5.1 Fourier trans

$$f(x)=\sum_{n=-\infty}^{\infty}\alpha_n e^{i\frac{n\pi}{L}x},\quad \alpha_n=\frac{1}{2L}\int_{-L}^{L}f(x)e^{-i\frac{n\pi}{L}x}\quad \frac{n\pi}{L}\to\mu_n,\ \Delta\mu_n=\frac{\pi}{L}$$

Let $\tilde f_L(\mu_n)=\frac{\alpha_n}{n\pi/L}=\frac{1}{2\pi}\int_{-L}^{L}f(x)e^{-i\mu x}dx$

$$\Rightarrow f(x)=\sum_{n=-\infty}^{\infty}\tilde f_L(\mu_n)e^{i\mu_n x}\Delta\mu_n\quad \boxed{L\to\infty\Rightarrow f(x)=\int_{-\infty}^{\infty}\tilde f(\mu)e^{i\mu x}d\mu}$$

where $\boxed{\hat f(\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-i\mu x}dx}$

Parseval: $\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)^2dx=\int_{-\infty}^{\infty}|\hat f(\mu)|^2d\mu$

ex

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(x-m)^2}{2\sigma^2}},\quad \tilde f(\mu)=\frac{1}{2\pi}e^{-im\mu}e^{-\frac{\mu^2\sigma^2}{2}}$$

$$f(x)=1=\lim_{\sigma\to\infty}e^{\frac{-x^2}{2\sigma^2}},\quad \tilde f(\mu)=\delta(\mu)=\begin{cases}\infty, & \mu=0\\ 0, & \mu\neq 0.\end{cases}$$

$\int_{-\infty}^{\infty}\delta(x)p(x)dx=p(0)$

在积分中选出 $x=0$ 处值

5.2

Heat eq on inf rod by Fourier trans

$$\begin{cases}u_t=Ku_{xx}, & t>0,\ -\infty<x<\infty\\ u(x,0)=f(x), & -\infty<x<\infty\end{cases}\quad \Rightarrow f(x)=\int_{-\infty}^{\infty}\tilde u(\mu,0)e^{i\mu x}d\mu$$

Step 1 Fourier trans:

$$u(x,t)=\int_{-\infty}^{\infty}\tilde u(\mu,t)e^{i\mu x}d\mu,\quad \tilde u(\mu,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}u(x,t)e^{-i\mu x}dx$$

$$\Rightarrow u_t(x,t)=\int_{-\infty}^{\infty}\tilde u_t(\mu,t)e^{-i\mu x}d\mu,\quad u_{xx}(x,t)=\int_{-\infty}^{\infty}\tilde u(\mu,t)(-\mu^2)e^{i\mu x}d\mu$$

$$\Rightarrow \widehat{u_{xx}}=(i\mu)^2\hat u \quad \Leftarrow \text{note: } \widehat{u_x}=i\mu\hat u$$

$$\Rightarrow \frac{\partial\hat u(\mu,t)}{\partial t}=-K\mu^2\hat u(\mu,t)$$

$$\Rightarrow \hat u(\mu,t)=e^{(-K\mu^2)t}\hat u(\mu,0)=e^{-K\mu^2t}\tilde f(\mu)$$

$$\Rightarrow u(x,t)=\int_{-\infty}^{\infty}\hat f(\mu)e^{-K\mu^2t}e^{i\mu x}d\mu \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)e^{-i\mu y}dy$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)\int_{-\infty}^{\infty}\exp\left[-Kt\left(\mu^2-\frac{2i(x-y)}{2Kt}-\frac{(x-y)^2}{4K^2t^2}\right)-\frac{(x-y)^2}{4Kt}\right]d\mu\, dy$$

$\left(\mu-\frac{i(x-y)}{2Kt}\right)^2$

$$=\sqrt{\frac{\pi}{Kt}}\frac{1}{2\pi}\int_{-\infty}^{\infty}f(y)e^{-\frac{(x-y)^2}{4Kt}}dy$$

note: $\int_{-\infty}^{\infty}e^{-a(x-b)^2}dx=\sqrt{\frac{\pi}{a}}$

$$=\frac{1}{\sqrt{4\pi Kt}}\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4Kt}}f(y)dy$$

5.2 变体

$$\begin{cases}u_t=Ku_{xx}+Du_x, & t>0,\ -\infty<x<\infty\\ u(x,0)=f(x), & -\infty<x<\infty\end{cases}$$

$$\hat u_t(\mu,t)=K(i\mu)^2\hat u(\mu,t)+Di\mu\hat u(\mu,t)$$

$$=(-K\mu^2+Di\mu)\hat u(\mu,t)$$

$$\hat u(\mu,t)=e^{(-K\mu^2+Di\mu)t}\hat u(\mu,0)=e^{(-K\mu^2+Di\mu)t}\hat f(\mu)$$

$$\Rightarrow u(x,t)=\int_{-\infty}^{\infty}e^{(-K\mu^2+Di\mu)t+i\mu x}\hat f(\mu)d\mu$$

$$=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-K\mu^2t+i(Dt+x-y)\mu}d\mu\, f(y)dy$$

$$-K\mu^2t+i(Dt+x-y)\mu=-Kt\left(\mu^2-\frac{i}{Kt}(Dt+x-y)\mu\right)$$

$$=-Kt\left(\mu-\frac{i}{2Kt}(Dt+x-y)\right)^2-\frac{1}{4Kt}(Dt+x-y)^2$$

$$\Rightarrow u(x,t)=\underbrace{\frac{1}{2\pi}\sqrt{\frac{\pi}{Kt}}}_{=\frac{1}{\sqrt{4\pi Kt}}}e^{-\frac{1}{4Kt}(Dt+x-y)^2}f(y)dy$$

