**Question 1.** Find the steady-state (time independent) solution of the heat equation $u_t = K u_{zz}$ in the slab $0 < z < L$, with boundary conditions $[u_z - h(u - T_0)](0) = 0$ and $[u_z + h(u - T_1)](L) = 0$. Assume that $K, h, T_0, T_1$ are all positive constants.
**Solution.** $u(x, y, z) = U(z) = \frac{T_1(1+hz)+T_0[1+h(L-z)]}{2+hL}$.

Sol Time independent $\Rightarrow U_t = 0, \forall t \Rightarrow K u_{zz} = 0$

So $U(z) = Az + B$ for some $A, B$

$\Rightarrow U_z = A$

When $z = 0$, $u = B$

$\Rightarrow A - h(B - T_0) = 0$ ①

When $z = L$, $u = LA + B$

$\Rightarrow A + h(LA + B - T_1) = 0$ ②

Combining ①② we have

$$(2 + hL)A - h(T_1 - T_0) = 0 \Rightarrow A = \frac{h(T_1 - T_0)}{2 + hL}$$

$$\Rightarrow \frac{h(T_1 - T_0)}{2 + hL} + hT_0 = hB \Rightarrow B = \frac{T_1 - T_0 + (2 + hL)T_0}{2 + hL}$$

$$= \frac{T_0 + T_1 + T_0 hL}{2 + hL}$$

$$\Rightarrow u = \frac{h(T_1 - T_0)z + (T_0 + T_1 + T_0 hL)}{2 + hL} = \frac{(1 + hz)T_1 + (1 + hL - hz)T_0}{2 + hL}$$

**Question 2.** Solve the initial-value problem $u_t = K u_{zz} (K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u(0, t) = u(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.
**Solution.** $u(z, t) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi z}{L} \exp\left[-\left(\frac{n\pi}{L}\right)^2 Kt\right]$.

Sol Let $u = Z(z)T(t) = K u_{zz}$

$\Rightarrow ZT' = KTZ''$

$$\frac{T'}{KT} = \frac{Z''}{Z} = -\lambda \Rightarrow \begin{cases} T' + \lambda K T = 0 \Rightarrow T = Ce^{-\lambda K t}, t > 0 \\ Z'' + \lambda Z = 0 \end{cases}$$

$$\Rightarrow Z = \begin{cases} A\cos(\sqrt{\lambda} z) + B\sin(\sqrt{\lambda} z), \lambda > 0 \\ Az + B, \lambda = 0 \\ Ae^{\sqrt{-\lambda} z} + Be^{-\sqrt{-\lambda} z}, \lambda < 0 \end{cases}$$

For $\lambda < 0$:

$u(0, t) = 0 \Rightarrow A + B = 0$
$u(L, t) = 0 \Rightarrow A = B = 0$
$\Rightarrow u = 0 \Rightarrow$ contradicts with initial condition

For $\lambda = 0$. $A, B = 0 \Rightarrow u = 0 \Rightarrow$ contradicts with initial condition

For $\lambda > 0$:

$u(0, t) = 0 \Rightarrow A = 0$

$u(L, t) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

$$\Rightarrow u = \sum_{n=0}^{\infty} B_n \sin_n e^{-\left(\frac{n\pi}{L}\right)^2 Kt}$$

Since $u(z, 0) = z$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L z \sin\frac{n\pi z}{L} dz = -\frac{2}{n\pi}\int_0^L z \, d\left(\cos\frac{n\pi z}{L}\right)$$

$$= \left[-\frac{2}{n\pi} z \cos\frac{n\pi z}{L}\right]_0^L + \frac{2}{n\pi}\int_0^L \cos\frac{n\pi z}{L} dz$$

$$= \frac{2L}{-n\pi}(-1)^n - 0 + \underbrace{\left[\dots \sin\frac{n\pi z}{L}\right]_0^L}_{=0}$$

$$= -\frac{2L}{n\pi}(-1)^n$$

$\Rightarrow$ Above all, the solution to this BVT is

$$u(z, t) = \sum_{n=1}^{\infty} \frac{-2L}{n\pi}(-1)^n \sin\frac{n\pi z}{L} e^{-Kt\left(\frac{n\pi}{L}\right)^2}$$

