Question 1. Compute the following quantities.

1) Compute $P_1'(0), P_2'(0), P_3'(0), P_4'(0)$ by differentiating Legendre polynomials.
2) Compute $P_{2n+1}'(0)$ and $P_{2n}'(0) (n = 0, 1, 2, \cdots)$ using Rodrigues' formula.
3) Compute $P_{2n}(0)$ using Rodrigues' formula.

Solution. 1) $P_1'(0) = 1$, $P_2'(0) = 0$, $P_3'(0) = -3/2$, $P_4'(0) = 0$. 2) $P_{2n+1}'(0) = (-1)^n \frac{(2n+2)!}{n!(n+1)!2^{2n+1}}$ and $P_{2n}'(0) = 0$. 3) $P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$.

Sol (1) $P_1(x) = x \Rightarrow P_1'(0) = 1$

$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow P_2'(0) = [3x]_{x=0} = 0$

$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow P_3'(0) = \left[\frac{1}{2}(15x^2-3)\right]_{x=0} = -\frac{3}{2}$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3) \Rightarrow P_4'(0) = \left[\frac{1}{8}(140x^3-60x)\right]_{x=0} = 0$

(2) P_{2n} is an even function $\Rightarrow P_{2n}'$ is odd $\Rightarrow P_{2n}'(0) = 0$

$$P_{2n+1}'(x) = \frac{1}{2^{2n+1}(2n+1)!} \frac{d}{dx}\left[\frac{d^{2n+1}}{dx^{2n+1}}(x^2-1)^{2n+1}\right]$$

$$= \frac{1}{2^{2n+1}(2n+1)!} \frac{d^{2n+2}}{dx^{2n+2}} \left(\sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^{2n+1-k} x^{2k}\right)$$

evaluated at $x=0 \Rightarrow$ all term of deg $> 2n+2$, and all terms of deg $< 2n+2$ vanishing

$$\text{So } P_{2n+1}'(0) = \frac{1}{2^{2n+1}(2n+1)!} \frac{d^{2n+2}}{dx^{2n+2}} \binom{2n+1}{n+1} (-1)^n x^{2n+2}$$

$$= \frac{1}{2^{2n+1}(2n+1)!} \binom{2n+1}{n+1} (-1)^n (2n+2)!$$

$$= \frac{(-1)^n (2n+2)!}{2^{2n+1}(n+1)!n!}$$

(3) (Same as (2))

$$P_{2n}(0) = \frac{1}{2^{2n}(2n)!} \frac{d^{2n}}{dx^{2n}} \binom{2n}{n} (-1)^{2n-n} x^{2n}$$

$$= \frac{1}{2^{2n}(2n)!} \binom{2n}{n} (-1)^n (2n)! = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$$

Question 2. Prove that different form of Legendre equations are equivalent.

1) $(\sin\theta\,\Theta'(\theta))' + k(k+1)\sin\theta\,\Theta(\theta) = 0$.
2) $\Theta''(\theta) + \cot\theta\,\Theta'(\theta) + k(k+1)\Theta(\theta) = 0$.
3) $((1-s^2)y')' + k(k+1)y = 0$.
4) $(1-s^2)y'' - 2sy' + k(k+1)y = 0$.

Solution. 1) $\Leftrightarrow$ 2) and 3) $\Leftrightarrow$ 4) can be proved using straightforward computation. 2) $\Leftrightarrow$ 3) can be proved using the change of variable $s = \cos\theta$.

(1) $\Leftrightarrow$ (2):

Pf $(\sin(\theta)\,\varphi'(\theta))' + k(k+1)\sin\theta\,\varphi(\theta) = 0$

$\Leftrightarrow \cos\theta\,\varphi'(\theta) + \sin\theta\,\varphi''(\theta) + k(k+1)\sin\theta\,\varphi(\theta) = 0$

$\Leftrightarrow \frac{\cos\theta}{\sin\theta}\varphi'(\theta) + \varphi''(\theta) + k(k+1)\varphi(\theta) = 0$

$\Leftrightarrow \varphi''(\theta) + \cot\theta\,\varphi'(\theta) + k(k+1)\varphi(\theta) = 0$

(3) $\Leftrightarrow$ (4):

