$$\begin{cases} x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ z \end{cases} \Leftrightarrow \begin{cases} \rho=\sqrt{x^2+y^2} \\ \varphi=\arctan\frac{y}{x} \\ z \end{cases}$$

$$\partial_x = \cos\varphi\,\partial_\rho - \frac{\sin\varphi}{\rho}\partial_\varphi$$

$$\partial_y = \sin\varphi\,\partial_\rho - \frac{\cos\varphi}{\rho}\partial_\varphi$$

$$\nabla^2 f = \Delta f = f_{xx}+f_{yy} = \frac{1}{\rho}(\rho f_\rho)_\rho + \frac{1}{\rho^2} f_{\varphi\varphi}$$

## 3.1 Solving homo Laplace

$$\begin{cases} \nabla^2 u = 0, \ 0\le\rho\le R, \ \varphi\in[-\pi,\pi] \\ u(R,\varphi)=u_2(\varphi), \ \varphi\in[-\pi,\pi] \end{cases}$$

$$u = R(\rho)\Phi(\varphi) \Rightarrow \frac{1}{\rho}(\rho R')'\Phi + \frac{1}{\rho^2}R\Phi'' \Rightarrow \rho\frac{(\rho R')'}{R} = -\frac{\Phi''}{\Phi} = \lambda$$

$$\begin{cases} \Phi''+\lambda\Phi=0, \ \Phi(-\pi)=\Phi(\pi), \ \Phi'(-\pi)=\Phi'(\pi) \\ \rho(\rho R)'-\lambda R=0 \end{cases}$$

let $\lambda=m^2 \Rightarrow \Phi_m(\varphi)=A_m\cos m\varphi + B_m\sin m\varphi$

let $\rho=e^s$

$$\Rightarrow \rho(\rho R)' - m^2 R = 0 \Rightarrow R_m(s) = Ae^{ms}+Be^{-ms} = C_m\rho^m + D_m\rho^{-m}$$

$$\Rightarrow u_m(\rho,\varphi) = \begin{cases} (C_m\rho^m + D_m\rho^{-m})(A_m\cos m\varphi + B_m\sin m\varphi), & m\in\mathbb{N} \\ C_0+D_0\ln\rho, & m=0 \ (\text{const}) \end{cases}$$

$D_m=0$ since otherwise $|u|\to\infty$ as $\rho\to 0$, not smooth

$$\Rightarrow u(\rho,\varphi) = \sum_{m=0}^{\infty}\rho^m(A_m\cos m\varphi + B_m\sin m\varphi)$$

$$u_2(\varphi) = u(R,\varphi) = A_0 + \sum_{m=1}^{\infty}R^m(A_m\cos m\varphi + B_m\sin m\varphi)$$

$\Rightarrow R^m A_m, R^m B_m$ are fourier coeff

solved by $R^m A_m = \frac{1}{\pi}\int_{-\pi}^{\pi} u_2\cos mx$, $B_m = \frac{1}{\pi}\int_{-\pi}^{\pi} u_2\sin mx$

## 3.2 non-homo case and Bessels

$$\begin{cases} \Delta u = -\lambda u \\ u(R,\varphi)=u_2(\varphi) \end{cases}$$

$$-\lambda R\Phi = \frac{1}{\rho}(\rho R')'\Phi + \frac{1}{\rho^2}R\Phi''$$

$$-\lambda\rho^2 = \frac{\rho(\rho R')'}{R} + \frac{\Phi''}{\Phi} \Rightarrow \frac{\rho(\rho R')'+\lambda\rho^2 R}{R} = -\frac{\Phi''}{\Phi} = m^2$$

$$\Rightarrow \begin{cases} \Phi''+m^2\Phi=0, \ \Phi(-\pi)=\Phi(\pi), \ \Phi'(-\pi)=\Phi'(\pi) \Rightarrow \varphi_m = A_m\cos m\varphi + B_m\sin m\varphi \\ \boxed{\rho(\rho R')' + (\lambda\rho^2 - m^2)R = 0} \end{cases}$$

