hw2

Question 1. Compute the Fourier series of $f(x)=x^2, -L<x<L$.
Solution: $\frac{L^2}{3}+\sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{L}$.

Sol Since it is an even function there are only cos terms in the Fourier series.

$f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos_n$

$=\frac{1}{2L}\int_{-L}^{L}x^2dx+\frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L}x^2\cos\frac{n\pi x}{L}dx\right)\cos\frac{n\pi x}{L}$

$=\frac{1}{3}L^2+\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_{0}^{L}x^2\cos\frac{n\pi x}{L}dx\right)\cos\frac{n\pi x}{L}$

$\int_0^L x^2\cos\frac{n\pi x}{L}dx=\left[x^2\left(\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right)\right]_0^L-\frac{2L}{n\pi}\int_0^L x\sin\frac{n\pi x}{L}dx$

$\int_0^L x\sin\frac{n\pi x}{L}dx=\left[x\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\right]_0^L+\underbrace{\frac{L}{n\pi}\int_0^L\cos\frac{n\pi x}{L}dx}_{=\frac{L^2}{(n\pi)^2}\int_0^{n\pi}\cos t\,dt=}$

$\Rightarrow \int_0^L x\sin\frac{n\pi x}{L}dx=\frac{-L^2}{n\pi}(-1)^n$

$\Rightarrow \int_0^L x^2\cos\frac{n\pi x}{L}dx=0-\frac{2L}{n\pi}\left(\frac{-L^2}{n\pi}(-1)^n\right)=\underline{\frac{2L^3}{n^2\pi^2}(-1)^n}$

$\Rightarrow \underline{f(x)=\frac{1}{3}L^2+\sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{L}}$

Question 2. Compute the Fourier series of $f(x)=e^x, -L<x<L$.
Solution: $\frac{\sinh L}{L}\left[1+2\sum_{n=1}^{\infty}(-1)^n\frac{\cos(n\pi x/L)-(n\pi/L)\sin(n\pi x/L)}{1+(n\pi/L)^2}\right]$.

Sol $f(x)=A_0+\sum_{n=1}^{\infty}(A_n\cos_n+B_n\sin_n)$

$A_0=\frac{1}{2L}\int_{-L}^{L}e^xdx=\frac{1}{2L}(e^L-e^{-L})=\frac{\sinh L}{L}$

$A_n=\frac{1}{L}\int_{-L}^{L}e^x\cos\frac{n\pi x}{L}dx$

$I_{\cos}=\int_{-L}^{L}\overset{f}{e^x}\,\overset{g'}{\cos\frac{n\pi x}{L}}dx=\underbrace{\left[\overset{g}{\frac{L}{n\pi}\sin\frac{n\pi x}{L}}\,\overset{f}{e^x}\right]_{-L}^{L}}_{=0}-\frac{L}{n\pi}I_{\sin}$

$I_{\sin}=\int_{0}^{L}\overset{f}{e^x}\,\overset{g'}{\sin\frac{n\pi x}{L}}dx=\underbrace{\left[\overset{g}{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}}\,\overset{f}{e^x}\right]_{-L}^{L}}_{=\frac{-L}{n\pi}(-1)^n(e^L-e^{-L})}+\frac{L}{n\pi}I_{\cos}$

$\Rightarrow I_{\sin}=\frac{-L}{n\pi}(-1)^n2\sinh(L)+\frac{L}{n\pi}\left(-\frac{L}{n\pi}\right)I_{\sin}$

$\Rightarrow I_{\sin}=\frac{\frac{-L}{n\pi}(-1)^n2\sinh(L)}{1+\left(\frac{L}{n\pi}\right)^2}$

$\Rightarrow I_{\cos}=-\frac{L}{n\pi}I_{\sin}=\frac{\left(\frac{L}{n\pi}\right)^2(-1)^n2\sinh(L)}{1+\left(\frac{L}{n\pi}\right)^2}$

$\Rightarrow f(x)=\frac{\sinh L}{L}+\sum_{n=1}^{\infty}\left(\frac{1}{L}\frac{\left(\frac{L}{n\pi}\right)^2(-1)^n2\sinh(L)}{1+\left(\frac{L}{n\pi}\right)^2}\cos_n+\frac{1}{L}\frac{\frac{-L}{n\pi}(-1)^n2\sinh L}{1+\left(\frac{L}{n\pi}\right)^2}\sin_n\right)$

$=\frac{\sinh L}{L}\left(1+2\sum_{n=1}^{\infty}(-1)^n\frac{\left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L}-\frac{L}{n\pi}\sin\frac{n\pi x}{L}}{1+\left(\frac{L}{n\pi}\right)^2}\right)$

(multiplying $\left(\frac{n\pi}{L}\right)^2$ on nominator & denominator) $=\frac{\sinh L}{L}\left(1+2\sum_{n=1}^{\infty}(-1)^n\frac{\cos\frac{n\pi x}{L}-\frac{n\pi}{L}\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2+1}\right)$

Question 3. Compute the Fourier series of $f(x)=\sin^2 2x, -\pi<x<\pi$.
Solution: $\frac{1}{2}-\frac{1}{2}\cos 4x$.

