hw 1

Question 1. Classify each of the following second-order equations as elliptic, parabolic, or hyperbolic.

1) $u_{xx} + 3u_{xy} + u_{yy} + 2u_x - u_y = 0$
2) $u_{xx} + 3u_{xy} + 8u_{yy} + 2u_x - u_y = 0$
3) $u_{xx} - 2u_{xy} + u_{yy} + 2u_x - u_y = 0$
4) $u_{xx} + xu_{yy} = 0$

Solution: 1) hyperbolic, 2) elliptic, 3) parabolic, 4) elliptic if $x > 0$ and hyperbolic if $x < 0$.

1) $\Delta = b^2 - 4ac = 9 - 4 = 5 > 0 \Rightarrow$ hyperbolic

2) $\Delta = b^2 - 4ac = 9 - 32 = -23 < 0 \Rightarrow$ elliptic

3) $\Delta = b^2 - 4ac = 4 - 4 = 0 \Rightarrow$ parabolic

4) $\Delta = b^2 - 4ac = -4x \Rightarrow$ elliptic for $x > 0$
and hyperbolic for $x < 0$
and parabolic for $x = 0$

Question 2. Prove the following claims.

1) Prove the formulas $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ and $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.
2) Use Euler's formula to prove $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\cos(x+y) = \cos x \cos y - \sin x \sin y$ similarly.
3) Use Euler's formula to prove that all the real functions in $C_+e^{ix} + C_-e^{-ix}$ is of the form $A \cos x + B \sin x$.

Solution: 1) $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2} = \sinh(x+y)$. $\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x - e^{2}}{2}\frac{e^y - e^{-y}}{2} = \cosh(x+y)$.

2) $\sin x \cos y + \cos x \sin y = \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} - e^{-iy}}{2i} = \sin(x+y)$. $\cos x \cos y + \sin x \sin y = \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} - e^{-iy}}{2i} = \cos(x+y)$.

Pf 1) $\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2}\frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + \cancel{e^{x-y}} - \cancel{e^{y-x}} - e^{-y-x} + e^{x+y} - \cancel{e^{x-y}} + \cancel{e^{y-x}} - e^{-y-x}}{4}$

$= \frac{2e^{x+y} - 2e^{-y-x}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$

$\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2}\frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2}\frac{e^y - e^{-y}}{2}$

$= \frac{e^{x+y} + \cancel{e^{x-y}} + \cancel{e^{y-x}} + e^{-x-y} + e^{x+y} - \cancel{e^{x-y}} - \cancel{e^{y-x}} + e^{-x-y}}{4}$

$= \frac{2e^{x+y} + 2e^{-y-x}}{4} = \frac{e^{x+y} + e^{-(x+y)}}{2} = \cosh(x+y)$ $\square$

2) $\sin x \cos y + \cos x \sin y = \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2i}\frac{e^{iy} - e^{-iy}}{2}$

$= \frac{e^{i(x+y)} + \cancel{e^{i(x-y)}} - \cancel{e^{i(y-x)}} - e^{i(-x-y)} + e^{i(x+y)} - \cancel{e^{i(x-y)}} + \cancel{e^{i(y-x)}} - e^{i(-x-y)}}{4i}$

$= \frac{2e^{i(x+y)} - 2e^{i(-x-y)}}{4i} = \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} = \sin(x+y)$

$\cos x \cos y - \sin x \sin y = \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} + e^{-iy}}{2} - \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} - e^{-iy}}{2i}$

$= \frac{e^{i(x+y)} + \cancel{e^{i(x-y)}} + \cancel{e^{i(y-x)}} + e^{i(-x-y)} + e^{i(x+y)} - \cancel{e^{i(x-y)}} - \cancel{e^{i(y-x)}} + e^{i(-x-y)}}{4}$

$= \frac{2e^{i(x+y)} + 2e^{-i(x+y)}}{4} = \frac{e^{i(x+y)} + e^{-i(x+y)}}{2} = \cos(x+y)$ $\square$

3) Let $f: \mathbb{R} \to \mathbb{R}$
sending $x \mapsto C_+e^{ix} + C_-e^{-ix}$ for some C_+, C_- be a real function

$\Rightarrow f(x) = C_+(\cos x + i \sin x) + C_-(\cos(-x) + i \sin(-x))$
$= C_+ \cos x + C_+ i \sin x + C_- \cos x - C_- i \sin x$
$= (C_+ + C_-) \cos x + (iC_+ - iC_-) \sin x$

Question 3. Find the general solutions.

