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Math 454: Boundary Value Problems for Partial Differential Equations
No deadline

Review 1

Question 1. Redo all the problems in HW1 - HW4.

Question 2. Classify each PDE as parabolic, hyperbolic and elliptic,

1) $u_{tt} + 3u_{xx} = u$,
2) $u_{xx} + 2u_{xy} - 4u_{yy} = u_x$,
3) $u_x = 2u_{yy}$,
4) $u_{xx} + x^3 u_{yy} = 0$.

Solution. 1) Elliptic. 2) Hyperbolic 3) Parabolic 4) Elliptic if $x > 0$ and hyperbolic if $x < 0$.

Question 3. Prove the following facts

1) Orthogonality of Fourier, complex Fourier, cosine Fourier and sine Fourier series.
2) Parseval's theorem for Fourier, complex Fourier, cosine Fourier and sine Fourier series.
3) The following operator is symmetric

$$A = -\frac{d^2}{dx^2}.$$

4) Odd functions only have sin terms and even functions only have cos terms.

Solution. TODO:

Question 4. Compute the Fourier, complex Fourier, cosine Fourier and sine Fourier series for the following functions (domain for the first two is $[-\pi, \pi]$, for the last two is $[0, \pi]$)

1) $f(x) = \begin{cases} 1 & \text{if} \quad -\pi < x < 0, \\ 3 & \text{if} \quad 0 < x < \pi. \end{cases}$
2) $f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < h \\ 0 & \text{if} \quad h < x < \pi \ \textit{or} \ -\pi < x < 0 \end{cases}$

Solution. 1) Fourier: $f(x) = 2 + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$, complex Fourier: $f(x) = 2 - \frac{2i}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{(2n+1)xi}}{2n+1}$, cosine Fourier: $f(x) = 3$, sine Fourier: $f(x) = \frac{12}{\pi}\sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$. 2) Fourier: $f(x) = \frac{h}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(nh)}{nh}\cos nx\right\}$, complex Fourier: $f(x) = \frac{h}{2\pi}\left\{1 + \sum_{n\in\mathbb{Z}, n\neq 0} \frac{\sin(nh)}{nh} e^{inx}\right\}$, cosine Fourier: $f(x) = \frac{h}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin nh}{n}\cos nx$, sine Fourier: $f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos nh}{n}\sin nx$.

Question 5. Solve the following problems,

1) Sketch the graph on $[-\pi, \pi]$ of Fourier, complex Fourier, cosine Fourier and sine Fourier series of Question 4 1)
2) Use Parseval's theorem to show that $\sigma_{2N} = O\left(\frac{1}{\sqrt{N}}\right)$ for the Fourier series of Question 4 1).

Solution.

Question 6. Solve the following differential equation

1) $y' = y - x$, $y(1) = 2$.
2) $y' = 4y + 3x$, $y(0) = -1$.
3) $y' = 2xy + x$, $y(0) = 3$.
4) $y'' + 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$.
5) $y'' + 2y' + 2y = 0$ (in terms of cosine and sine).
6) Reduce $y'' + y' + (\lambda x - 1)y = 0$ to the Sturm-Liouville form.

Solution. 1) $y(x) = x + 1$, 2) $y(x) = \frac{1}{16}(-3 - 13e^{4x} - 12x)$, 3) $y(x) = \frac{1}{2}(-1 + 7e^{x^2})$, 4) $y(x) = e^{-x} - e^{-2x}$, 5) $y(x) = Ae^{-x}\cos x + Be^{-x}\sin x$, 6) $(e^x y')' + (\lambda x e^x - e^x)y$, $s(x) = e^x$, $\rho(x) = xe^x$ and $q(x) = e^x$.

Question 7. Using separation of variables to solve the following PDEs

1) Solve the initial-value problem $u_t = Ku_{zz}(K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u_z(0,t) = u_z(L,t) = 0$ and the initial condition $u(z,0) = \frac{1}{2}\cos\frac{\pi z}{L} + \cos^2\frac{\pi z}{L}$, $0 < z < L$.
2) Find the separated solutions $u(x,y)$ of Laplace's equation $u_{xx} + u_{yy} = 0$ in the region $0 < x < L, y > 0$, that satisfy the boundary conditions $u(0,y) = 0$, $u(L,y) = 0$, $u(x,0) = 1$ and the boundedness condition $|u(x,y)| \leq M$ for $y > 0$, where M is a constant independent of (x,y).

Solution. 1) $u(z,t) = \frac{1}{2}\left(1 + e^{-\left(\frac{\pi}{L}\right)^2 Kt}\cos\frac{\pi x}{L} + e^{-\left(\frac{2\pi}{L}\right)^2 Kt}\cos\frac{2\pi x}{L}\right)$.

2) $u(x,y) = \sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi}\sin\frac{n\pi x}{L}e^{-n\pi y/L}$.