**Question 3.** Solve the initial-value problem $u_t = K u_{zz} (K > 0)$ for $t > 0, 0 < z < L$, with the boundary conditions $u_z(0, t) = u_z(L, t) = 0$ and the initial condition $u(z, 0) = z$, $0 < z < L$.
**Solution.** $u(z, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos[(2n-1)\pi z/L]}{(2n-1)^2} \exp\left[-\frac{(2n-1)^2\pi^2 Kt}{L^2}\right]$.

Sol General (separated) solution solved before:

$$u = \begin{cases} (A\cos(\sqrt{\lambda} z) + B\sin(\sqrt{\lambda} z))e^{-K\lambda t}, \lambda > 0 \\ Az + B, \lambda = 0 \\ (Ae^{\sqrt{-\lambda} z} + Be^{-\sqrt{-\lambda} z})e^{-K\lambda t}, \lambda > 0 \end{cases}$$

For $\lambda > 0$:

$u_z(0, t) = 0 \Rightarrow B\sqrt{\lambda} e^{-\lambda K t} = 0 \ \forall t \Rightarrow B = 0$

$u_z(L, t) = 0 \Rightarrow A\sqrt{\lambda} \sin\sqrt{\lambda} z \, e^{-\lambda K t} = 0 \ (\forall t)$

$\Rightarrow \sqrt{\lambda} A \sin(L\sqrt{\lambda}) e^{-\lambda K t} = 0 \ \forall t \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2$

For $\lambda = 0$, $u_z(0, t) = 0, \forall z \Rightarrow A = 0$,
$u(z, 0) = z, \forall z \Rightarrow A = 1$
$\Rightarrow$ contradicts, impossible

For $\lambda < 0$, $u_z(z, t) = \left(A\sqrt{-\lambda} e^{\sqrt{-\lambda} z} - B\sqrt{-\lambda} e^{-\sqrt{-\lambda} z}\right)e^{-K\lambda t}$

$u_z(0, t) = \sqrt{-\lambda} A - \sqrt{-\lambda} B = 0 \ \forall t \Rightarrow A = B$

$u(z, 0) = 0 \ \forall z \Rightarrow Ae^{\sqrt{-\lambda} z} + Ae^{-\sqrt{-\lambda} z} = 0 \ \forall z \Rightarrow A = 0$

$\Rightarrow u = 0 \ \forall z, t \Rightarrow$ contradicts with initial condition

Thus only the $\lambda > 0$ case holds, with $u(z, 0) = \sum_{n=0}^{\infty} A_n \cos_n = z$

We calculate the Fourier cosine series:

$$A_n = \frac{2}{L}\int_0^L z \cos\frac{n\pi z}{L} dz \qquad A_0 = \frac{1}{L}\int_0^L x \, dx = \frac{L}{2}$$

$$= \frac{2}{n\pi}\int_0^L z \, d\sin\left(\frac{n\pi z}{L}\right)$$

$$= \underbrace{\frac{2}{n\pi}\left[z \sin\frac{n\pi z}{L}\right]_0^L}_{=0} - \frac{2}{n\pi}\int_0^L \sin\frac{n\pi z}{L} dz$$

$$= \frac{2L}{n^2\pi^2}\left[\cos\frac{n\pi z}{L}\right]_0^L = \frac{2L}{n^2\pi^2}((-1)^n - 1)$$

$$\text{Thus } u(z, t) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}((-1)^n - 1)\exp\left(-\frac{n^2\pi^2 Kt}{L^2}\right)$$

$$= \frac{L}{2} + \frac{4L}{\pi^2}\sum_{m=1}^{\infty} \frac{\cos\frac{(2m-1)\pi z}{L}}{(2m-1)^2}\exp\left(-\frac{(2m-1)^2\pi^2 Kt}{L^2}\right)$$

**Question 4.** Let $\varphi_1 = 1, \varphi_2 = x, \varphi_3 = x^2$ on the interval $0 \le x \le 1$. Compute the following quantities
1) $\langle \varphi_1, \varphi_2 \rangle$,
2) $\langle \varphi_1, \varphi_3 \rangle$,
3) $\|\varphi_1 - \varphi_2\|^2$,
4) $\|\varphi_1 + 3\varphi_2\|^2$.