1) $y' = ky(1-y)$,
2) $xy' + 4y = x^2$,
3) $y'' + 4y' + 4y = 0$,
4) $y'' + 2y' - 15y = 0$.

c 1) $y(x) = 1/\left(1 + Ce^{-kx}\right)$, 2) $y(x) = x^2/6 + C/x^4$, 3) $y(x) = C_1e^{-2x} + C_2xe^{-2x}$, and 4) $y(x) = C_1e^{3x} + C_2e^{-5x}$.

1) $\int \frac{dy}{y(1-y)} = \int k\,dx \Rightarrow \int \left(\frac{1}{y} - \frac{1}{1-y}\right) dy = \int k\,dx$

$\Rightarrow \ln\left|\frac{y}{1-y}\right| = kx + c \Rightarrow \frac{y}{1-y} = e^{kx+c} = Ce^{kx}$

$\Rightarrow y(x) = \frac{1}{1 + Ce^{-kx}}$ as general solution

2) $y' + \frac{4}{x}y = x$

integrating factor: $\mu(x) = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = x^4$

$\Rightarrow x^4y' + 4x^3y = x^5 \Rightarrow \frac{d}{dx}(x^4y) = x^5 \Rightarrow x^4y = \frac{x^6}{6} + C$

$\Rightarrow y(x) = \frac{x^2}{6} + \frac{C}{x^4}$ as general solution

3) characteristic equation: $\lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = -2$

$\Rightarrow y(x) = (C_1 + C_2x)e^{-2x}$ as general solution.

4) characteristic equation: $\lambda^2 + 2\lambda - 15 = 0$

$\Rightarrow \lambda_1 = -3, \lambda_2 = 5$

$\Rightarrow y(x) = C_1e^{3x} + C_2e^{-5x}$ as general solution

Question 4. Find the separated equations satisfied by $X(x), Y(y)$ for the following PDEs.

1) $u_{xx} - 2u_{yy} = 0$,
2) $u_{xx} + u_{yy} + 2u_x = 0$,
3) $x^2u_{xx} - 2yu_y = 0$,
4) $u_{xx} + u_x + u_y - u = 0$.

Solution: 1) $X'' - 2\lambda X = 0, Y'' - \lambda Y = 0$, 2) $X'' + 2X' + \lambda X = 0, Y'' - \lambda Y = 0$, 3) $x^2X'' - \lambda X = 0, 2yY' - \lambda Y = 0$ 4) $X'' + X' - \lambda X = 0, Y' + (\lambda - 1)Y = 0$.

1) Assume $u = X(x)Y(y)$

$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial^2 u}{\partial y^2} \Rightarrow YX'' = 2XY'' \Rightarrow \frac{X''}{2X} = \frac{Y''}{Y} = \lambda$

$\Rightarrow X'' - 2\lambda X = 0,\ Y'' - \lambda Y = 0$

2) Assume $u = X(x)Y(y)$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0 \Rightarrow YX'' + XY'' + 2YX' = 0$

$\Rightarrow YX'' + 2YX' = -XY'' \Rightarrow \frac{X'' + 2X'}{X} = \frac{Y''}{-Y} = \lambda$

$\Rightarrow X'' + 2X' - \lambda X = 0,\ Y'' + \lambda Y = 0$

3) Assume $u = X(x)Y(y)$

$x^2\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial u}{\partial y} = 0 \Rightarrow x^2YX'' - 2yXY' = 0 \Rightarrow \frac{x^2X''}{X} = \frac{2yY'}{Y} = \lambda$

$\Rightarrow x^2X'' - \lambda X = 0,\ 2yY' - \lambda Y = 0$

4) Assume $u = X(x)Y(y)$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0 \Rightarrow YX'' + YX' + XY' - XY = 0$