Pf $((1-s^2)y')' + k(k+1)y = 0$

Since $((1-s^2)y')' = \frac{d}{ds}[(1-s^2)y'] = -2sy' + (1-s^2)y''$

$\Leftrightarrow (1-s^2)y'' - 2sy' + k(k+1)y = 0$

(2) $\Leftrightarrow$ (3):

Pf Let $s = \cos\theta \Rightarrow \theta = \arccos s$, $y(s) = \varphi(\theta)$

$\frac{dy}{d\theta} = y'(s)(-\sin\theta)$, $\frac{d^2y}{d\theta^2} = -\cos\theta\, y'(s) - \sin\theta \frac{dy'(s)}{d\theta}$ $\quad \frac{dy'(s)}{d\theta} = y''(s)(-\sin\theta)$

$= -\cos\theta\, y'(s) + \sin^2(\theta)\, y''(s)$

So eq(2) $\Leftrightarrow$

$y''(s)\sin^2\theta - y'(s)\cos\theta + \cot\theta(-\sin\theta\, y'(s)) + k(k+1)y(s) = 0$

$\Leftrightarrow y''(s)\sin^2(\theta) - 2y'(s)\cos\theta + k(k+1)y(s) = 0$

$\Leftrightarrow (1-s^2)y''(s) - 2sy'(s) + k(k+1)y(s) = 0$

note $((1-s^2)y')' = \frac{d}{ds}[(1-s^2)y'] = -2sy' + (1-s^2)y''$

$\Leftrightarrow (1-s^2)y'' - 2sy' + k(k+1)y = 0$

Question 3. Find the expansion of the following functions in a series of Legendre polynomials.

1) Let $f(s) = 0$ for $-1 < s < 0$ and $f(s) = 1$ for $0 < s < 1$.
2) Let $f(s) = 3s^2 + 4s + 1$.

Solution. 1) Using Rodrigue's formula and integration by parts, we get $f(s) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2k+1}{2k(k+1)} P_k'(0) P_k(s)$, $-1 < s < 1$. 2) $f(s) = 2P_0(s) + 4P_1(s) + 2P_1(s)$. You get this using Rodrigue's formula, but a simpler method is to apply method of undetermined coefficients, set $f(s) = \sum_{i=0}^{2} a_i P_i(s) = a_0 + a_1 s + a_2 \cdot \frac{1}{2}(3s^2 - 1)$ and compare the coefficient of s^i.

(1) $f(s) = \begin{cases} 0, & -1<s<0 \\ 1, & 0<s<1 \end{cases}$

$$f(s) = \frac{1}{2} + \sum_{k=1}^{\infty} C_k P_k(s)$$

$$C_k = \frac{2k+1}{2}\int_{-1}^{1} f(s)P_k(s)\,ds = \frac{2k+1}{2}\int_0^1 P_k(s)\,ds$$

Claim $P_k'(0) = k(k+1)\int_0^1 P_k(s)\,ds$

Pf $\int_0^s \frac{d}{ds}\left[(1-s^2)\frac{dP_k(s)}{ds}\right]ds + k(k+1)\int_0^s P_k(s)\,ds = 0$

$\Rightarrow \left[(1-s^2)\frac{dP_k(s)}{ds}\right]_0^s + k(k+1)\int_0^s P_k(s)\,ds = 0$

$\xRightarrow{\text{eval at } 0} P_k'(0) = k(k+1)\int_0^1 P_k(s)\,ds$

Therefore $C_k = \frac{2k+1}{2k(k+1)} \cdot \frac{2k+1}{2}\int_0^1 P_k(s)\,ds = \frac{2k+1}{2k(k+1)} P_k'(0)$

So $f(s) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2k+1}{2k(k+1)} P_k'(0) P_k(s)$, $-1<s<1$

(2) Set $f(s) = \sum_{i=0}^{2} a_i P_i(s) \Rightarrow a_0 + a_1 s + a_2 \frac{3s^2-1}{2} = 3s^2+4s+1$

$\Rightarrow \left(\frac{3a_2}{2}\right)s^2 + a_1 s + \left(a_0 - \frac{a_2}{2}\right) = 3s^2+4s+1$

$\Rightarrow a_0 = 2, a_1 = 4, a_2 = 2$

So $f(s) = 2P_0(s) + 4P_1(s) + 2P_2(s)$

Question 4. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < a$ satisfying the boundary condition $u(a,\theta) = 1$ if $0 < \theta < \pi/2$ and $u(a,\theta) = 0$ otherwise.

