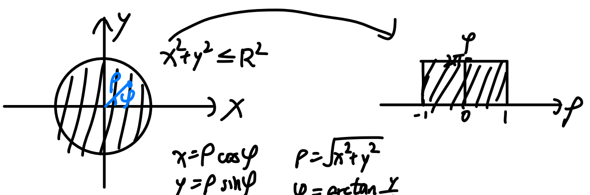
PDE in cylindrical coord

$$(x,y,z) \underset{}{\xrightleftharpoons{\text{change of var}}} (\rho,\varphi,z)$$



$x=\rho\cos\varphi$ $\quad \rho=\sqrt{x^2+y^2}$
$y=\rho\sin\varphi$ $\quad \varphi=\arctan\frac{y}{x}$
$\rho\le R$
$\varphi\in[0,2\pi)$

Compute Δf in terms of ρ,φ
$(\Delta f=\partial_x^2 f+\partial_y^2 f)$

Note $\begin{cases}\partial_x = \cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi \\ \partial_y = \sin\varphi\,\partial_\rho + \frac{\cos\varphi}{\rho}\partial_\varphi\end{cases}$

Pf $\begin{cases}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x} \\ \frac{\partial f}{\partial y} = \cdots \text{ (the same)}\end{cases}$

$\rho_x = \frac{x}{\sqrt{x^2+y^2}} = \cos\varphi, \quad \rho_y = \frac{y}{\sqrt{x^2+y^2}} = \sin\varphi$

$\varphi_x = \partial_x\left(\arctan\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x}\right)^2 = \frac{-y}{x^2+y^2} = \underline{\frac{-\sin\varphi}{\rho}}$

$\varphi_y = \cdots = \underline{\frac{\cos\varphi}{\rho}}$ $\qquad\square$

Prop

$$\Delta f = \frac{1}{\rho}\frac{\partial\left(\rho\frac{\partial f}{\partial\rho}\right)}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}$$

Pf $\Delta f = (\partial_x f)^2 + (\partial_y f)^2$

$$(\partial_x f)^2 = \left(\cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi\right)\left(\cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi\right)f$$
$$= \cos^2\varphi\, f_{\rho\rho} - \frac{\sin\varphi}{\rho}\partial_\varphi(\cos\varphi f_\rho) - \cos\varphi\,\partial_\rho\left(\frac{\sin\varphi}{\rho}f_\varphi\right) + \frac{\sin\varphi}{\rho}\partial_\varphi\left(\frac{\sin\varphi}{\rho}f_\varphi\right)$$
$$= \cos^2\varphi\, f_{\rho\rho} + \cancel{\frac{2\cos\varphi\sin\varphi}{\rho^2}f_\varphi} - \cancel{\frac{2\sin\varphi\cos\varphi}{\rho}f_{\rho\varphi}} + \frac{\sin^2\varphi}{\rho}f_\rho + \frac{\sin^2\varphi}{\rho^2}f_{\varphi\varphi}$$

Dually $\quad +\updownarrow = f_{\rho\rho} \quad +\updownarrow = 0 \quad +\updownarrow = 0 \quad +\updownarrow = \frac{1}{\rho}f_\rho \quad +\updownarrow = \frac{1}{\rho^2}f_{\varphi\varphi}$

$$(\partial_y f)^2 = \sin^2\varphi\, f_{\rho\rho} - \cancel{\frac{2\sin\varphi\cos\varphi}{\rho^2}}f_\varphi + \cancel{\frac{2\sin\varphi\cos\varphi}{\rho}f_{\rho\varphi}} + \frac{\cos^2\varphi}{\rho}f_\rho + \frac{\cos^2\varphi}{\rho^2}f_{\varphi\varphi}$$

So $\Delta f = f_{\rho\rho} + \frac{1}{\rho}f_\rho + \frac{1}{\rho^2}f_{\varphi\varphi} = \underline{\frac{1}{\rho}(\rho f_\rho)_\rho + \frac{1}{\rho^2}f_{\varphi\varphi}}$

