Question 1

Find the asymptotic function for Question 4 in the Homework Problem Set 5 and justify your answer.

Solution.

$$u(t,z) \to U(z) \quad \text{where} \quad U(z) \text{ is the steady solution.}$$

$$\lim_{t\to\infty} u(z,t) = T_1 + \phi_2 z + 0 = U(z)$$

Question 2

Find the solution of the nonhomogeneous heat equation

$$u_t = K u_{zz} + v e^{-at} \sin \frac{\pi z}{L}, \quad 0 < z < L, \quad t > 0,$$

with $u(0,t) = u(L,t) = u(z,0) = 0$. Here a, v, K are positive constants.

Solution. If $a \neq \frac{\pi^2 K}{L^2}$, then

$$u(z,t) = -v \sin \frac{\pi z}{L} \frac{e^{-at} - e^{-\frac{\pi^2 K t}{L^2}}}{a - \frac{\pi^2 K}{L^2}}.$$

If $a = \frac{\pi^2 K}{L^2}$, then

$$u(z,t) = v \sin \frac{\pi z}{L} t e^{-\frac{\pi^2 K t}{L^2}}.$$

Step 1 Solve $\begin{cases} U_{zz}(z,t) = 0 \Rightarrow U(z,t) = A(t) + B(t) z \\ U(0,t) = 0 \\ U(L,t) = 0 \end{cases}$

$\Rightarrow A(t) = 0, \ B(t)\cdot L = 0 \Rightarrow B(t) = 0 \Rightarrow U(z,t) = 0$

$\Rightarrow U_t(z,t) = 0, \ U(z,0) = 0$

$F(z) = f(z)$

Step 2

Define $v(z,t) = r(z,t) - U(z,t)$

$R(z,t) = r(z,t) - U_t(z,t) = r(z,t) = v e^{-at} \sin \frac{\pi z}{L}$

$F(z) = f(z) - U(z,0) = f(z) = 0$

We solve $\begin{cases} V_t = K V_{zz} + V^{-\alpha t} \sin \frac{\pi z}{L} \\ V(0,t) = 0 \\ V(L,t) = 0 \\ V(z,0) = 0 \end{cases}$ for $V(z,t)$

Step 3 We compute the eigenvalue & eigenfunctions of

$$A: \varphi \mapsto -\partial_{zz}\varphi, \quad \mathrm{Dom}\, A = \{\varphi(z) \mid \varphi(0) = \varphi(L) = 0\}$$

Here $\rho(x) = 1$, $s(x) = 1$,

$$SL: A\varphi = \lambda \varphi$$

$$\Rightarrow -\varphi'' = \lambda \varphi, \quad \underbrace{\varphi'' + \lambda \varphi = 0}_{\begin{cases} \varphi(0) = 0 \\ \varphi(L) = 0 \end{cases}}$$

$$\Rightarrow \varphi_n(z) = \sin \frac{n\pi z}{L}, \quad \varphi_n''(z) = \left(\frac{n\pi}{L}\right)^2 \varphi$$

$$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Step 4 expand $R = Ve^{-\alpha t} \sin \frac{\pi z}{L}$ and $F = 0$ by eigenfunctions

We have $V(z,t) = \sum_{n=1}^{\infty} V_n(t) \varphi_n(z)$

$$R(z,t) = \sum_{n=1}^{\infty} R_n(t) \varphi_n(z)$$

$$F(z) = \sum_{n=1}^{\infty} F_n(t) \varphi_n(z)$$

where $R_n(t) = \dfrac{\int_0^L \left(Ve^{-\alpha t} \sin \frac{\pi z}{L}\right) \sin \frac{n\pi z}{L}\, dz}{\int_0^L \left(\sin \frac{n\pi z}{L}\right)^2 dz}$, $\quad F_n = 0$

$$= \begin{cases} Ve^{-\alpha t}, & n = 1 \\ 0, & n \neq 1 \end{cases}$$ by orthogonality

$$\Rightarrow V_n(t) = F_n e^{-\lambda_n K t} + \int_0^t R_n(s) e^{-\lambda_n K(t-s)} ds$$

$$= \int_0^t R_n(s) e^{-\lambda_n K(t-s)} ds$$

$$= \begin{cases} 0, & n \neq 1 \\ \int_0^t v e^{-at - K(\frac{\pi}{L})^2 (t-s)} ds, & n = 1 \end{cases}$$

$$\Rightarrow \underline{u(z,t)} = v(z,t) + 0 = \sum_{n=1}^{\infty} V_n(t) \varphi_n(z)$$

$$= \underline{V_1(t) \sin \frac{\pi z}{L}}$$

$$a = \frac{\pi^2 K}{L^2} \Rightarrow V_1(t) = vt e^{-\frac{\pi^2 K t}{L^2}}$$

$$a \neq \frac{\pi^2 K}{L^2} \Rightarrow V_1(t) = \frac{v}{\frac{\pi^2 K}{L^2} - a}\left(e^{-at} - e^{-\frac{\pi^2 K t}{L^2}}\right)$$

