Question 1

Find the solution of the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 1,\ t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = x(1-x), \quad 0 < x < 1,$$
$$u_t(x,0) = 0, \quad 0 < x < 1.$$

Solution.

$$u(x,t) = \sum_{k=1, k \text{ odd}}^{\infty} \frac{8}{k^3\pi^3} \sin(k\pi x)\cos(ck\pi t).$$

Sol Suppose $u = \varphi(x)T(t)$

$\Rightarrow \varphi T'' = c^2 T \varphi''$

$\frac{\varphi''}{\varphi} = c^2 \frac{T''}{T} = -\lambda$ for some λ

① $\begin{cases} \varphi'' + \lambda\varphi = 0 \\ \varphi(0) = \varphi(L) = 0 \end{cases}$, ② $\begin{cases} T'' + \lambda c^2 T = 0 \\ T'(0) = 0 \end{cases}$

By ① we have $\varphi_n(x) = \sin(\sqrt{\lambda_n} x)$, $\lambda_n = (\frac{n\pi}{L})^2 = (n\pi)^2$

Then By ② we have $T_n(t) = C_1 \sin(\sqrt{\lambda_n} ct) + C_2 \cos(\sqrt{\lambda_n} ct)$

By $T'(0) = 0 \Rightarrow C_1 = 0 \Rightarrow T_n(t) = \cos(\sqrt{\lambda_n} ct)$
(omitting coeff)

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \cos(n\pi ct)\sin(n\pi x)$

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = x(1-x) \Rightarrow B_n = \frac{2}{L}\int_0^L x(1-x)\sin(n\pi x)\,dx$

$B_n = 2\left(\int_0^1 x\sin(n\pi x)dx - \int_0^1 x^2 \sin(n\pi x)dx\right)$

$= 2\left(\frac{(-1)^{n+1}}{\pi n} - \left(\frac{-1}{n\pi} + \frac{2}{(n\pi)^3}\right)(-1)^n + \frac{2}{(n\pi)^3}\right) = \frac{4}{(n\pi)^3}\left(-(-1)^n + 1\right)$

Therefore $u(x,t) = \sum_{n=1, n \text{ odd}}^{\infty} \frac{8}{(n\pi)^3} \cos(n\pi ct)\sin(n\pi x)$

Question 2

Compute the multidimensional sine Fourier series of $f(x,y) = xy$, $x, y \in [-\pi,\pi] \times [-\pi,\pi]$.

Solution.

$$f(x,y) = \sum_{n,m=1}^{\infty} \frac{4}{nm}(-1)^{n+m}\sin(nx)\sin(my).$$

We only need to consider $x \in [0,\pi]$ and $y \in [0,\pi]$

Since $\forall x \in [-\pi,0]$, $f(x,y) = -f(-x,y)$

$\forall y \in [-\pi,0]$, $f(x,y) = -f(x,-y)$

So $\forall x,y \in [0,\pi]$, $f(x,y) = \sum_{n=1}^{\infty} B_{mn} \sin(mx)\sin(ny) = xy$

where $B_{mn} = \frac{4}{\pi^2}\int_0^\pi \int_0^\pi xy \sin(mx)\sin(ny)\,dx\,dy$

$= \frac{4}{\pi^2}\int_0^\pi x\sin mx\,dx \int_0^\pi y \sin ny\,dy$

$= \frac{4}{\pi^2}\frac{1}{nm}(1-(-1)^m)\pi(1-(-1)^n)\pi$

$= \frac{4}{nm}(1-(-1)^m)(1-(-1)^n)$

Therefore $f(x,y) = \sum_{n,m=1}^{\infty} \frac{4}{nm}(-1)^{n+m}\sin(mx)\sin(ny)$

Question 3

Solve the initial-value problem for the heat equation $u_t = K\Delta u$ in the column $0 < x < L_1$, $0 < y < L_2$ with the boundary conditions $u(0,y,t) = 0$, $u_x(L_1,y,t) = 0$, $u(x,0,t) = 0$, $u_y(x,L_2,t) = 0$, and the initial condition $u(x,y,0) = 1$.