**Def** Bessel's functions $J_m(x)$ are sols to

$$x(xJ_m')' + (x^2-m^2)J_m = 0$$

(or say $J_m'' + \frac{1}{x}J_m' + (1-\frac{m^2}{x^2})J_m = 0$)

Let $\gamma = \sqrt{\lambda}\rho \Rightarrow J_m(\gamma) = R_m(\rho)$ is sol to it

$$\Rightarrow u(\rho,\varphi) = \sum_{m=0}^{\infty} J_m(\sqrt{\lambda}\rho)(A_m\cos m\varphi + B_m\sin m\varphi)$$

and $u_2(\varphi) = \sum_{m=0}^{\infty} J_m(\sqrt{\lambda}R)(A_m\cos m\varphi + B_m\sin m\varphi)$

$A_m J_m(\sqrt{\lambda}R), B_m J_m(\sqrt{\lambda}R)$ are Fourier coeff of $u_2(\varphi)$

(solved by $\frac{1}{\pi}\int_{-\pi}^{\pi}\dots$)

## 3.3 Integral formula

$$J_m(x) = \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{ix\cos\theta}e^{-im\theta}d\theta, \quad m\in\mathbb{N}\cup\{0\}$$

Pf $e^{i\rho\cos\varphi}$ is sol to $-\Delta u = u$

$$\Rightarrow e^{i\rho\cos\varphi} = \sum_{m=0}^{\infty} J_m(\rho)(\alpha_m e^{im\varphi} + \beta_m e^{-im\varphi})$$

$\Rightarrow \alpha_m J_m(\rho)$ is Fourier coeff of $e^{i\rho\cos\varphi}$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\cos\varphi}e^{-im\varphi}d\varphi \Rightarrow \text{proportional to } J_m$$

Taylor expansion

$$J_m(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+m}}{2^{m+2n}(m+n)!\,n!}$$

## 3.4 Properties of $J_m$

(1) $J_m(0) = \begin{cases} 1, & m=0 \\ 0, & m\neq 0 \end{cases}$

Pf (integral formula)

(2) $J_m(x) = \frac{x}{2m}[J_{m-1}(x) + J_{m+1}(x)]$, $m\in\mathbb{N}$

Pf
$$J_{m-1}+J_{m+1} = \frac{1}{2\pi i^m}\left(i\int_{-\pi}^{\pi} e^{ix\cos\varphi}e^{-i(m-1)\varphi} - i\int_{-\pi}^{\pi}e^{ix\cos\varphi}e^{-i(m+1)\varphi}\right)$$
$$= \frac{1}{2\pi i^m}\left(i\int_{-\pi}^{\pi} e^{ix\cos\varphi}\underbrace{(ie^{i\varphi}-ie^{-i\varphi})}_{=2(\cos\varphi)'}e^{-im\varphi}\right)$$
$$= \frac{2}{ix}\frac{1}{2\pi i^m}\int_{-\pi}^{\pi}(e^{ix\cos\varphi})'e^{-im\varphi}d\varphi$$
$$= \underbrace{\left[e^{ix\cos\varphi}e^{-im\varphi}\right]_{-\pi}^{\pi}}_{=e^{-ix}-e^{-ix}=0} - \int_{-\pi}^{\pi}e^{ix\cos\varphi}(e^{-im\varphi})'$$
$$= \frac{2m}{x}\left(\frac{1}{2\pi i^m}\int_{-\pi}^{\pi}e^{-ix\cos\varphi}e^{-im\varphi}\right) \quad \square$$