$= \frac{1}{\rho}(f_\rho + \rho f_{\rho\rho}) + \frac{1}{\rho^2}f_{\varphi\varphi}$

$= \underline{\frac{1}{\rho}f_\rho + f_{\rho\rho} + \frac{1}{\rho^2}f_{\varphi\varphi}}$

Separation of vars for 2D Laplace

Laplace
$\underline{\Delta u = 0,}\quad a \le \sqrt{x^2+y^2} \le b$

Case $\underline{a=0,\ b=R_2}$

① $\begin{cases}\Delta u = \underline{\frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} = 0}, & 0\le\rho\le R,\ \varphi\in[-\pi,\pi] \\ \underline{u(R_2,\varphi) = u_2(\varphi)}\end{cases}$

$\boxed{u(\rho,\varphi) = R(\rho)\Phi(\varphi)}$

$\Rightarrow 0 = \frac{1}{\rho}(\rho R')'\Phi + \frac{1}{\rho^2}R\Phi''$

$\Rightarrow \rho\frac{(\rho R')'}{R} = \frac{\Phi''}{\Phi} = \lambda$ $\qquad$ (cylindrical 的好处)
↑ $\quad \Rightarrow$ SL

$\begin{cases}\Phi''+\lambda\Phi = 0,\ \underline{\Phi(-\pi)=\Phi(\pi),\ \Phi'(\pi)=\Phi'(-\pi)} & ① \\ \underline{\rho(\rho R')' - \lambda R = 0} & ② \quad R''+\frac{1}{\rho}R' - \frac{\lambda}{\rho^2}R = 0\end{cases}$

① is SL problem

$\Rightarrow \Phi(\pi) = A\cos(\sqrt{\lambda}\varphi) + B\sin(\sqrt{\lambda}\varphi)$

Let $\sqrt{\lambda} = m$ $(\lambda = m^2)$ then $\Phi(\pi) = A\cos m\varphi + B\sin m\varphi$

② $\rho(\rho R')' - m^2 R = 0$

$\frac{df}{d\rho} = \frac{df}{ds}\frac{ds}{d\rho} = \frac{df}{ds}(\ln\rho)' = \frac{1}{\rho}\frac{df}{ds}$

Let $\rho = e^s \Rightarrow \rho\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}R\right) = \frac{d}{d(\underbrace{\ln\rho}_{s})}\left(\frac{d}{d(\underbrace{\ln\rho}_{s})}R\right)$

$\Rightarrow R''(s) - m^2R(s) = 0$

$\underline{R(s) = Ce^{ms} + De^{-ms} = C\rho^m + D\rho^{-m}}$

$\Rightarrow u_m(\rho,\varphi) = \begin{cases}(C_m\rho^m + D_m\rho^{-m})(A_m\cos m\varphi + B_m\sin m\varphi), & m\ne 0 \\ C_0 + D_0\ln\rho, & m=0\end{cases}$

if $D_m \ne 0 \Rightarrow u(0,\varphi) = \infty$
So $D_m = 0$

$\Rightarrow \underline{u(\rho,\varphi) = \sum_{m=0}^{\infty}\rho^m(A_m\cos m\varphi + B_m\sin m\varphi)}$

③ Apply BV:

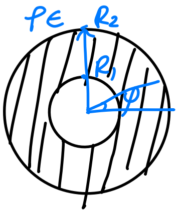
$u(R_2,\varphi) = u_2(\varphi)$

$\Rightarrow u(R_2,\varphi) = \sum_{m=0}^{\infty}R_2^m(A_m\cos m\varphi + B_m\sin m\varphi)$

where $A_m = \frac{1}{\pi R_2^m}\int_{-\pi}^{\pi}u_2(\varphi)\cos m\varphi\,d\varphi$, $A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}u_2(\varphi)\,d\varphi$