**Solution.** 1) 1/2, 2) 1/3, 3) 1/3, 4) 7.

1) $\langle \varphi_1, \varphi_2 \rangle = \int_0^1 \varphi_1(x)\varphi_2(x) dx = \int_0^1 x \, dx = \frac{1}{2}$

2) $\langle \varphi_1, \varphi_3 \rangle = \int_0^1 \varphi_1(x)\varphi_3(x) dx = \int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

3) $\|\varphi_1 - \varphi_2\|^2 = \int_0^1 (x - 1)^2 dx = \left[\frac{1}{3}x^3 - x^2 + x\right]_0^1 = \frac{1}{3}$

4) $\|\varphi_1 + 3\varphi_2\|^2 = \int_0^1 (3x + 1)^2 dx = \left[3x^3 + 3x^2 + x\right]_0^1 = 7$

**Question 5.** Check if the following operator is symmetric on its domain with respect to given inner product.
1) $A = -\frac{d^2}{dx^2} + 1$ on domain $\{\varphi(x) : \varphi(0) = \varphi(L) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx$.
2) $A = -\frac{d^2}{dx^2} + 1$ on domain $\{\varphi(x) : \varphi(0) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)dx$.
3) $A = \frac{d}{dx}$ on domain $\{\varphi(x) : \varphi(0) = \varphi(L) = 0\}$. $\langle \varphi, \psi \rangle = \int_0^L \varphi(x)\psi(x)x dx$.

**Solution.** 1) True, 2) False, 3) False. For 1) try integration by parts as in the class. For 2), 3) try to find counter-examples.

1) True

$A\varphi = -\frac{d^2\varphi}{dx^2} + \varphi$, $A\psi = -\frac{d^2\psi}{dx^2} + \psi$

$$\Rightarrow \langle A\varphi, \psi \rangle = -\int_0^L \frac{d^2\varphi}{dx^2}\psi \, dx + \int_0^L \varphi\psi \, dx$$

$$= -\int_0^L \psi \, d\left(\frac{d\varphi(x)}{dx}\right) + \int_0^L \varphi\psi \, dx$$

$$= \left[-\psi(x)\frac{d\varphi(x)}{dx}\right]_{x=0}^{x=L} + \int_0^L \frac{d\varphi}{dx} d\psi + \int_0^L \varphi\psi \, dx$$

$$= \underbrace{-\psi(L)\frac{d\varphi}{dx}(L) + \psi(0)\frac{d\varphi}{dx}(0)}_{=0 \text{ since } \psi(0) = \psi(L) = 0} + \int_0^L \frac{d\varphi}{dx}\frac{d\psi}{dx} dx + \int_0^L \varphi\psi \, dx$$

$$\langle \varphi, A\psi \rangle = -\int_0^L \frac{d^2\psi}{dx^2}\varphi \, dx + \int_0^L \psi\varphi \, dx$$

$$= -\int_0^L \varphi \, d\left(\frac{d\psi}{dx}\right) + \int_0^L \varphi\psi \, dx$$

$$= \underbrace{\left[-\varphi(x)\frac{d\psi}{dx}\right]_{x=0}^{x=L}}_{=0 \text{ since } \varphi(0)=\varphi(L)=0} + \int_0^L \frac{d\psi}{dx}\frac{d\varphi}{dx}dx + \int_0^L \varphi\psi\, dx$$

$$= \int_0^L \frac{d\psi}{dx}\frac{d\varphi}{dx}dx + \int_0^L \varphi\psi\, dx = \langle A\varphi, \psi\rangle$$

Thus it is symmetric.