Sol $f(x)=\sin^2 2x=\frac{1-\cos4x}{2}$, $-\pi<x<\pi$ is an even function, thus only has even terms.

$\Rightarrow f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos_n$ by Fourier cosine series on $(-\pi,\pi)$

$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\cos4x}{2}dx=\frac{1}{16\pi}\int_{-\pi}^{\pi}(1-\cos4x)\,d(4x)$

$=\frac{1}{16\pi}\left[4x-\sin4x\right]_{-\pi}^{\pi}=\frac{8\pi}{16\pi}=\frac{1}{2}$

$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos4x-1}{2}\cos nx\,dx$

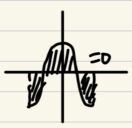
$=\frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos4x\cos nx\,dx}-\frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi}\frac{1}{2}\cos nx\,dx}_{=0}$

$=\int_{-1}^{1}\cos4\pi x\cos n\pi x\,d(\pi x)$

$=0$ unless $n=4$, by orthogonality on $[-\pi,\pi]$

$\int_{-\pi}^{\pi}\cos^2 4x\,dx=\int_{-\pi}^{\pi}\frac{\cos8x-1}{2}dx$

$=\frac{1}{16}\underbrace{\int_{-\pi}^{\pi}\cos t\,dt}_{=0}-\frac{1}{2}\int_{-\pi}^{\pi}1\,dx=-\frac{1}{2}(2\pi)$



$\Rightarrow A_n=-\frac{1}{2}$ for $n=4$ and $A_n=0$ for $n\neq4$

$\Rightarrow f(x)=\frac{1}{2}-\frac{1}{2}\cos\frac{4\pi}{\pi}x=\frac{1}{2}-\frac{1}{2}\cos4x=\frac{1-\cos4x}{2}$ is the Fourier series of f.

(which is itself since f is trigonometric)

Question 4. Prove the orthogonality relations

1) $\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx=\begin{cases}0, & n\neq m \text{ or } n=m=0,\\ L, & n=m\neq 0.\end{cases}$

2) $\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx=0$ for all n,m.

Solution: Recall $\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$ and $\sin(a\pm b)=\sin a\cos b\pm\cos a\sin b$. From these relations, we can derive the trigonometric identities: $\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$, $\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$, and $\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$. By using these identities, we can carry out the integrals in the orthogonality relations.

(1) Pf ① let $n\neq m\in\mathbb{N}$ be arbitrary

$\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx=\int_{-L}^{L}\frac{1}{2}\left(\cos\frac{(n-m)\pi x}{L}-\cos\frac{(n+m)\pi x}{L}\right)dx$

by product-to-sum identity

$=\frac{1}{2}\frac{L}{(n-m)\pi}\int_{-(n-m)\pi}^{(n-m)\pi}\cos\frac{(n-m)\pi x}{L}\,d\frac{(n-m)\pi x}{L}-\frac{1}{2}\frac{L}{(n+m)\pi}\int_{-(n+m)\pi}^{(n+m)\pi}\cos\frac{(n+m)\pi x}{L}\,d\frac{(n+m)\pi x}{L}$

$=\frac{1}{2(n-m)\pi}\int_{-(n-m)\pi}^{(n-m)\pi}\cos t\,dt-\frac{1}{2}\frac{L}{(n+m)\pi}\int_{-(n+m)\pi}^{(n+m)\pi}\cos t\,dt$

Since the upper and lower limit is integer multiples of π, the integrals evaluate to 0

Thus $\int_{-L}^{L}\sin_n\sin_m dx=0$

② If $n=m\neq0\Rightarrow\int_{-L}^{L}\sin_n\sin_m dx=\int_{-L}^{L}\sin_n^2dx=\frac{1}{2}\int_{-L}^{L}\left(1-\cos\frac{2n\pi x}{L}\right)dx$ $=\frac{1}{2}\int_{-L}^{L}1\,dx=\frac{2L}{2}=L$

③ If $n=m=0\Rightarrow\int_{-L}^{L}\sin_n\sin_m dx=\int_{-L}^{L}0\,dx=0$

This finishes the proof that $\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx=\begin{cases}0, & n\neq m \text{ or } n=m=0\\ L, & n=m\neq0\end{cases}$ □

(2) Pf let $n,m\in\mathbb{N}\cup\{0\}$ be arbitrary

$\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx=\int_{-L}^{L}\frac{1}{2}\left(\sin\frac{(n-m)\pi x}{L}-\sin\frac{(n+m)\pi x}{L}\right)dx$

by product-to-sum identity

$$= \frac{1}{2}\int_{-L}^{L} \sin\frac{(n-m)\pi}{L}x\,dx - \frac{1}{2}\int_{-L}^{L}\sin\frac{(n+m)\pi x}{L}\,dx$$

both functions are odd, so the intergrals from $-L$ to L evaluate to 0.