Solution.

$$u(x,y,t) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty} u_{mn}(x,y,t),$$

where

$$u_{mn}(x,y,t) = \sin\left(\left(m - \frac{1}{2}\right)\frac{\pi x}{L_1}\right)\frac{1}{m-\frac{1}{2}}\sin\left(\left(n-\frac{1}{2}\right)\frac{\pi y}{L_2}\right)\frac{1}{n-\frac{1}{2}}e^{-\lambda_{mn}Kt}.$$

Here,

$$\lambda_{mn} = \left(m - \frac{1}{2}\right)^2\left(\frac{\pi}{L_1}\right)^2 + \left(n - \frac{1}{2}\right)^2\left(\frac{\pi}{L_2}\right)^2.$$

Sol suppose $u = X(x)Y(y)T(t)$

$u_t = K\Delta u \Rightarrow XYT' = K(X''Y + XY'')T$

$\Rightarrow \frac{T'}{KT} = \frac{X''}{X} + \frac{Y''}{Y}$

$\Rightarrow \frac{T'}{KT} = -\lambda$, $\frac{X''}{X} = -\lambda_X$, $\frac{Y''}{Y} = -\lambda_Y$ for some const $\lambda = \lambda_X + \lambda_Y$

$\Rightarrow$ ① $T = e^{-\lambda Kt}$, ② $X'' + \lambda_X X = 0$, ③ $Y'' + \lambda_Y Y = 0$

②: $X_n = A_n \sin(\sqrt{\lambda_{X_n}} x) + B\cos(\sqrt{\lambda_{X_n}} x)$

By $u(0,y,t) = 0 \Rightarrow X(0) = 0 \Rightarrow B_n = 0 \Rightarrow X_n = A_n \sin(\sqrt{\lambda_{X_n}} x)$

And by $u_x(L_1,y,t) = 0 \Rightarrow \sqrt{\lambda_{X_n}} L_1 = (n-\frac{1}{2})\pi$

$\lambda_{X_n} = (n-\frac{1}{2})^2(\frac{\pi}{L_1})^2$

③: $Y_m = C_m \sin(\sqrt{\lambda_{Y_m}} x) + D_m \cos(\sqrt{\lambda_{Y_m}} x)$

By $u(y,0,t) = 0 \Rightarrow Y_m(0) = 0 \Rightarrow D_m = 0 \Rightarrow Y_m = C_m \sin(\sqrt{\lambda_{Y_m}} x)$

By $u_y(y,L_2,t) = 0 \Rightarrow \lambda_{Y_m} = (m-\frac{1}{2})^2(\frac{\pi}{L_2})^2$

$\Rightarrow u(x,y,t) = \sum_{m,n=1} a_{mn} \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\frac{1}{2})\frac{\pi y}{L_2}\right)e^{-\lambda_{mn}Kt}$

Apply the initial condition:

$1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\frac{1}{2})\frac{\pi y}{L_2}\right)$

$\Rightarrow a_{mn} = \frac{4}{L_1L_2}\int_0^{L_1}\int_0^{L_2} 1 \cdot \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\frac{1}{2})\frac{\pi y}{L_2}\right)dx\,dy$

$\Rightarrow a_{mn} = \left(\int_0^{L_1} \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)dx\right)\left(\int_0^{L_2}\sin\left((m-\frac{1}{2})\frac{\pi x}{L_2}\right)dy\right)$

$= 4\frac{1}{(m-\frac{1}{2})\pi}\frac{1}{(n-\frac{1}{2})\pi} = \frac{4}{\pi^2}\frac{1}{(m-\frac{1}{2})(n-\frac{1}{2})}$

$\left(\text{Since } \int_0^{L_1} \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)dx = \int_0^{(n-\frac{1}{2})\pi} \sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)d\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right) = [-\cos t]_0^{(n-\frac{1}{2})\pi} = 1\right)$

$\Rightarrow u(x,y,t) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty} \frac{1}{(n-\frac{1}{2})(m-\frac{1}{2})}\sin\left((n-\frac{1}{2})\frac{\pi x}{L_1}\right)\sin\left((m-\frac{1}{2})\frac{\pi x}{L_2}\right)e^{-\lambda_{mn}Kt}$

where $\lambda_{mn} = \lambda_{X_n} + \lambda_{Y_m} = (n-\frac{1}{2})^2(\frac{\pi}{L_1})^2 + (m-\frac{1}{2})^2(\frac{\pi}{L_2})^2$