$B_m = \frac{1}{\pi R_2^m}\int_{-\pi}^{\pi}u_2(\varphi)\sin m\varphi\,d\varphi$ $\qquad$ ✓

但 if $R_1 \neq 0$:



$$\begin{cases} u(R_1, \varphi) = u_1(\varphi) \\ u(R_2, \varphi) = u_2(\varphi) \end{cases}$$

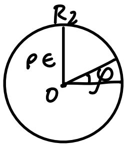
$$\Rightarrow u(\rho, \varphi) = C_0 + D_0 \ln \rho + \sum_{m=1}^{\infty} \left[(C_m \rho^m + D_m \rho^{-m}) \cos m\varphi + (C_m \rho^m + D_m \rho^{-m}) \sin m\varphi \right]$$

Solving $u_m(\rho, \varphi)$

$$m=0: \begin{cases} C_0 + D_0 \ln R_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_1(\varphi) d\varphi \\ C_0 + D_0 \ln R_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_2(\varphi) d\varphi \end{cases}$$

$m \geq 1$: C_m, D_m can be solved by higher order Fourier series

Nonhomogeneous Laplace eq



$$\Delta u = -\lambda u$$

$$\begin{cases} \frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} = -\lambda u \\ u(R_2, \varphi) = u_2(\varphi) \end{cases}$$

$$\Rightarrow \frac{\rho(\rho R')' + \lambda^2 \rho^2 R}{R} = -\frac{\Phi''}{\Phi} = m^2$$

$$\begin{cases} \rho(\rho R')' + (\lambda\rho^2 - m^2) R = 0 & ① \\ \Phi'' + m^2 \Phi = 0, \quad \Phi(\pi) = \Phi(-\pi), \ \Phi'(\pi) = \Phi'(-\pi) & ② \end{cases}$$

$\Rightarrow \Phi(\varphi) = A\cos m\varphi + B\sin m\pi$ by SL (easy)

Def $\rho(\rho J_m')' + (\rho^2 - m^2) J_m = 0$ called Bessel's eq

The sols $J_m(\rho)$ satisfying $J_m(0) < \infty$ is referred to as the Bessel's function.

For ①: $\rho(\rho R')' + (\lambda\rho^2 - m^2) R = 0$

let $x = \sqrt{\lambda}\rho$

$$\Rightarrow \rho \frac{d}{d\rho}\left(\rho \frac{d}{d\rho} R\right) + (x^2 - m^2) R = 0$$

note : $\rho \frac{df}{d\rho} = x \frac{df}{dx}$

since $\frac{df}{d\rho} = \frac{df}{dx} \frac{dx}{d\rho} = \sqrt{\lambda} \frac{df}{dx}$

$$\Rightarrow x \frac{d}{dx}\left(x \frac{d}{dx} R\right) + (x^2 - m^2) R = 0$$

which is exactly the Bessel's eq

So $R_m = J_m(x) = J_m(\sqrt{\lambda}\rho)$

$$\Rightarrow u_m(\rho, \varphi) = J_m(\sqrt{\lambda}\rho)(A_m \cos m\varphi + B_m \sin m\varphi)$$

$$u(\rho, \varphi) = \sum_{m=0}^{\infty} J_m(\sqrt{\lambda}\rho)(A_m \cos m\varphi + B_m \sin m\varphi)$$

BV:

$$u_2(\varphi) = \sum_{m=0}^{\infty} J_m(\sqrt{\lambda} R_2)(A_m \cos m\varphi + B_m \sin m\varphi)$$

$\Rightarrow A_m J_m(\sqrt{\lambda} R)$ is the Fourier coeff of $u_2(\varphi)$

本课 summary: ① homogeneous Laplace eq in cylin coord: solved by SL + Fourier expansion

② nonhomogeneous Laplace eq ～ : sols are the Bessel's functions