(3) $J_m'(x) = \frac{1}{2}[J_{m-1}(x) - J_{m+1}(x)]$, $m=0,1,\dots$

Pf
$$\overset{\shortparallel}{\frac{1}{2\pi i^m}}\int_{-\pi}^{\pi}\frac{d}{dx}(e^{ix\cos\varphi})e^{-im\varphi}d\varphi$$
$$= \frac{1}{2\pi i^m}\int_{-\pi}^{\pi}e^{ix\cos\varphi}\,i\,\frac{e^{i\varphi}+e^{-i\varphi}}{2}e^{-im\varphi}d\varphi$$
$$= \frac{1}{2}\left(\frac{1}{2\pi i^{m-1}}\int_{-\pi}^{\pi}e^{-ix\cos\varphi}e^{-i(m-1)\varphi}d\varphi - \frac{1}{2\pi i^{m+1}}\int_{-\pi}^{\pi}e^{ix\cos\varphi}e^{-i(m+1)\varphi}d\varphi\right)$$
$$= \frac{1}{2}(J_{m-1}(x) - J_{m+1}(x)) \quad \square$$

(4) $\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x)$, $m=1,2,\dots$

Pf
$$\overset{\shortparallel}{m}x^{m-1}J_m + x^m J_m'$$
$$= mx^{m-1}\frac{x}{2m}(J_{m-1}+J_{m+1}) + \frac{x^m}{2}(J_{m-1}-J_{m+1})$$
$$= x^m J_{m-1}$$

(5) $\frac{d}{dx}[x^{-m}J_m(x)] = -x^{-m}J_{m+1}(x)$, $m=0,1,\dots$

$$\overset{\shortparallel}{-m}x^{-m-1}J_m + x^{-m}J_m'$$
$$= -mx^{-m-1}\frac{x}{2m}(J_{m-1}+J_{m+1}) + \frac{x^{-m}}{2}(J_{m-1}-J_{m+1})$$

## 3.5 Orthogonality

We know $R(x) = J_m(\sqrt{\lambda} x)$ is sol to $x(xR')' + (\lambda x^2 - m^2)R = 0$

But if together with B.C. $R(1)\cos\beta + R'(1)\cos\beta + R'(1)\sin\beta = 0$

$\Rightarrow J_m(\sqrt{\lambda})\cos\beta + \sqrt{\lambda} J_m'(\sqrt{\lambda})\sin\beta = 0$

Def $\{x_n^{(m)}\}_{n\in\mathbb{N}}$ are nonneg sol to $\boxed{J_m(x_n^{(m)})\cos\beta + x_n^{(m)} J_m'(x_n^{(m)})\sin\beta = 0}$

$\Rightarrow$ ① $\int_0^1 J_m(x x_{n_1}^{(m)}) J_m(x x_{n_2}^{(m)})\, x\, dx = 0, \ \forall n_1 \neq n_2$

② $\int_0^1 J_m(x x_n^{(m)})^2 x\, dx = \begin{cases} \frac{1}{2} J_{m+1}(x_n^{(m)})^2, & \beta = 0 \\ \frac{x_n^2 - m^2 + \cot^2\beta}{2x_n^2} J_m(x_n^{(m)})^2, & 0 < \beta \leq \frac{\pi}{2} \end{cases}$

## 3.6 Fourier-Bessel expansion

of pw-sm on $x \in (0,1)$

Let $J_m$ be a Bessel, and $\{x_n^{(m)}\}$ be nonneg sols.

of $J_m(x)\cos\beta + x J_m'(x)\sin\beta = 0$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} A_n J_m(x x_n^{(m)}), \ x \in (0,1)$ $\left(\to \frac{1}{2}[f(x+0) + f(x-0)]\right)$

$$A_n = \frac{\int_0^1 f(x) J_m(x x_n^{(m)})\, x\, dx}{\int_0^1 J_m(x x_n^{(m)})^2 x\, dx}$$

ex $f(x) = 1, \ 0 < x < 1, \quad m = 0, \beta = 0$ (eq: $J_0(x) = 0$)