$\Rightarrow \frac{X'' + X'}{X} = \frac{Y'}{Y} - 1 = \lambda$

$\Rightarrow X'' + X' - \lambda X = 0,\ Y' - \lambda Y = 0$

Question 5. Find the separated solutions of $u_{xx}+yu_y+u=0$.
Solution:

$$u(x,y)=\begin{cases}\left(A_1e^{x\sqrt{\lambda}}+A_2e^{-x\sqrt{\lambda}}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda>0,\\ (A_1+A_2x)\left(1/|y|\right) & \text{for } \lambda=0,\\ \left(A_1\cos x\sqrt{-\lambda}+A_2\sin x\sqrt{-\lambda}\right)\left(1/|y|^{1+\lambda}\right) & \text{for } \lambda<0.\end{cases}$$

Sol $\frac{\partial^2 u}{\partial x^2}+y\frac{\partial u}{\partial y}+u=0$

Suppose $u=X(x)Y(y)$

The equation becomes $YX''+yXY'+XY=0$

$\Rightarrow YX''=-yXY'-XY$

$\Rightarrow \frac{X''}{X}=-y\frac{Y'}{Y}-1=\lambda,\ \lambda\in\mathbb{R}$

$\Rightarrow \underline{X''-\lambda X=0}, \quad \underline{\frac{Y'}{Y}=-(\lambda+1)\frac{1}{y}}$

$\Rightarrow$ characteristic equation: $t^2-\lambda=0$

$\Rightarrow \int\frac{1}{Y}dY=-(\lambda+1)\int\frac{1}{y}dy$

for $\lambda>0 \Rightarrow t_{1,2}=\pm\sqrt{\lambda}$

$\ln|Y|=-(\lambda+1)\ln|y|$

for $\lambda=0 \Rightarrow t_{1,2}=0$

$Y=|y|^{-(1+\lambda)}$

for $\lambda<0$, $t_{1,2}=\sqrt{-\lambda}\,i$

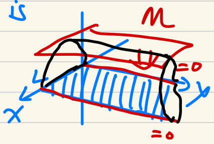
Therefore the separation solutions are:

$$u(x,y)=\begin{cases}\left(A_1e^{x\sqrt{\lambda}}+A_2e^{-x\sqrt{\lambda}}\right)y^{-(1+\lambda)}, & \lambda>0\\ (A_1+A_2x)\frac{1}{|y|}, & \lambda=0\\ \left(A_1e^{ix\sqrt{-\lambda}}+A_2e^{-ix\sqrt{-\lambda}}\right)y^{-(1+\lambda)}, & \lambda<0\end{cases}$$

Question 6. Find the separated solutions $u(x,y)$ of Laplace's equation $u_{xx}+u_{yy}=0$ in the region $0<x<L, y>0$, that satisfy the boundary conditions $u(0,y)=0, u(L,y)=0$ and the boundedness condition $|u(x,y)|\le M$ for $y>0$, where M is a constant independent of (x,y).
Solution:

$$u(x,y)=C\sin\frac{n\pi x}{L}e^{-n\pi y/L}\quad (n=1,2,\ldots)$$

The separated solutions of Laplace's equation is



$$u(x,y)=\begin{cases}(A_1x+A_2)(B_1y+B_2) & ①\\ \left(A_1e^{ikx}+A_2e^{-ikx}\right)\left(B_1e^{ky}+B_2e^{-ky}\right), k>0 & ②\\ \left(A_1e^{lx}+A_2e^{-lx}\right)\left(B_1e^{ily}+B_2e^{-ily}\right), l>0 & ③\end{cases}$$

In case ①, to satisfy $(\forall y,\ u(0,y)=0 \text{ and } u(L,y)=0)$

We have $A_1=B_1=0$ and $A_2B_2=0 \Rightarrow u=0$

In case ②, $\forall y$ $u(0,y)$ and $u(L,y)=0$ implies $X(0)=0, X(L)=0$

we chose a different linearly independently solution form for

X s.t. $\underline{X=A_1\cos(kx)+A_2\sin(kx)}$

$\Rightarrow k=\frac{n\pi}{L}, n\in\mathbb{N} \Rightarrow \underline{X=A_2\sin\frac{n\pi x}{L}, n\in\mathbb{N}}$

