases}$$

$\Rightarrow x_0^2 + \frac{2}{e^{2y}-1}x_0 - \frac{1}{e^{2y}-1}x = 0$

$$x_0 = \frac{-2(e^{2y}-1)^{-1} + \sqrt{\frac{4}{(e^{2y}-1)^2} + 4\frac{x}{e^{2y}-1}}}{2}$$ $\left(\frac{-b \pm \sqrt{\Delta}}{2a}, \text{ judge } \pm\right)$

$$x_0 = \frac{-1 + \sqrt{1 + x(e^{2y}-1)}}{e^{2y}-1}$$

$u = x_0 e^{s_1}$
$= \sim e^{y}$

Thm Let $\tilde{u}(s) = u(\tilde{x},\tilde{y}) \Rightarrow \frac{d\tilde{u}}{ds} = c(\tilde{x},\tilde{y},\tilde{u})$

Pf $\frac{d\tilde{u}}{ds} = u_x\frac{d\tilde{x}}{ds} + u_y\frac{d\tilde{y}}{ds} = u_x a + u_y b = c(\tilde{x},\tilde{y},u(\tilde{x},\tilde{y}))$