$\Rightarrow 1 = \sum_{n=1}^{\infty} A_n J_0(x x_n^{(0)}), \ A_n = \dfrac{\int_0^1 J_0(x x_n^{(0)})\, dx}{\int_0 J_0(x x_n^{(0)})\, x\, dx}$

$(t = x x_n^{(0)}) = \dfrac{\frac{1}{x_n^2}\int_0^{x_n} t J_0(x)\, dt}{\int_0^1 \underbrace{J_0(x x_n^{(0)})^2 x\, dx}_{=\frac{1}{2}J_1(x_n)^2}} = \dfrac{2}{x_n J_1(x_n)}$ $\quad = (t J_1(x))'$ by (4)

## vibrating mebrane in cylinder

$$\begin{cases} u_{tt} = c^2 \Delta u \\ u(\rho = a, \varphi, t) = 0 \\ u(\rho, \varphi, 0) = 1, \\ u_t(\rho, \varphi, 0) = 0 \end{cases} \overset{u = R\Phi T}{\Longrightarrow} \underbrace{\frac{1}{c^2}\frac{T''}{T}}_{=-\lambda} = \underbrace{\frac{R'' + \frac{1}{\rho}R'}{R} + \frac{1}{\rho^2}\frac{\Phi_{\varphi\varphi}}{\Phi}}_{=\lambda}$$

$$\Downarrow$$

$$\underbrace{\frac{\rho^2 R'' + \rho R'}{R} - \lambda\rho^2}_{=\mu} = \underbrace{\frac{\Phi_{\varphi\varphi}}{\Phi}}_{=-\mu} \Rightarrow \begin{cases} \frac{1}{c^2}\frac{T''}{T} = -\lambda \\ \frac{\rho^2 R'' + \rho R'}{R} - \lambda\rho^2 = \mu \\ -\frac{\Phi_{\varphi\varphi}}{\Phi} = \mu \end{cases}$$

$\Rightarrow$ ① $\Phi'' + \mu\Phi = 0, \ \Phi(-\pi) = \Phi(\pi)$

② $R'' + \frac{1}{\rho}R' + (\lambda - \frac{\mu}{\rho^2})R = 0$, $R(a) = 0$ here

③ $T'' + \lambda c^2 T = 0$

① $\Rightarrow \Phi_m = A\cos m\varphi + B\sin m\varphi, \ \mu_m = m^2$

(Fix m), put $\mu_m$ into ②: $R'' + \frac{1}{\rho}R' + (\lambda - \frac{m^2}{\rho^2})R = 0$

sol is exactly $R(\rho) = J_m(\rho\sqrt{\lambda})$

$R(a) = 0 \Rightarrow a\sqrt{\lambda}$ is picked from nonneg roots of $\underset{(\beta=0)}{J_m(x) = 0}$

Let $\{x_n^{(m)}\}$ be nonneg roots of $J_m(x) = 0$

$\Rightarrow \sqrt{\lambda}_{m,n} = \frac{x_n^{(m)}}{a}$

③ $\Rightarrow T = C\cos c\sqrt{\lambda}t + D\sin c\sqrt{\lambda}t$

$\Rightarrow T_{mn} = C\cos\left(\frac{ct x_n^{(m)}}{a}\right) + D\sin\left(\frac{ct x_n^{(m)}}{a}\right)$

$\Rightarrow u_{mn}(\rho, \varphi, t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\left(\cos\frac{ct x_n^{(m)}}{a} + D\sin\frac{ct x_n^{(m)}}{a}\right)$

$u_t(\rho, \varphi, 0) = 0 \Rightarrow D = 0$

$$\Rightarrow u(\rho, \varphi, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\cos m\varphi + B_{mn} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\sin m\varphi\right]\cos\frac{ct x_n^{(m)}}{a}$$

bring into $u(\rho, \varphi, 0) = 1 \Rightarrow 1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(A_{mn}J_m(\cdots)\cos m\varphi + B_{mn}J_m(\cdots)\sin m\varphi)$

we multiply both sides by $J_m\left(\frac{\rho x_n^{m}}{a}\right)\cos m\varphi$, take integral over $[-\pi, \pi]$