Question 4

Find the solution of the homogeneous wave equation

$$u_{tt} = u_{xx} + u_{yy}, \quad t > 0,\ (x,y) \in [0,\pi]\times[0,\pi],$$
$$u(0,y,t) = 0, \quad u(\pi,y,t) = 0, \quad t>0,\ y\in[0,\pi],$$
$$u(x,0,t) = 0, \quad u(x,\pi,t) = 0, \quad t>0,\ x\in[0,\pi],$$
$$u(x,y,0) = xy, \quad u_t(x,y,0) = 0, \quad x,y \in [0,\pi]\times[0,\pi].$$

Solution.

$$u(x,y,t) = \sum_{n,m=1}^{\infty} \frac{4}{nm}(-1)^{n+m}\cos\left(\sqrt{n^2+m^2}\,t\right)\sin(nx)\sin(my).$$

Sol Suppose $U = XYT$

$$\Rightarrow XYT'' = X''YT + XY''T$$
$$\Rightarrow \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \text{ for some const } \lambda$$
$$\Rightarrow \begin{cases} X'' + \lambda_x X = 0, \\ X(0) = X(\pi) = 0 \end{cases} \quad \begin{cases} Y'' + \lambda_y Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases}$$
$$\Rightarrow X_n = \sin(nx), \quad Y_m = \sin(my)$$
$$\lambda_n = n^2 \qquad \lambda_m = m^2$$

So $T_{nm}(t) = A_{nm}\cos\left(\sqrt{n^2+m^2}\,t\right) + B_{nm}\sin\left(\sqrt{n^2+m^2}\,t\right)$

$$U_t(x,y,t) = XYT' = \sum_{m,n=1}^{\infty}\left(-A_{nm}\sqrt{n^2+m^2}\sin\left(\sqrt{n^2+m^2}\,t\right) + B_{nm}\sqrt{n^2+m^2}\cos\left(\sqrt{n^2+m^2}\,t\right)\right)\sin(nx)\sin(my)$$

$U_t(x,y,0) = 0 \Rightarrow B_{nm} = 0$

$$U(x,y,0) = xy \Rightarrow \sum_{m,n=1}^{\infty} A_{nm}\cos\left(\sqrt{n^2+m^2}\,t\right)\sin(nx)\sin(my) = xy$$

$$A_{nm} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi xy\sin(nx)\sin(my)\,dx\,dy$$
$$= \frac{4}{\pi^2}\int_0^\pi x\sin(nx)\,dx \int_0^\pi y\sin(my)\,dy = \frac{4}{\pi^2}$$
$$= \frac{4}{\pi^2}\left(\frac{\pi^2}{n}(-1)^{n+1}\right)\left(\frac{\pi^2}{m}(-1)^{m+1}\right) = \frac{4}{nm}(-1)^{n+m}$$

$$\Rightarrow U(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{4}{nm}(-1)^{n+m}\cos\left(\sqrt{n^2+m^2}\,t\right)\sin(nx)\sin(my)$$

Question 5

Find the solution of the homogeneous wave equation

$$u_{tt} = u_{xx} + u_{yy}, \quad t>0,\ (x,y)\in[0,\pi]\times[0,\pi],$$
$$u_x(0,y,t) = 0, \quad u_x(\pi,y,t) = 0, \quad t>0,\ y\in[0,\pi],$$
$$u(x,0,t) = 0, \quad u(x,\pi,t) = 0, \quad t>0,\ x\in[0,\pi],$$
$$u(x,y,0) = x(\pi-x)y, \quad u_t(x,y,0) = 0, \quad x,y\in[0,\pi]\times[0,\pi].$$

Solution.

$$u(x,y,t) = -\sum_{m=1}^{\infty}\frac{\pi^2(-1)^m}{3m}\cos(mt)\sin(my) + \sum_{n,m=1}^{\infty}\frac{4(1+(-1)^n)(-1)^m}{n^2 m}\cos\left(\sqrt{n^2+m^2}\,t\right)\cos(nx)\sin(my).$$