2) False

By calculation in 1) we already have

$$\langle \varphi, A\psi\rangle - \langle A\varphi, \psi\rangle = -\varphi(L)\frac{d\psi}{dx}(L) + \varphi(0)\frac{d\psi}{dx}(0) - \left(-\psi(L)\frac{d\varphi}{dx}(L) + \psi(0)\frac{d\varphi}{dx}(0)\right)$$

$$= -\varphi(L)\frac{d\psi}{dx}(L) + \psi(L)\frac{d\varphi}{dx}(L) \quad \text{since } \varphi(0)=\psi(0)=0$$

Consider $\varphi(x)=x^2$, $\psi(x)=x$

$\Rightarrow \langle \varphi, A\psi\rangle - \langle A\varphi, \psi\rangle = -L^2 + 2L^2 = L^2 \neq 0$, so $\langle \varphi, A\psi\rangle \neq \langle A\varphi, \psi\rangle$.

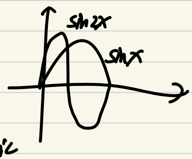
3) False

$\langle A\varphi, \psi\rangle = \int_0^L (x\varphi'\psi)dx$, $\langle \varphi, A\psi\rangle = \int_0^L (x\psi'\varphi)dx$

Consider $L=\pi$, $\varphi(x)=\sin x$, $\psi(x)=\sin 2x$ which satisfies the domain

$\langle A\varphi, \psi\rangle = \int_0^\pi x\cos x \sin 2x\, dx = \frac{2}{3}\pi$

$\langle \varphi, A\psi\rangle = \int_0^\pi 2x\cos 2x \sin x\, dx = -\frac{2}{3}\pi$




This counterex. suffices to show that $A$ is not symmetric

**Question 6.** Convert the following ODE into Sturm-Liouville form and write the $s(x)$, $\rho(x)$ and $q(x)$ functions.

1) $y'' + 2xy' + \lambda y = 0$.
2) $x^2y'' + xy' + (\lambda x^2 - 1)y = 0$.
3) $y'' + \frac{1}{x}y' + \lambda y = 0$.

**Solution.** 1) $(e^{x^2}y')' + \lambda e^{x^2}y = 0$, $s(x) = e^{x^2}$, $\rho(x) = e^{x^2}$ and $q(x) = 0$. 2) $(xy')' + (\lambda x - \frac{1}{x})y = 0$, $s(x) = x$, $\rho(x) = x$ and $q(x) = \frac{1}{x}$. 3) $(xy')' + \lambda xy = 0$, $s(x) = x$, $\rho(x) = x$ and $q(x) = 0$.

Recall Sturm-Liouville form: $\frac{d}{dx}\left(s(x)\frac{dy}{dx}\right) + (\lambda\rho(x) - q(x))y = 0$

1) Note that $\frac{d}{dx}(e^{x^2}y') = e^{x^2}y'' + 2xe^{x^2}y'$

So we set $s(x) = e^{x^2}$

Multiply the ODE by $e^{x^2}$ on both sides $\Rightarrow e^{x^2}y'' + 2xe^{x^2}y' + \lambda e^{x^2}y = 0$

Then the ODE becomes $\frac{d}{dx}\left(e^{x^2}\frac{dy}{dx}\right) + \lambda e^{x^2}y = 0$

$\Rightarrow \rho(x) = e^{x^2}, q(x) = 0$

2) Dividing both sides by $x \Rightarrow xy'' + y' + (\lambda x - \frac{1}{x})y = 0$

Note that $\frac{d}{dx}(xy') = xy'' + y'$

So set $s(x) = x$

So the ODE becomes $\frac{d}{dx}(xy') + (\lambda x - \frac{1}{x})y = 0$

So $\rho(x) = x$, $q(x) = \frac{1}{x}$

3) multiply both sides by $x \Rightarrow xy'' + y' + \lambda xy = 0$

Note that $\frac{d}{dx}(xy') = xy'' + y'$

Then the ODE becomes $\frac{d}{dx}(xy') + \lambda xy = 0$

$\Rightarrow \rho(x) = x, q(x) = 0$