Since $\boxed{\int_{-\pi}^{\pi}\cos mx\, dx = \int_{-\pi}^{\pi}\sin mx\, dx = 0 \text{ if } m \neq 0} \Rightarrow A_{mn}, B_{mn} \neq 0$ iff $m = 0$

$\Rightarrow u(\rho, \varphi, t) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\cos\frac{ct x_n^{(0)}}{a}$

Take into $u(\rho, \varphi, 0) = 1 \Rightarrow 1 = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\rho x_n^{(0)}}{a}\right)$

Since $1 = 2\sum_{n=1}^{\infty}\frac{J_0(x x_n^{(0)})}{x_n^{(0)} J_1(x_n^{(0)})} \Rightarrow A_n = \frac{2}{x_n^{(0)} J_1(x_n^{(0)})}$

$\Rightarrow u = \sum_{n=1}^{\infty}\frac{2}{x_n^{(0)} J_1(x_n^{(0)})} J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\cos\frac{ct x_n^{(0)}}{a}$

## heat eq. in cylinder.

$$\begin{cases} u_t = K\nabla^2 u + \sigma, & t>0,\ 0\le\rho<a,\ -\pi\le\varphi\le\pi \\ u(a,\varphi,t) = T_1, & t>0,\ -\pi\le\varphi\le\pi \\ u(\rho,\varphi,0) = T_2, & 0\le\rho<a,\ -\pi\le\varphi\le\pi \end{cases}$$

Step 1 $u = v(\rho,\varphi,t) + U(\rho)$

Find steady state sol $U(\rho)$ st $K\Delta U + \sigma = u_t = 0$

$K\Delta U = K\left(U'' + \frac{1}{\rho}U'\right) = -\sigma \Rightarrow U'' + \frac{1}{\rho}U' = -\frac{\sigma}{K}$

$\Rightarrow U' = -\frac{\sigma}{2K}\rho + \frac{\text{const}}{\rho} \Rightarrow U = -\frac{\sigma}{4K}\rho^2 + C_1\ln\rho + C_2$

To bound $\ln\rho \Rightarrow C_1 = 0$

$U(a) = T_1 \Rightarrow C_2 = T_1 + \frac{\sigma}{4K}a^2 \Rightarrow U(\rho) = T_1 + \frac{\sigma}{4K}(a^2-\rho^2)$

Then we solve for $v(\rho,\varphi,t)$: $\begin{cases} v_t = K\Delta v \\ v(a,\varphi,t) = 0 \\ v(\rho,\varphi,0) = T_2 - T_1 \end{cases}$

By Separation: $\frac{1}{K}\frac{T'}{T} = -\lambda,\ \frac{\rho^2R''+\rho R'}{R} - \lambda\rho^2 = \mu,\ \frac{\Phi''}{\Phi} = -\mu$

$$\Rightarrow \begin{cases} \Phi'' + \mu\Phi = 0,\ \Phi(-\pi) = \Phi(\pi),\ \Phi'(-\pi) = \Phi'(\pi) \\ R'' + \frac{1}{\rho}R' + \left(\lambda - \frac{\mu}{\rho^2}\right)R = 0,\ R(a) = 0 \\ T' + \lambda K T = 0 \Rightarrow T = \exp\left(-\frac{(x_n^{(m)})^2 K t}{a^2}\right) \end{cases}$$

(Same as drumhead, except $T'$ 而非 $T''$, $T_2 - T_1$ 而非 $1$, $k$ 而非 $c^2$)

$$\Rightarrow u = T_1 + \frac{\sigma}{4K}(a^2-\rho^2) + \sum_{n=1}^{\infty} \frac{2(T_2-T_1)}{x_n^{(0)} J_1(x_n^{(0)})} J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\exp\left(-\frac{(x_n^{(0)})^2 K t}{a^2}\right)$$