Therefore $u(z,t) = \begin{cases} vt e^{-\frac{\pi^2 K t}{L^2}} \sin\left(\frac{\pi z}{L}\right), & \text{if } a \neq \frac{\pi^2 K}{L^2} \\ \frac{v}{\frac{\pi^2 K}{L^2} - a}\left(e^{-at} - e^{-\frac{\pi^2 K t}{L^2}}\right) \sin\left(\frac{\pi z}{L}\right), & \text{if } a \neq \frac{\pi^2 K}{L^2} \end{cases}$

Question 3

Find the solution of the nonhomogeneous heat equation

$$u_t = u_{xx} + \frac{1}{2}e^{-t}, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = \frac{1}{2}e^{-t}, \quad u(1,t) = \frac{1}{2}e^{-t},$$
$$u(x,0) = x + \frac{1}{2}.$$

Solution.

$$u(x,t) = \frac{1}{2}e^{-t} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t} \sin n\pi x + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)} \left(e^{-t} - e^{-n^2\pi^2 t}\right) \sin n\pi x.$$

Step 1 Solve $\begin{cases} U_{xx}(x,t) = 0, \ 0<x<1 \Rightarrow U(x,t) = A(t)x + B(t) \\ U(0,t) = \frac{1}{2}e^{-t} \Rightarrow B(t) = \frac{1}{2}e^{-t} \\ U(1,t) = \frac{1}{2}e^{-t} \Rightarrow A(t) + B(t) = \frac{1}{2}e^{-t} \Rightarrow A(t) = 0 \end{cases}$

$\Rightarrow U(x,t) = \frac{1}{2}e^{-t}$

Step 2 Define: $v(x,t) = u(x,t) - U(x,t) = u(x,t) - \frac{1}{2}e^{-t}$

$R(x,t) = r(x,t) - U_t(x,t) = e^{-t}$

$F(x) = u(x,0) - U(x,0) = x + \frac{1}{2} - \frac{1}{2} = x$

We solve $\begin{cases} v_t = v_{xx} + e^{-t}, \ 0<x<1, \ t>0 \\ v(0,t) = v(1,t) = 0, \ t>0 \\ v(x,0) = x, \ 0<x<1 \end{cases}$

Step 3 Define $A: \varphi \mapsto -\varphi''$

Consider SL problem $\begin{cases} \varphi'' + \lambda\varphi = 0, \ 0<x<1 \\ \varphi(0) = \varphi(1) = 0 \end{cases}$

We get $\varphi_n(x) = \sin(n\pi x) \Rightarrow \lambda_n = (n\pi)^2$

Step 4 expand $v(x,t)$, $R(x,t)$, $F(x)$ by eigenfunction

$$v(x,t)=\sum_{n=1}^{\infty} v_n(t)\varphi_n(x), \quad R(x,t)=\sum_{n=1}^{\infty} R_n(t)\varphi_n(x),$$

$$F(x)=\sum_{n=1}^{\infty} F_n\varphi_n(x)$$

$$\text{where } F_n=\frac{\int_0^1 F(x)\varphi_n(x)\,dx}{\int_0^1 \varphi_n^2(x)\,dx} = 2\int_0^1 x\sin(n\pi x)\,dx$$

$$\int_0^1 x\sin(n\pi x)\,dx = \frac{-x\cos(n\pi x)}{n\pi}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{\cos(n\pi)}{-n\pi} + \frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1$$

$$= -\frac{(-1)^n}{n\pi}$$

$$\Rightarrow F_n = \frac{2(-1)^{n+1}}{n\pi}$$

$$\text{Similarly, } R_n(t) = 2e^{-t}\int_0^1 \sin(n\pi x)\,dx = \frac{2(1-(-1)^n)e^{-t}}{n\pi}$$

$$\text{So } v_n(t)= F_n e^{-\lambda_n k t} + \int_0^t R_n(s) e^{-\lambda_n k(t-s)}\,ds$$

$$= \frac{2(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t} + \int_0^t e^{-(n\pi)^2(t-s)}\,ds$$

$$= \frac{2(-1)^{n+1} e^{-n^2\pi^2 t}}{n\pi} + \frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)}\left(e^{-t}-e^{-n^2\pi^2 t}\right)$$

Therefore

$$u(x,t)= U(x,t)+v(x,t) = U(x,t)+\sum_{n=1}^{\infty} v_n(t)\varphi_n(x)$$

$$= \frac{1}{2}e^{-t} + \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1} e^{-n^2\pi^2 t}}{n\pi} + \frac{2(1-(-1)^n)}{n\pi(n^2\pi^2-1)}\left(e^{-t}-e^{-n^2\pi^2 t}\right)\right)$$

Question 4

The energy of a vibrating string of tension T_0 and density $\rho = \frac{m}{L}$ is defined by

$$E = \frac{1}{2}\int_0^L \left(\rho y_t^2 + T_0 y_s^2\right) ds.$$

Let

$$y(s,t) = \sum_{n=1}^{\infty} \left(A_n \cos\omega_n t + B_n \sin\omega_n t\right) \sin\frac{n\pi s}{L}$$

be a solution of the wave equation with $\omega_n = \frac{n\pi c}{L}$, where $c^2 = \frac{T_0}{\rho}$. Show that E is independent of t (conservation of energy) by using Parseval's theorem to write E as an infinite series involving A_n, B_n.