Solution. $u(r,\theta) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(2k+1)P_k'(0)}{2k(k+1)} \left(\frac{r}{a}\right)^k P_k(\cos\theta)$.

Sol $\frac{1}{r^2}\left(r^2\frac{\partial u}{\partial r}\right)_r + \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)_\theta = 0$

Suppose u separable, $u(r,\theta) = R(r)\varphi(\theta)$

get $\frac{(r^2R')'}{R} = -\frac{(\sin\theta\,\varphi')'}{\sin\theta\,\varphi}$

$\begin{cases} (\sin\theta\,\varphi')' + \lambda\sin\theta\,\varphi = 0 \\ (r^2R')' - \lambda R = 0 \end{cases}$

The radical eq: $(r^2R')' - n(n+1)R = 0 \Rightarrow r^2R'' + 2rR' - n(n+1)R = 0$

sol: $R_n = A_n r^n + B_n r^{-(n+1)}$

Since $r^{-(n+1)} \to \infty$ as $r \to 0$ for $n \ge 0 \Rightarrow B_n$ must be 0.

$\Rightarrow$ the general sol is $u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

Apply BV: $f(\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) \Rightarrow A_n = \frac{a_n}{a^n}$

$a_n = \frac{2n+1}{2}\int_0^\pi f(\theta)P_n(\cos\theta)\sin\theta\,d\theta$

Since $f(\theta)$ is 0 from 0 to $\frac{\pi}{2} \Rightarrow a_n = \frac{2n+1}{2}\int_0^{\frac{\pi}{2}} P_n(\cos\theta)\sin\theta\,d\theta$

$= \frac{2n+1}{2}\int_0^1 P_n(x)\,dx = \frac{2n+1}{2}\frac{P_n'(0)}{n(n+1)}$ by problem 3

So $A_n = \frac{(2n+1)P_n'(0)}{2n(n+1)a^n}$, $n \ge 2$

$A_0 = \frac{a_0}{a^0} = \frac{1}{2}$

Therefore $u(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(2n+1)P_n'(0)}{2n(n+1)} \left(\frac{r}{a}\right)^n P_n(\cos\theta)$

Question 5. Find the solution of Laplace's equation $\nabla^2 u = 0$ in the sphere $0 \le r < 1$ satisfying the boundary condition $u = 3z^2 + 4z + 1$ if $r = 1$.

Solution. $u(r,\theta) = 2+4r\cos\theta+r^2(3\cos^2\theta-1)$ in spherical coordinates and $u(x,y,z) = 2+4z+2z^2-x^2-y^2$ in Euclidean coordinates.

Sol Same as problem 4 we get:

$$\begin{cases} (\sin\theta\,\varphi')' + \lambda \sin\theta\,\varphi = 0 \\ (r^2R')' - \lambda R = 0 \end{cases}$$

The radical eq: $(r^2R')' - n(n+1)R = 0 \Rightarrow r^2R'' + 2rR' - n(n+1)R = 0$

sol: $R_n = A_n r^n + B_n r^{-(n+1)}$

Since $r^{-(n+1)} \to \infty$ as $r \to 0$ for $n \geq 0 \Rightarrow B_n$ must be 0.

$\Rightarrow$ the general sol is $u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

Apply BV: $u(1,\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)$

$= f(\theta) = 3\cos^2\theta + 4\cos\theta + 1 = 2P_2(\cos\theta) + 4P_1(\cos\theta) + 2P_0(\cos\theta)$

by problem 3

$= 2 + 4r\cos\theta + r^2(3\cos^2\theta - 1)$

So $A_0 = 2$, $A_1 = 4$, $A_2 = 2$,

$A_n = 0$ for $n \geq 3$

Therefore the general sol is $u(r,\theta) = 2 + 4r\cos\theta + r^2(3\cos^2\theta - 1)$

in Euclidean coord it is: $u(x,y,z) = 2 + 4z + 3z^2 - z^2 - r^2$

$= 2 + 4z + 2z^2 - x^2 - y^2$