Sol Suppose $U = XYT$

$$\Rightarrow XYT'' = X''YT + XY''T$$
$$\Rightarrow \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \text{ for some const } \lambda$$
$$\Rightarrow \begin{cases} X'' + \lambda_x X = 0, \\ X'(0) = X'(\pi) = 0 \end{cases} \quad \begin{cases} Y'' + \lambda_y Y = 0 \\ Y(0) = Y(\pi) = 0 \end{cases}$$
$$\Rightarrow X_n = \cos(nx), \qquad Y_m = \sin(my)$$
$$\lambda_n = n^2, \qquad \lambda_m = m^2,\ m\in\mathbb{N}$$
$$n \in \mathbb{N}\cup\{0\}$$

$$\Rightarrow T_{nm}(t) = A_{nm}\cos\left(\sqrt{n^2+m^2}\,t\right) + B_{nm}\sin\left(\sqrt{n^2+m^2}\,t\right)$$

$\to$ note as w_{nm}

$$\text{So } u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(A_{nm}\cos(w_{nm}t) + B_{nm}\sin(w_{nm}t)\right)\cos nx \sin my$$

$$u_t(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(-A_{nm}w_{nm}\sin(w_{nm}t) + B_{nm}w_{nm}\cos(w_{nm}t)\right)\cos nx\sin my$$

$u_t(x,y,0) = 0 \Rightarrow B_{nm} = 0 \quad \forall n,m$

Now we determine A_{nm}:

$$x(\pi-x)y = \sum_{n,m=1}^{\infty} A_{nm}\cos nx\sin my$$

$$\text{where } A_{nm} = \frac{\int_0^\pi\int_0^\pi x(\pi-x)y\cos nx\sin my\,dx\,dy}{\int_0^\pi\cos^2 nx\,dx\int_0^\pi\sin^2 my\,dy}$$

$$\int_0^\pi \cos^2 nx\,dx = \begin{cases}\pi, & n=0\\ \frac{\pi}{2}, & n\geq 1\end{cases}, \qquad \int_0^\pi \sin^2 my\,dy = \frac{\pi}{2}$$

for $n=0$, $\int_0^\pi\int_0^\pi x(\pi-x)y\cos nx\sin my\,dx\,dy$

$$= \left(\int_0^\pi x(\pi-x)\,dx\right)\left(\int_0^\pi y\sin my\,dy\right)$$
$$= \frac{\pi^3}{6}\left(\frac{-\pi(-1)^m}{m}\right)$$
$$\Rightarrow A_{0m} = \frac{\frac{\pi^3}{6}\left(-\frac{\pi(-1)^m}{m}\right)}{\frac{1}{2}\pi^2} = \frac{-\pi^2(-1)^m}{3m}$$

for $n\geq 1$, $\int_0^\pi\int_0^\pi x(\pi-x)y\cos nx\sin my\,dx\,dy$

$$= \left(\int_0^\pi x(\pi-x)\cos nx\,dx\right)\left(\int_0^\pi y\sin my\,dy\right)$$
$$= \left(-\pi\left(\left(\frac{\pi}{n\pi}\right)^2 - (-1)^n\left(\frac{\pi}{n\pi}\right)^2 - \frac{2\pi^3}{(n\pi)^2}(-1)^n\right)\right)\left(\frac{\pi^2(-1)^{m+1}}{m\pi}\right)$$
$$= \left(-\pi\frac{1}{n^2}(1-(-1)^n) - \frac{2\pi}{n^2}(-1)^n\right)\frac{\pi}{m}(-1)^{m+1}$$
$$= \left(\frac{-\pi}{n^2} - \frac{\pi}{n^2}(-1)^n\right)\frac{\pi}{m}(-1)^{m+1} = \frac{-\pi^2(1+(-1)^n)(-1)^{m+1}}{n^2 m}$$

$$\Rightarrow A_{nm} = \frac{4}{\pi^2}\,\frac{-\pi^2(1+(-1)^n)(-1)^{m+1}}{n^2 m} = \frac{4(1+(-1)^n)(-1)^m}{n^2 m}$$

Therefore

$$u(x,y,t) = -\sum_{m=1}^{\infty}\frac{\pi^2(-1)^m}{3m}\cos mt\sin my + \sum_{n,m=1}^{\infty}\frac{4(-1)^n(-1)^m}{n^2 m}\cos\left(\sqrt{n^2+m^2}\,t\right)\cos nx\sin my$$