Since $|u|$ is bounded and $e^{ky}(k>0)$ is unbounded for $y>0$

$\Rightarrow \underline{B_1=0} \Rightarrow \underline{Y=B_2e^{-\frac{n\pi}{L}y}}$

$\Rightarrow \underline{u=C\sin\frac{n\pi x}{L}e^{-\frac{n\pi y}{L}},\ C\in\mathbb{R}, n\in\mathbb{N}}$

In case ③, same as case ② we have $X(0)=0$, $X(L)=0$,

it only happens when $A_1=A_2=0 \Rightarrow u=0$

Hence overall, the separated solutions to the Laplace's equation for this boundary value problem is

$$u=C\sin\frac{n\pi x}{L}e^{-\frac{n\pi y}{L}},\ C\in\mathbb{R}, n\in\mathbb{N}$$

Question 7. Find the separated solutions $u(x,t)$ of the heat equation $u_t-u_{xx}=0$ in the region $0<x<L, t>0$, that satisfy the boundary conditions $u(0,t)=0, u(L,t)=0$.
Solution:

$$u(x,t)=\begin{cases}\left(A_1e^{kx}+A_2e^{-kx}\right)e^{k^2t} & \text{for } \lambda=k^2, k>0,\\ A_1x+A_2 & \text{for } \lambda=0,\\ \left(A_1e^{ilx}+A_2e^{-ilx}\right)e^{-l^2t} & \text{for } \lambda=-l^2, l>0.\end{cases}$$

The case $\lambda<0$ satisfies the boundary conditions. We obtain

$$u(x,t)=C\sin\frac{n\pi x}{L}e^{-(n\pi/L)^2t}\quad (n=1,2,\ldots).$$

Sol First we look for the general separated solution:

Suppose $u=X(x)T(t)$

Then $\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0 \Rightarrow XT'=TX'' \Rightarrow \frac{X''}{X}=\frac{T'}{T}=\lambda$

$\Rightarrow \underline{X''-\lambda X=0}, \quad \underline{T'-\lambda T=0}$

$\Downarrow \qquad \Downarrow$

$\Rightarrow$ characteristic equation: $x^2-\lambda=0$

$\int\frac{1}{T}dT=\int\lambda dt$

for $\lambda>0 \Rightarrow x_{1,2}=\pm\sqrt{\lambda}$

$\ln|T|=\lambda t+C$

for $\lambda=0 \Rightarrow x_{1,2}=0$

$T=Ce^{\lambda t}$

for $\lambda<0 \Rightarrow x_{1,2}=\sqrt{-\lambda}\,i$

So the general solutions are:

$$u(x,t)=\begin{cases}\left(A_1e^{\sqrt{\lambda}x}+A_2e^{-\sqrt{\lambda}x}\right)e^{\lambda t}, & \lambda>0 \quad ①\\ A_1x+A_2, & \lambda=0 \quad ②\\ \left(A_1e^{i\sqrt{-\lambda}x}+A_2e^{i\sqrt{-\lambda}x}\right)e^{\lambda t}, & \lambda<0 \quad ③\end{cases}$$

In case ①, $(\forall t, u(0,t)=u(L,t)=0)$ implies $X(0)=X(L)=0$,

this can only be true when $A_1=A_2=0 \Rightarrow u=0$

In case ②, $(\forall t, u(0,t)=u(L,t)=0)$ implies $A_1=A_2=0 \Rightarrow u=0$

In case ③, $(\forall t, u(0,t)=u(L,t)=0)$ implies $X(0)=X(L)=0$

we chose a different linearly independently solution form for

X s.t. $\underline{X=A_1\cos(\sqrt{-\lambda}x)+A_2\sin(\sqrt{-\lambda}x)}$

$X(0)=X(L)=0 \Rightarrow A_1=0,\ \sqrt{-\lambda}=\frac{n\pi}{L}, n\in\mathbb{N}$

Hence overall, the separated solutions to the heat equation for this boundary value problem is

$$u=C\sin\frac{n\pi x}{L}e^{-\left(\frac{n\pi}{L}\right)^2t}\ \text{ where } C\in\mathbb{R}, n\in\mathbb{N}$$