Solution.

$$E = \frac{L}{4}\sum_{n=1}^{\infty}\left(\rho\omega_n^2 B_n^2 + T_0\left(\frac{n\pi}{L}\right)^2 A_n^2\right).$$

Pf $\quad y_t = \sum_{n=1}^{\infty}\left(-A_n\omega_n \sin(\omega_n t) + B_n\omega_n\cos(\omega_n t)\right)\sin\left(\frac{n\pi s}{L}\right)$

By Parseval's Thm for Forier sine series.

① $\frac{1}{L}\int_0^L y_t^2 ds = \frac{1}{2}\sum_{n=1}^{\infty}\left(-A_n\omega_n\sin(\omega_n t) + B_n\omega_n\cos(\omega_n t)\right)^2$

$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\omega_n^2 A_n^2\sin^2(\omega_n t) + \omega_n^2 B_n^2\cos^2(\omega_n t) - \underbrace{2A_nB_n\sin\omega_n t\cos\omega_n t}_{=0 \text{ by orthogonality}}\right)$

$= \frac{1}{2}\sum_{n=1}^{\infty}\omega_n^2\left(A_n^2\sin^2(\omega_n t) + B_n^2\cos^2(\omega_n t)\right)$

$y_s = \sum_{n=1}^{\infty}\left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)\frac{n\pi}{L}\cos\frac{n\pi s}{L}$

By Parseval's Thm for Forier cosine series.

② $\frac{1}{L}\int_0^L y_s^2 ds = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)^2$

$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2\left(A_n^2\cos\omega_n t + B_n^2\sin\omega_n t\right)$

$\Rightarrow E = \frac{L}{2}\left(\rho ① + T_0 ②\right) = \frac{L}{4}\sum_{n=1}^{\infty}\left(T_0\left(\frac{n\pi}{L}\right)^2 A_n^2 + \rho\omega_n^2 B_n^2\right)$

Question 5

Consider the following initial-value problem for the wave equation $y_{tt} = c^2 y_{ss}$ for $t > 0$, $0 < s < L$ with $y(0,t) = y(L,t) = 0$ for $t > 0$ and $y(s,0) = 0$, $y_t(s,0) = 1$ for $0 < s < L$. Find the Fourier representation of the solution.

Solution.

$$y(s,t) = \frac{2L}{\pi^2 c} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \sin\frac{n\pi s}{L} \sin\frac{n\pi ct}{L}.$$

Sol Suppose $y(s,t) = X(s)T(t)$

$$\Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \text{ for some } \lambda$$

① $X''(s) + \lambda X(s) = 0$, ② $T''(t) + c^2\lambda T(t) = 0$

For ①, by the boundary conditions $X(0) = X(L) = 0$

$$\Rightarrow X_n(s) = \sin\left(\frac{n\pi s}{L}\right),\ \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Then for ② we have $T_n(t) = A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct)$

$$= A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

By $T(0) = 0 \Rightarrow A_n = 0$

$$\Rightarrow y(s,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi s}{L}\right)$$

$$y_t(s,t) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi s}{L}\right)$$

$$y_t(s,0) = 1 \Rightarrow \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\frac{n\pi s}{L} = 1$$

Let $m \in \mathbb{N}$. Multiply both sides by $\sin\frac{m\pi s}{L}$ and integrate $\Rightarrow$

$$\int_0^L y_t(s,0) \sin\frac{m\pi s}{L}\, ds = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \int_0^L \sin\frac{n\pi s}{L} \sin\frac{m\pi s}{L}$$

By orthogonality $\Rightarrow \frac{L}{2} B_m \frac{m\pi c}{L} = \int_0^L \sin\left(\frac{m\pi s}{L}\right) ds$

$$= \frac{L}{m\pi}(1-(-1)^m)$$

$$\Rightarrow B_m = \frac{2L}{(m\pi)^2 c}(1-(-1)^m)$$

Thus $$y(s,t) = \sum_{n=1}^{\infty} \frac{2L}{(n\pi)^2 c} (1-(-1)^n) \sin\frac{n\pi s}{L} \sin\frac{n\pi ct}{L}.$$

$$= \frac{2L}{\pi^2 c} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \sin\frac{n\pi s}{L} \sin\frac{n\pi ct}{L}$$