# Solving 1st order PDE by MoC

linear (no u)

## linear homo

ex: $a = x+1$, $b = -(y+1)$

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = 0 \\ u(x,0) = f(x) \end{cases}$$

homo

ex: $f(x) = x+1$

**Sol** **Step 1** Characteristic curve

$$\begin{cases} \dfrac{d\tilde{x}}{ds} = a(\tilde{x},\tilde{y}) = \tilde{x}+1 \\ \dfrac{d\tilde{y}}{ds} = b(\tilde{x},\tilde{y}) = -(\tilde{y}+1) \end{cases} \Rightarrow \begin{cases} \tilde{x}(s) = \ldots = -1+Ae^{s}, & \tilde{x}(0) = x_0 \\ \tilde{y}(s) = \ldots = -1+Be^{-s}, & \tilde{y}(0) = 0 \end{cases}$$

$\int \frac{1}{y+1} = \int 1$

$-\ln|y+1| = -s$

$y+1 = Be^{-s}$

$A = x_0+1$

$B = 1$

**Step 2** for $(x,y)$ on the curve

$$\begin{cases} x = x(s_1) = -1+(x_0+1)e^{s_1} \\ y = y(s_1) = -1+e^{-s_1} \end{cases} \Rightarrow x_0 = \ldots(x,y)$$

$(x+1)(y+1) = x_0+1$

$\Rightarrow x_0 = (x+1)(y+1)-1$

$$u(x,y) = u(x_0,0) = f(x_0) = f(\ldots(x,y))$$

$$= x_0+1 = (x+1)(y+1)$$

**Thm** u is const on one characteristic curve

Linear $\begin{cases} \overset{\gamma}{a(x,t)}u_x + \overset{1}{b(x,t)}u_t + \overset{\alpha}{c(x,t)}u = 0 & \text{for ex} \\ u(x,0) = f(x) & \sin x \text{ for ex} \end{cases}$

Step 1 Characteristic curve:

$$\begin{cases} \frac{d\tilde{x}}{ds} = a^{\gamma} \\ \frac{d\tilde{t}}{ds} = b^{1} \end{cases} \Rightarrow \begin{cases} \tilde{x} = \overset{\gamma s + A}{\cdots(s)}, \text{ and } \tilde{x}(0) = x_0 & \Rightarrow A = x_0 \quad \underline{\tilde{x} = \gamma s + x_0} \\ \tilde{t} = \underset{s+B}{\cdots(s)}, \text{ and } \tilde{t}(0) = 0 & \Rightarrow B = 0 \quad \underline{\tilde{t} = s} \end{cases}$$

On curve $x = \underset{= \gamma s_1 + x_0}{\tilde{x}(s_1)}$, $t = \underset{= s_1}{\tilde{t}(s_1)}$ $\Rightarrow$ express $x_0, s$

$x_0 = x - \gamma s_1 = x - \gamma t$, $s = t$

we have $\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = b\,u_t + a\,u_x = -cu$
$= -\alpha u$

$$\Rightarrow \begin{cases} \frac{du}{ds} + cu = 0, & s > 0 \\ u = f(x_0), & s = 0 \end{cases} \Rightarrow \text{solve } u(s)$$

$u(s) = \sin(x_0)e^{-\alpha s}$

$\Rightarrow u(x,t) = \sin(x - \gamma t)e^{-\alpha t}$

(if $c = 0$, then

$u(x,t) = f(x_0) = \sin(x_0) = \sin(x - \gamma t)$

Quasi-linear

$$\begin{cases} \underbrace{a(x,y,u)}_{} u_x + \underbrace{b(x,y,u)}_{} u_y = \underbrace{c(x,y,u)}_{} \\ u(x, V(x)) = f(x) \end{cases}$$

ex $a = u^2$, $b = 1$, $c = u$; ex $u(x,0) = x$

Step 1
$$\begin{cases} \frac{d\tilde{x}}{ds} = a \quad (\tilde{u}^2) \\ \frac{d\tilde{y}}{ds} = b \quad (1) \\ \frac{d\tilde{u}}{ds} = c \quad (\tilde{u}) \end{cases} \qquad \begin{cases} \tilde{x}(0) = x_0 \\ \tilde{y}(0) = y_0 \quad (0) \\ \tilde{u}(0) = f(x_0) \quad (x_0) \end{cases} \Rightarrow \begin{cases} \tilde{x}(s) = x_0 + \frac{1}{2}x_0^2(e^{2s}-1) \\ \tilde{y}(s) = s \\ \hat{u}(s) = x_0 e^{s} \end{cases}$$

on curve
$$\begin{cases} x = \tilde{x}(s_1) \\ y = \tilde{y}(s_1) \\ u = \hat{u}(s_1) \end{cases} \Rightarrow \begin{cases} x = x_0 + \frac{1}{2}x_0^2(e^{2s_1}-1) \\ y = s_1, \quad u = x_0 e^{s_1} \end{cases}$$

$$\Rightarrow x_0^2 + \frac{2}{e^{2y}-1}x_0 - \frac{1}{e^{2y}-1}x = 0$$

$$x_0 = \frac{-2(e^{2y}-1) + \sqrt{\frac{4}{(e^{2y}-1)^2} + 4\frac{x}{e^{2y}-1}}}{2} \quad \left(\frac{-b \pm \sqrt{\Delta}}{2a}, \text{ judge } \pm\right)$$

$$x_0 = \frac{-1 + \sqrt{1 + x(e^{2y}-1)}}{e^{2y}-1}$$

$$u = x_0 e^{s_1}$$
$$= \sim e^{y}$$

Thm Let $\tilde{u}(s) = u(\tilde{x}, \tilde{y}) \Rightarrow \frac{d\tilde{u}}{ds} = c(\tilde{x}, \tilde{y}, \tilde{u})$

Pf $\frac{d\tilde{u}}{ds} = u_x \frac{d\tilde{x}}{ds} + u_y \frac{d\tilde{y}}{ds} = u_x a + u_y b = c(\tilde{x}, \tilde{y}, u(\tilde{x}, \tilde{y}))$