5.3 method of image

$$\begin{cases}u_t=Ku_{xx}\\ u(0,t)=0\\ u(x,0)=f(x)\end{cases}\qquad \begin{cases}u_t=Ku_{xx}\\ u_x(0,t)=0\\ u(x,0)=f(x)\end{cases}$$

extend $f_O(x)=\begin{cases}f(x)\\ 0\\ -f(-x)\end{cases}$; extend $f_E(x)=\begin{cases}f(x)\\ f(-x)\end{cases}$

$$\Rightarrow\begin{cases}u_t=Ku_{xx}\\ u(x,0)=f_O(x)\end{cases}\qquad \Rightarrow\begin{cases}u_t=Ku_{xx}\\ u(x,0)=f_E(x)\end{cases}$$

$$\xRightarrow{\text{both}} u(x,t)=\frac{1}{\sqrt{4\pi Kt}}\left(\int_{-\infty}^{0}+\int_{0}^{\infty}\right)e^{\frac{-(x-y)^2}{4Kt}}f_{O/E}(y)dy$$

$$=\int_0^{\infty}\frac{\exp\left(\frac{-(x-y)^2}{4Kt}\right)+\exp\left(\frac{-(x-y)^2}{4Kt}\right)}{\sqrt{4\pi Kt}}f(y)dy$$

5.4 d'Alembert's formula for wave eq.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t>0, -\infty<x<\infty \\ u(x,0) = f_1(x), & -\infty<x<\infty \\ u_t(x,0) = f_2(x), & -\infty<x<\infty \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{u}_{tt} + c^2\mu^2\tilde{u} = 0 \\ \tilde{u}(\mu,0) = \tilde{f}_1(\mu) \\ \tilde{u}_t(\mu,0) = \tilde{f}_2(\mu) \end{cases}$$

Since $\tilde{u}(\mu,t) = A(\mu)\cos(\mu ct) + B(\mu)\sin(\mu ct)$

By initial cond $A(\mu) = \tilde{f}_1(\mu)$, $B(\mu) = \frac{\tilde{f}_2(\mu)}{\mu c}$

$$u(x,t) = \int_{-\infty}^{\infty} \left[\tilde{f}_1(\mu)\cos(\mu ct) + \tilde{f}_2(\mu)\frac{\sin(\mu ct)}{\mu c}\right] e^{i\mu x}\, d\mu$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}_1(\mu)\left[e^{i\mu(x+ct)} + e^{i\mu(x-ct)}\right] d\mu + \frac{1}{2c}\int_{-\infty}^{\infty} \hat{f}_2(\mu)\int_{x-ct}^{x+ct} e^{i\mu y}\, dy\, d\mu$$

$$= \frac{1}{2}\left[f_1(x+ct) + f_1(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} f_2(y)\, dy$$

5.5 Green's function

$$G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}}$$

heat eq with $u(x,0) = f(x)$

soled with $u(x,t) = \int_{-\infty}^{\infty} G(x,y,t) f(y)\, dy$

Solving $\begin{cases} u_t - ku_{xx} = h(x,t), & 0<t<T, x\in\mathbb{R} \\ u(x,t) = 0, & x\in\mathbb{R} \end{cases}$

$$u(x,t) = \int_{-\infty}^{\infty} G(x,y,t) f(y)\, dy + \int_0^t \int_{-\infty}^{\infty} G(x,y,t-s) h(y,s)\, dy\, ds$$

Solving $\begin{cases} u_t - ku_{xx} = h(x,t), & 0<t<T, 0<x<\infty \\ ① u(0,t) = 0 \ (\text{or } u_x(0,t) = 0 ②) \\ u = f(x), & t=0, 0<x<\infty \end{cases}$

To solve for ① (Dirichlet's BC)

Define $G_D(x,y,t) = G(x,y,t) - G(x,-y,t)$

$$u(x,t) = \int_0^{\infty} G_D(x,y,t) f(y)\, dy + \int_0^t \int_0^{\infty} G_D(x,y,t-s) h(y,s)\, dy\, ds$$

To solve for ② (Neumann's BC)

Define $G_N(x,y,t) = G(x,y,t) + G(x,-y,t)$

$\Rightarrow$ same except G_N 代替 G_D

Solving $\begin{cases} u_t - ku_{xx} = 0, & 0<t<T, 0<x<L \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

Define $G_L(x,y,t) = \sum_{m=-\infty}^{\infty} \left(G(x,y+2mL,t) - G(x,-y+(2m+2)L,t)\right)$

$$\Rightarrow u(x,t) = \int_0^L G_L(x,y,t) f(y)\, dy$$

(if non-homo with $h(x,t)$ then

$u(x,t) = \int_0^L G_L(x,y,t) f(y)\, dy + \int_0^t \int_0^L G_L(x,y,t-s) h(y,s)\, dy\, ds$