Thus for all $n, m \in \mathbb{N} \cup \{0\}$, $\int_{-L}^{L} \sin_n \cos_m \,dx = 0$ □

Question 5. Which of the following functions are even, odd, or neither? Explain the reason.

1) $f(x) = x^3 - 3x$,
2) $f(x) = x^2 + 4$,
3) $f(x) = \cos 3x$,
4) $f(x) = x^3 - 3x^2$.

Solution: 1): odd; 2), 3): even; 4): neither.

(1) $f(-x) = -x^3 + 3x = -f(-x) \Rightarrow$ odd

(2) $f(-x) = x^2 + 4 = f(x) \Rightarrow$ even

(3) $f(-x) = \cos(-3x) = \cos 3x = f(x) \Rightarrow$ even

(4) $f(-x) = -x^3 - 3x^2 \neq f(x)$ on domain (equal iff $x=0$) $\Rightarrow$ neither

Question 6.

1) Find the Fourier sine series for $f(x) = e^x, 0 < x < L$.
2) Find the Fourier cosine series for $f(x) = e^x, 0 < x < L$.

Solution: 1) We obtain the Fourier sine series $\frac{2\pi}{L^2}\sum_{n=1}^{\infty} n\left[\frac{1-e^L(-1)^n}{1+(n\pi/L)^2}\right]\sin\frac{n\pi x}{L}$ by either directly apply the B_n formula or multiply e^x by $\sin\frac{n\pi x}{L}$ and then apply the orthogonality 2) We obtain the Fourier cosine series $\frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\left[\frac{(-1)^n e^L - 1}{1+(n\pi/L)^2}\right]\cos\frac{n\pi x}{L}$ by either directly apply the A_n formula or multiply e^x by $\cos\frac{n\pi x}{L}$ and then apply the orthogonality.

Sol

$$I_{\cos_n} = \int_0^L e^x \cos\frac{n\pi x}{L}\,dx = \underbrace{\left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}e^x\right]_0^L}_{=0} - \frac{L}{n\pi}I_{\sin_n}$$

$$I_{\sin_n} = \int_0^L e^x \sin\frac{n\pi x}{L}\,dx = \underbrace{\left[\frac{-L}{n\pi}\cos\frac{n\pi x}{L}e^x\right]_0^L}_{=\frac{-L}{n\pi}(-1)^n e^L + \frac{L}{n\pi}} + \frac{L}{n\pi}I_{\cos_n}$$

$$\Rightarrow I_{\sin_n} = \frac{-L}{n\pi}(-1)^n e^L + \frac{L}{n\pi} + \frac{L}{n\pi}\left(-\frac{L}{n\pi}\right)I_{\sin_n}$$

$$\Rightarrow I_{\sin_n} = \frac{\frac{-L}{n\pi}(-1)^n e^L + \frac{L}{n\pi}}{1+\left(\frac{L}{n\pi}\right)^2}$$

$$\Rightarrow I_{\cos_n} = -\frac{L}{n\pi}I_{\sin_n} = \frac{\left(\frac{L}{n\pi}\right)^2(-1)^n e^L - \left(\frac{L}{n\pi}\right)^2}{1+\left(\frac{L}{n\pi}\right)^2}$$

$$A_0 = \frac{1}{L}\int_0^L e^x\,dx = \frac{e^L - 1}{L}$$

$\Rightarrow$ Fourier sine series $f_{\sin}(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos_n$

$$= \frac{e^L-1}{L} + \sum_{n=1}^{\infty}\frac{2}{L}I_{\cos_n}\cos\frac{n\pi x}{L}$$

$$= \frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{\left(\frac{L}{n\pi}\right)^2(-1)^n e^L - \left(\frac{L}{n\pi}\right)^2}{1+\left(\frac{L}{n\pi}\right)^2}\cos\frac{n\pi x}{L}$$

$$= \frac{e^L-1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\frac{(-1)^n e^L - 1}{1+\left(\frac{L}{n\pi}\right)^2}\cos\frac{n\pi x}{L}, \quad x\in(0,L)$$

Fourier cosine series $f_{\cos}(x) = \sum_{n=1}^{\infty} B_n \sin_n$

$$= \frac{2}{L}\sum_{n=1}^{\infty} I_{\sin_n}\sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L}\sum_{n=1}^{\infty}\frac{-\frac{n\pi}{L}(-1)^n e^L + \frac{n\pi}{L}}{\left(\frac{n\pi}{L}\right)^2+1}$$

$$= \frac{2\pi}{L^2}\sum_{n=1}^{\infty} n\,\frac{1-(-1)^n e^L}{\left(\frac{n\pi}{L}\right)^2+1}, \quad x\in(0,L)$$